Available online at www.sciencedirect.com
SCIENCE DIRECT®
PHYSICS LETTERS A
Physics Letters A 313 (2003) 55–61
www.elsevier.com/locate/pla

Spatial prisoner's dilemma optimally played in small-world networks

Naoki Masuda [a,*], Kazuyuki Aihara [a,b]

[a] *Department of Complexity Science and Engineering, Graduate School of Frontier Sciences, The University of Tokyo, 7-3-1 Hongo, Bunkyo-ku, Tokyo 113-8656, Japan*
[b] *Core Research for Evolutional Science and Technology, Japan Science and Technology Corporation, 4-1-8 Hon-cho, Kawaguchi, Saitama 332-0012, Japan*

Received 5 December 2002; received in revised form 17 April 2003; accepted 18 April 2003

Communicated by C.R. Doering

Abstract

Cooperation is commonly found in ecological and social systems even when it apparently seems that individuals can benefit from selfish behavior. We investigate how cooperation emerges with the spatial prisoner's dilemma played in a class of networks ranging from regular lattices to random networks. We find that, among these networks, small-world topology is the optimal structure when we take into account the speed at which cooperative behavior propagates. Our results may explain why the small-world properties are self-organized in real networks.

PACS: 87.23.G; 87.23.C; 05.45.X

Keywords: Game theory; Spatial prisoner's dilemma; Small-world networks; Evolution of cooperation

1. Introduction

Cooperation among individuals is everywhere in ecological and social systems such as animal groups and human societies. How altruistic behavior emerges in the situations where each individual is apparently tempted to defect has been debated for a long time. In the game theory, this situation is typically formulated as the prisoner's dilemma (PD) [3,4]. To resolve the dilemma and explain the actually found altruism, various mechanisms such as kin selection [6], reciprocal altruism in iterated games [3,4], and spatial games [4, 13–16] have been proposed.

For example, the iterated PD with evolutionary dynamics assumes a population of players randomly interacting with each other. A round of interaction between two players consists of the repetition of PD, in which a next PD is played with probability w where $0 < w < 1$. Consequently, they are provided with the payoff summed over the round. After playing one round of the game with all the other players, the players that have gained higher total payoff are more likely to survive in the next round. When w is sufficiently large, cooperative strategies

* Corresponding author.
E-mail address: masuda@sat.t.u-tokyo.ac.jp (N. Masuda).

0375-9601/03/$ – see front matter
doi:10.1016/S0375-9601(03)00693-5

such as Tit-for-Tat outperform the mean players such as pure defectors. Only the former can elicit reciprocal altruism from other good players to benefit both of them and is not exploited by mean players [3,4].

Another solution is to introduce spatial structure as found in the real world so that a player interacts with a limited number of neighbors in each round. If defection is not extremely advantageous, clusters of cooperators can survive in the sea of defectors when the network is clustered, that is, when one can expect that a friend of my friend is another friend of mine with a significantly high probability. Clustered cooperators of a certain size reciprocate each other to surpass defectors that get payoff just by exploiting cooperators. Two-dimensional square lattice networks each of whose vertex has immediate eight neighbors (Moore neighborhood), for example, are equipped with the clustering property and facilitate altruism [7,13–16]. Simple good strategies such as pure cooperators can be territorially stable in the noniterated PD even if they cannot persist in random networks [4]. The results have also been extended to more general cases involving stochasticity, asynchronous updating, irregularity in lattices [15,16], and the spatial Hawk–Dove games [8]. Moreover, interesting phenomena such as chaotic fluctuation in the proportion of cooperators and dynamic fractals of population patterns are reported with this framework [13,14].

