6–7. A double pendulum consists of two massless rods of length l and two particles of mass m which can move in a given vertical plane, as shown.

(a) Assuming frictionless joints, and using θ and ϕ as coordinates, obtain the differential equations of motion.

(b) What are the linearized equations for small θ and ϕ?

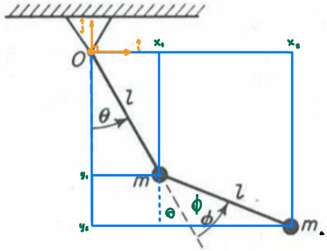


Figure P6-7

(a). Obtain the differential equation of motion.

Fundamental form of lagrange's equation. $\boxed{\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\right] - \frac{\partial L}{\partial q} = Q}$ $\cos A = cA$, $\sin A = sA$

We have : $L[q,\dot{q}] = T - V$ $L[q,\dot{q}] = \frac{1}{2}mv^2 - mgy$

From: $v^2 = (\sqrt{\dot{x}^2+\dot{y}^2})^2 = \dot{x}^2 + \dot{y}^2$ and then $s = x + y$

We get $x_1 = l\sin\theta$, $y_1 = -l\cos\theta$, $x_2 = l\sin\theta + l\sin\theta+\phi$, $y_2 = -l\cos\theta - l\cos\theta+\phi$

And then $s_1 = x_1\hat{i} + y_1\hat{j} = (l\sin\theta)\hat{i} - (l\cos\theta)\hat{j} \Rightarrow \dot{s}_1 = (l\dot{\theta}\cos\theta)\hat{i} + l\dot{\theta}\sin\theta) \Rightarrow v_1^2 = \dot{s}_1^2 = l^2\dot{\theta}^2(\cancelto{1}{\cos^2\theta + \sin^2\theta}) = l^2\dot{\theta}^2$ #

$s_2 = (l\sin\theta + l\sin\theta+\phi)i - (l\cos\theta + l\cos\theta+\phi)\hat{j} \Rightarrow \dot{s}_2 = l(\dot{\theta}\cos\theta + (\dot{\theta}+\dot{\phi})\cos\theta+\phi)\hat{i} + l(\dot{\theta}\sin\theta + (\dot{\theta}+\dot{\phi})\sin\theta+\phi)\hat{i}$

$\dot{s}_2^2 = v_2^2 = l^2\left[\left((\dot{\theta}^2c^2\theta + 2\dot{\theta}c\theta(\dot{\theta}+\dot{\phi})c\theta+\phi + ((\dot{\theta}+\dot{\phi})c\theta+\phi)^2\right)\hat{i} + \left(\dot{\theta}^2s^2\theta + 2\dot{\theta}s\theta(\dot{\theta}+\dot{\phi})s\theta+\phi + (\dot{\theta}+\dot{\phi})^2s^2\theta+\phi\right)\right]$

$v_2^2 = l^2\left[\dot{\theta}^2c^2\theta + 2\dot{\theta}c\theta(\dot{\theta}+\dot{\phi})c\theta+\phi + (\dot{\theta}+\dot{\phi})^2c^2\theta+\phi + \dot{\theta}^2 \cdot s^2\theta + 2\dot{\theta}s\theta(\dot{\theta}+\dot{\phi})s\theta+\phi + (\dot{\theta}+\dot{\phi})^2s^2\theta+\phi\right]$

$v_2^2 = l^2\left[\dot{\theta}^2(\cancelto{1}{c^2\theta+s^2\theta}) + (\dot{\theta}+\dot{\phi})^2(\cancelto{1}{c^2\theta+\phi+s^2\theta+\phi}) + 2\dot{\theta}(\dot{\theta}+\dot{\phi})(c\theta c\theta+\phi + s\theta s\theta+\phi)\right] \Rightarrow l^2\left[\dot{\theta}^2 + (\dot{\theta}+\dot{\phi})^2 + 2\dot{\theta}(\dot{\theta}+\dot{\phi})(c\theta c\theta+\phi + s\theta s\theta+\phi)\right]$

From $\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$ and $\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$

Thus we can obtain $v_2^2 = l^2\left[\dot{\theta}^2 + (\dot{\theta}+\dot{\phi})^2 + 2\dot{\theta}(\dot{\theta}+\dot{\phi})(c\theta(c\theta c\phi - s\theta s\phi) + s\theta(s\theta c\phi + c\theta s\phi)\right]$

