2

Limits and Continuity

OVERVIEW In this chapter we develop the concept of a limit, first intuitively and then formally. We use limits to describe the way a function varies. Some functions vary *continuously*; small changes in x produce only small changes in $f(x)$. Other functions can have values that jump, vary erratically, or tend to increase or decrease without bound. The notion of limit gives a precise way to distinguish among these behaviors.

2.1 Rates of Change and Tangent Lines to Curves

Average and Instantaneous Speed

HISTORICAL BIOGRAPHY
Galileo Galilei
(1564–1642)
www.goo.gl/QFpvl0

In the late sixteenth century, Galileo discovered that a solid object dropped from rest (initially not moving) near the surface of the earth and allowed to fall freely will fall a distance proportional to the square of the time it has been falling. This type of motion is called **free fall**. It assumes negligible air resistance to slow the object down, and that gravity is the only force acting on the falling object. If y denotes the distance fallen in feet after t seconds, then Galileo's law is

$$y = 16t^2 \text{ ft}, \quad \text{Free fall.}$$

where 16 is the (approximate) constant of proportionality. (If y is measured in meters instead, then the constant is close to 4.9.)

More generally, suppose that a moving object has traveled distance $f(t)$ at time t. The object's **average speed** during an interval of time $[t_1, t_2]$ is found by dividing the distance traveled $f(t_2) - f(t_1)$ by the time elapsed $t_2 - t_1$. The unit of measure is length per unit time: kilometers per hour, feet (or meters) per second, or whatever is appropriate to the problem at hand.

Average Speed

When $f(t)$ measures the distance traveled at time t,

$$\text{Average speed over } [t_1, t_2] = \frac{\text{distance traveled}}{\text{elapsed time}} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

EXAMPLE 1 A rock breaks loose from the top of a tall cliff. What is its average speed

(a) during the first 2 sec of fall?

(b) during the 1-sec interval between second 1 and second 2?

Δ is the capital Greek letter Delta

Solution The average speed of the rock during a given time interval is the change in distance, Δy, divided by the length of the time interval, Δt. (The capital Greek letter Delta, written Δ, is traditionally used to indicate the increment, or change, in a variable. Increments like Δy and Δt are reviewed in Appendix 3, and pronounced "delta y" and "delta t.") Measuring distance in feet and time in seconds, we have the following calculations:

(a) For the first 2 sec: $\quad \dfrac{\Delta y}{\Delta t} = \dfrac{16(2)^2 - 16(0)^2}{2 - 0} = 32\,\dfrac{\text{ft}}{\text{sec}}$

(b) From sec 1 to sec 2: $\quad \dfrac{\Delta y}{\Delta t} = \dfrac{16(2)^2 - 16(1)^2}{2 - 1} = 48\,\dfrac{\text{ft}}{\text{sec}}$ ■

We want a way to determine the speed of a falling object at a single instant t_0, instead of using its average speed over an interval of time. To do this, we examine what happens when we calculate the average speed over shorter and shorter time intervals starting at t_0. The next example illustrates this process. Our discussion is informal here but will be made precise in Chapter 3.

EXAMPLE 2 Find the speed of the falling rock in Example 1 at $t = 1$ and $t = 2$ sec.

Solution We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$, having length $\Delta t = (t_0 + h) - (t_0) = h$, as

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^{\,2}}{h}\,\frac{\text{ft}}{\text{sec}}. \tag{1}$$

We cannot use this formula to calculate the "instantaneous" speed at the exact moment t_0 by simply substituting $h = 0$, because we cannot divide by zero. But we *can* use it to calculate average speeds over shorter and shorter time intervals starting at either $t_0 = 1$ or $t_0 = 2$. When we do so, by taking smaller and smaller values of h, we see a pattern (Table 2.1).

TABLE 2.1 Average speeds over short time intervals $[t_0, t_0 + h]$

Average speed: $\dfrac{\Delta y}{\Delta t} = \dfrac{16(t_0 + h)^2 - 16t_0^{\,2}}{h}$

Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

The average speed on intervals starting at $t_0 = 1$ seems to approach a limiting value of 32 as the length of the interval decreases. This suggests that the rock is falling at a speed of 32 ft/sec at $t_0 = 1$ sec. Let's confirm this algebraically.

If we set $t_0 = 1$ and then expand the numerator in Equation (1) and simplify, we find that

$$\frac{\Delta y}{\Delta t} = \frac{16(1 + h)^2 - 16(1)^2}{h} = \frac{16(1 + 2h + h^2) - 16}{h}$$
$$= \frac{32h + 16h^2}{h} = 32 + 16h. \quad \text{Can cancel } h \text{ when } h \neq 0$$

For values of h different from 0, the expressions on the right and left are equivalent and the average speed is $32 + 16h$ ft/sec. We can now see why the average speed has the limiting value $32 + 16(0) = 32$ ft/sec as h approaches 0.

Similarly, setting $t_0 = 2$ in Equation (1), for values of h different from 0 the procedure yields

$$\frac{\Delta y}{\Delta t} = 64 + 16h.$$

As h gets closer and closer to 0, the average speed has the limiting value 64 ft/sec when $t_0 = 2$ sec, as suggested by Table 2.1. ■

The average speed of a falling object is an example of a more general idea, an average rate of change.

Average Rates of Change and Secant Lines

Given any function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y, $\Delta y = f(x_2) - f(x_1)$, by the length $\Delta x = x_2 - x_1 = h$ of the interval over which the change occurs. (We use the symbol h for Δx to simplify the notation here and later on.)

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

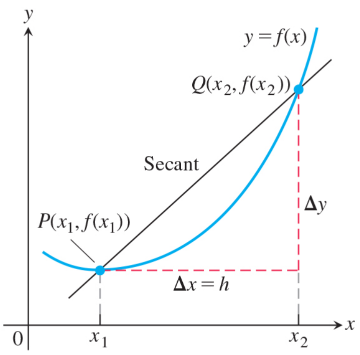


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y/\Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ (Figure 2.1). In geometry, a line joining two points of a curve is called a **secant line**. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of secant line PQ. As the point Q approaches the point P along the curve, the length h of the interval over which the change occurs approaches zero. We will see that this procedure leads to the definition of the slope of a curve at a point.

Defining the Slope of a Curve

We know what is meant by the slope of a straight line, which tells us the rate at which it rises or falls—its rate of change as a linear function. But what is meant by the *slope of a curve* at a point P on the curve? If there is a *tangent line* to the curve at P—a line that grazes the curve like the tangent line to a circle—it would be reasonable to identify the slope of the tangent line as the slope of the curve at P. We will see that, among all the lines that pass through the point P, the tangent line is the one that gives the best approximation to the curve at P. We need a precise way to specify the tangent line at a point on a curve.

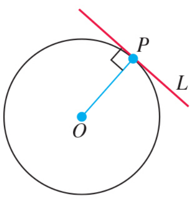


FIGURE 2.2 L is tangent to the circle at P if it passes through P perpendicular to radius OP.

Specifying a tangent line to a circle is straightforward. A line L is tangent to a circle at a point P if L passes through P and is perpendicular to the radius at P (Figure 2.2). But what does it mean to say that a line L is tangent to a more general curve at a point P?

HISTORICAL BIOGRAPHY
Pierre de Fermat
(1601–1665)
www.goo.gl/1YO4DR

To define tangency for general curves, we use an approach that analyzes the behavior of the secant lines that pass through P and nearby points Q as Q moves toward P along the curve (Figure 2.3). We start with what we *can* calculate, namely the slope of the secant line PQ. We then compute the limiting value of the secant line's slope as Q approaches P along the curve. (We clarify the limit idea in the next section.) If the limit exists, we take it to be the slope of the curve at P and *define* the tangent line to the curve at P to be the line through P with this slope.

The next example illustrates the geometric idea for finding the tangent line to a curve.

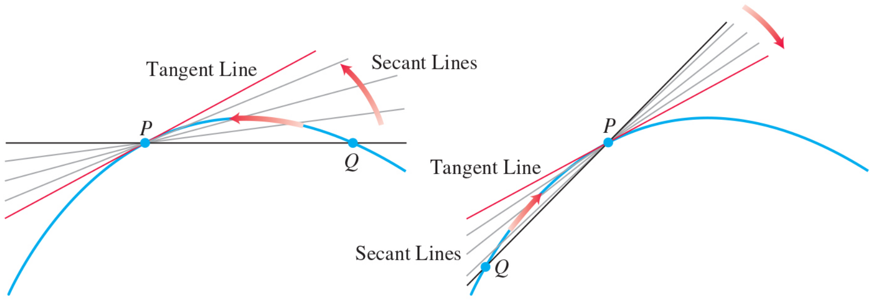


FIGURE 2.3 The tangent line to the curve at P is the line through P whose slope is the limit of the secant line slopes as $Q \rightarrow P$ from either side.

EXAMPLE 3 Find the slope of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$ by analyzing the slopes of secant lines through $(2, 4)$. Write an equation for the tangent line to the parabola at this point.

Solution We begin with a secant line through $P(2, 4)$ and a nearby point $Q(2 + h, (2 + h)^2)$. We then write an expression for the slope of the secant line PQ and investigate what happens to the slope as Q approaches P along the curve:

$$\begin{aligned}\text{Secant line slope} = \frac{\Delta y}{\Delta x} &= \frac{(2 + h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h} \\ &= \frac{h^2 + 4h}{h} = h + 4.\end{aligned}$$

If $h > 0$, then Q lies above and to the right of P, as in Figure 2.4. If $h < 0$, then Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero and the secant line slope $h + 4$ approaches 4. We take 4 to be the parabola's slope at P.

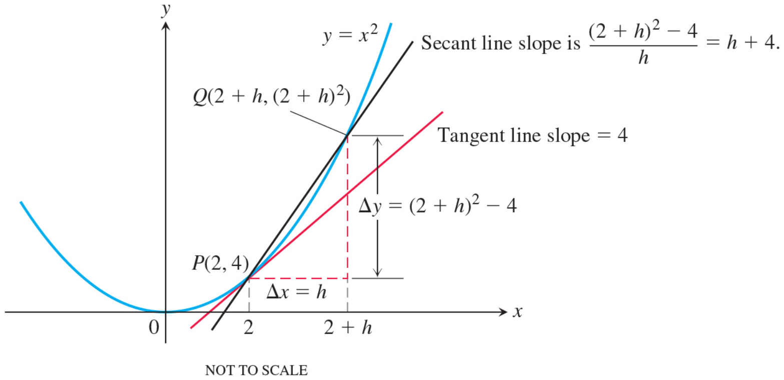


FIGURE 2.4 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant line slopes (Example 3).

The tangent line to the parabola at P is the line through P with slope 4:

$$y = 4 + 4(x - 2) \qquad \text{Point-slope equation}$$
$$y = 4x - 4.$$

Rates of Change and Tangent Lines

The rates at which the rock in Example 2 was falling at the instants $t = 1$ and $t = 2$ are called *instantaneous rates of change*. Instantaneous rates of change and slopes of tangent lines are closely connected, as we see in the following examples.

EXAMPLE 4 Figure 2.5 shows how a population p of fruit flies (*Drosophila*) grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to the number of elapsed days t, and the points joined by a smooth curve (colored blue in Figure 2.5). Find the average growth rate from day 23 to day 45.

Solution There were 150 flies on day 23 and 340 flies on day 45. Thus the number of flies increased by $340 - 150 = 190$ in $45 - 23 = 22$ days. The average rate of change of the population from day 23 to day 45 was

$$\text{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day}.$$

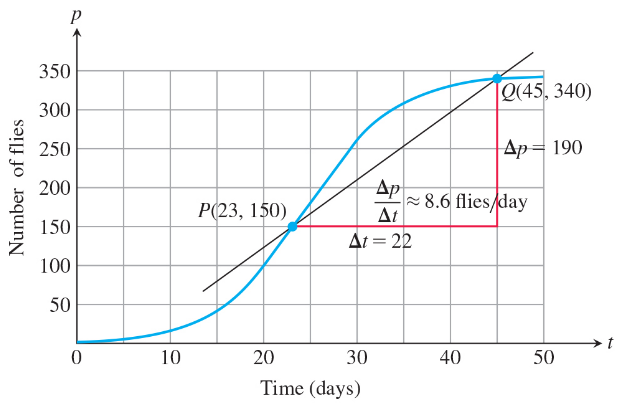


FIGURE 2.5 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line (Example 4).

This average is the slope of the secant line through the points P and Q on the graph in Figure 2.5. ■

The average rate of change from day 23 to day 45 calculated in Example 4 does not tell us how fast the population was changing on day 23 itself. For that we need to examine time intervals closer to the day in question.

EXAMPLE 5 How fast was the number of flies in the population of Example 4 growing on day 23?

Solution To answer this question, we examine the average rates of change over shorter and shorter time intervals starting at day 23. In geometric terms, we find these rates by calculating the slopes of secant lines from P to Q, for a sequence of points Q approaching P along the curve (Figure 2.6).

Q	Slope of $PQ = \Delta p/\Delta t$ (flies / day)
(45, 340)	$\dfrac{340 - 150}{45 - 23} \approx 8.6$
(40, 330)	$\dfrac{330 - 150}{40 - 23} \approx 10.6$
(35, 310)	$\dfrac{310 - 150}{35 - 23} \approx 13.3$
(30, 265)	$\dfrac{265 - 150}{30 - 23} \approx 16.4$

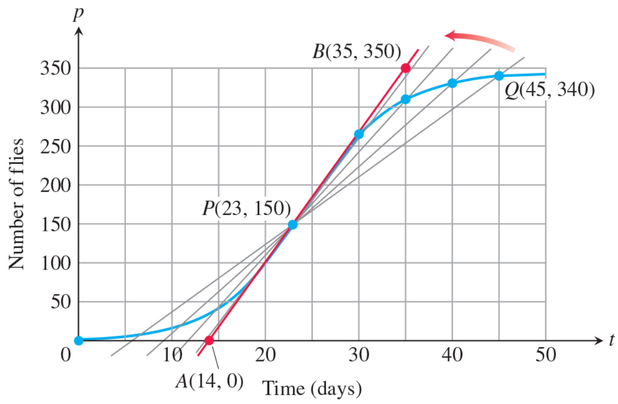


FIGURE 2.6 The positions and slopes of four secant lines through the point P on the fruit fly graph (Example 5).

The values in the table show that the secant line slopes rise from 8.6 to 16.4 as the t-coordinate of Q decreases from 45 to 30, and we would expect the slopes to rise slightly higher as t continued decreasing toward 23. Geometrically, the secant lines rotate counterclockwise about P and seem to approach the red tangent line in the figure. Since the line appears to pass through the points (14, 0) and (35, 350), its slope is approximately

$$\frac{350 - 0}{35 - 14} = 16.7 \text{ flies/day.}$$

On day 23 the population was increasing at a rate of about 16.7 flies / day. ■

The instantaneous rate of change is the value the average rate of change approaches as the length h of the interval over which the change occurs approaches zero. The average rate of change corresponds to the slope of a secant line; the instantaneous rate corresponds to the slope of the tangent line at a fixed value. So instantaneous rates and slopes of tangent lines are closely connected. We give a precise definition for these terms in the next chapter, but to do so we first need to develop the concept of a *limit*.

EXERCISES 2.1

$\dfrac{\overset{f x_2}{f(x+h)} - \overset{f x_1}{f(x)}}{h}$ $h = x_2 - x_1$

Average Rates of Change

In Exercises 1–6, find the average rate of change of the function over the given interval or intervals.

1. $f(x) = x^3 + 1$
   a. $[2, 3]$ (9, 28) $\rightsquigarrow \frac{28-9}{3-2} = 19$
   b. $[-1, 1]$ (0, 2) $\rightsquigarrow \frac{2-0}{1-(-1)} = 1$
2. $g(x) = x^2 - 2x$
   a. $[1, 3]$ (−1, 3)
   b. $[-2, 4]$ (8, 8)
3. $h(t) = \cot t$
   a. $[\pi/4, 3\pi/4]$ (1, −1) $\rightsquigarrow \frac{-1-1}{\frac{3\pi}{4} - \frac{\pi}{4}} = \frac{-2}{\frac{2\pi}{4}} = \frac{-4}{\pi}$
   b. $[\pi/6, \pi/2]$ ($\frac{1}{\sqrt{3}}$, 0) $\rightsquigarrow \frac{0 - \frac{1}{\sqrt{3}}}{\frac{\pi}{2} - \frac{\pi}{6}} = \frac{-\frac{1}{\sqrt{3}}}{\frac{2\pi}{6}} = \frac{-3}{\pi\sqrt{3}}$
4. $g(t) = 2 + \cos t$
   a. $[0, \pi]$
   b. $[-\pi, \pi]$
5. $R(\theta) = \sqrt{4\theta + 1}$; $[0, 2]$ (1, 3)
6. $P(\theta) = \theta^3 - 4\theta^2 + 5\theta$; $[1, 2]$

Slope of a Curve at a Point

$y - y_0 = m(x - x_0)$

In Exercises 7–18, use the method in Example 3 to find **(a)** the slope of the curve at the given point P, and **(b)** an equation of the tangent line at P.

7. $y = x^2 - 5$, $P(2, -1)$

   $* \dfrac{f(2+h) - f(2)}{h} = \dfrac{h^2 + 4h - 1 - (-1)}{h}$; slope $= \dfrac{\cancel{h}(h+4)}{\cancel{h}}$ $\therefore y - (-1) = 4(x - 2)$, $y = 4x - 9$
8. $y = 7 - x^2$, $P(2, 3)$
9. $y = x^2 - 2x - 3$, $P(2, -3)$

   $* \dfrac{(2+h)^2 - 2(2+h) - 3 - (-3)}{h} = \dfrac{h^2 + 4h \cancel{+4-4} - 2h \cancel{-3+3}}{h}$ $h^2 + 2h = h(h+2)$ $\therefore y + 3 = 2(x - 2)$, $y = 2x - 7$
10. $y = x^2 - 4x$, $P(1, -3)$
11. $y = x^3$, $P(2, 8)$
12. $y = 2 - x^3$, $P(1, 1)$
13. $y = x^3 - 12x$, $P(1, -11)$
14. $y = x^3 - 3x^2 + 4$, $P(2, 0)$
15. $y = \dfrac{1}{x}$, $P(-2, -1/2)$

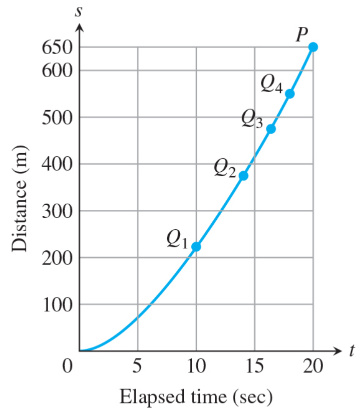
16. $y = \dfrac{x}{2 - x}, \quad P(4, -2)$

17. $y = \sqrt{x}, \quad P(4, 2)$

18. $y = \sqrt{7 - x}, \quad P(-2, 3)$

Instantaneous Rates of Change

19. Speed of a car The accompanying figure shows the time-to-distance graph for a sports car accelerating from a standstill.



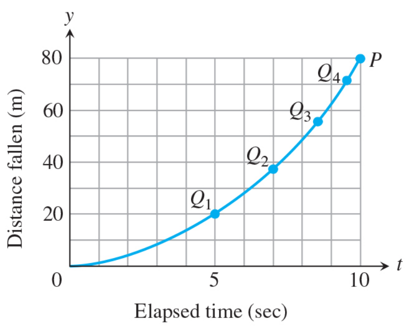
a. Estimate the slopes of secant lines PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in order in a table like the one in Figure 2.6. What are the appropriate units for these slopes?

b. Then estimate the car's speed at time $t = 20$ sec.

20. The accompanying figure shows the plot of distance fallen versus time for an object that fell from the lunar landing module a distance 80 m to the surface of the moon.

a. Estimate the slopes of the secant lines PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in a table like the one in Figure 2.6.

b. About how fast was the object going when it hit the surface?



T **21.** The profits of a small company for each of the first five years of its operation are given in the following table:

Year	Profit in $1000s
2010	6
2011	27
2012	62
2013	111
2014	174

a. Plot points representing the profit as a function of year, and join them by as smooth a curve as you can.

b. What is the average rate of increase of the profits between 2012 and 2014?

c. Use your graph to estimate the rate at which the profits were changing in 2012.

T **22.** Make a table of values for the function $F(x) = (x + 2)/(x - 2)$ at the points $x = 1.2$, $x = 11/10$, $x = 101/100$, $x = 1001/1000$, $x = 10001/10000$, and $x = 1$.

a. Find the average rate of change of $F(x)$ over the intervals $[1, x]$ for each $x \neq 1$ in your table.

b. Extending the table if necessary, try to determine the rate of change of $F(x)$ at $x = 1$.

T **23.** Let $g(x) = \sqrt{x}$ for $x \geq 0$.

a. Find the average rate of change of $g(x)$ with respect to x over the intervals $[1, 2]$, $[1, 1.5]$ and $[1, 1 + h]$.

b. Make a table of values of the average rate of change of g with respect to x over the interval $[1, 1 + h]$ for some values of h approaching zero, say $h = 0.1, 0.01, 0.001, 0.0001, 0.00001$, and 0.000001.

c. What does your table indicate is the rate of change of $g(x)$ with respect to x at $x = 1$?

d. Calculate the limit as h approaches zero of the average rate of change of $g(x)$ with respect to x over the interval $[1, 1 + h]$.

T **24.** Let $f(t) = 1/t$ for $t \neq 0$.

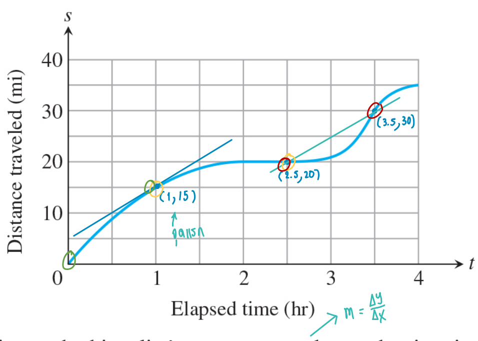
a. Find the average rate of change of f with respect to t over the intervals (i) from $t = 2$ to $t = 3$, and (ii) from $t = 2$ to $t = T$.

b. Make a table of values of the average rate of change of f with respect to t over the interval $[2, T]$, for some values of T approaching 2, say $T = 2.1, 2.01, 2.001, 2.0001, 2.00001$, and 2.000001.

c. What does your table indicate is the rate of change of f with respect to t at $t = 2$?

d. Calculate the limit as T approaches 2 of the average rate of change of f with respect to t over the interval from 2 to T. You will have to do some algebra before you can substitute $T = 2$.

25. The accompanying graph shows the total distance s traveled by a bicyclist after t hours.

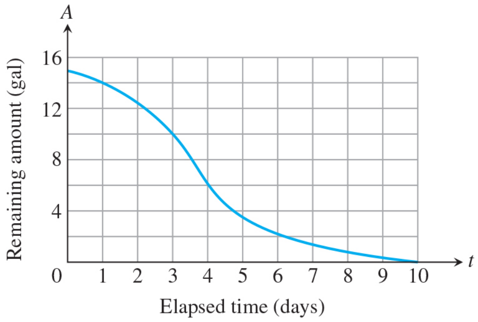


a. Estimate the bicyclist's average speed over the time intervals $[0, 1]$, $[1, 2.5]$, and $[2.5, 3.5]$. 15, 3.3, 10

b. Estimate the bicyclist's instantaneous speed at the times $t = \frac{1}{2}$, $t = 2$, and $t = 3$.

c. Estimate the bicyclist's maximum speed and the specific time at which it occurs.

26. The accompanying graph shows the total amount of gasoline A in the gas tank of an automobile after being driven for t days.



a. Estimate the average rate of gasoline consumption over the time intervals $[0, 3]$, $[0, 5]$, and $[7, 10]$.

b. Estimate the instantaneous rate of gasoline consumption at the times $t = 1$, $t = 4$, and $t = 8$.

c. Estimate the maximum rate of gasoline consumption and the specific time at which it occurs.

2.2 Limit of a Function and Limit Laws

In Section 2.1 we saw how limits arise when finding the instantaneous rate of change of a function or the tangent line to a curve. We begin this section by presenting an informal definition of the limit of a function. We then describe laws that capture the behavior of limits. These laws enable us to quickly compute limits for a variety of functions, including polynomials and rational functions. We present the precise definition of a limit in the next section.

HISTORICAL ESSAY
Limits
www.goo.gl/5V45iK

Limits of Function Values

Frequently when studying a function $y = f(x)$, we find ourselves interested in the function's behavior *near* a particular point c, but not *at* c itself. An important example occurs when the process of trying to evaluate a function at c leads to division by zero, which is undefined. We encountered this when seeking the instantaneous rate of change in y by considering the quotient function $\Delta y/h$ for h closer and closer to zero. In the next example we explore numerically how a function behaves near a particular point at which we cannot directly evaluate the function.

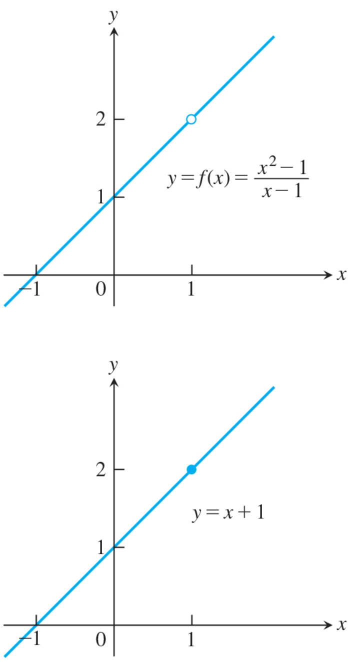


FIGURE 2.7 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

EXAMPLE 1 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution The given formula defines f for all real numbers x except $x = 1$ (since we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \qquad \text{for} \qquad x \neq 1.$$

The graph of f is the line $y = x + 1$ with the point $(1, 2)$ *removed*. This removed point is shown as a "hole" in Figure 2.7. Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ as close as we want to 2 by choosing x close enough to 1 (Table 2.2). ■

An Informal Description of the Limit of a Function

We now give an informal definition of the limit of a function f at an interior point of the domain of f. Suppose that $f(x)$ is defined on an open interval about c, *except possibly at c*

TABLE 2.2 As x gets closer to 1, $f(x)$ gets closer to 2.

x	$f(x) = \dfrac{x^2-1}{\underset{\neq 1}{x}-1}$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

itself. If $f(x)$ is arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to c, other than c itself, then we say that f approaches the **limit** L as x approaches c, and write

$$\lim_{x \to c} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches c is L." In Example 1 we would say that $f(x)$ approaches the *limit* 2 as x approaches 1, and write

แทนค่า x

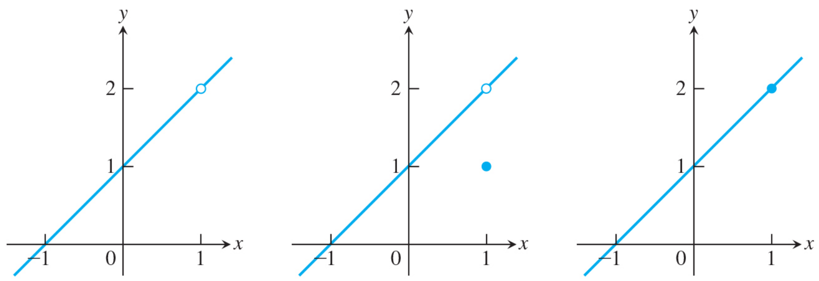
$(\cancel{x-1})(x+1)$

$$\lim_{x \to 1} f(x) = 2, \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{\cancel{x - 1}} = 2.$$

Essentially, the definition says that the values of $f(x)$ are close to the number L whenever x is close to c. The value of the function at c itself is not considered.

Our definition here is informal, because phrases like *arbitrarily close* and *sufficiently close* are imprecise; their meaning depends on the context. (To a machinist manufacturing a piston, *close* may mean *within a few thousandths of an inch*. To an astronomer studying distant galaxies, *close* may mean *within a few thousand light-years*.) Nevertheless, the definition is clear enough to enable us to recognize and evaluate limits of many specific functions. We will need the precise definition given in Section 2.3, when we set out to prove theorems about limits or study complicated functions. Here are several more examples exploring the idea of limits.

EXAMPLE 2 The limit of a function does not depend on how the function is defined at the point being approached. Consider the three functions in Figure 2.8. The function f has limit 2 as $x \to 1$ even though f is not defined at $x = 1$. The function g has limit 2 as $x \to 1$ even though $2 \neq g(1)$. The function h is the only one of the three functions in Figure 2.8 whose limit as $x \to 1$ equals its value at $x = 1$. For h, we have $\lim_{x \to 1} h(x) = h(1)$. This equality of limit and function value has an important meaning. As illustrated by the three examples in Figure 2.8, equality of limit and function value captures the notion of "continuity." We study this in detail in Section 2.5. ■



(a) $f(x) = \dfrac{x^2-1}{x-1}$ (b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

FIGURE 2.8 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 2).

The process of finding a limit can be broken up into a series of steps involving limits of basic functions, which are combined using a sequence of simple operations that we will develop. We start with two basic functions.

EXAMPLE 3 We find the limits of the identity function and of a constant function as x approaches $x = c$.

(a) If f is the **identity function** $f(x) = x$, then for any value of c (Figure 2.9a),

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c.$$

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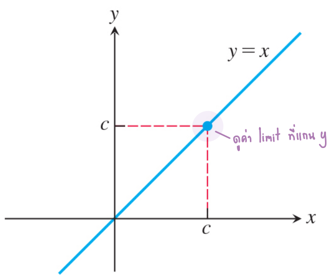
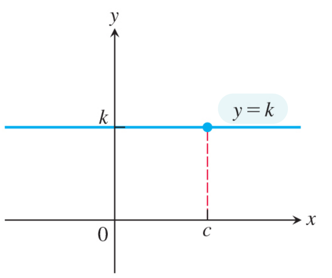


(a) Identity function



(b) Constant function

FIGURE 2.9 The functions in Example 3 have limits at all points c.

(b) If f is the **constant function** $f(x) = k$ (function with the constant value k), then for any value of c (Figure 2.9b),

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k.$$

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For instances of each of these rules we have

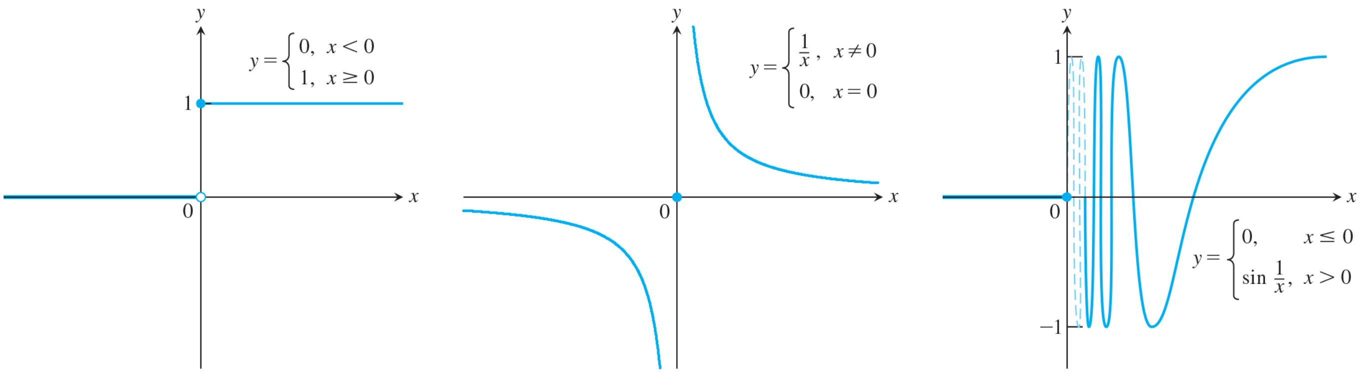
$$\lim_{x \to 3} x = 3 \qquad \text{Limit of identity function at } x = 3$$

and

$$\lim_{x \to -7} (4) = \lim_{x \to 2} (4) = 4. \qquad \text{Limit of constant function } f(x) = 4 \text{ at } x = -7 \text{ or at } x = 2$$

We prove these rules in Example 3 in Section 2.3. ■

A function may not have a limit at a particular point. Some ways that limits can fail to exist are illustrated in Figure 2.10 and described in the next example.



(a) Unit step function $U(x)$ (b) $g(x)$ (c) $f(x)$

FIGURE 2.10 None of these functions has a limit as x approaches 0 (Example 4).

EXAMPLE 4 Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \ge 0 \end{cases}$

(b) $g(x) = \begin{cases} \frac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \le 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$

Solution

(a) The function *jumps*: The **unit step function** $U(x)$ has no limit as $x \to 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \to 0$ (Figure 2.10a).

(b) The function *grows too "large" to have a limit*: $g(x)$ has no limit as $x \to 0$ because the values of g grow arbitrarily large in absolute value as $x \to 0$ and therefore do not stay close to *any* fixed real number (Figure 2.10b). We say the function is *not bounded*.

(c) The function *oscillates too much to have a limit*: $f(x)$ has no limit as $x \to 0$ because the function's values oscillate between $+1$ and -1 in every open interval containing 0. The values do not stay close to any single number as $x \to 0$ (Figure 2.10c). ■

The Limit Laws

A few basic rules allow us to break down complicated functions into simple ones when calculating limits. By using these laws, we can greatly simplify many limit computations.

THEOREM 1—Limit Laws
If L, M, c, and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x \to c} [f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing c.)

The Sum Rule says that the limit of a sum is the sum of the limits. Similarly, the next rules say that the limit of a difference is the difference of the limits; the limit of a constant times a function is the constant times the limit of the function; the limit of a product is the product of the limits; the limit of a quotient is the quotient of the limits (provided that the limit of the denominator is not 0); the limit of a positive integer power (or root) of a function is the integer power (or root) of the limit (provided that the root of the limit is a real number).

There are simple intuitive arguments for why the properties in Theorem 1 are true (although these do not constitute proofs). If x is sufficiently close to c, then $f(x)$ is close to L and $g(x)$ is close to M, from our informal definition of a limit. It is then reasonable that $f(x) + g(x)$ is close to $L + M$; $f(x) - g(x)$ is close to $L - M$; $kf(x)$ is close to kL; $f(x)g(x)$ is close to LM; and $f(x)/g(x)$ is close to L/M if M is not zero. We prove the Sum Rule in Section 2.3, based on a rigorous definition of the limit. Rules 2–5 are proved in Appendix 4. Rule 6 is obtained by applying Rule 4 repeatedly. Rule 7 is proved in more advanced texts. The Sum, Difference, and Product Rules can be extended to any number of functions, not just two.

EXAMPLE 5 Use the observations $\lim_{x \to c} k = k$ and $\lim_{x \to c} x = c$ (Example 3) and the limit laws in Theorem 1 to find the following limits.