Generally speaking, the network topology significantly influences the dynamical behavior in ecological and social networks [1,2,9,10,19]. Regular lattices employed in spatial games have an important feature of social networks, that is, the cluster property that enables a group of players to cooperate and benefit each other [7]. Randomly connected networks lack this property. On the other hand, real networks also have the property that, on the average, any two individuals are connected via a path much shorter than is expected for regular lattices [2,12,18,19]. In a relationship network among individuals, one mainly interacts with other individuals in its own cluster or group, but it also tends to interacts with some individuals that belong to other groups. The small-world properties comprising the clustered property and the average short path length are realized in the pioneering mathematical model by Watts and Strogatz, which suitably describes many types of biological, physical, and social networks [18,19]. To generate one-dimensional small-world networks, let us start with a ring of n vertices each of which has $k/2$ nearest neighbors on each side. Then we rewire a proportion p of the total edges by removing $pkn/2$ edges and creating $pkn/2$ new edges each of whose initial vertex is the initial vertex of a removed edge and its terminal vertex is randomly chosen so that the generated graph does not have multiple edges or once removed edges. Graphs with small positive rewiring probabilities satisfy the small-world properties with small average path lengths $L(p)$ and large clustering coefficients $C(p)$, where $L(p)$ is defined as the shortest path length between two points averaged over all point pairs, and $C(p)$ is the normalized number of triangles [18,19]. The regular lattice $(p = 0)$ and the random graph $(p = 1)$ are generated by this procedure as the two extremes.

The spatial PD in small-world networks, which may be more plausible models for social interaction, has been examined by a few authors. Watts [19] showed that generalized Tit-for-Tat and Win–Stay–Lose–Shift strategies survive in the population mixed with defectors if $C(p)$ is large enough, in other words, if p is small enough. There exists a critical rewiring probability p_c above which good strategies die out. As clusters collapse with an increasing p, similar kinds of transition appear in epidemic models with small-world networks [10]. Watts has also measured the time necessary for the convergence to stable proportions of good strategies, and related it to $L(p)$. In [9], an influential node is endowed with an increased number of shortcuts, and how instantaneous strategy change of the influential node affects the macroscopic dynamics is studied. In another work, proportions of defectors in the small-world spatial PD is examined [1]. In this Letter, we examine the spatial PD in small-world networks and find how the combined effects of $C(p)$ and $L(p)$ result in emergence and development of cooperation. Especially, we show that the proportion of cooperators determined by $C(p)$ and the speed of convergence determined by $L(p)$ can be balanced, but rapid convergence and total dominance of cooperators, though it may seem socially preferable, are not simultaneously realized even in the small-world regime. Instead, small-world networks realize rapid emergence of slightly suboptimal states with many cooperators in a global scale.

2. Models and results

We assume that each vertex of a network, which ranges from a regular lattice to a random graph by changing p, is occupied by a player. In each round, a player interacts with its immediate neighbors. We consider only memoryless strategies; each player chooses just cooperation (denoted by C) or defection (denoted by D) in each round [13–16]. When a player chooses C, it receives payoff R (reward) or S (sucker) as the opponent chooses C or D, respectively. A player that chooses D receives T (temptation) or P (punishment) as the opponent chooses C or D, respectively. Given $T > R > P > S$, a player is always tempted to defect no matter whether the opponent takes C or D. The combination of D and D, with which both of the players get unsatisfactory payoff P, is the unique Nash equilibrium in a single game. Each player sums the payoff received by playing a single PD with its neighbors and compares the sum with those of the neighbors. Among them, the strategy with the maximal payoff is copied as the player's strategy in the next round. This imitation procedure is considered to stem from genetic evolution or social learning. Following the earlier work, we assume $T = b > 1$, $4R = 1$, and $P = S = 0$ [13–16]. Though the condition $P > S$ is violated, its influence on the dynamics is negligible.

Let us denote the dimension of the underlying network by d, and we set $n = 3600$ and $k = 8$. We examine the case of $d = 1$ in which n vertices are aligned in a ring [1,17,19] and the case of $d = 2$ in which 60×60 vertices are aligned in a square lattice with periodic boundary conditions. The connection structure is parameterized by p. When $p = 0$, each vertex is connected to four nearest neighbors on each side for $d = 1$ or to the eight players in the Moore neighborhood for $d = 2$. For positive values of p, we randomly rewire the proportion p of the edges. If the conventional rewiring procedure were used, the vertex degree would vary from vertex to vertex after rewiring. To avoid artificial normalization of payoff due to the dispersed number of neighbors, we randomly rewire the edges keeping the degree of every vertex constant [11].

