Sharp tail estimates for exponential Lévy models

Martin Forde

DCU, Dept of Mathematics

20th April 2011

Outline of talk

- Using saddlepoint methods, we compute tail estimates for the density, cdf and call option prices for an exponential Lévy model when the cumulant generating function $\kappa(p) = \log \mathbb{E}(e^{pX_t})$ of the log stock price X_t tends to infinity as $p \nearrow p_+$, where $p_+ = \sup\{p : \mathbb{E}(e^{pX_t}) < \infty\}$.
- This can include infinite activity Lévy processes with finite or infinite variation, and jump diffusion processes.
- Using the recent work of Gao&Lee[GL11] which refines the previous estimates of Gulisashvili[Gul10a] and Benaim&Friz[BF08] for implied volatility at large strikes, we characterize the behaviour of implied volatility smile as the strike tends to infinity.
- We give numerical examples for the double exponential Kou and CGMY models.
- Similar tail estimates are given in Friz,Gerhold,Gulisashvili&Sturm [FGGS10] for call options under the Heston model, and in Jensen[Jen95] for the cdf of a compound Poisson sum of iid random variables

The modelling framework

Let $(X_t)_{t\geq 0}$ be Lévy process defined on a probability space $(\Omega, \mathbb{P}, \mathcal{F})$ with Lévy triple (b, σ^2, ν) such that $\nu(\mathbb{R}\setminus\{0\}) > 0$.

- Let ψ denote the characteristic exponent of X_1 given by the Lévy-Khintchine formula as

$$\psi(u) = \log \mathbb{E}(e^{iuX_1}) = ibu - \frac{1}{2}\sigma^2 u^2 + \int (e^{iuy} - 1 - iuy\, 1_{\{|y|\leq 1\}})\nu(dy) \;\; (u \in \mathbb{R}).$$

- We assume that ψ is analytic with a strip of analyticity of the form $\{z \in \mathbb{C} : \mathrm{Im}(z) \in (p_-, p_+)\}$ for some $p_- < 0 < 1 < p_+ < \infty$.
- Let $\kappa(p) = \psi(-ip) = bp + \frac{\sigma^2}{2}p^2 + \int \left(e^{py} - 1 - py 1_{\{|y|\leq 1\}}\right)\nu(dy)$.
- κ' is differentiable and strictly increasing since

$$\begin{aligned}\kappa'(p) &= b + \sigma^2 p + \int y(e^{py} - 1_{\{|y|\leq 1\}})\nu(dy), \\ \kappa''(p) &= \sigma^2 + \int y^2 e^{py}\nu(dy) > 0\end{aligned}$$

for any $p \in (p_-, p_+)$.

Technical conditions on κ

Assumption 1. Assume that

$$\kappa(p) = \frac{A}{(p_+ - p)^\alpha} + \Gamma + O(p_+ - p) \qquad (p \nearrow p_+) \qquad (1)$$

for some constants Γ, $A > 0$, $\alpha > 0$.
Assumption 1 is satisfied by the Kou model but not the CGMY or NIG models because $\kappa(p) \nrightarrow \infty$ as $p \nearrow p_+$ for the latter two models.[1]
Assumption 2. For some β, $Y > 0$,

$$|\,\mathbb{E}(e^{(p+iu)X_1})\,| \;=\; O(e^{-\beta|u|^Y})$$

as $u \to \infty$, for $u, p \in \mathbb{R}$ with p fixed and $1 < p < p_+$.

- From the Lévy-Khintchine formula, Assumption 1 is satisfied if $\nu(x) \sim \frac{1}{x^{1-\alpha}} e^{-p_+ x}$ as $x \to \infty$.
- Let $V(p) = \frac{A}{(p_+ - p)^\alpha}$ denote the leading order term in the expansion for $\kappa(p)$ in (1).

[1] See later slide for analysis of CGMY model.

The saddlepoint tail estimate

Proposition 1. For $\alpha \in (0,1)$, we have the following tail behaviour for the density $p_t(x)$ of X_t

$$
\begin{aligned}
p_t(x) &= \frac{e^{-p^* x + t(V(p^*)+\Gamma)}}{\sqrt{2\pi t V''(p^*)}}\,[1 + (-\frac{5}{24}\gamma_3(x)^2 + \frac{1}{8}\gamma_4(x))(1+o(1))] \\
&= \frac{1}{\sqrt{2\pi t\, A_1\,(\frac{x}{t})^{\beta_2}}}\, e^{-p_+ x + t(c(\frac{x}{t})^{\beta_1}+\Gamma)}\,[1 + A_2\,(\frac{x}{t})^{-\beta_1}(1+o(1))]
\end{aligned}
$$