(a) $\lim_{x \to c} (x^3 + 4x^2 - 3)$

(b) $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$

(c) $\lim_{x \to -2} \sqrt{4x^2 - 3}$

Solution

(a) $\lim_{x \to c}(x^3 + 4x^2 - 3) = \lim_{x \to c} x^3 + \lim_{x \to c} 4x^2 - \lim_{x \to c} 3$ — Sum and Difference Rules

$= c^3 + 4c^2 - 3$ — Power and Multiple Rules

(b) $\lim_{x \to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5} = \dfrac{\lim_{x \to c}(x^4 + x^2 - 1)}{\lim_{x \to c}(x^2 + 5)}$ — Quotient Rule

$= \dfrac{\lim_{x \to c} x^4 + \lim_{x \to c} x^2 - \lim_{x \to c} 1}{\lim_{x \to c} x^2 + \lim_{x \to c} 5}$ — Sum and Difference Rules

$= \dfrac{c^4 + c^2 - 1}{c^2 + 5}$ — Power or Product Rule

(c) $\lim_{x \to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \to -2}(4x^2 - 3)}$ — Root Rule with $n = 2$

$= \sqrt{\lim_{x \to -2} 4x^2 - \lim_{x \to -2} 3}$ — Difference Rule

$= \sqrt{4(-2)^2 - 3}$ — Product and Multiple Rules and limit of a constant function

$= \sqrt{16 - 3}$

$= \sqrt{13}$ ■

Evaluating Limits of Polynomials and Rational Functions

Theorem 1 simplifies the task of calculating limits of polynomials and rational functions. To evaluate the limit of a polynomial function as x approaches c, just substitute c for x in the formula for the function. To evaluate the limit of a rational function as x approaches a point *c at which the denominator is not zero*, substitute c for x in the formula for the function. (See Examples 5a and 5b.) We state these results formally as theorems.

THEOREM 2—Limits of Polynomials
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions
If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

EXAMPLE 6 The following calculation illustrates Theorems 2 and 3:

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

Since the denominator of this rational expression does not equal 0 when we substitute -1 for x, we can just compute the value of the expression at $x = -1$ to evaluate the limit. ■

Eliminating Common Factors from Zero Denominators

Theorem 3 applies only if the denominator of the rational function is not zero at the limit point c. If the denominator is zero, canceling common factors in the numerator and

Identifying Common Factors
If $Q(x)$ is a polynomial and $Q(c) = 0$, then $(x - c)$ is a factor of $Q(x)$. Thus, if the numerator and denominator of a rational function of x are both zero at $x = c$, they have $(x - c)$ as a common factor.

denominator may reduce the fraction to one whose denominator is no longer zero at c. If this happens, we can find the limit by substitution in the simplified fraction.

EXAMPLE 7 Evaluate

$$\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

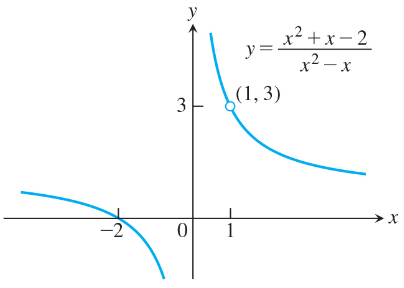
Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling this common factor gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \qquad \text{if } x \neq 1.$$

Using the simpler fraction, we find the limit of these values as $x \to 1$ by evaluating the function at $x = 1$, as in Theorem 3:

$$\lim_{x\to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x\to 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

See Figure 2.11. ■



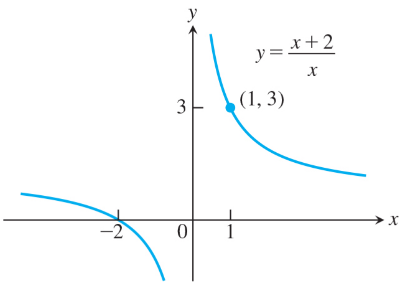


FIGURE 2.11 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \to 1$ (Example 7).

Using Calculators and Computers to Estimate Limits

We can try using a calculator or computer to guess a limit numerically. However, calculators and computers can sometimes give false values and misleading evidence about limits. Usually the problem is associated with rounding errors, as we now illustrate.

EXAMPLE 8 Estimate the value of $\lim_{x\to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$.

Solution Table 2.3 lists values of the function obtained on a calculator for several points approaching $x = 0$. As x approaches 0 through the points ± 1, ± 0.5, ± 0.10, and ± 0.01, the function seems to approach the number 0.05.

As we take even smaller values of x, ± 0.0005, ± 0.0001, ± 0.00001, and ± 0.000001, the function appears to approach the number 0.

Is the answer 0.05 or 0, or some other value? We resolve this question in the next example. ■

TABLE 2.3 Computed values of $f(x) = \frac{\sqrt{x^2 + 100} - 10}{x^2}$ near $x = 0$

x	$f(x)$	
± 1	0.049876	approaches 0.05?
± 0.5	0.049969	
± 0.1	0.049999	
± 0.01	0.050000	
± 0.0005	0.050000	approaches 0?
± 0.0001	0.000000	
± 0.00001	0.000000	
± 0.000001	0.000000	

Using a computer or calculator may give ambiguous results, as in Example 8. A computer cannot always keep track of enough digits to avoid rounding errors in computing the values of $f(x)$ when x is very small. We cannot substitute $x = 0$ in the problem, and the numerator and denominator have no obvious common factors (as they did in Example 7). Sometimes, however, we can create a common factor algebraically.

EXAMPLE 9 Evaluate

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}.$$

Solution This is the limit we considered in Example 8. We can create a common factor by multiplying both numerator and denominator by the conjugate radical expression $\sqrt{x^2 + 100} + 10$ (obtained by changing the sign after the square root). The preliminary algebra rationalizes the numerator:

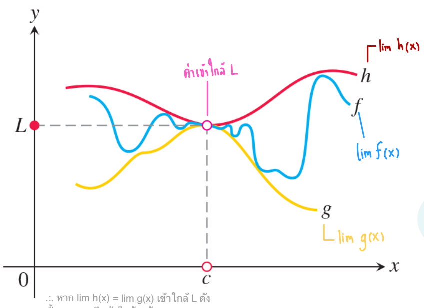
$$\begin{aligned}
\frac{\sqrt{x^2 + 100} - 10}{x^2} &= \frac{\sqrt{x^2 + 100} - 10}{x^2} \cdot \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10} && \text{Multiply and divide by the conjugate.} \\
&= \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)} && \text{Simplify} \\
&= \frac{x^2}{x^2(\sqrt{x^2 + 100} + 10)} && \text{Common factor } x^2 \\
&= \frac{1}{\sqrt{x^2 + 100} + 10}. && \text{Cancel } x^2 \text{ for } x \neq 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 100} + 10} \\
&= \frac{1}{\sqrt{0^2 + 100} + 10} && \text{Limit Quotient Rule: Denominator not 0 at } x = 0 \text{ so can substitute.} \\
&= \frac{1}{20} = 0.05.
\end{aligned}$$

This calculation provides the correct answer, resolving the ambiguous computer results in Example 8. ■

We cannot always manipulate the terms in an expression to find the limit of a quotient where the denominator becomes zero. In some cases the limit might then be found with geometric arguments (see the proof of Theorem 7 in Section 2.4), or through methods of calculus (developed in Section 4.5). The next theorem shows how to evaluate difficult limits by comparing them with functions having known limits.



∴ หาก lim h(x) = lim g(x) เข้าใกล้ L ดังนั้น lim f(x) จึงเข้าใกล้ L ด้วย

FIGURE 2.12 The graph of f is sandwiched between the graphs of g and h.

The Sandwich Theorem

The following theorem enables us to calculate a variety of limits. It is called the Sandwich Theorem because it refers to a function f whose values are sandwiched between the values of two other functions g and h that have the same limit L at a point c. Being trapped between the values of two functions that approach L, the values of f must also approach L (Figure 2.12). A proof is given in Appendix 4.

THEOREM 4—The Sandwich Theorem
Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.

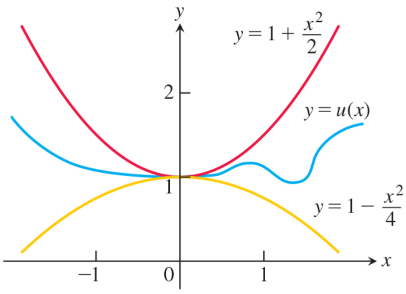


FIGURE 2.13 Any function $u(x)$ whose graph lies in the region between $y = 1 + (x^2/2)$ and $y = 1 - (x^2/4)$ has limit 1 as $x \to 0$ (Example 10).

The Sandwich Theorem is also called the Squeeze Theorem or the Pinching Theorem.

EXAMPLE 10 Given a function u that satisfies

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \qquad \text{for all } x \ne 0,$$

find $\lim_{x\to 0} u(x)$, no matter how complicated u is.

Solution Since

$$\lim_{x\to 0} (1 - (x^2/4)) = 1 \qquad \text{and} \qquad \lim_{x\to 0} (1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x\to 0} u(x) = 1$ (Figure 2.13). ■

EXAMPLE 11 The Sandwich Theorem helps us establish several important limit rules:

(a) $\lim_{\theta\to 0} \sin\theta = 0$

(b) $\lim_{\theta\to 0} \cos\theta = 1$

(c) For any function f, $\lim_{x\to c} |f(x)| = 0$ implies $\lim_{x\to c} f(x) = 0$.

Solution

(a) In Section 1.3 we established that $-|\theta| \le \sin\theta \le |\theta|$ for all θ (see Figure 2.14a). Since $\lim_{\theta\to 0}(-|\theta|) = \lim_{\theta\to 0} |\theta| = 0$, we have

$$\lim_{\theta\to 0} \sin\theta = 0.$$

(b) From Section 1.3, $0 \le 1 - \cos\theta \le |\theta|$ for all θ (see Figure 2.14b), and we have $\lim_{\theta\to 0}(1 - \cos\theta) = 0$ so

$$\lim_{\theta\to 0} 1 - (1 - \cos\theta) = 1 - \lim_{\theta\to 0} (1 - \cos\theta) = 1 - 0,$$

$$\lim_{\theta\to 0} \cos\theta = 1. \qquad \text{Simplify}$$

(c) Since $-|f(x)| \le f(x) \le |f(x)|$ and $-|f(x)|$ and $|f(x)|$ have limit 0 as $x \to c$, it follows that $\lim_{x\to c} f(x) = 0$. ■

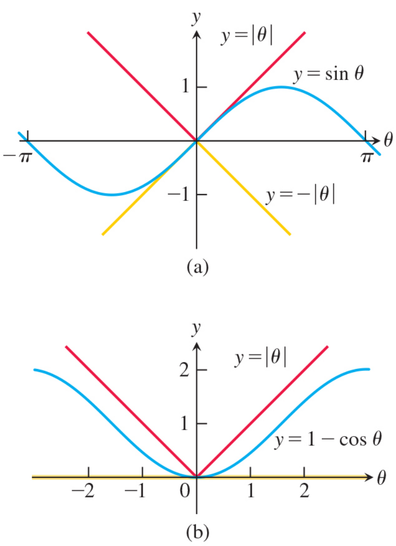


FIGURE 2.14 The Sandwich Theorem confirms the limits in Example 11.

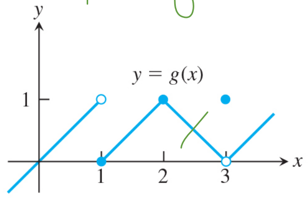
Example 11 shows that the sine and cosine functions are equal to their limits at $\theta = 0$. We have not yet established that for any c, $\lim_{\theta\to c} \sin\theta = \sin c$, and $\lim_{\theta\to c} \cos\theta = \cos c$. These limit formulas do hold, as will be shown in Section 2.5.

EXERCISES 2.2

Limits from Graphs

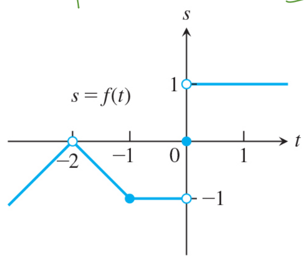
1. For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

   a. $\lim_{x\to 1} g(x)$ ✗ b. $\lim_{x\to 2} g(x)$ 1 c. $\lim_{x\to 3} g(x)$ 0 d. $\lim_{x\to 2.5} g(x)$ 1.5



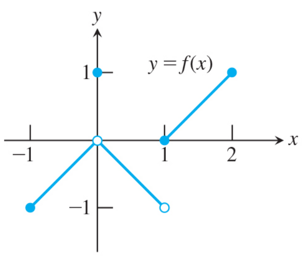
2. For the function $f(t)$ graphed here, find the following limits or explain why they do not exist.

   a. $\lim_{t\to -2} f(t)$ 0 b. $\lim_{t\to -1} f(t)$ −1 c. $\lim_{t\to 0} f(t)$ ✗ d. $\lim_{t\to -0.5} f(t)$ −1



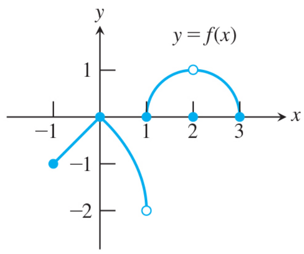
3. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

   a. $\lim_{x\to 0} f(x)$ exists. ✓
   b. $\lim_{x\to 0} f(x) = 0$ ✓
   c. $\lim_{x\to 0} f(x) = 1$ ✗
   d. $\lim_{x\to 1} f(x) = 1$ ✗
   e. $\lim_{x\to 1} f(x) = 0$ ✗
   f. $\lim_{x\to c} f(x)$ exists at every point c in $(-1, 1)$. ✓
   g. $\lim_{x\to 1} f(x)$ does not exist. ✓
   h. $f(0) = 0$ ✗
   i. $f(0) = 1$ ✓
   j. $f(1) = 0$ ✓
   k. $f(1) = -1$ ✗



4. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?

   a. $\lim_{x\to 2} f(x)$ does not exist.
   b. $\lim_{x\to 2} f(x) = 2$
   c. $\lim_{x\to 1} f(x)$ does not exist.
   d. $\lim_{x\to c} f(x)$ exists at every point c in $(-1, 1)$.
   e. $\lim_{x\to c} f(x)$ exists at every point c in $(1, 3)$.
   f. $f(1) = 0$
   g. $f(1) = -2$
   h. $f(2) = 0$
   i. $f(2) = 1$



Existence of Limits

In Exercises 5 and 6, explain why the limits do not exist.

5. $\lim_{x\to 0} \dfrac{x}{|x|}$

6. $\lim_{x\to 1} \dfrac{1}{x-1}$

7. Suppose that a function $f(x)$ is defined for all real values of x except $x = c$. Can anything be said about the existence of $\lim_{x\to c} f(x)$? Give reasons for your answer.

8. Suppose that a function $f(x)$ is defined for all x in $[-1, 1]$. Can anything be said about the existence of $\lim_{x\to 0} f(x)$? Give reasons for your answer.

9. If $\lim_{x\to 1} f(x) = 5$, must f be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude *anything* about the values of f at $x = 1$? Explain.

10. If $f(1) = 5$, must $\lim_{x\to 1} f(x)$ exist? If it does, then must $\lim_{x\to 1} f(x) = 5$? Can we conclude *anything* about $\lim_{x\to 1} f(x)$? Explain.

Calculating Limits

Find the limits in Exercises 11–22.

11. $\lim_{x\to -3} (x^2 - 13)$ -4

12. $\lim_{x\to 2} (-x^2 + 5x - 2)$

13. $\lim_{t\to 6} 8(t-5)(t-7)$ -8

14. $\lim_{x\to -2} (x^3 - 2x^2 + 4x + 8)$

15. $\lim_{x\to 2} \dfrac{2x+5}{11-x^3}$ $\frac{9}{3} = 3$

16. $\lim_{s\to 2/3} (8-3s)(2s-1)$

17. $\lim_{x\to -1/2} 4x(3x+4)^2$ $= -\frac{25}{2}$

18. $\lim_{y\to 2} \dfrac{y+2}{y^2+5y+6}$

19. $\lim_{y\to -3} (5-y)^{4/3}$ 8 $\sqrt[3]{64}$ $= 16$

20. $\lim_{z\to 4} \sqrt{z^2 - 10}$

21. $\lim_{h\to 0} \dfrac{3}{\sqrt{3h+1}+1}$ $\frac{3}{2}$

22. $\lim_{h\to 0} \dfrac{\sqrt{5h+4}-2}{h}$

Limits of quotients Find the limits in Exercises 23–42.

23. $\lim_{x\to 5} \dfrac{x-5}{x^2-25}$ $\dfrac{x-5}{(x-5)(x+5)} = \dfrac{1}{10}$

24. $\lim_{x\to -3} \dfrac{x+3}{x^2+4x+3}$

25. $\lim_{x\to -5} \dfrac{x^2+3x-10}{x+5}$ $\dfrac{\cancel{(x+5)}(x-2)}{\cancel{x+5}} = -7$

26. $\lim_{x\to 2} \dfrac{x^2-7x+10}{x-2}$

27. $\lim_{t\to 1} \dfrac{t^2+t-2}{t^2-1}$ $\dfrac{(t+2)\cancel{(t-1)}}{\cancel{(t-1)}(t+1)} = \dfrac{3}{2}$

28. $\lim_{t\to -1} \dfrac{t^2+3t+2}{t^2-t-2}$

29. $\lim_{x\to -2} \dfrac{-2x-4}{x^3+2x^2}$ $\dfrac{-2\cancel{(x+2)}}{x^2\cancel{(x+2)}} = \dfrac{-2}{x^2} = -\dfrac{1}{2}$

30. $\lim_{y\to 0} \dfrac{5y^3+8y^2}{3y^4-16y^2}$

31. $\lim\limits_{x\to 1} \dfrac{x^{-1}-1}{x-1}$

32. $\lim\limits_{x\to 0} \dfrac{\frac{1}{x-1}+\frac{1}{x+1}}{x}$

33. $\lim\limits_{u\to 1} \dfrac{u^4-1}{u^3-1}$

34. $\lim\limits_{v\to 2} \dfrac{v^3-8}{v^4-16}$

35. $\lim\limits_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$

36. $\lim\limits_{x\to 4} \dfrac{4x-x^2}{2-\sqrt{x}}$

37. $\lim\limits_{x\to 1} \dfrac{x-1}{\sqrt{x+3}-2}$

38. $\lim\limits_{x\to -1} \dfrac{\sqrt{x^2+8}-3}{x+1}$

39. $\lim\limits_{x\to 2} \dfrac{\sqrt{x^2+12}-4}{x-2}$

40. $\lim\limits_{x\to -2} \dfrac{x+2}{\sqrt{x^2+5}-3}$

41. $\lim\limits_{x\to -3} \dfrac{2-\sqrt{x^2-5}}{x+3}$

42. $\lim\limits_{x\to 4} \dfrac{4-x}{5-\sqrt{x^2+9}}$

Limits with trigonometric functions Find the limits in Exercises 43–50.

43. $\lim\limits_{x\to 0}(2\sin x - 1)$ $= -1$

44. $\lim\limits_{x\to 0}\sin^2 x$

45. $\lim\limits_{x\to 0}\sec x$ $= 1$

46. $\lim\limits_{x\to 0}\tan x$

47. $\lim\limits_{x\to 0} \dfrac{1+x+\cancel{\sin x}^{\,0}}{3\cancel{\cos x}_{\,1}}$ $\frac{1}{3}$

48. $\lim\limits_{x\to 0}(x^2-1)(2-\cos x)$

49. $\lim\limits_{x\to -\pi}\sqrt{x+4}\,\cos(x+\pi)$ $\sqrt{4-\pi}$

50. $\lim\limits_{x\to 0}\sqrt{7+\sec^2 x}$

Using Limit Rules

51. Suppose $\lim_{x\to 0} f(x) = 1$ and $\lim_{x\to 0} g(x) = -5$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 0}\frac{2f(x)-g(x)}{(f(x)+7)^{2/3}} = \frac{\lim\limits_{x\to 0}(2f(x)-g(x))}{\lim\limits_{x\to 0}(f(x)+7)^{2/3}} \quad \text{(a)}$$

$$= \frac{\lim\limits_{x\to 0}2f(x) - \lim\limits_{x\to 0}g(x)}{\left(\lim\limits_{x\to 0}(f(x)+7)\right)^{2/3}} \quad \text{(b)}$$

$$= \frac{2\lim\limits_{x\to 0}f(x) - \lim\limits_{x\to 0}g(x)}{\left(\lim\limits_{x\to 0}f(x) + \lim\limits_{x\to 0}7\right)^{2/3}} \quad \text{(c)}$$

$$= \frac{(2)(1)-(-5)}{(1+7)^{2/3}} = \frac{7}{4}$$

52. Let $\lim_{x\to 1} h(x) = 5$, $\lim_{x\to 1} p(x) = 1$, and $\lim_{x\to 1} r(x) = 2$. Name the rules in Theorem 1 that are used to accomplish steps (a), (b), and (c) of the following calculation.

$$\lim_{x\to 1}\frac{\sqrt{5h(x)}}{p(x)(4-r(x))} = \frac{\lim\limits_{x\to 1}\sqrt{5h(x)}}{\lim\limits_{x\to 1}(p(x)(4-r(x)))} \quad \text{(a)}$$

$$= \frac{\sqrt{\lim\limits_{x\to 1}5h(x)}}{\left(\lim\limits_{x\to 1}p(x)\right)\left(\lim\limits_{x\to 1}(4-r(x))\right)} \quad \text{(b)}$$

$$= \frac{\sqrt{5\lim\limits_{x\to 1}h(x)}}{\left(\lim\limits_{x\to 1}p(x)\right)\left(\lim\limits_{x\to 1}4 - \lim\limits_{x\to 1}r(x)\right)} \quad \text{(c)}$$

$$= \frac{\sqrt{(5)(5)}}{(1)(4-2)} = \frac{5}{2}$$

53. Suppose $\lim_{x\to c} f(x) = 5$ and $\lim_{x\to c} g(x) = -2$. Find

a. $\lim\limits_{x\to c} f(x)g(x)$ **b.** $\lim\limits_{x\to c} 2f(x)g(x)$

c. $\lim\limits_{x\to c}(f(x)+3g(x))$ **d.** $\lim\limits_{x\to c}\dfrac{f(x)}{f(x)-g(x)}$

54. Suppose $\lim_{x\to 4} f(x) = 0$ and $\lim_{x\to 4} g(x) = -3$. Find

a. $\lim\limits_{x\to 4}(g(x)+3)$ **b.** $\lim\limits_{x\to 4} xf(x)$

c. $\lim\limits_{x\to 4}(g(x))^2$ **d.** $\lim\limits_{x\to 4}\dfrac{g(x)}{f(x)-1}$

55. Suppose $\lim_{x\to b} f(x) = 7$ and $\lim_{x\to b} g(x) = -3$. Find

a. $\lim\limits_{x\to b}(f(x)+g(x))$ **b.** $\lim\limits_{x\to b} f(x)\cdot g(x)$

c. $\lim\limits_{x\to b} 4g(x)$ **d.** $\lim\limits_{x\to b} f(x)/g(x)$

56. Suppose that $\lim_{x\to -2} p(x) = 4$, $\lim_{x\to -2} r(x) = 0$, and $\lim_{x\to -2} s(x) = -3$. Find

a. $\lim\limits_{x\to -2}(p(x)+r(x)+s(x))$

b. $\lim\limits_{x\to -2} p(x)\cdot r(x)\cdot s(x)$

c. $\lim\limits_{x\to -2}(-4p(x)+5r(x))/s(x)$

Limits of Average Rates of Change

Because of their connection with secant lines, tangents, and instantaneous rates, limits of the form

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

occur frequently in calculus. In Exercises 57–62, evaluate this limit for the given value of x and function f.

57. $f(x) = x^2$, $x = 1$ $y = 1 \rightsquigarrow \dfrac{(1+h)^2-1^2}{h} = \dfrac{h^2+2h}{h} = 2$

58. $f(x) = x^2$, $x = -2$

59. $f(x) = 3x-4$, $x = 2$ $y = 2 \rightsquigarrow \dfrac{3(2+h)-4-2}{h} = \dfrac{6+3h-6}{h} = 3$

60. $f(x) = 1/x$, $x = -2$

61. $f(x) = \sqrt{x}$, $x = 7$ $y = \sqrt{7} \rightsquigarrow \dfrac{\sqrt{7+h}-\sqrt{7}}{h} \rightsquigarrow \dfrac{\cancel{7}+\cancel{h}-\cancel{7}}{\cancel{h}(\sqrt{7+h}+\sqrt{7})} = \dfrac{1}{\sqrt{7+h}+\sqrt{7}}$

62. $f(x) = \sqrt{3x+1}$, $x = 0$

Using the Sandwich Theorem

63. If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$. $= \sqrt{5}$

64. If $2 - x^2 \le g(x) \le 2\cos x$ for all x, find $\lim_{x\to 0} g(x)$.

65. a. It can be shown that the inequalities

$$1 - \frac{x^2}{6} < \frac{x\sin x}{\underbrace{2-2\cos x}_{2(1-\cos x)}} < 1$$

hold for all values of x close to zero. What, if anything, does this tell you about

$$\lim_{x\to 0}\frac{x\sin x}{2-2\cos x}?$$

Give reasons for your answer.

T **b.** Graph $y = 1-(x^2/6)$, $y = (x\sin x)/(2-2\cos x)$, and $y = 1$ together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

66. **a.** Suppose that the inequalities

$$\frac{1}{2} - \frac{x^2}{24} < \frac{1 - \cos x}{x^2} < \frac{1}{2}$$

hold for values of x close to zero. (They do, as you will see in Section 9.9.) What, if anything, does this tell you about

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2}?$$

Give reasons for your answer.

T **b.** Graph the equations $y = (1/2) - (x^2/24)$, $y = (1 - \cos x)/x^2$, and $y = 1/2$ together for $-2 \le x \le 2$. Comment on the behavior of the graphs as $x \to 0$.

Estimating Limits

-3

T You will find a graphing calculator useful for Exercises 67–76.

67. Let $f(x) = (x^2 - 9)/(x + 3)$. $(x-3)(x+3)$

a. Make a table of the values of f at the points $x = -3.1$, $-3.01, -3.001$, and so on as far as your calculator can go. Then estimate $\lim_{x \to -3} f(x)$. What estimate do you arrive at if you evaluate f at $x = -2.9, -2.99, -2.999, \ldots$ instead?

b. Support your conclusions in part (a) by graphing f near $c = -3$ and using Zoom and Trace to estimate y-values on the graph as $x \to -3$.

c. Find $\lim_{x \to -3} f(x)$ algebraically, as in Example 7.

68. Let $g(x) = (x^2 - 2)/(x - \sqrt{2})$.

a. Make a table of the values of g at the points $x = 1.4, 1.41$, 1.414, and so on through successive decimal approximations of $\sqrt{2}$. Estimate $\lim_{x \to \sqrt{2}} g(x)$.

b. Support your conclusion in part (a) by graphing g near $c = \sqrt{2}$ and using Zoom and Trace to estimate y-values on the graph as $x \to \sqrt{2}$.

c. Find $\lim_{x \to \sqrt{2}} g(x)$ algebraically.

69. Let $G(x) = (x + 6)/(x^2 + 4x - 12)$.

a. Make a table of the values of G at $x = -5.9, -5.99, -5.999$, and so on. Then estimate $\lim_{x \to -6} G(x)$. What estimate do you arrive at if you evaluate G at $x = -6.1, -6.01$, $-6.001, \ldots$ instead?

b. Support your conclusions in part (a) by graphing G and using Zoom and Trace to estimate y-values on the graph as $x \to -6$.

c. Find $\lim_{x \to -6} G(x)$ algebraically.

70. Let $h(x) = (x^2 - 2x - 3)/(x^2 - 4x + 3)$.

a. Make a table of the values of h at $x = 2.9, 2.99, 2.999$, and so on. Then estimate $\lim_{x \to 3} h(x)$. What estimate do you arrive at if you evaluate h at $x = 3.1, 3.01, 3.001, \ldots$ instead?

b. Support your conclusions in part (a) by graphing h near $c = 3$ and using Zoom and Trace to estimate y-values on the graph as $x \to 3$.

c. Find $\lim_{x \to 3} h(x)$ algebraically.

71. Let $f(x) = (x^2 - 1)/(|x| - 1)$. $\frac{-(x-1)(x+1)}{-x-1}$, $\frac{(x-1)(x+1)}{x-1}$

a. Make tables of values of f at values of x that approach $c = -1$ from above and below. Then estimate $\lim_{x \to -1} f(x)$.

$\frac{(-1.5)^2-1}{|-1.5|-1} = \frac{1.25}{0.5}$ 2.25 2.5 $+2, 0$

b. Support your conclusion in part (a) by graphing f near $c = -1$ and using Zoom and Trace to estimate y-values on the graph as $x \to -1$.

c. Find $\lim_{x \to -1} f(x)$ algebraically.

72. Let $F(x) = (x^2 + 3x + 2)/(2 - |x|)$.

a. Make tables of values of F at values of x that approach $c = -2$ from above and below. Then estimate $\lim_{x \to -2} F(x)$.

b. Support your conclusion in part (a) by graphing F near $c = -2$ and using Zoom and Trace to estimate y-values on the graph as $x \to -2$.

c. Find $\lim_{x \to -2} F(x)$ algebraically.

73. Let $g(\theta) = (\sin \theta)/\theta$.

a. Make a table of the values of g at values of θ that approach $\theta_0 = 0$ from above and below. Then estimate $\lim_{\theta \to 0} g(\theta)$.

b. Support your conclusion in part (a) by graphing g near $\theta_0 = 0$.

74. Let $G(t) = (1 - \cos t)/t^2$.

a. Make tables of values of G at values of t that approach $t_0 = 0$ from above and below. Then estimate $\lim_{t \to 0} G(t)$.

b. Support your conclusion in part (a) by graphing G near $t_0 = 0$.

75. Let $f(x) = x^{1/(1-x)}$.

a. Make tables of values of f at values of x that approach $c = 1$ from above and below. Does f appear to have a limit as $x \to 1$? If so, what is it? If not, why not?

b. Support your conclusions in part (a) by graphing f near $c = 1$.

76. Let $f(x) = (3^x - 1)/x$.

a. Make tables of values of f at values of x that approach $c = 0$ from above and below. Does f appear to have a limit as $x \to 0$? If so, what is it? If not, why not?

b. Support your conclusions in part (a) by graphing f near $c = 0$.

Theory and Examples

77. If $x^4 \le f(x) \le x^2$ for x in $[-1, 1]$ and $x^2 \le f(x) \le x^4$ for $x < -1$ and $x > 1$, at what points c do you automatically know $\lim_{x \to c} f(x)$? What can you say about the value of the limit at these points?

78. Suppose that $g(x) \le f(x) \le h(x)$ for all $x \ne 2$ and suppose that

$$\lim_{x \to 2} g(x) = \lim_{x \to 2} h(x) = -5.$$

Can we conclude anything about the values of f, g, and h at $x = 2$? Could $f(2) = 0$? Could $\lim_{x \to 2} f(x) = 0$? Give reasons for your answers.

79. If $\displaystyle\lim_{x \to 4} \frac{f(x) - 5}{x - 2} = 1$, find $\displaystyle\lim_{x \to 4} f(x)$.

80. If $\displaystyle\lim_{x \to -2} \frac{f(x)}{x^2} = 1$, find

a. $\displaystyle\lim_{x \to -2} f(x)$ **b.** $\displaystyle\lim_{x \to -2} \frac{f(x)}{x}$

81. **a.** If $\displaystyle\lim_{x \to 2} \frac{f(x) - 5}{x - 2} = 3$, find $\displaystyle\lim_{x \to 2} f(x)$.

b. If $\displaystyle\lim_{x \to 2} \frac{f(x) - 5}{x - 2} = 4$, find $\displaystyle\lim_{x \to 2} f(x)$.

82. If $\lim_{x\to 0} \dfrac{f(x)}{x^2} = 1$, find

a. $\lim_{x\to 0} f(x)$ **b.** $\lim_{x\to 0} \dfrac{f(x)}{x}$

T **83.** **a.** Graph $g(x) = x\sin(1/x)$ to estimate $\lim_{x\to 0} g(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

T **84.** **a.** Graph $h(x) = x^2\cos(1/x^3)$ to estimate $\lim_{x\to 0} h(x)$, zooming in on the origin as necessary.

b. Confirm your estimate in part (a) with a proof.

COMPUTER EXPLORATIONS

Graphical Estimates of Limits

In Exercises 85–90, use a CAS to perform the following steps:

a. Plot the function near the point c being approached.

b. From your plot guess the value of the limit.

85. $\lim_{x\to 2} \dfrac{x^4 - 16}{x - 2}$

86. $\lim_{x\to -1} \dfrac{x^3 - x^2 - 5x - 3}{(x+1)^2}$

87. $\lim_{x\to 0} \dfrac{\sqrt[3]{1+x} - 1}{x}$

88. $\lim_{x\to 3} \dfrac{x^2 - 9}{\sqrt{x^2+7} - 4}$

89. $\lim_{x\to 0} \dfrac{1 - \cos x}{x\sin x}$

90. $\lim_{x\to 0} \dfrac{2x^2}{3 - 3\cos x}$

2.3 The Precise Definition of a Limit

We now turn our attention to the precise definition of a limit. The early history of calculus saw controversy about the validity of the basic concepts underlying the theory. Apparent contradictions were argued over by both mathematicians and philosophers. These controversies were resolved by the precise definition, which allows us to replace vague phrases like "gets arbitrarily close to" in the informal definition with specific conditions that can be applied to any particular example. With a rigorous definition, we can avoid misunderstandings, prove the limit properties given in the preceding section, and establish many important limits.

To show that the limit of $f(x)$ as $x \to c$ equals the number L, we need to show that the gap between $f(x)$ and L can be made "as small as we choose" if x is kept "close enough" to c. Let us see what this requires if we specify the size of the gap between $f(x)$ and L.

EXAMPLE 1 Consider the function $y = 2x - 1$ near $x = 4$. Intuitively it seems clear that y is close to 7 when x is close to 4, so $\lim_{x\to 4}(2x - 1) = 7$. However, how close to $x = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by, say, less than 2 units?

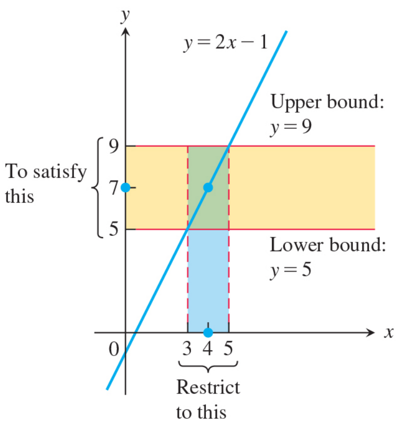


FIGURE 2.15 Keeping x within 1 unit of $x = 4$ will keep y within 2 units of $y = 7$ (Example 1).