For various values of p and b, the proportion of cooperators after the transient is shown in Fig. 1. The initial proportion of cooperators that are randomly and independently chosen for all the sites is equal to $c(0) = 0.98$. It is known for $d = 2$ and $p = 0$ that cooperators dominate for small b, defectors occupy the whole space for sufficiently large b, and intermediate values of b ($1.8 < b < 2.0$) result in chaotic fluctuation [13,14]. Fig. 1(b) shows that these regimes persist for any values of p although a larger value of p lowers the critical values of b for the transitions. Moreover, as we will see shortly, we can identify three ranges of b based on the combination of final proportions of cooperators and dependence of the results on p. These observations hold true also for $d = 1$ as shown in Fig. 1(a) just with slight shifts of the critical values of b dividing different regimes. Such kinds of quantitative change in different dimensions are also observed with regular lattices [15,16]. The three regimes are as follows.

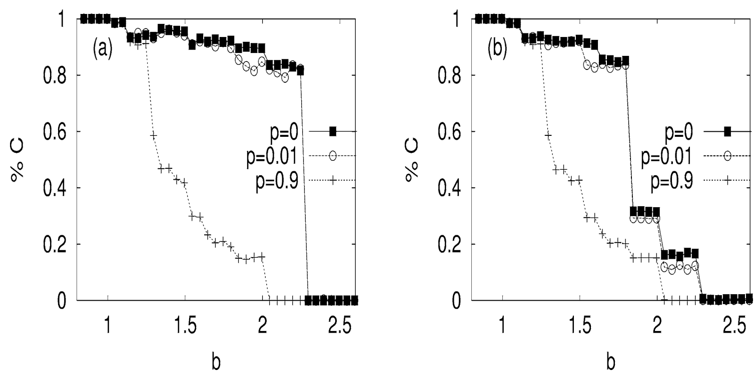

Fig. 1. The proportion of cooperators after transient for (a) $d = 1$ and (b) $d = 2$. We set $c(0) = 0.98$, and the statistics are based on averaging over 10 trials.

(i) For small b, it is not so tempting for players to exploit cooperators. Consequently, the proportion of cooperators converges to a value close to 1 regardless of p.
(ii) The number of cooperators highly depends on p roughly for $1.3 \leqslant b \leqslant 2.3$. In this case, $C(p)$ needs to be larger for cooperators to survive [19]. The chaotic situation is also included in this regime.
(iii) For larger b, players are inclined to betray. Even if cooperators happen to form tight clusters, they cannot survive once they face defectors. Finally, the cooperators eventually extinguish whatever values p takes.

In regime (ii), which is of our particular interest, the clustering coefficient and the number of cooperators are in close relation. Actually, the clustering property of networks is related to the measure of assortment r which, for example, represents the relatedness coefficient associated with kin selection [5]. The proportion r of the total players are supposed to side with the decision of a reference player, and cooperators are more likely to survive in more assortative populations. In accordance with this observation, simple calculation leads to $p = 1 - r$. Fig. 2(b) and (e) show temporal profiles of the proportion of cooperators $c(n)$ after n rounds in this regime for $d = 1$ and $d = 2$, respectively. We set $b = 1.7$ and the initial condition $c(0) = 0.50$. We find more cooperators for smaller values of p. The cooperators effectively form clusters and occupy a large part of the network for small p, while cooperators and defectors are balanced when p is larger. However, the convergence is faster for smaller $L(p)$, that is, for larger p. Fig. 2(b) and (e) together with further parameter search reveal that rapid convergence to the states with as many cooperators as in the case of $p = 0$, which might be desirable from a social