as $x \to \infty$, where $c = A^{\frac{1}{\alpha+1}}(\alpha^{\frac{1}{\alpha+1}} + \alpha^{-\frac{\alpha}{\alpha+1}})$, $\beta_1 = \frac{\alpha}{\alpha+1}$, $\beta_2 = \frac{\alpha+2}{\alpha+1}$, two constants A_1, A_2 and $p^* = p^*(\frac{x}{t})$, where $p^*(x)$ is the unique solution to the *approximate* saddlepoint equation $V'(p^*) = x$ given explicitly by

$$
p^*(x) = p_+ - (\frac{A\alpha}{x})^{\frac{1}{1+\alpha}} \to p_+ \qquad (x \to \infty),
$$

and

$$
\gamma_3(x) = \frac{tV'''(p^*)}{[tV''(p^*)]^{\frac{3}{2}}} = O(x^{-\frac{1}{2}\frac{\alpha}{\alpha+1}})\ ,\ \gamma_4(x) = \frac{tV''''(p^*)}{[tV''(p^*)]^2} = O(x^{-\frac{\alpha}{\alpha+1}})\,.
$$

Proposition 2. For $1 \leq \alpha < \infty$, we have

$$p_t(x) \sim \frac{e^{-p^* x + t(V(p^*)+\Gamma)}}{\sqrt{2\pi t V''(p^*)}} \qquad (x \to \infty).$$

Proof. We first set $t = 1$. The density of X_t is given by the inverse Laplace transform

$$p_t(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{-px + \kappa(p)} dp \qquad (2)$$

for $1 < \gamma < p_+$. Choose $\gamma = p^*(x) = p_+ - (\frac{A\alpha}{x})^{\frac{1}{1+\alpha}} \to p_+ \quad (x \to \infty)$ (the approximate saddlepoint) so $p = p^* + iy$ for $y \in \mathbb{R}$ (this is the vertical contour through p^*). Then we have the following expansion for the exponent in (2) around $p = p^*$

$$\begin{aligned} -px + \kappa(p) &= -p^* x + V(p^*) - \frac{1}{2!} V'' y^2 - \frac{1}{3!} i\, V''' y^3 + \frac{1}{4!} V'''' y^4 + O(y^5) \\ &+ \Gamma + O(|p - p_+|) \end{aligned}$$

where all derivatives of V are evaluated at $p = p^*$. Define the standardized cumulant γ_n associated with V as follows

$$\gamma_n = V^{(n)}(p^*)/[V''(p^*)]^{\frac{1}{2}n} = O(x^{-\frac{(\frac{1}{2}n-1)\alpha}{\alpha+1}})$$

and let $\sigma^2 = V''(p^*) \to \infty$ as $x \to \infty$. $\gamma_n \to 0$ as $x \to \infty$ for $n \geq 3$ because $\alpha > 0$ [2].

[2] this is why we impose that $\alpha > 0$

Now set $\delta = \delta(x) = x^{\frac{\frac{1}{2}\alpha}{\alpha+1} - q} = O(\frac{1}{\gamma_3} x^{-q})$ for $0 < q < \frac{\frac{1}{2}\alpha}{\alpha+1}$, so $\gamma_3 \delta = O(x^{-q}) \to 0$. Then using the series expansion we have

$$
\begin{aligned}
p_t(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-px+\kappa(p)} dy \\
&= \frac{e^{-p^*x+V(p^*)+\Gamma+O(x^{-\frac{1}{1+\alpha}})+O(\frac{\delta}{\sigma})}}{2\pi} \int_{-\frac{\delta}{\sigma}}^{\frac{\delta}{\sigma}} e^{-\frac{V''}{2!}y^2 - \frac{V'''}{3!}y^3 + \frac{V^{(4)}}{4!}y^4 + \ldots)} dy + I \\
&= \frac{1}{2\pi\sigma} e^{-p^*x+V(p^*)+\Gamma+O(x^{-\frac{1}{1+\alpha}})} \int_{-\delta}^{\delta} e^{-\frac{z^2}{2}(1+\frac{i\gamma_3 z}{3} - \frac{\gamma_4 z^2}{12} + \ldots + O(\gamma_n z^{n-2}))} dz + I \\
&= (.) \int_{-\delta}^{\delta} e^{-\frac{z^2}{2}(1+\frac{i\gamma_3 z}{3} - \frac{\gamma_4 z^2}{12} + \ldots + O(x^{-(n-2)q}))} dz + I \\
&= (.) \int_{-\delta}^{\delta} e^{-\frac{z^2}{2}} [1 - \frac{1}{2}\frac{\gamma_3^2}{36} z^6 + \frac{\gamma_4 z^4}{24} + \ldots + O(x^{-(n-2)q})] dz + I \\
&= \frac{e^{-p^*x+V(p^*)+\Gamma}}{\sqrt{2\pi V''(p^*)}} [1 - \frac{5}{24}\gamma_3^2 + \frac{1}{8}\gamma_4 + O(x^{-\frac{2\alpha \wedge 1}{\alpha+1}}) + O(e^{-\frac{1}{2}\delta^2})] + I
\end{aligned}
$$