Solution We are asked: For what values of x is $|y - 7| < 2$? To find the answer we first express $|y - 7|$ in terms of x:

$$|y - 7| = |(2x - 1) - 7| = |2x - 8|.$$

The question then becomes: what values of x satisfy the inequality $|2x - 8| < 2$? To find out, we solve the inequality:

$$\begin{aligned}
|2x - 8| &< 2 \\
-2 < 2x - 8 &< 2 && \text{Removing absolute value gives two inequalities.} \\
6 < 2x &< 10 && \text{Add 8 to each term.} \\
3 < x &< 5 && \text{Solve for } x. \\
-1 < x - 4 &< 1. && \text{Solve for } x - 4.
\end{aligned}$$

Keeping x within 1 unit of $x = 4$ will keep y within 2 units of $y = 7$ (Figure 2.15). ■

In the previous example we determined how close x must be to a particular value c to ensure that the outputs $f(x)$ of some function lie within a prescribed interval about a limit value L. To show that the limit of $f(x)$ as $x \to c$ actually equals L, we must be able to show that the gap between $f(x)$ and L can be made less than *any prescribed error*, no matter how

δ is the Greek letter delta
ε is the Greek letter epsilon

small, by holding x close enough to c. To describe arbitrary prescribed errors, we introduce two constants, δ (delta) and ε (epsilon). These Greek letters are traditionally used to represent small changes in a variable or a function.

Definition of Limit

Suppose we are watching the values of a function $f(x)$ as x approaches c (without taking on the value c itself). Certainly we want to be able to say that $f(x)$ stays within one-tenth of a unit from L as soon as x stays within some distance δ of c (Figure 2.16). But that in itself is not enough, because as x continues on its course toward c, what is to prevent $f(x)$ from jumping around within the interval from $L - (1/10)$ to $L + (1/10)$ without tending toward L? We can be told that the error can be no more than $1/100$ or $1/1000$ or $1/100{,}000$. Each time, we find a new δ-interval about c so that keeping x within that interval satisfies the new error tolerance. And each time the possibility exists that $f(x)$ might jump away from L at some later stage.

The figures on the next page illustrate the problem. You can think of this as a quarrel between a skeptic and a scholar. The skeptic presents ε-challenges to show there is room for doubt that the limit exists. The scholar counters every challenge with a δ-interval around c which ensures that the function takes values within ε of L.

How do we stop this seemingly endless series of challenges and responses? We can do so by proving that for *every* error tolerance ε that the challenger can produce, we can present a matching distance δ that keeps x "close enough" to c to keep $f(x)$ within that ε-tolerance of L (Figure 2.17). This leads us to the precise definition of a limit.

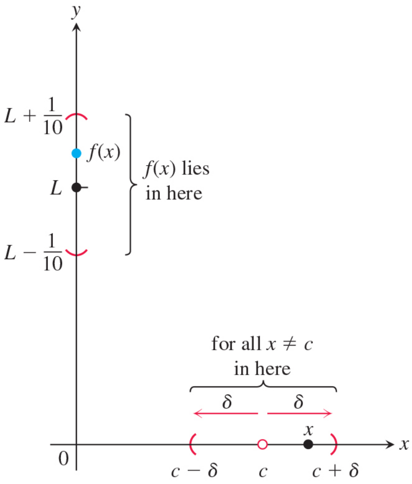


FIGURE 2.16 How should we define $\delta > 0$ so that keeping x within the interval $(c - \delta, c + \delta)$ will keep $f(x)$ within the interval $\left(L - \frac{1}{10}, L + \frac{1}{10}\right)$?

DEFINITION Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L**, and write

$$\lim_{x \to c} f(x) = L,$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

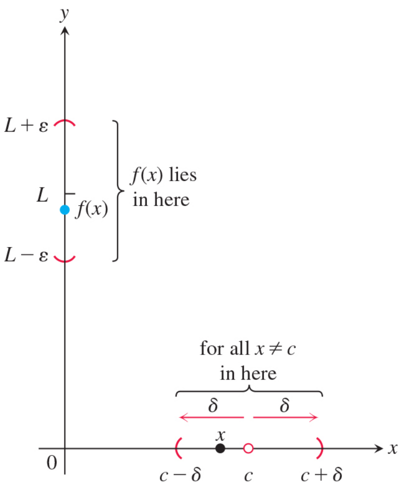


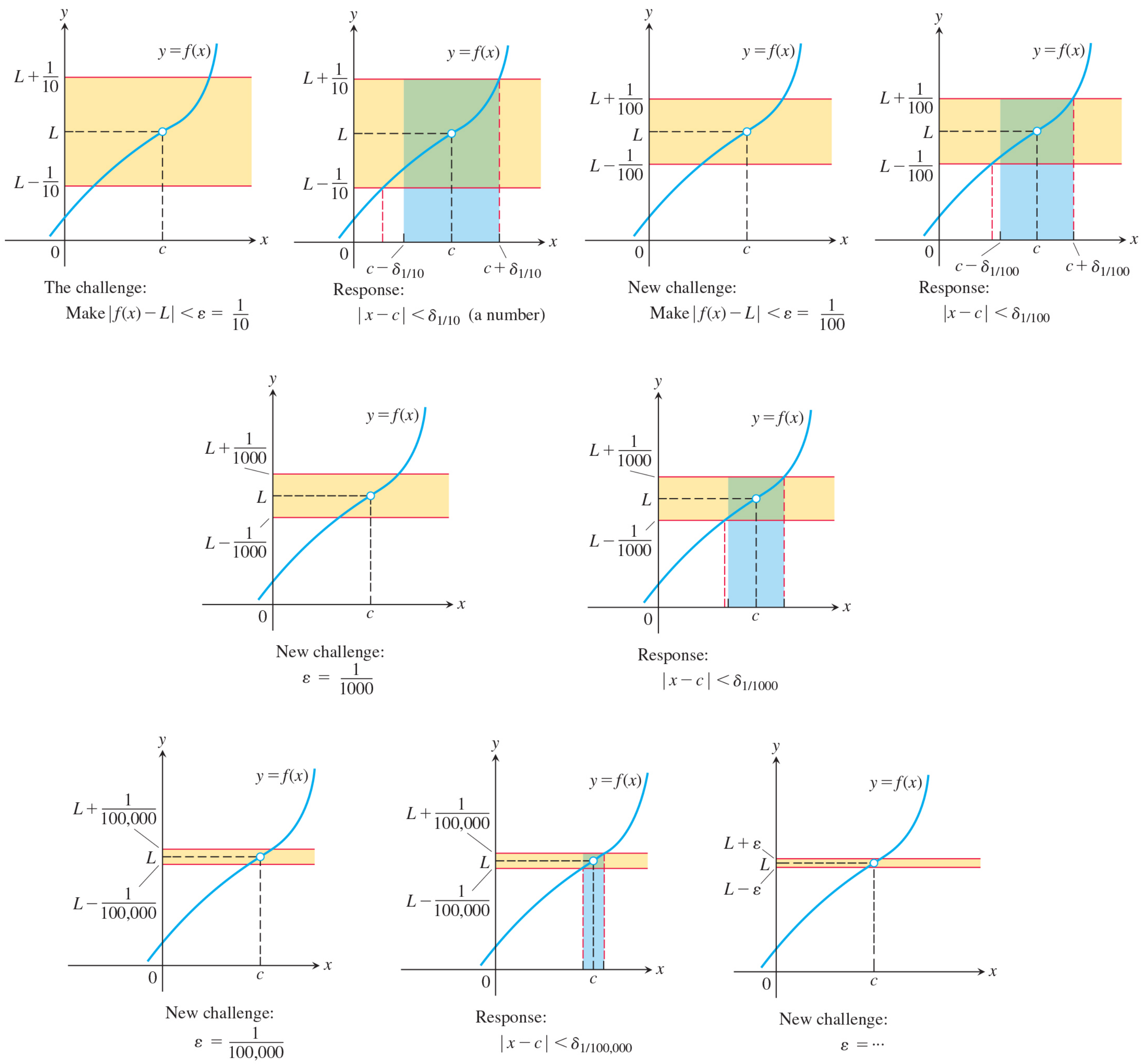
FIGURE 2.17 The relation of δ and ε in the definition of limit.

To visualize the definition, imagine machining a cylindrical shaft to a close tolerance. The diameter of the shaft is determined by turning a dial to a setting measured by a variable x. We try for diameter L, but since nothing is perfect we must be satisfied with a diameter $f(x)$ somewhere between $L - \varepsilon$ and $L + \varepsilon$. The number δ is our control tolerance for the dial; it tells us how close our dial setting must be to the setting $x = c$ in order to guarantee that the diameter $f(x)$ of the shaft will be accurate to within ε of L. As the tolerance for error becomes stricter, we may have to adjust δ. The value of δ, how tight our control setting must be, depends on the value of ε, the error tolerance.

The definition of limit extends to functions on more general domains. It is only required that each open interval around c contains points in the domain of the function other than c. See Additional and Advanced Exercises 39–43 for examples of limits for functions with complicated domains. In the next section we will see how the definition of limit applies at points lying on the boundary of an interval.

Examples: Testing the Definition

The formal definition of limit does not tell how to find the limit of a function, but it does enable us to verify that a conjectured limit value is correct. The following examples show how the definition can be used to verify limit statements for specific functions. However, the real purpose of the definition is not to do calculations like this, but rather to prove general theorems so that the calculation of specific limits can be simplified, such as the theorems stated in the previous section.



EXAMPLE 2 Show that

$$\lim_{x\to 1}(5x - 3) = 2.$$

Solution Set $c = 1$, $f(x) = 5x - 3$, and $L = 2$ in the definition of limit. For any given $\varepsilon > 0$, we have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ of $c = 1$, that is, whenever

$$0 < |x - 1| < \delta,$$

it is true that $f(x)$ is within distance ε of $L = 2$, so

$$|f(x) - 2| < \varepsilon.$$

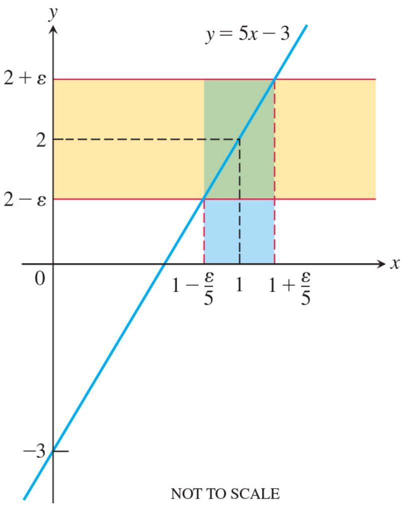


FIGURE 2.18 If $f(x) = 5x - 3$, then $0 < |x - 1| < \varepsilon/5$ guarantees that $|f(x) - 2| < \varepsilon$ (Example 2).

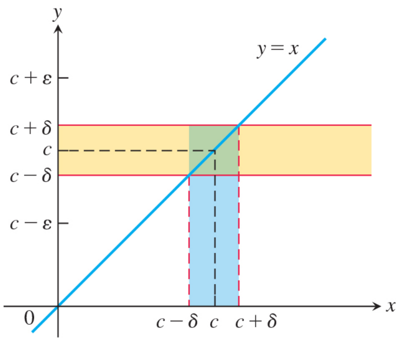


FIGURE 2.19 For the function $f(x) = x$, we find that $0 < |x - c| < \delta$ will guarantee $|f(x) - c| < \varepsilon$ whenever $\delta \le \varepsilon$ (Example 3a).

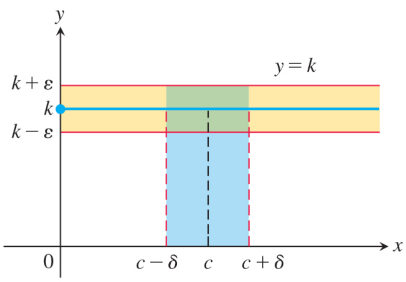


FIGURE 2.20 For the function $f(x) = k$, we find that $|f(x) - k| < \varepsilon$ for any positive δ (Example 3b).

We find δ by working backward from the ε-inequality:

$$\begin{aligned} |(5x - 3) - 2| = |5x - 5| &< \varepsilon \\ 5|x - 1| &< \varepsilon \\ |x - 1| &< \varepsilon/5. \end{aligned}$$

Thus, we can take $\delta = \varepsilon/5$ (Figure 2.18). If $0 < |x - 1| < \delta = \varepsilon/5$, then

$$|(5x - 3) - 2| = |5x - 5| = 5\,|x - 1| < 5(\varepsilon/5) = \varepsilon,$$

which proves that $\lim_{x\to 1}(5x - 3) = 2$.

The value of $\delta = \varepsilon/5$ is not the only value that will make $0 < |x - 1| < \delta$ imply $|5x - 5| < \varepsilon$. Any smaller positive δ will do as well. The definition does not ask for the "best" positive δ, just one that will work. ■

EXAMPLE 3 Prove the following results presented graphically in Section 2.2.

(a) $\lim_{x\to c} x = c$

(b) $\lim_{x\to c} k = k$ (k constant)

Solution

(a) Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$|x - c| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - c| < \delta.$$

The implication will hold if δ equals ε or any smaller positive number (Figure 2.19). This proves that $\lim_{x\to c} x = c$.

(b) Let $\varepsilon > 0$ be given. We must find $\delta > 0$ such that

$$|k - k| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - c| < \delta.$$

Since $k - k = 0$, we can use any positive number for δ and the implication will hold (Figure 2.20). This proves that $\lim_{x\to c} k = k$. ■

Finding Deltas Algebraically for Given Epsilons

In Examples 2 and 3, the interval of values about c for which $|f(x) - L|$ was less than ε was symmetric about c and we could take δ to be half the length of that interval. When the interval around c on which we have $|f(x) - L| < \varepsilon$ is not symmetric about c, we can take δ to be the distance from c to the interval's *nearer* endpoint.

EXAMPLE 4 For the limit $\lim_{x\to 5}\sqrt{x - 1} = 2$, find a $\delta > 0$ that works for $\varepsilon = 1$. That is, find a $\delta > 0$ such that

$$\left|\sqrt{x - 1} - 2\right| < 1 \quad \text{whenever} \quad 0 < |x - 5| < \delta.$$

Solution We organize the search into two steps.

1. *Solve the inequality* $\left|\sqrt{x - 1} - 2\right| < 1$ *to find an interval containing* $x = 5$ *on which the inequality holds for all* $x \ne 5$.

$$\begin{gathered} \left|\sqrt{x - 1} - 2\right| < 1 \\ -1 < \sqrt{x - 1} - 2 < 1 \\ 1 < \sqrt{x - 1} < 3 \\ 1 < x - 1 < 9 \\ 2 < x < 10 \end{gathered}$$

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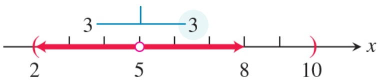

FIGURE 2.21 An open interval of radius 3 about $x = 5$ will lie inside the open interval (2, 10).

The inequality holds for all x in the open interval (2, 10), so it holds for all $x \neq 5$ in this interval as well.

2. *Find a value of $\delta > 0$ to place the centered interval* $5 - \delta < x < 5 + \delta$ (centered at $x = 5$) *inside the interval* (2, 10). The distance from 5 to the nearer endpoint of (2, 10) is 3 (Figure 2.21). If we take $\delta = 3$ or any smaller positive number, then the inequality $0 < |x - 5| < \delta$ will automatically place x between 2 and 10 and imply that $|\sqrt{x-1} - 2| < 1$ (Figure 2.22):

$$|\sqrt{x-1} - 2| < 1 \qquad \text{whenever} \qquad 0 < |x - 5| < 3. \qquad \blacksquare$$

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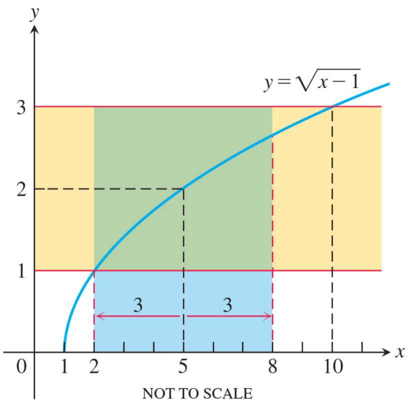


FIGURE 2.22 The function and intervals in Example 4.

How to Find Algebraically a δ for a Given f, L, c, and $\varepsilon > 0$

The process of finding a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - c| < \delta$$

can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \varepsilon$ to find an open interval (a, b) containing c on which the inequality holds for all $x \neq c$. Note that we do not require the inequality to hold at $x = c$. It may hold there or it may not, but the value of f at $x = c$ does not influence the existence of a limit.
2. *Find a value of* $\delta > 0$ that places the open interval $(c - \delta, c + \delta)$ centered at c inside the interval (a, b). The inequality $|f(x) - L| < \varepsilon$ will hold for all $x \neq c$ in this δ-interval.

EXAMPLE 5 Prove that $\lim_{x \to 2} f(x) = 4$ if

$$f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2. \end{cases}$$

Solution Our task is to show that given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - 4| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - 2| < \delta.$$

lim f(x)

1. *Solve the inequality $|f(x) - 4| < \varepsilon$ to find an open interval containing $x = 2$ on which the inequality holds for all $x \neq 2$.*

For $x \neq c = 2$, we have $f(x) = x^2$, and the inequality to solve is $|x^2 - 4| < \varepsilon$:

$$\begin{aligned} |x^2 - 4| &< \varepsilon \\ -\varepsilon < x^2 - 4 &< \varepsilon \\ 4 - \varepsilon < x^2 &< 4 + \varepsilon \\ \sqrt{4 - \varepsilon} < |x| &< \sqrt{4 + \varepsilon} \qquad \text{Assumes } \varepsilon < 4\text{; see below.} \\ \sqrt{4 - \varepsilon} < x &< \sqrt{4 + \varepsilon}. \qquad \text{An open interval about } x = 2 \text{ that solves the inequality.} \end{aligned}$$

The inequality $|f(x) - 4| < \varepsilon$ holds for all $x \neq 2$ in the open interval $\left(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon}\right)$ (Figure 2.23).

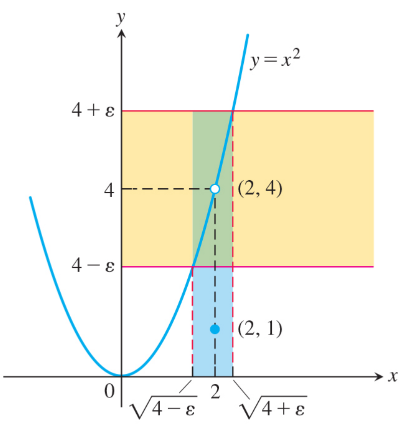


FIGURE 2.23 An interval containing $x = 2$ so that the function in Example 5 satisfies $|f(x) - 4| < \varepsilon$.

2. *Find a value of $\delta > 0$ that places the centered interval $(2 - \delta, 2 + \delta)$ inside the interval* $\left(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon}\right)$.

Take δ to be the distance from $x = 2$ to the nearer endpoint of $\left(\sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon}\right)$. In other words, take $\delta = \min\left\{2 - \sqrt{4 - \varepsilon}, \sqrt{4 + \varepsilon} - 2\right\}$, the *minimum* (the smaller) of the two numbers $2-\sqrt{4-\varepsilon}$ and $\sqrt{4+\varepsilon}-2$. If δ has this or any smaller positive value, the inequality $0<|x-2|<\delta$ will automatically place x between $\sqrt{4-\varepsilon}$ and $\sqrt{4+\varepsilon}$ to make $|f(x)-4|<\varepsilon$. For all x,

$$|f(x)-4|<\varepsilon \qquad \text{whenever} \qquad 0<|x-2|<\delta.$$

This completes the proof for $\varepsilon<4$.

If $\varepsilon \geq 4$, then we take δ to be the distance from $x=2$ to the nearer endpoint of the interval $\left(0, \sqrt{4+\varepsilon}\right)$. In other words, take $\delta=\min\left\{2, \sqrt{4+\varepsilon}-2\right\}$. (See Figure 2.23.) ■

Using the Definition to Prove Theorems

We do not usually rely on the formal definition of limit to verify specific limits such as those in the preceding examples. Rather, we appeal to general theorems about limits, in particular the theorems of Section 2.2. The definition is used to prove these theorems (Appendix 5). As an example, we prove part 1 of Theorem 1, the Sum Rule.

EXAMPLE 6 Given that $\lim_{x\to c} f(x)=L$ and $\lim_{x\to c} g(x)=M$, prove that

$$\lim_{x\to c}(f(x)+g(x))=L+M.$$

Solution Let $\varepsilon>0$ be given. We want to find a positive number δ such that

$$|f(x)+g(x)-(L+M)|<\varepsilon \qquad \text{whenever} \qquad 0<|x-c|<\delta.$$

Regrouping terms, we get

$$\begin{aligned}|f(x)+g(x)-(L+M)| &= |(f(x)-L)+(g(x)-M)| \\ &\leq |f(x)-L|+|g(x)-M|.\end{aligned}$$

Triangle Inequality: $|a+b| \leq |a|+|b|$

Since $\lim_{x\to c} f(x)=L$, there exists a number $\delta_1>0$ such that

$$|f(x)-L|<\varepsilon/2 \qquad \text{whenever} \qquad 0<|x-c|<\delta_1.$$

Can find δ_1 since $\lim_{x\to c} f(x)=L$

Similarly, since $\lim_{x\to c} g(x)=M$, there exists a number $\delta_2>0$ such that

$$|g(x)-M|<\varepsilon/2 \qquad \text{whenever} \qquad 0<|x-c|<\delta_2.$$

Can find δ_2 since $\lim_{x\to c} g(x)=M$

Let $\delta=\min\{\delta_1, \delta_2\}$, the smaller of δ_1 and δ_2. If $0<|x-c|<\delta$ then $|x-c|<\delta_1$, so $|f(x)-L|<\varepsilon/2$, and $|x-c|<\delta_2$, so $|g(x)-M|<\varepsilon/2$. Therefore

$$|f(x)+g(x)-(L+M)|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.$$

This shows that $\lim_{x\to c}(f(x)+g(x))=L+M$. ■

EXERCISES 2.3

Centering Intervals About a Point

In Exercises 1–6, sketch the interval (a, b) on the x-axis with the point c inside. Then find a value of $\delta>0$ such that $a<x<b$ whenever $0<|x-c|<\delta$.

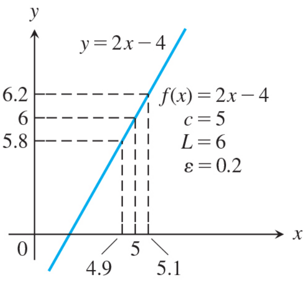
1. $a=1, \quad b=7, \quad c=5$ $1<x<7,\ 0<|x-5|<\delta \rightsquigarrow \delta=2$
   ตัวที่ใกล้ที่สุด +2
2. $a=1, \quad b=7, \quad c=2$
3. $a=-7/2, \quad b=-1/2, \quad c=-3$
4. $a=-7/2, \quad b=-1/2, \quad c=-3/2$
5. $a=4/9, \quad b=4/7, \quad c=1/2$
6. $a=2.7591, \quad b=3.2391, \quad c=3$

Finding Deltas Graphically

In Exercises 7–14, use the graphs to find a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

7.



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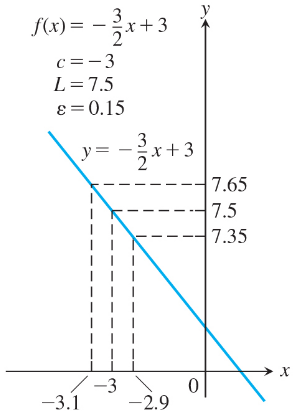
8.



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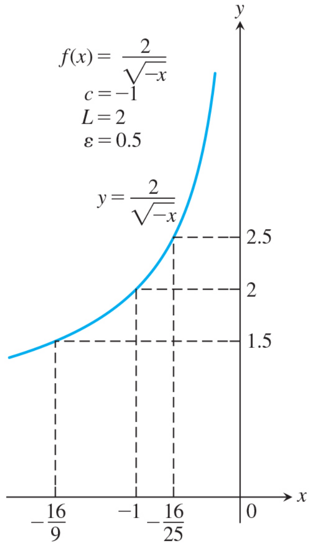

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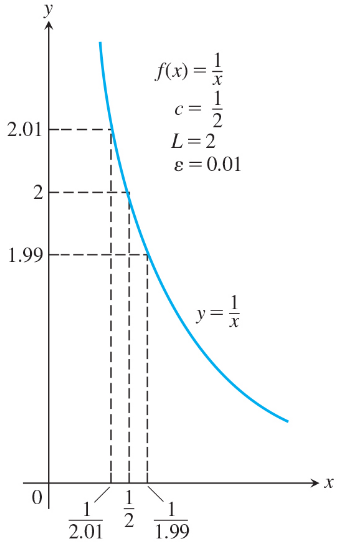

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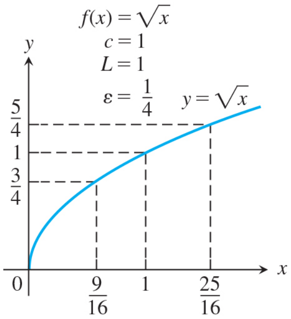

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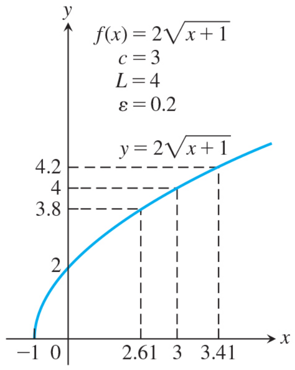

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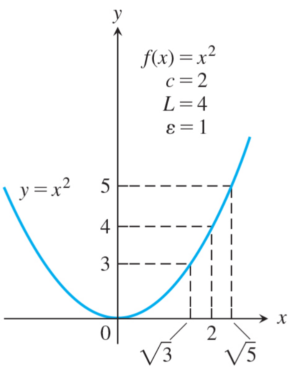

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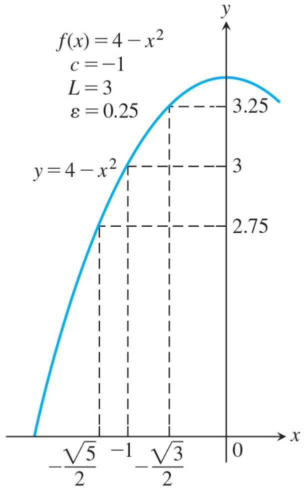

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Finding Deltas Algebraically

Each of Exercises 15–30 gives a function $f(x)$ and numbers L, c, and $\varepsilon > 0$. In each case, find an open interval about c on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that for all x satisfying $0 < |x - c| < \delta$ the inequality $|f(x) - L| < \varepsilon$ holds.

15. $f(x) = x + 1, \quad L = 5, \quad c = 4, \quad \varepsilon = 0.01$ $|x+1-5| < 0.01, 0<$ $|x-4| < 4$

16. $f(x) = 2x - 2, \quad L = -6, \quad c = -2, \quad \varepsilon = 0.02$

17. $f(x) = \sqrt{x + 1}, \quad L = 1, \quad c = 0, \quad \varepsilon = 0.1$

18. $f(x) = \sqrt{x}, \quad L = 1/2, \quad c = 1/4, \quad \varepsilon = 0.1$

19. $f(x) = \sqrt{19 - x}, \quad L = 3, \quad c = 10, \quad \varepsilon = 1$

20. $f(x) = \sqrt{x - 7}, \quad L = 4, \quad c = 23, \quad \varepsilon = 1$

21. $f(x) = 1/x, \quad L = 1/4, \quad c = 4, \quad \varepsilon = 0.05$

22. $f(x) = x^2, \quad L = 3, \quad c = \sqrt{3}, \quad \varepsilon = 0.1$

23. $f(x) = x^2, \quad L = 4, \quad c = -2, \quad \varepsilon = 0.5$

24. $f(x) = 1/x, \quad L = -1, \quad c = -1, \quad \varepsilon = 0.1$

25. $f(x) = x^2 - 5, \quad L = 11, \quad c = 4, \quad \varepsilon = 1$

26. $f(x) = 120/x, \quad L = 5, \quad c = 24, \quad \varepsilon = 1$

27. $f(x) = mx, \quad m > 0, \quad L = 2m, \quad c = 2, \quad \varepsilon = 0.03$

28. $f(x) = mx, \quad m > 0, \quad L = 3m, \quad c = 3, \quad \varepsilon = c > 0$

29. $f(x) = mx + b, \quad m > 0, \quad L = (m/2) + b,$
$c = 1/2, \quad \varepsilon = c > 0$

30. $f(x) = mx + b, \quad m > 0, \quad L = m + b, \quad c = 1, \quad \varepsilon = 0.05$

Using the Formal Definition

Each of Exercises 31–36 gives a function $f(x)$, a point c, and a positive number ε. Find $L = \lim_{x \to c} f(x)$. Then find a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

31. $f(x) = 3 - 2x, \quad c = 3, \quad \varepsilon = 0.02$

32. $f(x) = -3x - 2, \quad c = -1, \quad \varepsilon = 0.03$

33. $f(x) = \dfrac{x^2 - 4}{x - 2}, \quad c = 2, \quad \varepsilon = 0.05$

34. $f(x) = \dfrac{x^2 + 6x + 5}{x + 5}, \quad c = -5, \quad \varepsilon = 0.05$

35. $f(x) = \sqrt{1 - 5x}, \quad c = -3, \quad \varepsilon = 0.5$

36. $f(x) = 4/x, \quad c = 2, \quad \varepsilon = 0.4$

Prove the limit statements in Exercises 37–50.

37. $\lim_{x\to 4} (9 - x) = 5$

38. $\lim_{x\to 3} (3x - 7) = 2$

39. $\lim_{x\to 9} \sqrt{x - 5} = 2$

40. $\lim_{x\to 0} \sqrt{4 - x} = 2$

41. $\lim_{x\to 1} f(x) = 1 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq 1 \\ 2, & x = 1 \end{cases}$

42. $\lim_{x\to -2} f(x) = 4 \quad \text{if} \quad f(x) = \begin{cases} x^2, & x \neq -2 \\ 1, & x = -2 \end{cases}$

43. $\lim_{x\to 1} \dfrac{1}{x} = 1$

44. $\lim_{x\to \sqrt{3}} \dfrac{1}{x^2} = \dfrac{1}{3}$

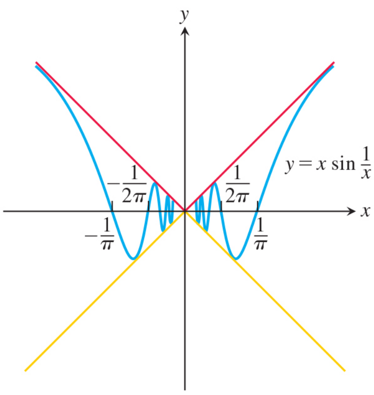
45. $\lim_{x\to -3} \dfrac{x^2 - 9}{x + 3} = -6$

46. $\lim_{x\to 1} \dfrac{x^2 - 1}{x - 1} = 2$

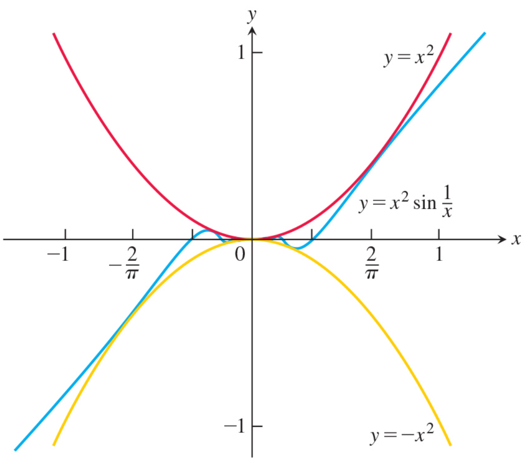
47. $\lim_{x\to 1} f(x) = 2 \quad \text{if} \quad f(x) = \begin{cases} 4 - 2x, & x < 1 \\ 6x - 4, & x \geq 1 \end{cases}$

48. $\lim_{x\to 0} f(x) = 0 \quad \text{if} \quad f(x) = \begin{cases} 2x, & x < 0 \\ x/2, & x \geq 0 \end{cases}$

49. $\lim_{x\to 0} x \sin \dfrac{1}{x} = 0$



50. $\lim_{x\to 0} x^2 \sin \dfrac{1}{x} = 0$



Theory and Examples

51. Define what it means to say that $\lim_{x\to 0} g(x) = k$.

52. Prove that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{h\to 0} f(h + c) = L$.

53. A wrong statement about limits Show by example that the following statement is wrong.

> The number L is the limit of $f(x)$ as x approaches c if $f(x)$ gets closer to L as x approaches c.

Explain why the function in your example does not have the given value of L as a limit as $x \to c$.

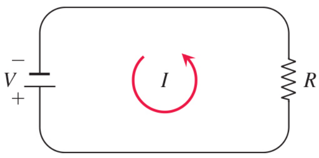
54. Another wrong statement about limits Show by example that the following statement is wrong.

> The number L is the limit of $f(x)$ as x approaches c if, given any $\varepsilon > 0$, there exists a value of x for which $|f(x) - L| < \varepsilon$.

Explain why the function in your example does not have the given value of L as a limit as $x \to c$.

T **55. Grinding engine cylinders** Before contracting to grind engine cylinders to a cross-sectional area of 9 in^2, you need to know how much deviation from the ideal cylinder diameter of $c = 3.385$ in. you can allow and still have the area come within 0.01 in^2 of the required 9 in^2. To find out, you let $A = \pi(x/2)^2$ and look for the interval in which you must hold x to make $|A - 9| \leq 0.01$. What interval do you find?

56. Manufacturing electrical resistors Ohm's law for electrical circuits like the one shown in the accompanying figure states that $V = RI$. In this equation, V is a constant voltage, I is the current in amperes, and R is the resistance in ohms. Your firm has been asked to supply the resistors for a circuit in which V will be 120 volts and I is to be 5 ± 0.1 amp. In what interval does R have to lie for I to be within 0.1 amp of the value $I_0 = 5$?



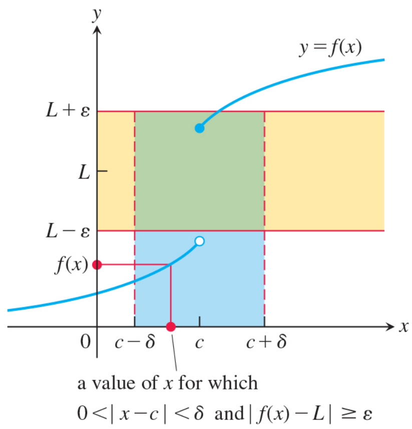
When Is a Number L Not the Limit of $f(x)$ as $x \to c$?

Showing L is not a limit We can prove that $\lim_{x\to c} f(x) \neq L$ by providing an $\varepsilon > 0$ such that no possible $\delta > 0$ satisfies the condition

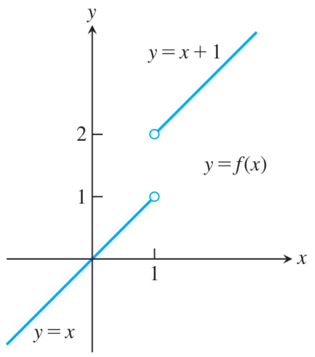
$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

We accomplish this for our candidate ε by showing that for each $\delta > 0$ there exists a value of x such that

$$0 < |x - c| < \delta \quad \text{and} \quad |f(x) - L| \geq \varepsilon.$$



57. Let $f(x) = \begin{cases} x, & x < 1 \\ x + 1, & x > 1. \end{cases}$



a. Let $\varepsilon = 1/2$. Show that no possible $\delta > 0$ satisfies the following condition:

$$|f(x) - 2| < 1/2 \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

That is, for each $\delta > 0$ show that there is a value of x such that

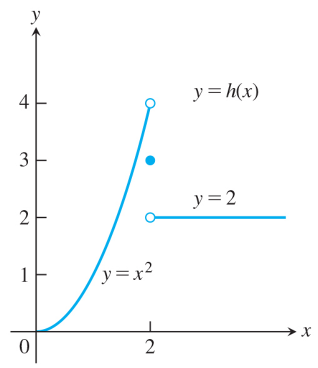
$$0 < |x - 1| < \delta \qquad \text{and} \qquad |f(x) - 2| \geq 1/2.$$

This will show that $\lim_{x \to 1} f(x) \neq 2$.

b. Show that $\lim_{x \to 1} f(x) \neq 1$.

c. Show that $\lim_{x \to 1} f(x) \neq 1.5$.

