Optimal signal trading with temporary, resilient and power law price impact and transaction costs - existence/uniqueness for mean-field predictive FBSDEs

Martin Forde[*] Benjamin Smith[†]

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Abstract

We characterize the optimal trading speed ϕ_t^* for the classic optimal liquidation problem for an agent subject to linear temporary price impact and resilient price impact via a propagator term, and a general square integrable stock price process satisfying a conditional growth assumption. We find that a trading strategy is optimal if and only if it satisfies a non-standard linear predictive mean-field FBSDE, and we prove existence/uniqueness for T sufficiently small using the fixed point theorem applied to a certain Banach space of admissible trading strategies, so in particular the stock price process itself is now a functional of the solution to this FBSDE. The FBSDE can be viewed as a stochastic Fredholm equation of the second kind, and the model is a regularized stochastic version of the continuous time propagator models considered in Gatheral,Schied&Slynko[GSS12] and Lehalle&Neuman[LN18], but unlike [LN18] we construct the true stochastic optimal strategy as opposed to just minimizing over deterministic strategies which is somewhat unnatural given that the stock process in [LN18] has a stochastic drift. We also consider the partially constrained problem with a quadratic terminal penalty, which allows us to work with a general non semi-martingale price process and incorporate power-law market impact, which is consistently reported with empirical studies. We later consider the unconstrained problem with inside information and/or transaction costs of size ϵ for which the optimal trading speed is shown to be proportional to $\frac{1}{2k}(\xi_t - \epsilon\,\mathrm{sgn}(\xi_t))\,1_{|\xi_t|\geq\epsilon}$ where $\xi_t = \mathbb{E}(S_T - S_t|\mathcal{F}_t)$ is the agent's "edge" on the market, i.e. we have a no-trade region and the filtration here can be non-standard which allows one to incorporate inside information or latency effects, and ϕ_t^* can be computed explicitly if e.g. S is fractional Brownian motion. We also show how ϕ can be computed explicitly for the constrained problem (with no propagator term) in the presence of a running quadratic inventory penalty (which generalizes an earlier result in Schied[Sch13] for the case with no quadratic penalty) by modifying the proof in Muhle-Karbe et al.[BMO18] using the matrix exponential function, and we also incorporate (stochastic) limit order flow. Finally we show how ϕ_t^* can be computed explicitly for various models which have appeared recently in the literature: fractional Gaussian models, an OU model, a target zone model with reflecting barriers using the Skorokhod map and the Wiener-Hopf decomposition, and limit order flow driven by a self-exciting generalized Hawkes process as in Gatheral&Keller-Ressel[GK18].

1 Introduction

A critical problem for algorithmic traders is how to optimally split a large trade so as to minimize trading costs and market impact. The seminal article of Almgren&Chriss[AC01] formulates this problem as trade-off between expected execution cost and risk; more specifically, they assume the stock price is a martingale and execution costs are linear in the trading rate and the choice of risk criterion is variance. Under these assumptions, there is a closed-form analytical solution for the optimal liquidation problem which is deterministic.

More recently, [BMO18] derive the optimal trading strategy for a linear price impact model with a partial liquidation penalty of the form ΓX_T^2 for $\Gamma > 0$, when the stock price is a general unspecified semimartingale. Using a similar variational argument to [BSV17], they show that $(X_t, \dot{X}_t)$ satisfies a coupled linear FBSDE, which can be re-written in a matrix form and solved explicitly using the same trick that is used to compute the solution for a standard OU process. The BMO argument can be very easily adapted to deal with the infinite penalty case $\Gamma = 0$ by simply replacing the vector $(\frac{\Gamma}{\lambda}, -1)$ with $(1, 0)$ which multiplies their FBSDE on page 4, to enforce that $X_T = 0$.

Lehalle&Neuman[LN18] consider a similar problem for a *propagator model* where the price paid per unit stock is $S_t + \int_0^t G(t-s)dX_s$ for some (positive-definite) decay kernel G, and they impose that X is deterministic, and left continuous with finite variation (but not necessarily differentiable). This type of model is a middle ground between purely temporary price impact i.e. $G(t) = k\delta_{\{0\}}(dt)$ and purely permanent price impact where $G(t) = 1$). [LN18] further assume the unaffected stock price is a martingale plus a Markovian drift term (known as the "signal"), and the non-zero drift here extends the driftless propagator model considered in Gatheral et al.[GSS12]). By considering dirac round trip perturbations and using a simple variational argument, [LN18] show that X is optimal if and only if X is a (measure-valued) solution to a Fredholm-type integral equation; they go on to prove uniqueness for this equation but not existence, but in general even if we have

[*]Dept. Mathematics, King's College London, Strand, London, WC2R 2LS (Martin.Forde@kcl.ac.uk)
[†]Dept. Mathematics, King's College London, Strand, London, WC2R 2LS (Benjamin.Smith@kcl.ac.uk)

establish existence, this will not be the true optimal strategy unless the signal is zero or takes a special form, e.g. an Ornstein-Uhlenbeck process. For the case of zero signal, the true optimal strategy will be deterministic, and [GSS12] prove existence in this case for a non-constant nonincreasing convex decay kernel G, if G is bounded and $\lim_{t\to 0} G(t)$ exists; under these conditions they prove that there is no *transaction-triggered price manipulation* i.e. buying as part of a sell program and vice versa. Moreover, the Fredhom equation can be solved explicitly in the case of exponential and power decay kernels; for the former the solution contains a block i.e. impulse response sell trade at time zero and at the final maturity with continuous selling in between proportional to the resilience parameter ρ (see example 2.12 in [GSS12], and for the latter the integral equation reduces to the well known Abel integral equation which also has an explicit solution which is U-shaped and symmetric, c.f. section 2.2 in [CGL17]. The Fredholm equation becomes a weakly singular Urysohn equation of the first kind if the temporary price impact component is non-linear, i.e. the price paid per unit stock is $S_t + \int_0^t G(t-s)f(\dot{X}_s)dt$ for some non-linear impact function f, and X is assumed to be absolutely continuous (see [Dan14] and [CGL17] for more on this, and numerical schemes for solving such non-linear integral equations).

General BSDEs and FBSDEs with Lipschitz continuous generators are well understood, see e.g. chapter 6 in [Pham09] and Delarue[Del02], and existence and uniqueness is typically proved using a Picard scheme via the fixed point theorem on a certain Banach space, which (for a given filtration) is the space of progressively measurable processes whose squared supremum has finite expectation.

In section 2 of this article, we consider a model with linear temporary and resilient price impact and a general square integrable stock price process satisfying a conditional growth assumption. We characterize the optimal trading strategy for the optimal liquidation problem via a linear mean-field FBSDE with a predictive term (i.e. involving a conditional expectation looking into the future), and an existence/uniqueness result is established subject to a certain condition on the model parameters, using a Picard argument. In section 3 we consider the partially constrained problem with a quadratic terminal penalty for both exponential and power-decay kernels. In section 4.1, we shift attention to the unconstrained problem, and for this we show how to incorporate inside information or latency effects and/or transaction costs using a new type of no-trade region. In section 4.2 show how ϕ can be computed explicitly for the constrained problem (wit no propagator component) with a running quadratic inventory penalty (which generalizes an earlier result of Schied[Sch13]), by making suitable modifications to the proof in Muhle-Karbe et al.[BMO18], and we later also allow the agent to post limit orders at a fixed depth δ on either side of the limit order book, where the limit order flow is modelled by a general finite variation process.

