PARTITIONING THE VARIANCE OF Y

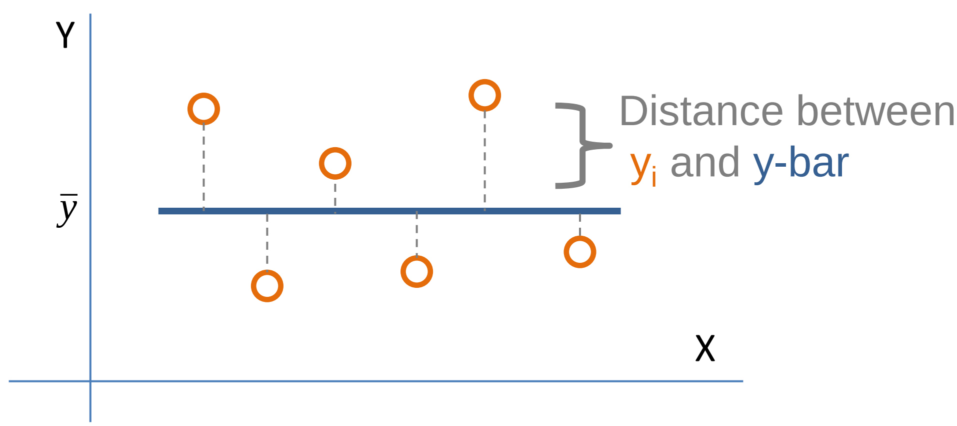
Fundamentals of

PROGRAM EVALUATION

JESSE LECY

THE VARIANCE CALCULATION

$$\mathrm{var}(y) = \frac{\sum \overbrace{(y_i - \bar{y})}^{\text{Distance between } y_i \text{ and y-bar}}{}^2}{n-1}$$



Variance: square the distances, add them up, divide by n-1

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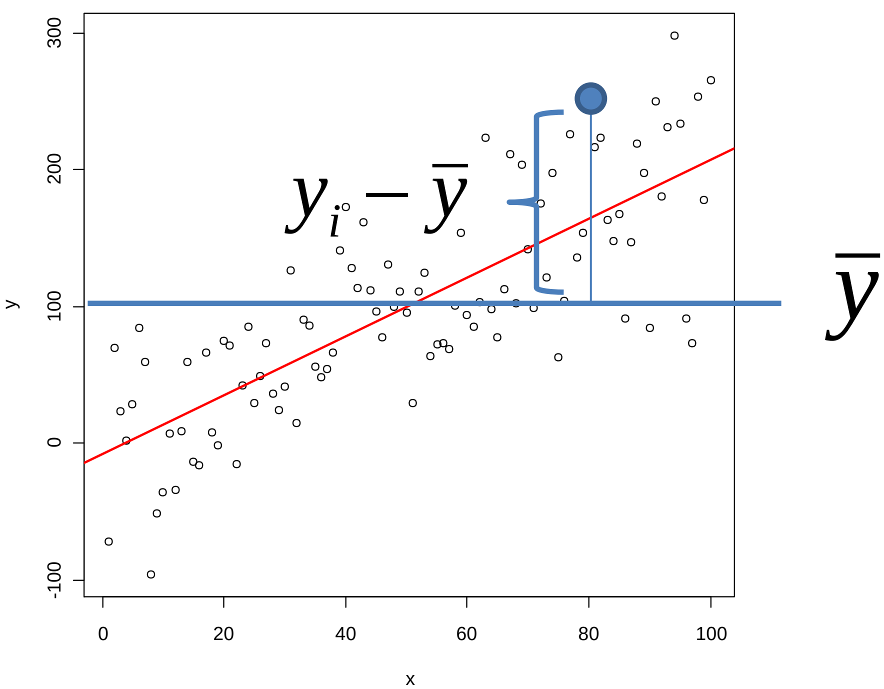

$$\mathrm{var}(y) = \frac{\sum (y_i - \bar{y})^2}{n-1}$$

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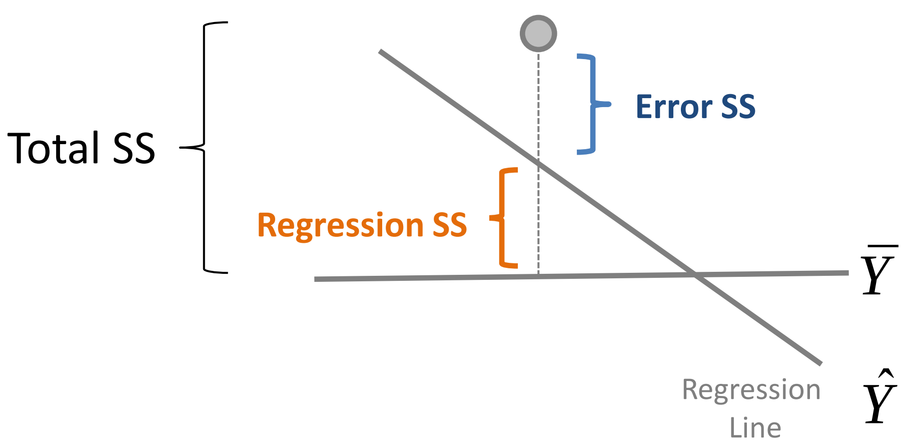
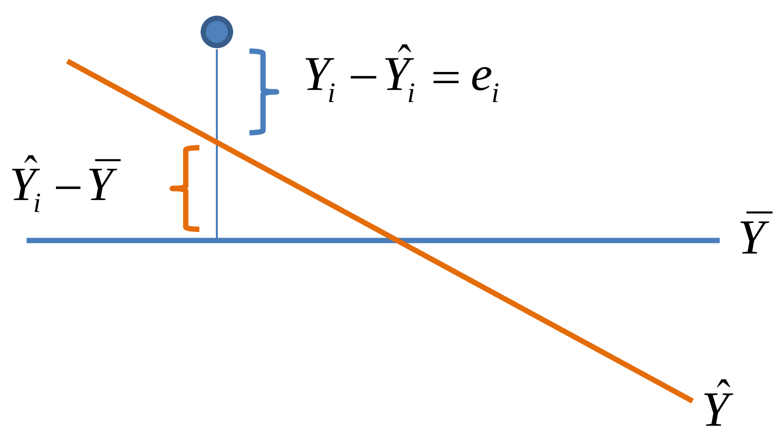
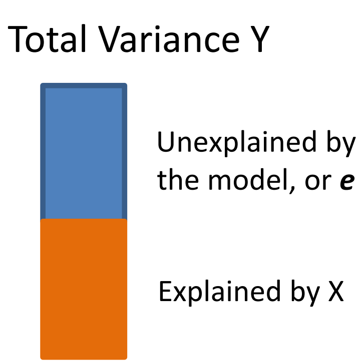


