Math 347 HW 6

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Problem 1

Show that any two ordered fields with the least upper bound property are order isomorphic.

Proof. Let A, B be ordered fields with the least upper bound property

First we show that A, B contain the natural numbers:

Since A is a field it contains $0_A, 1_A$. 1_A is distinct from $1_A + 1_A$, since $0_A \neq 1_A$ (property of fields). Let $n_A = \sum_{i=0}^{i=n} 1_A$. All n_A are distinct by induction. Define a map $f : \mathbb{N} \to A$ as $f(n) = n_A$. Since A is a field all of it's elements have additive inverses. Extend the map f to the integers. $f : \mathbb{Z} \to A$ as $f(z) = z_A$.

Any field that contains the integers contains the rationals as a subfield (Exercise 3.0.1.)

Extend the map to the rationals $f : \mathbb{Q} \to A$ as $f(\frac{a}{b}) = \frac{f(a)}{f(b)}$

There are injective homomorphisms from the rationals to A. Therefore A contain the rationals.

Proof is similar for B.

Let $r_A \in A$. By O1 $\forall q_A \in Q_A, q_A < r \vee q_A > r_A \vee q = r$

Consider $L_{r_A} = \{q_A \in \mathbb{Q}_A \mid q_A < r_A\}$

Define a map $\phi : A \to B$ as $\phi(r_A) = lub(L_{r_b}) = lub(\{q_B \in Q_B \mid q_B < r_B)\}$

$$\begin{aligned}
\phi(x_A + y_A) = lub(L_{x_B + y_B}) &= lub(q \in \mathbb{Q}_B \mid q < x + y) \\
&= lub(L_{x_B} + L_{y_B})(\text{as defined in problem 3 ii}) \\
&= lub(L_{x_B}) + lub(L_{y_B})(\text{Problem 3 ii}) \\
&= \phi(x_A) + \phi(y_A)
\end{aligned}$$

$$\begin{aligned}\phi(x_A y_A) &= lub(L_{x_B y_B}) \\ &= lub(L_{x_B} * L_{y_B}) \\ &= lub(L_{x_B}) * lub(L_{y_B}) \\ &= \phi(x)\phi(y)\end{aligned}$$

□

Problem 2

Prove that an ordered field has the least upper bound property iff it has the greatest lower bound property.

Proof. Let F be an ordered field with the least upper bound property, and let A be a non empty subset of F that is bouded below.

Consider the subset of F, $B = \{\text{Lower bounds of } A\}$. B is non empty by assumption that A is bounded below. Since every element of A is an upper bound on B, B is bounded above, and thus B has a least upper bound s.
We now show that s is the greatest lower bound of A.

i. s is a lower bound of A.

Every element in A is an upper bound of B. By definition of least upper bound, $s \leq$ every other upper bound of B Therefore $\forall a \in A, s \leq a$ which means s is a lower bound of A

ii. If l is some other lower bound of $A, l \leq s$
B is the set of lowers bounds of A. By defintion of lub, $\forall b \in B, b \leq s$ Therefore every other lower bound of A is $\leq s$ □

Proof. Let F be an ordered field with the greatest lower bound property, and let A be a non empty subset of F that is bouded above.

Consider the subset of F, $B = \{\text{Upper bounds of } A\}$. B is non empty by assumption that A is bounded above. Since every element of A is an lower bound on B, B is bounded below, and thus B has a greatest lower bound s.
We now show that s is the least upper bound of A.

i. s is an upper bound of A.

Every element in A is an lower bound of B. By definition of greatest upper bound, $s \leq$ every other lower bound of B Therefore $\forall a \in A, s \geq a$ which means s is an upper bound of A

ii. If l is some other upper bound of $A, l \geq s$
B is the set of upper bounds of A. By defintion of glb, $\forall b \in B, b \geq s$ Therefore every other lower bound of A is $\geq s$ □

Problem 3

Suppose that A and B are bounded sets in R. Prove or disprove the following:
(i) $lub(A \cup B) = max\{lub(A), lub(B)\}$.