2 Existence/uniqueness for the optimal liquidation strategy

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ throughout, with a filtration $(\mathcal{F}_t)_{t\geq 0}$ which satisfies the usual conditions, and $\mathbb{E}_t(.)$ will denote $\mathbb{E}(.|\mathcal{F}_t)$ throughout. We consider a model with temporary and (semi) permanent price impact where the midprice process of an asset satisfies

$$S_t \quad = \quad P_t + Y_t$$

for some $\mathcal{F}_t$-progressively measurable process P_t with $\mathbb{E}(P_t^2) < \infty$ for all $t \in [0,T]$ (which we refer to as the *unaffected* price process). Note that we do not assume that S is a semimartingale (as is usually assumed in the literature). Y is defined as

$$Y_t \quad = \quad \int_0^t G(t-s)\phi_s ds$$

where $u_t := -\phi_t$ denotes the *selling speed*, and we let $\mathcal{A}$ denote the Banach space $\mathcal{A}$ of $\mathcal{F}_t$-progressively measurable processes such that $\mathbb{E}(\int_0^T u_t^2 dt) < \infty$ with norm $\|u\| = \mathbb{E}(\int_0^T u_t^2 dt)^{\frac{1}{2}}$ (see also e.g. chapter 6 of [Pham09] who works with the same space), and $\xi_t = \mathbb{E}_t(P_T - P_t)$.

$X_t = X_0 + \int_0^t \phi_u du$ is the number of shares held at time t, which we assume is absolutely continuous in t so ϕ_t is the trading speed, and we assume G satisfies the Hölder growth condition

$$|G(t) - G(s)| \quad \leq \quad \gamma_G |t-s|^\beta$$

for some $\beta > \frac{1}{2}$ (this condition is clearly satisfied by the exponential kernel $G(t) = \eta e^{-\rho t}$ for $\gamma_G = \rho$ and $\beta = 1$, see e.g. [CGL17] for more on this case). Y represents the cumulative effect of our trading activities on the current stock price, and G is the *decay kernel* which characterizes resilience of price impact between trades. We further assume that the price we actually pay per unit of stock is

$$\tilde{S}_t \quad = \quad S_t + k\phi_t$$

for $k > 0$ i.e. we are also subject to *temporary* price impact.

Our performance criterion is to maximize

$$V(u) \quad = \quad \mathbb{E}(\int_0^T (P_t - ku_t)u_t dt + P_T X_T - \int_0^T \phi_t \int_0^t G(t-s)\phi_s ds dt) \quad = \quad \mathbb{E}(\int_0^T (-\xi_t u_t - ku_t^2)dt + \mathbb{E}(\int_0^T u_t Y_t dt)$$

over $u \in \mathcal{A}$ (where we have used the tower property to obtain the second expression), subject to the singular terminal condition

$$X_T = 0$$

i.e. we must liquidate all inventory by time T.

Proposition 2.1 *If the following conditional Hölder growth assumption on P is satisfied:*

$$|\mathbb{E}_t(P_u - P_t)| = |\mathbb{E}_t(\xi_t - \xi_u)| \leq c_1|u-t|^\alpha \tag{1}$$

for $\alpha > \frac{1}{2}$ and

$$\frac{\gamma_G^2 T^{2\beta+2}}{4k^2}\left(\frac{1}{4\beta^2-1} + \frac{2}{\sqrt{(4\beta^2-1)(1+2\beta)}} + \frac{1}{\sqrt{(1+2\beta)}}\right) < 1 \tag{2}$$

then a unique optimal trading strategy $\hat{u}$ exists which satisfies

$$\xi_t + 2k(\hat{u}_t - \mathbb{E}_t(\hat{u}_T)) + \mathbb{E}_t\left(\int_0^T (G(|t-s|) - G(|T-s|))\hat{u}_s ds\right) = 0\,.$$

Remark 2.1 (1) is satisfied if S is a Lévy process with $\alpha = 1$, see section 2.1 below for a separate discussion on the case when S is an OU process.

Proof. From the same variational arguments in Lemma 5.2 in Bank et al.[BSV17], we can easily check that

$$-\xi_t - 2ku_t - \mathbb{E}_t\left(\int_0^T G(|t-s|)u_s ds\right) + 2kM_t = 0 \tag{3}$$

is a necessary and sufficient condition for optimality, for some $M \in \mathcal{M}_2$ (the Hilbert space of square integrable $\mathcal{F}_t$-adapted martingales) to be determined. This clearly implies that $-2ku_T + Y_T + 2kM_T = 0$ so M can be written explicitly as $M_t = \frac{1}{2k}\mathbb{E}_t(2ku_T - Y_T) = \frac{1}{2k}\mathbb{E}_t(2ku_T + \int_0^T G(T-s)u_s ds)$.

To prove existence of a solution to (3), we first consider an arbitrary $u \in \mathcal{A}$ such that $X_0 - \int_0^T u_t dt = 0$, and define $(\Phi(u), M)$ (with $M \in \mathcal{M}_2$) as the unique solution to

$$-Z_t - 2kv_t + 2kM_t = 0 \tag{4}$$

subject to $X_0 - \int_0^T v_s ds = 0$, where $v_t = \Phi(u)_t$, and

$$Z_t = \xi_t + \mathbb{E}_t\left(\int_0^T G(|t-s|)u_s ds\right).$$

Using a similar variational argument to Lemma 5.2 in [BSV17], we know that $X \in \mathcal{A}$ minimizes J if and only if

$$\begin{cases} M_t - v_t = \frac{1}{2k}Z_t \\ X_T = 0 \end{cases} \tag{5}$$

for some $M \in \mathcal{M}_2$. If we now replace t with a dummy variable u and integrate both sides of (5) from $u = t$ to T and take conditional expectations with respect to $\mathcal{F}_t$, we see that (5) implies

$$\mathbb{E}_t\left(\int_t^T M_u du\right) - \mathbb{E}_t\left(\int_t^T v_u du\right) = \frac{1}{2k}\mathbb{E}_t\left(\int_t^T Z_u du\right)$$

and using the martingale property of M we can re-write this as

$$(T-t)M_t - X_t = \frac{1}{2k}\mathbb{E}_t\left(\int_t^T Z_u du\right) \tag{6}$$

where we have also now imposed the terminal condition $X_t - \int_t^T v_u du = 0$ on the left hand side. Putting this in differential form, we see that

$$(T-t)dM_t - M_t dt + v_t dt = \frac{1}{2k}d\mathbb{E}_t\left(\left(\int_0^T - \int_0^t\right)Z_u du\right) = \frac{1}{2k}d\left(\mathbb{E}_t\left(\int_0^T Z_u du\right) - \int_0^t Z_u du\right) = dN_t - \frac{1}{2k}Z_t dt$$

where $N \in \mathcal{M}_2$ is defined by $N_t = \frac{1}{2k}\mathbb{E}_t(\int_0^T Z_u du)$. Equating the drift and martingale terms, we see that (5) is satisfied, and $M_t = M_0 + \int_0^t \frac{1}{T-s}dN_s$. Hence the single equation in (6) is equivalent to the two equations in (5). In particular if P

is a martingale, then M is a constant (this is the case in e.g. Theorem 2.4 in [LN18] and Eq 2.6 in [CGL17], although for their problem $k=0$).

Substituting the expression for M_t back into (5), we see that

$$\phi_t = -\frac{X_t}{T-t} + \frac{1}{2k}Z_t - \frac{1}{T-t}\frac{1}{2k}\int_t^T \mathbb{E}_t(Z_u)du = -\frac{X_t}{T-t} - \frac{1}{T-t}\frac{1}{2k}\int_t^T \mathbb{E}_t(Z_u - Z_t)du \tag{7}$$

which is a random linear ODE for X_t, with a solution given by

$$X_t = X_0(1-\frac{t}{T}) - \frac{1}{2k}(T-t)\int_0^t \frac{\int_s^T \mathbb{E}_s(Z_u-Z_s)du}{(T-s)^2}ds \tag{8}$$

$$v_t = -\dot{X}_t = \Phi(u)_t = \frac{X_0}{T} - \frac{1}{2k}\int_0^t \frac{\int_s^T \mathbb{E}_s(Z_u-Z_s)du}{(T-s)^2}ds + \frac{1}{2k}\frac{1}{T-t}\int_t^T \mathbb{E}_t(Z_u-Z_t)du\,. \tag{9}$$

Since (7) is linear in X_t with additive noise, we can trivially verify that it has a unique solution by considering the difference between two solutions to (7) and then showing that the difference must be the zero function. Note that (1) implies that

$$|\mathbb{E}_s(\xi_u-\xi_s)| = |\mathbb{E}_s(\mathbb{E}_u(P_T-P_u)) - \mathbb{E}_s(\mathbb{E}_s(P_T-P_s)| = |\mathbb{E}_s(P_s-P_u)| \le c_1|u-s|^\alpha\,.$$

