The quadratic rough Heston model

Consider the following stochastic volatility model

$$\begin{aligned} dS_t &= S_t\sqrt{V_t}dW_t \\ Z_t &= g_0(t) + \int_0^t K(t-s)\sqrt{V_s}dW_s \end{aligned} \tag{1}$$

where $V_t = aZ_t^2 + c$ and W is a standard Brownian motion. Note there is only 1 Brownian motion here so the model is **complete**, i.e. any contingent claim can be perfectly replicated with dynamic trading in S, but we gain no benefit from using the Renault-Touzi conditioning trick from Chapter 1. This is known as a **quadratic Volterra Heston** model, and an equation like this for V_t is known as a **Stochastic Volterra Equation** (SVE). Recall that for the standard Heston model

$$V_t = V_0 + \int_0^t \lambda(\theta - V_s)ds + \int_0^t \nu\sqrt{V_s}dW_s$$

A good choice for K in (1) is the **Gamma kernel**

$$K(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}e^{-\lambda t}$$

with $\alpha = H + \frac{1}{2}$ ($H \in (0, \frac{1}{2}]$) and $g_0(t) = \theta\lambda^{\frac{1}{2}+H}\int_0^t K(t-s)ds = \theta(1 - \frac{\Gamma(\alpha,t\lambda)}{\Gamma(\alpha)}) \to \theta$ as $t \to \infty$ and $g_0(0) = 0$, which is known as the **quadratic rough Heston** model since in this case we expect that V has **rough** sample paths when $H \in (0, \frac{1}{2})$, i.e $|V_t - V_s| \le const. \times |t-s|^{H-\varepsilon}$ for any $\varepsilon > 0$, for some random const. λ controls the extent of **mean reversion**, similar to the **Ornstein Uhlenbeck** process: $Y_t = Y_0 + \int_0^t e^{-\lambda(t-s)}dW_s$.

We can also consider a variant of the model where

$$Z_t = g_0(t) + \int_0^t \nu(t-s)^{H-\frac{1}{2}}\,(\lambda(\theta - V_s)ds + \nu\sqrt{V_s}dW_s)$$

but we will not be looking at code for this model today.

We can approximate Z in (1) numerically with a time step of size $\frac{1}{n}$ as

$$Z_t = g_0(t) + \int_0^t K(t-s)\sqrt{V_{\frac{[ns]}{n}}}\,dW_s$$

where $[x]$ denotes the integer part of x, for which a basic **Euler**-type Monte Carlo scheme is given by:

$$Z_{j\Delta t} = g_0(j\Delta t) + \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{j-1} K((j-k)\Delta t)\sqrt{V_{k\Delta t}}\,\Delta W_k$$

for $j = 1, 2, ...$, with $\Delta t = \frac{1}{n}$, where ΔW_k is a sequence of i.i.d. $N(0, \Delta t)$ random variables. Since this sum has to be computed for each j, simulating 1 path of Z requires a **double loop**, since we have to repeat this for $j = 1..N$, where N is the number of **time steps** for the Monte Carlo scheme. Since V is "frozen" over each interval $[t_k, t_{k+1}]$ (where $t_k = k/n$ here), we can improve the accuracy of the scheme by computing the following expectation exactly:

$$\sigma_{j,k}^2 = \mathbb{E}((\int_{t_k}^{t_{k+1}} K(j\Delta t - s)dW_s)^2) = \int_{t_k}^{t_{k+1}} K(t-s)^2 ds = F(t_{k+1}) - F(t_k)$$