$v_2^2 = l^2\left[\dot{\theta}^2 + (\dot{\theta}+\dot{\phi})^2 + 2\dot{\theta}(\dot{\theta}+\dot{\phi})(c^2\theta c\phi - \cancel{c\theta s\theta s\phi} + s^2\theta c\phi + \cancel{s\theta c\theta s\phi})\right] \Rightarrow l^2\left[\dot{\theta}^2 + (\dot{\theta}+\dot{\phi})^2 + 2\dot{\theta}(\dot{\theta}+\dot{\phi})(\cancelto{1}{s^2\theta+c^2\theta})c\phi\right]$

$v_2^2 = l^2\left[\dot{\theta}^2 + (\dot{\theta}+\dot{\phi})^2 + 2\dot{\theta}(\dot{\theta}+\dot{\phi})\cos\phi\right]$ #

We can find kinetic energy from $T = \frac{mv_1^2}{2} + \frac{mv_2^2}{2} = \frac{m}{2}\left(l^2\dot{\theta}^2 + l^2\left[\dot{\theta}^2 + (\dot{\theta}+\dot{\phi})^2 + 2\dot{\theta}(\dot{\theta}+\dot{\phi})\cos\phi\right]\right)$ #

$T = \frac{ml^2}{2}(2\dot{\theta}^2 + \dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2 + 2\dot{\theta}^2\cos\phi + 2\dot{\theta}\dot{\phi}\cos\phi) \Rightarrow \frac{ml^2}{2}(3\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}(1+\cos\phi) + 2\dot{\theta}^2\cos\phi)$ #

And then. We can find potentail energy from $V = mgy_1 + mgy_2 = mg(-l\cos\theta - l\cos\theta - l\cos\theta+\phi) = -mgl(2\cos\theta + \cos\theta+\phi)$ #

We have $L = T - V \Rightarrow \frac{ml^2}{2}\left(3\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}(1+\cos\phi) + 2\dot{\theta}^2\cos\phi\right) + mgl(2\cos\theta + \cos\theta+\phi)$

And then $\frac{\partial L}{\partial \theta} = -mgl(2\sin\theta + \sin\theta+\phi)$, $\frac{\partial L}{\partial \dot{\theta}} = \frac{ml^2}{\cancel{2}_1}\left(\cancel{6}^{3}\dot{\theta} + \cancel{2}^{1}\dot{\phi}(1+\cos\phi) + \cancel{4}^{2}\dot{\theta}\cos\phi\right) = ml^2(3\dot{\theta} + \dot{\phi}(1+\cos\phi) + 2\dot{\theta}\cos\phi)$

$\frac{\partial L}{\partial \phi} = -\frac{ml^2}{\cancel{2}_1}(\cancel{2}^{1}\dot{\theta}\dot{\phi}\sin\phi + \cancel{2}^{1}\dot{\theta}^2\sin\phi) - mgl\sin\theta+\phi \Rightarrow -ml^2(\dot{\theta}\dot{\phi}\sin\phi + \dot{\theta}^2\sin\phi) - mgl\sin\theta+\phi$

$\frac{\partial L}{\partial \dot{\phi}} = \frac{ml^2}{\cancel{2}}(\cancel{2}\dot{\phi} + \cancel{2}\dot{\theta}(1+\cos\phi) \Rightarrow ml^2(\dot{\phi} + \dot{\theta}(1+\cos\phi))$

$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\theta}}\right] = ml^2(3\ddot{\theta} + (1+\cos\phi)\ddot{\phi} - (\dot{\phi}\sin\phi)\dot{\phi} - 2\dot{\theta}\dot{\phi}\sin\phi + 2\ddot{\theta}\cos\phi) \Rightarrow ml^2[(3+2\cos\phi)\ddot{\theta} + (1+\cos\phi)\ddot{\phi} - \dot{\phi}^2\sin\phi - 2\dot{\theta}\dot{\phi}\sin\phi]$

$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\phi}}\right] = ml^2(\ddot{\phi} + \dot{\theta}(-\dot{\phi}\sin\phi) + \ddot{\theta}(1+\cos\phi)) \Rightarrow ml^2(\ddot{\phi} - \dot{\theta}\dot{\phi}\sin\phi + (1+\cos\phi)\ddot{\theta}$