We want to split total variance into explained and unexplained portions.

PARTITIONING THE VARIANCE OF Y





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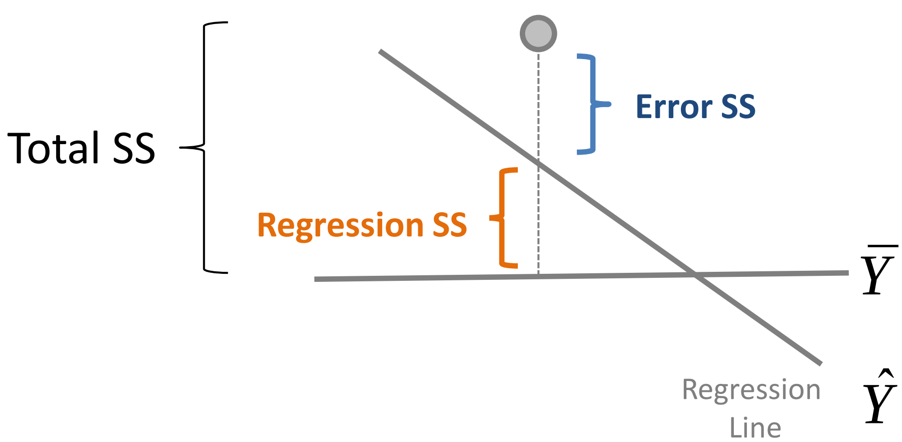


