Applications of large deviations in finance and mathematical physics.

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Outline

- Motivation and definition of the large deviation principle (LDP).
- Examples - Brownian motion, Cramér's theorem, Lévy processes, Sanov's theorem.
- The Brownian sheet.
- Saddlepoint methods; the Feynman path integral.
- The Donsker-Varadhan LDP for the occupation measure of the Ornstein Uhlenbeck process $dY_t = -\theta Y_t dt + dW_t$ for $\theta > 0$.
- Applications to stochastic volatility models - the Ornstein-Uhlenbeck and CEV-Heston models.
- Large deviations for the maximum likelihood estimator of θ.
- Application to SPDEs - Freidlin-Wentzell theory for the stochastic heat equation.

The Large deviation principle (LDP): motivation

- Suppose we have sequence of random variables (X_n) such that X_n is concentrated around x_0 as $n \to \infty$, and for sets A away from x_0, $\mathbb{P}(X_n \in A)$ tends to zero exponentially rapidly in n:

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \in A) \;=\; -I(A)$$

$$\text{i.e.} \quad \forall \delta > 0, \quad e^{-n(I(A)+\delta)} \;\leq\; \mathbb{P}(X_n \in A) \;\leq\; e^{-n(I(A)-\delta)}$$

  for $n = n(\delta)$ sufficiently large, and some *rate function* $I \geq 0$.

- **Example**: for standard Brownian motion (W_t), $W_t \to 0$ a.s. as $t \to 0$ and (by SLLN) $\frac{W_t}{t} \to 0$ a.s. as $t \to \infty$, but

$$\begin{aligned} \lim_{t\to 0} t \log \mathbb{P}(W_t > x) &= -\frac{1}{2}x^2\,, \\ \lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(\frac{W_t}{t} > x) &= -\frac{1}{2}x^2 \end{aligned}$$

  for $x > 0$.

The Large deviation principle (LDP): definition

Definition. A sequence of random variables (X_n) in a topological space S satisfies the LDP with non-negative lower semicontinuous rate function I if we have the following exponential upper/lower bounds for $A \in \mathcal{B}(S)$:

$$\begin{aligned} -\inf_{x \in A^\circ} I(x) \quad &\leq \quad \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in A) \\ &\leq \quad \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in A) \leq -\inf_{x \in \bar{A}} I(x)\,. \end{aligned}$$

Definition. X_n is said to satisfy the weak LDP if

$$\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in B_\delta) = -I(x)\,.$$

Examples

- **Cramér's theorem**. Let (X_i) be an i.i.d. sequence of random variables with finite mean $\mathbb{E}(X_1) < \infty$ and cumulant generating function

$$V(p) = \log \mathbb{E}(e^{pX_1})\,.$$

  Then $\bar{S}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ satisfies the LDP with rate function equal to the *Fenchel-Legendre transform* $V^*(x) = \sup_{p\in\mathbb{R}}\{px - V(p)\}$. For Brownian motion, $V(p) = \frac{1}{2}p^2$, $V^*(x) = \frac{1}{2}x^2$.

- A **Lévy process** (X_t) has i.i.d. increments, so $(\frac{X_t}{t})$ satisfies an LDP as $t \to \infty$ with rate function $V^*(x)$.

Sketch proof of Cramér's theorem

- Cramér upper bound proved using a simple Chebychev argument:

$$\mathbb{P}(\bar{S}_n \geq x) = \mathbb{E}(1_{\{S_n \geq nx\}}) \leq \mathbb{E}(e^{-\theta nx}\, e^{\theta S_n}) = e^{-n\theta x} e^{nV(\theta)}\,.$$

  We then tighten the bound by taking the inf over θ on the right hand side:

$$\mathbb{P}(\bar{S}_n \geq x) \leq e^{-n \sup_\theta [\theta x - V(\theta)]} = e^{-nV^*(x)}\,.$$

- Lower bound is obtained by changing to a different measure $\mathbb{P}_{\theta^*(x)}$ under which $\{\bar{S}_n \geq x\}$ is no longer a rare, large deviation event.