From lagrange equation. $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{q_i} = Q_i$, $Q_1 = Q_2 = 0$

equation of motion of θ

$$ml^2[(3+2\cos\phi)\ddot{\theta} + (1+\cos\phi)\ddot{\phi} - \dot{\phi}^2\sin\phi - 2\dot{\theta}\dot{\phi}\sin\phi] + mgl(2\sin\theta + \sin\theta+\phi) = 0 \quad \underline{\text{Ans}}$$

$$ml^2(3+2\cos\phi)\ddot{\theta} = ml^2[\dot{\phi}^2\sin\phi - (1+\cos\phi)\ddot{\phi} + 2\dot{\theta}\dot{\phi}\sin\phi] - 2mgl\sin\theta - mgl\sin\theta+\phi$$

or
$$\boxed{\ddot{\theta} = \frac{[\dot{\phi}^2\sin\phi - (1+\cos\phi)\ddot{\phi} + 2\dot{\theta}\dot{\phi}\sin\phi]}{(3+2\cos\phi)} - \frac{2g\sin\theta - g\sin\theta+\phi}{l(3+2\cos\phi)}}$$

equation of motion of ϕ $Q_2 = 0$

$$ml^2[(\ddot{\phi} - \dot{\theta}\dot{\phi}\sin\phi + (1+\cos\phi)\ddot{\theta} + \dot{\theta}\dot{\phi}\sin\phi + \dot{\theta}^2\sin\phi - mgl\sin\theta+\phi] = 0$$

$$ml^2[\ddot{\phi} + (1+\cos\phi)\ddot{\theta} + \dot{\theta}^2\sin\phi] + mgl\sin\theta+\phi = 0 \quad \underline{\text{Ans}}$$

$$ml^2\ddot{\phi} = ml^2[-(1+\cos\phi)\ddot{\theta} - \dot{\theta}^2\sin\phi] - mgl\sin\theta+\phi$$

Or
$$\boxed{\ddot{\phi} = -(1+\cos\phi)\ddot{\theta} - \dot{\theta}^2\sin\phi - \frac{g\sin\theta+\phi}{l}}$$

(b). equation for small θ and ϕ

set $\cos\theta, \cos\phi \approx 1$ and $\sin\theta \approx \theta$, $\sin\phi \approx \phi$ and $\sin\theta+\phi \approx \theta+\phi$, nonlinear $(\dot{\theta}^2, \dot{\phi}^2, \dot{\theta}\dot{\phi})$ set to zero

Then the linearized equation of motion are

From $ml^2[(3+2\cos\phi)\ddot{\theta} + (1+\cos\phi)\ddot{\phi} - \dot{\phi}^2\sin\phi - 2\dot{\theta}\dot{\phi}\sin\phi] + mgl(2\sin\theta + \sin\theta+\phi) = 0$

We get $ml^2[5\ddot{\theta} + 2\ddot{\phi}] + mgl(2\theta + (\theta+\phi)) = 0$ $\underline{\text{Ans}}$

And then. from $ml^2[\ddot{\phi} + (1+\cos\phi)\ddot{\theta} + \dot{\theta}^2\sin\phi] + mgl\sin\theta+\phi = 0$

We get $ml^2[\ddot{\phi} + 2\ddot{\theta}] + mgl(\theta+\phi) = 0$ $\underline{\text{Ans}}$

6–25. Two wheels, each of mass m, are connected by a massless axle of length l. Each wheel is considered to have its mass concentrated as a particle at its hub. The wheels can roll without slipping on a horizontal plane. The hub of wheel A is attached by a spring of stiffness k and unstressed length l to a fixed point O. Using r, θ, and ϕ as generalized coordinates, obtain the differential equations of motion.

