Fractional Brownian motion

Fractional Brownian motion (fBM) is a natural generalization of standard Brownian motion.

A **zero-mean Gaussian process** is uniquely characterized by its **covariance function** $R(s,t) := \mathbb{E}(X_s X_t)$. $R(s,t)$ uniquely defines the process, since it determines the **covariance matrix** of $(X_{t_1},...,X_{t_n})$ for any ordered pair of time values $0 < t_1 < ... < t_n$, with (i,j)th element $R_{ij} := R(t_i,t_j)$, and for a zero-mean Gaussian vector, we only need its covariance to describe its **joint pdf**. Specifically, the joint density of a n-dimensional Gaussian random vector $\mathbf{X} = (X_1,...,X_n)$ (with zero mean vector) is given by

$$p(\mathbf{x}) \quad = \quad p(x_1,..,x_n) \quad = \quad (2\pi)^{-n/2}\det(\Sigma)^{-\frac{1}{2}}\, e^{-\frac{1}{2}\mathbf{x}^T\mathbf{\Sigma}^{-1}\mathbf{x}} \tag{1}$$

where $\mathbf{\Sigma}$ is the matrix with i,jth element $\Sigma_{ij} = \mathbb{E}(X_i X_j)$, and det denotes the **determinant** of a matrix (if $n = 1$ this reduces to the density of a one-dimensional $N(0,\sigma^2)$ random variable, i.e. $p(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\frac{x^2}{\sigma^2}}$).

We can **sample** a Gaussian process X at $(t_1,...,t_n)$ as $\mathbf{X} = \mathbf{CZ}$, where Z is a **column vector** $(Z_1,..,Z_n)$ of standard $N(0,1)$ random variables and C is the **unique lower triangular** $n \times n$ matrix such that $\mathbf{CC}^\top = \mathbf{\Sigma}$ (C is known as the **Cholesky decomposition** of Σ, see also FM06). Lower triangular means that $C_{ij} = 0$ if $i < j$, so the matrix looks like

$$C \quad = \quad \begin{bmatrix} C_{11} & 0 & .. & 0 \\ C_{21} & C_{22} & ... & 0 \\ ... & & & \\ C_{n1} & C_{n2} & ... & C_{nn} \end{bmatrix}$$

Note for any (non-zero) vector $\mathbf{x} = (x_1,...,x_n)$:

$$\mathbb{E}((\sum_{i=1}^n x_i X_{t_i})^2) \quad = \quad \sum_{i=1}^n\sum_{j=1}^n R(t_i,t_j)x_i x_j \quad = \quad \mathbf{x}^\top\mathbf{\Sigma}\,\mathbf{x} \quad \geq \quad \mathbf{0}\,.$$

If we make the natural non-degeneracy assumption that $X_{t_1},...,X_{t_n}$ are **linearly independent** i.e. that $\sum_{i=1}^n x_i X_{t_i} \neq 0$ a.s. when at least one of the x_i's are non-zero, then clearly the square of this quantity $(\sum_{i=1}^n x_i X_{t_i})^2 > 0$ a.s. and hence its expectation $\mathbb{E}((\sum_{i=1}^n x_i X_{t_i})^2) = \sum_{i=1}^n\sum_{j=1}^n R(t_i,t_j)x_i x_j > \quad 0$. Hence Σ is **positive definite**, which (from standard results in linear algebra) implies that Σ has **positive determinant** and is **invertible** (which is needed for (1)). If we don't have linear independence it means $\mathrm{Corr}(X_{t_i},X_{t_j}) = 1$ for some i,j, we just remove some of the X_{t_i}'s until this is no longer the case.

The Cholesky method gives the correct covariance for X because

$$\mathbb{E}(\mathbf{XX}^\top) \quad = \quad \mathbf{C}\mathbb{E}(\mathbf{ZZ}^\top)\mathbf{C}^\top \quad = \quad \mathbf{CC}^\top \quad = \quad \mathbf{\Sigma}$$

(where here we interpret $\mathbf{X}$ as a column vector), and we have used that

$$\mathbf{ZZ}^\top \quad = \quad \begin{bmatrix} Z_1^2 & Z_1Z_2 & .. & 0 \\ Z_2Z_1 & Z_2^2 & ... & 0 \\ ... & & & \\ Z_nZ_1 & ... & ... & Z_n^2 \end{bmatrix}$$

so $\mathbb{E}(\mathbf{ZZ}^\top) = \mathbf{I}$, i.e. the **identity matrix**.