Using this and the Lipschitz property of G we see that for $u\ge s$ we have

$$\begin{aligned}
|\mathbb{E}_s(Z_u-Z_s))| &\le c_1(u-s)^\alpha + \mathbb{E}_s(\int_0^T (G(|u-v|)-G(|s-v|))u_v dv)\\
&\le c_1(u-s)^\alpha + \mathbb{E}_s(\int_0^T (G(|u-v|)-G(|s-v|)^2dv)^{\frac12}(\int_0^T u_v^2 dv)^{\frac12})\\
&\le c_1(u-s)^\alpha + \gamma_G(\int_0^T (u-s)^{2\beta}dv)^{\frac12}\,\mathbb{E}_s(\|u\|_{L^2[0,T]})\\
&= c_1(u-s)^\alpha + \gamma_G(u-s)^\beta\sqrt{T}\,\mathbb{E}_s(\|u\|_{L^2[0,T]})
\end{aligned}$$

which implies that

$$(\mathbb{E}_s(Z_u-Z_s))^2 \le c_1^2|u-s|^{2\alpha} + 2c_1\gamma_G|u-s|^{\alpha+\beta}\mathbb{E}_s(\|u\|_{L^2[0,T]})\sqrt{T} + \gamma_G^2|u-s|^{2\beta}\,T\mathbb{E}_s(\|u\|^2_{L^2[0,T]})$$

where we have used the conditional Jensen inequality for the final term. Applying conditional Jensen again, we find that

$$\begin{aligned}
&\mathbb{E}((\int_0^t \frac{\int_s^T \mathbb{E}_s(Z_u-Z_s)du}{(T-s)^2}ds)^2)\\
\le\ &\mathbb{E}(t\int_0^t (\frac{\int_s^T \mathbb{E}_s(Z_u-Z_s)du}{(T-s)^2})^2 ds)\\
\le\ &\mathbb{E}(t\int_0^t \frac{(T-s)\int_s^T(\mathbb{E}_s(Z_u-Z_s))^2du}{(T-s)^4}ds)\\
\le\ &\mathbb{E}(t\int_0^t \frac{\int_s^T(c_1^2(u-s)^{2\alpha} + 2c_1\gamma_G(u-s)^{\alpha+\beta}\mathbb{E}_s(\|u\|_{L^2[0,T]})\sqrt{T} + \gamma_G^2(u-s)^{2\beta}\,T\mathbb{E}_s(\|u\|^2_{L^2[0,T]}))du}{(T-s)^3}ds)\\
=\ &\mathbb{E}(t\int_0^t \frac{\int_s^T(c_1^2(u-s)^{2\alpha} + 2c_1\gamma_G(u-s)^{\alpha+\beta}\|u\|_{L^2[0,T]}\sqrt{T} + \gamma_G^2(u-s)^{2\beta}\,T\|u\|^2_{L^2[0,T]})du}{(T-s)^3}ds)\\
&\text{(using the tower property and Fubini's theorem)}\\
\le\ &\frac{c_1^2T^{2\alpha}}{4\alpha^2-1} + \frac{2c_1\gamma_G T^{\alpha+\beta+\frac12}}{(\alpha+\beta)^2-1}\mathbb{E}(\|u\|_{L^2[0,T]}) + \frac{\gamma_G^2T^{2\beta+1}}{4\beta^2-1}\mathbb{E}(\|u\|^2_{L^2[0,T]})
\end{aligned}$$

(note this bound is independent of t), and

$$\begin{aligned}
\mathbb{E}((\frac{1}{T-t}\int_t^T \mathbb{E}_t(Z_u-Z_t)du)^2) &\le \mathbb{E}(\frac{1}{T-t}\int_t^T(\mathbb{E}_t(Z_u-Z_t))^2du)\\
&\le \frac{c_1^2T^{2\alpha}}{1+2\alpha} + \frac{2c_1\gamma_G T^{\alpha+\beta+\frac12}}{1+\alpha+\beta}\mathbb{E}(\|u\|_{L^2[0,T]}) + \frac{\gamma_G^2T^{2\beta+1}}{1+2\beta}\mathbb{E}(\|u\|^2_{L^2[0,T]})\,.
\end{aligned}$$

which is also clearly independent of t. Applying these upper bounds to the integral terms in (9), we see that $\mathbb{E}(v_t^2)$ has an upper bound which is independent of t and finite if $\mathbb{E}(\|u\|^2_{L^2[0,T]}<\infty$, so we see that $\mathbb{E}(\int_0^T v_t^2dt)<\infty$ and v is $\mathcal{F}_t$-adapted, so v has a modification which lies in $\mathcal{A}$ (check). Then choosing another $u'\in\mathcal{A}$ with $X_0-\int_0^T u_t dt=0$ and letting $Z'_t=\xi_t+\mathbb{E}_t(\int_0^T G(|t-s|)u'_s ds)$, the ξ contributions to Z' and Z cancel each other out, and we see that

$$|\mathbb{E}_s(Z'_u-Z'_s-(Z_u-Z_s))| \le \gamma_G(u-s)^\beta\sqrt{T}\,\mathbb{E}_s(\|u'-u\|_{L^2[0,T]})\,.$$

Equation (9) may be used to compute the norm of the difference $v' - v$:

$$v'_t - v_t = \frac{1}{2k}\underbrace{\int_0^t \frac{\int_s^T \mathbb{E}_s[Z'_u - Z'_t - (Z_u - Z_t)]du}{(T-s)^2}ds}_{:=A} - \frac{1}{2k}\underbrace{\frac{1}{T-t}\int_t^T \mathbb{E}_t[Z'_u - Z'_t - (Z_u - Z_t)]du}_{:=B}$$

Squaring, taking expectations and applying Cauchy-Schwarz:

$$\begin{aligned}
\mathbb{E}[(v'_t - v_t)^2] &\leq \frac{1}{4k^2}(\mathbb{E}[A^2] + \mathbb{E}[B^2] + 2\mathbb{E}[A^2]^{1/2}\mathbb{E}[B^2]^{1/2}) \\
&= \underbrace{\frac{\gamma_G^2 T^{2\beta+1}}{4k^2}(\frac{1}{4\beta^2-1} + \frac{2}{\sqrt{(4\beta^2-1)(1+2\beta)}} + \frac{1}{\sqrt{(1+2\beta)}})}_{:=C}\mathbb{E}[||u'-u||^2_{L^2[0,T]}]
\end{aligned}$$

Using the definition of the norm on the space of admissible strategies

$$\begin{aligned}
||v' - v|| &= (\int_0^T \mathbb{E}[((v'_t - v_t)^2)]dt)^{1/2} \\
&\leq (CT)^{1/2}||u' - u||
\end{aligned}$$

This is a contraction if $CT < 1$ or, equivalently:

$$\frac{\gamma_G^2 T^{2\beta+2}}{4k^2}(\frac{1}{4\beta^2-1} + \frac{2}{\sqrt{(4\beta^2-1)(1+2\beta)}} + \frac{1}{\sqrt{(1+2\beta)}}) < 1 \tag{10}$$

In this case (from the Banach fixed point theorem), Φ has a unique fixed point u which is the solution to (3). ■

2.1 An Ornstein-Uhlenbeck price process

If P is an Ornstein-Uhlenbeck process $dP_t = -\alpha(P_t - \bar{P})dt + \sigma dW_t$ and $G \equiv 0$ (see also section 2.3 in [LN18]), then

$$\begin{aligned}
\mathbb{E}_s(P_u - P_s) &= (e^{-\alpha(u-s)} - 1)P_s + (1 - e^{-\alpha(u-s)})\bar{P} \\
\mathbb{E}((\mathbb{E}_s(P_u - P_s))^2) &= (e^{-\alpha(u-s)} - 1)^2[\frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha s}) + (P_0 e^{-\alpha s} + \bar{P}(1 - e^{-\alpha s}))^2] \\
&+ 2(e^{-\alpha(u-s)} - 1)(P_0 e^{-\alpha s} + \bar{P}(1 - e^{-\alpha s}))(1 - e^{-\alpha(u-s)})\bar{P} + (1 - e^{-\alpha(u-s)})^2\bar{P}^2 \\
&\leq const. \times (u-s)^2 .
\end{aligned}$$

In this case, from (8) and the definition of Z, the conjectured optimal strategy takes the explicit form

$$\hat{X}_t = X_0(1 - \frac{t}{T}) - \frac{1}{2k}(T-t)\int_0^t \frac{\int_s^T \mathbb{E}_s(P_u - P_s)du}{(T-s)^2}ds$$

but using the bound on $\mathbb{E}((\mathbb{E}_s(P_u - P_s))^2)$ we can easily verify that $\hat{X} \in \mathcal{A}$, so in fact this is the optimal strategy.