(where we have set $t = j\Delta t$) and $F(s) = 4^{-H}\lambda^{-2H}\Gamma(2H, 2(t-s)\lambda)$ ($\Gamma(a,z) = \int_z^\infty t^{\alpha-1}e^{-t}dt$ here is the **incomplete Gamma function** for which MATLAB and Python have in-built functions), and now approximate the Z process as

$$Z_{j\Delta t} = g_0(j\Delta t) + \frac{1}{\Gamma(\alpha)}\sum_{k=0}^{j-1} \sigma_{j,k}\sqrt{V_{k\Delta t}}\,\tilde{Z}_k$$

where the $\tilde{Z}_k$'s are i.i.d standard Normals. Python code to simulate a single V path will then look like:

```
Z[0]=g0(0)
for j in range(1,N+1)
    t=j*dt
    Z[j]=g0(t)
    for k in range(1,j+1)
        s=(k-1)*dt
        sigmajk=...
       Z[j]= Z[j]+sigmajk*sqrt[V[k-1]]*Ztilde[k-1]
    V[j]= a*Z[j]**2+c
```

Note that these loops actually stop at $j = N$ and $k = j$ not $N+1$ and $j+1$, due to Python's strange loop convention

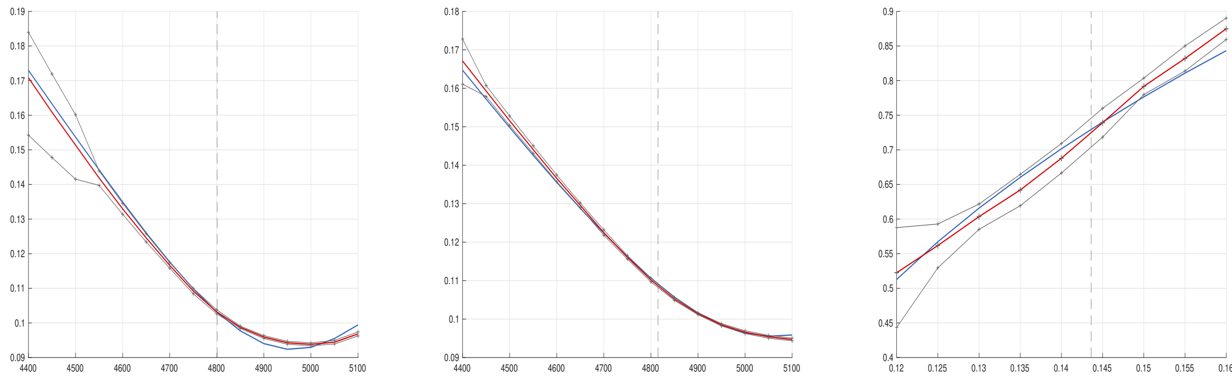

Figure 1: Here we have plotted market mid-implied vol (red, computed from average of bid and ask option prices) vs calibrated model (blue) implied volatility as a function of the strike K at 1545 EST on Fri 12th Jan 2024 for SPX smiles with $T = 25/251$ (left), 45/251 (middle), and VIX smile for $T = 22/251$ (right) using data from CBOE datashop. The dashed lines are the bid and ask implied vols, and the vertical lines are the forward prices. Calibrated parameters are $\lambda = 3.927$, $\theta = -0.0769$, $a = 0.400$, $Z_0 = -0.0387$, $c = 0.00525$, $H = 0.0857$ using 280000 paths and 512 time steps with antithetic sampling