A zero-mean Gaussian process B_t^H is called standard **fractional Brownian motion** (fBM) with **Hurst exponent** $H \in (0,1)$ if it has covariance function

$$R_H(s,t) \quad = \quad \mathbb{E}(B_t^H B_s^H) - \mathbb{E}(B_t^H)\mathbb{E}(B_s^H) \quad = \quad \mathbb{E}(B_t^H B_s^H) \quad = \quad \frac{1}{2}(t^{2H} + s^{2H} - (t-s)^{2H}) \tag{2}$$

$0 \leq s \leq t$ (note B^H can be defined for all $t \in \mathbb{R}$ or just $t \in [0,\infty)$).

- For $H = \frac{1}{2}$ and $0 \leq s \leq t$, we see that $R_H(s,t) = \frac{1}{2}(t + s - (t-s)) = s$, so we see that for $H = \frac{1}{2}$, $R_H(s,t) = \min(s,t)$, i.e. when $H = \frac{1}{2}$, fBM is just a **standard Brownian motion**.
- When $H \in (0,\frac{1}{2})$, B^H is **rougher** than standard BM, and when $H \in (\frac{1}{2},1)$, B^H is smoother than standard BM (see simulations in Figure 1 below); more specifically B^H is $H-\varepsilon$ **Hölder continuous** which means that $|B_t^H - B_s^H| \leq c_1(\omega)|t-s|^{H-\varepsilon}$ a.s. for any $\varepsilon \in [0,H)$ where $c_1(\omega)$ is a (in general random) constant depending on B^H itself (this comes partly from the **Kolmogorov Continuity Theorem**, see below for full statement and application to fBM.

We now prove some basic fundamental properties of fBM:

- $R(as,at) = a^{2H}R(s,t)$, so

$$X_{a(.)} \sim a^H X_{(.)}$$

  (i.e. both processes on the left and right here have the same joint distribution at $(t_1,...,t_n)$), so the process X is said to be **self-similar**, and in particular for a single fixed t-value we have $B^H_{at} \sim a^H B^H_t$. Note for $H=\frac{1}{2}$ this reduces to well known property of BM that $B_{at} \sim \sqrt{a}B_t$.

- From (2), for $0 \le s \le t$, we see that

$$\begin{aligned}\mathbb{E}((B^H_t - B^H_s)^2) &= \mathbb{E}((B^H_t)^2) + \mathbb{E}((B^H_s)^2) - 2\mathbb{E}(B^H_s B^H_t) &= t^{2H} + s^{2H} - (t^{2H} + s^{2H} - (t-s)^{2H}) \\ & &= (t-s)^{2H}\end{aligned}$$

  so $B^H_t - B^H_s \sim N(0,|t-s|^{2H})$; since the answer only depends on the difference $t-s$, we say that B^H has **stationary increments**.

- There exists a function $k(s,t)$ such that B^H can be realized as $B^H_t = \int_0^t k(s,t)dB_s$ where B is standard Brownian motion, and $k(s,t) \sim const.(t-s)^{H-\frac{1}{2}}$ as $s \nearrow t$, so k blows up as $s \nearrow t$ when $H \in (0,\frac{1}{2})$.

  The Cholesky matrix above approximates the function k such that $B^H_t = \int_0^t k(s,t)dB_s$ where B is a standard Brownian motion **if you use the same Z vector to generate B and B^H, which you should do for this Task in Part 2 of the rough vol project**. In particular, for $0<t<u$ we have the **conditional decomposition**:

$$B^H_u = \int_0^t k(s,u)dB_s + \int_t^u k(s,u)dB_s\,.$$

  The two expressions on the right hand side are **independent**, and conditioned on B up to time t, B^H has conditional distribution which is $N(\int_0^t k(s,u)dB_s, \int_t^u k(s,u)^2 ds)$. In this sense we see that the process B^H has **memory**. Since $\int_0^t k(s,u)dB_s \ne B_t$ when $H \ne \frac{1}{2}$ and not just a simple function of B_t, we see that B^H is not a martingale, nor is Markov.

- $\mathbb{E}((Z^H_t)^2) = \mathbb{E}((B^H_t)^2)$. A commonly used simpler version of this process is the **Riemann-Liouville** process $Z^H_t = \sqrt{2H}\int_0^t (t-s)^{H-\frac{1}{2}}dB_s$, which is also self-similar, but no longer has stationary increments. Note that $Z^H_t \sim B^H_t$, but B^H and Z do not have the same covariance function and Z does not have stationary increments.