3 The partially unconstrained problem

3.1 Exponential resilience

If we now replace the singular terminal condition $X_T = 0$ in the previous section with a terminal penalty of the form $-\Gamma X_T^2$ as in [BMO18], and assume exponential resilience, i.e. set $G(t) = \eta e^{-\beta t}$ for some $\beta \geq 0$, and we drop the assumption in (1) (but retain all other assumptions on P), we obtain the following variant of Proposition 2.1:

Proposition 3.1 *The optimal trading strategy $\hat{u}$ satisfies*

$$\xi_t + 2k\hat{u}_t - 2\Gamma\mathbb{E}_t(\hat{X}_T) + \mathbb{E}_t(\int_0^T G(|t-s|)\hat{u}_s ds) = 0 \tag{11}$$

where $\hat{X}_t = X_0 - \int_0^t u_s ds$, and this equation has a unique solution in $\mathcal{A}$ if

$$\frac{2\Gamma + \eta}{2k}T < 1 . \tag{12}$$

Proof. From the variational arguments in Lemma 5.2 in [BSV17], we can show that is a necessary and sufficient condition for optimality. We now define a mapping Φ on $\mathcal{A}$ by the following relation:

$$-\xi_t \;-\; 2kv_t \;+\; 2\Gamma\mathbb{E}_t(X_T) \;-\; \eta\mathbb{E}_t(\int_0^T e^{-\rho|t-s|}u_s ds) \quad=\quad 0 \tag{13}$$

where $X_t = X_0 - \int_0^t u_s ds$ and $v_t = \Phi(u)_t$. Then if u' is also in $\mathcal{A}$ with $X'_t = x - \int_0^t u'_s ds$, then

$$v'_t - v_t \quad=\quad \frac{1}{2k}\,[2\Gamma\mathbb{E}_t(X'_T - X_T) \;-\; \eta\mathbb{E}_t(\int_0^T e^{-\rho|t-s|}(u'_s - u_s)ds)]\,. \tag{14}$$

But

$$|X'_T - X_T| \vee |\int_0^T e^{-\rho|t-s|}(u'_s - u_s)ds| \quad\leq\quad \int_0^T |u'_s - u_s|ds \quad\leq\quad \sqrt{T}\,(\int_0^T (u'_s - u_s)^2 ds)^{\frac{1}{2}} \tag{15}$$

Applying these bounds to (14) and using the conditional Jensen inequality, we see that

$$(v'_t - v_t)^2 \quad\leq\quad \frac{1}{4k^2}T(2\Gamma+\eta)^2\,(\mathbb{E}_t[(\int_0^T (u'_s - u_s)^2 ds)^{\frac{1}{2}}])^2 \quad\leq\quad \frac{1}{4k^2}T(2\Gamma+\eta)^2\,\mathbb{E}_t(\int_0^T (u'_s - u_s)^2 ds)$$

which (from the tower property) implies that

$$\mathbb{E}(\int_0^T (v'_t - v'_t)^2 dt) \quad\leq\quad \frac{1}{4k^2}(2\Gamma+\eta)^2\,T^2\mathbb{E}(\int_0^T (u'_s - u_s)^2 ds)\,.$$

Taking the square root of both sides we see that $\|v\| = \|\Phi u\| \leq \frac{1}{2k}(2\Gamma+\eta)T\,\|u\|$. Moreover, if ξ_t is continuous, $\Phi(u)$ is progressively measurable, since $\Phi(u)$ is continuous and adapted. Thus Φ is a strict contraction on the Banach space $\mathcal{A}$ if (12) is satisfied. In this case (from the Banach fixed point theorem), Φ has a unique fixed point $\hat{u}$ which is the solution to (3). ■

3.2 Power-law market impact

Many empirical studies suggest that G is a power function of the form $G(t) = \eta t^{-\beta}$ for some $\beta \in (0, \frac{1}{2})$ and $\eta > 0$ (see e.g. Gatheral et al.[GSS12] and references therein). For this choice of G, most of the arguments of Proposition 3.1 remain unscathed, except now we can use the Cauchy-Schwarz inequality to replace (15) with

$$|\int_0^T \eta|t-s|^{-\beta}(u'_s - u_s)ds| \quad\leq\quad \frac{\eta}{\sqrt{1-2\beta}}(t^{1-2\beta} + (T-t)^{1-2\beta})^{\frac{1}{2}}(\int_0^T |u'_s - u_s|^2 ds)^{\frac{1}{2}} \quad\leq\quad \frac{2^\beta\eta}{\sqrt{1-2\beta}}T^{\frac{1}{2}-\beta}(\int_0^T |u'_s - u_s|^2 ds)^{\frac{1}{2}}$$

where we have used that $t^{1-2\beta} + (T-t)^{1-2\beta}$ attains its maximum at $t = \frac{1}{2}T$. Thus we now have

$$\mathbb{E}(\int_0^T (v'_t - v'_t)^2 dt) \quad\leq\quad \frac{1}{4k^2}T(2\Gamma\sqrt{T} + \frac{2^\beta\eta}{\sqrt{1-2\beta}}T^{\frac{1}{2}-\beta})^2 \cdot \int_0^T (u'_s - u_s)^2 ds\,.$$

Thus Proposition 3.1 still holds with the condition (12) replaced by

$$\frac{T}{4k^2}(2\Gamma\sqrt{T} + \frac{2^\beta}{\sqrt{1-2\beta}}T^{\frac{1}{2}-\beta})^2 \quad<\quad 1\,.$$

4 Examples with explicit solutions

4.1 The unconstrained problem with zero resilience

If we set $G \equiv 0$ and consider the unconstrained problem (i.e. remove the restriction that $X_0 = 0$) and only assume that $\mathbb{E}(S_t^2) < \infty$ for all $t \in [0, t]$ (so in particular we make no growth or semimartingale assumption on S and we do not assume that S is $\mathcal{F}_t$-adapted), then our performance criterion is now $V(\phi) = \mathbb{E}(\int_0^T(\xi_t\phi_t - k\phi_t^2)dt)$ where now $\xi_t := \mathbb{E}_t(S_T - S_t)$ since $S_t = P_t$ for this problem. We can now maximize the integrand $\xi_t\phi_t - k\phi_t^2$ *pointwise* to obtain the optimal ϕ as

$$\phi^*_t \quad=\quad \frac{1}{2k}\xi_t \tag{16}$$

and note that $\phi^*_T = \xi_T = 0$, as in Lemma 5.2 in [BSV17]. Moreover from the conditional Jensen inequality, we see that

$$2k^2\mathbb{E}(\int_0^T (\phi^*_t)^2 dt) \quad=\quad \mathbb{E}(\int_0^T (\mathbb{E}_t(S_T - S_t))^2 dt) \quad\leq\quad \mathbb{E}(\int_0^T \mathbb{E}_t((S_T - S_t)^2)dt) \quad=\quad \int_0^T \mathbb{E}((S_T - S_t)^2)dt) \quad<\quad \infty \tag{17}$$

from the assumption that $\mathbb{E}(S_t^2) < \infty$ for all $t \in [0, T]$. Thus $\phi^* \in \mathcal{A}$.

Remark 4.1 Note that $\mathcal{F}_t$ may be strictly larger or smaller than $\mathcal{F}^S_t$, e.g. $\mathcal{F}_t = \mathcal{F}^S_{t+\delta}$ which allows us to incorporate **inside information** or held by informed traders, or conversely traders with imperfect information/**latency** who do not know the whole history of S up to time t.