Sanov's theorem

- Let (X_i) be a sequence of n i.i.d. random variables in $\mathbb{R}$ with common probability measure μ. The sample distribution:

$$L^n = \frac{1}{n}\sum_{i=1}^{n} \delta_{X_i}$$

is a random probability measure (a.k.a. the *empirical measure*).
- Let $P_n = \mu^n \circ (L^n)^{-1}$ denote the distribution of L^n, where μ^n is the product measure. P_n is a probability measure on $(\mathcal{P}(\mathbb{R}), \mathcal{B}(\mathcal{P}(\mathbb{R}))$ - i.e. $P_n \in \mathcal{P}(\mathcal{P}(\mathbb{R}))$. By SLLN, we can show that $P_n \stackrel{w}{\to} \delta_\mu$.
- **Theorem** (Sanov). (L^n) satisfies an LDP in the topology of weak convergence[1] as $n \to \infty$ with rate function given by the infinite dimensional counterpart of $V^*(x)$:

$$R(\nu|\mu) = \sup_{p \in B(\mathbb{R})} \left[\int p d\nu - \log \int e^p d\mu \right],$$

where $B(\mathbb{R})$ is the space of bounded, measurable functions on $\mathbb{R}$. (see [Var10]), so $\mathbb{P}(L^n \in A) \approx e^{-t \inf_{\nu \in A} R(\nu|\mu)}$ for $\mu \notin A$.

[1] $A \subseteq \mathcal{P}(\mathbb{R})$ is closed iff for any $(\mu_n) \in A$ with $\mu_n \stackrel{w}{\to} \mu \in \mathcal{P}(\mathbb{R})$, we have $\mu \in A$.

By solving the variational problem on the previous slide, we can show that the rate function simplifies to

$$R(\nu|\mu) = \begin{cases} \int_{-\infty}^{\infty} (\log \frac{d\nu}{d\mu})\, d\nu = \int_{-\infty}^{\infty} \frac{d\nu}{d\mu} (\log \frac{d\nu}{d\mu})\, d\mu & \text{if } (\nu \ll \mu)\,, \\ \infty & \text{otherwise} \end{cases}$$

The Brownian sheet

- Let (Z_t) be the *Brownian sheet*, i.e. the centred Gaussian process on $[0,1]^2$ with zero mean and covariance structure

$$\mathbb{E}(Z_t Z_s) = (s_1 \wedge t_1)(s_2 \wedge t_2)$$

where $t = (t_1, t_2)$, $s = (s_1, s_2)$. Z is a "two-parameter" Brownian motion.

- Then $\sqrt{\epsilon}\, Z$ satisfies the LDP on $C_0([0,1]^2)$ with rate function

$$I(f) \quad = \quad \begin{cases} \frac{1}{2}\int_{[0,1]^2} (\frac{\partial^2 f}{\partial s \partial t})^2 \, dsdt & \text{if } \frac{\partial^2 f}{\partial s \partial t} \in L^2 \,, \\ +\infty & \text{otherwise} \,. \end{cases}$$

The LDP gives crude exponential bounds. For a Lévy process (X_t) with density $p_t(x)$, we can sharpen these bounds using saddlepoint methods, proved using contour integration:

- **Large-time estimate**

$$p_t(xt) \quad \sim \quad \frac{e^{-t(p^* x - V(p^*))}}{\sqrt{2\pi t V''(p^*)}} = \frac{e^{-tV^*(x)}}{\sqrt{2\pi t V''(p^*)}} \qquad (t \to \infty)$$

  (see F-López,Forde&Jacquier[FLFJ11]).

- **Tail estimate**

$$p_t(x) \quad \sim \quad \frac{e^{-p^*(\frac{x}{t})x + tV(p^*(\frac{x}{t}))}}{\sqrt{2\pi t V''(p^*(\frac{x}{t}))}} \qquad (x \to \infty).$$

  (see F-López,Forde[FLF11]). $p^* = p^*(x)$ is the unique solution to the saddlepoint equation $V'(p^*) = x$.

- Similar saddlepoint estimates can be obtained for the well known **Heston** stochastic volatility model for large-time[FJ09],[FJM10], small-time[FJL10] and tail regimes (see Friz et al.[FGGS10]).

