Math 347 HW 9

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Exercise 3.6.31.

Show that a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Proof. <= is proved in Theorem 3.6.27 (Heine-Borel)

We now prove =>. If $S = \varnothing$ the proof is trivial. $S^C = \mathbb{R}$ is obviously open and S is vacuously bounded. Then let $S \subseteq \mathbb{R}$ be a non empty compact set. First we show that S is bounded. Pick some point $x \in S$ and consider $A = \bigcup_{i \in \mathbb{N}} B_i(x)$. A is an open cover of S. Since S is compact we can pick a finite subcover $A' = \bigcup_{i=1}^{n} B_i(x)$. Then S is bounded above by $x + n$ and bounded below by $x - n$.
Now we show that S is closed or equivalently, S^c is open. Pick some point $a \in S^c$. We can construct an open cover of S as follows. Define $r_b = \frac{|a-b|}{2}$ for a point $b \in S$. Now $\bigcup_{b \in S} B_{r_b}(b)$ is an open cover of S. By compactness of S we can pick a finite subcover. Consider the set $A = \{r_b | B_{r_b}(b) \in \text{our finite subcover}\}$ and $r = min(A)$. This is justified since A is finite. Then $B_r(a)$ is a neighborhood of a that is completely contained in S^c. Since a was chosen arbirartily S^c is open $\implies$ S is closed. ∎

Exercise 3.6.32.

A subset of R is compact if and only if it is sequentially compact.

Proof. => Suppose $S \subseteq \mathbb{R}$ is a compact set. By exercise 3.6.31, S is closed and bounded. Let $(a_k)_{k \in \mathbb{N}}$ be a sequence in S. By lemma 3.6.10, $(a_k)_{k \in \mathbb{R}}$ has a convergent subsequence we call (a_{k_n}). We aim to show that (a_{k_n}) converges to an element in S. Let a be the limit of (a_{k_n}). Then the definition of convergence states $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t $\forall n \geq N |a_{k_n} - a| < \epsilon$ Then a is an accumulation point of S. But S is closed iff S contains all of its accumulation points. Thus S contains a and (a_{k_n}) converges to $a \in S$

<= Suppose $S \subseteq \mathbb{R}$ is a sequentially compact set i.e every sequence in S has a subsequence that converges to an element in S. Assume S is not bounded to derive a contradiction. Pick some element $a_1 \in S$. Since S is not bounded $\exists a_2 \in S, |a_2 - a_1| > 2$. Continue selecting a sequence $(a_n)_{n \in \mathbb{N}}$ such that $|a_n - a_{n-1}| > n$ Then there is a convergent subsequence $(a_{n_k}) \to a$. Since (a_{n_k}) is convergent, it is also Cauchy. $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t $\forall n, m \geq N, |a_{n_k} - a_{m_k}| < \epsilon$ But this is a contradiction since $|a_{n_k} - a_{m_k}|$ is unbounded.
Now we show that S is closed. Assume not to derive a contradiction. Then S does not contain all of its acculumation points. Let x be an accumulation point of S that is not in S. Then $\forall \epsilon > 0, B_\epsilon(x) \backslash \{x\} \cap S \neq \varnothing$ Pick a sequence (a_k) as follows. $a_1 = p \in B_1(x) \backslash \{x\} \cap S$. Then pick $a_2 = p \in B_{1/2}(x) \backslash \{x, a_1\} \cap S$ Continuing inductively we select a sequence (a_k) that has a subsequence $(a_{k_n}) \to a \in S^c$. Every convergent subsequence has the same limit, therefore (a_k) does not have a subsequence that converges in S. But this contradicts our assumption that S is sequentially compact. Thus S must be closed.

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Problem 3

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Suppose $(a_k)_{k \in \mathbb{N}} \to a$. Prove $(f(a_k)) \to f(a)$.

Proof. Fix $\epsilon > 0$. Then by definition of continuity

$$\exists \delta > 0 \text{ s.t } a_k \in B_\delta(a) \implies f(a_k) \in B_\epsilon(f(a))$$

$(a_n) \to a$ means

$$\forall \delta > 0, \exists N \in \mathbb{N} \text{ s.t } \forall n \geq N, a_n \in B_\delta(a)$$

Using continuity, $\exists N \in \mathbb{N}$ s.t $\forall n \geq N, f(a_n) \in B_\epsilon(f(a))$. But this is simply the definition of $(f(a_n)) \to f(a)$.

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Problem 4

Let (X, d) be a metric space. Prove that the open ball $B_r(x_0) := \{x \in X : d(x, x_0) < r\}$ is indeed open.

Proof. Let x_1 be a point in $B_r(x_0)$. Let $\phi =$

$$\begin{cases} d(x_0, x_1) & d(x_0, x_1) \leq r/2 \\ r - d(x_0, x_1) & d(x_0, x_1) > r/2 \end{cases}$$

Case 1: $d(x_0, x_1) \leq r/2$. Let $x \in B_\phi(x_1)$.

$$d(x_0, x) \leq d(x_0, x_1) + d(x_1, x) < r/2 + r/2 = r$$

Case 2: $d(x, x_0) > r/2$. Let $x \in B_\phi(x_1)$.

$$d(x_0, x) \leq d(x_0, x_1) + d(x_1, x) < d(x_0, x_1) + r - d(x_0, x_1) = r$$

In both cases $B_\phi(x_1) \subseteq B_r(x_0)$ ∎

Exercise 4.2.8.

Let p be a real number such that $p \geq 1$. For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we define

$$||x||_p = (\sum_{j=1}^{n} |x_j|^p)^{1/p}$$

$d_p(x, y) = ||x - y||$

Prove that d_p is a metric on $\mathbb{R}^n$.

Proof. d_p is clearly positive definite due to the $| * |$ in the summation.
d_p is symmetric by commutativity of $\mathbb{R}$
Now we show triangle inequality.

$$||x + y||_p^p = \sum_i |x_i + y_i|^p \leq \sum_i |x_i + y_i|^{p-1}|x_i| + \sum_i |x_i + y_i|^{p-1}|y_i|$$

Now we apply Holder's inequality to both summations individually.

$$\leq (\sum |x_i + y_i|^{(p-1)*\frac{1}{(1-1/p)}})^{1-1/p}(\sum |x_i|^p)^{1/p} + (\sum |x_i + y_i|^{(p-1)*\frac{1}{(1-1/p)}})^{1-1/p}(\sum |y_i|^p)^{1/p}$$

$$\leq (\sum |x_i + y_i|^p)^{1-1/p}((\sum |x_i|^p)^{1/p} + (\sum |y_i|^p)^{1/p})$$

$$\leq (\sum |x_i + y_i|^p)^{p-1}((\sum |x_i|^p)^{1/p} + (\sum |y_i|^p)^{1/p})$$

$$||x + y||_p \leq (\sum |x_i|^p)^{1/p} + (\sum |y_i|^p)^{1/p}$$

∎