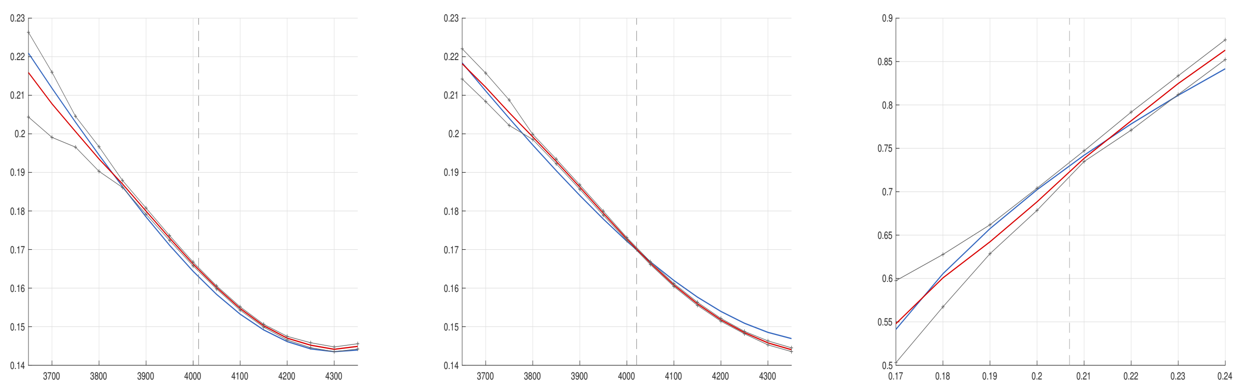

Figure 2: Fit to the same smiles 1 year before on Fri 13th Jan 2023: $\lambda = 0.292$, $\theta = -0.473$, $a = 0.400$, $Z_0 = -0.0955$, $c = 0.01681$, $H = 0.144$

Pricing VIX options

Consider a stochastic volatility model of the usual form

$$dS_t \quad = \quad S_t\sqrt{V}_t dW_t$$

Now let $\xi_t(u) = \mathbb{E}(V_u|\mathcal{F}_t)$ for $u \geq t$. Note if V is a martingale, then $\xi_t(u) = V_t$, which will be the case if e.g.

$$dV_t \quad = \quad \sigma V_t^p dB_t$$

for $p \in (0,1]$ (V is not a martingale for $p > 1$). Now consider a more general model where

$$d\xi_t(u) \quad = \quad \xi_t(u)\kappa(u-t)dB_t$$

where B is another Brownian motion with $\mathbb{E}(B_t W_t) = \rho t$, and κ is some function such that $\int_0^t \kappa(u)^2 du < \infty$ for all t. The κ function we will be interested in is $\kappa(t) = \nu t^{H-\frac{1}{2}}$, for $H \in (0, \frac{1}{2})$, which will lead to what we call a **rough Bergomi model**. Then for each fixed u, $\xi_t(u)$ is just **Geometric Brownian motion** but with a time-dependent volatility function $\kappa(u-t)$. Then the VIX index at time t is theoretically defined as

$$\text{VIX}_t^2 \quad = \quad \frac{1}{\Delta}\int_t^{t+\Delta} \xi_t(u)du$$

Similar to the Black-Scholes model, the solution to the SDE for $\xi_t(u)$ is given by

$$\xi_t(u) \quad = \quad \xi_0(u)e^{\int_0^t \kappa(u-s)dB_s - \frac{1}{2}\mathbb{E}((\int_0^t \kappa(u-s)dB_s)^2)}$$

so

$$\xi_t(t) \quad = \quad V_t \quad = \quad \xi_0(u)e^{\int_0^t \kappa(t-s)dB_s - \frac{1}{2}\mathbb{E}((\int_0^t \kappa(t-s)dB_s)^2)}$$

i.e. the rough Bergomi model. If

$$dV_t \quad = \quad \kappa(\theta - V_t)dt \,+\, \sigma V_t^p dB_t$$

then one can show that

$$\mathbb{E}(V_u|V_t) \quad = \quad \theta + e^{-\kappa(u-t)}(V_t - \theta)$$

for $u \geq t$ i.e. we just replace t with $u - t$ and V_0 with V_t in $f(t)$, so setting $t = T$ we see that

$$\text{VIX}_T^2 \quad = \quad \frac{1}{\Delta}\int_T^{T+\Delta} (\theta + e^{-\kappa(u-T)}(V_T - \theta))du\,.$$

We can compute the integral here explicitly since V_T does not depend on u, and we obtain

$$\text{VIX}_T^2 \quad = \quad \frac{1-e^{-\kappa\Delta}}{\kappa\Delta}V_T \,+\, \frac{\theta}{\kappa\Delta}(e^{-\kappa\Delta} + \kappa\Delta - 1) \quad = \quad aV_T \,+\, b\,.$$