Saddlepoint methods in infinite dimensions - the Feynman path integral

- Consider the Feynman path integral for a **wavefunction** $\psi(x,t)$:

$$\begin{aligned}\psi(x,t) &= (2\pi i)^{-\frac{n}{2}} \int_{\gamma:\gamma_t=x} e^{\frac{i}{\hbar}[\frac{1}{2}m\int_0^t \dot{\gamma}^2 d\tau - \int_0^t V(\gamma_\tau)d\tau]}\,\psi(\gamma_0,0)\,\mathcal{D}\gamma \\ &= (2\pi i)^{-\frac{n}{2}} \int_{\mathcal{H}} e^{-\frac{i}{2}\frac{m}{\hbar}\int_0^t \dot{\gamma}^2 d\tau} d\mu(\gamma)\end{aligned}$$

  for $x \in \mathbb{R}^n$, with $\psi(x,0) = e^{\frac{i}{\hbar}f(y)}\chi(y)$.
- The first line is the formal expression for the path integral which we define rigorously via the **Fresnel integral** in the second line over $\mathcal{H} = \{\gamma \in C[0,t] : \dot{\gamma} \in L^2[0,t], \gamma_t = x\}$ for $V, \psi(.,0) \in \mathcal{F}(\mathcal{H})$.
- $f(\gamma) = e^{-\frac{i}{\hbar}\int_0^t V(\gamma_\tau)d\tau}\psi(\gamma_0,0)$ is the Fourier transform:

$$f(\gamma) = \int_{\mathcal{H}} e^{(\gamma,\gamma_1)} d\mu(\gamma_1)$$

  of $\mu \in \mathcal{M}(\mathcal{H})$ with B.V., where $(\gamma,\gamma_1) = \frac{m}{\hbar}\int_0^1 \dot{\gamma}\dot{\gamma_2}d\tau$, [AHM08][2].
- $\psi(x,t)$ satisfies the **Schrödinger eq**: $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi$.

[2] Feynman integral can also be defined via analytic continuation of Wiener measure.

Letting $\hbar \to 0$ - the semi-classical expansion

- The integrand $e^{\frac{i}{\hbar}S_t} = e^{\frac{i}{\hbar}[\frac{1}{2}m\int_0^t \dot{\gamma}^2 d\tau - \int_0^t V(\gamma_\tau)d\tau]}$ is an infinite-dimensional oscillatory integral. If we let $\hbar \to 0$, we tend towards classical everyday Newtonian mechanics and the integral becomes highly oscillatory, so we expect the main contribution to come from the classical path γ^* which make S_t stationary (in analogy with the finite-dimensional *method of stationary phase*).

- From this we can compute the *semi-classical expansion*:

$$\psi(x,t) \sim (2\pi i)^{-\frac{n}{2}} \frac{1}{\sqrt{\det(\ldots)}} e^{\frac{i}{\hbar}[\frac{1}{2}m\int_0^t (\dot{\gamma}^*)^2 d\tau - \int_0^t V(\gamma_\tau^*)d\tau]} \chi(y)$$

  as $\hbar \to 0$ (see [AHM08]).

- The stationary path γ^* is just the classical path $m\ddot{\gamma} = -\nabla V$ followed by a particle moving under the potential $V(x)$, which goes from y to x in time t with initial momentum $f'(y)$ (IF there is a unique non-degenerate stationary path γ^* with this property).

The Donsker-Varadhan LDP for the occupation measure of the Ornstein Uhlenbeck process

Let $dY_t = -\theta Y_t dt + dW_t$ be an OU process for $\theta > 0$. Let

$$\mu_t(A) = \frac{1}{t}\int_0^t 1_A(Y_s)ds$$

denote the proportion of time that Y spends in A, for $A \in \mathcal{B}(\mathbb{R})$. For each $t > 0$ and ω, $\mu_t(\omega, .) \in \mathcal{P}(\mathbb{R})$. Then from [DV76] (or [Str84]) $\mu_t(.)$ satisfies the LDP as $t \to \infty$ in the topology of weak convergence, with a good[3], convex, lower semicontinuous rate function given by:

$$I_B(\mu) = -\inf_{u\in\mathcal{D}^+}\int_{-\infty}^{\infty}\frac{Lu}{u}\,d\mu$$

where $L = -\theta y\frac{d}{dy} + \frac{1}{2}\frac{d^2}{dy^2}$ is the infinitesimal generator for Y and $\mathcal{D}^+$ is the set of u in the domain $\mathcal{D}$ of L with $u > \epsilon$ for some $\epsilon > 0$.

