Jump Diffusion models

In this chapter, we enrich the Black-Scholes model by adding jumps to the model, so the stock price sample path is no longer continuous.

- To begin with, we let

$$X_t = W_t + N_t$$

  where W_t is standard Brownian motion, and N_t is an independent Poisson process with parameter $\lambda > 0$, which means that

$$\mathbb{P}(N_t = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

  i.e. N_t is a Poisson random variable with parameter λt (rather than the usual λ).

- Recall that a Poisson process has the property that the times between jumps are i.i.d. exponential $\text{Exp}(\lambda)$ random variables and the probability of a jump in a small time interval Δt tends to $\lambda \Delta t$ as $\Delta t \to 0$.

- We now wish to compute $\mathbb{P}(X_t > x)$. We first note that we can re-write the event $\{X_t > x\}$ as follows:

$$\{X_t > x\} = \cup_{n=0}^{\infty} \{X_t > x\} \cap \{N_t = n\}\,.$$

  But the (infinite) union on the right hand side is a union of *disjoint* events. Thus the probability of the union is equal to the sum of the individual probabilities:

$$\begin{aligned}
\mathbb{P}(X_t > x) &= \sum_{n=0}^{\infty} \mathbb{P}(X_t > x, N_t = n) \\
&= \sum_{n=0}^{\infty} \mathbb{P}(X_t > x \,|\, N_t = n)\, \mathbb{P}(N_t = n)
\end{aligned}$$

  where we have used the usual rule of conditional probability that $\mathbb{P}(A \cap B) = \mathbb{P}(A \,|\, B)\mathbb{P}(B)$ in the last line.

- We can now compute the distribution function of X_t by *conditioning* on the independent N_t process first:

$$\begin{aligned}
\mathbb{P}(X_t > x) &= \sum_{n=0}^{\infty} \mathbb{P}(X_t > x \,|\, N_t = n)\, \mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} \mathbb{P}(W_t + n > x \,|\, N_t = n)\, \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= \sum_{n=0}^{\infty} \mathbb{P}(W_t + n > x)\, \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= \sum_{n=0}^{\infty} \mathbb{P}(\frac{W_t}{\sqrt{t}} > \frac{x-n}{\sqrt{t}})\, \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= \sum_{n=0}^{\infty} \Phi^c(\frac{x-n}{\sqrt{t}})\, \frac{(\lambda t)^n e^{-\lambda t}}{n!}\,.
\end{aligned}$$

- X_t has a Brownian motion component, and a jump component, and is a simple example of a **Lévy process**.

6.1 The Merton jump diffusion model

- Now let $S_t = e^{X_t}$ denote the stock price process, where

$$X_t \quad = \quad \mu t + \sigma W_t + \sum_{i=1}^{N_t} \xi_i$$

where W_t is standard Brownian motion, the ξ_i's are independent $N(\alpha, \delta^2)$ random variables i.e. the jumps are now of **random** size, and N_t is a Poisson process with parameter $\lambda > 0$. We assume that N_t, W_t and all the ξ_i's are independent of one another.

- The sum $\sum_{i=1}^{N_t} \xi_i$ is known as a compound Poisson process. Note that if $\xi_i \equiv 1$, then $\sum_{i=1}^{N_t} \xi_i = N_t$ and we are back to the simple model discussed in the previous section.

- As before, we compute the distribution function of X_t by *conditioning* on the independent N_t process first:

$$\begin{aligned} \mathbb{P}(X_t > x) &= \sum_{n=0}^{\infty} \mathbb{P}(X_t > x \,|\, N_t = n)\, \mathbb{P}(N_t = n) \\ &= \sum_{n=0}^{\infty} \mathbb{P}(\mu t + \sigma W_t + \sum_{i=1}^{n} \xi_i > x)\, \frac{(\lambda t)^n e^{-\lambda t}}{n!} \end{aligned}$$

- But $\mu t + \sigma W_t + \sum_{i=1}^{n} \xi_i$ is just a constant (i.e. μt) plus a sum of $n+1$ independent Normal random variables. But a sum of independent Normal random variables is still a Normal random variable, with mean and variance given by the sum of the individual means and variances. So in this case

$$\mu t + \sigma W_t + \sum_{i=1}^{n} \xi_i \quad \sim \quad N(\mu t + n\alpha, \sigma^2 t + n\delta^2)\,. \tag{1}$$

- Thus we have

$$\mathbb{P}(X_t > x) \quad = \quad e^{-\lambda t} \sum_{n=0}^{\infty} \Phi^c\Big(\frac{x - \mu t - n\alpha}{\sqrt{\sigma^2 t + n\delta^2}}\Big)\, \frac{(\lambda t)^n}{n!}$$

This series converges rapidly and be easily computed in VBA (see the Excel sheet JumpDiffusionModel.xls).