58. Let $h(x) = \begin{cases} x^2, & x < 2 \\ 3, & x = 2 \\ 2, & x > 2. \end{cases}$



Show that

a. $\lim_{x \to 2} h(x) \neq 4$

b. $\lim_{x \to 2} h(x) \neq 3$

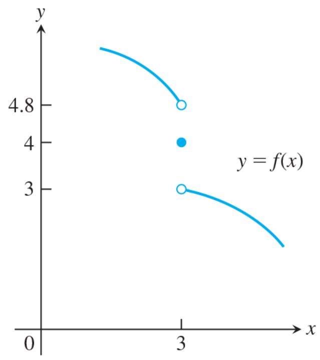
c. $\lim_{x \to 2} h(x) \neq 2$

59. For the function graphed here, explain why

a. $\lim_{x \to 3} f(x) \neq 4$

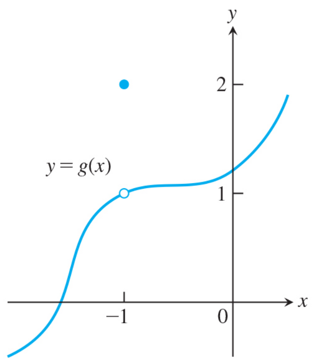
b. $\lim_{x \to 3} f(x) \neq 4.8$

c. $\lim_{x \to 3} f(x) \neq 3$



60. **a.** For the function graphed here, show that $\lim_{x \to -1} g(x) \neq 2$.

b. Does $\lim_{x \to -1} g(x)$ appear to exist? If so, what is the value of the limit? If not, why not?



COMPUTER EXPLORATIONS

In Exercises 61–66, you will further explore finding deltas graphically. Use a CAS to perform the following steps:

a. Plot the function $y = f(x)$ near the point c being approached.

b. Guess the value of the limit L and then evaluate the limit symbolically to see if you guessed correctly.

c. Using the value $\varepsilon = 0.2$, graph the banding lines $y_1 = L - \varepsilon$ and $y_2 = L + \varepsilon$ together with the function f near c.

d. From your graph in part (c), estimate a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Test your estimate by plotting f, y_1, and y_2 over the interval $0 < |x - c| < \delta$. For your viewing window use $c - 2\delta \leq x \leq c + 2\delta$ and $L - 2\varepsilon \leq y \leq L + 2\varepsilon$. If any function values lie outside the interval $[L - \varepsilon, L + \varepsilon]$, your choice of δ was too large. Try again with a smaller estimate.

e. Repeat parts (c) and (d) successively for $\varepsilon = 0.1, 0.05$, and 0.001.

61. $f(x) = \dfrac{x^4 - 81}{x - 3}, \quad c = 3$

62. $f(x) = \dfrac{5x^3 + 9x^2}{2x^5 + 3x^2}, \quad c = 0$

63. $f(x) = \dfrac{\sin 2x}{3x}, \quad c = 0$

64. $f(x) = \dfrac{x(1 - \cos x)}{x - \sin x}, \quad c = 0$

65. $f(x) = \dfrac{\sqrt[3]{x} - 1}{x - 1}, \quad c = 1$

66. $f(x) = \dfrac{3x^2 - (7x + 1)\sqrt{x} + 5}{x - 1}, \quad c = 1$

2.4 One-Sided Limits

In this section we extend the limit concept to *one-sided limits*, which are limits as x approaches the number c from the left-hand side (where $x < c$) or the right-hand side ($x > c$) only. These allow us to describe functions that have different limits at a point, depending on whether we approach the point from the left or from the right. One-sided limits also allow us to say what it means for a function to have a limit at an endpoint of an interval.

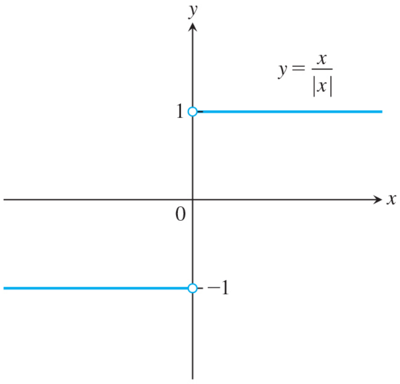


FIGURE 2.24 Different right-hand and left-hand limits at the origin.

Approaching a Limit from One Side

Suppose a function f is defined on an interval that extends to both sides of a number c. In order for f to have a limit L as x approaches c, the values of $f(x)$ must approach the value L as x approaches c from either side. Because of this, we sometimes say that the limit is **two-sided**.

If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit** or **limit from the right**. From the left, it is a **left-hand limit** or **limit from the left**.

The function $f(x) = x/|x|$ (Figure 2.24) has limit 1 as x approaches 0 from the right, and limit -1 as x approaches 0 from the left. Since these one-sided limit values are not the same, there is no single number that $f(x)$ approaches as x approaches 0. So $f(x)$ does not have a (two-sided) limit at 0.

Intuitively, if we only consider the values of $f(x)$ on an interval (c, b), where $c < b$, and the values of $f(x)$ become arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. In this case we write

$$\lim_{x\to c^+} f(x) = L.$$

The notation "$x \to c^+$" means that we consider only values of $f(x)$ for x greater than c. We don't consider values of $f(x)$ for $x \le c$.

Similarly, if $f(x)$ is defined on an interval (a, c), where $a < c$ and $f(x)$ approaches arbitrarily close to M as x approaches c from within that interval, then f has **left-hand limit** M at c. We write

$$\lim_{x\to c^-} f(x) = M.$$

The symbol "$x \to c^-$" means that we consider the values of f only at x-values less than c.

These informal definitions of one-sided limits are illustrated in Figure 2.25. For the function $f(x) = x/|x|$ in Figure 2.24 we have

$$\lim_{x\to 0^+} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to 0^-} f(x) = -1.$$

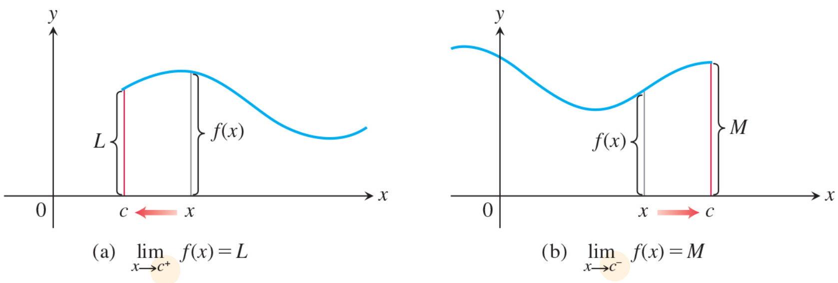


FIGURE 2.25 (a) Right-hand limit as x approaches c. (b) Left-hand limit as x approaches c.

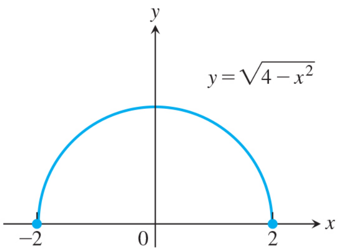
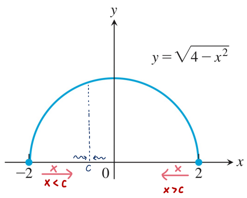


FIGURE 2.26 The function $f(x) = \sqrt{4 - x^2}$ has a right-hand limit 0 at $x = -2$ and a left-hand limit 0 at $x = 2$ (Example 1).

$y = \sqrt{4 - x^2}$
$y = 4 - x^2$
$x^2 + y^2 = 2^2$ —— สมการวงกลม



Function has two side limit at each point in (-2,2)

We now give the definition of the limit of a function at a boundary point of its domain. This definition is consistent with limits at boundary points of regions in the plane and in space, as we will see in Chapter 14. When the domain of f is an interval lying to the left of c, such as $(a, c]$ or (a, c), then we say that f has a limit at c if it has a left-hand limit at c. Similarly, if the domain of f is an interval lying to the right of c, such as $[c, b)$ or (c, b), then we say that f has a limit at c if it has a right-hand limit at c.

EXAMPLE 1 The domain of $f(x) = \sqrt{4 - x^2}$ is $[-2, 2]$; its graph is the semicircle in Figure 2.26. We have

$$\lim_{x \to -2^+} \sqrt{4 - x^2} = 0 \qquad \text{and} \qquad \lim_{x \to 2^-} \sqrt{4 - x^2} = 0.$$

This function has a two-sided limit at each point in $(-2, 2)$. It has a left-hand limit at $x = 2$ and a right-hand limit at $x = -2$. The function does not have a left-hand limit at $x = -2$ or a right-hand limit at $x = 2$. It does not have a two-sided limit at either -2 or 2 because f is not defined on both sides of these points. At the domain boundary points, where the domain is an interval on one side of the point, we have $\lim_{x \to -2} \sqrt{4 - x^2} = 0$ and $\lim_{x \to 2} \sqrt{4 - x^2} = 0$. The function f does have a limit at $x = -2$ and at $x = 2$. ■

One-sided limits have all the properties listed in Theorem 1 in Section 2.2. The right-hand limit of the sum of two functions is the sum of their right-hand limits, and so on. The theorems for limits of polynomials and rational functions hold with one-sided limits, as does the Sandwich Theorem. One-sided limits are related to limits at interior points in the following way.

THEOREM 6 Suppose that a function f is defined on an open interval containing c, except perhaps at c itself. Then $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Theorem 6 applies at interior points of a function's domain. At a boundary point of its domain, a function has a limit when it has an appropriate one-sided limit.

EXAMPLE 2 For the function graphed in Figure 2.27,

At $x = 0$:	$\lim_{x \to 0^-} f(x)$ does not exist,	f is not defined to the left of $x = 0$.
	$\lim_{x \to 0^+} f(x) = 1$,	f has a right-hand limit at $x = 0$.
	$\lim_{x \to 0} f(x) = 1$.	f has a limit at domain endpoint $x = 0$.
At $x = 1$:	$\lim_{x \to 1^-} f(x) = 0$,	Even though $f(1) = 1$.
	$\lim_{x \to 1^+} f(x) = 1$,	
	$\lim_{x \to 1} f(x)$ does not exist.	Right- and left-hand limits are not equal.
At $x = 2$:	$\lim_{x \to 2^-} f(x) = 1$,	
	$\lim_{x \to 2^+} f(x) = 1$,	
	$\lim_{x \to 2} f(x) = 1$.	Even though $f(2) = 2$.
At $x = 3$:	$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) = \lim_{x \to 3} f(x) = f(3) = 2$.	
At $x = 4$:	$\lim_{x \to 4^-} f(x) = 1$,	Even though $f(4) \neq 1$.
	$\lim_{x \to 4^+} f(x)$ does not exist,	f is not defined to the right of $x = 4$.
	$\lim_{x \to 4} f(x) = 1$.	f has a limit at domain endpoint $x = 4$.

At every other point c in $[0, 4]$, $f(x)$ has limit $f(c)$. ■

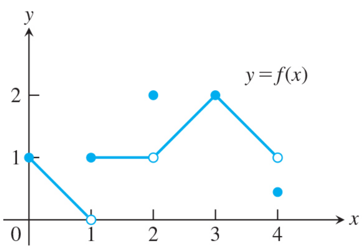


FIGURE 2.27 Graph of the function in Example 2.

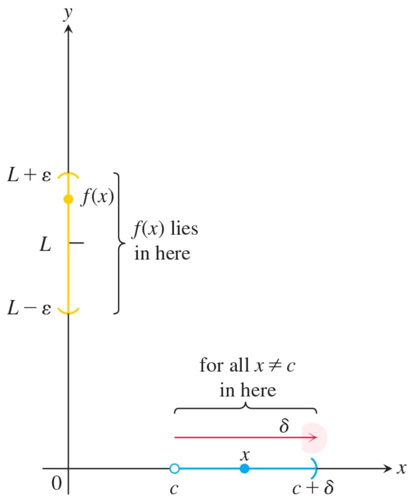


FIGURE 2.28 Intervals associated with the definition of right-hand limit.

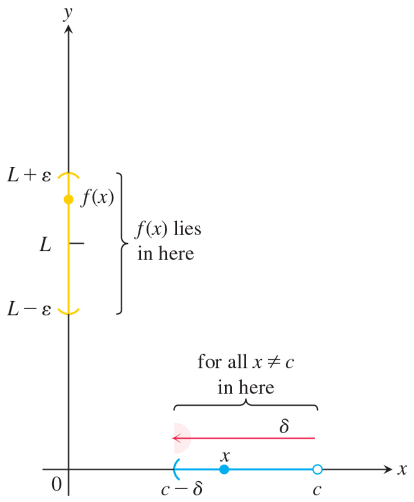


FIGURE 2.29 Intervals associated with the definition of left-hand limit.

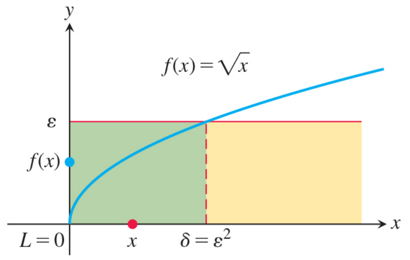


FIGURE 2.30 $\lim_{x\to 0^+} \sqrt{x} = 0$ in Example 3.

Precise Definitions of One-Sided Limits

The formal definition of the limit in Section 2.3 is readily modified for one-sided limits.

> **DEFINITIONS** (a) Assume the domain of f contains an interval (c, d) to the right of c. We say that $f(x)$ has **right-hand limit L at c**, and write
>
> $$\lim_{x\to c^+} f(x) = L$$
>
> if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that
>
> $$|f(x) - L| < \varepsilon \quad \text{whenever} \quad c < x < c + \delta.$$
>
> (b) Assume the domain of f contains an interval (b, c) to the left of c. We say that f has **left-hand limit L at c**, and write
>
> $$\lim_{x\to c^-} f(x) = L$$
>
> if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that
>
> $$|f(x) - L| < \underbrace{\varepsilon}_{\text{ebsilon}} \quad \text{whenever} \quad c - \underbrace{\delta}_{\text{delta}} < x < c.$$

The definitions are illustrated in Figures 2.28 and 2.29.

EXAMPLE 3 Prove that

$$\lim_{x\to 0^+} \sqrt{x} = 0.$$

Solution Let $\varepsilon > 0$ be given. Here $c = 0$ and $L = 0$, so we want to find a $\delta > 0$ such that

$$|\sqrt{x} - 0| < \varepsilon \qquad \text{whenever} \qquad 0 < x < \delta,$$

or

$$\sqrt{x} < \varepsilon \qquad \text{whenever} \qquad 0 < x < \delta.$$

$\sqrt{x} \gtrsim 0$ so $|\sqrt{x}| = \sqrt{x}$

Squaring both sides of this last inequality gives

$$x < \varepsilon^2 \qquad \text{if} \qquad 0 < x < \delta.$$

If we choose $\delta = \varepsilon^2$ we have

$$\sqrt{x} < \varepsilon \qquad \text{whenever} \qquad 0 < x < \delta = \varepsilon^2,$$

or

$$|\sqrt{x} - 0| < \varepsilon \qquad \text{whenever} \qquad 0 < x < \varepsilon^2.$$

According to the definition, this shows that $\lim_{x\to 0^+} \sqrt{x} = 0$ (Figure 2.30). ■

The functions examined so far have had some kind of limit at each point of interest. In general, that need not be the case.

EXAMPLE 4 Show that $y = \sin(1/x)$ has no limit as x approaches zero from either side (Figure 2.31).

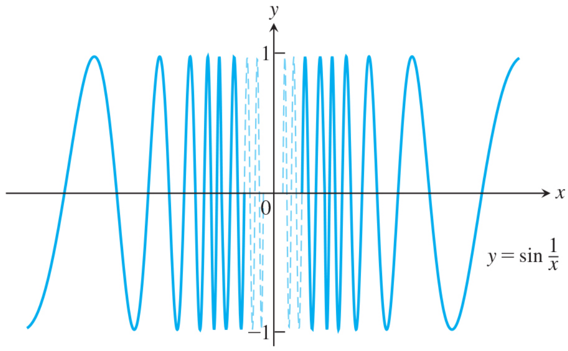


FIGURE 2.31 The function $y = \sin(1/x)$ has neither a right-hand nor a left-hand limit as x approaches zero (Example 4). The graph here omits values very near the y-axis.

Solution As x approaches zero, its reciprocal, $1/x$, grows without bound and the values of $\sin(1/x)$ cycle repeatedly from -1 to 1. There is no single number L that the function's values stay increasingly close to as x approaches zero. This is true even if we restrict x to positive values or to negative values. The function has neither a right-hand limit nor a left-hand limit at $x = 0$. ■

Limits Involving $(\sin\theta)/\theta$

A central fact about $(\sin\theta)/\theta$ is that in radian measure its limit as $\theta \to 0$ is 1. We can see this in Figure 2.32 and confirm it algebraically using the Sandwich Theorem. You will see the importance of this limit in Section 3.5, where instantaneous rates of change of the trigonometric functions are studied.

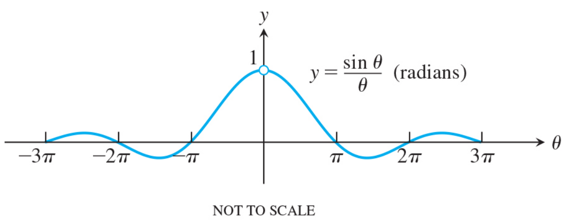


FIGURE 2.32 The graph of $f(\theta) = (\sin\theta)/\theta$ suggests that the right- and left-hand limits as θ approaches 0 are both 1.

THEOREM 7—Limit of the Ratio $\sin\theta/\theta$ as $\theta \to 0$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \qquad (1)$$

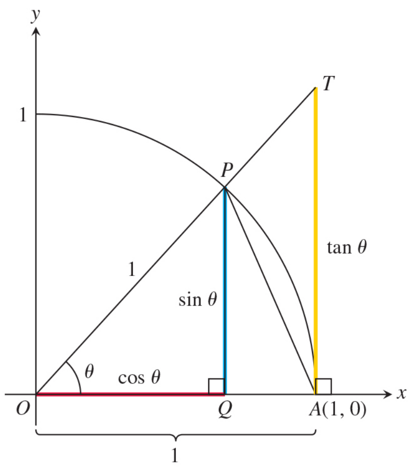


FIGURE 2.33 The ratio $TA/OA = \tan\theta$, and $OA = 1$, so $TA = \tan\theta$.

Proof The plan is to show that the right-hand and left-hand limits are both 1. Then we will know that the two-sided limit is 1 as well.

To show that the right-hand limit is 1, we begin with positive values of θ less than $\pi/2$ (Figure 2.33). Notice that

$$\text{Area } \Delta OAP < \text{ area sector } OAP < \text{ area } \Delta OAT.$$

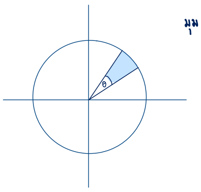
We can express these areas in terms of θ as follows:

$$\text{Area } \Delta OAP = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\sin\theta) = \frac{1}{2}\sin\theta$$

$$\text{Area sector } OAP = \frac{1}{2}r^2\theta = \frac{1}{2}(1)^2\theta = \frac{\theta}{2} \qquad (2)$$

$$\text{Area } \Delta OAT = \frac{1}{2}\text{base} \times \text{height} = \frac{1}{2}(1)(\tan\theta) = \frac{1}{2}\tan\theta.$$

The use of radians to measure angles is essential in Equation (2): The area of sector OAP is $\theta/2$ only if θ is measured in radians.



มุม 2π ; พื้นที่วงกลม $= \pi r^2$

$\theta = \dfrac{\pi r^2 \times \theta}{2\pi}$

$= \frac{1}{2}r^2\theta$

Thus,

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

This last inequality goes the same way if we divide all three terms by the number $(1/2)\sin\theta$, which is positive, since $0 < \theta < \pi/2$:

$$1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking reciprocals reverses the inequalities:

$$1 > \frac{\sin\theta}{\theta} > \cos\theta.$$

Since $\lim_{\theta\to 0^+}\cos\theta = 1$ (Example 11b, Section 2.2), the Sandwich Theorem gives

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1.$$

To consider the left-hand limit, we recall that $\sin\theta$ and θ are both *odd functions* (Section 1.1). Therefore, $f(\theta) = (\sin\theta)/\theta$ is an *even function*, with a graph symmetric about the y-axis (see Figure 2.32). This symmetry implies that the left-hand limit at 0 exists and has the same value as the right-hand limit:

$$\lim_{\theta\to 0^-}\frac{\sin\theta}{\theta} = 1 = \lim_{\theta\to 0^+}\frac{\sin\theta}{\theta},$$

so $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ by Theorem 6.

$\sin^2\theta = \dfrac{1-\cos 2\theta}{2}$

$\cos 2\theta = 1 - 2\sin^2\theta$

$\therefore \cos y = 1 - 2\sin^2\left(\frac{y}{2}\right)$

EXAMPLE 5 Show that **(a)** $\displaystyle\lim_{y\to 0}\frac{\cos y - 1}{y} = 0$ and **(b)** $\displaystyle\lim_{x\to 0}\frac{\sin 2x}{5x} = \frac{2}{5}$.

Solution

(a) Using the half-angle formula $\cos y = 1 - 2\sin^2(y/2)$, we calculate

$\displaystyle\lim \frac{2\sin^2(y/2)}{y}$

$\displaystyle= \lim_{\theta\to 0}\frac{2\sin^2\theta}{2\theta}$

$\displaystyle= \lim \frac{\sin^2\theta}{\theta}$

$\displaystyle= \lim \frac{\sin\theta}{\theta}\ 1 \ \lim_{\theta\to 0}\sin\theta\ 0$

$$\lim_{h\to 0}\frac{\cos y - 1}{y} = \lim_{h\to 0} -\frac{2\sin^2(y/2)}{y}$$

$$= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\sin\theta \qquad \text{Let } \theta = y/2.$$

$$= -(1)(0) = 0. \qquad \text{Eq. (1) and Example 11a in Section 2.2}$$

; $\displaystyle\lim_{y\to 0}\frac{\cos y - 1}{y} = \lim \frac{(\cancel{1} - 2\sin^2(y/2)) \cancel{-1}}{y}$

$\displaystyle= \lim_{y\to 0}\frac{-2\sin^2(y/2)}{y}$

(b) Equation (1) does not apply to the original fraction. We need a $2x$ in the denominator, not a $5x$. We produce it by multiplying numerator and denominator by $2/5$:

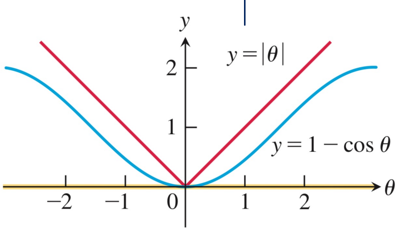
$$\lim_{x\to 0}\frac{\sin 2x}{5x} = \lim_{x\to 0}\frac{(2/5)\cdot\sin 2x}{(2/5)\cdot 5x} \rightarrow \frac{\frac{2}{5}\cdot\sin 2x}{\frac{2}{\cancel{5}}\cdot\cancel{5}x} \rightsquigarrow \frac{2}{5}\,\cancel{\frac{\sin 2x}{x}}\ 1$$

พยายามทำให้เท่ากัน

$$= \frac{2}{5}\lim_{x\to 0}\frac{\sin 2x}{2x} \qquad \text{Eq. (1) applies with } \theta = 2x.$$

$$= \frac{2}{5}(1) = \frac{2}{5}. \qquad ■$$

ทฤษฎีแซนวิส



$0 \le 1 - \cos\theta \le |\theta|$

$0 < 1 - \cos\theta < |\theta|$

$-|\theta| < \sin\theta < |\theta|$

$\displaystyle\lim_{\theta\to 0} 1 - \lim\cos\theta = C$

$1 - \lim\cos\theta$

$1 = \lim\cos\theta$

สูตรตรีโกณมิติ

Double - Angle Formulas

- $\cos 2\theta = \cos^2\theta - \sin^2\theta$
- $\sin 2\theta = 2\sin\theta\cos\theta$

Half - Angle Formulas

- $\cos^2\theta = \frac{1 + \cos 2\theta}{2}$
- $\sin^2\theta = \frac{1 - \cos 2\theta}{2}$

* Additional Formulus come from combining the equations

- $\cos^2\theta + \sin^2\theta = 1$
- $\cos^2\theta - \sin^2\theta = \cos 2\theta$

EXAMPLE 6 Find $\lim_{t\to 0} \frac{\tan t \sec 2t}{3t}$.

Solution From the definition of $\tan t$ and $\sec 2t$, we have

$$\lim_{t\to 0} \frac{\tan t \sec 2t}{3t} = \lim_{t\to 0} \frac{1}{3}\cdot\frac{1}{t}\cdot\frac{\sin t}{\cos t}\cdot\frac{1}{\cos 2t}$$

$\tan = \frac{\sin}{\cos}$, $\sec = \frac{1}{\cos}$

$$= \frac{1}{3}\lim_{t\to 0} \frac{\sin t}{t}\cdot\frac{1}{\cos t}\cdot\frac{1}{\cos 2t}$$

; $\cos 0 = 1$

$\cos 0$ $\quad$ $\cos 2(0)$ ใช้กฎการหารก่อน

$$= \frac{1}{3}(1)(1)(1) = \frac{1}{3}.$$

Eq. (1) and Example 11b in Section 2.2 ■

$\cos 0$ $\quad$ $\cos 2(0)$ ใช้กฎการหาร ก่อน

EXAMPLE 7 Show that for nonzero constants A and B.

$$\lim_{\theta\to 0} \frac{\sin A\theta}{\sin B\theta} = \frac{A}{B}.$$

Solution

$$\lim_{\theta\to 0} \frac{\sin A\theta}{\sin B\theta} = \lim_{\theta\to 0} \frac{\sin A\theta}{A\theta} A\theta \frac{B\theta}{\sin B\theta}\frac{1}{B\theta}$$

Multiply and divide by $A\theta$ and $B\theta$.

คูณ $\frac{A\theta}{A\theta}\cdot\frac{B\theta}{B\theta}$ เพิ่ม เพื่อให้เลขนั้นสมดุล

$$= \lim_{\theta\to 0} \frac{\sin A\theta}{A\theta}\frac{B\theta}{\sin B\theta}\frac{A}{B}$$

$\lim_{u\to 0} \frac{\sin u}{u} = 1$, with $u = A\theta$

$$\frac{\cancel{\frac{B\theta}{B\theta}}^{1}}{\cancel{\frac{\sin B\theta}{B\theta}}^{1}}$$

$$= \lim_{\theta\to 0} (1)(1)\frac{A}{B}$$

$\lim_{v\to 0} \frac{v}{\sin v} = 1$, with $v = B\theta$

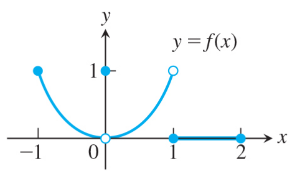
$$= \frac{A}{B}.$$

■

EXERCISES 2.4

Finding Limits Graphically

1. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x\to -1^+} f(x) = 1$ ✓ b. $\lim_{x\to 0^-} f(x) = 0$ ✓

c. $\lim_{x\to 0^-} f(x) = 1$ ✗ d. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$ ✓

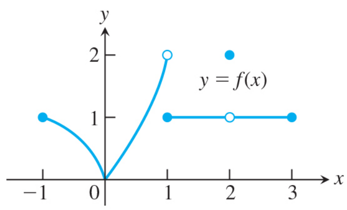
e. $\lim_{x\to 0} f(x)$ exists. ✓ f. $\lim_{x\to 0} f(x) = 0$ ✓

g. $\lim_{x\to 0} f(x) = 1$ ✗ h. $\lim_{x\to 1} f(x) = 1$ ✗

i. $\lim_{x\to 1} f(x) = 0$ ✗ j. $\lim_{x\to 2^-} f(x) = 2$ ✗

k. $\lim_{x\to -1^-} f(x)$ does not exist. ✓ l. $\lim_{x\to 2^+} f(x) = 0$ ✗

2. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x\to -1^+} f(x) = 1$ b. $\lim_{x\to 2} f(x)$ does not exist.

c. $\lim_{x\to 2} f(x) = 2$ d. $\lim_{x\to 1^-} f(x) = 2$

e. $\lim_{x\to 1^+} f(x) = 1$ f. $\lim_{x\to 1} f(x)$ does not exist.

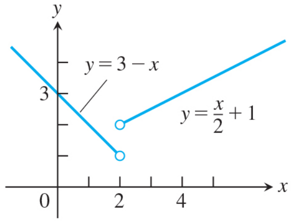
g. $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

h. $\lim_{x\to c} f(x)$ exists at every c in the open interval $(-1, 1)$.

i. $\lim_{x\to c} f(x)$ exists at every c in the open interval $(1, 3)$.

j. $\lim_{x\to -1^-} f(x) = 0$ k. $\lim_{x\to 3^+} f(x)$ does not exist.

3. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ \dfrac{x}{2} + 1, & x > 2. \end{cases}$



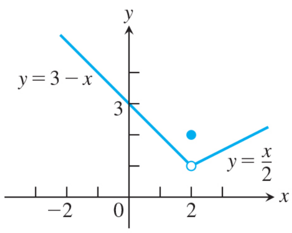
a. Find $\lim_{x\to 2^+} f(x)$ and $\lim_{x\to 2^-} f(x)$.

b. Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x\to 4^-} f(x)$ and $\lim_{x\to 4^+} f(x)$.

d. Does $\lim_{x\to 4} f(x)$ exist? If so, what is it? If not, why not?

4. Let $f(x) = \begin{cases} 3 - x, & x < 2 \\ 2, & x = 2 \\ \dfrac{x}{2}, & x > 2. \end{cases}$



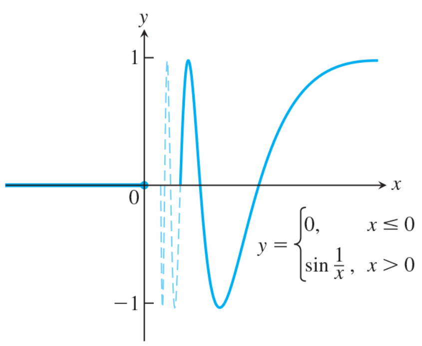
a. Find $\lim_{x\to 2^+} f(x)$, $\lim_{x\to 2^-} f(x)$, and $f(2)$.

b. Does $\lim_{x\to 2} f(x)$ exist? If so, what is it? If not, why not?

c. Find $\lim_{x\to -1^-} f(x)$ and $\lim_{x\to -1^+} f(x)$.

d. Does $\lim_{x\to -1} f(x)$ exist? If so, what is it? If not, why not?

5. Let $f(x) = \begin{cases} 0, & x \le 0 \\ \sin \dfrac{1}{x}, & x > 0. \end{cases}$

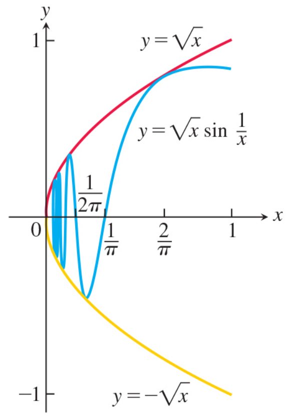


a. Does $\lim_{x\to 0^+} f(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x\to 0^-} f(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x\to 0} f(x)$ exist? If so, what is it? If not, why not?

6. Let $g(x) = \sqrt{x}\sin(1/x)$.



a. Does $\lim_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

b. Does $\lim_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

c. Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

7. **a.** Graph $f(x) = \begin{cases} x^3, & x \ne 1 \\ 0, & x = 1. \end{cases}$

b. Find $\lim_{x\to 1^-} f(x)$ and $\lim_{x\to 1^+} f(x)$.

c. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

8. **a.** Graph $f(x) = \begin{cases} 1 - x^2, & x \ne 1 \\ 2, & x = 1. \end{cases}$

b. Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.

c. Does $\lim_{x\to 1} f(x)$ exist? If so, what is it? If not, why not?

Graph the functions in Exercises 9 and 10. Then answer these questions.

a. What are the domain and range of f?

b. At what points c, if any, does $\lim_{x\to c} f(x)$ exist?

c. At what points does the left-hand limit exist but not the right-hand limit?

d. At what points does the right-hand limit exist but not the left-hand limit?

9. $f(x) = \begin{cases} \sqrt{1 - x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$

10. $f(x) = \begin{cases} x, & -1 \le x < 0, \text{ or } 0 < x \le 1 \\ 1, & x = 0 \\ 0, & x < -1 \text{ or } x > 1 \end{cases}$

Finding One-Sided Limits Algebraically

Find the limits in Exercises 11–20.

11. $\displaystyle \lim_{x\to -0.5^-} \sqrt{\frac{x+2}{x+1}}$

12. $\displaystyle \lim_{x\to 1^+} \sqrt{\frac{x-1}{x+2}}$

13. $\displaystyle \lim_{x\to -2^+} \left(\frac{x}{x+1}\right)\left(\frac{2x+5}{x^2+x}\right)$

14. $\lim_{x\to 1^-}\left(\dfrac{1}{x+1}\right)\left(\dfrac{x+6}{x}\right)\left(\dfrac{3-x}{7}\right)$

15. $\lim_{h\to 0^+}\dfrac{\sqrt{h^2+4h+5}-\sqrt{5}}{h}$

16. $\lim_{h\to 0^-}\dfrac{\sqrt{6}-\sqrt{5h^2+11h+6}}{h}$

17. a. $\lim_{x\to -2^+}(x+3)\dfrac{|x+2|}{x+2}$ **b.** $\lim_{x\to -2^-}(x+3)\dfrac{|x+2|}{x+2}$

18. a. $\lim_{x\to 1^+}\dfrac{\sqrt{2x}(x-1)}{|x-1|}$ **b.** $\lim_{x\to 1^-}\dfrac{\sqrt{2x}(x-1)}{|x-1|}$

19. a. $\lim_{x\to 0^+}\dfrac{|\sin x|}{\sin x}$ **b.** $\lim_{x\to 0^-}\dfrac{|\sin x|}{\sin x}$

20. a. $\lim_{x\to 0^+}\dfrac{1-\cos x}{|\cos x-1|}$ **b.** $\lim_{x\to 0^-}\dfrac{\cos x-1}{|\cos x-1|}$

Use the graph of the greatest integer function $y=\lfloor x\rfloor$, Figure 1.10 in Section 1.1, to help you find the limits in Exercises 21 and 22.

