Robust hedging of forward starting options

We now let $\mathcal{M}(\mu,\nu)$ denote the collection of **joint distributions** ρ for X and Y such that $X \sim \mu$, $Y \sim \nu$ and $\mathbb{E}(Y|X) = X$. Then we can consider the problem

$$P \quad := \quad \min_{\rho \in \mathcal{M}(\mu,\nu)} \mathbb{E}^{\rho}(|X-Y|) \quad = \quad \min_{\rho \in \mathcal{M}(\mu,\nu)} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |y-x|\rho(dx,dy) \tag{1}$$

where $\mathbb{E}^{\rho}(.)$ means compute the expectation under the assumption that X and Y have joint distribution ρ.

The explicit Hobson-Klimmek solution

Assume μ and ν both have densities, and $\mu(x) \geq \nu(x)$ for $x \in E = [a,b]$ and $\mu(x) \leq \nu(x)$ otherwise, and assume $\nu(x) > 0$ if $x \in [\ell, r]$, and zero otherwise, with $[a,b] \subset [\ell, r]$. For the minimization problem in (1), the minimizing "model" can then be characterized as follows:

- Let $X \sim \mu$ (which we can simulate by setting $X = F_X^{-1}(U)$ where $U \sim U[0,1]$ if X has a density, see first FM02 lecture). Then set $Y = X$ with probability $\frac{\nu(X)}{\mu(X)}$ if $X \in E$, or probability 1 if $x \notin E$.
- If $X \in E$ and $Y \neq X$, then there are only two possibilities: $Y = q(X) > X$ or $Y = p(X) < X$ for two decreasing functions q and p such that $q : E \to [b, r)$ and $p : E \to [\ell, a]$ (see first graph below). $\mathbb{P}(Y = q(X)|X \neq Y) = \mathfrak{q}(X)$, where $\mathfrak{q}(x)$ is uniquely determined by the martingale condition:

  $$\mathbb{E}(Y|X=x) \quad = \quad x \quad = \quad \mathfrak{q}(x)q(x) \,+\, (1-\mathfrak{q}(x))p(x)\,.$$

  We can use a **Bernoulli** random variable Z_1 with success rate $\frac{\nu(X)}{\mu(X)}$ to decide whether $X = Y$ (or not), and if $X \neq Y$ we can use an additional independent Bernoulli random variable Z_2 with success rate $\mathfrak{q}$ to determine whether $Y = q(X)$ or $p(X)$. Z_1 and Z_2 can in turned by simulated using a $U[0,1]$ random variable (why?)
- Thus we see the optimal model is a **trinomial tree** from time T_1 to T_2.
- q and p can be computed as the explicit solution to the pair of **coupled ODEs**:

  $$q'(x) \quad = \quad \frac{(p(x)-x)\mu(x)}{(q(x)-p(x))\nu(q(x))} \quad , \quad p'(x) \quad = \quad \frac{(x-q(x))\mu(x)}{(q(x)-p(x))\nu(q(x))}$$

  for $x \in E$ with $p(a) = a$ and $q(a) = r$ (see first plot below), which can be easily solved numerically using an Euler scheme (note one can incur numerical problems if $\nu(q(x)) = 0$ which may require adjustments to the code).
- To verify that $Y \sim \nu$, note that for $y > b$

  $$\begin{aligned}
  \mathbb{P}(Y \geq y) &= \int_a^{q^{-1}(y)} \frac{x-p(x)}{q(x)-p(x)}\mu(x)(1-\frac{\nu(x)}{\mu(x)})dx \\
  &= \int_a^{q^{-1}(y)} q'(x)(\mu(q(x)) - \nu(q(x)))dz \\
  &= (F_\mu((q(x)) - F_\nu(q(x)))|_a^{q^{-1}(y)} \\
  &= F_\mu(y) - F_\nu(y) - (F_\mu(r) - F_\nu(r)) \quad = \quad 1 - F_\nu(y)
  \end{aligned} \tag{2}$$

