High frequency trading and optimal trade execution notes

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ throughout this final chapter, with a filtration $(\mathcal{F}_t)_{t\geq 0}$ which satisfies the usual conditions. We consider a generalization of the classical **Almgren-Chriss** price impact model where the price we pay for a stock is given by

$$\tilde{S}_t = S_t + f(\phi_t)$$

for some general stock price process S_t, where

$$X_t = X_0 + \int_0^t \phi_u du$$

is the number of shares held at time t which we assume is differentiable in t (note this is not the case for e.g. the usual Black-Scholes Δ-hedging strategy), and $k > 0$ which measures the illiquidity of the stock. At this point, the only condition we impose on the stock is that $\mathbb{E}(S_t^2) < \infty$ for all $t \in [0,T]$. To begin with, we will assume **linear price impact** i.e. $f(x) = kx$.

Note that the filtration $\mathcal{F}_t$ may be strictly larger or smaller than $\mathcal{F}_t^S$, which allows us to incorporate **inside information** held by informed traders, or conversely traders with **imperfect information/latency** who do not know the whole history of S up to time t at time t, if we wish. The $k\phi_t$ term corresponds to the temporary or **transient linear price impact**, which effectively penalizes a trader for trading too fast, but goes away as soon as he stops trading.

Unconstrained problems

We assume $X_0 = 0$ to begin with, and we say that a trading strategy $\phi \in \mathcal{A}$ (the set of admissible trading strategies) if ϕ is $\mathcal{F}_t$-adapted and $\mathbb{E}(\int_0^T \phi_u^2 du) < \infty$ a.s. Then the investor's total wealth at the final time T is given by

$$\int_0^T X_t dS_t - \int_0^T k\phi_t dX_t = S_T X_T - \int_0^T S_t\phi_t dt - \int_0^T k\phi_t^2 dt\,. \tag{1}$$

Suppose the investor wishes to maximize expected terminal wealth $J(\phi)$ over all admissible trading strategies ϕ, i.e.

$$\begin{aligned} J(\phi) &= \mathbb{E}(S_T X_T - \int_0^T S_t\phi_t dt - \int_0^T k\phi_t^2 dt) = \mathbb{E}(\int_0^T (S_T - S_t - k\phi_t)\phi_t dt) \\ &= \int_0^T \mathbb{E}((S_T - S_t - k\phi_t)\phi_t) dt \\ &= \int_0^T \mathbb{E}((\mathbb{E}(S_T - S_t|\mathcal{F}_t) - k\phi_t)\phi_t) dt \\ &= \mathbb{E}(\int_0^T (\xi_t\phi_t - k\phi_t^2) dt) \end{aligned}$$

where $\xi_t := \mathbb{E}(S_T - S_t|\mathcal{F}_t)$.

- We can now maximize the integrand $\xi_t\phi_t - k\phi_t^2$ with respect to ϕ_t to obtain the optimal trading speed explicitly as

  $$\phi_t^* = \frac{1}{2k}\mathbb{E}(S_T - S_t|\mathcal{F}_t)$$

  and note that $\phi_T^* = 0$, i.e. the trading speed at expiration is zero.

- If $(S_t)_{t\geq 0}$ is $\mathcal{F}_t$-adapted as is usually the case for previous problems we have looked at, then clearly $\mathbb{E}(S_t|\mathcal{F}_t) = S_t$ so ϕ_t^* simplifies to $\phi_t^* = \frac{1}{2k}(\mathbb{E}(S_T|\mathcal{F}_t) - S_t)$. Thus we increase our position when the expected final stock price minus its current price is positive, and vice versa, with intensity inversely proportional to k.

- Of course $\mathbb{E}(S_T|\mathcal{F}_t) = S_t$ if S is a martingale wrt $\mathcal{F}_t$, but we have also seen how to calculate $\mathbb{E}(S_T|\mathcal{F}_t)$ if e.g. S is a Lévy process (since we looked at how to compte first and second moments of a Levy process and a Levy process has independent increments) or fractional Brownian motion (recall we computed $\mathbb{E}(W_t^H|\mathcal{F}_u)$).

- If S is an $\mathcal{F}_t$-martingale (which is often assumed for market microstructure problems with short time horizons) then $\mathbb{E}(S_T - S_t|\mathcal{F}_t) = 0$ but $\mathbb{E}(S_T - S_t|\mathcal{F}_{t+\delta}) = S_{t+\delta} - S_t$; $\delta > 0$ means we can look ε into the future (i.e. inside information), $\delta < 0$ means we only know where the stock was ε in the past (could be caused by latency effects, time it takes light to travel from e.g. London to Singapore exchange).