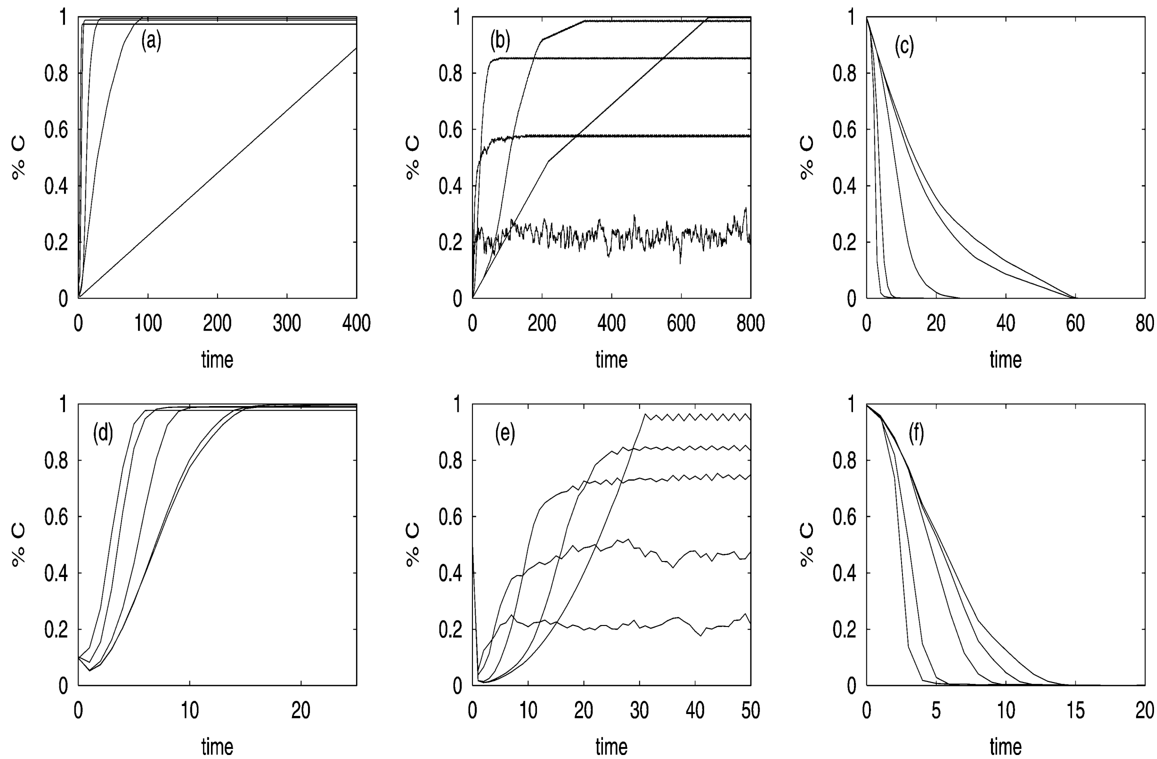

Fig. 2. The proportion of cooperators in the spatial PD with $d = 1$ (a), (b), (c) and $d = 2$ (d), (e), (f). We set $(b, c(0)) = (1.1, 0.1)$, $(b, c(0)) = (1.7, 0.5)$, $(b, c(0)) = (3.0, 0.995)$, for (a), (d), (b), (e), and (c), (f), respectively. The initial conditions are set randomly and independently for all the players. The results for $p = 0$, $p = 0.001$, $p = 0.01$, $p = 0.1$, and $p = 0.8$ are shown, with more rapidly converging lines corresponding to larger values of p.

point of view, is not realized even in the small-world schemes. This suggests that the range of p yielding $C(p)$ large enough for the maintenance of total governance of cooperators and that guaranteeing the rapid convergence to the equilibrium proportion of cooperators do not exactly overlap. However, in this situation, we observe rapid convergence to the proportion of cooperators just slightly less than that for $p = 0$. Small-world properties are generally found in social networks [2,9,12,18,19], and our results suggest that small-world networks may be socially preferred.

Rapid convergence to equilibria is more clearly understood in regimes (i) and (iii), with Fig. 2(a) and (d) (or Fig. 2(c) and (f)) for regime (i) (or regime (iii)) showing the convergence to the all-C (or all-D) states. Comparing the results for these two regimes, we cannot comment on whether small $L(p)$ is 'good' or 'bad' in terms of payoff. Small values of $L(p)$ just guarantee fast convergence to final states whatever they are [9], and this fact agrees with our daily experience. In these cases with extreme values of b the players are just imitating the neighbors in a coordinated manner, and this game dynamics can be considered as a simple epidemic model with permanent-removal dynamics that consists only of the susceptible and infectious populations [19]. In this situation, the infectious individuals, which is C for regime (i) and D for regime (iii), devour the susceptible individuals, and this process is faster for smaller $L(p)$. As a remark, convergence to chaotic states for $1.8 < b < 2.0$ [13,14] is also rapid with larger p while the size of $\lim_{n\to\infty} c(n)$ is severely sacrificed compared with the case when $b = 1.7$ (data not shown).