where I is the tail integral $I = \frac{1}{2\pi} \int_{|y|>\delta/\sigma} e^{-px+\kappa(p)} dy$ and we have used that $\frac{\delta}{\sigma} = O(x^{-\frac{1}{1+\alpha}})$, and we have ignored odd powers of z. If $\alpha \geq 1$, the error may be larger than γ_3^2 and γ_4. For $t \neq 1$, we set $p \mapsto \frac{p}{t}$.

Dealing with the tail integrals

- Recall that
$$\delta(x) = O(x^r)$$
where $r = \frac{1}{2}\frac{\alpha}{\alpha+1} - q > 0$.
- Proceeding as in [FGGS10], from Assumption 2, we know that if $R = O(x)$ then
$$\left|\int_{p^*+iR}^{p^*+i\infty} e^{-px+\kappa(p)}dp\right| \leq e^{-p^*x}\int_{p^*+iR}^{p^*+i\infty} e^{-\beta|y|^Y}dp = O(e^{-p^*x}e^{-\beta x^Y})$$
which is a higher order term than the inner integral around $y = 0$.
- If $|z| > \delta$ then $|y| > \delta/\sigma$. We can prove that $\mathrm{Re}(-px + V(p))$ is strictly decreasing for $y > 0$ (and increasing for $y < 0$) along the vertical contour $p = p^* + iy$, so we have
$$\begin{aligned}\int_{p^*+\frac{\delta}{\sigma}}^{p^*+R} e^{-px+V(p)}dp &= e^{-p^*x+V(p^*)}\,O(R)O(e^{-\frac{1}{2}V''\frac{\delta^2}{\sigma^2}}) \\ = e^{-p^*x+V(p^*)}\,O(R)O(e^{-\frac{1}{2}\delta^2}) &= e^{-p^*x+V(p^*)}O(x)\,O(e^{-\frac{1}{2}x^{2r}})\end{aligned}$$
which is also a higher order term.

Tail estimates for the CDF and call option prices

- By a similar Fourier argument, we can show that

$$\mathbb{P}(X_t > x) \;=\; \frac{e^{-p^* x + t(V(p^*)+\Gamma)}}{p^*\sigma\sqrt{2\pi t}}\Big[1 + -\frac{5}{24}\gamma_3^2 + \frac{1}{8}\gamma_4 - \frac{1}{p^*\sigma}\Big(\frac{1}{2}\gamma_3 + \frac{1}{\sigma^2 p^*}\Big)(1+o(1))\Big] \qquad (x \to \infty)$$

  The terms in the second line are lower order terms.
- A similar expansion is derived for the cdf of a compound Poisson sum in chapter 7 of Jensen[Jen95].
- If (S_t) is a martingale, we have the following estimate for call options

$$\frac{1}{S_0}\mathbb{E}(S_t - K)^+ = \frac{e^{-(p^*-1)x + t(V(p^*)+\Gamma)}}{(p^{*2} - p^*)\sigma\sqrt{2\pi t}}\Big[1 + \Big(-\frac{5}{24}\gamma_3^2 + \frac{1}{8}\gamma_4\Big) + \Big(\frac{1}{V''^2(p^* - p^{*2})}\Big[\frac{1}{6}(1-2p^*)V''' + \Big(\frac{-1+4p^*-4p^{*2}}{p^{*2}-p^*} + 1\Big)V''\Big] + \frac{1}{3}\frac{(1-2p^*)V'''}{(p^*-p^{*2})V''^2}\Big)(1+o(1))\Big] \qquad (x \to \infty) \tag{3}$$

for $K = S_0 e^x$ where $p^* = p^*(\frac{x}{t})$ (note the change in the exponent).

The Kou model

- The double exponential Kou model has characteristic function

$$\mathbb{E}(\exp(iuX_t)) = \exp[t(ibu - \frac{1}{2}\sigma^2 u^2 + iu\lambda(\frac{p}{\lambda_+ - iu} - \frac{1-p}{\lambda_- + iu}))],$$

so we see that $\alpha = 1$ for this model.

