On the hyper-rough Heston model

MRK-Gath paper, Eduardo Wishart paper, variance curve tilting, add local vol, try f as shrinking normal dist, thick pts, multifrac Kolmogorov continuity of the integral, check why we have Holder regularity,

Consider standard driftless rough Heston model

$$\begin{aligned} dX_t &= -\frac{1}{2}V_t dt + \sqrt{V}_t dW_t \\ V_t &= V_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\nu\sqrt{V_s}dW_s \end{aligned}$$

and let $A_t = \int_0^t V_s ds$ and $\tilde{X}_t = X_t + \frac{1}{2}\int_0^t V_s ds$. Then for $\alpha > \frac{1}{2}$

$$\begin{aligned} A_t - V_0 t &= \int_0^t\int_0^s (s-u)^{\alpha-1}\nu\sqrt{V_u}dW_u ds = \int_0^t \nu\sqrt{V_u}dW_u\int_u^t (s-u)^{\alpha-1}ds \\ &= \frac{\nu}{\alpha\Gamma(\alpha)}\int_0^t (t-u)^{\alpha}\sqrt{V_u}dW_u \\ &= \frac{\nu}{\alpha\Gamma(\alpha)}\int_0^t (t-u)^{\alpha}dB_{A_u} \\ &\quad \text{(where } B_t = X_{T_t},\ T_t = \inf\{s : A_s > t\}\text{) and } B \text{ is a BM)} \\ &= \frac{\nu}{\alpha\Gamma(\alpha)}B_{A_u}(t-u)^{\alpha-1}|_{u=0}^t + \frac{\nu}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}B_{A_u}du \\ &= \nu I^\alpha B_{A_t}\,. \end{aligned} \tag{1}$$

Schauder theorem, We have uniqueness in law for $\alpha > \frac{1}{2}$, we now take

$$A_t = \nu I^\alpha B_{A_t} \tag{2}$$

(where $c_\alpha := \nu\Gamma(\alpha)$) as the *definition* of the model for $\alpha \in [\frac{1}{2}, 1)$, where B is now a given Brownian motion. Then for a given sample path $B_t(\omega)$, we can regard (2) as an fractional ODE of the form:

$$A(t) = V_0 t + I^\alpha f(A(t)) \tag{3}$$

0.1 Skew as $\alpha \to 0$

$$\begin{aligned} \mathbb{E}(X_T^3) = 3\mathbb{E}(X_T\langle X\rangle_T) &= 3\mathbb{E}(\int_0^T \sqrt{V}_s(\rho dW_s + \bar{\rho}dB_s)\int_0^T V_t dt) \\ &= 3\mathbb{E}(\int_0^T \sqrt{V}_s\rho dW_s\int_0^T V_t dt) \\ &= 3\mathbb{E}(\int_0^T \sqrt{V}_s\rho dW_s\int_0^T V_t dt) \end{aligned}$$

so formally we need to compute

$$\begin{aligned} \mathbb{E}(\sqrt{V}_s V_t dW_s) &= \mathbb{E}(\sqrt{V}_s(V_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\nu\sqrt{V_u}dW_u)dW_s) \\ &= \mathbb{E}(\sqrt{V}_s\frac{\nu}{\Gamma(\alpha)}(t-s)^{\alpha-1}1\sqrt{V_s}ds\,1_{s<t}) \\ &= \frac{\nu}{\Gamma(\alpha)}(t-s)^{\alpha-1}\nu\,1_{s<t}\,\mathbb{E}(V_s)ds \\ &= \frac{\nu}{\Gamma(\alpha)}(t-s)^{\alpha-1}\nu\,1_{s<t}\,V_0 ds\,. \end{aligned}$$

Thus

$$\mathbb{E}(X_T^3) = 3\rho\int_0^T\int_0^t \mathbb{E}(\sqrt{V}_s V_t dW_s) = \frac{3c_\alpha T^{1+\alpha}\nu\rho}{\alpha(1+\alpha)}$$

From this we see that

$$\lim_{\alpha\to\frac{1}{2}} \mathbb{E}((\frac{X_T}{\sqrt{T}})^3) = 4c_\alpha\nu\rho$$

calibrate skew term structure