Math 347 HW 8

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Exercise 3.6.13

If $(a_k)_{k \in \mathbb{N}}$ is a Cauchy seqence in $\mathbb{R}$, show that, for any $\epsilon > 0$, there exists a subsequence $(a_{k_j})_{j \in \mathbb{N}}$ so that $|a_{k_j} - a_{k_j+1}| < \frac{\epsilon}{2^{j+1}}$ for $j \in \mathbb{N}$

Proof. fix $\epsilon > 0$

Since (a_k) is Cauchy, $\exists N_1 \in \mathbb{N}, \forall n, m \geq N_1, |a_n - a_m| < \frac{\epsilon}{2^{1+1}}$

Pick $a_{k_1} = a_{N_1}$. Similarly, $\exists N_2 \in \mathbb{N}, \forall m \geq N_2, |a_{k_1} - a_m| < \frac{\epsilon}{2^{2+1}}$

Pick $a_{k_2} = a_{N_2}$. Again, $\exists N_3 \in \mathbb{N}, \forall m \geq N_3, |a_{k_2} - a_m| < \frac{\epsilon}{2^{3+1}}$ and pick $a_{k_3} = a_{N_3}$

Continuing inductively we select $a_{k_j} = N_j$

■

Exercise 3.6.25

Show that a subset of $\mathbb{R}$ is closed iff it contains all of its accumulation points.

Proof. $\rightarrow$ Suppose $S \subseteq \mathbb{R}$ and S is closed.

Assume S does not contain all of its accumulation points to derive a contradiction.

Since S is closed S^c is open.

This means $\forall x \in S^c, \exists \epsilon > 0 s.t B_\epsilon(x) \subseteq S^c$

Let x be an accumulation point of S that is in S^c

Then $\forall \epsilon > 0, B_\epsilon(x) \backslash \{x\} \cap S \neq \varnothing$

Let $y \in B_\epsilon(x) \backslash \{x\} \cap S$

But since S^c is open $y \in S^c$, a contradiction.

$\leftarrow$ Suppose $S \subseteq \mathbb{R}$ and S contains all of its accumulation points.

Assume S is not closed to derive a contradiction

Then S^c is not open which means $\exists x \in S^c$ such that $\forall \epsilon > 0$, $B_\epsilon(x) \nsubseteq S^c$

This is equivalent to $\exists x \in S^c, \forall \epsilon > 0$, $B_\epsilon(x) \cap S = B_\epsilon(x) \backslash \{x\} \cap S \neq \varnothing$ (note $x \notin S$)

But then x is an accumulation point of S that is not in S, a contradiction. ■

Exercise 3.6.26

Let C be the Cantor set.

1. *C set is closed*

   Proof. C^c is a union of open intervals between $[0,1]$ which implies C^c is open. (Exercise 3.6.24)

   Therefore C is closed ∎

2. *C is uncountable*

   Proof. We proceed with a typical diagonalization argument.

   Suppose that C is countable. Then we can create a list of the elements of C.
   C consists of all numbers in [0,1] whose ternary expansions have only 0's and (possibly infinite) 2's. (part 3)

   $x_1 = 0.d_1^1 d_2^1 d_3^1 d_4^1 \ldots$
   $x_2 = 0.d_1^2 d_2^2 d_3^2 d_4^2 \ldots$
   $x_3 = 0.d_1^3 d_2^3 d_3^3 d_4^3 \ldots$
   $\vdots$

   Construct $x = d_1 d_2 \ldots$ not in the list by setting $d_n = 0$ if $d_n^n = 2$ and $d_n = 2$ if $d_n^n = 0$
   But this is contradicts our assumption that we can list the elements of C.
   Therefore C is not countable ∎

3. *C consists of all numbers in the closed interval [0, 1] whose ternary expansion consists of only 0's and 2's and may end in infinitely many 2's*

   Proof. Consider the ternary representation of $x \in (1/3, 2/3)$

   Write 1/3 as $0.1 = 0.0222\ldots$ and 2/3 as 0.2

   Then $x = 0.1d_2d_3\ldots$

   Removing this middle third means we have now removed all x where x has a $d_1 = 1$

   Meaning $x \in C_1 = [0,1] \backslash (1,3) \implies x = 0.0d_2d_3d_4\ldots$ or $x = 0.2d_2d_3d_4\ldots$

   Now consider $x \in (1/9, 2/9)$. $x = 0.01d_3d_4\ldots$

   Simiilarly, $x \in (5/9, 8/9)$ has the form $x = 0.21d_3d_4\ldots$

   Removing these two open intervals removes all x where x has the form $0.d_1 1 d_3 d_4 \ldots$

   Continuing inductively, on the nth step we remove all x where $d_n = 1$.

   ∎

4. *Every point of C is an accumulation point of C*

Proof. Consider $x \in C$ and $(x - \epsilon, x + \epsilon) \backslash \{x\}$ where $\epsilon > 0$

Construct an element of C in $(x - \epsilon, x + \epsilon) \backslash \{x\}$ as follows:

Note that $2 * 3^{-n}$ is a number in ternary with $d_i = 0$ for i from 1 to n-1 and $d_n = 2$
For example, $2 * 3^{-4} = 0.0002$
Also note that the ternary expansion of x consists only of 0's and 2's.

Case 1: Ternary expansion of x terminates

Idea is to append arbitrarily many 0's to x and then a 2
Let d_n^ϵ be the first non zero digit of ϵ and m be the length of the ternary form of x.
$k = max(m, n)$
$x + 2 * 3^{-k-1} < x + \epsilon$ and it is an element of C since it contains only 0s and 2s

Case 2: x ends in infinite 2's

Idea is to turn a 2 from the "tail" of x into a 0
Let d_n^ϵ be the first non zero digit of ϵ
Let d_m^x be the last 0 of the ternary expansion of x.
$k = max(m, n)$
$a = 2 * 3^{k+1}$
Then $x - a > x - \epsilon$ and it is an element of C

Every neighborhood of x contains an element in C. Therefore x is an accumulation point.

∎

5. *The set* $[0, 1]\backslash$*Cantor set is a dense subset of* $[0, 1]$.

Proof. Let $a, b \in [0, 1]$ and $b > a$.
Let C_n be the nth iteration of the Cantor set construction
C_n is a union of open interverals of length $\frac{1}{3^n}$
Choose n large enough that $b - a < \frac{1}{3^n}$
Then $[a, b]$ is not completely contained in any one interval of C_n
Then there is $x \in [a, b]$ but $x \notin C_n$
$\implies x \notin C$
$\implies x \in [0, 1] \backslash C$

∎