The expected profit/loss is

$$\begin{aligned} V(\phi^*) &= \mathbb{E}[\int_0^T \frac{1}{2k}(S_T - S_t)\mathbb{E}_t(S_T - S_t)dt \,-\, k\int_0^T \frac{1}{4k^2}(\mathbb{E}_t(S_T-S_t))^2 dt] &= \frac{1}{4k}\mathbb{E}[\int_0^T (\mathbb{E}_t(S_T-S_t))^2 dt] \\ & &\leq \frac{1}{4k}\mathbb{E}[\int_0^T \mathbb{E}_t((S_T-S_t)^2)dt] \\ & &= \frac{1}{4k}\int_0^T \mathbb{E}((S_T-S_t)^2)dt \quad < \quad \infty \end{aligned}$$

by the assumption that $\mathbb{E}(S_t^2) < \infty$ for all $t \in [0,T]$.

Remark 4.2 If $k\phi_t$ is replaced by a non-linear price impact function $f(\phi_t)$, then if $g(x) = xf(x)$ and we assume g is e.g. smooth and convex, then pointwise optimization yields that $\phi^*_t = (g')^{-1}(\mathbb{E}_t(S_T - S_t))$.

4.2 Transaction costs

Setting $f(x) = kx + \epsilon\,\mathrm{sgn}(x)$ for $k, \epsilon > 0$ corresponds to linear temporary price impact plus fixed transaction costs, i.e. a fixed bid/offer spread of ϵ. In this case, using that $\xi x - kx^2 - \epsilon\,|x|$ is maximized at $x = \frac{\xi-\epsilon}{2k}$ if $\xi - \epsilon \geq 0$, at $x = \frac{\xi+\epsilon}{2k}$ if $\xi + \epsilon \leq 0$ and at $x = 0$ otherwise and a similar pointwise optimization argument as before, we find that

$$\phi^*_t \quad = \quad \frac{1}{2k}(\xi_t - \epsilon\,\mathrm{sgn}(\xi_t))\,1_{|\xi_t|\geq\epsilon}$$

and we see that there is now a *no-trade region* defined by $|\mathrm{sgn}(\xi_t)| < \epsilon$, and outside this region, the effect of ϵ is to bet on the trends as before, but more sluggishly.

4.3 The constrained case with zero resilience and running quadratic inventory penalty

If we now return to the constrained problem, and we still assume $G = 0$ but we add a running quadratic inventory penalty term $-\gamma\int_0^T X_t^2 dt$ into our performance criterion and now assume S is a semimartingale in $\mathcal{H}^2$ (see [BMO18] for definition), this corresponds to setting $\Gamma = \infty$ in Theorem 3.1 in [BMO18]; we cannot simply modify their final answer since their optimal trading strategy is undefined for $\Gamma = \infty$, but rather we pick up their proof half way through and then modify a few steps steps accordingly. From their proof of Theorem 3.1 in [BMO18] we have

$$\begin{pmatrix}\hat{X}_T \\ \hat{u}_T\end{pmatrix} \quad = \quad e^{B(T-t)}\begin{pmatrix}\hat{X}_t \\ \hat{u}_t\end{pmatrix} \,+\, \frac{1}{2k}\int_t^T e^{B(t-s)}dZ_s$$

where $\hat{u}_t = -\phi_t$ and our k parameter corresponds to their λ parameter, and $Z_t = \begin{pmatrix}0 \\ S_t - N_t\end{pmatrix}$ for some square integrable martingale N_t. Then applying the terminal condition $X_T = 0$, we see that

$$\begin{aligned} 0 &= (1,0)\,.\,e^{B(T-t)}\begin{pmatrix}\hat{X}_t \\ \hat{u}_t\end{pmatrix} \,+\, (1,0)\,.\,\frac{1}{2k}\mathbb{E}_t(\int_t^T e^{B(T-u)}\begin{pmatrix}0 \\ dS_u\end{pmatrix}) \\ &= a_{11}(t)\hat{X}_t \,+\, a_{12}(t)\hat{u}_t \,+\, \frac{1}{2k}\mathbb{E}_t(\int_t^T a_{12}(u)dS_u) \end{aligned} \tag{18}$$

where $B = \begin{pmatrix}0 & -1 \\ -\frac{\gamma}{k} & 0\end{pmatrix}$ in our notation, $a_{ij}(t)$ denotes the elements of the matrix-valued function $A(t) := e^{B(T-t)}$, and $e^{B(T-t)}$ is easily computed explicitly in e.g. Mathematica, since $B(T-t)$ is easily diagonalized. (18) is now a random linear ODE for $\hat{X}_t$ which is easily solved numerically for many models or we can do this analytically in terms of cosh/sinh functions by examining the entries of $A(t)$ as in [BMO18] (we defer the details for the sake of brevity). Technically of course we have to verify that $\hat{X}$ here is admissible, but we defer the tedious details of checking that here for the sake of brevity.

If we let $\gamma \to 0$, then $e^{B(T-t)} \to \begin{pmatrix}1 & t-T \\ 0 & 1\end{pmatrix}$ and (18) simplifies to

$$\hat{X}_t \,-\, (T-t)\hat{u}_t \,-\, \frac{1}{2k}\mathbb{E}_t(\int_t^T (T-u)dS_u) \quad = \quad 0 \tag{19}$$

which agrees with Eq 19 in Schied[Sch13], and we can further re-arrange this as

$$\hat{u}_t = \frac{\hat{X}_t}{T-t} + \frac{1}{2k}\frac{1}{T-t}\mathbb{E}_t\Big(\int_t^T (T-u)dS_u\Big) = \frac{\hat{X}_t}{T-t} - \frac{1}{2k}\frac{1}{T-t}[(T-u)S_u|_{u=t}^T + \mathbb{E}_t\Big(\int_t^T S_u du\Big)]$$
$$= \frac{\hat{X}_t}{T-t} - \frac{1}{2k}\frac{1}{T-t}\mathbb{E}_t\Big(\int_t^T (S_u - S_t)du\Big)$$

If we instead assume $\xi_t \equiv 0$, then $\hat{X}_t$ simplifies to $\hat{u}_t = b\coth(b(T-t))\hat{X}_t$ where $b = \sqrt{\gamma/k}$, which can be solved explicitly to obtain the classical Almgren-Chriss solution $\hat{X}_t = X_0 \frac{\sinh(b(T-t))}{\sinh(bT)}$ which is of course deterministic.

4.4 Adding limit order flow

We can enrich the setup in the previous subsection by assume the agent can also place limit orders on both side of the limit order book at fixed depth δ which are filled randomly over time (i.e. he places sell orders at price $S_t + \delta$ and buy orders at price $S_t - \delta$). To this end, we now let Q_t denote our total stock holding which evolves as

$$Q_t = X_t - J_t$$

where $X_t = X_0 + \int_0^t \phi_s ds$ is the cumulative stock purchases from market orders only, as before. We still assume that $X \in \mathcal{A}$ but we now impose that $Q_T = 0$ so $X_T = J_T$ (i.e. this is now what is known as a *stochastic target* problem, see [BSV17] for more on these type of problems), and we assume J is an $\mathcal{F}_t$-progressively measurable finite variation process with finite second moments which models limit order flow. If $J_t = J^{\uparrow} - J^{\downarrow}$ denotes the canonical decomposition of J as the difference of two increasing process, then the trader makes a spread of δ per unit of stock bought or sold as a limit order, and thus his value function $\tilde{J}$ now has an additional term given by $-\int_0^T (S_T - S_t)dJ_t + \delta(J_T^{\uparrow} + J_T^{\downarrow})$. The value of δ will not affect the optimal trading strategy X^* since we are not assuming that δ is a stochastic control, but J does affect X^*. Then we can easily adapt the analysis above to show that (20) is now replaced with

$$(\hat{X}_t - \mathbb{E}_t(J_T)) - (T-t)\hat{u}_t - \frac{1}{2k}\mathbb{E}_t\Big(\int_t^T (T-u)dS_u\Big) = 0 \tag{20}$$

5 Example stock price and order flow processes

In this section we discuss some candidate stock price processes for which ξ_t can be computed and simulated (see the following page for Monte Carlo simulations of these processes and the associated optimal trading strategies with price impact/transaction costs for the unconstrained and constrained problem).