[3] good means that the level set $\{x : I(x) \leq \alpha\}$ is compact.

Simplifying the rate function

- We can simplify I_B to the following:

$$I_B(\mu) = \frac{1}{2}\int_{-\infty}^{\infty} |\partial_y \sqrt{(}\frac{d\mu}{d\mu_\infty})(y)|^2 \, \mu_\infty(dy)$$

  for $\mu \ll \mu_\infty$, where $\mu_\infty(y) = (\frac{\theta}{\pi})^{\frac{1}{2}} \, e^{-\theta y^2}$ is the unique stationary distribution for Y, i.e. $N(0, 1/(2\theta))$. $\frac{d\mu}{d\mu_\infty}$ is the Radon-Nikodým derivative. If μ is not absolutely cts wrt μ_∞, then $I_B(\mu) = \infty$.

- $\mathcal{P}(\mathbb{R})$ can be made into a (non-compact) metric space using the **Prokhorov metric**.

- $I_B(\mu)$ clearly attains its minimum value of zero at $\mu = \mu_\infty$, and we can show that μ_∞ is the unique minimizer of $I_B(\mu)$

An uncorrelated Stochastic volatility model

- Consider a stochastic volatility model for a log stock price process $X_t = \log S_t$:

$$\begin{cases} dX_t = -\frac{1}{2}\sigma^2(Y_t)dt + \sigma(Y_t)dW_t^1 \,, \\ dY_t = -\theta Y_t dt + dW_t^2 \end{cases} \tag{1}$$

  for $\theta > 0$, where $f(y) = \sigma^2(y)$ is a continuous non-decreasing function with $0 < f_{\min} \leq f(y) \leq f_{\max}$ and $d\langle W_1, W_2 \rangle = 0$ with $x_0 = 0$.

- The distribution of X_t, *conditional* on $A_t = \frac{1}{t}\int_0^t \sigma^2(Y_s)ds$, is $N(-\frac{1}{2}A_t t, A_t t)$.

Using the contraction principle

- Let $F(\mu)=\int_{-\infty}^{\infty} f(y)\mu(dy)$ for $\mu\in\mathcal{P}(\mathbb{R})$. Then we can re-write A_t as
$$A_t=F(\mu_t)=\int_{-\infty}^{\infty} f(y)\mu_t(dy)=\frac{1}{t}\int_0^t f(Y_s)ds\,.$$

- $F:\mathcal{P}(\mathbb{R})\mapsto[f_{\min},f_{\max}]$ is a bounded, continuous functional [4], because if $\mu_n\stackrel{w}{\to}\mu$ then $\int f(y)\mu^n(dy)\to\int f(y)\mu(dy)$, because $f\in C_b$.

- Thus, by the *contraction principle* from large deviations theory, A_t also satisfies the LDP, with rate function
$$I_f(a)=\inf_{\mu\in\mathcal{P}(\mathbb{R})\,:\,F(\mu)=a} I_B(\mu)\quad,\quad a\in[f_{\min},f_{\max}]\,. \qquad (2)$$

[4] in the topology of weak convergence.

A joint LDP for $(X_t/t, A_t)$

Proposition [Forde11a]. $(X_t/t, A_t)$ satisfies a LDP on $\mathbb{R} \times [f_{\min}, f_{\max}]$ as $t \to \infty$ with rate function

$$I(x, a) = aV^*(\frac{x}{a}) + I_f(a)$$

where $V^*(x) = \frac{1}{2}(x + \frac{1}{2})^2$.

Sketch proof. Let $Z_t = X_t/t$. We first note that $(Z_t, A_t) \stackrel{d}{=} (\frac{1}{t} W_{tA_t} - \frac{1}{2}A_t, A_t)$. Conditioning on A_t, formally we have

$$\mathbb{P}(|Z_t - x| < \delta, |A_t - a| < \delta) \;\approx\; (.) \times \; e^{-aV^*(\frac{y}{a})/t} \, e^{-I_f(a)/t}$$

as $t \to \infty$, where $aV^*(\frac{x}{a})$ is the rate function of $W_{ta} - \frac{1}{2}a$, for a *fixed*. This argument can be made rigorous.