- Differentiating this expression with respect to x, and multiplying by -1, we obtain the density of X_t (we omit the details for the same of brevity).

6.2 A general jump diffusion model

Now consider a more general jump diffusion model where $X_t = \mu t + \sigma W_t + Y_t$, where $Y_t = \sum_{i=1}^{N_t} \xi_i$ and the ξ_i's are independent and identically distributed (i.i.d) random variables with density $\mu(x)$ and N_t is a Poisson process with parameter $\lambda > 0$, and W, N and the ξ_i are all independent of each other. Using independence we have

$$\mathbb{E}(e^{iuX_t}) \quad = \quad \mathbb{E}(e^{iu(\mu t + \sigma W_t)})\, \mathbb{E}(e^{iuY_t}) \quad = \quad \exp\big[i\mu u t \,-\, \frac{1}{2}\sigma^2 u^2 t\big]\, \mathbb{E}(e^{iuY_t}) \tag{2}$$

where $i = \sqrt{-1}$. Here we have used that $\mathbb{E}(e^{pZ}) = e^{\mu_1 p + \frac{1}{2}\sigma_1^2 p^2}$ if $Z \sim N(\mu_1, \sigma_1^2)$; here $Z = \mu t + \sigma W_t$ so $\mu_1 = \mu t$, $\sigma_1^2 = \sigma^2 t$ and $p = iu$.

$$\begin{aligned} \mathbb{E}(e^{iuY_t}) \quad = \quad \sum_{n=0}^{\infty} \mathbb{E}(e^{iuY_t} \,|\, N_t = n)\, \mathbb{P}(N_t = n) &= \sum_{n=0}^{\infty} \mathbb{E}(e^{iu(\xi_1 + \ldots \xi_n)})\, \mathbb{P}(N_t = n) \\ &= \sum_{n=0}^{\infty} \mathbb{E}(e^{iu\xi_1})\mathbb{E}(e^{iu\xi_2}) \ldots \mathbb{E}(e^{iu\xi_n})\, \mathbb{P}(N_t = n) \\ &= \sum_{n=0}^{\infty} \mathbb{E}(e^{iu\xi_1})^n\, \mathbb{P}(N_t = n) \end{aligned}$$

where we have used that $\xi_1, \xi_2, ...$ are i.i.d. Now let $\phi(u) = \mathbb{E}(e^{iu\xi_1})$. Then we have

$$\begin{aligned}
\mathbb{E}(e^{iuY_t}) &= \sum_{n=0}^{\infty} \phi(u)^n \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= \exp(-\lambda t + \phi(u)\lambda t) \\
&= \exp(-\lambda t + \lambda t \int e^{iux} \mu(x) dx) \\
&= \exp[\lambda t \int_{-\infty}^{\infty} (e^{iux} - 1)\mu(x) dx]
\end{aligned}$$

using that $\int \mu(x)dx = 1$ because $\mu(x)$ is a density. In general, if we know the characteristic function $\phi(k)$ of a random variable, if its density exists, we can compute its density using an **inverse Fourier transform**:

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \phi(k) dk\,.$$

7. Lévy models

7.1 Definition of a Lévy process

A Lévy process is a generalization of the jump diffusion process $X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} \xi_i$ discussed in the previous lecture. X is said to be a Lévy process if:

- $X_0 = 0$;
- X has independent increments;
- $X_t - X_s$ has the same distribution as X_{t-s} for any $0 < s < t$;
- X can only jump at random times.

Remark 0.1 Examples of Lévy processes: standard Brownian motion W_t, the Poisson process N_t, the sum $X_t = W_t + N_t$, the general jump diffusion $X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} \xi_i$ from the previous lecture with general jump size distribution $\mu(dx)$ and $(H_b)_{b \geq 0}$ (the first hitting time process for standard Brownian motion) where now b is the time variable.