Pricing barrier options under the Black-Scholes model

Let $(W_t)_{t\geq 0}$ denote a standard Brownian motion, and let $\bar{W}_t = M_t = \max_{0\leq s\leq t} W_s$ denote the **running maximum process** of W, and we use this notation for the runninig maximum of other processes as well. The **reflection principle** states that for $b \geq 0$ and $x \leq b$

$$\mathbb{P}(M_t > b, W_t < x) \quad = \quad \mathbb{P}(W_t > 2b - x) \quad = \quad \Phi^c(\frac{2b-x}{\sqrt{t}})$$

From this one can also show that

$$\mathbb{P}(\tau_b \leq t) \quad = \quad \mathbb{P}(M_t \geq b) \quad = \quad 2\mathbb{P}(W_t \geq b) \quad = \quad \Phi^c(\frac{b}{\sqrt{t}}) \tag{2}$$

and clearly $\mathbb{P}(\tau_b \leq t) = \mathbb{P}(M_t > b)$, where $\tau_b = \min\{t : W_t = b$ is the first hitting time of W to b. We can therefore compute the density of τ_b by differentiating the right hand side of (2) with respect to t and again using the chain rule to obtain

$$f_{\tau_b}(t) \quad = \quad 2\frac{d}{dt}\Phi^c(\frac{b}{\sqrt{t}}) \quad = \quad 2\cdot\frac{1}{2}bt^{-\frac{3}{2}}n(\frac{b}{\sqrt{t}}) \quad = \quad \frac{b}{\sqrt{2\pi t^3}}e^{-\frac{b^2}{2t}}$$

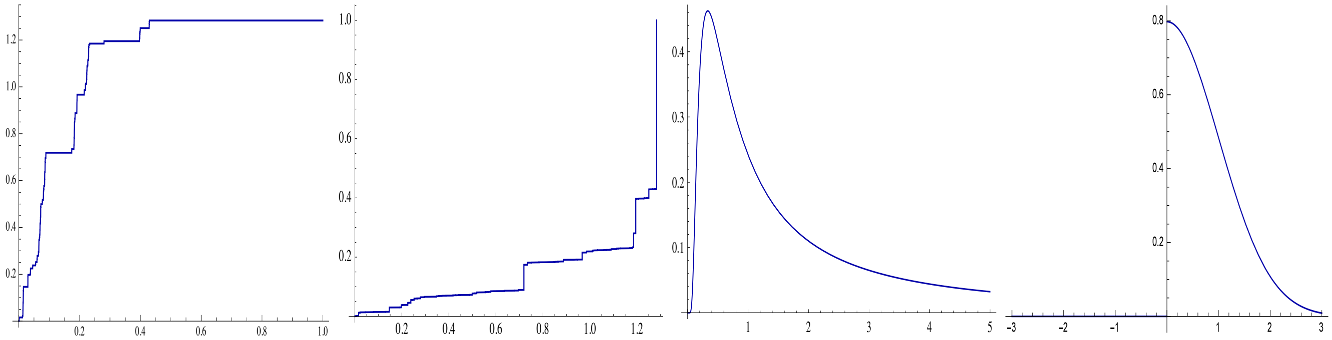

Figure 3: Here we have plotted (in order) a Monte Carlo simulation of M_t, its inverse process τ_b, the density of the hitting time density $f_{\tau_b}(t)$ for $b=1$, and the density of M_t for $t=1$.

for $t \geq 0$, and $f_{\tau_b}(t) = 0$ otherwise (see plot below). Density of M_t is $p(b) = \frac{2}{\sqrt{2\pi t}} e^{-b^2/2t}$ for $b \geq 0$.

From the Reflection Principle we can also compute the **joint density** of W_t and M_t as

$$f(x,b;t) \;=\; \frac{2(2b-x)}{\sqrt{2\pi t^3}}\, e^{-\frac{(2b-x)^2}{2t}}$$

and this will be key to pricing barrier options.