The comparison of Fig. 2(b) and (e) reveals that rapid convergence to the equilibria with considerable cooperators is more emphasized for $d = 1$ than for $d = 2$. This is because the dynamic ranges of $C(p)$ and $L(p)$ with respect to p are wider for $d = 1$, which pronounces the differences in the speed of convergence and the final proportions of cooperators. Actually, if the number of neighbors is fixed, networks with less dimensions have wider dynamical ranges of $C(p)$ and $L(p)$. Let us suppose $k = 8$. Then straightforward calculations show $C(0) = 9/14$ for $d = 1$ and $C(0) = 3/7$ for $d = 2$, whereas $C(1) = k/n$ for both cases. Similarly, $L(0) \cong n/2k$ for $d = 1$, and $L(0) \cong \sqrt{n}/2k$ for $d = 2$, whereas $L(1) \cong \log(n)/\log(k)$ for both dimensions [18]. For better presentations of the results, we set $d = 1$ in the remainder of this Letter. However, qualitatively similar results are obtained for $d = 2$ as well.

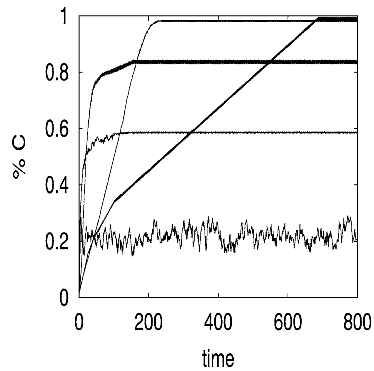

Fig. 3. The proportion of cooperators with clustered initial conditions. We set $d = 1$ and $(b, c(0)) = (1.7, 0.01)$.

To demonstrate generality of our results, let us note the initial sharp drop of $c(n)$ in Fig. 2(b) and (e) due to the random initial conditions. In the beginning, defectors devour cooperators that do not form clusters. Then only tight clusters of cooperators survive and then begin to grow. A more relevant initial condition is clusters of cooperators in the sea of defectors. Therefore we examine the dynamics starting with randomly placed clusters each of which contains nine successively aligned cooperators in the background of defectors. The positions of the clusters are determined randomly and independently. The results for $b = 1.7$ and $c(0) = 0.01$ shown in Fig. 3 are similar to those in Fig. 2. The same is also true for regime (i) and for $d = 2$ (data not shown).

Finally, we consider effects of noise. In the real world, decision of each player can accompany errors because information available for decision making is imperfect and the behavior can be inherently unpredictable to some extent. Now we introduce stochastic decision rules in which the probability that a player chooses C in the next round is $\sum_{i=0}^{k} A_i^m s_i / \sum_{i=0}^{k} A_i^m$ [15]. Here A_i is the payoff of the ith neighbor, with $i = 0$ corresponding to the player itself. When the current strategy of the ith neighbor is C (or D), s_i takes one (or zero). The value of m specifies the degree of stochasticity. The deterministic case corresponds to $m = \infty$, and smaller m indicates more randomness. The results for $m = 20$,

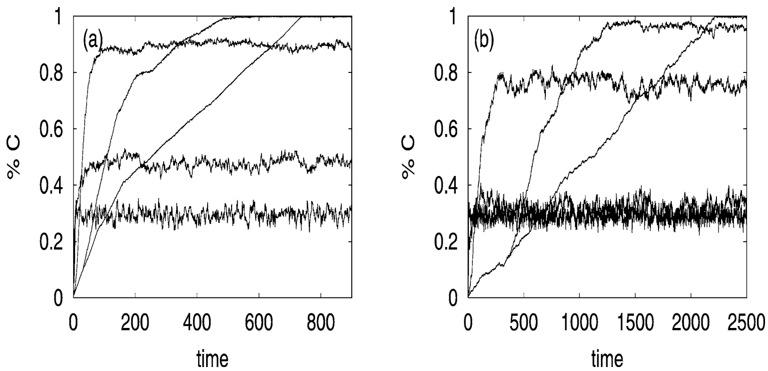

Fig. 4. The proportion of cooperators in the case of $d = 1$ with independent initial conditions and noise. We set $(b, c(0), m) = (1.6, 0.5, 20)$ and $(b, c(0), m) = (1.55, 0.5, 10)$ for (a) and (b), respectively.