- The Kou model is the Black-Scholes model perturbed by a compound Poisson process, with jump distribution given by a two-sided exponential distribution, and Lévy density equal to

$$\nu(x) = \lambda[p\lambda_+ e^{-\lambda_+ x} 1_{x>0} + (1-p)\lambda_- e^{\lambda_- x} 1_{x<0}].$$

Numerics for the Kou model

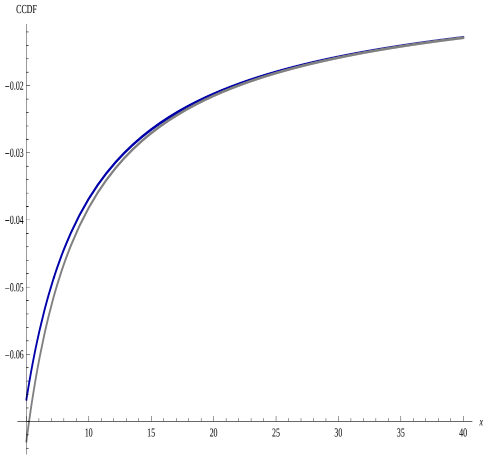

Figure: Here we have plotted TrueCCDF/LeadingOrderTerm - 1 (blue) vs $-\frac{5}{24}\gamma_3^2 + \frac{1}{8}\gamma_4 - \frac{1}{p^*\sigma}(\frac{1}{2}\gamma_3 + \frac{1}{\sigma^2 p^*})$ (grey) for the Kou model with $\sigma = 1, \lambda = \lambda_\pm = 5, p = \frac{1}{2}, t = 1$, computed numerically using an inverse Fourier transform along the horizontal contour going through the saddlepoint.

Translating call option prices into implied volatility

- From Eq 11 in Gulisashvili[Gul10a], we have the following large-strike behaviour for the dimensionless Black-Scholes implied volatility V at log-moneyness k

$$|G_-(k, L - \frac{1}{2}\log L) - V| = O(L^{-\frac{1}{2}}) \qquad (k \to \infty) \qquad (4)$$

where $L = \log \frac{1}{C(k)}$ and $G_-(k,u) = \sqrt{2}\,[\sqrt{u+k} - \sqrt{k}]$.

- Corollary 6.3 in Gao&Lee[GL11] sharpens the Gulisashvili estimate to the following

$$|G_-(k, L - \log \frac{\sqrt{4\pi L}}{1-(k+L)^{-\frac{1}{2}}}) - V| = O(\frac{\log L}{L^{\frac{3}{2}}}) \qquad (k \to \infty)\,.$$

Large-strike smile for the Kou model

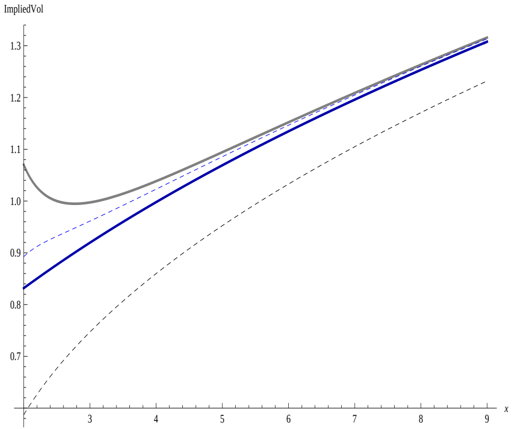

Figure: Here we have plotted the large-strike implied volatility smile obtained numerically (navy blue), the Gulisashvili approximation (black dashed), the Gao-Lee approximation using just the leading order term (grey) and the Gao-Lee approximation using the subleading order term as well (blue dashed) for the Kou model with $\sigma = 1, \lambda = \lambda_{\pm} = 5, p = \frac{1}{2}, t = 1$.