21. a. $\lim_{\theta\to 3^+}\dfrac{\lfloor\theta\rfloor}{\theta}$ **b.** $\lim_{\theta\to 3^-}\dfrac{\lfloor\theta\rfloor}{\theta}$

22. a. $\lim_{t\to 4^+}(t-\lfloor t\rfloor)$ **b.** $\lim_{t\to 4^-}(t-\lfloor t\rfloor)$

Using $\lim_{\theta\to 0}\dfrac{\sin\theta}{\theta}=1$

Find the limits in Exercises 23–46.

23. $\lim_{\theta\to 0}\dfrac{\sin\sqrt{2}\theta}{\sqrt{2}\theta}$

24. $\lim_{t\to 0}\dfrac{\sin kt}{t}$ (k constant)

25. $\lim_{y\to 0}\dfrac{\sin 3y}{4y}$

26. $\lim_{h\to 0^-}\dfrac{h}{\sin 3h}$

27. $\lim_{x\to 0}\dfrac{\tan 2x}{x}$

28. $\lim_{t\to 0}\dfrac{2t}{\tan t}$

29. $\lim_{x\to 0}\dfrac{x\csc 2x}{\cos 5x}$

30. $\lim_{x\to 0}6x^2(\cot x)(\csc 2x)$

31. $\lim_{x\to 0}\dfrac{x+x\cos x}{\sin x\cos x}$

32. $\lim_{x\to 0}\dfrac{x^2-x+\sin x}{2x}$

33. $\lim_{\theta\to 0}\dfrac{1-\cos\theta}{\sin 2\theta}$

34. $\lim_{x\to 0}\dfrac{x-x\cos x}{\sin^2 3x}$

35. $\lim_{t\to 0}\dfrac{\sin(1-\cos t)}{1-\cos t}$

36. $\lim_{h\to 0}\dfrac{\sin(\sin h)}{\sin h}$

37. $\lim_{\theta\to 0}\dfrac{\sin\theta}{\sin 2\theta}$

38. $\lim_{x\to 0}\dfrac{\sin 5x}{\sin 4x}$

39. $\lim_{\theta\to 0}\theta\cos\theta$

40. $\lim_{\theta\to 0}\sin\theta\cot 2\theta$

41. $\lim_{x\to 0}\dfrac{\tan 3x}{\sin 8x}$

42. $\lim_{y\to 0}\dfrac{\sin 3y\cot 5y}{y\cot 4y}$

43. $\lim_{\theta\to 0}\dfrac{\tan\theta}{\theta^2\cot 3\theta}$

44. $\lim_{\theta\to 0}\dfrac{\theta\cot 4\theta}{\sin^2\theta\cot^2 2\theta}$

45. $\lim_{x\to 0}\dfrac{1-\cos 3x}{2x}$

46. $\lim_{x\to 0}\dfrac{\cos^2 x-\cos x}{x^2}$

Theory and Examples

47. Once you know $\lim_{x\to a^+}f(x)$ and $\lim_{x\to a^-}f(x)$ at an interior point of the domain of f, do you then know $\lim_{x\to a}f(x)$? Give reasons for your answer.

48. If you know that $\lim_{x\to c}f(x)$ exists, can you find its value by calculating $\lim_{x\to c^+}f(x)$? Give reasons for your answer.

49. Suppose that f is an odd function of x. Does knowing that $\lim_{x\to 0^+}f(x)=3$ tell you anything about $\lim_{x\to 0^-}f(x)$? Give reasons for your answer.

50. Suppose that f is an even function of x. Does knowing that $\lim_{x\to 2^-}f(x)=7$ tell you anything about either $\lim_{x\to -2^-}f(x)$ or $\lim_{x\to -2^+}f(x)$? Give reasons for your answer.

Formal Definitions of One-Sided Limits

51. Given $\varepsilon>0$, find an interval $I=(5,5+\delta)$, $\delta>0$, such that if x lies in I, then $\sqrt{x-5}<\varepsilon$. What limit is being verified and what is its value?

52. Given $\varepsilon>0$, find an interval $I=(4-\delta,4)$, $\delta>0$, such that if x lies in I, then $\sqrt{4-x}<\varepsilon$. What limit is being verified and what is its value?

Use the definitions of right-hand and left-hand limits to prove the limit statements in Exercises 53 and 54.

53. $\lim_{x\to 0^-}\dfrac{x}{|x|}=-1$

54. $\lim_{x\to 2^+}\dfrac{x-2}{|x-2|}=1$

55. Greatest integer function Find **(a)** $\lim_{x\to 400^+}\lfloor x\rfloor$ and **(b)** $\lim_{x\to 400^-}\lfloor x\rfloor$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x\to 400}\lfloor x\rfloor$? Give reasons for your answer.

56. One-sided limits Let $f(x)=\begin{cases}x^2\sin(1/x), & x<0\\ \sqrt{x}, & x>0.\end{cases}$

Find **(a)** $\lim_{x\to 0^+}f(x)$ and **(b)** $\lim_{x\to 0^-}f(x)$; then use limit definitions to verify your findings. **(c)** Based on your conclusions in parts (a) and (b), can you say anything about $\lim_{x\to 0}f(x)$? Give reasons for your answer.

2.5 Continuity

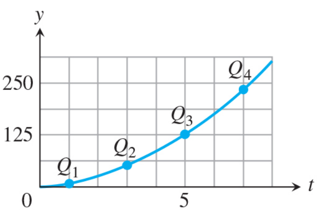


FIGURE 2.34 Connecting plotted points.

When we plot function values generated in a laboratory or collected in the field, we often connect the plotted points with an unbroken curve to show what the function's values are likely to have been at the points we did not measure (Figure 2.34). In doing so, we are assuming that we are working with a *continuous function*, so its outputs vary regularly and consistently with the inputs, and do not jump abruptly from one value to another without taking on the values in between. Intuitively, any function $y=f(x)$ whose graph can be sketched over its domain in one unbroken motion is an example of a continuous function. Such functions play an important role in the study of calculus and its applications.

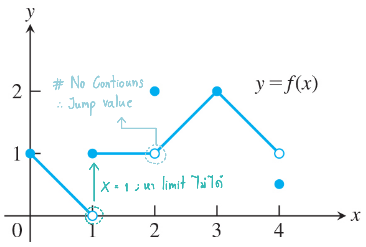


FIGURE 2.35 The function is not continuous at $x = 1$, $x = 2$, and $x = 4$ (Example 1).

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Continuity at a Point

To understand continuity, it helps to consider a function like that in Figure 2.35, whose limits we investigated in Example 2 in the last section.

EXAMPLE 1 At which numbers does the function f in Figure 2.35 appear to be not continuous? Explain why. What occurs at other numbers in the domain?

Solution First we observe that the domain of the function is the closed interval $[0, 4]$, so we will be considering the numbers x within that interval. From the figure, we notice right away that there are breaks in the graph at the numbers $x = 1$, $x = 2$, and $x = 4$. The break at $x = 1$ appears as a jump, which we identify later as a "jump discontinuity." The break at $x = 2$ is called a "removable discontinuity" since by changing the function definition at that one point, we can create a new function that is continuous at $x = 2$. Similarly $x = 4$ is a removable discontinuity.

Numbers at which the graph of f has breaks:

At the interior point $x = 1$, the function fails to have a limit. It does have both a left-hand limit, $\lim_{x\to 1^-} f(x) = 0$, as well as a right-hand limit, $\lim_{x\to 1^+} f(x) = 1$, but the limit values are different, resulting in a jump in the graph. The function is not continuous at $x = 1$. However the function value $f(1) = 1$ is equal to the limit from the right, so the function *is* continuous from the right at $x = 1$.

At $x = 2$, the function does have a limit, $\lim_{x\to 2} f(x) = 1$, but the value of the function is $f(2) = 2$. The limit and function values are not the same, so there is a break in the graph and f is not continuous at $x = 2$.

At $x = 4$, the function does have a left-hand limit at this right endpoint, $\lim_{x\to 4^-} f(x) = 1$, but again the value of the function $f(4) = \frac{1}{2}$ differs from the value of the limit. We see again a break in the graph of the function at this endpoint and the function is not continuous from the left.

Numbers at which the graph of f has no breaks:

At $x = 3$, the function has a limit, $\lim_{x\to 3} f(x) = 2$. Moreover, the limit is the same value as the function there, $f(3) = 2$. The function is continuous at $x = 3$.

At $x = 0$, the function has a right-hand limit at this left endpoint, $\lim_{x\to 0^+} f(x) = 1$, and the value of the function is the same, $f(0) = 1$. The function is continuous from the right at $x = 0$. Because $x = 0$ is a left endpoint of the function's domain, we have that $\lim_{x\to 0} f(x) = 1$ and so f is continuous at $x = 0$.

At all other numbers $x = c$ in the domain, the function has a limit equal to the value of the function, so $\lim_{x\to c} f(x) = f(c)$. For example, $\lim_{x\to 5/2} f(x) = f\left(\frac{5}{2}\right) = \frac{3}{2}$. No breaks appear in the graph of the function at any of these numbers and the function is continuous at each of them. ■

The following definitions capture the continuity ideas we observed in Example 1.

DEFINITIONS Let c be a real number that is either an interior point or an endpoint of an interval in the domain of f.

The function f is **continuous at c** if

$$\lim_{x\to c} f(x) = f(c).$$

The function f is **right-continuous at c (or continuous from the right)** if

$$\lim_{x\to c^+} f(x) = f(c).$$

The function f is **left-continuous at c (or continuous from the left)** if

$$\lim_{x\to c^-} f(x) = f(c).$$

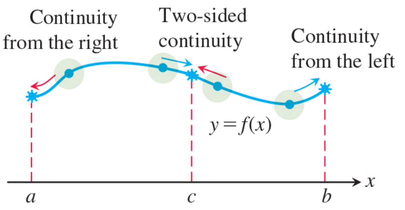

FIGURE 2.36 Continuity at points a, b, and c.

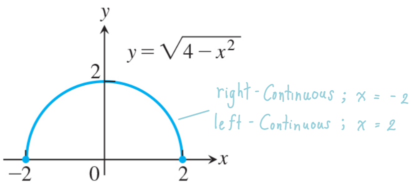

FIGURE 2.37 A function that is continuous over its domain (Example 2).

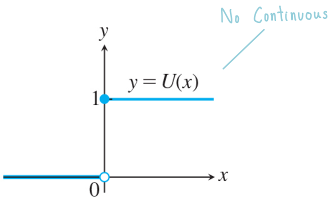

FIGURE 2.38 A function that has a jump discontinuity at the origin (Example 3).

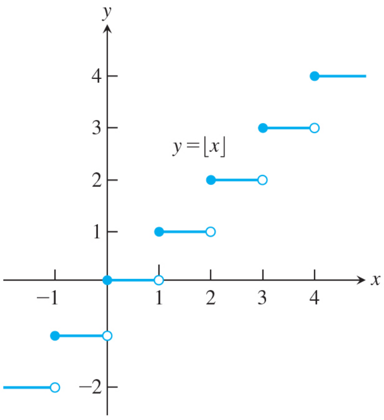

FIGURE 2.39 The greatest integer function is continuous at every noninteger point. It is right-continuous, but not left-continuous, at every integer point (Example 4).

The function f in Example 1 is continuous at every x in $[0, 4]$ except $x = 1, 2,$ and 4. It is right-continuous but not left-continuous at $x = 1$, neither right- nor left-continuous at $x = 2$, and not left-continuous at $x = 4$.

From Theorem 6, it follows immediately that a function f is continuous at an *interior* point c of an interval in its domain if and only if it is both right-continuous and left-continuous at c (Figure 2.36). We say that a function is **continuous over a closed interval** $[a, b]$ if it is right-continuous at a, left-continuous at b, and continuous at all interior points of the interval. This definition applies to the infinite closed intervals $[a, \infty)$ and $(-\infty, b]$ as well, but only one endpoint is involved. If a function is not continuous at point c of its domain, we say that f is **discontinuous at c**, and that f has a discontinuity at c. Note that a function f can be continuous, right-continuous, or left-continuous only at a point c for which $f(c)$ is defined.

EXAMPLE 2 The function $f(x) = \sqrt{4 - x^2}$ is continuous over its domain $[-2, 2]$ (Figure 2.37). It is right-continuous at $x = -2$, and left-continuous at $x = 2$. ■

EXAMPLE 3 The unit step function $U(x)$, graphed in Figure 2.38, is right-continuous at $x = 0$, but is neither left-continuous nor continuous there. It has a jump discontinuity at $x = 0$. ■

At an interior point or an endpoint of an interval in its domain, a function is continuous at points where it passes the following test.

Continuity Test

A function $f(x)$ is continuous at a point $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists หาค่าได้ (c lies in the domain of f).
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value).

For one-sided continuity, the limits in parts 2 and 3 of the test should be replaced by the appropriate one-sided limits.

EXAMPLE 4 The function $y = \lfloor x \rfloor$ introduced in Section 1.1 is graphed in Figure 2.39. It is discontinuous at every integer n, because the left-hand and right-hand limits are not equal as $x \to n$:

$$\lim_{x \to n^-} \lfloor x \rfloor = n - 1 \quad \text{and} \quad \lim_{x \to n^+} \lfloor x \rfloor = n.$$

Since $\lfloor n \rfloor = n$, the greatest integer function is right-continuous at every integer n (but not left-continuous).

The greatest integer function is continuous at every real number other than the integers. For example,

$$\lim_{x \to 1.5} \lfloor x \rfloor = 1 = \lfloor 1.5 \rfloor.$$

In general, if $n - 1 < c < n$, n an integer, then

$$\lim_{x \to c} \lfloor x \rfloor = n - 1 = \lfloor c \rfloor. \quad ■$$

Figure 2.40 displays several common ways in which a function can fail to be continuous. The function in Figure 2.40a is continuous at $x = 0$. The function in Figure 2.40b does not contain $x = 0$ in its domain. It would be continuous if its domain extended so that $f(0) = 1$. The function in Figure 2.40c would be continuous if $f(0)$ were 1 instead of 2. The discontinuity in Figure 2.40c is **removable**. The function has a limit as $x \to 0$, and we can remove the discontinuity by setting $f(0)$ equal to this limit.

The discontinuities in Figure 2.40d through f are more serious: $\lim_{x\to 0} f(x)$ does not exist, and there is no way to improve the situation by appropriately defining f at 0. The step function in Figure 2.40d has a **jump discontinuity**: The one-sided limits exist but have different values. The function $f(x) = 1/x^2$ in Figure 2.40e has an **infinite discontinuity**. The function in Figure 2.40f has an **oscillating discontinuity**: It oscillates so much that its values approach each number in $[-1, 1]$ as $x \to 0$. Since it does not approach a single number, it does not have a limit as x approaches 0.

$* \; y = \frac{1}{u} \; ; \; x \neq 0$

Domain $= (-\infty, 0) \cup (0, \infty)$

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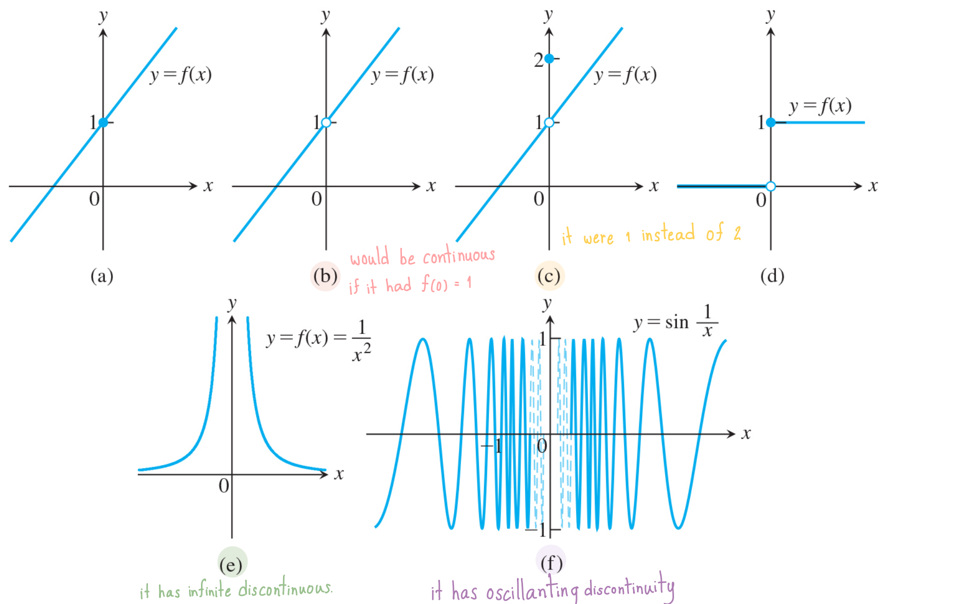


FIGURE 2.40 The function in (a) is continuous at $x = 0$; the functions in (b) through (f) are not.

Continuous Functions

We now describe the continuity behavior of a function throughout its entire domain, not only at a single point. We define a **continuous function** to be one that is continuous at every point in its domain. This is a property of the *function*. A function always has a specified domain, so if we change the domain then we change the function, and this may change its continuity property as well. If a function is discontinuous at one or more points of its domain, we say it is a **discontinuous** function.

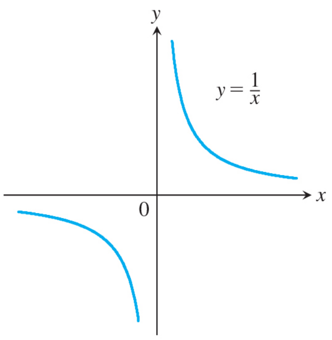


FIGURE 2.41 The function $f(x) = 1/x$ is continuous over its natural domain. It is not defined at the origin, so it is not continuous on any interval containing $x = 0$ (Example 5).

EXAMPLE 5

(a) The function $f(x) = 1/x$ (Figure 2.41) is a continuous function because it is continuous at every point of its domain. The point $x = 0$ is not in the domain of the function f, so f is not continuous on any interval containing $x = 0$. Moreover, there is no way to extend f to a new function that is defined and continuous at $x = 0$. The function f does not have a removable discontinuity at $x = 0$.

(b) The identity function $f(x) = x$ and constant functions are continuous everywhere by Example 3, Section 2.3. ■

Algebraic combinations of continuous functions are continuous wherever they are defined.

THEOREM 8—Properties of Continuous Functions
If the functions f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Constant multiples:* $k \cdot f$, for any number k
4. *Products:* $f \cdot g$
5. *Quotients:* f/g, provided $g(c) \neq 0$
6. *Powers:* f^n, n a positive integer
7. *Roots:* $\sqrt[n]{f}$, provided it is defined on an interval containing c, where n is a positive integer

Most of the results in Theorem 8 follow from the limit rules in Theorem 1, Section 2.2. For instance, to prove the sum property we have

$$\begin{aligned}
\lim_{x \to c}(f + g)(x) &= \lim_{x \to c}(f(x) + g(x)) \\
&= \lim_{x \to c} f(x) + \lim_{x \to c} g(x) && \text{Sum Rule, Theorem 1} \\
&= f(c) + g(c) && \text{Continuity of } f, g \text{ at } c \\
&= (f + g)(c).
\end{aligned}$$

This shows that $f + g$ is continuous.

EXAMPLE 6

(a) Every polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is continuous because $\lim_{x \to c} P(x) = P(c)$ by Theorem 2, Section 2.2.

(b) If $P(x)$ and $Q(x)$ are polynomials, then the rational function $P(x)/Q(x)$ is continuous wherever it is defined ($Q(c) \neq 0$) by Theorem 3, Section 2.2. ■

EXAMPLE 7 The function $f(x) = |x|$ is continuous. If $x > 0$, we have $f(x) = x$, a polynomial. If $x < 0$, we have $f(x) = -x$, another polynomial. Finally, at the origin, $\lim_{x \to 0} |x| = 0 = |0|$. ■

The functions $y = \sin x$ and $y = \cos x$ are continuous at $x = 0$ by Example 11 of Section 2.2. Both functions are continuous everywhere (see Exercise 72). It follows from Theorem 8 that all six trigonometric functions are continuous wherever they are defined. For example, $y = \tan x$ is continuous on $\cdots \cup (-\pi/2, \pi/2) \cup (\pi/2, 3\pi/2) \cup \cdots$.

Inverse Functions and Continuity

When a continuous function defined on an interval has an inverse, the inverse function is itself a continuous function over its own domain. This result is suggested by the observation that the graph of f^{-1}, being the reflection of the graph of f across the line $y = x$, cannot have any breaks in it when the graph of f has no breaks. A rigorous proof that f^{-1} is continuous whenever f is continuous on an interval is given in more advanced texts. As an example, the inverse trigonometric functions are all continuous over their domains.

We defined the exponential function $y = a^x$ in Section 1.5 informally. The graph was obtained from the graph of $y = a^x$ for x a rational number by "filling in the holes" at the irrational points x, so as to make the function $y = a^x$ continuous over the entire real line. The inverse function $y = \log_a x$ is also continuous. In particular, the natural exponential function $y = e^x$ and the natural logarithm function $y = \ln x$ are both continuous over their domains. Proofs of continuity for these functions will be given in Chapter 7.

Continuity of Compositions of Functions

Functions obtained by composing continuous functions are continuous. If $f(x)$ is continuous at $x = c$ and $g(x)$ is continuous at $x = f(c)$, then $g \circ f$ is also continuous at $x = c$ (Figure 2.42). In this case, the limit of $g \circ f$ as $x \to c$ is $g(f(c))$.

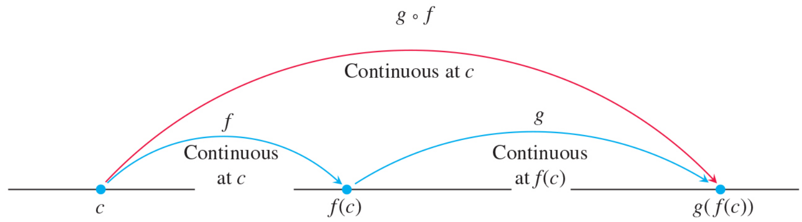


FIGURE 2.42 Compositions of continuous functions are continuous.

> **THEOREM 9—Compositions of Continuous Functions**
> If f is continuous at c and g is continuous at $f(c)$, then the composition $g \circ f$ is continuous at c.

Intuitively, Theorem 9 is reasonable because if x is close to c, then $f(x)$ is close to $f(c)$, and since g is continuous at $f(c)$, it follows that $g(f(x))$ is close to $g(f(c))$.

The continuity of compositions holds for any finite number of compositions of functions. The only requirement is that each function be continuous where it is applied. An outline of a proof of Theorem 9 is given in Exercise 6 in Appendix 4.

$(g \cdot f)(x)$

a $y = \sqrt{x^2 - 2x - 5}$ $\begin{cases} f(x) = x^2 - 2x - 5 \\ g(x) = \sqrt{x} \end{cases}$

$f(x)$ …… ถ้าเป็นฟังก์ชันต่อเนื่อง

$\therefore g(x)$ จะเป็นฟังก์ชันต่อเนื่องด้วย

EXAMPLE 8 Show that the following functions are continuous on their natural domains.

(a) $y = \sqrt{x^2 - 2x - 5}$

(b) $y = \dfrac{x^{2/3}}{1 + x^4}$ $\sqrt[3]{x^2}$ ต่อเนื่องในทุกๆค่า

(c) $y = \left|\dfrac{x - 2}{x^2 - 2}\right|$ $x \neq \pm\sqrt{2}$; โดยต่อเนื่องที่ $R - \{\pm\sqrt{2}\}$

(d) $y = \left|\dfrac{x \sin x}{x^2 + 2}\right|$ sine function is everywhere-Continuous

Solution

(a) The square root function is continuous on $[0, \infty)$ because it is a root of the continuous identity function $f(x) = x$ (Part 7, Theorem 8). The given function is then the composition of the polynomial $f(x) = x^2 - 2x - 5$ with the square root function $g(t) = \sqrt{t}$, and is continuous on its natural domain.

(b) The numerator is the cube root of the identity function squared; the denominator is an everywhere-positive polynomial. Therefore, the quotient is continuous.

(c) The quotient $(x - 2)/(x^2 - 2)$ is continuous for all $x \neq \pm\sqrt{2}$, and the function is the composition of this quotient with the continuous absolute value function (Example 7).

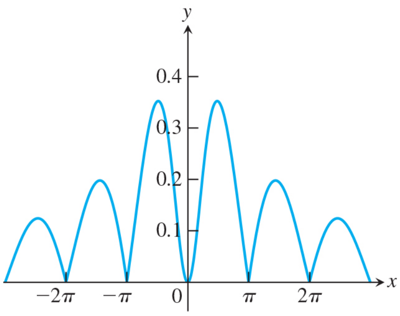


FIGURE 2.43 The graph suggests that $y = |(x \sin x)/(x^2 + 2)|$ is continuous (Example 8d).

(d) Because the sine function is everywhere-continuous (Exercise 72), the numerator term $x \sin x$ is the product of continuous functions, and the denominator term $x^2 + 2$ is an everywhere-positive polynomial. The given function is the composite of a quotient of continuous functions with the continuous absolute value function (Figure 2.43). ■

Theorem 9 is actually a consequence of a more general result, which we now prove. It states that if the limit of $f(x)$ as x approaches c is equal to b, then the limit of the composition function $g \circ f$ as x approaches c is equal to $g(b)$.

> **THEOREM 10—Limits of Continuous Functions**
>
> If $\lim_{x\to c} f(x) = b$ and g is continuous at the point b, then
>
> $$\lim_{x\to c} g(\underbrace{f(x)}_{\lim_{x\to c} f(x) = b}) = g(b).$$

Proof Let $\varepsilon > 0$ be given. Since g is continuous at b, there exists a number $\delta_1 > 0$ such that

$$|g(y) - g(b)| < \varepsilon \quad \text{whenever} \quad 0 < |y - b| < \delta_1.$$

$\lim_{y\to b} g(y) = g(b)$ since g is continuous at $y = b$.

Since $\lim_{x\to c} f(x) = b$, there exists a $\delta > 0$ such that

$$|f(x) - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

Definition of $\lim_{x\to c} f(x) = b$

If we let $y = f(x)$, we then have that

$$|y - b| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta,$$

which implies from the first statement that $|g(y) - g(b)| = |g(f(x)) - g(b)| < \varepsilon$ whenever $0 < |x - c| < \delta$. From the definition of limit, it follows that $\lim_{x\to c} g(f(x)) = g(b)$. This gives the proof for the case where c is an interior point of the domain of f. The case where c is an endpoint of the domain is entirely similar, using an appropriate one-sided limit in place of a two-sided limit. ■

EXAMPLE 9 As an application of Theorem 10, we have the following calculations.

(a) $$\lim_{x\to\pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right) = \cos\left(\lim_{x\to\pi/2} 2\overset{\frac{\pi}{2}}{\cancel{x}} + \lim_{x\to\pi/2} \sin\left(\underbrace{\frac{3\pi}{2} + \overset{\frac{\pi}{2}}{\cancel{x}}}_{\frac{4\pi}{2} = 2\pi}\right)\right)$$

$$= \cos(\pi + \sin 2\pi) = \cos \pi = -1.$$

(b) $$\lim_{x\to 1} \sin^{-1}\left(\frac{1 - x}{1 - x^2}\right) = \sin^{-1}\left(\lim_{x\to 1} \frac{\cancel{1 - x}}{\underbrace{1 - x^2}_{(\cancel{1-x})(1+x)}}\right)$$ Arcsine is continuous.

$$= \sin^{-1}\left(\lim_{x\to 1} \frac{1}{1 + x}\right)$$ Cancel common factor $(1 - x)$.

$$= \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$$

We sometimes denote e^u by $\exp(u)$ when u is a complicated mathematical expression.

(c) $$\lim_{x\to 0} \sqrt{x + 1}\, e^{\tan x} = \lim_{x\to 0} \sqrt{\underset{0}{\cancel{x}} + 1} \cdot \exp\left(\lim_{x\to 0} \underbrace{\tan x}_{\frac{\sin 0}{\cos 0} = 0}\right)$$ exp is continuous.

$$= 1 \cdot e^0 = 1$$ ■

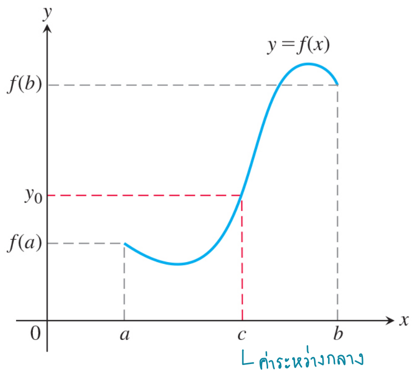
Intermediate Value Theorem for Continuous Functions

A function is said to have the **Intermediate Value Property** if whenever it takes on two values, it also takes on all the values in between.

* ฟังก์ชันต่อเนื่อง
หา $f(c)$ ได้, $\lim f(x) = L$
และ $f(c) = \lim f(x)$

THEOREM 11—The Intermediate Value Theorem for Continuous Functions

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



ที่หาค่าระหว่าง a-b
และต้องเป็นฟังก์ชันต่อเนื่อง

Theorem 11 says that continuous functions over *finite closed* intervals have the Intermediate Value Property. Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the y-axis between the numbers $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.

The proof of the Intermediate Value Theorem depends on the completeness property of the real number system. The completeness property implies that the real numbers have no holes or gaps. In contrast, the rational numbers do not satisfy the completeness property, and a function defined only on the rationals would not satisfy the Intermediate Value Theorem. See Appendix 7 for a discussion and examples.

The continuity of f on the interval is essential to Theorem 11. If f fails to be continuous at even one point of the interval, the theorem's conclusion may fail, as it does for the function graphed in Figure 2.44 (choose y_0 as any number between 2 and 3).

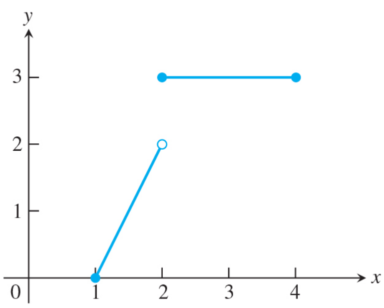


FIGURE 2.44 The function

$$f(x) = \begin{cases} 2x - 2, & 1 \le x < 2 \\ 3, & 2 \le x \le 4 \end{cases}$$

does not take on all values between $f(1) = 0$ and $f(4) = 3$; it misses all the values between 2 and 3.

A Consequence for Graphing: Connectedness Theorem 11 implies that the graph of a function that is continuous on an interval cannot have any breaks over the interval. It will be **connected**—a single, unbroken curve. It will not have jumps such as the ones found in the graph of the greatest integer function (Figure 2.39), or separate branches as found in the graph of $1/x$ (Figure 2.41).

เมื่อค่าฟังก์ชัน = 0

A Consequence for Root Finding We call a solution of the equation $f(x) = 0$ a **root** of the equation or **zero** of the function f. The Intermediate Value Theorem tells us that if f is continuous, then any interval on which f changes sign contains a zero of the function. Somewhere between a point where a continuous function is positive and a second point where it is negative, the function must be equal to zero.

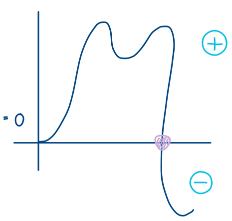
In practical terms, when we see the graph of a continuous function cross the horizontal axis on a computer screen, we know it is not stepping across. There really is a point where the function's value is zero.

EXAMPLE 10 Show that there is a root of the equation $x^3 - x - 1 = 0$ between 1 and 2.

สุ่มเลข

Solution Let $f(x) = x^3 - x - 1$. Since $f(1) = 1 - 1 - 1 = -1 < 0$ and $f(2) = 2^3 - 2 - 1 = 5 > 0$, we see that $y_0 = 0$ is a value between $f(1)$ and $f(2)$. Since f is a polynomial, it is continuous, and the Intermediate Value Theorem says there is a zero of f between 1 and 2. Figure 2.45 shows the result of zooming in to locate the root near $x = 1.32$. ■

$f(x) = 0$



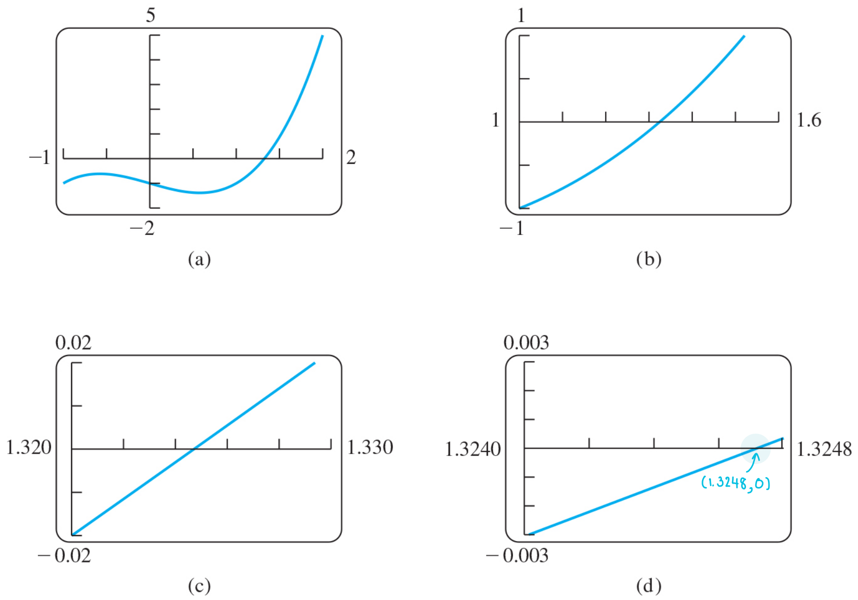


FIGURE 2.45 Zooming in on a zero of the function $f(x) = x^3 - x - 1$. The zero is near $x = 1.3247$ (Example 10).

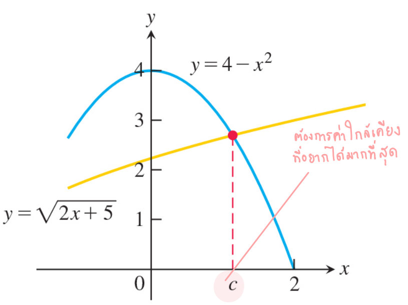


FIGURE 2.46 The curves $y = \sqrt{2x + 5}$ and $y = 4 - x^2$ have the same value at $x = c$ where $\underbrace{\sqrt{2x + 5} + x^2 - 4}_{f(x)} = 0$ (Example 11). กราฟพาราโบลา

$\therefore f(1) = 3.828$

EXAMPLE 11 Use the Intermediate Value Theorem to prove that the equation

$$\sqrt{2x + 5} = 4 - x^2$$

has a solution (Figure 2.46).