Optimal liquidation problems

If we now consider the more realistic/interesting problem where we impose the additional terminal constraint $X_T = 0$ and allow any $X_0 \in \mathbb{R}$ (the so-called optimal liquidation problem); Then it turns out that X now minimizes J if and only if

$$M_t + \phi_t = \frac{1}{2k}\mathbb{E}(S_T - S_t|\mathcal{F}_t) \tag{2}$$

for some square integrable martingale M, i.e. the same eq as before but with an additional martingale which we have to compute. If we now integrate from t to T, and then taken the conditional expectation with respect to $\mathcal{F}_t$ and used the tower property, we obtain

$$(T-t)M_t - X_t = \frac{1}{2k}\int_t^T \mathbb{E}(S_T - S_u|\mathcal{F}_t)du$$

Solving for M_t we find that

$$M_t = \frac{1}{T-t}(X_t + \frac{1}{2k}\int_t^T \mathbb{E}(S_T - S_u|\mathcal{F}_t)du)\,.$$

Substituting this expression into the first equation, we find that

$$\begin{aligned}
\phi_t^* &= M_t + \frac{1}{2k}\mathbb{E}(S_T - S_t|\mathcal{F}_t) \\
&= -\frac{X_t^*}{T-t} + \frac{1}{2k}\mathbb{E}(S_T - S_t|\mathcal{F}_t) - \frac{1}{T-t}\frac{1}{2k}\int_t^T \mathbb{E}(S_T - S_u|\mathcal{F}_t)du \\
&= -\frac{X_t^*}{T-t} + \frac{1}{T-t}\frac{1}{2k}\int_t^T \mathbb{E}(S_u - S_t|\mathcal{F}_t)du \\
&= -\frac{X_t^*}{T-t} + \frac{1}{T-t}\frac{1}{2k}\int_t^T \mathbb{E}(S_u - S_t|\mathcal{F}_t)du
\end{aligned}$$

(**solve explicitly**) and clearly $J(\phi^*) < \infty$ for this problem since this is also true for the unconstrained problem with $\lambda = 0$.

- The optimal trading speed is the usual **VWAP** drift term plus $\frac{1}{2k}$ times a (history dependent) correction term proportional to the average of expected change in the stock price at each time up to the terminal time. Note that if $\frac{1}{2k}\int_t^T(\mathbb{E}(S_u - S_t|\mathcal{F}_t))|\mathcal{F}_t) > X_t + \frac{\lambda}{2k}\int_t^T(T-u)\mathbb{E}(S_u|\mathcal{F}_t)dt$, it is optimal to buy as part of an overall sell program (and vice versa). In particular for $X_0 = 0$ (i.e. a **round trip**), it is still optimal to trade if S is not a martingale, although the risk aversion term does not make financial sense if X goes negative. If S is a martingale, ϕ_t^* simplifies to the VWAP solution:

$$\phi_t^* = -\frac{X_t^*}{T-t}$$

which is satisfied if $X_t^* = X_0(1-\frac{t}{T})$, i.e. we trade at constant speed so that $X_T^* = 0$.

- To see why Eq (2) is satisfied, assume (2) is satisfied; then : $\phi_t = M_t + \frac{1}{2k}\xi_t$ then $N_t + \xi_t - 2k\phi_t = 0$ for some other martingale $N_t = 2kM_t$. To see why this is a sufficient condition for optimality, consider an admissible *perturbation* of this strategy ϕ^1, which satisfies $\int_0^T \phi_t^1 dt = 0$ (this ensures that $\phi_t + \varepsilon\phi_t^1$ is still an admissible strategy for the optimal liquidation problem since $X_0 + \int_0^T(\phi_t + \varepsilon\phi_t^1)dt = 0$). Then

$$\begin{aligned}
J(\phi + \varepsilon\phi^1) &= \mathbb{E}(\int_0^T (\xi_t(\phi_t + \varepsilon\phi_t^1) - k(\phi_t + \varepsilon\phi_t^1)^2)dt) \\
&= J(\phi) + \varepsilon\mathbb{E}(\int_0^T \phi_t^1(\xi_t - 2k\phi_t)dt) - k\varepsilon^2\int_0^T(\phi^1)_t^2 dt\,.
\end{aligned}$$

Letting $\Phi_t^1 = \int_0^t \phi_u^1 du$, and using the Ito product rule, we see that

$$\Phi_T^1 N_T - \Phi_0^1 N_0 = 0 = \mathbb{E}(\int_0^T \Phi_t^1 dN_t) + \mathbb{E}(\int_0^T N_t\phi_t^1 dt) = \mathbb{E}(\int_0^T N_t\phi_t^1 dt) \tag{3}$$