- **Corollary** [Forde11a]. (X_t/t) satisfies the LDP as $t \to \infty$ with a good rate function given by

$$I(x) = \inf_{a \in [f_{\min}, f_{\max}]} \left\{ \frac{(x + \frac{1}{2}a)^2}{2a} + I_f(a) \right\} \leq \frac{(x + \frac{1}{2}\bar{\sigma}^2)^2}{2\bar{\sigma}^2} \quad (3)$$

  Proof The LDP with a good rate function just follows from the contraction principle.

- This can be applied to price call options with value $\mathbb{E}(e^{X_t} - K)^+$.

- We can relax the assumption that σ is bounded to a **sublinear** growth condition $\sigma(y) \leq A(1 + |y|^p)$, $A > 0, p \in (0, 1)$; in this case we take the infimum over all $a \in (0, \infty)$ in (3) (see [Forde11b]).

- The LDP can be also be extended to a Lévy process or a CEV process evaluated the OU time-change $\int_0^t f(Y_s)ds$.

For the case of sublinear growth, the following lemma is the key observation:

Lemma. If $I_B(\mu) \leq \alpha$ and $k \in (0,1)$, we have

$$\int_{-\infty}^{\infty} y^2 \mu(dy) \leq \frac{\alpha + k}{2k(1-k)} .$$

Proof. If we consider the test function $u = e^{ky^2}$ in $I_B(\mu) = -\inf_{u \in \mathcal{D}^+} \int_{-\infty}^{\infty} \frac{Lu}{u}\, d\mu$, then $-\frac{Lu}{u}(y) = k\,[2(1-k)y^2 - 1]$. From this we obtain

$$\begin{aligned} \alpha \geq I_B(\mu) = -\inf_{u \in \mathcal{D}^+} \int_{-\infty}^{\infty} \frac{Lu}{u}\, d\mu \quad &= \quad \sup_{u \in \mathcal{D}^+} -\int_{-\infty}^{\infty} \frac{Lu}{u}\, d\mu \\ &\geq \quad \int_{\infty}^{\infty} k\,[2(1-k)y^2 - 1]\mu(dy) \\ &= \quad 2k(1-k) \int_{-\infty}^{\infty} y^2 \mu(dy) - k\,. \end{aligned}$$

The CEV model

- The CEV model is defined by the SDE
$$dS_t = \delta S_t^{\beta}\, dW_t \tag{4}$$
with $\beta \in (0,1)$, $\delta > 0$ and $S = 0$ absorbing so (S_t) is a martingale.

- The transition density is
$$p(t, S_0, S) = \frac{S^{-2\bar{\beta}-\frac{3}{2}} S_0^{\frac{1}{2}}}{\delta^2 |\bar{\beta}| t} \exp\Big(-\frac{S_0^{-2\bar{\beta}} + S^{-2\bar{\beta}}}{2\delta^2 \bar{\beta}^2 t}\Big) I_\nu\Big(\frac{S_0^{-\bar{\beta}} S^{-\bar{\beta}}}{\delta^2 \bar{\beta}^2 t}\Big) \qquad (S > 0), \tag{5}$$
where $\bar{\beta} = \beta - 1$, $\nu = \frac{1}{2|\bar{\beta}}|$, and $I_\nu(.)$ is the modified Bessel function of the first kind (see [DavLin01]).

- **Proposition**. Let $\gamma = 1/|\bar{\beta}|$. Then using (5) we can show that (S_t/t^γ) satisfies the LDP on $[0,\infty)$ as $t \to \infty$ with continuous rate function
$$I_{\mathrm{CEV}}(K) = \frac{K^{2|\bar{\beta}|}}{2\delta^2 \bar{\beta}^2} \qquad (K \geq 0)\,.$$

The CEV-Heston model

- Combining the CEV model with a CIR time-change, we can define the uncorrelated *CEV-Heston model*, governed by the following SDEs

$$\begin{cases} dS_t = S_t^\beta \sqrt{Y_t} dW_t^1 \,, \\ dY_t = \kappa(\theta - Y_t)dt + \sigma\sqrt{Y_t} dW_t^2 \end{cases}$$

with $dW_t^1 dW_t^2 = 0,\ Y_0 = y_0 > 0$.