7.2 A Lévy process

7.2 A Poisson random measure

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $E = \mathbb{R} \times [0, t]$ and let $\mathcal{E} = \mathcal{B}(\mathbb{E})$ denote the Borel σ-algebra on E. Then N is said to be a **Poisson random measure** on E with intensity $\mu(dx, dt) = \nu(dx) \times dt$ if

- For any $A \in \mathcal{E}$, i.e. any Borel subset of $\mathbb{R} \times [0, t]$, $N(A)$ is a Poisson random variable with parameter $\mu(A)$.
- For any disjoint sets $A_1, .., A_n$, $N(A_1), .., N(A_n)$ are independent (in fact they will be n independent Poisson random variables with parameters $\mu(A_1), .., \mu(A_n)$).

Remark 0.2 Such a measure can be constructed (see e.g. Tankov's notes). $N(\omega, .)$ is a *measure* on $\mathbb{R} \times [0, t]$, and $N(., A) = N(A)$ is a random variable.

Remark 0.3 All we really care about is a set A of the form $A = B \times [0, t]$ where $B \in \mathcal{B}(\mathbb{R})$. Then the number of points that fall inside B is just a Poisson random variable with parameter $t\nu(A)$.

How does this tie in with a Lévy process? Well for a Lévy process X, we can define the **jump measure** J_X as

$$J_X(A) = \sharp\{t : \Delta X_t \neq 0 \text{ and } (t, \Delta X_t) \in A\}$$

where $\sharp$ means "the number of", so $J_X(A)$ is the number of times t that X undergoes a jump and $(t, \Delta X_t) \in A$, and we define the **Lévy measure** on $\mathbb{R}$ by

$$\nu(A) \quad = \quad \mathbb{E}(\sharp\{t \in [0,1] : \Delta X_t \neq 0 \text{ and } \Delta X_t \in A\} \,.$$

ν is a measure but it need not be a probability measure unless X is a jump diffusion of the form $X_t = \gamma t + \sigma W_t + \sum_{i=1}^{N_t} \xi_i$, in which case $\nu(dx)$ is just the jump size distribution.

Theorem 0.1 *(The Lévy-Itô decomposition).*

- *J_X defined above is a Poisson random measure on* $(\mathbb{R} \times [0,t], \mathcal{B}(\mathbb{R} \times [0,t]))$.
- *If the Lévy measure ν satisfies $\int_{-\infty}^{\infty}(x^2 \wedge 1)\nu(dx) < \infty$, we can decompose X as*

$$X_t \quad = \quad \gamma t \,+\, \sigma W_t \,+\, \int_0^t \int_{|x|>1} x J_X(dx, ds) \,+\, \int_0^t \int_{|x|\leq 1} x(J_X(dx, ds) - \nu(dx)ds) \,.$$

This formula shows how all the jump at different sizes are added together to give the whole process X. The triple (γ, σ, ν) is known as the **characteristic triple** of X.

The arrival rate (i.e. the expected number of jumps) whose size falls in $[x, x+\Delta x]$ is $\nu[x, x+\Delta x]$, and the number of jumps of size $[x, x+\Delta x]$ over $[0,t]$ is a Poisson r.v. with parameter $t\nu[x, x+\Delta x]$. For any two distinct x values x, x_1, the number of jumps over $[0,t]$ whose jump size falls in $[x, x+\Delta x]$ and the number of jumps which fall in $[x_1, x_1+\Delta x]$ are independent if Δx is sufficiently small so $[x, x+\Delta x]$ and $[x_1, x_1+\Delta x]$ don't overlap.

A Lévy process has an associated **Lévy measure** $\nu(dx)$ which is such that for any n disjoint sets $A_1, A_1, ..., A_n$ in $\mathcal{B}(\mathbb{R})$, the number of jumps which fall in $A_1, ..., A_n$ over $[0,t]$ is a vector of n independent **Poisson random variables** with parameters $\nu(A_1)t, ..., \nu(A_n)t$. A jump diffusion process X_t (which we saw in the last lecture) is a special simple type of Levy process for which $\nu(dx) = \lambda\mu(x)$, where $\mu(x)$ is the jump size distribution and λ is the arrival rate for the Poisson process. For some models e.g. the well known CGMY model discussed below, $\nu(0, x] = \infty$ for any $x > 0$; this means there is an *infinite* number of positive jumps almost surely, and this is known as an **infinite-activity** Lévy process.