- If we set $X_n = B^H_n - B^H_{n-1}$; then X_n is a **discrete-time Gaussian process**; in fact from the stationary increments property above we know that $X_k \sim N(0,1)$ for all k i.e. (X_k) is a sequence of $N(0,1)$ random variables which are not independent of each other. Thus X is a discrete-time stationary process, and X is known as **fractional Gaussian noise (fGn)**); then $\rho_n = \mathbb{E}(X_{k+n}X_k)$ depends only on n (not k) and has **autocovariance** function

$$\begin{aligned}\rho(n) := \mathbb{E}(X_{k+n}X_k) &= \mathbb{E}((B^H_{k+n} - B^H_{k+n-1})(B^H_k - B^H_{k-1})) \\ &= R_H(k+n,k) + R_H(k+n-1,k-1) - R_H(k+n,k-1) - R_H(k+n-1,k) \\ &= \frac{1}{2}[(n+1)^{2H} - n^{2H} - (n^{2H} - (n-1)^{2H})] \sim const. \times n^{2H-2} \qquad (n \to \infty)\end{aligned}$$

  and thus (by convexity of the function $g(n) := n^{2H}$), we see that $\mathbb{E}(X_{k+n}X_k) > 0$ if $H \in (\frac{1}{2},1)$ (which we call **persistent**) and $\mathbb{E}(X_{k+n}X_k) < 0$ for $H \in (0,\frac{1}{2})$ (which we call **anti-persistent**). Loosely speaking, for $H > \frac{1}{2}$, if B^H was increasing in the past, it is more likely to increase in the future, and vice versa. Similarly for $H < \frac{1}{2}$, if B^H was increasing in the past, it is more likely to decrease in the future, and vice versa.

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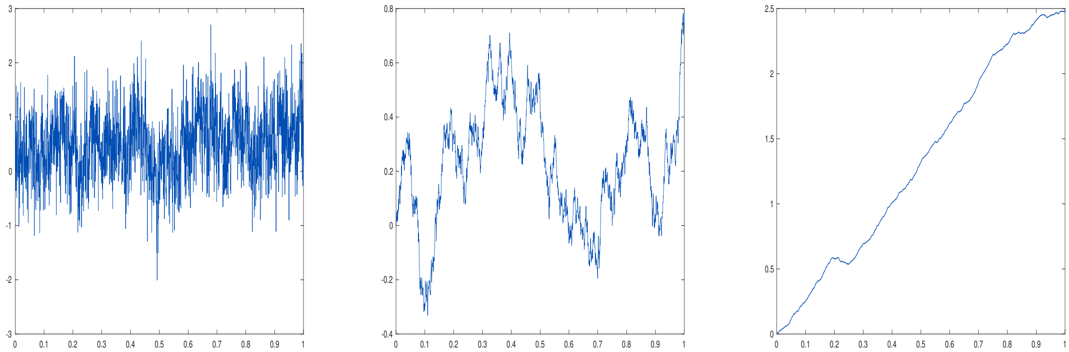

Figure 1: (i) Here we have plotted a Monte Carlo simulation of B^H using the Cholesky method for $H = .05$, $H = .5$ and $H = 0.9$

We now recall the **Kolmogorov continuity theorem**:

Theorem 0.1 *Let $\alpha, \varepsilon, c > 0$ and X be a random process which satisfies*

$$\mathbb{E}(|X_t - X_s|^\alpha) \quad \leq \quad c|t-s|^{1+\varepsilon}\,.$$

*Then X is γ-**Hölder continuous** for all $\gamma \in [0, \frac{\varepsilon}{\alpha})$.*

Application to fBM: From above, we know that

$$B_t^H - B_s^H \;\sim\; N(0,(t-s)^{2H}) \;\sim\; (t-s)^H Z$$

where $Z \sim N(0,1)$, so $\mathbb{E}(|B_t^H - B_s^H|^q) = \mathbb{E}(|Z|^q)(t-s)^{qH}$. Then applying the Kolmogorov continuity theorem to fBM with $\alpha = q$ and $1+\varepsilon = qH$, we see that B^H is γ-Hölder continuous for all $0 < \gamma < \frac{\varepsilon}{\alpha} = \frac{qH-1}{q}$ for any $q > 1/H$ which ensures that $qH - 1 > 0$.

But $\frac{qH-1}{q} \nearrow H$ as $q \to \infty$ because the qH term dominates the 1, so we can make the stronger statement that B^H is γ-Hölder continuous for all $0 < \gamma < H$. Note the theorem does not tell us that B^H isn't smooth, but in FM14 last year we proved the more precise statement that fBM is H-Hölder continuous but not $H+\varepsilon$-Hölder continuous.

In Task 2 of the rough volatility project in the summer, you are asked to consider a sample path of the process $X = \nu B^H$ with ν and H unknown, so we can set $\Sigma_{ij} = \nu^2 \mathbb{E}(B_{i/n}^H B_{j/n}^H) = \mathbb{E}(X_{i/n} X_{j/n})$. You need to numerically maximize the log of the **likelihood function** in (1) over H and ν (note MATLAB and Python minimize not maximize so one has to minimize minus the log likelihood function to get a maximizer).

Since $B_t^H = 0$, do not include $t = 0$ in your $\Sigma_{i,j}$ matrix, or else you will get a zero determinant for Σ. For Cholesky, do not include $t = 0$ in your set of time points $t_1, ..., t_n$.