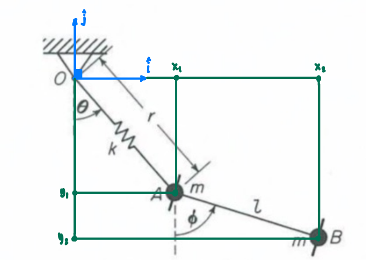


Figure P6-25

Fundamental form of Lagrange's equation. $\boxed{\frac{d}{dt}\left[\frac{\partial L}{\partial \dot q}\right] - \frac{\partial L}{\partial q} = Q}$

Define: $L[q,\dot q] = T - V$ $\quad L[q,\dot q] = \frac{1}{2}mv^2 - mgy$

From: $v^2 = (\sqrt{\dot x^2 + \dot y^2})^2 = \dot x^2 + \dot y^2$

We get $x_1 = r\sin\theta$, $y_1 = -r\cos\theta$, $x_2 = r\sin\theta + l\sin\phi$, $y_2 = -r\cos\theta - l\cos\phi$

And then $v_1^2 = (r\dot\theta\cos\theta + \dot r\sin\theta)^2 + (r\dot\theta\sin\theta - \dot r\cos\theta)^2$

$$v_1^2 = r^2\dot\theta^2\cos^2\theta + \cancel{2(r\dot\theta\cos\theta)(\dot r\sin\theta)} + \dot r^2\sin^2\theta + r^2\dot\theta^2\sin^2\theta - \cancel{2(r\dot\theta\sin\theta)(\dot r\cos\theta)} + \dot r^2\cos^2\theta$$

$$v_1^2 = (r^2\dot\theta^2 + \dot r^2)\cos^2\theta + (\dot r^2 + r^2\dot\theta^2)\sin^2\theta = (\dot r^2 + r^2\dot\theta^2)\overset{1}{\cancel{(\cos^2\theta + \sin^2\theta)}} = \dot r^2 + r^2\dot\theta^2$$

$x_2 = r\sin\theta + l\sin\phi$, $y_2 = -r\cos\theta - l\cos\phi$

$s_2 = (r\sin\theta + l\sin\phi)\hat i - (r\cos\theta + l\cos\phi)\hat j$

$$v_2^2 = \dot s_2^2 = (r\dot\theta\cos\theta + \dot r\sin\theta + l\dot\phi\cos\phi)^2 + (r\dot\theta\sin\theta - \dot r\cos\theta + l\dot\phi\sin\phi)^2$$

$$v_2^2 = (r\dot\theta\cos\theta + \dot r\sin\theta + l\dot\phi\cos\phi)(r\dot\theta\cos\theta + \dot r\sin\theta + l\dot\phi\cos\phi) + (r\dot\theta\sin\theta - \dot r\cos\theta + l\dot\phi\sin\phi)(r\dot\theta\sin\theta - \dot r\cos\theta + l\dot\phi\sin\phi)$$

$$v_2^2 = r^2\dot\theta^2c^2\theta + \cancel{r\dot r\dot\theta c\theta s\theta} + rl\dot\theta\dot\phi c\theta c\phi + \cancel{\dot r r\dot\theta s\theta c\theta} + \dot r^2s^2\theta + l\dot r\dot\phi s\theta c\phi + rl\dot\theta\dot\phi c\phi c\theta + l\dot r\dot\phi c\phi s\theta + l^2\dot\phi^2c^2\phi + r^2\dot\theta^2s^2\theta - \cancel{\dot r r\dot\theta s\theta c\theta} + rl\dot\theta\dot\phi s\theta s\phi - \cancel{\dot r r\dot\theta c\theta s\theta}$$
$$+ \dot r^2c^2\theta - l\dot r\dot\phi c\theta s\phi + lr\dot\theta\dot\phi s\phi s\theta - l\dot r\dot\phi s\phi c\theta + l^2\dot\phi^2\sin^2\phi$$

$$v_2^2 = r^2\dot\theta^2(\overset{1}{\cancel{c^2\theta + s^2\theta}}) + \dot r^2(\overset{1}{\cancel{c^2\theta + s^2\theta}}) + l^2\dot\phi^2(\overset{1}{\cancel{c^2\phi + s^2\phi}}) + rl\dot\theta\dot\phi(c\theta c\phi + s\theta s\phi) + l\dot r\dot\phi(s\theta c\phi - c\theta s\phi) + rl\dot\theta\dot\phi(c\phi c\theta + s\phi s\theta) + l\dot r\dot\phi(c\phi s\theta - s\phi c\theta)$$

$$v_2^2 = r^2\dot\theta^2 + \dot r^2 + l^2\dot\phi^2 + 2rl\dot\theta\dot\phi(c\phi c\theta + s\phi s\theta) + 2l\dot r\dot\phi(s\phi c\theta - c\phi s\theta)$$