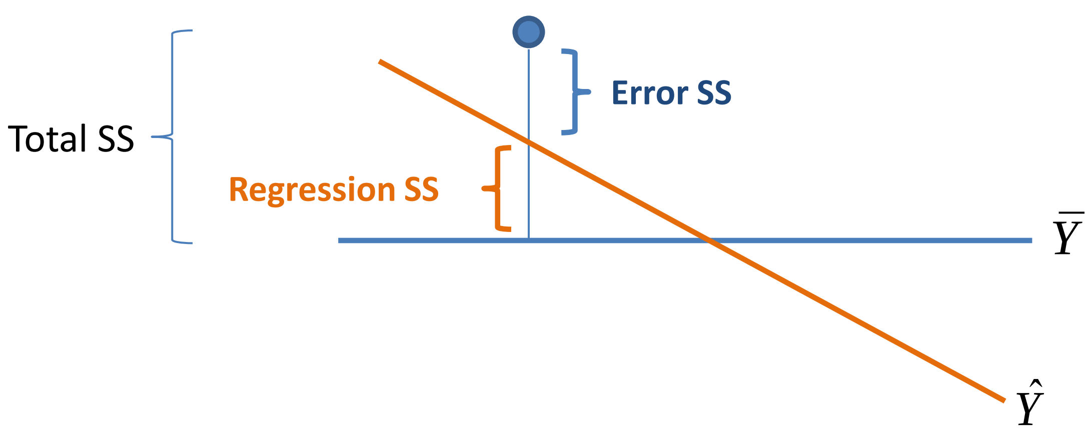
$$TotalSS = \sum (y_i - \bar{y})^2$$

$$RegressionSS = \sum (\hat{y}_i - \bar{y})^2$$

$$ErrorSS = \sum (y_i - \hat{y}_i)^2$$

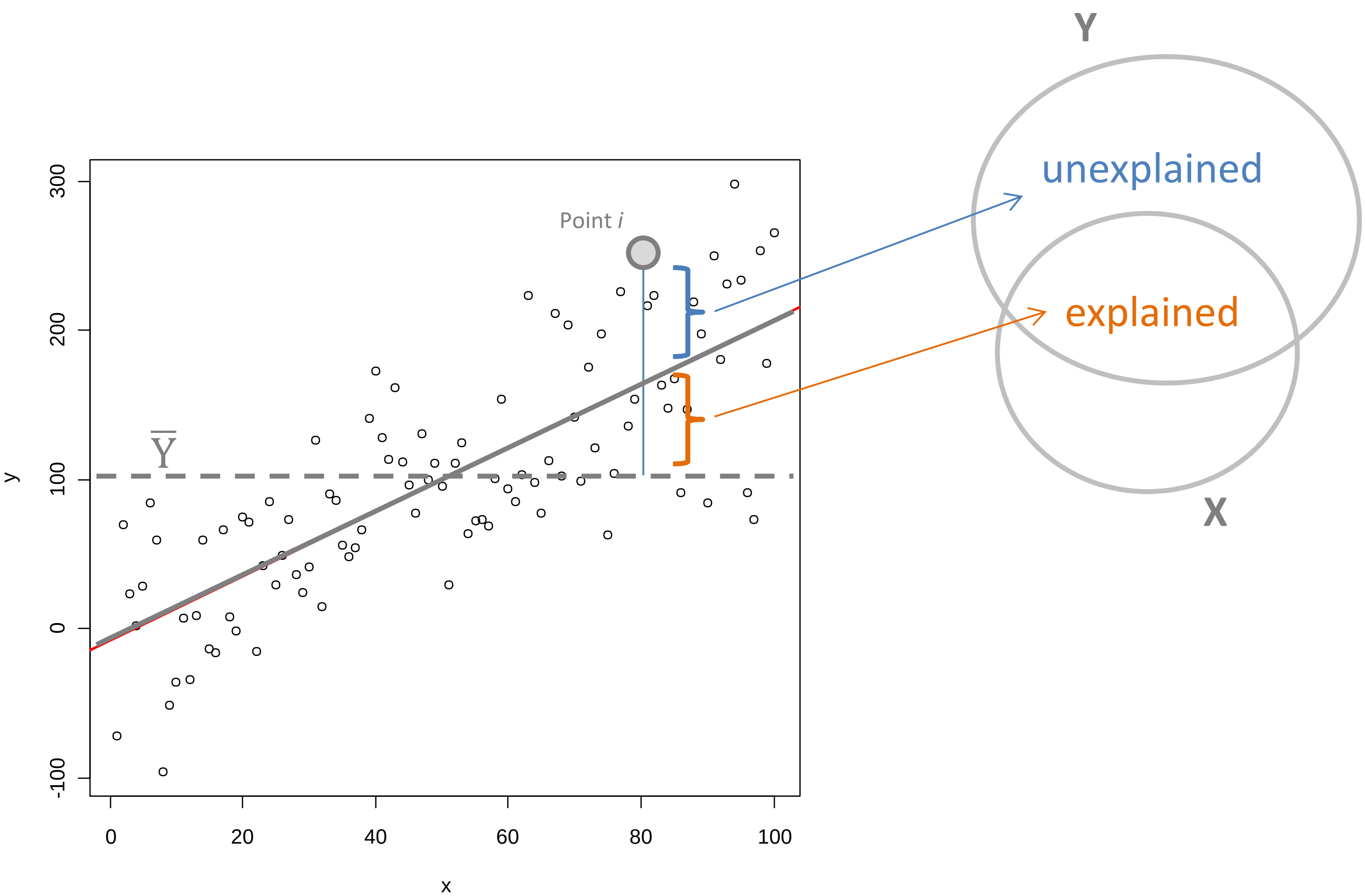
$$TSS = RSS + ESS$$

PARTITIONING THE VARIANCE OF Y



TSS = Regression/Explained SS + Error/Residual SS

$$R^2 = \frac{\hat{Y}_i - \bar{Y}}{Y_i - \bar{Y}} = \frac{Explained\ \ SS}{Total\ \ SS}$$



$$Unexplained: \quad Y_i - \hat{Y}_i = e_i$$

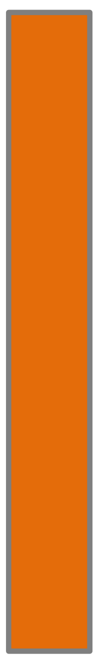
$$Explained: \quad \hat{Y}_i - \bar{Y}$$

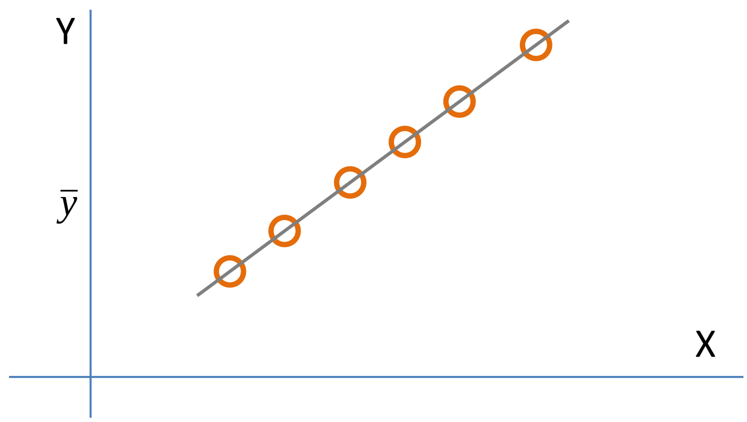
Two parts of the variance of Y

The Venn diagram is a simplified representation of the regression model. In our regression, the explained portion of the variance of outcome will always be the distance from the mean to the predicted value of Y (which always falls on the regression line), and the unexplained portion is the distance between the regression line and the actual data point, also called the residual or the error *e*.