5.1 Fractional Gaussian processes

(16) is most interesting when applied to a stock price process which is persistent or anti/persistent. If we let $Z_t = \sqrt{2H}\int_0^t (t-s)^{H-\frac{1}{2}}dW_s$ denote a Riemann-Liouville process for $H \in (0,1)$ which is Gaussian and self similar like fBM and has the same marginals as fBM at time zero but does not have stationary increments, then if $S_t = S_0 + \sigma Z_t + S_0$ and $\mathcal{F}_t = \mathcal{F}_t^S$ we see that

$$\mathbb{E}_t(S_u - S_t) = \sigma\sqrt{2H}\int_0^t [(u-s)^{H-\frac{1}{2}} - (t-s)^{H-\frac{1}{2}}]dW_s \tag{21}$$

which is a Gaussian Volterra process in t. Integrating this expression and using the Stochastic Fubini theorem, we obtain

$$\int_t^T \mathbb{E}_t((S_u - S_t))du = \sigma\sqrt{2H}\int_0^t\int_t^T [(u-s)^{H-\frac{1}{2}} - (t-s)^{H-\frac{1}{2}}]du\, dW_s = \int_0^t \bar{K}_H(s,t)dW_s$$

for some new kernel $\bar{K}_H(s,t)$.

In practice, we would observe the sample path of Z not W, but we can re-write W_t in terms of Z using the transformation formula:

$$W_t = \tilde{c}_H \int_0^t (t-s)^{\frac{1}{2}-H}dZ_s$$

for some constant $\tilde{c}_H$ depending on H (see Remark 5.5 in [Jost06] and Lemma 1.1 in Forde&Viitasaari [FV18] for a proof).

Similarly, if $W_t^H = c_H\Gamma(\frac{1}{2}-H)\int_{-\infty}^t [(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}]dW_s$ is the Mandelbrot-van Ness two-sided fBM, we can transform W^H to another fBM W^{H_2} using the formula:

$$W_t^{H_2} = c_{H_2,H}\int [(t-v)_+^{H_2-H} - (-v)_+^{H_2-H}]dW_v^{H_1}$$

(see Corollary 5.7 in [Jost06]), so there is a one-to-one mapping between the two fBMs when $t>0$ so in particular we can extract W from W^H in the MvN representation. Clearly this cannot hold for $t<0$ as the history of W over $(-\infty,0]$ is needed to compute W^H_t. If $S_t=\sigma W^H_t$, then we have the explicit *prediction formula*:

$$\mathbb{E}(S_T|\mathcal{F}^S_t)-S_t \;=\; \sigma\frac{\cos(\pi H)}{\pi}\,a^{H+\frac{1}{2}}\int_0^\infty \frac{W^H_{t-u}-W^H_t}{u^{H+\frac{1}{2}}(u+a)}du$$

where $a=T-t$ (see Theorem 4.2 in Nuzmman&Poor[NP00]), which yields explicit integral expressions for ϕ^*_t for both the constrained and unconstrained problems, in terms of the history of the stock price itself (see Proposition 2.9 in [FV18] for a similar prediction formula for the RL process). H and σ can be estimated from a time series using maximum likelihood methods (see e.g. Chang[Cha14] for details).

5.2 A general Ornstein-Uhlenbeck process - optimal mean reversion trading

If

$$dS_t \;=\; -\alpha(S_t-\bar{S})dt+dM_t$$

i.e. S_t is an Ornstein-Uhlenbeck process driven by a square integrable martingale M with $\alpha>0$, then

$$\mathbb{E}(S_u|S_t) \;=\; e^{-\alpha(u-t)}S_t+(1-e^{-\alpha(u-t)})\bar{S} \tag{22}$$

Then

$$\begin{aligned}|\mathbb{E}(S_u-S_t|S_t)| &= |(e^{-\alpha(u-t)}-1)S_t+(1-e^{-\alpha(u-t)})\bar{S}| \\ &= |(1-e^{-\alpha(u-t)})(\bar{S}-S_t)| \\ &\le |t-u|\cdot|\bar{S}-S_t|\end{aligned}$$

i.e. a weighted average of S_t and $\bar{S}_\infty$ so we can easily compute $\xi_t=\mathbb{E}(S_u-S_t|\mathcal{F}^S_t)$, and ϕ^*_t for the constrained simplifies to

$$\begin{aligned}\phi^*_t &= -\frac{X^*_t-\mathbb{E}_t(J_T)}{T-t}+\frac{1}{T-t}\frac{1}{2k}\int_t^T\mathbb{E}_t(S_u-S_t)du \\ &= -\frac{X^*_t-\mathbb{E}_t(J_T)}{T-t}-\frac{S_t-\bar{S}}{T-t}\frac{1}{2k}\int_t^T(1-e^{-\alpha(u-t)})dt\end{aligned}$$

This is often referred to as optimal mean reversion trading (see articles and book by Tim Leung for more on this).

5.3 Reflecting Target zone models and the Skorokhod map

In this section, we begin by defining the Brownian motion X with two reflecting boundaries. Let W_t be standard Brownian motion starting at 0. Then for any $x\in[a,b]$, there is a unique pair of non-decreasing, continuous adapted processes (L,U), starting at 0, such that

$$X_t \;=\; x+W_t+L_t-U_t \;\in\; [a,b], \qquad \forall t\ge 0.$$

such that L can only increase when $X=a$ and U_t can only increase when $X=b$. Existence and uniqueness follow easily from the more general work of Lions&Sznitman[LS84] the earlier work of Skorokhod[Sko62], or a bare-hands proof can be given by successive applications of the standard one-sided reflection mapping using a sequence of stopping times (see [Wil92].)

One can easily verify that the following Fourier series:

$$p(x,t;y,T) \;=\; \frac{1}{b-a}\sum_{n=0}^\infty e^{-\lambda_n(T-t)}\cos(\frac{n\pi(y-a)}{b-a})\cos(\frac{n\pi(x-a)}{b-a})$$

is the transition density of X, which satisfies the von Neuman boundary conditions $p_y(a,t;y,T)=0=p_y(b,t;y,T)$. Then if $S_t=X_t$ e.g. if S is an exchange rate which is forced to stay in $[a,b]$ via government intervention when S hits a or b, then we can easily compute $\mathbb{E}(S_T|S_t=x)$ by integrating term-by-term as

$$\mathbb{E}(S_T|S_t=x) \;=\; \int_{y=a}^b yp(x,t;y,T)dy \;=\; \frac{a+b}{2}+\sum_{n=1}^\infty e^{-\lambda_n(T-t)}\frac{1}{n^2\pi^2}(a-b+(b-a)\cos(n\pi)+bn\pi\sin(n\pi))\cos(\frac{n\pi(x-a)}{b-a})$$

Alteratively, we can consider a general target zone model for a stock price process S with one reflecting barrier at $S=b>S_0$ as in Muhle-Karbe et al.[BMO18], defined via the solution to the *Skorokhod map* as

$$S_t \;:=\; M_t-(\bar{M}_t-b)^+$$

where M is a martingale in $\mathcal{H}^2$ (see [BMO18] for definition) and $\bar{M}_t = \sup_{s\in[0,t]} M_s$. Then $\xi_t = \mathbb{E}(S_T|\mathcal{F}^M_t) = M_t - \mathbb{E}((\bar{M}_T - b)^+|M_t)$. If e.g. $M_t = X_t$ is a martingale spectrally negative Lévy process with $X_0 = 0$, then from the Wiener-Hopf decomposition it is known that

$$\mathbb{P}(\bar{X}_{e_q} > b) \quad = \quad e^{-\Phi(q)b}$$

where $e_q \sim \mathrm{Exp}(q)$ is independent of M and $\Phi(q) = \sup\{\theta : V(\theta) = q\}$ is the largest root of $V(p) := \log \mathbb{E}(e^{pX_t}) = q$ for $q > 0$ (see e.g. Eq 2.15 in Kyprianou et al.[KKR13]), so

$$\mathbb{E}(\bar{X}_{e_q} - b)^+ \quad = \quad \int_b^\infty \mathbb{P}(\bar{M}_{e_q} > b')db' \quad = \quad \frac{e^{-\Phi(q)b}}{\Phi(q)}\,. \tag{23}$$

$\mathbb{E}(\bar{X}_T - b)^+$ (for T fixed) can then be computed via Laplace inversion, and we can easily adapt this to compute $\mathbb{E}((\bar{M}_T - b)^+|M_t)$ since a Lévy process is a Markov process and $M_T - M_t \overset{(d)}{=} M_{T-t} - M_0$ (see numerical simulation in Figure 5 below). See also Neuman&Schied[NS16] for related target zone problems.