From $\cos(A-B) = \cos A\cos B + \sin A\sin B$ and $\sin(A-B) = \sin A\cos B - \cos A\sin B$

Thus we can obtain $v_2^2 = r^2\dot\theta^2 + \dot r^2 + l^2\dot\phi^2 + 2rl\dot\theta\dot\phi\cos(\phi-\theta) + 2l\dot r\dot\phi\sin(\phi-\theta)$

Calculate the kinetic energy from $T = \frac{mv_1^2}{2} + \frac{mv_2^2}{2} = \frac{m}{2}\left[\dot r^2 + r^2\dot\theta^2 + r^2\dot\theta^2 + \dot r^2 + l^2\dot\phi^2 + 2rl\dot\theta\dot\phi\cos(\phi-\theta) + 2l\dot r\dot\phi\sin(\phi-\theta)\right]$

$$T = \frac{m}{2}\left[2\dot r^2 + 2r^2\dot\theta^2 + l^2\dot\phi^2 + 2rl\dot\theta\dot\phi\cos(\phi-\theta) + 2l\dot r\dot\phi\sin(\phi-\theta)\right]$$

And then calculate the potential energy from $V = \frac{kx^2}{2} = \frac{k(r-l)^2}{2}$ #

We have the generalized coordinate is r, θ, ϕ but 2 Dof so "Nonholonomic system"

From $L = T - V = \frac{m}{2}\left[2\dot r^2 + 2r^2\dot\theta^2 + l^2\dot\phi^2 + 2rl\dot\theta\dot\phi\cos(\phi-\theta) - 2l\dot r\dot\phi\sin(\phi-\theta)\right] - \frac{k(r-l)^2}{2}$ #

From lagrange multiplier $\boxed{\frac{d}{dt}\left[\frac{\partial L}{\partial \dot q_i}\right]-\frac{\partial L}{\partial q_i}=\sum_j \lambda_j a_{ji}}$ $i = r, \dot\theta, \phi$

Find a_{ij} from equation of nonholonomic constraint

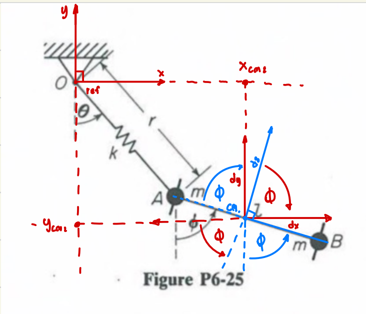
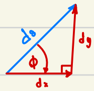


Figure P6-25

set $x_{cm} = x$, $y_{cm} = y$

$x = r\sin\theta + \frac{L}{2}\sin\phi$, $dx = r\dot\theta\cos\theta + \dot r\sin\theta + \frac{L}{2}\dot\phi\cos\phi$ ___ (1)

$y = -r\cos\theta - \frac{L}{2}\cos\phi$, $dy = r\dot\theta\sin\theta - \dot r\cos\theta + \frac{L}{2}\dot\phi\sin\phi$ ___ (2)



$\sin\phi = \frac{dy}{ds} \Rightarrow dy = ds\sin\phi$, $\cos\phi = \frac{dx}{ds} \Rightarrow dx = ds\cos\phi$

We get $\frac{dx}{\cos\phi} = \frac{dy}{\sin\phi} \Rightarrow dx\sin\phi - dy\cos\phi = 0$ ___ (3)

And then represent 1,2 in 3

$(r\dot\theta\cos\theta + \dot r\sin\theta + \frac{L}{2}\dot\phi\cos\phi)\sin\phi - (r\dot\theta\sin\theta - \dot r\cos\theta + \frac{L}{2}\dot\phi\sin\phi)\cos\phi = 0$

$r\dot\theta c\theta s\phi + \dot r s\theta s\phi + \frac{L}{2}\dot\phi \cancel{c\phi s\phi} - r\dot\theta s\theta c\phi + \dot r c\theta c\phi - \cancel{\frac{L}{2}\dot\phi s\phi c\phi} = 0 \Rightarrow r\dot\theta(c\theta s\phi - s\theta c\phi) + \dot r(s\theta s\phi + c\theta c\phi) = 0$

$\cos(\phi-\theta)\dot r + r\sin(\phi-\theta)\dot\theta = 0$ # $a_{11} = \cos(\phi-\theta)$, $a_{12} = r\sin(\phi-\theta)$, $a_{13} = 0$