- Conditioning on $\int_0^t Y_s ds$, we can write $S_t = X_{\int_0^t Y_s ds}$, where X is now just the standard CEV process $dX_t = \delta X_t^\beta\, dW_t$ with $\delta = 1$.

- By a similar argument to that used for the OU model, we have: **Proposition**. (S_t/t^γ) satisfies the LDP on $[0,\infty)$ as $t \to \infty$ with a good rate function given by

$$I_{\mathrm{CEVH}}(K) \;=\; \inf_{a\in(0,\infty)} [aI_{\mathrm{CEV}}(\frac{K}{a^\gamma}) + I_{\mathrm{CIR}}(a)] \;\leq\; \theta I_{\mathrm{CEV}}(\frac{K}{\theta^\gamma}) \qquad (K \geq 0)\,,$$

where $I_{\mathrm{CIR}}(a)$ is the rate function of $A_t = \frac{1}{t}\int_0^t Y_s ds$, and the infimum of I is attained uniquely at $K=0$, where $I(K)=0$.

Call options

- We can show that call options have the same large-time behaviour

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}(S_t - Kt^{\gamma})^+ = I_{\mathrm{CEVH}}(K)\,.$$

- For the large-time, fixed-strike regime, we can show that

$$S_0 - \mathbb{E}(S_t - K)^+ \;=\; cK\,(\theta t)^{-\frac{\gamma}{2}}(1+o(1)) \qquad (t\to\infty)$$

where $c = \frac{1}{\Gamma(1+\frac{\gamma}{2})}[\frac{1}{2}(\frac{S_0^{-2\bar{\beta}}}{\delta^2\bar{\beta}^2})]^{\frac{\gamma}{2}}$.

The large-maturity smile for the CEV-Heston model

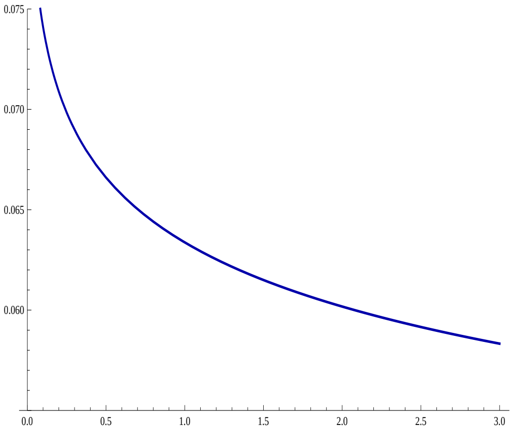

Figure: Here we have plotted the implied volatility for the CEV-Heston model in the large-time, large-strike regime for $t = 30$ years using Corollary 7.1 in Gao&Lee[GL11], with $\delta = 1, \beta = .7, S_0 = 1$ and $\kappa = 1.15, \theta = .04, \sigma = 0.2$. Working in the large-time, large-strike parameterizaton allows us to see the slope and the *convexity* effect.

The Maximum likelihood estimator of θ for the OU process

- Let θ_0 denote the true value of θ.
- Let $\mathbb{P}_\theta^T$ be the measure induced on $(\mathcal{C}[0,T], \mathcal{B}(\mathcal{C}[0,T]))$ by the solution of $dY_t = -\theta Y_t dt + dW_t$. Then, from Girsanov's theorem, we have the likelihood ratio

$$L(\theta) = \frac{d\mathbb{P}_\theta^T}{d\mathbb{P}_0^T} = e^{-\int_0^T \theta Y_t dY_t - \frac{1}{2}\int_0^T \theta^2 Y_t^2 dt} \tag{6}$$

(note that $\mathbb{P}_0^T$ is just the Wiener measure).
- Taking the log of $L(\theta)$, differentiating wrt θ and setting to zero, we obtain the classical maximum likelihood estimator for θ:

$$\hat{\theta}_T = -\frac{\int_0^T Y_t dY_t}{\int_0^t Y_t^2 dt},$$

(see [Kut04]) and $\hat{\theta}_T$ is a consistent estimator of θ_0 (i.e. $\hat{\theta}_T \to \theta_0$ in probability as $T \to \infty$).