For the Black-Scholes model we know that $dS_t = S_t(rdt + \sigma dW_t)$ where W is a Brownian motion under the risk-neutral measure $\mathbb{Q}$. From Ito's lemma, we know that $X_t = \log S_t$ satisfies

$$X_t \;=\; X_0 + (r - \frac{1}{2}\sigma^2)t + \sigma W_t \;=\; x_0 + \sigma(W_t + \gamma t)$$

where $\gamma = \frac{r - \frac{1}{2}\sigma^2}{\sigma}$. Then $X_t = X_0 + \sigma W_t^{(\gamma)}$ where $W_t^{(\gamma)} = W_t + \gamma t$, and $\bar{X}_t = X_0 + \sigma \bar{W}_t^{(\gamma)}$, and $S_t = e^{X_t}$ and $\bar{S}_t = e^{\bar{X}_t}$.

Then from **risk-neutral valuation** and **Girsanov's theorem**, the **unique no-arbitrage price** of a barrier option which pays $\phi(S_T, \bar{S}_T)$ at time T for some general function ϕ is

$$e^{-rT}\mathbb{E}(\phi(S_T,\bar{S}_T)) \;=\; e^{-rT}\int_{b=0}^{\infty}\int_{x=-\infty}^{b} f(x,b,T)\, e^{\gamma x - \frac{1}{2}\gamma^2 T}\; \phi(e^{X_0+\sigma x}, e^{X_0+\sigma b})\, dx\, db\,.$$

This double integral can be approximated using **Gaussian quadrature** with a double sum.

We can dynamically replicate this payoff as we would in FM02. Specifically we hold $\Delta_t = P_S(S_t,\bar{S}_t,t)$ units of stock at time t, where $P(S,\bar{S},t) = e^{-r(T-t)}\mathbb{E}(\phi(S_T,\bar{S}_T)|S_t = S, \bar{S}_t = \bar{S})$, and we place the our remaining wealth $X_t - \Delta_t S_t$ in the risk free bank account so our wealth evolves as $dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt$ with $X_T = \phi(S_T,\bar{S}_T)$ (i.e. a self-financing trading strategy).

We can **semi-statically hedge** (and hence also price) a **Knock Out (KO) call option** with barrier level $B < S_0$ by buying a K-strike call option at time zero, and selling a **shadow contract** which pays $(\frac{S_T}{B})^{2\gamma}\max(\frac{B^2}{S_T} - K, 0)$ at time zero, where $\gamma = \frac{1}{2} - r/\sigma^2$. This position is worth zero at the exact moment S hits B (if S hits B) and therefore mimics the behaviour of the KO call.

Conversely, if S does not hit B, the K-strike European call has the same terminal payoff as the KO call and the shadow contract expires worthless, as required. Hence initial unique no-arbitrage price of the KO call is the initial cost of the aforementioned replication strategy:

$$P(S_0,0) \;=\; e^{-rT}\mathbb{E}^{\mathbb{Q}}((S_T-K)^+ - (\frac{S_T}{B})^{2\gamma}\max(\frac{B^2}{S_T} - K, 0)) \;=\; e^{-rT}\int_0^{\infty}((S-K)^+ - (\frac{S}{B})^{2\gamma}(\frac{B^2}{S} - K)^+)\, p_{S_T}(S)dS$$

where $p_{S_T}(S)$ is the (lognormal) density of S_T under $\mathbb{Q}$, and the price is clearly less than the price of a standard European call with strike K. See the old 2022-23 FM02 Lecture Notes on KEATS for more details on this.

Stochastic Control project

We first consider a simplified version of the project assignment where

$$\begin{aligned} dS_t &= \kappa^S \alpha dt + \sigma^S dW_t \\ dQ_t &= v_t dt \\ dX_t &= (-S_t v_t - k v_t^2)dt \end{aligned}$$

where S is the asset price process, Q_t is the number of shares held at time t (also known as the **inventory process**), and X_t is the agent's **cash process**, and (unlike the project) α is constant here. We wish to compute

$$\begin{aligned} H(t,q,S,x) &= \sup_{v\in\mathcal{A}_t} \mathbb{E}_{t,x,S,q}(X_T + Q_T S_T - aQ_T^2) \\ &= \sup_{v\in\mathcal{A}_t} \mathbb{E}(X_T + Q_T S_T - aQ_T^2 | X_t = x, Q_t = q, S_t = S)\,. \end{aligned}$$