Proof. Let s, t be the $lub(A)$, $lub(B)$ respectively. $m = max\{lub(A), lub(B)\}$.
m is an upper bound on $A \cup B$ since $\forall a \in A, a \leq s \leq m$ and $\forall b \in B, b \leq t \leq m$
Let l be some other upper bound on $A \cup B, l \leq m$
Then $l \leq max\{s, t\}$
WLOG, assume $s \geq t$ Since s is a least upper bound, $l \leq s \implies l = s$
Therefore $l = m$ □

(ii) If $A + B = \{a + b \mid a \in A, b \in B\}$, then $lub(A + B) = lub(A) + lub(B)$.

Proof. Let $a \in A, b \in B$
$a \leq lub(A)$ and $b \leq lub(B) \implies a + b \leq lub(A) + lub(B)$
$lub(A) + lub(B)$ is an upper bound of $A + B$
$\implies lub(A + B) \leq lub(A) + lub(B)$
Now we show that no upper bound is smaller
$\forall \epsilon > 0, \exists a \in A, a > lub(A) - \frac{\epsilon}{2}$ and $\exists b \in B, b > lub(B) - \frac{\epsilon}{2}$
$a + b > lub(A) + lub(B) - \epsilon$
$\implies lub(A + B) > lub(A) + lub(B) - \epsilon$
$\implies lub(A + B) \geq lub(A) + lub(B)$
Therefore $lub(A + B) = lub(A) + lub(B)$

□

(iii) If the elements of A and B are positive and $A * B = \{ab|a \in A, b \in B\}$, then $lub(A * B) = lub(A) * lub(B)$.

Proof. Let $a \in A, b \in B$
$a \leq lub(A)$ and $b \leq lub(B)$
$ab \leq lub(A) * lub(B)$
$\implies lub(A + B) \leq lub(A) + lub(B)$
Now we show that no upper bound is smaller
$\forall \epsilon > 0, \exists a \in A, a > lub(A) - \sqrt{\epsilon}$ and $\exists b \in B, b > lub(B) - \sqrt{\epsilon}2$
$ab > lub(A)lub(B) - \epsilon * lub(A) - \epsilon * lub(B) - \epsilon^2$
$ab > lub(A)lub(B) - \epsilon(lub(A) + lub(B) + \epsilon)$ Note that $lub(A)$ and $lub(B) > 0$
$lub(AB) > lub(A)lub(B) - \epsilon(lub(A) + lub(B) + \epsilon)$
$lub(AB) \geq lub(A)lub(B)$

□

(iv) Formulate the analogous problems for the greatest lower bound.

$glb(A \cup B) = min\{glb(A), glb(B)\}$

If $A + B = \{a + b \mid a \in A, b \in B\}$, then $glb(A + B) = glb(A) + glb(B)$

If the elements of A and B are positive and $A * B = \{ab | a \in A, b \in B\}$, then $glb(A * B) = glb(A) * glb(B)$.

Problem 4

(i) *Show that any irrational number multiplied by any non-zero rational number is irrational.*

Proof. Let $a \in \mathbb{Q}^c, b \in \mathbb{Q} \backslash \{0\}$

Assume $ab \in \mathbb{Q}$ to derive a contradiction.

Then we can write $ab = \frac{p}{q}$ $p, q \in \mathbb{Z} p, q$ coprime

Then $a = \frac{p}{qb} \implies a \in \mathbb{Q}$

This is a contradiction to our assumption that a is irrational.

Thus $ab \in \mathbb{Q}^c$ □

(ii) *Show that the product of two irrational numbers may be rational or irrational.*

Proof. Case 1: Product of two irrationals is rational

$$\sqrt{2} \in \mathbb{Q}^c$$

$$\sqrt{2} * \sqrt{2} = 2 \in \mathbb{Q}$$

Case 2: Product of two irrationals is irrational

$$\sqrt{3}, \sqrt{6} \in \mathbb{Q}^c \text{ (proofs attached to back)}$$

$$\sqrt{2} * \sqrt{3} = \sqrt{6}$$

□

Problem 5

Show that, for any $a, b \in \mathbb{Q}$, we have $||a| - |b|| \leq |a - b|$

Proof. By the triangle inequality $|a| = |(a - b) + b| \leq |a - b| + |b|$

$|a| - |b| \leq |a - b|$

Consider the case: $|a| - |b| \geq 0$

Then $|a| - |b| = ||a| - |b|| \leq |a - b|$

Now consider the case $|a| - |b| < 0$

Then $||a| - |b|| = -(|a| - |b|) = |b| - |a| \leq |b - a| = |a - b|$

$\implies ||a| - |b|| \leq |a - b|$ □