PARTITIONING THE VARIANCE OF Y

Total Variance Y

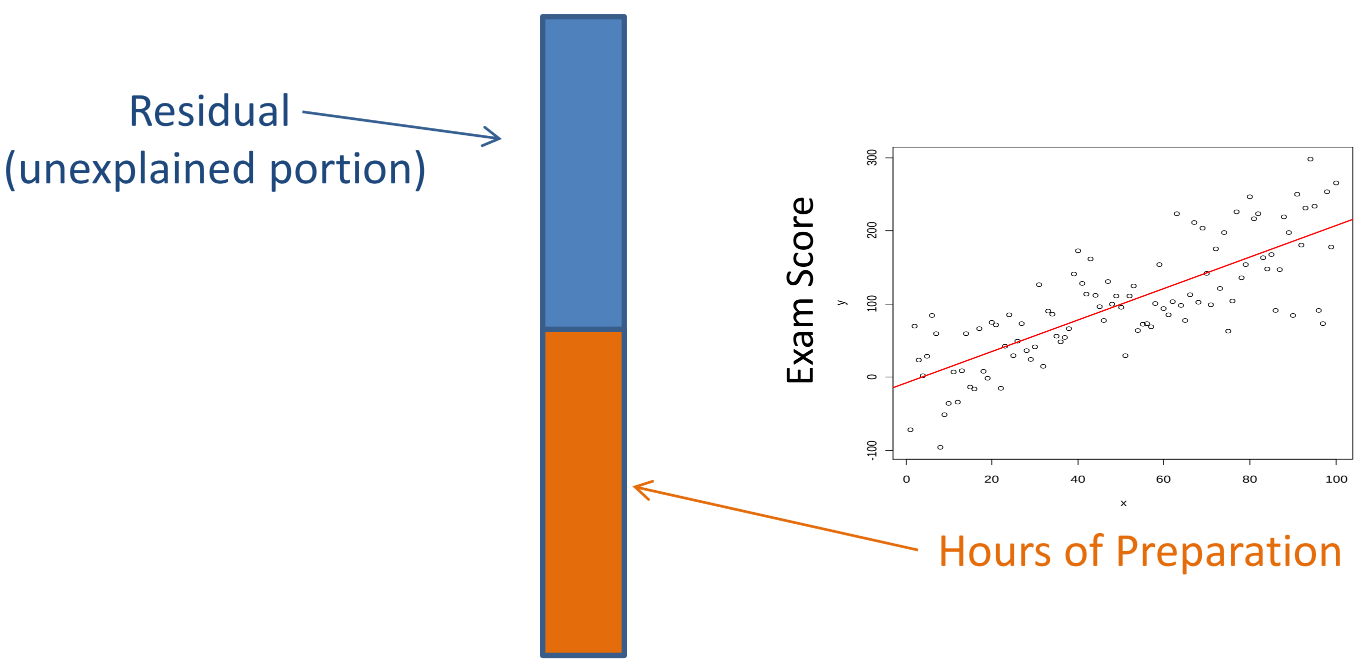




If all points lie on the regression line then we can explain everything about the variance with our model.

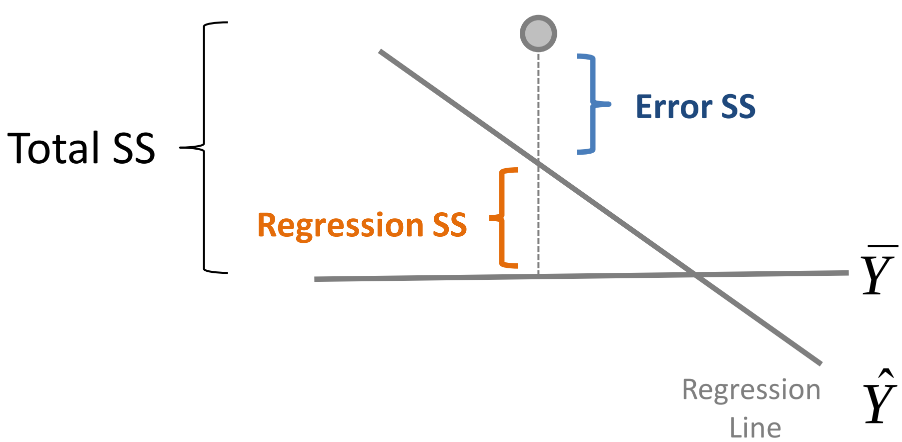
PARTITIONING THE VARIANCE OF Y

Final Exam Scores



The typical case is where the model explains some but not all of the variance.

PARTITIONING THE VARIANCE OF Y



Two parts of the variance of Y

Recall that the variance is just a sum of squared deviations from the mean divided by the sample size (minus a couple degrees of freedom). We sometimes just work with the sum of squares directly for the east of calculation.

We can split the total variance (TSS/n-1) into an explained and an error portion. These portions are then manipulated separately, and also used in important calculations like the R-square.