It can be shown (see [FLP99]) that $\hat{\theta}_T$ satisfies the LDP with good rate function

$$J(\theta) = \begin{cases} \frac{1}{4\theta}(\theta-\theta_0)^2 & (\theta \geq \frac{1}{3}\theta_0)\,, \\ -2\theta+\theta_0 & (\theta < \frac{1}{3}\theta_0)\,. \end{cases}$$

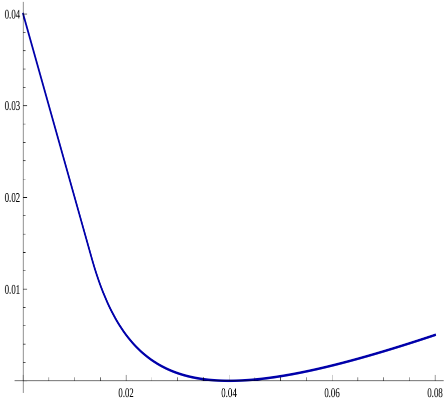

Figure: Here we have plotted $J(\theta)$ for $\theta = .04$.

SPDEs

- Consider the *stochastic* heat equation with small-noise:

$$\partial_t u^{\epsilon}_{t,x} = \frac{1}{2}\partial_{xx} u^{\epsilon}_{t,x} + \sqrt{\epsilon}\,\dot{W}_{t,x} \tag{7}$$

on $[0,T] \times [0,1]$, with Dirichlet boundary condition $u^{\epsilon}_{0,\cdot} \in C^{2\alpha}$, $0 \leq \alpha < \frac{1}{4}$ and $u^{\epsilon}_{t,0} = u^{\epsilon}_{t,1} = 1$. $\dot{W}$ is *space-time white noise*, which is a Gaussian random set function such that $W_A \sim N(0, \mathrm{Leb}(A))$ for $A \in \mathcal{B}([0,T] \times [0,1])$ and $\mathbb{E}(W_A W_B) = \mathrm{Leb}(A \cap B)$.

- $W_{t,x} := W_{[0,t]\times[0,x]}$ is the previously defined Brownian sheet.

- We can give a rigorous meaning to (7) by writing the solution in the integrated form

$$u^{\epsilon}_{t,x} = \int_0^1 G_t(x,y)u_0(y)dy + \sqrt{\epsilon}\int_0^t \int_0^1 G_{t-s}(x,y)W(ds,dy)$$

where the stochastic integral on the right is defined in a similar way to the classical Itō integral, and $G_t(x,y)$ is the usual Green kernel for the non-stochastic heat eq $\partial_t u = \frac{1}{2}\partial_{xx} u$ with the same Dirichlet boundary condition (see Pardoux[Par93]).

Large deviations for the stochastic heat equation

- The *skeleton* of $h = h(t, x)$ in the Cameron-Martin space for W is given by

$$Z^h_{t,x} = \int_0^1 G_t(x, y)u_0(y)dy + \sqrt{\epsilon}\int_0^t \int_0^1 G_{t-s}(x, y)\frac{\partial^2 h}{\partial t \partial x}(s, y)dsdy .$$

- By a generalized contraction principle, u^ϵ satisfies the LDP on $\chi = C^{\alpha,0}([0, T] \times [0, 1])$ with rate function

$$S(f) = \begin{cases} \inf\{I(h) : Z^h = f\} \ , & f \in \mathrm{Im}(Z) \\ +\infty & (otherwise) \end{cases} \tag{8}$$

(see [CM07]), where $I(h) = \frac{1}{2}\int_{[0,1]^2}(\frac{\partial^2 h}{\partial s \partial t})^2 \, dsdt$ is the previously defined rate function for the Brownian sheet.

- We can also compute a small-noise LDP for the alternative way of approaching SPDEs as a Hilbert-space valued SDE driven by a Hilbert-spaced valued Brownian motion (see [DPZ92]).

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