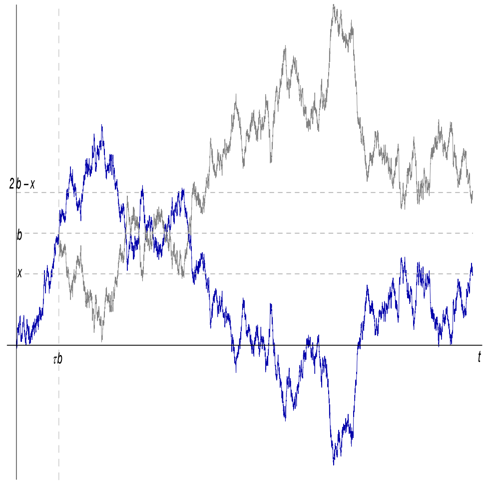

The reflection principle, the optimal stopping theorem, and applications to semi-static hedging

- Let $(W_t)_{t\geq 0}$ denote a standard Brownian motion and let $M_t = \max_{0\leq s\leq t} W_s$ denote the maximum process of the Brownian motion at time t (see Excel sheet on website for a Monte Carlo simulation of W and M). Let $\tau_b = \min(t : W_t = b)$ denote the **first hitting time** of W to b, and note that $W_{\tau_b} = b$. But if $\tau_b < t$ then $W_t - W_{\tau_b} = W_t - b$ is a new Brownian motion starting at zero which is independent of the history of W up to time τ_b; hence the conditional distribution of W_t at time τ_b is $W_t \sim N(b, t-\tau_b)$ and thus has a symmetric distribution. Thus by symmetry we have

$$\mathbb{P}(W_t < x \,|\, M_t > b) \quad = \quad \mathbb{P}(W_t > 2b - x \,|\, M_t > b) \tag{1}$$

  for $x \leq b$, because $2b - x = b + b - x$ is the same distance above b as x is below b, and the conditional distribution of W_T at τ_B is symmetric around the level b. Multiplying by $\mathbb{P}(M_t > b)$, we obtain

$$\mathbb{P}(W_t < x, M_t > b) \quad = \quad \mathbb{P}(M_t > b, W_t > 2b - x)\,. \tag{2}$$

  But if $W_t > 2b - x$, then W must have hit b before time t; thus we can re-write (2) as

$$\mathbb{P}(M_t > b, W_t > 2b - x) \quad = \quad \mathbb{P}(W_t > 2b - x) \quad = \quad \Phi^c(\frac{2b - x}{\sqrt{t}}) \tag{3}$$

  i.e. the event $\{M_t > b\}$ is unnecessary on the left hand side here.

- The result

$$\mathbb{P}(M_t > b, W_t < x) \quad = \quad \mathbb{P}(W_t > 2b - x)$$

  is known as the **reflection principle**, and is a very useful tool, both theoretically and in pricing **barrier options**, because it means we can expres the joint distribution of W_t and M_t in terms of the (simpler) known distribution of W_t. To get all the marks in the exam, you must mention that we are using symmetry AND the fact that the event $\{M_t > b\}$ is unnecessary if $W_t > 2b - x$.

Other important calculations using the reflection principle

- Setting $b = x$, and using (2) and (3), we obtain

$$\mathbb{P}(M_t > b, W_t < b) \;=\; \mathbb{P}(W_t > b) \;=\; \mathbb{P}(M_t > b, W_t > b)\,.$$

- But the events $\{M_t > b, W_t < b\}$ and $\{M_t > b, W_t > b\}$ are disjoint. Thus we can add both sides in this equation to obtain

$$\mathbb{P}(M_t > b) \quad = \quad 2\mathbb{P}(W_t > b) \quad = \quad 2\Phi^c(\frac{b}{\sqrt{t}})\,. \tag{4}$$

- Differentiating both sides with respect to b, we find that the density $p(b)$ of M_t is given by

$$p(b) = -\frac{d}{db}\mathbb{P}(M_t > b) = -\frac{d}{db}2\Phi^c(\frac{b}{\sqrt{t}}) = \frac{2}{\sqrt{t}}n(\frac{b}{\sqrt{t}}) = \frac{2}{\sqrt{2\pi t}}e^{-\frac{b^2}{2t}}$$

for $b \geq 0$, using that $(\Phi^c)'(z) = \frac{d}{dz}(1-\Phi(z)) = -\frac{1}{\sqrt{2\pi}}e^{-z^2/2} = -n(z)$. Thus M_t has what is known as a **one-sided Normal distribution**. The density is only defined for $b \geq 0$ because $M_t \geq 0$.