Feynmac-Kac formula where $\mathcal{A}_t$ is the space of $\mathcal{F}^W$-adapted process with $\mathbb{E}(\int_t^T v_s^2 ds|\mathcal{F}_t^W) < \infty$ (note there is no running inventory penalty here), and (from standard **FM20** arguments) $H(t,q,S,x)$ satisfies the **HJB equation**:

$$H_t + \kappa^S \alpha H_S + \frac{1}{2}(\sigma^S)^2 H_{SS} + \max_v(vH_q - v(S+kv)H_x) = 0$$

with **terminal condition**:

$$H(T,q,S,x) = x + qS - aq^2$$

since X_T is our final riskless wealth, $Q_T S_T$ is our final risky wealth, and $-aQ_T^2$ is the quadratic inventory penalty we are imposing. We will show how to solve this problem in **Mathematica** using a quadratic ansatz. Recall that $a > 0$ penalizes non liquidation at T, and if we take the limit $a \to \infty$, this corresponds to enforcing **perfect liquidation** at time t i.e. that $Q_T = 0$, since in this case we have an **infinite penalty** for non-liquidation (see chapter 6 in **Cartea** et al. book [CJP15] for more problems of this nature). Once we find the optimal $v = v^*(q,t)$, this gives us an **ODE** for Q_t^v of the form:

$$\frac{dQ_t}{dt} = v^*(Q_t, t)$$

and in this case there is no randomness here, since α was assumed constant. Can numerically solve an ODE with an **Euler scheme**.

More generally, for any choice of S process with $\mathbb{E}(S_t^2) < \infty$ for all $t \in [0,T]$ with S adapted to some filtration $\mathcal{F}_t$ (S need not be a **Markov process**), the optimal trading speed for the **full liquidation** (i.e. $Q_T = 0$) problem satisfies the (random) ODE

$$v_t = \frac{dQ_t}{dt} = -\frac{Q_t}{T-t} + \frac{1}{T-t}\frac{1}{2k}\int_t^T \mathbb{E}(S_u - S_t|\mathcal{F}_t)du \tag{3}$$

variational approach (we do not prove this here but email me if you want further details). In particular, if S is an $\mathcal{F}_t$-**martingale** then the second term on the right is clearly zero, so the ODE simplifies to $\frac{dQ_t}{dt} = -\frac{Q_t}{T-t}$ for which the solution is just the straight line $Q_t = Q_0(1 - \frac{t}{T})$, i.e. we liquidate at **constant speed** $v = -Q_0/T$. We can assume e.g. S is an OU process or even fractional Brownian motion, or some other non-martingale, non-Markov process with memory, which in particular can lead to non-constant (and random) **round trips** for the optimal strategy (i.e. $Q_0 = 0$, but it is still optimal to trade over $[0,T]$).

Potential topic for Part 3: optimal trading when the drift is unknown

In real life, if P is the price process for an asset which evolves as

$$dP_t = \mu dt + \sigma dW_t$$

with μ and σ constant (i.e. the same model as above with S replaced with P) then μ is unknown, but can be estimated with $\hat{\mu}_t = \frac{P_t - P_{t_0}}{t - t_0}$ for some reference time $t_0 < 0$ in the past, and we see that $\mathbb{E}(\hat{\mu}_t|\mathcal{F}_{t_0}^W) = \mu$, so $\hat{\mu}_t$ is an **unbiased estimator** of μ when we compute the expectation of $\hat{\mu}_t$ at t_0. Unlike μ, σ can be perfectly estimated over any window $[0,T]$ using that $\sigma^2 T = \lim_{n\to\infty}\sum_{i=1}^n (P_{iT/n} - P_{(i-1)T/n})^2$ (see e.g. FM02, FM04 or FM14) (assuming we have **continuous observations** of P over $[0,T]$), so for this reason we assume σ is known. Note $\hat{\mu}_T$ has **non-zero variance**, but this variance tends to zero as $t - t_0 \to \infty$.