  where F_ν is the distribution function of ν as required. We can perform a similar computation for $y < a$ and $y \in [a,b]$.
- We can compute the minimal price in terms of p, q, μ and ν as

  $$\begin{aligned}
  \mathbb{E}(|Y-X|) &= \int_a^b ((q(x)-x)\frac{x-p(x)}{q(x)-p(x)} + (x-p(x))\frac{q(x)-x}{q(x)-p(x)})(1-\frac{\nu(x)}{\mu(x)})\mu(x)dx \\
  &= 2\int_a^b \frac{(x-p(x))(q(x)-x)}{q(x)-p(x)}(\mu(x)-\nu(x))dx\,.
  \end{aligned} \tag{3}$$

- For the example in Figure 1, these simplify to $q'(x) = 2\frac{p(x)-x}{q(x)-p(x)}$, $p'(x) = 2\frac{x-q(x)}{q(x)-p(x)}$ for $x \in [-1,1]$, which can be solved explicitly.

The Hobson-Neuberger solution

For the maximization problem, the optimal solution is such that $X = q(X)$ or $Y = p(X)$ where q and p are both **increasing functions** with $q(x) > x$ and $p(x) < x$ (see second plot below). This is a **binomial** tree model from T_1 to T_2. For the special case when $X \sim U[-1,1]$ and $Y \sim U[-2,2]$, $q(x) = x+1$ and $p(x) = x-1$. See second plot below for a more interesting numerical example for this problem.

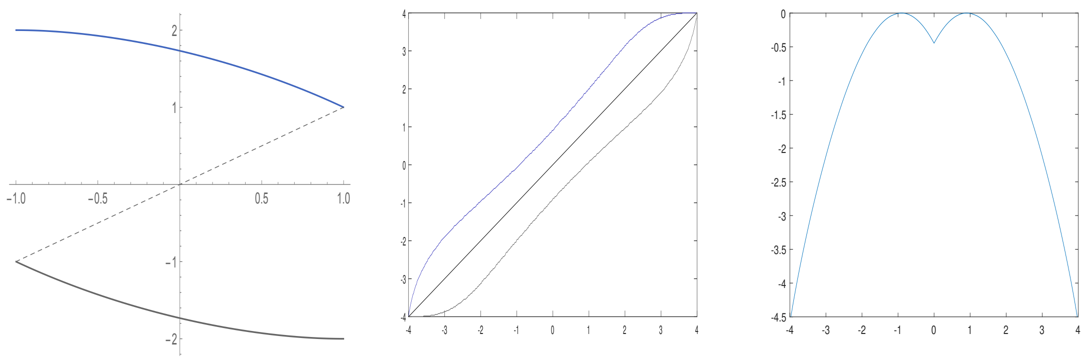

Figure 1: On the left we have plotted $q(x)$ (blue) and $p(x)$ (solid grey) for the case when $X \sim U[-1,1]$ and $Y \sim U[-2,2]$ for the minimization problem in (1), where $q(x) = \frac{1}{2}(\sqrt{12-3x^2} - x)$, $p(x) = \frac{1}{2}(-\sqrt{12-3x^2]} - x)$, and the dashed line is $y = x$, and we find that $P = D = 0.5931$. In the middle we have plotted $q(x)$ (blue), $p(x)$ (grey) and $y = x$ (black) for the maximization problem for the case when $X \sim N(0,1)$ and $Y \sim N(0,2)$ i.e. the marginals of Brownian motion at times 1 and 2, which we know are in convex order since W is a martingale, and we find (using Matlab) that $P = D \approx .958$. On the right we have plotted $L(x,y) = |x-y| - (\alpha(x) + \beta(y) + \gamma(x)(y-x))$ for the same problem as the middle graph, i.e. the forward-starter terminal payoff minus the superhedge payoff which is non-positive since we have a superhedge here, and note that L is tangential to zero at $p(x) = -.92$ and $q(x) = .92$ (we have set $x = 0$ for this example, but we can consider any x-value).