- Note that

$$\{M_t > b, W_t < x\} \cup \{M_t \leq b, W_t < x\} = \{W_t < x\}$$

But the left hand side is the union of two disjoint events, so we have

$$\mathbb{P}(M_t > b, W_t < x) + \mathbb{P}(M_t \leq b, W_t < x) = \mathbb{P}(W_t < x),$$

which we can re-arrange to obtain the joint distribution function of W_t, M_t as

$$\mathbb{P}(M_t \leq b, W_t < x) = \mathbb{P}(W_t < x) - \mathbb{P}(M_t > b, W_t < x). \tag{5}$$

and we know that $\mathbb{P}(M_t > b, W_t < x) = \Phi^c(\frac{2b-x}{\sqrt{t}})$ from the calculations above.

- The **joint density** of (W_t, M_t) is now obtained by differentiating the right hand side of (5) with respect to x and then with respect to b and again using that $(\Phi^c)'(z) = (1-\Phi)'(z) = -\frac{1}{\sqrt{2\pi}}e^{-z^2/2}$ and using the chain rule:

$$\begin{aligned} f(x,b;t) &= \frac{d}{dx}[\frac{d}{db}(\mathbb{P}(W_t < x) - \Phi^c(\frac{2b-x}{\sqrt{t}}))] \\ &= \frac{d}{dx}[\frac{2}{\sqrt{t}}n(\frac{2b-x}{\sqrt{t}})] \\ &= \frac{2}{\sqrt{t}} \cdot -\frac{1}{\sqrt{t}}n'(\frac{2b-x}{\sqrt{t}}) \\ &= \frac{2}{t} \cdot \frac{2b-x}{\sqrt{t}} n(\frac{2b-x}{\sqrt{t}}) \\ &= \frac{2(2b-x)}{\sqrt{2\pi t^3}}e^{-\frac{(2b-x)^2}{2t}} \end{aligned}$$

where $n(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$ is the standard Normal density, and we have used that $n'(z) = -\frac{z}{\sqrt{2\pi}}e^{-z^2/2} = -zn(z)$ in the fourth line.

- We also note that the event $\{M_t \geq b\}$ and $\{\tau_b \leq t\}$ are the same, and thus so are their probabilities:

$$\mathbb{P}(M_t \geq b) = \mathbb{P}(\tau_b \leq t). \tag{6}$$

because (common sense!) M_t can only exceed b if and only if the first hitting time τ_b to b is less than or equal to t.

- But we know that $\mathbb{P}(M_t \geq b) = 2\Phi^c(\frac{b}{\sqrt{t}})$. We can therefore compute the density of τ_b by differentiating the right hand side of (6) with respect to t and again using the chain rule

$$f_{\tau_b}(t) = 2\frac{d}{dt}\Phi^c(\frac{b}{\sqrt{t}}) = 2 \cdot \frac{1}{2}bt^{-\frac{3}{2}}n(\frac{b}{\sqrt{t}}) = \frac{b}{\sqrt{2\pi t^3}}e^{-\frac{b^2}{2t}}.$$

- Note that

$$\lim_{t\to\infty}\mathbb{P}(\tau_b < t) = \mathbb{P}(\tau_b < \infty) = \lim_{t\to\infty}\mathbb{P}(M_t > b) = \lim_{t\to\infty}2\Phi^c(\frac{b}{\sqrt{t}}) = 1 \tag{7}$$

which means that $\mathbb{P}(\tau_b = \infty) = 0$ and $\int_0^\infty f_{\tau_b}(t)dt = 1$.

- However, it turns out that $\mathbb{E}(\tau_b) = \int_0^\infty t f_{\tau_b}(t)dt = \infty$. To see this heuristically, note that $f_{\tau_b}(t) \sim \frac{b}{\sqrt{2\pi t^3}}$ as $t \to \infty$, but

$$\int_0^\infty t\frac{b}{\sqrt{2\pi t^3}}dt = \infty. \tag{8}$$

To make this rigorous, we just note that any $\delta > 0$, $e^{-\frac{b^2}{2t}} > 1-\delta$ for t sufficiently large, and use this as a lower bound for $f_{\tau_b}(t)$. The distribution of τ_b is a special case of a stable distribution.

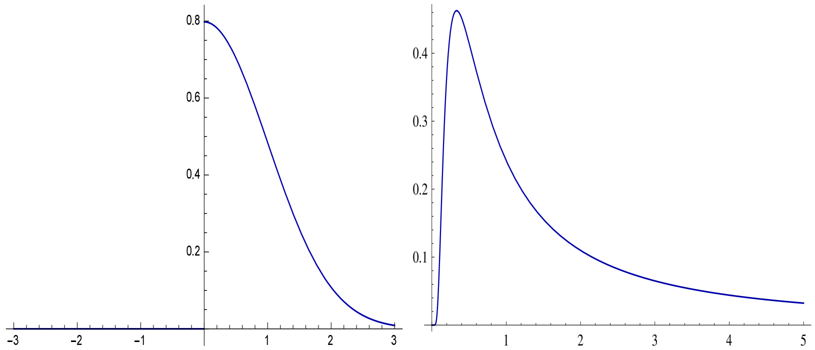

Figure 1: Here we have plotted the density of M_t for $t=1$, and the hitting time density $f_{\tau_b}(t)$ for $b=1$.