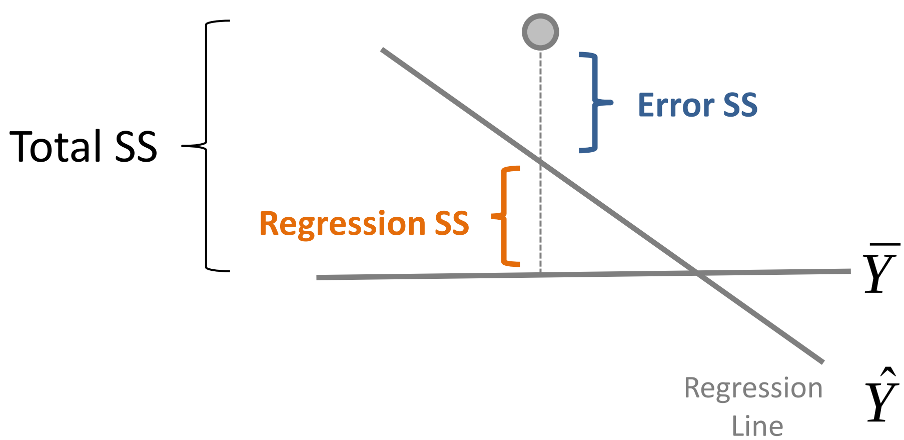
$$TotalSS = \sum (y_i - \bar{y})^2$$

$$RegressionSS = \sum (\hat{y}_i - \bar{y})^2$$

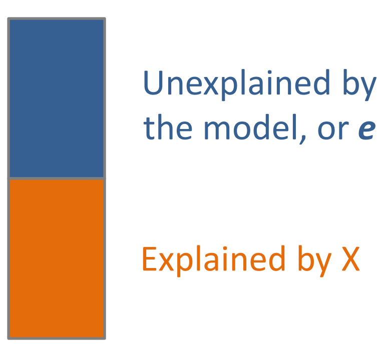
$$ErrorSS = \sum (y_i - \hat{y}_i)^2$$

$$TSS = RSS + ESS$$

PARTITIONING THE VARIANCE OF Y



Total Variance Y

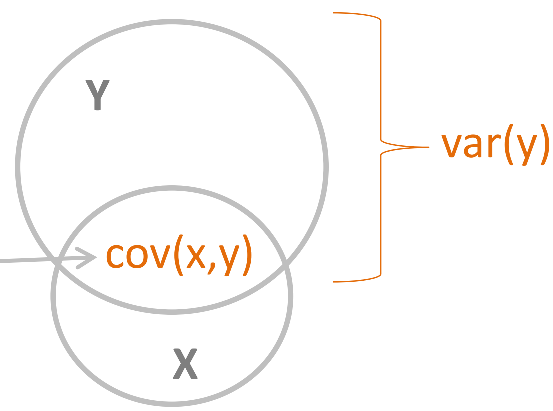


$$R^2 = \frac{Explained\ Variance\ of\ Y}{Total\ Variance\ of\ Y}$$

$$R^2 = \frac{RSS/n-1}{TSS/n-1}$$

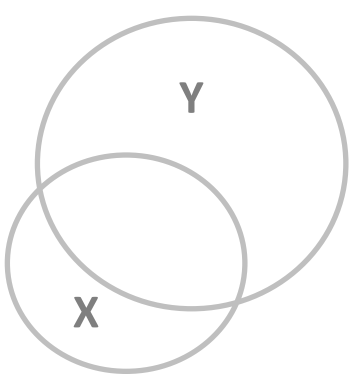
$$R^2 = \frac{RSS}{TSS} \quad = \quad 1 - \frac{ESS}{TSS}$$