Then if $N_t + \xi_t - 2k\phi_t = 0$ i.e. $\xi_t - 2k\phi_t = -N_t$, then we see that

$$\begin{aligned}
J(\phi + \varepsilon\phi^1) &= J(\phi) + \varepsilon\mathbb{E}(\int_0^T -\phi_t^1 N_t dt) - k\varepsilon^2\int_0^T(\phi^1)_t^2 dt \\
&= J(\phi) - k\varepsilon^2\int_0^T(\phi_t^1)^2 dt
\end{aligned}$$

where we have used (3) and that $-N_t$ is also a martingale. Thus we see that if (2) is satisfied, then any perturbation ϕ^1 of ϕ lowers our expected total profit/loss, so ϕ_t is optimal. I will give a proof of necessity next time.

- Now suppose S is a martingale and we wish to minimize $\mathbb{E}(\int_0^T (k\phi_t^2+\lambda X_t^2)dt)=\mathbb{E}(\int_0^T (k\dot{X}_t^2+\lambda X_t^2)dt)$ for $k,\lambda>0$ subject to $X_0=X$ and $X_T=0$. The λ term is known as a quadratic inventory penalty for holding too much stock. In this case, the optimal ϕ_t cannot be computed using the method above, but is deterministic because S is a martingale, and thus X_t^* satisfies the **Euler-Lagrange equation** from calculus of variations:

$$\frac{d}{dt}(2k\dot{X}_t) \quad = \quad 2\lambda X_t\,.$$

  We can also obtain this equation using a perturbation argument. This is just a 2nd order ODE, whose solution is the classical Almgren-Chriss solution

$$X_t^* \quad = \quad X_0\frac{\sinh\kappa(T-t)}{\sinh\kappa T}$$

  where $\kappa=\sqrt{\frac{\lambda}{k}}$.

- Example: if $dS_t=-\alpha(S_t-\bar{S})dt+\sigma dW_t$ is an **Ornstein-Uhlenbeck** process, then $\mathbb{E}(S_u|S_t)=e^{-\alpha(u-t)}S_t+(1-e^{-\alpha(u-t)})\bar{S}=(1-e^{-\alpha(u-t)})(S_t-\bar{S})$, i.e. a weighted average of S_t and $\bar{S}_\infty$ so we can easily compute $\xi_t=\mathbb{E}(S_u-S_t|\mathcal{F}_t^S)$, and

$$\phi_t^* \quad = \quad -\frac{X_t^*}{T-t}\;-\;\frac{S_t-\bar{S}}{T-t}\frac{1}{2k}\int_t^T(1-e^{-\alpha(u-t)})dt\,.$$

  This is simulated in the Excel sheet, and we sometimes find that X_t^* crosses zero before time T, which does not happen when $\alpha=0$ because for $\alpha=0$, S reduces to standard Brownian motion which is a martingale and thus $\xi_t=\mathbb{E}(S_T-S_t|\mathcal{F}_t^S)=0$.

Two-player games

Now consider two agents, and let the trading speed for agent 1 be ϕ_t with $X_t=X_0+\int_0^t\phi_u du$, and the trading speed for agent 2 be ψ_t with $Y_t=Y_0+\int_0^t\psi_u du$, and assume that

$$\tilde{S}_t \quad = \quad S_t\,+\,k(\phi_t+\psi_t)\,.$$

Then the expected P&L's for agents 1 and 2 are

$$\int_0^T(\xi_t\phi_t-k(\phi_t+\psi_t)\phi_t)dt \quad , \quad \int_0^T(\xi_t\psi_t-k(\psi_t+\phi_t)\psi_t)dt$$

respectively, and using the same variational arguments as above, we see that a (Nash) equilbrium exists if

$$\begin{aligned}\xi_t-k(2\phi_t+\psi_t) &= M_t\\ \xi_t-k(2\psi_t+\phi_t) &= N_t\end{aligned}$$

for two square integrable $\mathcal{F}_t^W$-martingales M and N, subject to $X_t=Y_t=0$. Setting $t=u$ and integrating from $u=t$ to T as before, we find that

$$\int_t^T\xi_u du\;-\;k\int_t^T(2\phi_u+\psi_u) \quad = \quad \int_t^T M_u du\,.$$