Figure 2: Here we have plotted the distribution function and the density of the last exit time g_t; this is the arcsin distribution.

2.2 The ArcSine law - the distribution of the last exit time of Brownian motion

- Let W_t be standard Brownian motion, $\bar{W}_t = \max_{0\leq s\leq t} W_s$ denote its maximum process, and let $g_t = \sup\{s \leq t : W_s = 0\}$ denote the last time that W hits zero before time t. g_t is known at the **last exit time**. Note that g_t is *not* a stopping time.

- Conditioning on the value of W_s, and using that $\mathbb{P}(\bar{W}_t > b) = 2\Phi^c(\frac{b}{\sqrt{t}})$ from the reflection principle, we see that

$$\mathbb{P}(g_t > s) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} \cdot \mathbb{P}(g_t > s | W_s = x)\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} \cdot 2\Phi^c(\frac{|x|}{\sqrt{t-s}}) dx\,. \tag{9}$$

  $\frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}}$ is just the density of W_s, and $2\Phi^c(\frac{|x|}{\sqrt{t-s}}) = \mathbb{P}(g_t > s | W_s = x)$ is the probability that W goes back to zero by time t, given that $W_s = x$ (this comes from the result that $\mathbb{P}(M_t > b) = 2\mathbb{P}(W_t > x)$ from the reflection principle). Note that the time remaining is $t - s$ not t and whether $x > 0$ or $x < 0$, the distance back to zero is $|x|$.

- Evaluating (9) (proof not required), we find that g_t has distribution function

$$\mathbb{P}(g_t \leq s) = \frac{2}{\pi} \arcsin \sqrt{\frac{s}{t}}$$

  for $s \in [0, t]$. This is known as the **arcsine law**, and differentiating we see that g_t has density

$$q(s) = \frac{1}{\pi \sqrt{t-s}\sqrt{s}}$$

  for $s \in [0, t]$, and we note that $q(0) = q(t) = +\infty$ (see plot of $q(s)$ below).

- It turns out that $\theta_t = \inf\{s : W_s = \bar{W}_t\}$, i.e. the first time s that W achieves maximum at time t also has density $q(s)$.

Application: semi-static hedging of barrier options

- Assume for simplicity that we have a stock price process which is just equal to standard Brownian motion

$$S_t = W_t\,.$$

- Now suppose we wish to hedge a **No Touch** option with barrier $B < 0$, which pays 1 if S does not hit B before time T.

- Recall that a digital call option of strike K is a contract which pays 1 at time T if $S_T > K$ and zero otherwise, similarly a digital put option of strike K is a contract which pays 1 at time T if $S_T \leq K$ and zero otherwise.

- Now consider a portfolio which consists of buying a digital call option of strike B and selling a digital put option of strike B. If we hit the barrier before T then (by **symmetry**) the digital call and the digital put are equal in value, so we can unwind (i.e. sell the digital call and buy back the digital put) this portfolio at zero cost.

- Conversely, if S does not hit B before time T then the digital put expires worthless, and the digital call pays 1.

- Thus this portfolio perfectly hedges the No Touch option whether the barrier is hit or not, **as long as we unwind the portfolio at the exact instant the barrier is hit, if the barrier is hit**. Thus the price of the No Touch is equal to the price of this replicating portfolio:

$$\begin{aligned}
\text{Price of No Touch} &= \mathbb{E}(1_{\tau_B > T}) = \mathbb{E}(1_{m_T > B}) \\
&= \mathbb{E}(1_{W_T > B}) - \mathbb{E}(1_{W_T < B}) \\
&= \mathbb{P}(W_T > B) - \mathbb{P}(W_T < B) \\
&= \Phi^c(\frac{B}{\sqrt{T}}) - \Phi(\frac{B}{\sqrt{T}})
\end{aligned}$$

  where $m_T = \min_{0 \leq s \leq t} W_s$, and we have used that m_T has the same distribution as the maximum $M_T = \max_{0 \leq s \leq t} W_s$ but reflected around the $x = 0$ (i.e. symmetry).

- Thus we have

$$\begin{aligned}
1 - \text{Price of No Touch} &= 1 - (\Phi^c(\frac{B}{\sqrt{T}}) - \Phi(\frac{B}{\sqrt{T}})) \\
&= 2\Phi(\frac{B}{\sqrt{T}}) \\
&= 2\mathbb{P}(W_t < B)\,.
\end{aligned}$$

  (note the similarity to (4)). But $1 -$ Price of No Touch is equal to the price of the **One Touch** contract which pays 1 dollar if we do hit B, and zero otherwise.

- This idea is known as **semi-static hedging**.

- This trick also works for the Black-Scholes model if $r = q = 0$, but in this case the replicating portfolio consists of buying a B strike digital call, selling a B strike digital put and buying $\frac{1}{B}$ European put options of strike B.