Solution We rewrite the equation as

$$\sqrt{2x + 5} + x^2 - 4 = 0,$$

and set $f(x) = \sqrt{2x + 5} + x^2 - 4$. Now $g(x) = \sqrt{2x + 5}$ is continuous on the interval $[-5/2, \infty)$ since it is formed as the composition of two continuous functions, the square root function with the nonnegative linear function $y = 2x + 5$. Then f is the sum of the function g and the quadratic function $y = x^2 - 4$, and the quadratic function is continuous for all values of x. It follows that $f(x) = \sqrt{2x + 5} + x^2 - 4$ is continuous on the interval $[-5/2, \infty)$. By trial and error, we find the function values $f(0) = \sqrt{5} - 4 \approx -1.76$ and $f(2) = \sqrt{9} = 3$. Note that f is continuous on the finite closed interval $[0, 2] \subset [-5/2, \infty)$. Since the value $y_0 = 0$ is between the numbers $f(0) = -1.76$ and $f(2) = 3$, by the Intermediate Value Theorem there is a number $c \in [0, 2]$ such that $f(c) = 0$. The number c solves the original equation. ■

Continuous Extension to a Point

เติมค่าให้เกิดความต่อเนื่อง

Sometimes the formula that describes a function f does not make sense at a point $x = c$. It might nevertheless be possible to extend the domain of f, to include $x = c$, creating a new function that is continuous at $x = c$. For example, the function $y = f(x) = (\sin x)/x$ is continuous at every point except $x = 0$, since $x = 0$ is not in its domain. Since $y = (\sin x)/x$ has a finite limit as $x \to 0$ (Theorem 7), we can extend the function's domain to include the point $x = 0$ in such a way that the extended function is continuous at $x = 0$. We define the new function

$$F(x) = \begin{cases} \dfrac{\sin x}{\underbrace{x}_{\neq 0}}, & x \neq 0 \\ 1, & x = 0. \end{cases}$$

Same as original function for $x \neq 0$

Value at domain point $x = 0$

สร้างฟังก์ชัน
ให้กราฟมีความต่อเนื่อง

The new function $F(x)$ is continuous at $x = 0$ because

$$\lim_{x \to 0} \frac{\sin x}{x} = F(0),$$

so it meets the requirements for continuity (Figure 2.47).

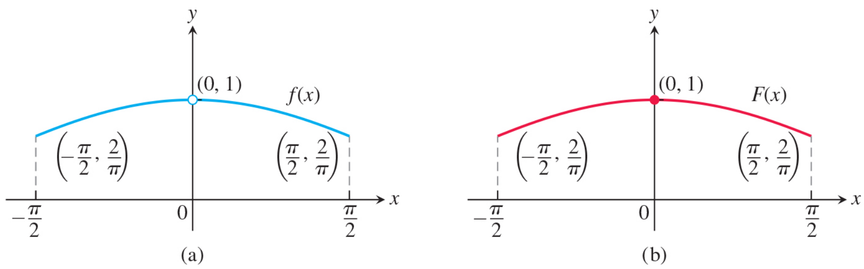


FIGURE 2.47 (a) The graph of $f(x) = (\sin x)/x$ for $-\pi/2 \le x \le \pi/2$ does not include the point (0, 1) because the function is not defined at $x = 0$. (b) We can extend the domain to include $x = 0$ by defining the new function $F(x)$ with $F(0) = 1$ and $F(x) = f(x)$ everywhere else. Note that $F(0) = \lim_{x \to 0} f(x)$ and $F(x)$ is a continuous function at $x = 0$.

More generally, a function (such as a rational function) may have a limit at a point where it is not defined. If $f(c)$ is not defined, but $\lim_{x \to c} f(x) = L$ exists, we can define a new function $F(x)$ by the rule

$$F(x) = \begin{cases} f(x), & \text{if } x \text{ is in the domain of } f \\ L, & \text{if } x = c. \end{cases}$$

The function F is continuous at $x = c$. It is called the **continuous extension of f** to $x = c$. For rational functions f, continuous extensions are often found by canceling common factors in the numerator and denominator.

EXAMPLE 12 Show that

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4}, \quad x \neq 2$$

has a continuous extension to $x = 2$, and find that extension.

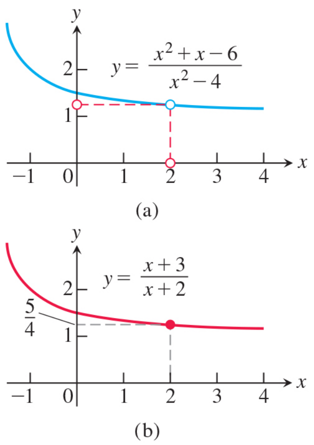


FIGURE 2.48 (a) The graph of $f(x)$ and (b) the graph of its continuous extension $F(x)$ (Example 12).

Solution Although $f(2)$ is not defined, if $x \neq 2$ we have

$$f(x) = \frac{x^2 + x - 6}{x^2 - 4} = \frac{\cancel{(x - 2)}(x + 3)}{\cancel{(x - 2)}(x + 2)} = \frac{x + 3}{x + 2}.$$

The new function

$$F(x) = \frac{x + 3}{x + 2}$$

is equal to $f(x)$ for $x \neq 2$, but is continuous at $x = 2$, having there the value of $5/4$. Thus F is the continuous extension of f to $x = 2$, and

∴ ทำด้วยวิธีการอะไรก็ได้ เพื่อตัดปัญหาจนทำให้ เป็นฟังก์ชันต่อเนื่อง

$$\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 4} = \lim_{x \to 2} f(x) = \frac{5}{4}.$$

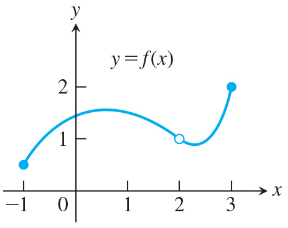
The graph of f is shown in Figure 2.48. The continuous extension F has the same graph except with no hole at $(2, 5/4)$. Effectively, F is the function f extended across the missing domain point at $x = 2$ so as to give a continuous function over the larger domain. ■

EXERCISES 2.5

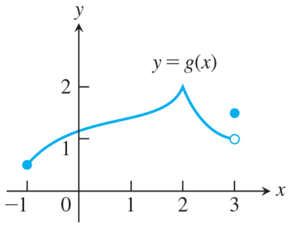
Continuity from Graphs

In Exercises 1–4, say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?

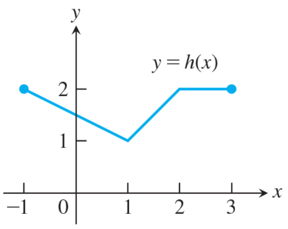
1.



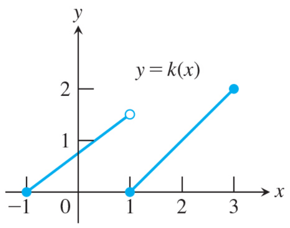
2.



3.



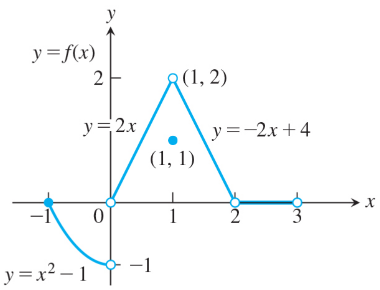
4.



Exercises 5–10 refer to the function

$$f(x) = \begin{cases} x^2 - 1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x + 4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

graphed in the accompanying figure.



The graph for Exercises 5–10.

5. a. Does $f(-1)$ exist?

b. Does $\lim_{x\to -1^+} f(x)$ exist?

c. Does $\lim_{x\to -1^+} f(x) = f(-1)$?

d. Is f continuous at $x = -1$?

6. a. Does $f(1)$ exist?

b. Does $\lim_{x\to 1} f(x)$ exist?

c. Does $\lim_{x\to 1} f(x) = f(1)$?

d. Is f continuous at $x = 1$?

7. a. Is f defined at $x = 2$? (Look at the definition of f.)

b. Is f continuous at $x = 2$?

8. At what values of x is f continuous?

9. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

10. To what new value should $f(1)$ be changed to remove the discontinuity?

Applying the Continuity Test

At which points do the functions in Exercises 11 and 12 fail to be continuous? At which points, if any, are the discontinuities removable? Not removable? Give reasons for your answers.

11. Exercise 1, Section 2.4

12. Exercise 2, Section 2.4

At what points are the functions in Exercises 13–32 continuous?

13. $y = \dfrac{1}{x-2} - 3x$

14. $y = \dfrac{1}{(x+2)^2} + 4$

15. $y = \dfrac{x+1}{x^2 - 4x + 3}$

16. $y = \dfrac{x+3}{x^2 - 3x - 10}$

17. $y = |x - 1| + \sin x$

18. $y = \dfrac{1}{|x| + 1} - \dfrac{x^2}{2}$

19. $y = \dfrac{\cos x}{x}$

20. $y = \dfrac{x+2}{\cos x}$

21. $y = \csc 2x$

22. $y = \tan \dfrac{\pi x}{2}$

23. $y = \dfrac{x \tan x}{x^2 + 1}$

24. $y = \dfrac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$

25. $y = \sqrt{2x + 3}$

26. $y = \sqrt[4]{3x - 1}$

27. $y = (2x - 1)^{1/3}$

28. $y = (2 - x)^{1/5}$

29. $g(x) = \begin{cases} \dfrac{x^2 - x - 6}{x - 3}, & x \ne 3 \\ 5, & x = 3 \end{cases}$

30. $f(x) = \begin{cases} \dfrac{x^3 - 8}{x^2 - 4}, & x \ne 2, x \ne -2 \\ 3, & x = 2 \\ 4, & x = -2 \end{cases}$

31. $f(x) = \begin{cases} 1 - x, & x < 0 \\ e^x, & 0 \le x \le 1 \\ x^2 + 2, & x > 1 \end{cases}$

32. $f(x) = \dfrac{x+3}{2 - e^x}$

Limits Involving Trigonometric Functions

Find the limits in Exercises 33–40. Are the functions continuous at the point being approached?

33. $\lim_{x\to\pi} \sin(x - \sin x)$

34. $\lim_{t\to 0} \sin\left(\dfrac{\pi}{2} \cos(\tan t)\right)$

35. $\lim_{y\to 1} \sec(y \sec^2 y - \tan^2 y - 1)$

36. $\lim_{x\to 0} \tan\left(\frac{\pi}{4}\cos(\sin x^{1/3})\right)$

37. $\lim_{t\to 0} \cos\left(\frac{\pi}{\sqrt{19 - 3\sec 2t}}\right)$

38. $\lim_{x\to\pi/6} \sqrt{\csc^2 x + 5\sqrt{3}\tan x}$

39. $\lim_{x\to 0^+} \sin\left(\frac{\pi}{2}e^{\sqrt{x}}\right)$

40. $\lim_{x\to 1} \cos^{-1}\left(\ln\sqrt{x}\right)$

Continuous Extensions

41. Define $g(3)$ in a way that extends $g(x) = (x^2 - 9)/(x - 3)$ to be continuous at $x = 3$.

42. Define $h(2)$ in a way that extends $h(t) = (t^2 + 3t - 10)/(t - 2)$ to be continuous at $t = 2$.

43. Define $f(1)$ in a way that extends $f(s) = (s^3 - 1)/(s^2 - 1)$ to be continuous at $s = 1$.

44. Define $g(4)$ in a way that extends

$$g(x) = (x^2 - 16)/(x^2 - 3x - 4)$$

to be continuous at $x = 4$.

45. For what value of a is

$$f(x) = \begin{cases} x^2 - 1, & x < 3 \\ 2ax, & x \geq 3 \end{cases}$$

continuous at every x?

46. For what value of b is

$$g(x) = \begin{cases} x, & x < -2 \\ bx^2, & x \geq -2 \end{cases}$$

continuous at every x?

47. For what values of a is

$$f(x) = \begin{cases} a^2x - 2a, & x \geq 2 \\ 12, & x < 2 \end{cases}$$

continuous at every x?

48. For what value of b is

$$g(x) = \begin{cases} \dfrac{x - b}{b + 1}, & x < 0 \\ x^2 + b, & x > 0 \end{cases}$$

continuous at every x?

49. For what values of a and b is

$$f(x) = \begin{cases} -2, & x \leq -1 \\ ax - b, & -1 < x < 1 \\ 3, & x \geq 1 \end{cases}$$

continuous at every x?

50. For what values of a and b is

$$g(x) = \begin{cases} ax + 2b, & x \leq 0 \\ x^2 + 3a - b, & 0 < x \leq 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuous at every x?

T In Exercises 51–54, graph the function f to see whether it appears to have a continuous extension to the origin. If it does, use Trace and Zoom to find a good candidate for the extended function's value at $x = 0$. If the function does not appear to have a continuous extension, can it be extended to be continuous at the origin from the right or from the left? If so, what do you think the extended function's value(s) should be?

51. $f(x) = \dfrac{10^x - 1}{x}$

52. $f(x) = \dfrac{10^{|x|} - 1}{x}$

53. $f(x) = \dfrac{\sin x}{|x|}$

54. $f(x) = (1 + 2x)^{1/x}$

Theory and Examples

55. A continuous function $y = f(x)$ is known to be negative at $x = 0$ and positive at $x = 1$. Why does the equation $f(x) = 0$ have at least one solution between $x = 0$ and $x = 1$? Illustrate with a sketch.

56. Explain why the equation $\cos x = x$ has at least one solution.

57. **Roots of a cubic** Show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

58. **A function value** Show that the function $F(x) = (x - a)^2 \cdot (x - b)^2 + x$ takes on the value $(a + b)/2$ for some value of x.

59. **Solving an equation** If $f(x) = x^3 - 8x + 10$, show that there are values c for which $f(c)$ equals **(a)** π; **(b)** $-\sqrt{3}$; **(c)** 5,000,000.

60. Explain why the following five statements ask for the same information.

   a. Find the roots of $f(x) = x^3 - 3x - 1$.

   b. Find the x-coordinates of the points where the curve $y = x^3$ crosses the line $y = 3x + 1$.

   c. Find all the values of x for which $x^3 - 3x = 1$.

   d. Find the x-coordinates of the points where the cubic curve $y = x^3 - 3x$ crosses the line $y = 1$.

   e. Solve the equation $x^3 - 3x - 1 = 0$.

61. **Removable discontinuity** Give an example of a function $f(x)$ that is continuous for all values of x except $x = 2$, where it has a removable discontinuity. Explain how you know that f is discontinuous at $x = 2$, and how you know the discontinuity is removable.

62. **Nonremovable discontinuity** Give an example of a function $g(x)$ that is continuous for all values of x except $x = -1$, where it has a nonremovable discontinuity. Explain how you know that g is discontinuous there and why the discontinuity is not removable.

63. **A function discontinuous at every point**

   a. Use the fact that every nonempty interval of real numbers contains both rational and irrational numbers to show that the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

   is discontinuous at every point.

   b. Is f right-continuous or left-continuous at any point?

64. If functions $f(x)$ and $g(x)$ are continuous for $0 \leq x \leq 1$, could $f(x)/g(x)$ possibly be discontinuous at a point of $[0, 1]$? Give reasons for your answer.

65. If the product function $h(x) = f(x) \cdot g(x)$ is continuous at $x = 0$, must $f(x)$ and $g(x)$ be continuous at $x = 0$? Give reasons for your answer.

66. Discontinuous composite of continuous functions Give an example of functions f and g, both continuous at $x = 0$, for which the composite $f \circ g$ is discontinuous at $x = 0$. Does this contradict Theorem 9? Give reasons for your answer.

67. Never-zero continuous functions Is it true that a continuous function that is never zero on an interval never changes sign on that interval? Give reasons for your answer.

68. Stretching a rubber band Is it true that if you stretch a rubber band by moving one end to the right and the other to the left, some point of the band will end up in its original position? Give reasons for your answer.

69. A fixed point theorem Suppose that a function f is continuous on the closed interval $[0, 1]$ and that $0 \le f(x) \le 1$ for every x in $[0, 1]$. Show that there must exist a number c in $[0, 1]$ such that $f(c) = c$ (c is called a **fixed point** of f).

70. The sign-preserving property of continuous functions Let f be defined on an interval (a, b) and suppose that $f(c) \neq 0$ at some c where f is continuous. Show that there is an interval $(c - \delta, c + \delta)$ about c where f has the same sign as $f(c)$.

71. Prove that f is continuous at c if and only if

$$\lim_{h \to 0} f(c + h) = f(c).$$

72. Use Exercise 71 together with the identities

$$\sin(h + c) = \sin h \cos c + \cos h \sin c,$$
$$\cos(h + c) = \cos h \cos c - \sin h \sin c$$

to prove that both $f(x) = \sin x$ and $g(x) = \cos x$ are continuous at every point $x = c$.

Solving Equations Graphically

T Use the Intermediate Value Theorem in Exercises 73–80 to prove that each equation has a solution. Then use a graphing calculator or computer grapher to solve the equations.

73. $x^3 - 3x - 1 = 0$

74. $2x^3 - 2x^2 - 2x + 1 = 0$

75. $x(x - 1)^2 = 1$ (one root)

76. $x^x = 2$

77. $\sqrt{x} + \sqrt{1 + x} = 4$

78. $x^3 - 15x + 1 = 0$ (three roots)

79. $\cos x = x$ (one root). Make sure you are using radian mode.

80. $2 \sin x = x$ (three roots). Make sure you are using radian mode.

2.6 Limits Involving Infinity; Asymptotes of Graphs

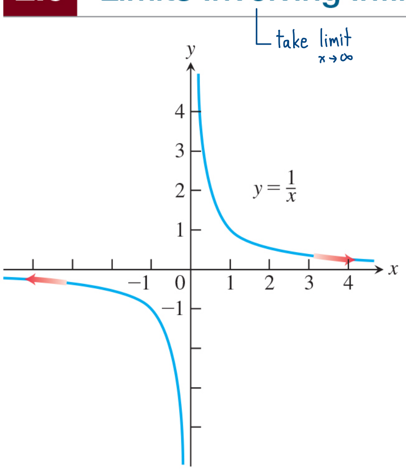


FIGURE 2.49 The graph of $y = 1/x$ approaches 0 as $x \to \infty$ or $x \to -\infty$.

In this section we investigate the behavior of a function when the magnitude of the independent variable x becomes increasingly large, or $x \to \pm\infty$. We further extend the concept of limit to *infinite limits*. Infinite limits provide useful symbols and language for describing the behavior of functions whose values become arbitrarily large in magnitude. We use these ideas to analyze the graphs of functions having *horizontal* or *vertical asymptotes*.

Finite Limits as $x \to \pm\infty$

The symbol for infinity (∞) does not represent a real number. We use ∞ to describe the behavior of a function when the values in its domain or range outgrow all finite bounds. For example, the function $f(x) = 1/x$ is defined for all $x \neq 0$ (Figure 2.49). When x is positive and becomes increasingly large, $1/x$ becomes increasingly small. When x is negative and its magnitude becomes increasingly large, $1/x$ again becomes small. We summarize these observations by saying that $f(x) = 1/x$ has limit 0 as $x \to \infty$ or $x \to -\infty$, or that 0 is a *limit of $f(x) = 1/x$ at infinity and at negative infinity*. Here are precise definitions for the limit of a function whose domain contains positive or negative numbers of unbounded magnitude.

DEFINITIONS

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x > M.$$

2. We say that $f(x)$ has the **limit L as x approaches negative infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

if, for every number $\varepsilon > 0$, there exists a corresponding number N such that for all x in the domain of f

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad x < N.$$

Intuitively, $\lim_{x\to\infty} f(x) = L$ if, as x moves increasingly far from the origin in the positive direction, $f(x)$ gets arbitrarily close to L. Similarly, $\lim_{x\to-\infty} f(x) = L$ if, as x moves increasingly far from the origin in the negative direction, $f(x)$ gets arbitrarily close to L.

The strategy for calculating limits of functions as $x \to +\infty$ or as $x \to -\infty$ is similar to the one for finite limits in Section 2.2. There we first found the limits of the constant and identity functions $y = k$ and $y = x$. We then extended these results to other functions by applying Theorem 1 on limits of algebraic combinations. Here we do the same thing, except that the starting functions are $y = k$ and $y = 1/x$ instead of $y = k$ and $y = x$.

The basic facts to be verified by applying the formal definition are

$$\lim_{x\to\pm\infty} k = k \qquad \text{and} \qquad \lim_{x\to\pm\infty} \frac{1}{x} = 0. \tag{1}$$

We prove the second result in Example 1, and leave the first to Exercises 93 and 94.

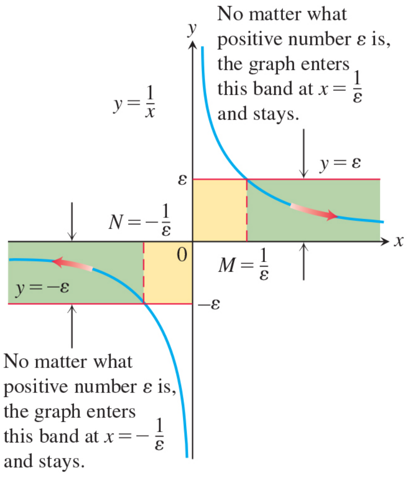
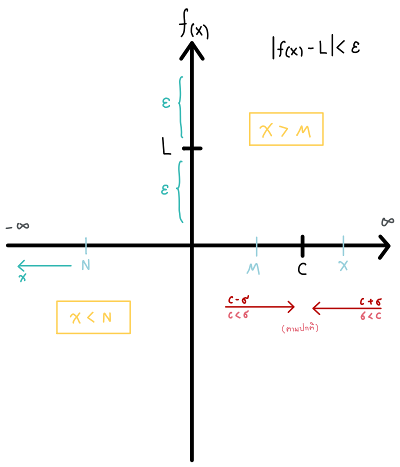


FIGURE 2.50 The geometry behind the argument in Example 1.



EXAMPLE 1 Show that

(a) $\lim_{x\to\infty} \frac{1}{x} = 0$ **(b)** $\lim_{x\to-\infty} \frac{1}{x} = 0.$

Solution

(a) Let $\varepsilon > 0$ be given. We must find a number M such that

$$\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon \qquad \text{whenever} \qquad x > M.$$

The implication will hold if $M = 1/\varepsilon$ or any larger positive number (Figure 2.50). This proves $\lim_{x\to\infty}(1/x) = 0$.

(b) Let $\varepsilon > 0$ be given. We must find a number N such that

$$\left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \varepsilon \qquad \text{whenever} \qquad x < N.$$

The implication will hold if $N = -1/\varepsilon$ or any number less than $-1/\varepsilon$ (Figure 2.50). This proves $\lim_{x\to-\infty}(1/x) = 0$. ■

Limits at infinity have properties similar to those of finite limits.

> **THEOREM 12** All the Limit Laws in Theorem 1 are true when we replace $\lim_{x\to c}$ by $\lim_{x\to\infty}$ or $\lim_{x\to-\infty}$. That is, the variable x may approach a finite number c or $\pm\infty$.

EXAMPLE 2 The properties in Theorem 12 are used to calculate limits in the same way as when x approaches a finite number c.

(a) $\lim_{x\to\infty}\left(5 + \frac{1}{x}\right) = \lim_{x\to\infty} 5 + \lim_{x\to\infty} \frac{1}{x}$ ~> มีค่า 0 — Sum Rule

$= 5 + 0 = 5$ — Known limits

(b) $\lim_{x\to-\infty} \frac{\pi\sqrt{3}}{x^2} = \lim_{x\to-\infty} \pi\sqrt{3}\cdot\frac{1}{x}\cdot\frac{1}{x}$

$= \lim_{x\to-\infty} \pi\sqrt{3}\cdot\lim_{x\to-\infty}\frac{1}{x}\cdot\lim_{x\to-\infty}\frac{1}{x}$ — Product Rule

$= \pi\sqrt{3}\cdot 0\cdot 0 = 0$ — Known limits ■

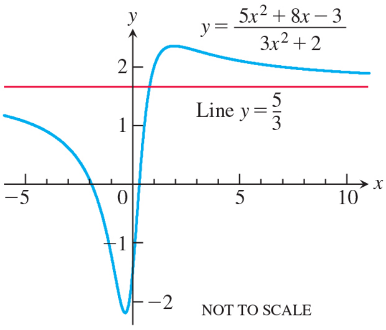


FIGURE 2.51 The graph of the function in Example 3a. The graph approaches the line $y = 5/3$ as $|x|$ increases.

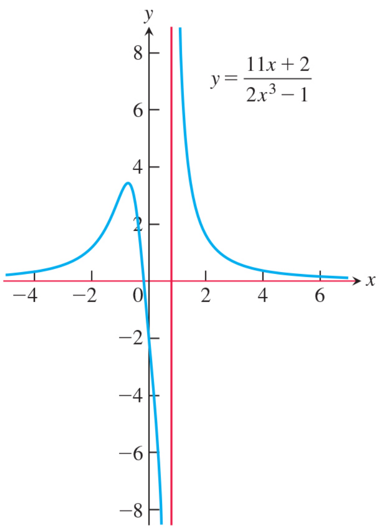


FIGURE 2.52 The graph of the function in Example 3b. The graph approaches the x-axis as $|x|$ increases.

Limits at Infinity of Rational Functions

To determine the limit of a rational function as $x \to \pm\infty$, we first divide the numerator and denominator by the highest power of x in the denominator. The result then depends on the degrees of the polynomials involved.

EXAMPLE 3 These examples illustrate what happens when the degree of the numerator is less than or equal to the degree of the denominator.

(a) $$\lim_{x\to\infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} = \lim_{x\to\infty} \frac{5 + (8/x) - (3/x^2)}{3 + (2/x^2)}$$ Divide numerator and denominator by x^3.

÷x^2

$$= \frac{5 + 0 - 0}{3 + 0} = \frac{5}{3}$$ See Fig. 2.51.

$\dfrac{\square\, x^{\text{MIN}}}{\square\, x^{\text{MAX}}} = \square$ Ans.

* เทคนิคลัด

เลขชี้กำลังสูงสุดบน = ล่าง

Ans เลขหน้าชี้กำลังสูงสุดตัวนั้น

เอาเลขชี้กำลังสูงสุด

หารตั้งเศษ + ส่วน

(b) $$\lim_{x\to-\infty} \frac{11x + 2}{2x^3 - 1} = \lim_{x\to-\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)}$$ Divide numerator and denominator by x^3.

÷x^3

$$= \frac{0 + 0}{2 - 0} = 0$$ See Fig. 2.52. ■

Cases for which the degree of the numerator is greater than the degree of the denominator are illustrated in Examples 10 and 14.

Horizontal Asymptotes

If the distance between the graph of a function and some fixed line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an *asymptote* of the graph.

Looking at $f(x) = 1/x$ (see Figure 2.49), we observe that the x-axis is an asymptote of the curve on the right because

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

and on the left because

$$\lim_{x\to-\infty} \frac{1}{x} = 0.$$

We say that the x-axis is a *horizontal asymptote* of the graph of $f(x) = 1/x$.

> **DEFINITION** A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either
>
> $$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b.$$
>
> เท่ากัน

The graph of a function can have zero, one, or two horizontal asymptotes, depending on whether the function has limits as $x \to \infty$ and as $x \to -\infty$.

The graph of the function

$$f(x) = \frac{5x^2 + 8x - 3}{3x^2 + 2}$$

sketched in Figure 2.51 (Example 3a) has the line $y = 5/3$ as a horizontal asymptote on both the right and the left because

$$\lim_{x\to\infty} f(x) = \frac{5}{3} \quad \text{and} \quad \lim_{x\to-\infty} f(x) = \frac{5}{3}.$$

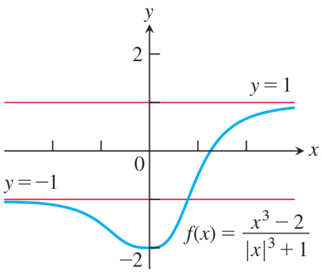


FIGURE 2.53 The graph of the function in Example 4 has two horizontal asymptotes.

EXAMPLE 4 Find the horizontal asymptotes of the graph of

$$f(x) = \frac{x^3 - 2}{|x|^3 + 1}.$$

Solution We calculate the limits as $x \to \pm\infty$.

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For $x \geq 0$: $\displaystyle \lim_{x\to\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to\infty} \frac{x^3 - 2}{x^3 + 1} = \lim_{x\to\infty} \frac{1 - (2/x^3)}{1 + (1/x^3)} = 1.$

For $x < 0$: $\displaystyle \lim_{x\to-\infty} \frac{x^3 - 2}{|x|^3 + 1} = \lim_{x\to-\infty} \frac{x^3 - 2}{(-x)^3 + 1} = \lim_{x\to-\infty} \frac{1 - (2/x^3)}{-1 + (1/x^3)} = -1.$

The horizontal asymptotes are $y = -1$ and $y = 1$. The graph is displayed in Figure 2.53. Notice that the graph crosses the horizontal asymptote $y = -1$ for a positive value of x. ■

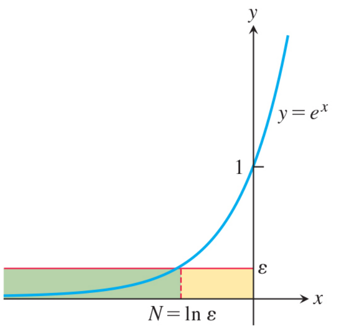

FIGURE 2.54 The graph of $y = e^x$ approaches the x-axis as $x \to -\infty$ (Example 5).

EXAMPLE 5 The x-axis (the line $y = 0$) is a horizontal asymptote of the graph of $y = e^x$ because

$$\lim_{x\to-\infty} \overbrace{e^x}^{f(x)} = \overbrace{0}^{L}.$$

c

To see this, we use the definition of a limit as x approaches $-\infty$. So let $\varepsilon > 0$ be given, but arbitrary. We must find a constant N such that

$$|e^x - 0| < \varepsilon \quad \text{whenever} \quad x < N.$$

Now $|e^x - 0| = e^x$, so the condition that needs to be satisfied whenever $x < N$ is

$$e^x < \varepsilon.$$

Let $x = N$ be the number where $e^x = \varepsilon$. Since e^x is an increasing function, if $x < N$, then $e^x < \varepsilon$. We find N by taking the natural logarithm of both sides of the equation $e^N = \varepsilon$, so $N = \ln \varepsilon$ (see Figure 2.54). With this value of N the condition is satisfied, and we conclude that $\lim_{x\to-\infty} e^x = 0$. ■

$e^x < \varepsilon$
$x < \log_e \varepsilon < \ln \varepsilon$; $y = e^x$
$N = \ln \varepsilon$

EXAMPLE 6 Find **(a)** $\displaystyle \lim_{x\to\infty} \sin(1/x)$ and **(b)** $\displaystyle \lim_{x\to\pm\infty} x\sin(1/x)$.

Solution

(a) We introduce the new variable $t = 1/x$. From Example 1, we know that $t \to 0^+$ as $x \to \infty$ (see Figure 2.49). Therefore,

$$\lim_{x\to\infty} \sin\frac{1}{x} = \lim_{t\to 0^+} \sin t = 0.$$

(b) We calculate the limits as $x \to \infty$ and $x \to -\infty$:

$$\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{t\to 0^+} \frac{\sin t}{t} = 1 \quad \text{and} \quad \lim_{x\to-\infty} x\sin\frac{1}{x} = \lim_{t\to 0^-} \frac{\sin t}{t} = 1.$$

The graph is shown in Figure 2.55, and we see that the line $y = 1$ is a horizontal asymptote. ■

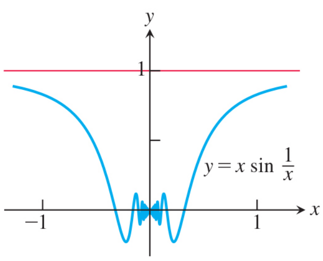

FIGURE 2.55 The line $y = 1$ is a horizontal asymptote of the function graphed here (Example 6b).

Similarly, we can investigate the behavior of $y = f(1/x)$ as $x \to 0$ by investigating $y = f(t)$ as $t \to \pm\infty$, where $t = 1/x$.

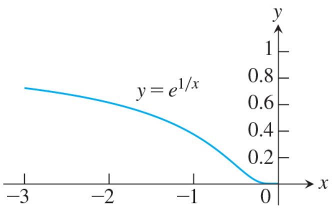

FIGURE 2.56 The graph of $y = e^{1/x}$ for $x < 0$ shows $\lim_{x \to 0^-} e^{1/x} = 0$ (Example 7).

EXAMPLE 7 Find $\lim\limits_{x \to 0^-} e^{1/x}$.

Solution We let $t = 1/x$. From Figure 2.49, we can see that $t \to -\infty$ as $x \to 0^-$. (We make this idea more precise further on.) Therefore,

$$\lim_{x \to 0^-} e^{1/x} = \lim_{t \to -\infty} e^t = 0 \qquad \text{Example 5}$$

(Figure 2.56). ■

The Sandwich Theorem also holds for limits as $x \to \pm\infty$. You must be sure, though, that the function whose limit you are trying to find stays between the bounding functions at very large values of x in magnitude consistent with whether $x \to \infty$ or $x \to -\infty$.

EXAMPLE 8 Using the Sandwich Theorem, find the horizontal asymptote of the curve

$$y = 2 + \frac{\sin x}{x}.$$

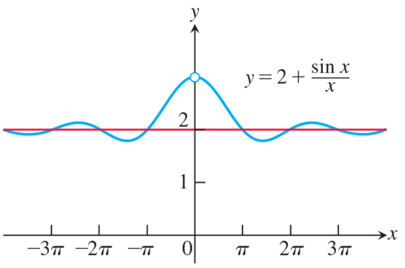

FIGURE 2.57 A curve may cross one of its asymptotes infinitely often (Example 8).

Solution We are interested in the behavior as $x \to \pm\infty$. Since

$$0 \le \left|\frac{\sin x}{x}\right| \le \left|\frac{1}{x}\right|$$

and $\lim_{x \to \pm\infty} |1/x| = 0$, we have $\lim_{x \to \pm\infty} (\sin x)/x = 0$ by the Sandwich Theorem. Hence,

$$\lim_{x \to \pm\infty} \left(2 + \frac{\sin x}{x}\right) = 2 + 0 = 2,$$

and the line $y = 2$ is a horizontal asymptote of the curve on both left and right (Figure 2.57).

This example illustrates that a curve may cross one of its horizontal asymptotes many times. ■

EXAMPLE 9 Find $\lim\limits_{x \to \infty} \left(x - \sqrt{x^2 + 16}\right)$.

Solution Both of the terms x and $\sqrt{x^2 + 16}$ approach infinity as $x \to \infty$, so what happens to the difference in the limit is unclear (we cannot subtract ∞ from ∞ because the symbol does not represent a real number). In this situation we can multiply the numerator and the denominator by the conjugate radical expression to obtain an equivalent algebraic expression:

Conjugate.

$$\lim_{x \to \infty} \left(x - \sqrt{x^2 + 16}\right) = \lim_{x \to \infty} \left(x - \sqrt{x^2 + 16}\right) \boxed{\frac{x + \sqrt{x^2 + 16}}{x + \sqrt{x^2 + 16}}} \qquad \text{Multiply and divide by the conjugate.}$$

$$= \lim_{x \to \infty} \frac{x^2 - (x^2 + 16)}{x + \sqrt{x^2 + 16}} = \lim_{x \to \infty} \frac{-16}{x + \sqrt{x^2 + 16}}.$$

÷x

As $x \to \infty$, the denominator in this last expression becomes arbitrarily large, while the numerator remains constant, so we see that the limit is 0. We can also obtain this result by a direct calculation using the Limit Laws:

$$\lim_{x \to \infty} \frac{-16}{x + \sqrt{x^2 + 16}} = \lim_{x \to \infty} \frac{-\frac{16}{x}}{1 + \sqrt{\frac{x^2}{x^2} + \frac{16}{x^2}}} = \frac{0}{1 + \sqrt{1 + 0}} = 0.$$

■

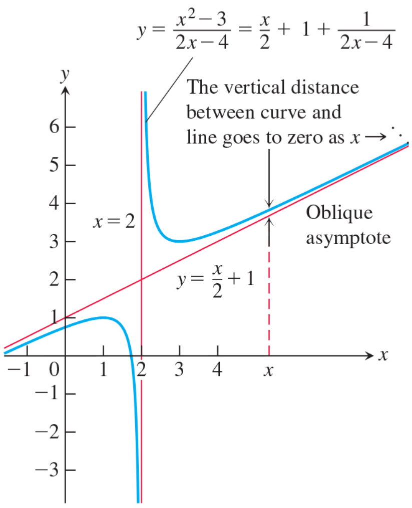

FIGURE 2.58 The graph of the function in Example 10 has an oblique asymptote.