Theorem 0.2 *Lévy-Khintchine representation). Let X be a infinite acticity Lévy process with characteristic triple (μ, σ, ν). Then*

$$\mathbb{E}(e^{iuX_t}) \quad = \quad \exp\left[t(i\mu u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux 1_{|x|\leq 1})\nu(dx)\,\right] \tag{3}$$

Notice that there is one additional term in the integral which is not there when X is a jump diffusion. We will not prove this here, but note that if $\nu(\mathbb{R}) = +\infty$, this term is needed to ensure that the integral here is finite.

7.3 Examples of Lévy processes

- The double exponential **Kou model** (as previously discussed) is a jump diffusion model for which the Lévy measure is just a multiple λ times a probability measure, given by a two-sided exponential distribution, so

  $$\nu(x) \quad = \quad \lambda[p\lambda_+ e^{-\lambda_+ x} 1_{x>0} \,+\, (1-p)\lambda_- e^{\lambda_- x} 1_{x<0}] \,.$$

  This is a **finite-activity** model, because $\nu(x)$ is just a multiple of a probability density, so $\nu(\mathbb{R}) < \infty$.

- As we have seen, the **Merton jump diffusion model** for which the Lévy measure is just a multiple λ times a normal density with parameters $N(\alpha, \delta^2)$, given by a two-sided exponential distribution, so in this case

  $$\nu(x) \quad = \quad \lambda \cdot \frac{1}{\delta\sqrt{2\pi}} e^{-(z-\alpha)^2/2\delta^2} \,.$$

  This is also a **finite-activity** model, because $\nu(x)$ is just a multiple of a probability density, so $\nu(\mathbb{R}) < \infty$.

- Let $\tau_b = \inf\{s : W_s = b\}$ denote the first hitting time (or **inverse maximum process**) for a Brownian W with $b \geq 0$. Then it turns out that $(\tau_b)_{b\geq 0}$ (with b the time variable) is a non-decreasing Lévy process. From the optional stopping theorem, recall that we proved that

$$\mathbb{E}(e^{-\lambda\tau_b}) \quad = \quad e^{-b\sqrt{2\lambda}}\,.$$

Setting $-\lambda = iu$ for $u \in \mathbb{R}$, we see that

$$\mathbb{E}(e^{iu\tau_b}) \quad = \quad e^{-b\sqrt{-2iu}}$$

and it turns out that we can re-write the right-hand side as

$$\mathbb{E}(e^{iu\tau_b}) \quad = \quad \exp[b\int_0^\infty (e^{iuz}-1)\nu(z)dz] \quad = \quad \exp[***+b\int_0^\infty (e^{iuz}-1-|z|1_{|z|\leq 1})\nu(z)dz]$$

where $\nu(z) = \frac{1}{\sqrt{2\pi z^3}}$, which we recognize as being one-half the rate of excursions which exceed time z. Thus τ_b is a Levy process (note that τ_b has no negative jumps, and the jump sizes correspond to the duration of the excursions of W away from its maximum process).

- The CGMY (Carr-Geman-Madan-Yor) model has a Lévy density of the form

$$\nu(x) \quad = \quad \frac{Ce^{-Mx}}{x^{1+Y}}\,\mathbf{1}_{\{x>0\}} + \frac{Ce^{Gx}}{|x|^{1+Y}}\,\mathbf{1}_{\{x<0\}}$$

for $C, G, M > 0$ and $Y \in (0,2)$, so we the see the jump rate tends to infinity as the jump size tends to zero. This is what we call an **infinite-activity** model, because $\nu(x)$ is not a multiple of a pdf.

- The characteristic function for the CGMY model is given by

$$\phi_t(u) = \mathbb{E}(e^{iuX_t}) = \exp\left[t\,C\Gamma(-Y)\left\{(M-iu)^Y + (G+iu)^Y - M^Y - G^Y\right\} + ibut\right],$$

for $Y \neq 1$ (proof not required) and some constant b which controls the drift, and from this we can compute the critical moments p_+, p_- (see homework/past exam questions).

- To compute the density of $p_t(x)$ of X_t, we use the inverse Fourier transform as before:

$$p(x) \quad = \quad \frac{1}{2\pi}\int_{-\infty}^\infty e^{-iux}\phi_t(u)du\,.$$