5.4 Order flow driven by a self-exciting process

Following e.g. section 3 in Gatheral&Keller-Ressel[GK18], we can consider a finite-activity right continuous self-exciting pure jump semi-martingale J with i.i.d. jump size distribution $\zeta(dx)$ and intensity λ_t which satisfies

$$\lambda_t \quad = \quad \mu + \int_{[0,t]} \phi(t-s)dJ_s$$

where μ is a positive constant and ϕ a positive function supported on $\mathbb{R}_+$ which satisfies the *stability condition*: $\int_0^\infty \phi(u)du < 1$. If we let $dM_t = dJ_t - \lambda_t dt$ denote the compensated process, then

$$\lambda_t \quad = \quad \mu + \int_{[0,t]} \phi(t-s)(dM_t + \lambda_t dt) \quad = \quad \mu + (\phi * dM)_t + (\phi * \lambda)_t\,.$$

Note that

$$\|\phi * \lambda\|_\infty \quad \leq \quad \|\lambda\|_\infty \int_{[0,t]} \phi(t-s)ds \quad = \quad \|\lambda\|_\infty \int_{[0,t]} \phi(u)du \quad < \quad \|\lambda\|_\infty \int_0^\infty \phi(u)du < \|\lambda\|_\infty$$

So $\phi*$ is a contraction on $C_b[0,\infty)$ under the sup norm, so its inverse is well defined, and also on $C_b[0,T]$. We can re-write this in operator notation as

$$(I - \phi*)\lambda \quad = \quad \mu + \phi * dM\,.$$

Inverting this expression, we see that

$$\begin{aligned}
\lambda \quad &= \quad (I-\phi*)^{-1}(\mu + \phi * dM) \quad = \quad (1+\kappa)*\mu + (I+\kappa)*(\phi * dM) \\
&= \quad \mu + \kappa * \mu + (\phi * + (\phi*)^2 + ...) * dM \\
&= \quad \mu + \kappa * \mu + \kappa * dM
\end{aligned}$$

where $\kappa = \sum_{k=1}^\infty (\phi*)^k$, so

$$\lambda_t \quad = \quad \mu + \mu \int_{[0,t]} \kappa(t-s)ds + \int_{[0,t]} \kappa(t-s)dM_s$$

κ is the resolvent of ϕ which satisfies the functional equation $(I+\kappa)*\phi = \kappa$ or $(I+\kappa)*\phi = \kappa$. Then

$$\xi_t(T) \quad := \quad \mathbb{E}(\lambda_T|\mathcal{F}_t) \quad = \quad \mu + \mu \int_{[0,T]} \kappa(T-s)ds + \int_{[0,t]} \kappa(T-s)dM_s$$

$d\xi_t(T) = \kappa(T-t)dM_t$. Then

$$\mathbb{E}(J_T|\mathcal{F}_t) \quad = \quad J_t + \int_{[t,T]} \mathbb{E}(\lambda_u|\mathcal{F}_t)du \quad = \quad J_t + \int_{[t,T]} (\mu + \mu \int_{[0,u]} \kappa(u-s)ds + \int_{[0,t]} \kappa(u-s)dM_s)du$$

where $\mathcal{F}_t = \mathcal{F}^J_t$, see e.g. [JR18] et al. for more on financial applications of Hawkes processes, and their link to rough stochastic volatility models.

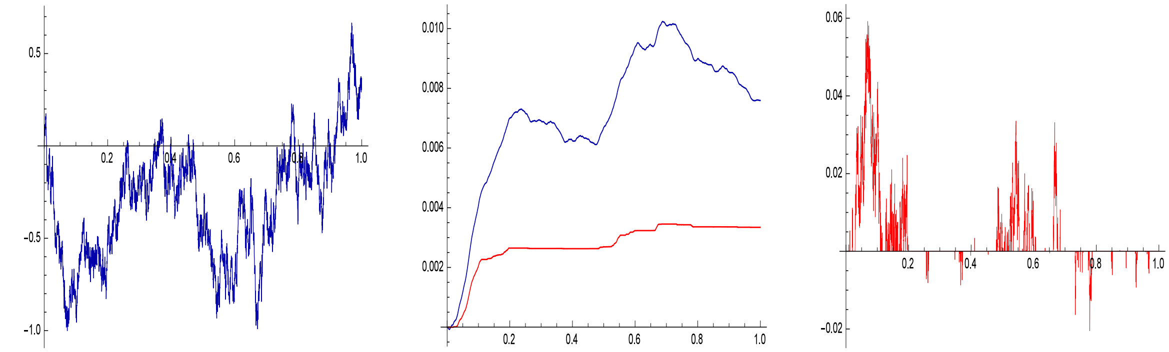

Figure 1: Unconstrained problem: Monte Carlo simulation of S_t, X^*_t and ϕ^*_t when $S_t = Z_t$ is the Riemann-Liouville process, with $H = .4$, $k = 1$ and $T = 1$ with and without transaction costs (red and blue respectively) for $\epsilon = .05$.

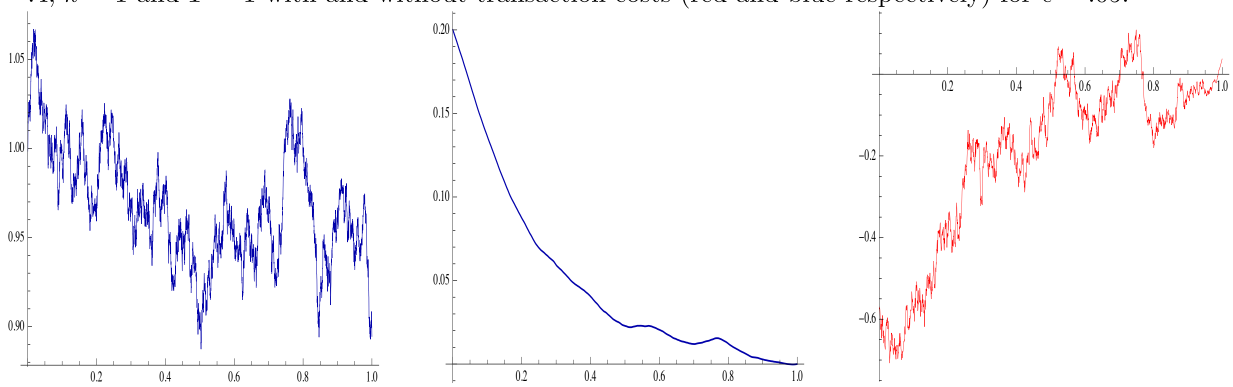

Figure 2: Constrained problem with quadratic inventory penalty: Monte Carlo simulation of S_t, X^*_t and ϕ^*_t with $X_0 = 1.0$ when $dS_t = -\alpha(S_t - \bar{S})dt + \sigma dW_t$ is a standard OU process with $S_0 = 1$, $\alpha = 10$, $\sigma = .2$ and $\gamma = 2$, $k = 1$.

Figure 3: Constrained problem: Monte Carlo simulation of S_t, X^*_t and ϕ^*_t with $X_0 = .25$ when $S_t = Z_t$ is the Riemann-Liouville process with $H = 0.6$, $k = 1$, $T = 1$.