Oblique Asymptotes

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**. We find an equation for the asymptote by dividing numerator by denominator to express f as a linear function plus a remainder that goes to zero as $x \to \pm\infty$.

EXAMPLE 10 Find the oblique asymptote of the graph of

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$$f(x) = \frac{x^2 - 3}{2x - 4}$$

in Figure 2.58.

Solution We are interested in the behavior as $x \to \pm\infty$. We divide $(2x - 4)$ into $(x^2 - 3)$:

$$\begin{array}{r} \frac{x}{2} + 1 \\ 2x - 4 \overline{\smash{)}\, x^2 + 0x - 3} \\ \underline{x^2 - 2x} \\ 2x - 3 \\ \underline{2x - 4} \\ 1 \end{array}$$

ระยะที่ห่าง
ให้แนวแกน y

วิธีหาร

$\frac{x}{2} + 1$ ...... f(x) linear

$2x - 4 \overline{\smash{)}\, x^2 + 0x - 3}$

$x^2 - 2x$

$2x - 3$

$2x - 4$

1 ...... เศษ

เอาอะไรก็ได้
คูณให้ได้เท่าตรงนี้

Linear function

Ans. $\boxed{\frac{x}{2} + 1} + \boxed{\frac{1}{2x - 4}}$

$y = \frac{1}{2}x + 1$

m

remainder

This tells us that

$$f(x) = \frac{x^2 - 3}{2x - 4} = \underbrace{\left(\frac{x}{2} + 1\right)}_{\text{linear } g(x)} + \underbrace{\left(\frac{1}{2x - 4}\right)}_{\text{remainder}}.$$

As $x \to \pm\infty$, the remainder, whose magnitude gives the vertical distance between the graphs of f and g, goes to zero, making the slanted line

$$g(x) = \frac{x}{2} + 1$$

an asymptote of the graph of f (Figure 2.58). The line $y = g(x)$ is an asymptote both to the right and to the left. ■

Notice in Example 10 that if the degree of the numerator in a rational function is greater than the degree of the denominator, then the limit as $|x|$ becomes large is $+\infty$ or $-\infty$, depending on the signs assumed by the numerator and denominator.

Infinite Limits

Let us look again at the function $f(x) = 1/x$. As $x \to 0^+$, the values of f grow without bound, eventually reaching and surpassing every positive real number. That is, given any positive real number B, however large, the values of f become larger still (Figure 2.59).

Thus, f has no limit as $x \to 0^+$. It is nevertheless convenient to describe the behavior of f by saying that $f(x)$ approaches ∞ as $x \to 0^+$. We write

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x} = \infty.$$

In writing this equation, we are *not* saying that the limit exists. Nor are we saying that there is a real number ∞, for there is no such number. Rather, this expression is just a concise way of saying that $\lim_{x \to 0^+} (1/x)$ *does not exist because* $1/x$ *becomes arbitrarily large and positive as* $x \to 0^+$.

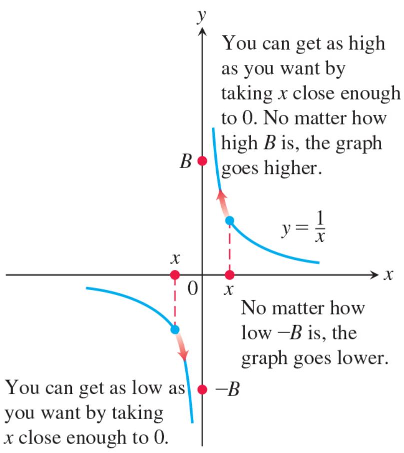

FIGURE 2.59 One-sided infinite limits:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

As $x \to 0^-$, the values of $f(x) = 1/x$ become arbitrarily large and negative. Given any negative real number $-B$, the values of f eventually lie below $-B$. (See Figure 2.59.) We write

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{1}{x} = -\infty.$$

Again, we are not saying that the limit exists and equals the number $-\infty$. There *is* no real number $-\infty$. We are describing the behavior of a function whose limit as $x \to 0^-$ *does not exist because its values become arbitrarily large and negative*.

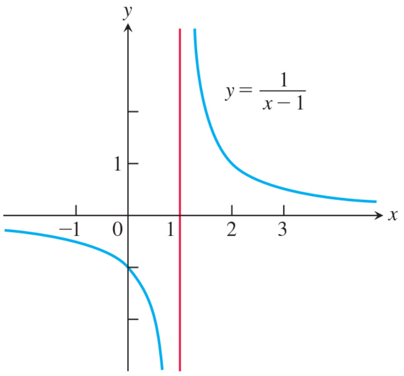

FIGURE 2.60 Near $x = 1$, the function $y = 1/(x - 1)$ behaves the way the function $y = 1/x$ behaves near $x = 0$. Its graph is the graph of $y = 1/x$ shifted 1 unit to the right (Example 11).

EXAMPLE 11 Find $\lim_{x\to 1^+} \frac{1}{x-1}$ and $\lim_{x\to 1^-} \frac{1}{x-1}$.

Geometric Solution The graph of $y = 1/(x - 1)$ is the graph of $y = 1/x$ shifted 1 unit to the right (Figure 2.60). Therefore, $y = 1/(x - 1)$ behaves near 1 exactly the way $y = 1/x$ behaves near 0:

$$\lim_{x\to 1^+} \frac{1}{x-1} = \infty \qquad \text{and} \qquad \lim_{x\to 1^-} \frac{1}{x-1} = -\infty.$$

shift Right 1 หน่วย

Analytic Solution Think about the number $x - 1$ and its reciprocal. As $x \to 1^+$, we have $(x - 1) \to 0^+$ and $1/(x - 1) \to \infty$. As $x \to 1^-$, we have $(x - 1) \to 0^-$ and $1/(x - 1) \to -\infty$. ■

EXAMPLE 12 Discuss the behavior of

$$f(x) = \frac{1}{x^2} \qquad \text{as} \qquad x \to 0.$$

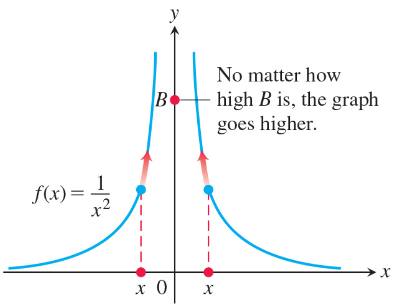

FIGURE 2.61 The graph of $f(x)$ in Example 12 approaches infinity as $x \to 0$.

Solution As x approaches zero from either side, the values of $1/x^2$ are positive and become arbitrarily large (Figure 2.61). This means that

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} = \infty.$$

The function $y = 1/x$ shows no consistent behavior as $x \to 0$. We have $1/x \to \infty$ if $x \to 0^+$, but $1/x \to -\infty$ if $x \to 0^-$. All we can say about $\lim_{x\to 0} (1/x)$ is that it does not exist. The function $y = 1/x^2$ is different. Its values approach infinity as x approaches zero from either side, so we can say that $\lim_{x\to 0} (1/x^2) = \infty$. ■

EXAMPLE 13 These examples illustrate that rational functions can behave in various ways near zeros of the denominator.

(a) $\displaystyle \lim_{x\to 2} \frac{(x-2)^2}{x^2-4} = \lim_{x\to 2} \frac{(x-2)^2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{x-2}{x+2} = 0$

Can substitute 2 for x after algebraic manipulation eliminates division by 0.

(b) $\displaystyle \lim_{x\to 2} \frac{x-2}{x^2-4} = \lim_{x\to 2} \frac{x-2}{(x-2)(x+2)} = \lim_{x\to 2} \frac{1}{x+2} = \frac{1}{4}$

Again substitute 2 for x after algebraic manipulation eliminates division by 0.

(c) $\displaystyle \lim_{x\to 2^+} \frac{x-3}{x^2-4} = \lim_{x\to 2^+} \frac{x-3}{(x-2)(x+2)} = -\infty$

The values are negative for $x > 2$, x near 2.

(d) $\displaystyle \lim_{x\to 2^-} \frac{x-3}{x^2-4} = \lim_{x\to 2^-} \frac{x-3}{(x-2)(x+2)} = \infty$

The values are positive for $x < 2$, x near 2.

(e) $\lim_{x\to 2}\dfrac{x-3}{x^2-4}=\lim_{x\to 2}\dfrac{x-3}{(x-2)(x+2)}$ does not exist. — Limits from left and from right differ.

(f) $\lim_{x\to 2}\dfrac{2-x}{(x-2)^3}=\lim_{x\to 2}\dfrac{-(x-2)}{(x-2)^3}=\lim_{x\to 2}\dfrac{-1}{(x-2)^2}=-\infty$ — Denominator is positive, so values are negative near $x = 2$.

In parts (a) and (b) the effect of the zero in the denominator at $x = 2$ is canceled because the numerator is zero there also. Thus a finite limit exists. This is not true in part (f), where cancellation still leaves a zero factor in the denominator. ■

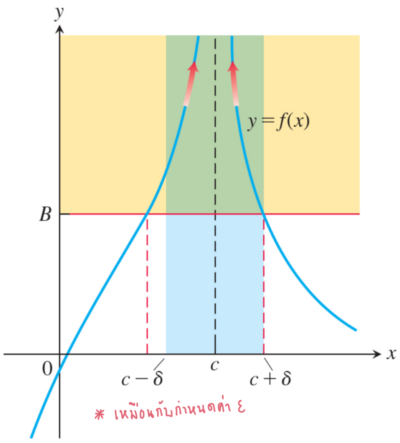

FIGURE 2.62 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies above the line $y = B$.

EXAMPLE 14 Find $\lim_{x\to -\infty}\dfrac{2x^5-6x^4+1}{3x^2+x-7}$.

Solution We are asked to find the limit of a rational function as $x \to -\infty$, so we divide the numerator and denominator by x^2, the highest power of x in the denominator:

$$* \lim_{x\to-\infty}\frac{2x^5-6x^4+1}{3x^2+x-7}=\lim_{x\to-\infty}\frac{2x^3-6x^2+x^{-2}}{3+x^{-1}-7x^{-2}}$$
$$=\lim_{x\to-\infty}\frac{2x^2(x-3)+x^{-2}}{3+x^{-1}-7x^{-2}}$$
$$=-\infty, \qquad x^{-n}\to 0,\ x-3\to-\infty$$

because the numerator tends to $-\infty$ while the denominator approaches 3 as $x \to -\infty$. ■

Precise Definitions of Infinite Limits

Instead of requiring $f(x)$ to lie arbitrarily close to a finite number L for all x sufficiently close to c, the definitions of infinite limits require $f(x)$ to lie arbitrarily far from zero. Except for this change, the language is very similar to what we have seen before. Figures 2.62 and 2.63 accompany these definitions.

สรุป : $-B < f(x) < B$ whenever $0 < |x - c| < \sigma$

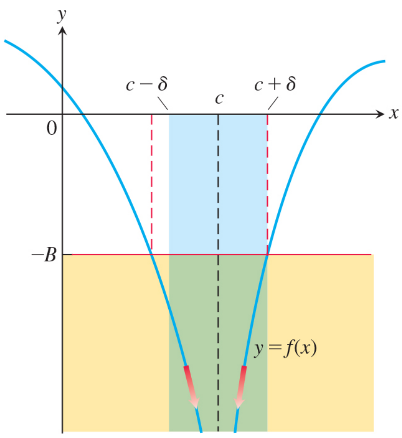

FIGURE 2.63 For $c - \delta < x < c + \delta$, the graph of $f(x)$ lies below the line $y = -B$.

DEFINITIONS

1. We say that **$f(x)$ approaches infinity as x approaches c**, and write
$$\lim_{x\to c} f(x) = \infty,$$
if for every positive real number B there exists a corresponding $\delta > 0$ such that
$$f(x) > B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

2. We say that **$f(x)$ approaches negative infinity as x approaches c**, and write
$$\lim_{x\to c} f(x) = -\infty,$$
if for every negative real number $-B$ there exists a corresponding $\delta > 0$ such that
$$f(x) < -B \quad \text{whenever} \quad 0 < |x - c| < \delta.$$

The precise definitions of one-sided infinite limits at c are similar and are stated in the exercises.

EXAMPLE 15 Prove that $\lim_{x\to 0}\dfrac{1}{x^2}=\infty$.

Solution Given $B > 0$, we want to find $\delta > 0$ such that

$$0 < |x - 0| < \delta \quad \text{implies} \quad \frac{1}{x^2} > B.$$

Now,

$$\frac{1}{x^2} > B \qquad \text{if and only if} \qquad x^2 < \frac{1}{B}$$

or, equivalently,

$$|x| < \frac{1}{\sqrt{B}}.$$

Thus, choosing $\delta = 1/\sqrt{B}$ (or any smaller positive number), we see that

$$|x| < \delta \quad \text{implies} \quad \frac{1}{x^2} > \frac{1}{\delta^2} \geq B.$$

Therefore, by definition,

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Vertical Asymptotes

Notice that the distance between a point on the graph of $f(x) = 1/x$ and the y-axis approaches zero as the point moves vertically along the graph and away from the origin (Figure 2.64). The function $f(x) = 1/x$ is unbounded as x approaches 0 because

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \text{and} \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty.$$

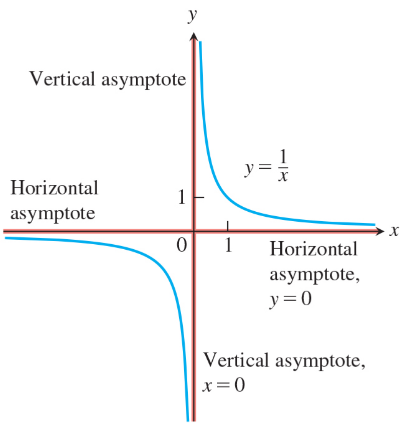

FIGURE 2.64 The coordinate axes are asymptotes of both branches of the hyperbola $y = 1/x$.

We say that the line $x = 0$ (the y-axis) is a *vertical asymptote* of the graph of $f(x) = 1/x$. Observe that the denominator is zero at $x = 0$ and the function is undefined there.

> **DEFINITION** A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either
>
> $$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

EXAMPLE 16 Find the horizontal and vertical asymptotes of the curve

$$y = \frac{x + 3}{x + 2}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and the behavior as $x \to -2$, where the denominator is zero.

The asymptotes are revealed if we recast the rational function as a polynomial with a remainder, by dividing $(x + 2)$ into $(x + 3)$:

$$\begin{array}{r} 1 \\ x + 2\overline{)x + 3} \\ \underline{x + 2} \\ 1 \end{array}$$

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This result enables us to rewrite y as:

$$y = 1 + \frac{1}{x + 2}.$$

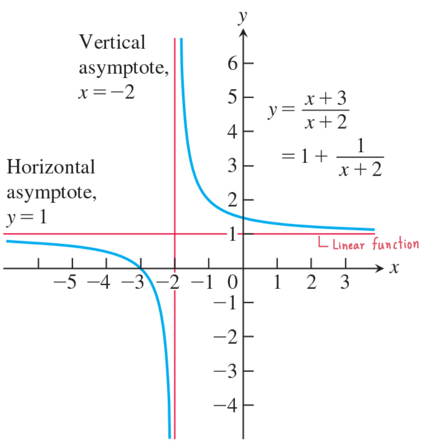


FIGURE 2.65 The lines $y = 1$ and $x = -2$ are asymptotes of the curve in Example 16.

As $x \to \pm\infty$, the curve approaches the horizontal asymptote $y = 1$; as $x \to -2$, the curve approaches the vertical asymptote $x = -2$. We see that the curve in question is the graph of $f(x) = 1/x$ shifted 1 unit up and 2 units left (Figure 2.65). The asymptotes, instead of being the coordinate axes, are now the lines $y = 1$ and $x = -2$.

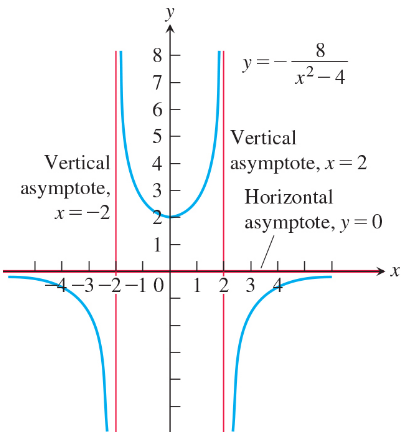


FIGURE 2.66 Graph of the function in Example 17. Notice that the curve approaches the x-axis from only one side. Asymptotes do not have to be two-sided.

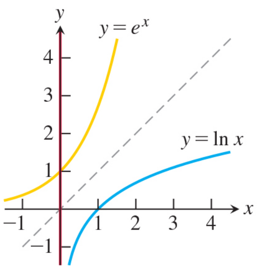


FIGURE 2.67 The line $x = 0$ is a vertical asymptote of the natural logarithm function (Example 18).

EXAMPLE 17 Find the horizontal and vertical asymptotes of the graph of

$$f(x) = -\frac{8}{x^2 - 4}.$$

Solution We are interested in the behavior as $x \to \pm\infty$ and as $x \to \pm 2$, where the denominator is zero. Notice that f is an even function of x, so its graph is symmetric with respect to the y-axis.

(a) *The behavior as* $x \to \pm\infty$. Since $\lim_{x\to\infty} f(x) = 0$, the line $y = 0$ is a horizontal asymptote of the graph to the right. By symmetry it is an asymptote to the left as well (Figure 2.66). Notice that the curve approaches the x-axis from only the negative side (or from below). Also, $f(0) = 2$.

(b) *The behavior as* $x \to \pm 2$. Since

$$\lim_{x\to 2^+} f(x) = -\infty \qquad \text{and} \qquad \lim_{x\to 2^-} f(x) = \infty,$$

the line $x = 2$ is a vertical asymptote both from the right and from the left. By symmetry, the line $x = -2$ is also a vertical asymptote.

There are no other asymptotes because f has a finite limit at all other points. ■

EXAMPLE 18 The graph of the natural logarithm function has the y-axis (the line $x = 0$) as a vertical asymptote. We see this from the graph sketched in Figure 2.67 (which is the reflection of the graph of the natural exponential function across the line $y = x$) and the fact that the x-axis is a horizontal asymptote of $y = e^x$ (Example 5). Thus,

$$\lim_{x\to 0^+} \ln x = -\infty.$$

The same result is true for $y = \log_a x$ whenever $a > 1$. ■

EXAMPLE 19 The curves

$$y = \sec x = \frac{1}{\cos x} \qquad \text{and} \qquad y = \tan x = \frac{\sin x}{\cos x}$$

both have vertical asymptotes at odd-integer multiples of $\pi/2$, where $\cos x = 0$ (Figure 2.68).

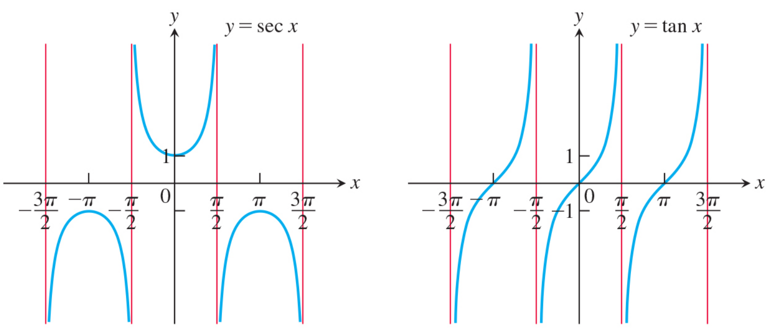


FIGURE 2.68 The graphs of $\sec x$ and $\tan x$ have infinitely many vertical asymptotes (Example 19). ■

Dominant Terms

In Example 10 we saw that by using long division, we can rewrite the function

$$f(x) = \frac{x^2 - 3}{2x - 4}$$

as a linear function plus a remainder term:

$$f(x) = \left(\frac{x}{2} + 1\right) + \left(\frac{1}{2x - 4}\right).$$

This tells us immediately that

$$f(x) \approx \frac{x}{2} + 1 \qquad \text{For } |x| \text{ large, } \frac{1}{2x-4} \text{ is near 0.}$$

$$f(x) \approx \frac{1}{2x - 4} \qquad \text{For } x \text{ near 2, this term is very large in absolute value.}$$

If we want to know how f behaves, this is the way to find out. It behaves like $y = (x/2) + 1$ when $|x|$ is large and the contribution of $1/(2x - 4)$ to the total value of f is insignificant. It behaves like $1/(2x - 4)$ when x is so close to 2 that $1/(2x - 4)$ makes the dominant contribution.

We say that $(x/2) + 1$ **dominates** when x approaches ∞ or $-\infty$, and we say that $1/(2x - 4)$ dominates when x approaches 2. **Dominant terms** like these help us predict a function's behavior.

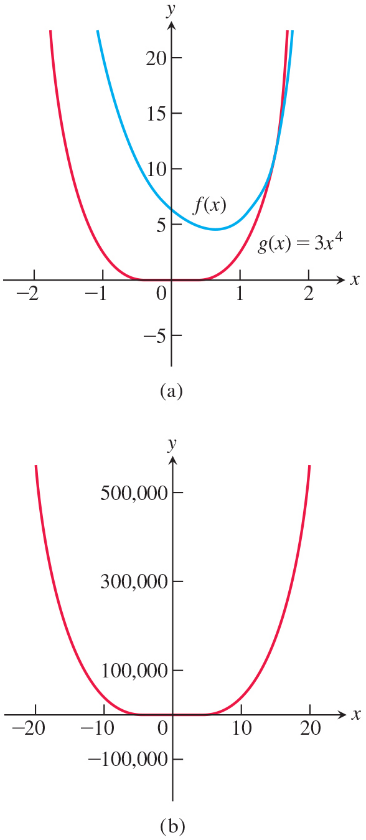


FIGURE 2.69 The graphs of f and g are (a) distinct for $|x|$ small, and (b) nearly identical for $|x|$ large (Example 20).

EXAMPLE 20 Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that although f and g are quite different for numerically small values of x, they behave similarly for $|x|$ very large, in the sense that their ratios approach 1 as $x \to \infty$ or $x \to -\infty$.

Solution The graphs of f and g behave quite differently near the origin (Figure 2.69a), but appear as virtually identical on a larger scale (Figure 2.69b).

We can test that the term $3x^4$ in f, represented graphically by g, dominates the polynomial f for numerically large values of x by examining the ratio of the two functions as $x \to \pm\infty$. We find that

$$\begin{aligned}
\lim_{x \to \pm\infty} \frac{f(x)}{g(x)} &= \lim_{x \to \pm\infty} \frac{3x^4 - 2x^3 + 3x^2 - 5x + 6}{3x^4} \\
&= \lim_{x \to \pm\infty} \left(1 - \frac{2}{3x} + \frac{1}{x^2} - \frac{5}{3x^3} + \frac{2}{x^4}\right) \\
&= 1,
\end{aligned}$$

which means that f and g appear nearly identical when $|x|$ is large. ■

EXERCISES 2.6

Finding Limits

1. For the function f whose graph is given, determine the following limits.

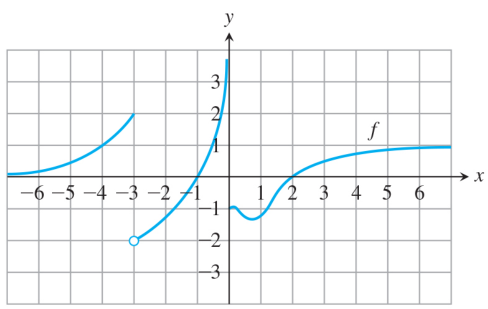
a. $\lim_{x \to 2} f(x)$ **b.** $\lim_{x \to -3^+} f(x)$

c. $\lim_{x \to -3^-} f(x)$ **d.** $\lim_{x \to -3} f(x)$

e. $\lim_{x \to 0^+} f(x)$ **f.** $\lim_{x \to 0^-} f(x)$

g. $\lim_{x \to 0} f(x)$ **h.** $\lim_{x \to \infty} f(x)$

i. $\lim_{x \to -\infty} f(x)$



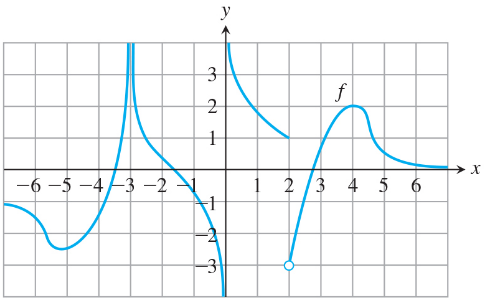
2. For the function f whose graph is given, determine the following limits.

a. $\lim_{x\to 4} f(x)$ **b.** $\lim_{x\to 2^+} f(x)$ **c.** $\lim_{x\to 2^-} f(x)$

d. $\lim_{x\to 2} f(x)$ **e.** $\lim_{x\to -3^+} f(x)$ **f.** $\lim_{x\to -3^-} f(x)$

g. $\lim_{x\to -3} f(x)$ **h.** $\lim_{x\to 0^+} f(x)$ **i.** $\lim_{x\to 0^-} f(x)$

j. $\lim_{x\to 0} f(x)$ **k.** $\lim_{x\to\infty} f(x)$ **l.** $\lim_{x\to -\infty} f(x)$



In Exercises 3–8, find the limit of each function **(a)** as $x\to\infty$ and **(b)** as $x\to -\infty$. (You may wish to visualize your answer with a graphing calculator or computer.)

3. $f(x) = \frac{2}{x} - 3$ **4.** $f(x) = \pi - \frac{2}{x^2}$

5. $g(x) = \frac{1}{2 + (1/x)}$ **6.** $g(x) = \frac{1}{8 - (5/x^2)}$

7. $h(x) = \frac{-5 + (7/x)}{3 - (1/x^2)}$ **8.** $h(x) = \frac{3 - (2/x)}{4 + (\sqrt{2}/x^2)}$

Find the limits in Exercises 9–12.

9. $\lim_{x\to\infty} \frac{\sin 2x}{x}$ **10.** $\lim_{\theta\to -\infty} \frac{\cos\theta}{3\theta}$

11. $\lim_{t\to -\infty} \frac{2 - t + \sin t}{t + \cos t}$ **12.** $\lim_{r\to\infty} \frac{r + \sin r}{2r + 7 - 5\sin r}$

Limits of Rational Functions

In Exercises 13–22, find the limit of each rational function **(a)** as $x\to\infty$ and **(b)** as $x\to -\infty$.

13. $f(x) = \frac{2x + 3}{5x + 7}$ **14.** $f(x) = \frac{2x^3 + 7}{x^3 - x^2 + x + 7}$

15. $f(x) = \frac{x + 1}{x^2 + 3}$ **16.** $f(x) = \frac{3x + 7}{x^2 - 2}$

17. $h(x) = \frac{7x^3}{x^3 - 3x^2 + 6x}$ **18.** $h(x) = \frac{9x^4 + x}{2x^4 + 5x^2 - x + 6}$

19. $g(x) = \frac{10x^5 + x^4 + 31}{x^6}$ **20.** $g(x) = \frac{x^3 + 7x^2 - 2}{x^2 - x + 1}$

21. $f(x) = \frac{3x^7 + 5x^2 - 1}{6x^3 - 7x + 3}$ **22.** $h(x) = \frac{5x^8 - 2x^3 + 9}{3 + x - 4x^5}$

Limits as $x\to\infty$ or $x\to -\infty$

The process by which we determine limits of rational functions applies equally well to ratios containing noninteger or negative powers of x: Divide numerator and denominator by the highest power of x in the denominator and proceed from there. Find the limits in Exercises 23–36.

23. $\lim_{x\to\infty} \sqrt{\frac{8x^2 - 3}{2x^2 + x}}$ **24.** $\lim_{x\to -\infty} \left(\frac{x^2 + x - 1}{8x^2 - 3}\right)^{1/3}$

25. $\lim_{x\to -\infty} \left(\frac{1 - x^3}{x^2 + 7x}\right)^5$ **26.** $\lim_{x\to\infty} \sqrt{\frac{x^2 - 5x}{x^3 + x - 2}}$

27. $\lim_{x\to\infty} \frac{2\sqrt{x} + x^{-1}}{3x - 7}$ **28.** $\lim_{x\to\infty} \frac{2 + \sqrt{x}}{2 - \sqrt{x}}$

29. $\lim_{x\to -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$ **30.** $\lim_{x\to\infty} \frac{x^{-1} + x^{-4}}{x^{-2} - x^{-3}}$

31. $\lim_{x\to\infty} \frac{2x^{5/3} - x^{1/3} + 7}{x^{8/5} + 3x + \sqrt{x}}$ **32.** $\lim_{x\to -\infty} \frac{\sqrt[3]{x} - 5x + 3}{2x + x^{2/3} - 4}$

33. $\lim_{x\to\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$ **34.** $\lim_{x\to -\infty} \frac{\sqrt{x^2 + 1}}{x + 1}$

35. $\lim_{x\to\infty} \frac{x - 3}{\sqrt{4x^2 + 25}}$ **36.** $\lim_{x\to -\infty} \frac{4 - 3x^3}{\sqrt{x^6 + 9}}$

Infinite Limits

Find the limits in Exercises 37–48.

37. $\lim_{x\to 0^+} \frac{1}{3x}$ **38.** $\lim_{x\to 0^-} \frac{5}{2x}$

39. $\lim_{x\to 2^-} \frac{3}{x - 2}$ **40.** $\lim_{x\to 3^+} \frac{1}{x - 3}$

41. $\lim_{x\to -8^+} \frac{2x}{x + 8}$ **42.** $\lim_{x\to -5^-} \frac{3x}{2x + 10}$

43. $\lim_{x\to 7} \frac{4}{(x - 7)^2}$ **44.** $\lim_{x\to 0} \frac{-1}{x^2(x + 1)}$

45. **a.** $\lim_{x\to 0^+} \frac{2}{3x^{1/3}}$ **b.** $\lim_{x\to 0^-} \frac{2}{3x^{1/3}}$

46. **a.** $\lim_{x\to 0^+} \frac{2}{x^{1/5}}$ **b.** $\lim_{x\to 0^-} \frac{2}{x^{1/5}}$

47. $\lim_{x\to 0} \frac{4}{x^{2/5}}$ **48.** $\lim_{x\to 0} \frac{1}{x^{2/3}}$

Find the limits in Exercises 49–52.

49. $\lim_{x\to(\pi/2)^-} \tan x$ **50.** $\lim_{x\to(-\pi/2)^+} \sec x$

51. $\lim_{\theta\to 0^-} (1 + \csc\theta)$ **52.** $\lim_{\theta\to 0} (2 - \cot\theta)$

Find the limits in Exercises 53–58.

53. $\lim \frac{1}{x^2 - 4}$ as

a. $x\to 2^+$ **b.** $x\to 2^-$

c. $x\to -2^+$ **d.** $x\to -2^-$

54. $\lim \frac{x}{x^2 - 1}$ as

a. $x\to 1^+$ **b.** $x\to 1^-$

c. $x\to -1^+$ **d.** $x\to -1^-$

55. $\lim \left(\frac{x^2}{2} - \frac{1}{x}\right)$ as

a. $x \to 0^+$ **b.** $x \to 0^-$

c. $x \to \sqrt[3]{2}$ **d.** $x \to -1$

56. $\lim \frac{x^2 - 1}{2x + 4}$ as

a. $x \to -2^+$ **b.** $x \to -2^-$

c. $x \to 1^+$ **d.** $x \to 0^-$

57. $\lim \frac{x^2 - 3x + 2}{x^3 - 2x^2}$ as

a. $x \to 0^+$ **b.** $x \to 2^+$

c. $x \to 2^-$ **d.** $x \to 2$

e. What, if anything, can be said about the limit as $x \to 0$?

58. $\lim \frac{x^2 - 3x + 2}{x^3 - 4x}$ as

a. $x \to 2^+$ **b.** $x \to -2^+$

c. $x \to 0^-$ **d.** $x \to 1^+$

e. What, if anything, can be said about the limit as $x \to 0$?

Find the limits in Exercises 59–62.

59. $\lim \left(2 - \frac{3}{t^{1/3}}\right)$ as

a. $t \to 0^+$ **b.** $t \to 0^-$

60. $\lim \left(\frac{1}{t^{3/5}} + 7\right)$ as

a. $t \to 0^+$ **b.** $t \to 0^-$

61. $\lim \left(\frac{1}{x^{2/3}} + \frac{2}{(x - 1)^{2/3}}\right)$ as

a. $x \to 0^+$ **b.** $x \to 0^-$

c. $x \to 1^+$ **d.** $x \to 1^-$

62. $\lim \left(\frac{1}{x^{1/3}} - \frac{1}{(x - 1)^{4/3}}\right)$ as

a. $x \to 0^+$ **b.** $x \to 0^-$

c. $x \to 1^+$ **d.** $x \to 1^-$

Graphing Simple Rational Functions

Graph the rational functions in Exercises 63–68. Include the graphs and equations of the asymptotes and dominant terms.

63. $y = \frac{1}{x - 1}$ **64.** $y = \frac{1}{x + 1}$

65. $y = \frac{1}{2x + 4}$ **66.** $y = \frac{-3}{x - 3}$

67. $y = \frac{x + 3}{x + 2}$ **68.** $y = \frac{2x}{x + 1}$

Domains, Ranges, and Asymptotes

Determine the domain and range of each function. Use various limits to find the asymptotes and the ranges.

69. $y = 4 + \frac{3x^2}{x^2 + 1}$ **70.** $y = \frac{2x}{x^2 - 1}$

71. $y = \frac{8 - e^x}{2 + e^x}$ **72.** $y = \frac{4e^x + e^{2x}}{e^x + e^{2x}}$

73. $y = \frac{\sqrt{x^2 + 4}}{x}$ **74.** $y = \frac{x^3}{x^3 - 8}$

Inventing Graphs and Functions

In Exercises 75–78, sketch the graph of a function $y = f(x)$ that satisfies the given conditions. No formulas are required—just label the coordinate axes and sketch an appropriate graph. (The answers are not unique, so your graphs may not be exactly like those in the answer section.)