Equation of motion of r

We have $L = \frac{M}{2}\left[2\dot r^2 + 2r^2\dot\theta^2 + l^2\dot\phi^2 + 2rl\dot\theta\dot\phi\cos(\phi-\theta) - 2l\dot r\dot\phi\sin(\phi-\theta)\right] - \frac{K(r-l)^2}{2}$

$\frac{\partial L}{\partial r} = \frac{M}{2}(4r\dot\theta^2 + 2l\dot\theta\dot\phi\cos(\phi-\theta)) - \frac{\cancel{2}K(r-l)}{\cancel{2}} = M(2r\dot\theta^2 + l\dot\theta\dot\phi\cos(\phi-\theta)) - K(r-l)$ #

$\frac{\partial L}{\partial \dot r} = \frac{M}{\cancel{2}}(\cancel{4}\dot r - \cancel{2}l\dot\phi\sin(\phi-\theta)) = M(2\dot r - l\dot\phi\sin(\phi-\theta))$ #

$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot r}\right] = M\left(2\ddot r - l\dot\phi(\dot\phi-\dot\theta)\cos(\phi-\theta) - l\ddot\phi\sin(\phi-\theta)\right)$ #

From $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot q_i}\right]-\frac{\partial L}{\partial q_i}=\sum_j \lambda_j a_{ji}$

$2m\ddot r - ml\dot\phi(\dot\phi-\dot\theta)\cos(\phi-\theta) - ml\ddot\phi\sin(\phi-\theta) - 2mr\dot\theta^2 - ml\dot\theta\dot\phi\cos(\phi-\theta) + K(r-l) = \lambda a_{11}$

$\ddot r = \frac{ml\dot\phi(\dot\phi-\dot\theta)\cos(\phi-\theta) + ml\ddot\phi\sin(\phi-\theta) + 2mr\dot\theta^2 + ml\dot\theta\dot\phi\cos(\phi-\theta) + K(r-l) + \lambda\cos(\phi-\theta)}{2m}$ Ans

Equation of motion of θ

$\frac{\partial L}{\partial \theta} = -\frac{M}{\cancel{2}}(\cancel{2}rl\dot\theta\dot\phi\sin(\phi-\theta) + \cancel{2}l\dot r\dot\phi\cos(\phi-\theta)) = -M(rl\dot\theta\dot\phi\sin(\phi-\theta) + l\dot r\dot\phi\cos(\phi-\theta))$

$\frac{\partial L}{\partial \dot\theta} = \frac{M}{\cancel{2}}\left[\cancel{4}r^2\dot\theta + \cancel{2}rl\dot\phi\cos(\phi-\theta)\right] = M\left[2r^2\dot\theta + rl\dot\phi\cos(\phi-\theta)\right]$

$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot\theta}\right] = M\left(2r^2\ddot\theta + 4r\dot r\dot\theta + rl(\dot\phi\,(-\sin(\phi-\theta))(\dot\phi-\dot\theta) + \ddot\phi\cos(\phi-\theta)) + \dot r l\dot\phi\cos(\phi-\theta)\right) = M\left(2r^2\ddot\theta + 4r\dot r\dot\theta - rl\dot\phi(\dot\phi-\dot\theta)\sin(\phi-\theta) + rl\ddot\phi\cos(\phi-\theta) + l\dot r\dot\phi\cos(\phi-\theta)\right)$

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From $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial q_i} = \sum_j \lambda_j a_{ji}$

$$m\left(2r^2\ddot{\theta} + 4r\dot{r}\dot{\theta} - rl\dot{\phi}(\dot{\phi}-\dot{\theta})\sin(\phi-\theta) + rl\ddot{\phi}\cos(\phi-\theta) + l\dot{r}\dot{\phi}\cos(\phi-\theta)\right) + m\left(rl\dot{\theta}\dot{\phi}\sin(\phi-\theta) + l\dot{r}\dot{\phi}\cos(\phi-\theta)\right) = \lambda a_{12}$$

$$m\left(2r^2\ddot{\theta} + 4r\dot{r}\dot{\theta} - rl\dot{\phi}(\dot{\phi}-\dot{\theta})\sin(\phi-\theta) + rl\ddot{\phi}\cos(\phi-\theta) + l\dot{r}\dot{\phi}\cos(\phi-\theta) + rl\dot{\theta}\dot{\phi}\sin(\phi-\theta) + l\dot{r}\dot{\phi}\cos(\phi-\theta)\right) = \lambda r\sin\phi-\theta \quad \underline{\text{Ans}}$$