Venn Diagram Version

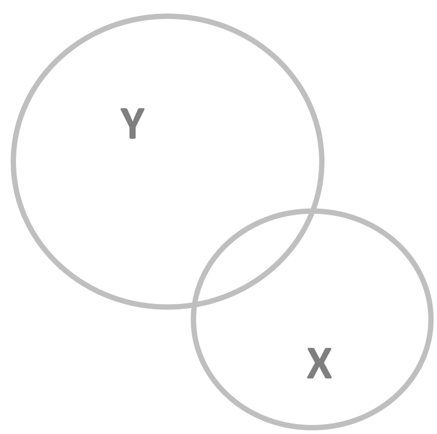


X "explains" this part of Y

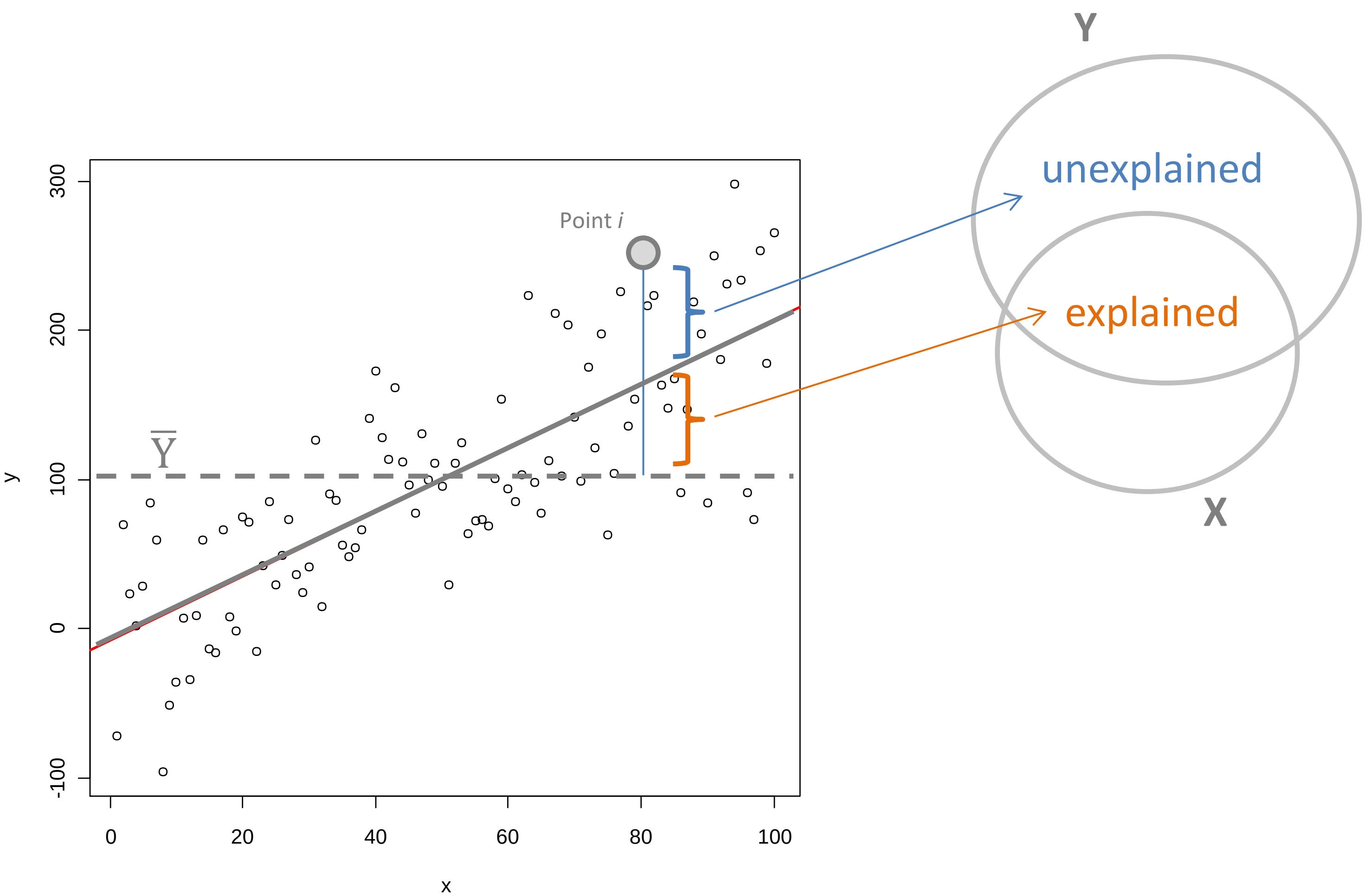
We can partition Y into the explained and unexplained portions. The portions that we can explain with the independent variable X will be the portion of Y that co-varies with X. We can often refer to the overlap region also as the correlation between X and Y. When two variables have more covariance, the correlation is stronger. Less covariance equates to weaker correlation.



more correlation



less correlation



$$Unexplained:\ \ Y_i - \hat{Y}_i = e_i$$

$$Explained:\ \ \hat{Y}_i - \bar{Y}$$

Two parts of the variance of Y

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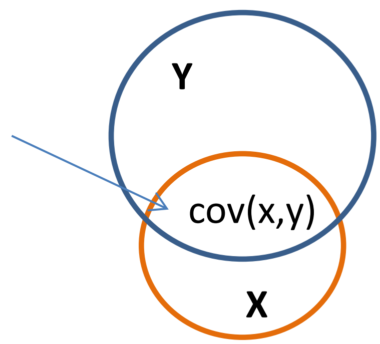
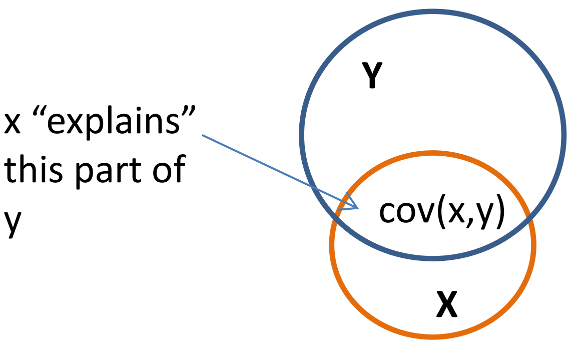
$$R^2 = \frac{RSS}{TSS}$$