75. $f(0) = 0$, $f(1) = 2$, $f(-1) = -2$, $\lim_{x \to -\infty} f(x) = -1$, and $\lim_{x \to \infty} f(x) = 1$

76. $f(0) = 0$, $\lim_{x \to \pm\infty} f(x) = 0$, $\lim_{x \to 0^+} f(x) = 2$, and $\lim_{x \to 0^-} f(x) = -2$

77. $f(0) = 0$, $\lim_{x \to \pm\infty} f(x) = 0$, $\lim_{x \to 1^-} f(x) = \lim_{x \to -1^+} f(x) = \infty$, $\lim_{x \to 1^+} f(x) = -\infty$, and $\lim_{x \to -1^-} f(x) = -\infty$

78. $f(2) = 1$, $f(-1) = 0$, $\lim_{x \to \infty} f(x) = 0$, $\lim_{x \to 0^+} f(x) = \infty$, $\lim_{x \to 0^-} f(x) = -\infty$, and $\lim_{x \to -\infty} f(x) = 1$

In Exercises 79–82, find a function that satisfies the given conditions and sketch its graph. (The answers here are not unique. Any function that satisfies the conditions is acceptable. Feel free to use formulas defined in pieces if that will help.)

79. $\lim_{x \to \pm\infty} f(x) = 0$, $\lim_{x \to 2^-} f(x) = \infty$, and $\lim_{x \to 2^+} f(x) = \infty$

80. $\lim_{x \to \pm\infty} g(x) = 0$, $\lim_{x \to 3^-} g(x) = -\infty$, and $\lim_{x \to 3^+} g(x) = \infty$

81. $\lim_{x \to -\infty} h(x) = -1$, $\lim_{x \to \infty} h(x) = 1$, $\lim_{x \to 0^-} h(x) = -1$, and $\lim_{x \to 0^+} h(x) = 1$

82. $\lim_{x \to \pm\infty} k(x) = 1$, $\lim_{x \to 1^-} k(x) = \infty$, and $\lim_{x \to 1^+} k(x) = -\infty$

83. Suppose that $f(x)$ and $g(x)$ are polynomials in x and that $\lim_{x \to \infty} (f(x)/g(x)) = 2$. Can you conclude anything about $\lim_{x \to -\infty} (f(x)/g(x))$? Give reasons for your answer.

84. Suppose that $f(x)$ and $g(x)$ are polynomials in x. Can the graph of $f(x)/g(x)$ have an asymptote if $g(x)$ is never zero? Give reasons for your answer.

85. How many horizontal asymptotes can the graph of a given rational function have? Give reasons for your answer.

Finding Limits of Differences When $x \to \pm\infty$

Find the limits in Exercises 86–92. (*Hint*: Try multiplying and dividing by the conjugate.)

86. $\lim_{x \to \infty} (\sqrt{x + 9} - \sqrt{x + 4})$

87. $\lim_{x \to \infty} (\sqrt{x^2 + 25} - \sqrt{x^2 - 1})$

88. $\lim_{x \to -\infty} (\sqrt{x^2 + 3} + x)$

89. $\lim_{x \to -\infty} (2x + \sqrt{4x^2 + 3x - 2})$

90. $\lim_{x \to \infty} (\sqrt{9x^2 - x} - 3x)$

91. $\lim_{x\to\infty} \left(\sqrt{x^2+3x} - \sqrt{x^2-2x}\right)$

92. $\lim_{x\to\infty} \left(\sqrt{x^2+x} - \sqrt{x^2-x}\right)$

Using the Formal Definitions

Use the formal definitions of limits as $x \to \pm\infty$ to establish the limits in Exercises 93 and 94.

93. If f has the constant value $f(x) = k$, then $\lim_{x\to\infty} f(x) = k$.

94. If f has the constant value $f(x) = k$, then $\lim_{x\to-\infty} f(x) = k$.

Use formal definitions to prove the limit statements in Exercises 95–98.

95. $\lim_{x\to 0} \dfrac{-1}{x^2} = -\infty$

96. $\lim_{x\to 0} \dfrac{1}{|x|} = \infty$

97. $\lim_{x\to 3} \dfrac{-2}{(x-3)^2} = -\infty$

98. $\lim_{x\to -5} \dfrac{1}{(x+5)^2} = \infty$

99. Here is the definition of **infinite right-hand limit**.

> Suppose that an interval (c, d) lies in the domain of f. We say that $f(x)$ approaches infinity as x approaches c from the right, and write
>
> $$\lim_{x\to c^+} f(x) = \infty,$$
>
> if, for every positive real number B, there exists a corresponding number $\delta > 0$ such that
>
> $$f(x) > B \quad \text{whenever} \quad c < x < c + \delta.$$

Modify the definition to cover the following cases.

a. $\lim_{x\to c^-} f(x) = \infty$

b. $\lim_{x\to c^+} f(x) = -\infty$

c. $\lim_{x\to c^-} f(x) = -\infty$

Use the formal definitions from Exercise 99 to prove the limit statements in Exercises 100–104.

100. $\lim_{x\to 0^+} \dfrac{1}{x} = \infty$

101. $\lim_{x\to 0^-} \dfrac{1}{x} = -\infty$

102. $\lim_{x\to 2^-} \dfrac{1}{x-2} = -\infty$

103. $\lim_{x\to 2^+} \dfrac{1}{x-2} = \infty$

104. $\lim_{x\to 1^-} \dfrac{1}{1-x^2} = \infty$

Oblique Asymptotes

Graph the rational functions in Exercises 105–110. Include the graphs and equations of the asymptotes.

105. $y = \dfrac{x^2}{x-1}$

106. $y = \dfrac{x^2+1}{x-1}$

107. $y = \dfrac{x^2-4}{x-1}$

108. $y = \dfrac{x^2-1}{2x+4}$

109. $y = \dfrac{x^2-1}{x}$

110. $y = \dfrac{x^3+1}{x^2}$

Additional Graphing Exercises

T Graph the curves in Exercises 111–114. Explain the relationship between the curve's formula and what you see.

111. $y = \dfrac{x}{\sqrt{4-x^2}}$

112. $y = \dfrac{-1}{\sqrt{4-x^2}}$

113. $y = x^{2/3} + \dfrac{1}{x^{1/3}}$

114. $y = \sin\left(\dfrac{\pi}{x^2+1}\right)$

T Graph the functions in Exercises 115 and 116. Then answer the following questions.

a. How does the graph behave as $x \to 0^+$?

b. How does the graph behave as $x \to \pm\infty$?

c. How does the graph behave near $x = 1$ and $x = -1$?

Give reasons for your answers.

115. $y = \dfrac{3}{2}\left(x - \dfrac{1}{x}\right)^{2/3}$

116. $y = \dfrac{3}{2}\left(\dfrac{x}{x-1}\right)^{2/3}$

CHAPTER 2 Questions to Guide Your Review

1. What is the average rate of change of the function $g(t)$ over the interval from $t = a$ to $t = b$? How is it related to a secant line?
2. What limit must be calculated to find the rate of change of a function $g(t)$ at $t = t_0$?
3. Give an informal or intuitive definition of the limit

$$\lim_{x\to c} f(x) = L.$$

   Why is the definition "informal"? Give examples.
4. Does the existence and value of the limit of a function $f(x)$ as x approaches c ever depend on what happens at $x = c$? Explain and give examples.
5. What function behaviors might occur for which the limit may fail to exist? Give examples.
6. What theorems are available for calculating limits? Give examples of how the theorems are used.
7. How are one-sided limits related to limits? How can this relationship sometimes be used to calculate a limit or prove it does not exist? Give examples.
8. What is the value of $\lim_{\theta\to 0} ((\sin\theta)/\theta)$? Does it matter whether θ is measured in degrees or radians? Explain.

9. What exactly does $\lim_{x\to c} f(x) = L$ mean? Give an example in which you find a $\delta > 0$ for a given f, L, c, and $\varepsilon > 0$ in the precise definition of limit.

10. Give precise definitions of the following statements.

a. $\lim_{x\to 2^-} f(x) = 5$ **b.** $\lim_{x\to 2^+} f(x) = 5$

c. $\lim_{x\to 2} f(x) = \infty$ **d.** $\lim_{x\to 2} f(x) = -\infty$

11. What conditions must be satisfied by a function if it is to be continuous at an interior point of its domain? At an endpoint?

12. How can looking at the graph of a function help you tell where the function is continuous?

13. What does it mean for a function to be right-continuous at a point? Left-continuous? How are continuity and one-sided continuity related?

14. What does it mean for a function to be continuous on an interval? Give examples to illustrate the fact that a function that is not continuous on its entire domain may still be continuous on selected intervals within the domain.

15. What are the basic types of discontinuity? Give an example of each. What is a removable discontinuity? Give an example.

16. What does it mean for a function to have the Intermediate Value Property? What conditions guarantee that a function has this property over an interval? What are the consequences for graphing and solving the equation $f(x) = 0$?

17. Under what circumstances can you extend a function $f(x)$ to be continuous at a point $x = c$? Give an example.

18. What exactly do $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = L$ mean? Give examples.

19. What are $\lim_{x\to\pm\infty} k$ (k a constant) and $\lim_{x\to\pm\infty} (1/x)$? How do you extend these results to other functions? Give examples.

20. How do you find the limit of a rational function as $x \to \pm\infty$? Give examples.

21. What are horizontal and vertical asymptotes? Give examples.

CHAPTER 2 Practice Exercises

Limits and Continuity

1. Graph the function

$$f(x) = \begin{cases} 1, & x \le -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \ge 1. \end{cases}$$

Then discuss, in detail, limits, one-sided limits, continuity, and one-sided continuity of f at $x = -1, 0$, and 1. Are any of the discontinuities removable? Explain.

2. Repeat the instructions of Exercise 1 for

$$f(x) = \begin{cases} 0, & x \le -1 \\ 1/x, & 0 < |x| < 1 \\ 0, & x = 1 \\ 1, & x > 1. \end{cases}$$

3. Suppose that $f(t)$ and $f(t)$ are defined for all t and that $\lim_{t\to t_0} f(t) = -7$ and $\lim_{t\to t_0} g(t) = 0$. Find the limit as $t \to t_0$ of the following functions.

a. $3f(t)$ **b.** $(f(t))^2$

c. $f(t)\cdot g(t)$ **d.** $\dfrac{f(t)}{g(t) - 7}$

e. $\cos(g(t))$ **f.** $|f(t)|$

g. $f(t) + g(t)$ **h.** $1/f(t)$

4. Suppose the functions $f(x)$ and $g(x)$ are defined for all x and that $\lim_{x\to 0} f(x) = 1/2$ and $\lim_{x\to 0} g(x) = \sqrt{2}$. Find the limits as $x \to 0$ of the following functions.

a. $-g(x)$ **b.** $g(x)\cdot f(x)$

c. $f(x) + g(x)$ **d.** $1/f(x)$

e. $x + f(x)$ **f.** $\dfrac{f(x)\cdot \cos x}{x - 1}$

In Exercises 5 and 6, find the value that $\lim_{x\to 0} g(x)$ must have if the given limit statements hold.

5. $\lim\limits_{x\to 0}\left(\dfrac{4 - g(x)}{x}\right) = 1$ **6.** $\lim\limits_{x\to -4}\left(x \lim\limits_{x\to 0} g(x)\right) = 2$

7. On what intervals are the following functions continuous?

a. $f(x) = x^{1/3}$ **b.** $g(x) = x^{3/4}$

c. $h(x) = x^{-2/3}$ **d.** $k(x) = x^{-1/6}$

8. On what intervals are the following functions continuous?

a. $f(x) = \tan x$

b. $g(x) = \csc x$

c. $h(x) = \dfrac{\cos x}{x - \pi}$

d. $k(x) = \dfrac{\sin x}{x}$

Finding Limits

In Exercises 9–28, find the limit or explain why it does not exist.

9. $\lim \dfrac{x^2 - 4x + 4}{x^3 + 5x^2 - 14x}$

a. as $x \to 0$ **b.** as $x \to 2$

10. $\lim \dfrac{x^2 + x}{x^5 + 2x^4 + x^3}$

a. as $x \to 0$ **b.** as $x \to -1$

11. $\lim\limits_{x\to 1} \dfrac{1 - \sqrt{x}}{1 - x}$ **12.** $\lim\limits_{x\to a} \dfrac{x^2 - a^2}{x^4 - a^4}$

13. $\lim\limits_{h\to 0} \dfrac{(x + h)^2 - x^2}{h}$ **14.** $\lim\limits_{x\to 0} \dfrac{(x + h)^2 - x^2}{h}$

15. $\lim\limits_{x\to 0} \dfrac{\dfrac{1}{2+x} - \dfrac{1}{2}}{x}$

16. $\lim\limits_{x\to 0} \dfrac{(2+x)^3 - 8}{x}$

17. $\lim\limits_{x\to 1} \dfrac{x^{1/3} - 1}{\sqrt{x} - 1}$

18. $\lim\limits_{x\to 64} \dfrac{x^{2/3} - 16}{\sqrt{x} - 8}$

19. $\lim\limits_{x\to 0} \dfrac{\tan(2x)}{\tan(\pi x)}$

20. $\lim\limits_{x\to \pi^-} \csc x$

21. $\lim\limits_{x\to \pi} \sin\left(\dfrac{x}{2} + \sin x\right)$

22. $\lim\limits_{x\to \pi} \cos^2(x - \tan x)$

23. $\lim\limits_{x\to 0} \dfrac{8x}{3\sin x - x}$

24. $\lim\limits_{x\to 0} \dfrac{\cos 2x - 1}{\sin x}$

25. $\lim\limits_{t\to 3^+} \ln(t - 3)$

26. $\lim\limits_{t\to 1} t^2 \ln(2 - \sqrt{t})$

27. $\lim\limits_{\theta\to 0^+} \sqrt{\theta}e^{\cos(\pi/\theta)}$

28. $\lim\limits_{z\to 0^+} \dfrac{2e^{1/z}}{e^{1/z} + 1}$

In Exercises 29–32, find the limit of $g(x)$ as x approaches the indicated value.

29. $\lim\limits_{x\to 0^+} (4g(x))^{1/3} = 2$

30. $\lim\limits_{x\to \sqrt{5}} \dfrac{1}{x + g(x)} = 2$

31. $\lim\limits_{x\to 1} \dfrac{3x^2 + 1}{g(x)} = \infty$

32. $\lim\limits_{x\to -2} \dfrac{5 - x^2}{\sqrt{g(x)}} = 0$

T Roots

33. Let $f(x) = x^3 - x - 1$.

a. Use the Intermediate Value Theorem to show that f has a zero between -1 and 2.

b. Solve the equation $f(x) = 0$ graphically with an error of magnitude at most 10^{-8}.

c. It can be shown that the exact value of the solution in part **(b)** is

$$\left(\frac{1}{2} + \frac{\sqrt{69}}{18}\right)^{1/3} + \left(\frac{1}{2} - \frac{\sqrt{69}}{18}\right)^{1/3}.$$

Evaluate this exact answer and compare it with the value you found in part (b).

T **34.** Let $f(\theta) = \theta^3 - 2\theta + 2$.

a. Use the Intermediate Value Theorem to show that f has a zero between -2 and 0.

b. Solve the equation $f(\theta) = 0$ graphically with an error of magnitude at most 10^{-4}.

c. It can be shown that the exact value of the solution in part (b) is

$$\left(\sqrt{\frac{19}{27}} - 1\right)^{1/3} - \left(\sqrt{\frac{19}{27}} + 1\right)^{1/3}.$$

Evaluate this exact answer and compare it with the value you found in part (b).

Continuous Extension

35. Can $f(x) = x(x^2 - 1)/|x^2 - 1|$ be extended to be continuous at $x = 1$ or -1? Give reasons for your answers. (Graph the function—you will find the graph interesting.)

36. Explain why the function $f(x) = \sin(1/x)$ has no continuous extension to $x = 0$.

T In Exercises 37–40, graph the function to see whether it appears to have a continuous extension to the given point a. If it does, use Trace and Zoom to find a good candidate for the extended function's value at a. If the function does not appear to have a continuous extension, can it be extended to be continuous from the right or left? If so, what do you think the extended function's value should be?

37. $f(x) = \dfrac{x - 1}{x - \sqrt[4]{x}}, \quad a = 1$

38. $g(\theta) = \dfrac{5\cos\theta}{4\theta - 2\pi}, \quad a = \pi/2$

39. $h(t) = (1 + |t|)^{1/t}, \quad a = 0$

40. $k(x) = \dfrac{x}{1 - 2^{|x|}}, \quad a = 0$

Limits at Infinity

Find the limits in Exercises 41–54.

41. $\lim\limits_{x\to \infty} \dfrac{2x + 3}{5x + 7}$

42. $\lim\limits_{x\to -\infty} \dfrac{2x^2 + 3}{5x^2 + 7}$

43. $\lim\limits_{x\to -\infty} \dfrac{x^2 - 4x + 8}{3x^3}$

44. $\lim\limits_{x\to \infty} \dfrac{1}{x^2 - 7x + 1}$

45. $\lim\limits_{x\to -\infty} \dfrac{x^2 - 7x}{x + 1}$

46. $\lim\limits_{x\to \infty} \dfrac{x^4 + x^3}{12x^3 + 128}$

47. $\lim\limits_{x\to \infty} \dfrac{\sin x}{\lfloor x \rfloor}$ (If you have a grapher, try graphing the function for $-5 \le x \le 5$.)

48. $\lim\limits_{\theta\to \infty} \dfrac{\cos\theta - 1}{\theta}$ (If you have a grapher, try graphing $f(x) = x(\cos(1/x) - 1)$ near the origin to "see" the limit at infinity.)

49. $\lim\limits_{x\to \infty} \dfrac{x + \sin x + 2\sqrt{x}}{x + \sin x}$

50. $\lim\limits_{x\to \infty} \dfrac{x^{2/3} + x^{-1}}{x^{2/3} + \cos^2 x}$

51. $\lim\limits_{x\to \infty} e^{1/x} \cos\dfrac{1}{x}$

52. $\lim\limits_{t\to \infty} \ln\left(1 + \dfrac{1}{t}\right)$

53. $\lim\limits_{x\to -\infty} \tan^{-1} x$

54. $\lim\limits_{t\to -\infty} e^{3t} \sin^{-1}\dfrac{1}{t}$

Horizontal and Vertical Asymptotes

55. Use limits to determine the equations for all vertical asymptotes.

a. $y = \dfrac{x^2 + 4}{x - 3}$

b. $f(x) = \dfrac{x^2 - x - 2}{x^2 - 2x + 1}$

c. $y = \dfrac{x^2 + x - 6}{x^2 + 2x - 8}$

56. Use limits to determine the equations for all horizontal asymptotes.

a. $y = \dfrac{1 - x^2}{x^2 + 1}$

b. $f(x) = \dfrac{\sqrt{x} + 4}{\sqrt{x + 4}}$

c. $g(x) = \dfrac{\sqrt{x^2 + 4}}{x}$

d. $y = \sqrt{\dfrac{x^2 + 9}{9x^2 + 1}}$

57. Determine the domain and range of $y = \dfrac{\sqrt{16 - x^2}}{x - 2}$.

58. Assume that constants a and b are positive. Find equations for all horizontal and vertical asymptotes for the graph of $y = \dfrac{\sqrt{ax^2 + 4}}{x - b}$.

CHAPTER 2 Additional and Advanced Exercises

T **1. Assigning a value to 0^0** The rules of exponents tell us that $a^0 = 1$ if a is any number different from zero. They also tell us that $0^n = 0$ if n is any positive number.

If we tried to extend these rules to include the case 0^0, we would get conflicting results. The first rule would say $0^0 = 1$, whereas the second would say $0^0 = 0$.

We are not dealing with a question of right or wrong here. Neither rule applies as it stands, so there is no contradiction. We could, in fact, define 0^0 to have any value we wanted as long as we could persuade others to agree.

What value would you like 0^0 to have? Here is an example that might help you to decide. (See Exercise 2 below for another example.)

a. Calculate x^x for $x = 0.1, 0.01, 0.001$, and so on as far as your calculator can go. Record the values you get. What pattern do you see?

b. Graph the function $y = x^x$ for $0 < x \le 1$. Even though the function is not defined for $x \le 0$, the graph will approach the y-axis from the right. Toward what y-value does it seem to be headed? Zoom in to further support your idea.

T **2. A reason you might want 0^0 to be something other than 0 or 1** As the number x increases through positive values, the numbers $1/x$ and $1/(\ln x)$ both approach zero. What happens to the number

$$f(x) = \left(\frac{1}{x}\right)^{1/(\ln x)}$$

as x increases? Here are two ways to find out.

a. Evaluate f for $x = 10, 100, 1000$, and so on as far as your calculator can reasonably go. What pattern do you see?

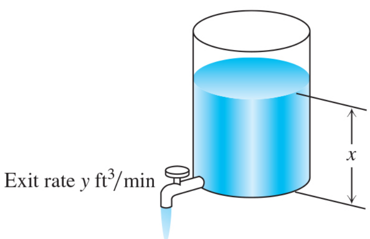
b. Graph f in a variety of graphing windows, including windows that contain the origin. What do you see? Trace the y-values along the graph. What do you find?

3. Lorentz contraction In relativity theory, the length of an object, say a rocket, appears to an observer to depend on the speed at which the object is traveling with respect to the observer. If the observer measures the rocket's length as L_0 at rest, then at speed v the length will appear to be

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}}.$$

This equation is the Lorentz contraction formula. Here, c is the speed of light in a vacuum, about 3×10^8 m/sec. What happens to L as v increases? Find $\lim_{v \to c^-} L$. Why was the left-hand limit needed?

4. Controlling the flow from a draining tank Torricelli's law says that if you drain a tank like the one in the figure shown, the rate y at which water runs out is a constant times the square root of the water's depth x. The constant depends on the size and shape of the exit valve.



Suppose that $y = \sqrt{x}/2$ for a certain tank. You are trying to maintain a fairly constant exit rate by adding water to the tank with a hose from time to time. How deep must you keep the water if you want to maintain the exit rate

a. within 0.2 ft^3/min of the rate $y_0 = 1$ ft^3/min?

b. within 0.1 ft^3/min of the rate $y_0 = 1$ ft^3/min?

5. Thermal expansion in precise equipment As you may know, most metals expand when heated and contract when cooled. The dimensions of a piece of laboratory equipment are sometimes so critical that the shop where the equipment is made must be held at the same temperature as the laboratory where the equipment is to be used. A typical aluminum bar that is 10 cm wide at 70°F will be

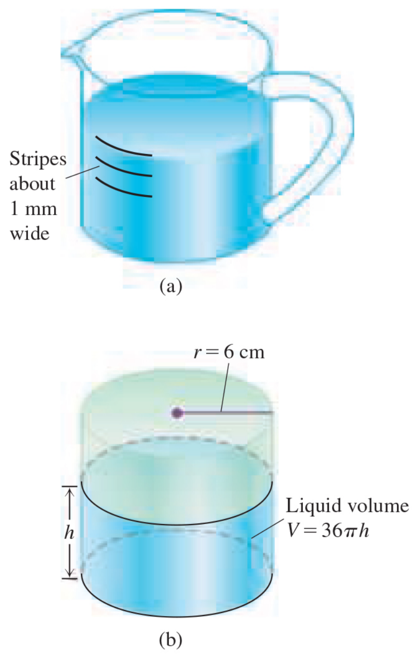
$$y = 10 + (t - 70) \times 10^{-4}$$

centimeters wide at a nearby temperature t. Suppose that you are using a bar like this in a gravity wave detector, where its width must stay within 0.0005 cm of the ideal 10 cm. How close to $t_0 = 70$°F must you maintain the temperature to ensure that this tolerance is not exceeded?

6. Stripes on a measuring cup The interior of a typical 1-L measuring cup is a right circular cylinder of radius 6 cm (see accompanying figure). The volume of water we put in the cup is therefore a function of the level h to which the cup is filled, the formula being

$$V = \pi 6^2 h = 36\pi h.$$

How closely must we measure h to measure out 1 L of water (1000 cm^3) with an error of no more than 1% (10 cm^3)?



A 1-L measuring cup (a), modeled as a right circular cylinder (b) of radius $r = 6$ cm

Precise Definition of Limit

In Exercises 7–10, use the formal definition of limit to prove that the function is continuous at c.

7. $f(x) = x^2 - 7, \quad c = 1$ **8.** $g(x) = 1/(2x), \quad c = 1/4$

9. $h(x) = \sqrt{2x - 3}, \quad c = 2$ **10.** $F(x) = \sqrt{9 - x}, \quad c = 5$

11. Uniqueness of limits Show that a function cannot have two different limits at the same point. That is, if $\lim_{x\to c} f(x) = L_1$ and $\lim_{x\to c} f(x) = L_2$, then $L_1 = L_2$.

12. Prove the limit Constant Multiple Rule:

$$\lim_{x\to c} kf(x) = k \lim_{x\to c} f(x) \quad \text{for any constant } k.$$

13. One-sided limits If $\lim_{x\to 0^+} f(x) = A$ and $\lim_{x\to 0^-} f(x) = B$, find

a. $\lim_{x\to 0^+} f(x^3 - x)$ **b.** $\lim_{x\to 0^-} f(x^3 - x)$

c. $\lim_{x\to 0^+} f(x^2 - x^4)$ **d.** $\lim_{x\to 0^-} f(x^2 - x^4)$

14. Limits and continuity Which of the following statements are true, and which are false? If true, say why; if false, give a counterexample (that is, an example confirming the falsehood).

a. If $\lim_{x\to c} f(x)$ exists but $\lim_{x\to c} g(x)$ does not exist, then $\lim_{x\to c}(f(x) + g(x))$ does not exist.

b. If neither $\lim_{x\to c} f(x)$ nor $\lim_{x\to c} g(x)$ exists, then $\lim_{x\to c} (f(x) + g(x))$ does not exist.

c. If f is continuous at x, then so is $|f|$.

d. If $|f|$ is continuous at c, then so is f.

In Exercises 15 and 16, use the formal definition of limit to prove that the function has a continuous extension to the given value of x.

15. $f(x) = \dfrac{x^2 - 1}{x + 1}, \quad x = -1$ **16.** $g(x) = \dfrac{x^2 - 2x - 3}{2x - 6}, \quad x = 3$

17. A function continuous at only one point Let

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

a. Show that f is continuous at $x = 0$.

b. Use the fact that every nonempty open interval of real numbers contains both rational and irrational numbers to show that f is not continuous at any nonzero value of x.

18. The Dirichlet ruler function If x is a rational number, then x can be written in a unique way as a quotient of integers m/n where $n > 0$ and m and n have no common factors greater than 1. (We say that such a fraction is in *lowest terms*. For example, $6/4$ written in lowest terms is $3/2$.) Let $f(x)$ be defined for all x in the interval $[0, 1]$ by

$$f(x) = \begin{cases} 1/n, & \text{if } x = m/n \text{ is a rational number in lowest terms} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

For instance, $f(0) = f(1) = 1$, $f(1/2) = 1/2$, $f(1/3) = f(2/3) = 1/3$, $f(1/4) = f(3/4) = 1/4$, and so on.

a. Show that f is discontinuous at every rational number in $[0, 1]$.

b. Show that f is continuous at every irrational number in $[0, 1]$. (*Hint*: If ε is a given positive number, show that there are only finitely many rational numbers r in $[0, 1]$ such that $f(r) \geq \varepsilon$.)

c. Sketch the graph of f. Why do you think f is called the "ruler function"?

19. Antipodal points Is there any reason to believe that there is always a pair of antipodal (diametrically opposite) points on Earth's equator where the temperatures are the same? Explain.

20. If $\lim_{x\to c} (f(x) + g(x)) = 3$ and $\lim_{x\to c} (f(x) - g(x)) = -1$, find $\lim_{x\to c} f(x)g(x)$.

21. Roots of a quadratic equation that is almost linear The equation $ax^2 + 2x - 1 = 0$, where a is a constant, has two roots if $a > -1$ and $a \neq 0$, one positive and one negative:

$$r_+(a) = \frac{-1 + \sqrt{1 + a}}{a}, \qquad r_-(a) = \frac{-1 - \sqrt{1 + a}}{a},$$

a. What happens to $r_+(a)$ as $a \to 0$? As $a \to -1^+$?

b. What happens to $r_-(a)$ as $a \to 0$? As $a \to -1^+$?

c. Support your conclusions by graphing $r_+(a)$ and $r_-(a)$ as functions of a. Describe what you see.

d. For added support, graph $f(x) = ax^2 + 2x - 1$ simultaneously for $a = 1, 0.5, 0.2, 0.1,$ and 0.05.

22. Root of an equation Show that the equation $x + 2\cos x = 0$ has at least one solution.

23. Bounded functions A real-valued function f is **bounded from above** on a set D if there exists a number N such that $f(x) \leq N$ for all x in D. We call N, when it exists, an **upper bound** for f on D and say that f is bounded from above by N. In a similar manner, we say that f is **bounded from below** on D if there exists a number M such that $f(x) \geq M$ for all x in D. We call M, when it exists, a **lower bound** for f on D and say that f is bounded from below by M. We say that f is **bounded** on D if it is bounded from both above and below.

a. Show that f is bounded on D if and only if there exists a number B such that $|f(x)| \leq B$ for all x in D.

b. Suppose that f is bounded from above by N. Show that if $\lim_{x\to c} f(x) = L$, then $L \leq N$.

c. Suppose that f is bounded from below by M. Show that if $\lim_{x\to c} f(x) = L$, then $L \geq M$.

24. Max $\{a, b\}$ and min $\{a, b\}$

a. Show that the expression

$$\max\{a, b\} = \frac{a + b}{2} + \frac{|a - b|}{2}$$

equals a if $a \geq b$ and equals b if $b \geq a$. In other words, $\max\{a, b\}$ gives the larger of the two numbers a and b.

b. Find a similar expression for $\min\{a, b\}$, the smaller of a and b.

Generalized Limits Involving $\dfrac{\sin\theta}{\theta}$

The formula $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$ can be generalized. If $\lim_{x\to c} f(x) = 0$ and $f(x)$ is never zero in an open interval containing the point $x = c$, except possibly at c itself, then

$$\lim_{x\to c} \frac{\sin f(x)}{f(x)} = 1.$$

Here are several examples.

a. $\displaystyle\lim_{x\to 0} \frac{\sin x^2}{x^2} = 1$

b. $\displaystyle\lim_{x\to 0} \frac{\sin x^2}{x} = \lim_{x\to 0} \frac{\sin x^2}{x^2} \lim_{x\to 0} \frac{x^2}{x} = 1 \cdot 0 = 0$

c. $\displaystyle\lim_{x\to-1}\frac{\sin(x^2-x-2)}{x+1}=\lim_{x\to-1}\frac{\sin(x^2-x-2)}{(x^2-x-2)}\cdot$

$\displaystyle\lim_{x\to-1}\frac{(x^2-x-2)}{x+1}=1\cdot\lim_{x\to-1}\frac{(x+1)(x-2)}{x+1}=-3$

d. $\displaystyle\lim_{x\to1}\frac{\sin\left(1-\sqrt{x}\right)}{x-1}=\lim_{x\to1}\frac{\sin\left(1-\sqrt{x}\right)}{1-\sqrt{x}}\frac{1-\sqrt{x}}{x-1}$

$\displaystyle=\lim_{x\to1}\frac{\left(1-\sqrt{x}\right)\left(1+\sqrt{x}\right)}{(x-1)\left(1+\sqrt{x}\right)}$

$\displaystyle=\lim_{x\to1}\frac{1-x}{(x-1)\left(1+\sqrt{x}\right)}=-\frac{1}{2}$

Find the limits in Exercises 25–30.

25. $\displaystyle\lim_{x\to0}\frac{\sin(1-\cos x)}{x}$

26. $\displaystyle\lim_{x\to0^+}\frac{\sin x}{\sin\sqrt{x}}$

27. $\displaystyle\lim_{x\to0}\frac{\sin(\sin x)}{x}$

28. $\displaystyle\lim_{x\to0}\frac{\sin(x^2+x)}{x}$

29. $\displaystyle\lim_{x\to2}\frac{\sin(x^2-4)}{x-2}$

30. $\displaystyle\lim_{x\to9}\frac{\sin\left(\sqrt{x}-3\right)}{x-9}$

Oblique Asymptotes

Find all possible oblique asymptotes in Exercises 31–34.

31. $y=\dfrac{2x^{3/2}+2x-3}{\sqrt{x}+1}$

32. $y=x+x\sin\dfrac{1}{x}$

33. $y=\sqrt{x^2+1}$

34. $y=\sqrt{x^2+2x}$

Showing an Equation Is Solvable

35. Assume that $1<a<b$ and $\dfrac{a}{x}+x=\dfrac{1}{x-b}$. Show that this equation is solvable.

More Limits

36. Find constants a and b so that each of the following limits is true.

a. $\displaystyle\lim_{x\to0}\frac{\sqrt{a+bx}-1}{x}=2$ **b.** $\displaystyle\lim_{x\to1}\frac{\tan(ax-a)+b-2}{x-1}=3$

37. Evaluate $\displaystyle\lim_{x\to1}\frac{x^{2/3}-1}{1-\sqrt{x}}$.

38. Evaluate $\displaystyle\lim_{x\to0}\frac{|3x+4|-|x|-4}{x}$.

Limits on Arbitrary Domains

The definition of the limit of a function at $x=c$ extends to functions whose domains near c are more complicated than intervals.

> ### General Definition of Limit
> Suppose every open interval containing c contains a point other than c in the domain of f. We say that $\lim_{x\to c}f(x)=L$ if for every number $\varepsilon>0$ there exists a corresponding number $\delta>0$ such that for all x in the domain of f, $|f(x)-L|<\varepsilon$ whenever $0<|x-c|<\delta$.

For the functions in Exercises 39–42,

a. Find the domain.

b. Show that at $c=0$ the domain has the property described above.

c. Evaluate $\lim_{x\to0}f(x)$.

39. The function f is defined as follows: $f(x)=x$ if $x=1/n$ where n is a positive integer, and $f(0)=1$.

40. The function f is defined as follows: $f(x)=1-x$ if $x=1/n$ where n is a positive integer, and $f(0)=1$.

41. $f(x)=\sqrt{x\sin(1/x)}$

42. $f(x)=\sqrt{\ln(\sin(1/x))}$

43. Let g be a function with domain the rational numbers, defined by

$$g(x)=\frac{2}{x-\sqrt{2}}\quad\text{for rational } x.$$

a. Sketch the graph of g as well as you can, keeping in mind that g is only defined at rational points.

b. Use the general definition of a limit to prove that $\lim_{x\to0}g(x)=-\sqrt{2}$.

c. Prove that g is continuous at the point $x=0$ by showing that the limit in part (b) equals $g(0)$.

d. Is g continuous at other points of its domain?

CHAPTER 2 Technology Application Projects

Mathematica/Maple Projects

Projects can be found within MyMathLab.

Take It to the Limit
Part I
Part II (Zero Raised to the Power Zero: What Does It Mean?)
Part III (One-Sided Limits)
Visualize and interpret the limit concept through graphical and numerical explorations.
Part IV (What a Difference a Power Makes)
See how sensitive limits can be with various powers of x.

Going to Infinity
Part I (Exploring Function Behavior as $x\to\infty$ or $x\to-\infty$)
This module provides four examples to explore the behavior of a function as $x\to\infty$ or $x\to-\infty$.
Part II (Rates of Growth)
Observe graphs that *appear* to be continuous, yet the function is not continuous. Several issues of continuity are explored to obtain results that you may find surprising.