$$\ddot{\theta} = \frac{-m\left(4r\dot{r}\dot{\theta} - rl\dot{\phi}(\dot{\phi}-\dot{\theta})\sin(\phi-\theta) + rl\ddot{\phi}\cos(\phi-\theta) + l\dot{r}\dot{\phi}\cos(\phi-\theta) + rl\dot{\theta}\dot{\phi}\sin(\phi-\theta) + l\dot{r}\dot{\phi}\cos(\phi-\theta)\right) + \lambda r\sin\phi-\theta}{2mr^2} \quad \underline{\text{Ans}}$$

Equation of motion of ϕ

$$\frac{\partial L}{\partial \phi} = -\frac{m}{\cancel{2}}\left(\cancel{2}rl\dot{\theta}\dot{\phi}\sin(\phi-\theta) + \cancel{2}l\dot{r}\dot{\phi}\cos(\phi-\theta)\right) = -ml\left(r\dot{\theta}\dot{\phi}\sin(\phi-\theta) + \dot{r}\dot{\phi}\cos(\phi-\theta)\right) \quad \#$$

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{m}{\cancel{2}}\left(\cancel{2}l^2\dot{\phi} + \cancel{2}rl\dot{\theta}\cos(\phi-\theta) - \cancel{2}l\dot{r}\sin(\phi-\theta)\right) = m\left(l^2\dot{\phi} + rl\dot{\theta}\cos(\phi-\theta) - l\dot{r}\sin(\phi-\theta)\right) \quad \#$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\phi}}\right] = m\left(l^2\ddot{\phi} - rl\dot{\theta}(\dot{\phi}-\dot{\theta})\sin(\phi-\theta) + rl\ddot{\theta}\cos(\phi-\theta) + l\dot{r}\dot{\theta}\cos(\phi-\theta) - l\dot{r}(\dot{\phi}-\dot{\theta})\cos(\phi-\theta) - l\ddot{r}\sin(\phi-\theta)\right)$$

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\phi}}\right] = m\left(l^2\ddot{\phi} - rl\dot{\theta}\dot{\phi}\sin(\phi-\theta) + rl\dot{\theta}^2\sin(\phi-\theta) + rl\ddot{\theta}\cos(\phi-\theta) + l\dot{r}\dot{\theta}\cos(\phi-\theta) - l\dot{r}\dot{\phi}\cos(\phi-\theta) + l\dot{r}\dot{\theta}\cos(\phi-\theta) - l\ddot{r}\sin(\phi-\theta)\right) \quad \#$$

From $\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial q_i} = \sum_j \lambda_j a_{ji}$

$$m\left(l^2\ddot{\phi} \cancel{- rl\dot{\theta}\dot{\phi}\sin(\phi-\theta)} + rl\dot{\theta}^2\sin(\phi-\theta) + rl\ddot{\theta}\cos(\phi-\theta) + l\dot{r}\dot{\theta}\cos(\phi-\theta) \cancel{- l\dot{r}\dot{\phi}\cos(\phi-\theta)} + l\dot{r}\dot{\theta}\cos(\phi-\theta) - l\ddot{r}\sin(\phi-\theta) \cancel{+ rl\dot{\theta}\dot{\phi}\sin(\phi-\theta)} \cancel{+ l\dot{r}\dot{\phi}\cos(\phi-\theta)}\right) = \lambda \cancel{a_{13}}^{\,0}$$

$$m\left(l^2\ddot{\phi} + r\dot{\theta}^2\sin(\phi-\theta) + rl\ddot{\theta}\cos(\phi-\theta) + 2l\dot{r}\dot{\theta}\cos(\phi-\theta) - l\ddot{r}\sin(\phi-\theta)\right) = 0$$

$$\ddot{\phi} = \frac{-\left(r\dot{\theta}^2\sin(\phi-\theta) + rl\ddot{\theta}\cos(\phi-\theta) + 2l\dot{r}\dot{\theta}\cos(\phi-\theta) - l\ddot{r}\sin(\phi-\theta)\right)}{l^2} \quad \underline{\text{Ans}}$$