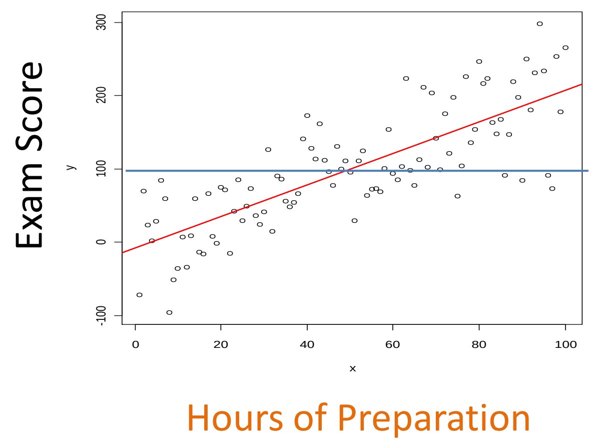
x “explains”
this part of y

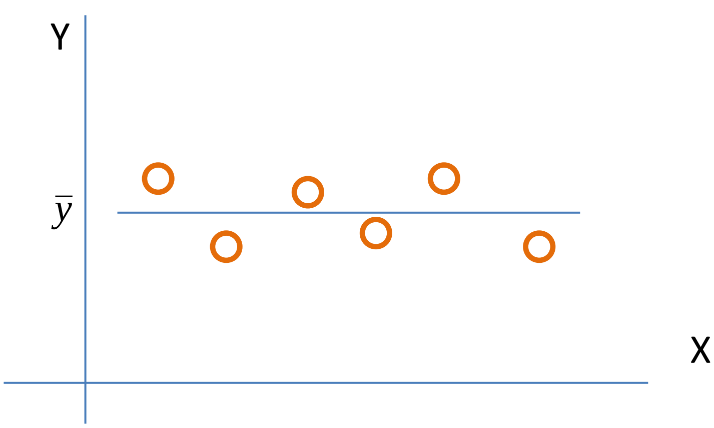


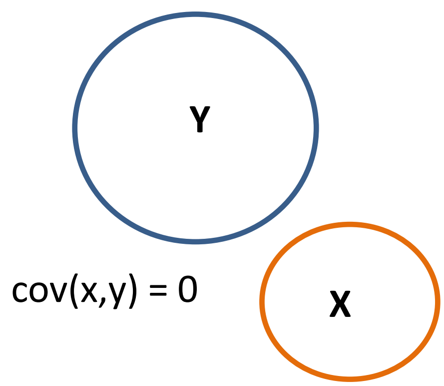
RSS (regression sum of squares)
TSS (total sum of squares)
ESS (error sum of squares)

* Note that sometimes RSS stands for “residual” SS and ESS sometimes for “explained” SS