$b = 1.6$, $c(0) = 0.5$ and those for $m = 10$, $b = 1.55$, $c(0) = 0.5$ are shown in Fig. 4(a) and (b), respectively. Similarity of the results to the deterministic cases in spite of apparently large amount of noise implies the robustness of the results against noise. However, larger noise requires smaller b for the same level of cooperation to be maintained because noise spoils the effect of clustering [15]. Noise also lengthen the transient time as manifested in Fig. 4(b). For still larger amount of noise with smaller m, we do not obtain quasi-convergent behavior [15].

3. Conclusions

We have shown that, in the spatial PD, three different kinds of dynamics and dependence on the network structure exist with respect to b. For intermediate values of b, small-world architecture realizes a quasi-optimal behavior in the sense of rapid convergence to a good equilibrium. Here the 'goodness' is implicitly measured by the following hierarchy of states. The best consequence is fast convergence to a equilibrium with many cooperators, the second best is slow convergence to this type of equilibria. The worst is fast convergence to a population with many defectors, and the second worst is slow convergence to such a state.

We have also investigated the effects of dimensionality, clustered initial conditions, and dynamical noise. The results are qualitatively the same for these cases, but networks with smaller dimensions generally lead to more enhanced effects of the small-world property because of the wider dynamic ranges of $C(p)$ and $L(p)$. In nongeometric social networks such as ecological networks, food webs, and friendship networks, the dimension of the underlying substratum is not really known whichever value of p is appropriate. This is in contrast with networks based on physical locations such as neural networks and power grids. Further investigation of the dimensionality effect is our future problem.

Acknowledgements

This work is supported by the Japan Society for the Promotion of Science, by the Advanced and Innovational Research program in Life Sciences from the Ministry of Education, Culture, Sports, Science, and Technology, the Japanese Government, and by Superrobust Computation Project in 21st Century COE Program on Information Science and Technology Strategic Core.

References

[1] G. Abramson, M. Kuperman, Phys. Rev. E 63 (2001) 030901(R).
[2] R. Albert, A.-L. Barabási, Rev. Mod. Phys. 74 (2002) 47.
[3] R. Axelrod, W.D. Hamilton, Science 211 (1981) 1390.
[4] R. Axelrod, Evolution of Cooperation, Basic Books, New York, 1984.
[5] R. Boyd, P.J. Richerson, J. Theor. Biol. 132 (1988) 337.
[6] W.D. Hamilton, J. Theor. Biol. 7 (1964) 1.
[7] Ch. Hauert, Proc. R. Soc. London B 268 (2001) 761.
[8] T. Killingback, M. Doebeli, Proc. R. Soc. London B 263 (1996) 1135.

[9] B.J. Kim, A. Trusina, P. Holme, P. Minnhagen, J.S. Chung, M.Y. Choi, Phys. Rev. E 66 (2002) 021907.
[10] M. Kuperman, G. Abramson, Phys. Rev. Lett. 86 (2001) 2909.
[11] N. Masuda, K. Aiharam, submitted for publication.
[12] M.E.J. Newman, Proc. Natl. Acad. Sci. USA 98 (2001) 404.
[13] M.A. Nowak, R.M. May, Nature 359 (1992) 826.
[14] M.A. Nowak, R.M. May, Int. J. Bifur. Chaos 3 (1993) 35.
[15] M.A. Nowak, S. Bonhoeffer, R.M. May, Proc. Natl. Acad. Sci. USA 91 (1994) 4877.
[16] M.A. Nowak, S. Bonhoeffer, R.M. May, Int. J. Bifur. Chaos 4 (1994) 33.
[17] G.B. Pollock, Social Networks 11 (1989) 175.
[18] D.J. Watts, S.H. Strogatz, Nature 393 (1998) 440.
[19] D.J. Watts, Small Worlds, Princeton Univ. Press, Princeton, NJ, 1999.