Basic Almgren-Chriss price impact problem

Let $(S_t)_{t\geq 0}$ denote a stock price process (which we assume is a Brownian motion for simplicity) and we make the (common) assumption that the amount of shares Q_t held at time t is **absolutely continuous**, so

$$\begin{aligned} dS_t &= \sigma dW_t \\ dQ_t &= v_t dt \end{aligned}$$

with Q_0 known, and we assume that the price paid per share at time t by an agent is $\tilde{S}_t = S_t + kv_t$ with $k > 0$ (one can think of S_t as the idealized **mid-price** and $\tilde{S}_t$ includes a penalty term kv_t so the agent pays more than S when they are buying, and receives less than S when selling. Then the **cash process** $(X_t)_{t\geq 0}$ of the agent evolves as

$$dX_t = -v_t(S_t + kv_t)dt\,.$$

The agent's Profit/Loss at time T is $X_T + Q_T S_T$, but we add an additional **quadratic penalty** term $-aQ_T^2$ to penalize non-liquidation, and a running quadratic penalty term $-\phi\int_0^T Q_s^2 ds$, with $a, \phi \geq 0$. Then we wish to maximize

$$H(S,q,x,t) = \mathbb{E}(X_T + Q_T S_T - aQ_T^2 - \phi\int_0^T Q_s^2 ds \,|\, S_t = S, Q_t = q, X_t = x)$$

over the space $\mathcal{A}$ of all adapted $\mathcal{F}_t^W$-adapted processes $(v_t)_{t\geq 0}$. Then by standard stochastic control arguments, V satisfies the (non-linear) **HJB equation**

$$H_t + \frac{1}{2}\sigma^2 H_{SS} + \sup_v (vH_q - (S+kv)H_x) - \phi q^2 = 0$$

which just comes from the infinitesimal generator for (S, Q, X) (note we are supping over all v terms here), subject to $H(S,q,x,T) = x + qS - aq^2$. From this we find that

$$v = \frac{-SH_x + H_q}{2kH_x}$$

which leads to a (messy) looking non-linear PDE for H. Substuting the quadratic ansatz $H(S,q,x,t) = x + qS + h(t,q)$, with $h(t,q) = q^2 h_2(t) + qh_1(t) + h_0(t)$, we find that h_0, h_1, h_2 satisfy a system of **coupled non-linear ODEs**:

$$\begin{aligned} \frac{h_1(t)^2}{4k} + h_0(t) &= 0 \\ \frac{h_1(t)h_2(t)}{k} + h_1'(t) &= 0 \\ -\phi + \frac{h_2(t)^2}{k} + h_2'(t) &= 0 \end{aligned}$$

subject to $h_2(T) = -a$, $h_0(T) = 0$ and $h_1(T) = 0$, and in terms of these quantities

$$v = \frac{h_1(t) + 2qh_2(t)}{2k}\,. \tag{1}$$

Recall that $v = \frac{dQ}{dt}$, so this is actually an **ODE** for $Q(t)$, so the optimal trading strategy here is **deterministic** in this case (not for the more advanced problem in the project where S has a **stochastic drift** driven by an **OU process**). If we set $\phi = 0$, the final ODE reduces to

$$\frac{h_2(t)^2}{k} + h_2'(t) = 0$$

(the same Eq appears in Ryan's AlgoTradingNotes4.pdf), with solution

$$h_2(t) = -\frac{k}{k/a + (T-t)}$$

and for this simple case clearly $h_1(t) \equiv 0$ and $h_0(t) \equiv 0$ (not so for the project problem). $h_2(t) \to -\frac{k}{T-t}$ as $a \to \infty$, i.e. the limit where we enforce **exact liquidation** $Q_T = 0$, for which the solution to the ODE (1) is the **straight line** solution: $Q_t = Q_0(1 - \frac{t}{T})$ (also known as the **VWAP strategy**), which clearly does not depend on k.

For $\phi > 0$, the solution to the final ODE for $h_2(t)$ is

$$h_2(t) = \sqrt{k\phi}\tanh(\frac{t\sqrt{\phi} + kc_1\sqrt{\phi}}{\sqrt{k}})$$

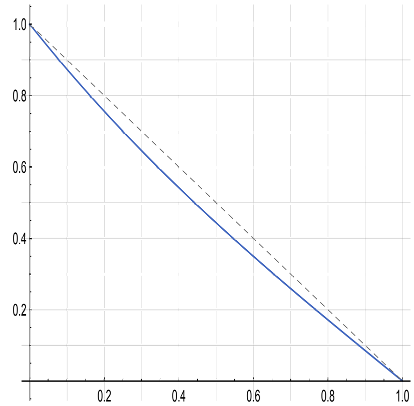

Figure 1: Blue curve here is the optimal inventory process $Q(t)$ in (2) when all parameters for the problem are 1 aside from $a = \infty$. The grey curve is the VWAP solution obtained when $\phi \to 0$.

where c_1 is chosen so $h_2(T) = -a$, again with $h_1(t) \equiv 0$ and $h_0(t) \equiv 0$. Again letting $a \to \infty$, we find that

$$\frac{dQ}{dt} = Q\sqrt{\frac{\phi}{k}}\coth((t-T)\sqrt{\frac{\phi}{k}})$$

and solving this ODE we get

$$Q_t = Q_0 \frac{\sinh((T-t)\sqrt{\frac{\phi}{k}})}{\sinh(T\sqrt{\frac{\phi}{k}})} \tag{2}$$

(see blue curve in plot below), and we recover the VWAP solution once more if we further let $\phi \to 0$.