The CGMY model

The standard CGMY process is a pure-jump Lévy process with

$$\nu(dx) = \left(\frac{Ce^{-G|x|}}{|x|^{1+Y}} 1_{\{x<0\}} + \frac{Ce^{-Mx}}{x^{1+Y}} 1_{\{x>0\}} \right) dx, \qquad (5)$$

for $C, G, M \geq 0$, $0 \leq Y < 2$, and has finite variation if $Y < 1$. If $Y < 0$, then X_t is a compound Poisson process. The characteristic exponent is

$$\psi(u) = C\Gamma(-Y)\left\{(M - iu)^Y + (G + iu)^Y - M^Y - G^Y\right\} + ibu. \qquad (6)$$

for $Y \neq 1$. For $S = e^{X_t}$ to be a martingale, we must have

$$b := -C\Gamma(-Y)\left\{(M-1)^Y + (G+1)^Y - M^Y - G^Y\right\},$$

so that $V(0) = V(1) = 0$. We first note that $p_+ = M$, $p_- = -G$ and $V'(p_\pm) \to \pm\infty$ as $p \to p_\pm$ if $Y \in (0,1)$. For the CGMY model, $\alpha = -Y$, so PropN 1 only applies if $Y < 0$. However, from Theorem 7.4 in Albin&Sunden[AS09], we have the following tail behaviour for X_t for $Y > 0$:

$$\mathbb{P}(X_t > x) \sim \frac{Ct}{M} e^{tV(M)} \frac{e^{-Mx}}{x^{1+Y}} \qquad (x \to \infty).$$

where here $V(p)$ is the true cgf.

For $M > 1$, the *Share measure* $\mathbb{P}^*(A) = \mathbb{E}(e^{X_t} 1_A)$ is well defined and under this condition, X is also a CGMY process under $\mathbb{P}^*$ with $C^* = C$, $Y^* = Y$, $M^* = M - 1$, $G^* = G + 1$ and a modified drift b^*. Then we have the following tail behaviour for call options of strike $K = S_0 e^k$

$$\begin{aligned}
C(k) &= \frac{1}{S_0}\mathbb{E}(S_t - K)^+ \\
&= \mathbb{P}^*(S_t > K) - e^k \mathbb{P}(S_t > K) \\
&\sim \frac{Ct}{M-1}\, e^{tV^*(M-1)}\, \frac{e^{-(M-1)k}}{k^{1+Y}} - \frac{Ct}{M}\, e^{tV(M)}\, \frac{e^{-(M-1)k}}{k^{1+Y}} \\
&= Ct\,[\frac{1}{M-1}\, e^{tV^*(M-1)} - \frac{1}{M}\, e^{tV(M)}]\, \frac{e^{-(M-1)k}}{k^{1+Y}} \qquad (x \to \infty)\,.
\end{aligned}$$

Large-strike smile for the CGMY model

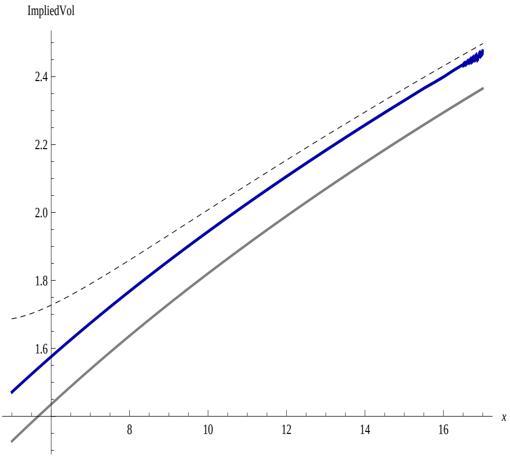

Figure: Here we have plotted the large-strike implied volatility squared obtained numerically using an inverse Fourier tranform (blue) verses the Gulisashvili approximation (grey) and the Gao-Lee refined approximation (dashed) for the CGMY model with $C = 1$, $Y = .5$, $M = 2$, $G = 1$, $b = 0$, $t = 1$.

References

beamericonarticle J.M.P.Albin and M.Sunden, "On the asymptotic behaviour of Levy processes, Part I: Subexponential and exponential processes", *Stochastic Processes and their Applications*, 119 (2009) 281304.

beamericonarticle Benaim, S., P.Friz, "Smile Asymptotics II: Models with Known MGFs", *J. Appl. Probab.*, 45 (2008), 16-32.

beamericonarticle P.Friz, S.Gerhold, A.Gulisashvili, S.Sturm, "On refined volatility smile expansion in the Heston model", forthcoming in *Quantitative Finance*.

beamericonarticle Gao, K. and R.Lee, (2011), "Asymptotics of Implied Volatility in Extreme Regimes", Preprint.

beamericonarticle Gulisashvili, A., "Asymptotic Formulas with Error Estimates for Call Pricing Functions and the Implied Volatility at Extreme Strikes", *SIAM J. Financial Math.*, Vol. 1, pp. 609-641, 2010.

beamericonarticle Gulisashvili, A., "Two-sided estimates for stock price distribution densities in jump-diffusion models", working paper.

beamericonarticle Jensen, J., 1995. Saddlepoint Approximations. Oxford University Press, Oxford, UK.