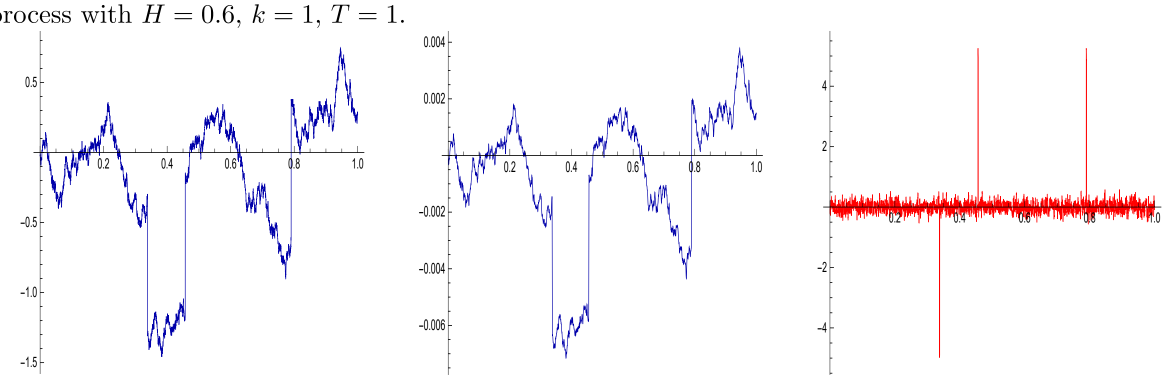

Figure 4: Unconstrained problem with inside information (with $\delta = .001$): Monte Carlo simulation of S_t, X^*_t and ϕ^*_t when $S_t = W_t + J(N^+_t - N^-_t)$ where W is Brownian motion and N^+ and N^- are two i.i.d. Poisson processes (independent of W) with intensity $\lambda = 2$, jump size $J = 2$, and $T = 1$, $k = 0.1$. Note that X^*_t is actually differentiable although this is not obvious to the naked eye.

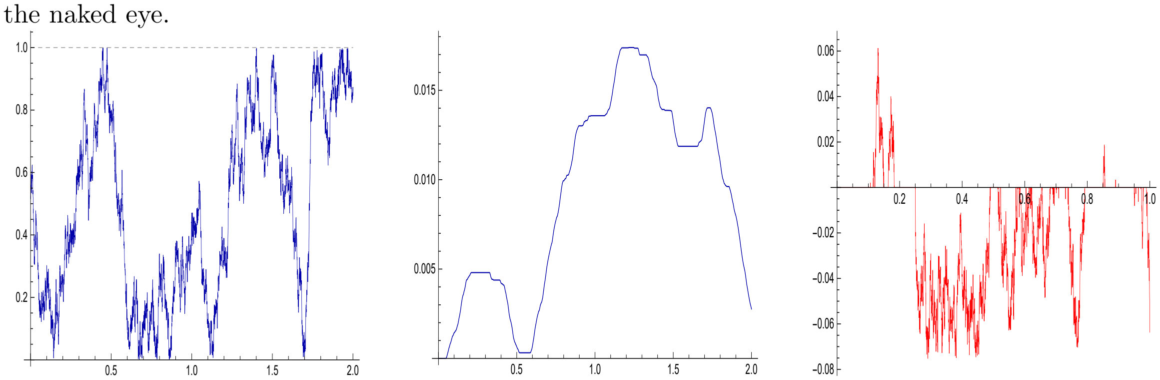

Figure 5: Unconstrained problem: Monte Carlo simulation of S_t, X^*_t and ϕ^*_t when S_t is Brownian motion with reflecting barriers at 0 at 1, with $k = 1$ and $T = 2$ and transaction costs with bid/offer spread $\epsilon = .1$.

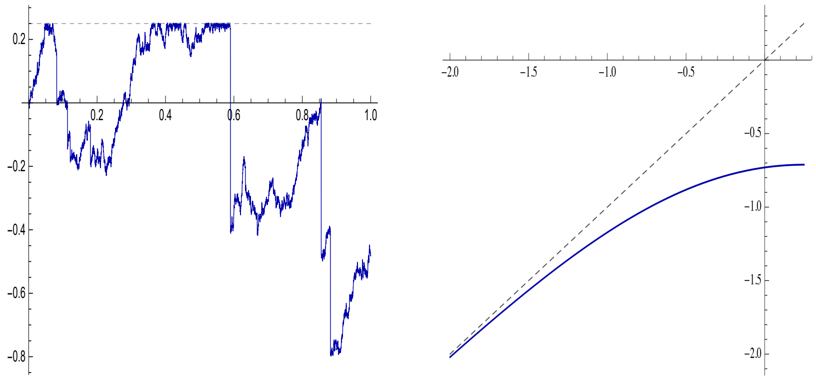

Figure 6: On the left we see a Monte Carlo simulation of a martingale spectrally negative Lévy process X with drift μ, volatility σ, (negative) exponentially distributed jumps with parameter λ_m and jump intensity λ (i.e. a Kou model with negative only jumps) for which $V(p) = p\mu + \frac{1}{2}p^2\sigma^2 - p\lambda/(p+\lambda_m)$, with $\sigma = .4$, $\lambda = 10$, $\lambda_m = 5$ and $\mu = \lambda/\lambda_m = 2$, and a reflecting upper barrier at $b = 0.25$ (using the Skorokhod map). On the right we have plotted $\mathbb{E}(X_T|X_t = x)$ (blue) by computing the inverse Laplace transform of (23) against x (dashed) for $t = 0$ and $T = 1$ as a function of x using Laplace inversion with $b = 3$ now, which is all we need to compute ϕ_t^*.

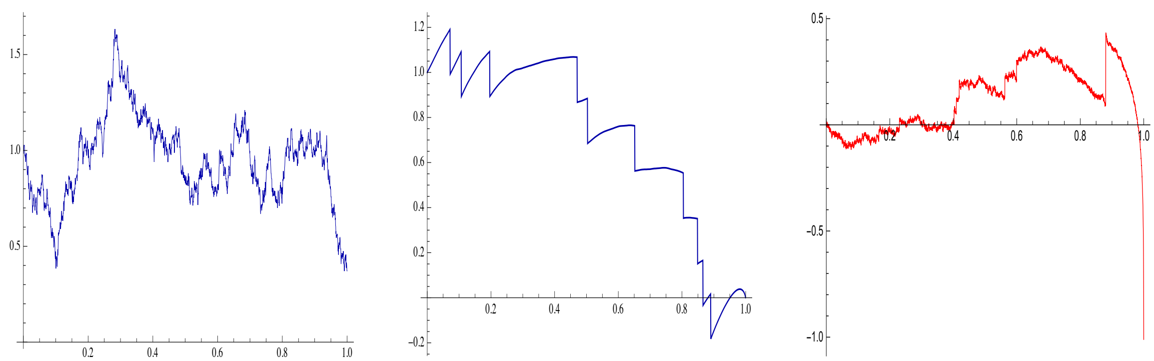

Figure 7: Constrained problem: Monte Carlo simulation of S_t, X_t^* and ϕ_t^* with $X_0 = .25$ when S_t is an OU process with the same parameters as before, and non-zero order flow with J a multiple of a Poisson process with rate $\lambda = 5$ and jump size .04.

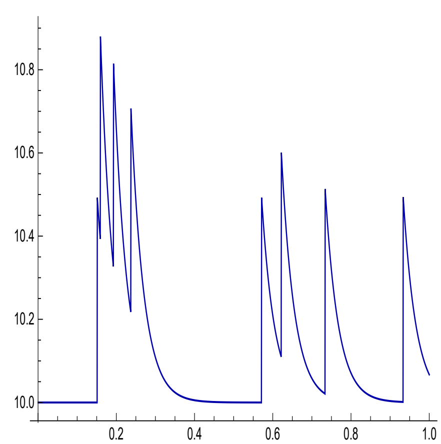

Figure 8: Here we have simulated the intensity process λ_t of a Hawkes process with $\lambda_0 = 0$ and kernel $\phi(t) = \frac{1}{2}e^{-30t}$, see similar graphs in Pakkanen&Moriau-Patrichi[PP18].

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