We can re-write the SDE for P as

$$dP_t = \hat{\mu}_t dt + \sigma d\bar{W}_t = \frac{P_t - P_{t_0}}{t - t_0}dt + \sigma d\bar{W}_t \tag{4}$$

i.e.

$$P_t = P_0 + \mu t + W_t = P_0 + \int_0^t \frac{P_u - P_{t_0}}{u - t_0}du + \sigma\bar{W}_t\,.$$

so

$$\bar{W}_t = \mu t + W_t - \int_0^t \frac{P_u - P_{t_0}}{u - t_0}du\,.$$

Then clearly $\bar{W}_t$ is not equal to the original Brownian motion W_t, and $\bar{W}$ is not a Brownian motion with respect to $\mathcal{F}^W$, but it can be shown that $\bar{W}$ *is* a Brownian motion with respect to $\mathcal{F}^P_t$, i.e. the **filtration** generated by P (recall that W is a Brownian motion with respect to $\mathcal{F}^W$), so e.g. the conditional distribution of $\bar{W}_t - \bar{W}_s$ given $\mathcal{F}^P_s$ (or in fact just P_s since P is Markov) is $N(0, t-s)$, and of course we know that the conditional distribution of $W_t - W_s$ given $\mathcal{F}^W_s$ (or just W_s) is also $N(0, t-s)$.

For those who did FM04 or FM14, (4) is the same SDE as satisfied by a **Brownian bridge** which starts at zero at time zero and is **conditioned** to end at P_{t_0} at time t_0, except in this case, t_0 is in the past not the future, so P can be viewed as the continuation of a Brownian bridge.

In particular, P is a Markov process with respect to $\mathcal{F}^P$, and we see that

$$\frac{d}{dt}\mathbb{E}(P_t|P_s) \quad = \quad \frac{\mathbb{E}(P_t|P_s) - P_{t_0}}{t - t_0}.$$

Solving this ODE for $f(t) := \mathbb{E}(P_t|P_s)$, we find that

$$\mathbb{E}(P_t|P_s) \quad = \quad P_{t_0} + \frac{P_s - P_{t_0}}{s - t_0}(t - t_0) \quad = \quad P_{t_0} + \hat{\mu}_s(t - t_0)$$

for $t_0 \leq 0 \leq s \leq t$. We can then plug this into (3) to obtain the optimal buying speed when μ is unknown as

$$v_t \quad = \quad \frac{dQ_t}{dt} \quad = \quad -\frac{Q_t}{T-t} + \frac{1}{T-t}\frac{1}{2k}\int_t^T \mathbb{E}(P_u - P_t|\mathcal{F}^P_t)du$$

which you can then numerically approximate and simulate. Note that since we have less information in this problem (since μ is unknown so we are an *uninformed* trader), the optimal value of the performance criterion will be lower.

Other ideas for Part 3:

- Look at **resilient** price impact where temporary price impact is replaced by a term of the form $\int_0^t G(t-s)v_s ds$ (which corresponds to **permanent price impact** when the G function here is constant), which is mathematically more interesting/challenging, e.g. $G(t) = ct^{-\gamma}$ or $G(t) = ce^{-\lambda t}$.
- Optimal **market making** (see FM20 and [CJP15] book).
- Optimal **pairs trading** (see e.g. articles by Tim Leung)
- Use of **Schrödinger bridges** to calibrate Markov local/stochastic volatility models using HJB eqs
- **Automated market makers** for crypto currencies, which is a kind of **stochastic permanent price impact**
- Use **deep learning** techniques from FM18 (e.g. **RNNs/LSTMs**) to learn optimal trading strategies with e.g. temporary/permanent price impact
- Look at **multi-agent problems**/Nash equilibria
- Look at other **free boundary problems**, e.g. **American options**, optimal stopping times for Brownian motion B such that $B_\tau \sim \mu$