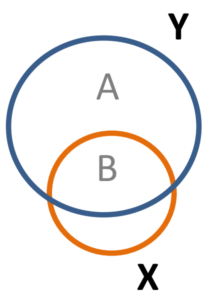

x "explains" this part of y
Y
cov(x,y)
X

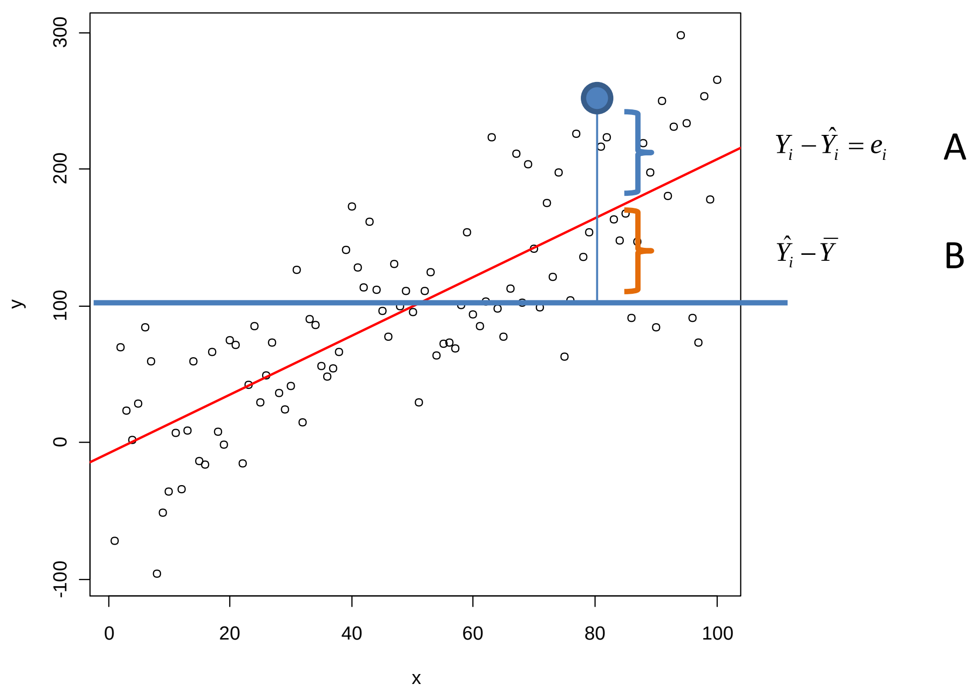

Exam Score
Hours of Preparation


Y
$\bar{y}$
X


Y
cov(x,y) = 0
X

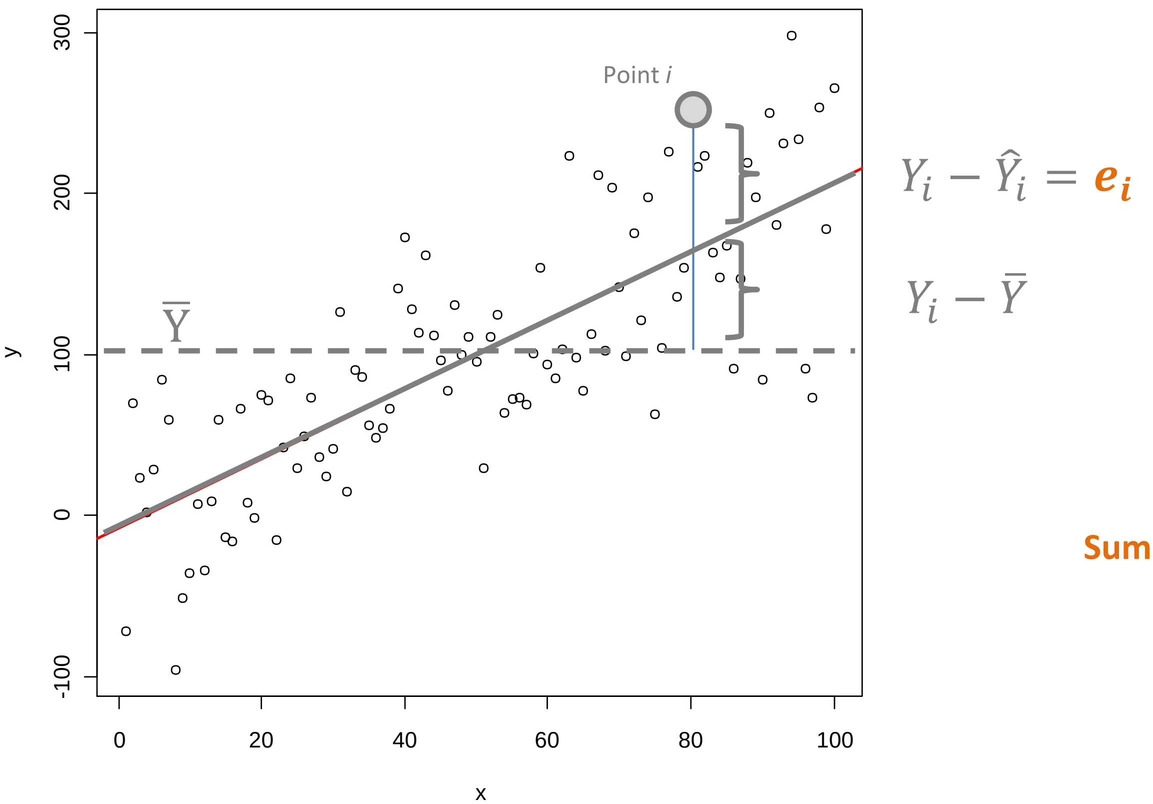
IN VENN DIAGRAM TERMS





STANDARD ERROR OF THE SLOPE

Standard Error in Regression



$$SSE = \sum e_i^2$$

Sum of Squared Error Terms

$$\hat{\sigma}_{\varepsilon}^2 = \frac{SSE}{n-2}$$

Variance of the residual

$$SE_{b_1} = \sqrt{\frac{\hat{\sigma}_{\varepsilon}^2}{\sum (x_i - \bar{x})^2}}$$

Standard error of the slope

The standard error of the slope is one of the most important concepts in regression because it determines the size of the confidence interval and thus the statistical significance of our study. We want standard errors to be as small as possible. We see here that the size of the standard error will be directly related to the amount of unexplained variance we have in our model, the residual *e*. The important thing to note is the unexplained portion shows up in the numerator of the standard error. As a result, the size of the standard error will be proportional to the amount of unexplained variance (plus a couple of other considerations to be covered later).

STANDARD ERROR IN REGRESSION

$$SE_{\bar{x}} = \frac{s_x}{\sqrt{n}}$$

The size of the standard error of the mean is driven by the variance of the variable, and the sample size.

$$\text{var}(x) = \frac{\sum (x_i - \bar{x})^2}{n-1} \quad \Rightarrow$$

$$\boxed{(n-1)\cdot \text{var}(x)} = \boxed{\sum (x_i - \bar{x})^2}$$

We can write the formula for the standard error of the slope in a couple of ways. I prefer the top because it is explicit about sample size and var(x).

$$SE_{b_1} = \frac{s_\varepsilon}{\sqrt{(n-1)\cdot \text{var}(x)}}$$

$$\Updownarrow$$

$$SE_{b_1} = \frac{s_\varepsilon}{\sqrt{\sum (x_i - \bar{x})^2}}$$

Similarly, the standard error of the slope is a function of the variance of the residual (the amount of unexplained variance in the outcome), the sample size (n-1), AND the variance of the explanatory variable.

$$SE_{b_1} = \sqrt{\frac{\hat{\sigma}_{\varepsilon}^{2}}{\sum (x_i - \bar{x})^2}}$$

Standard error of the Slope

Don't get too caught up with the math. The formula for the standard error of a regression coefficient is actually quite simple when you break it down. There are three moving parts – three things that can affect the size of the standard error. The portion of unexplained variance of the dependent variable (the residual), the sample size of the regression, and the amount of variance in the variable X associated with the regression slope.

$$\text{Standard Error of the Slope} \approx \frac{\text{residual}}{\text{sample size} \cdot \text{variance X}}$$

THE ROAD MAP

	Of the **Mean**:	Of the **Slope**:
Sampling Variance: ↓	$\sigma_x^2 = \frac{\sum (x_i - \bar{x})^2}{n-1}$ (for x)	$\sigma_\varepsilon^2 = \frac{SSE}{n-2} = \frac{\sum e_i^2}{n-2}$ (using the residual)
Standard Deviation: ↓	$\sigma_x = \sqrt{\sigma_x^2}$	$\sigma_\varepsilon = \sqrt{\sigma_\varepsilon^2}$
Standard Error: ↓	$SE_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}}$	$SE_{b_1} = \sqrt{\frac{\sigma_\varepsilon^2}{\sum (x_i - \bar{x})^2}}$
Confidence Interval	$\mu = \bar{x} \pm t \cdot SE_{\bar{x}}$ (of the mean)	$\beta_1 = b_1 \pm t \cdot SE_{b_1}$ (of the slope)

What should be clear in my mind?

1. **We split the variance of Y into explained and unexplained portions with a trick, inserting the regression line y-hat.**

2. **The standard error of the slope is derived from the unexplained portion of Y, the residual.**