Then taking conditional expectations at time t this leads to

$$\int_t^T\mathbb{E}_t(S_T-S_u)du\;+\;k(2X_t+Y_t) \quad = \quad (T-t)M_t$$

so we have a pair of coupled (random) ODEs for (ϕ_t,ψ_t):

$$\begin{aligned}\xi_t-k(2\phi_t+\psi_t) &= \frac{1}{T-t}\int_t^T\mathbb{E}_t(S_T-S_u)du\;+\;k(2X_t+Y_t))\\ \xi_t-k(2\psi_t+\phi_t) &= \frac{1}{T-t}\int_t^T\mathbb{E}_t(S_T-S_u)du\;+\;k(2Y_t+X_t))\end{aligned}$$

and for a known process of the form e.g. $P_t=\int_0^t g(t-s)dW_s$, we can compute these expressions and simulate X_t and Y_t, and we see there is interaction between the two agents' trading rates. This problem becomes more interesting if e.g. agent 1 has inside information with filtration $\mathcal{F}_{t+\delta}$ as above, in which case each agent has a different filtration and hence a different ξ_t, so the first equilibrium equation changes to

$$\xi_t^X-k(2\phi_t+\psi_t) \quad = \quad \frac{1}{T-t}\int_t^T\mathbb{E}(S_T-S_u|\mathcal{F}_{t+\delta})du\;+\;k(2X_t+Y_t))$$

where now $\xi_t^X=\mathbb{E}(S_T-S_t|\mathcal{F}_{t+\delta})$, and the first agent can "front-run" the second agent.

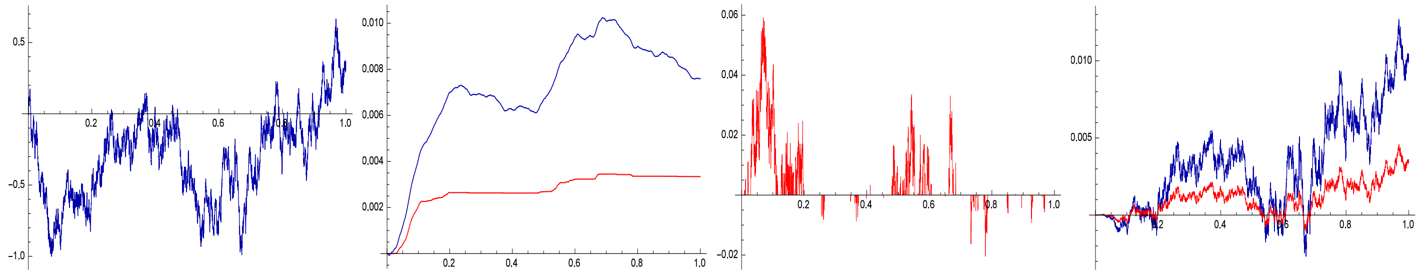

Figure 1: Unconstrained problem: Monte Carlo simulation of S_t, X_t^*, ϕ_t^* and total wealth when $S_t = Z_t$ is the Riemann-Liouville process, with $H = .4$, $k = 1$ and $T = 1$ with and without transaction costs (red and blue respectively) for $\varepsilon = .05$.

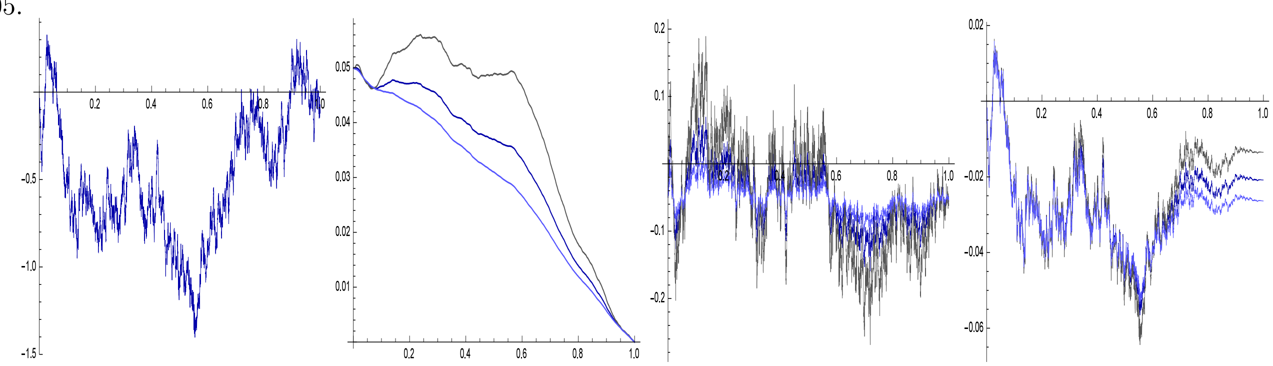

Figure 2: Constrained problem: Monte Carlo simulation of S_t, X_t^*, ϕ_t^* and total wealth with $X_0 = .05$ when $S_t = Z_t$ is the Riemann-Liouville process with $H = 0.4$, $k = 0.5, 1, 2$ and $T = 1$ (grey, dark blue and light blue resp.)

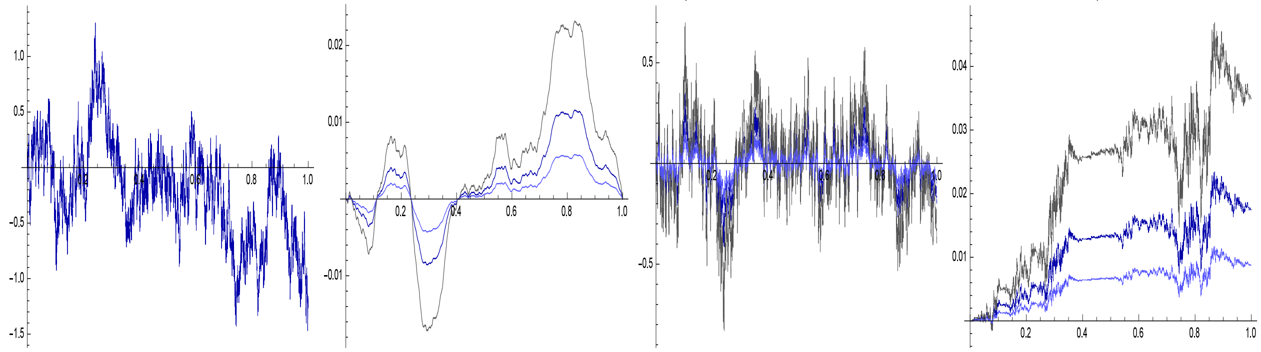

Figure 3: Constrained problem: Monte Carlo simulation of S_t, X_t^*, ϕ_t^* and total wealth with $X_0 = .0$ (i.e. a round trip) when $S_t = Z_t$ is the Riemann-Liouville process with $H = 0.4$ and $k = 0.5, 1, 2$ (grey, dark blue and light blue resp.).

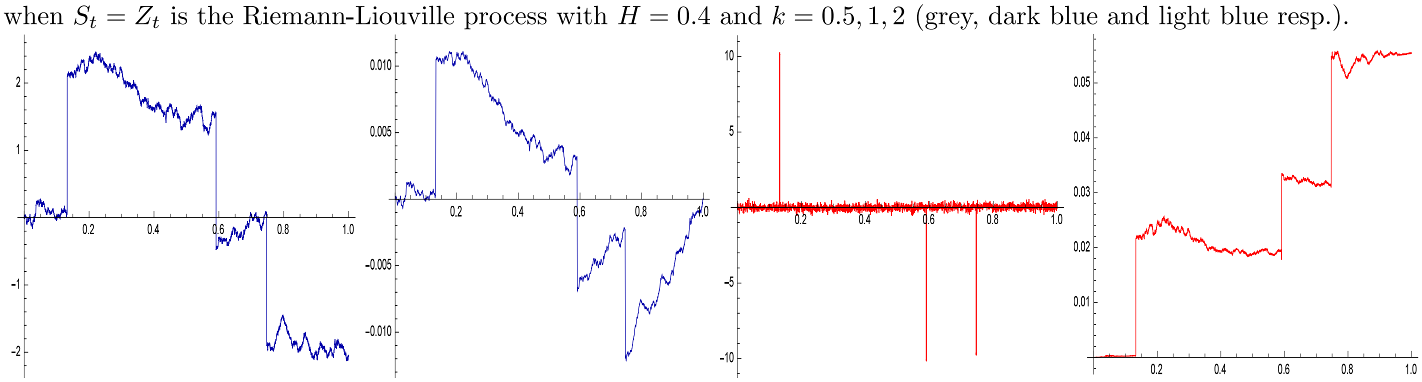

Figure 4: Unconstrained problem with inside information (with $\delta = .005$): Monte Carlo simulation of S_t, X_t, ϕ_t and total wealth when $S_t = W_t + J(N_t^+ - N_t^-)$ where W is Brownian motion and N^+ and N^- are two iid Poisson processes (independent of W) with intensity $\lambda = 2$, jump size $J = 3.5$, and $T = 1$, $k = 0.1$.