For Eq 10 in [F24] in the finite option case, set

$$
\begin{aligned}
u(x) &= \sum_{i=0}^{n} w_i^X (x - K_i^X)^+ \\
v(y) &= \sum_{i=0}^{n} w_i^Y (y - K_i^Y)^+ \\
w(z) &= \sum_{i=0}^{n} w_i^Z (z - K_i^Z)^+
\end{aligned}
$$

where $n = 5$ and $K_0^X = K_0^X = K_0^X = 0$ (which correspond to the forward contracts), we then maximize over the 3*6=18 weights to get e^{u+v+yw}. Also need to normalize e^{u+v+yw} as in Eq 11 in [F24]. Then if done correctly, $\mu^*(x,y) = \frac{e^{u+v+yw}}{\mathbb{E}(e^{u+v+yw})}\mu_X(x)\mu_Y(y)$ will correctly price all three smiles. This approach does not require Sinkhorn iterations or the SVI formula. Also remember the SVI formula is not a model.

$$0 \quad = \quad \int_0^\infty (x-y)e^{u_{n+1}(x)+yw_{n+1}(\frac{x}{y})+\sum_{i=1}^n \alpha_i\varphi_i(y)(x-y)}\bar{\mu}(x,y)dx$$

$$\mathbb{E}^{\mu_X}(u(X))+\mathbb{E}^{\mu_Y}(v(Y))+\mathbb{E}^{\mu_Z}(w(Z))-\log\mathbb{E}^{\bar{\mu}}(e^{u(X)+v(Y)+Yw(X/Y)+\sum_{j=1}^n \alpha_j\varphi_j(Y)(X-Y)})$$

Deriv wrt φ_i is

$$\mathbb{E}(\varphi_i(Y)(X-Y)e^{\cdots+\alpha_i\varphi_i(Y)(X-Y)}) \quad = \quad 0$$

Solve for each in turn, coordinate ascent ... same problem as before for root finding?

Galerkin/Ritz method, maximization doesnt always lead to nice first order condition

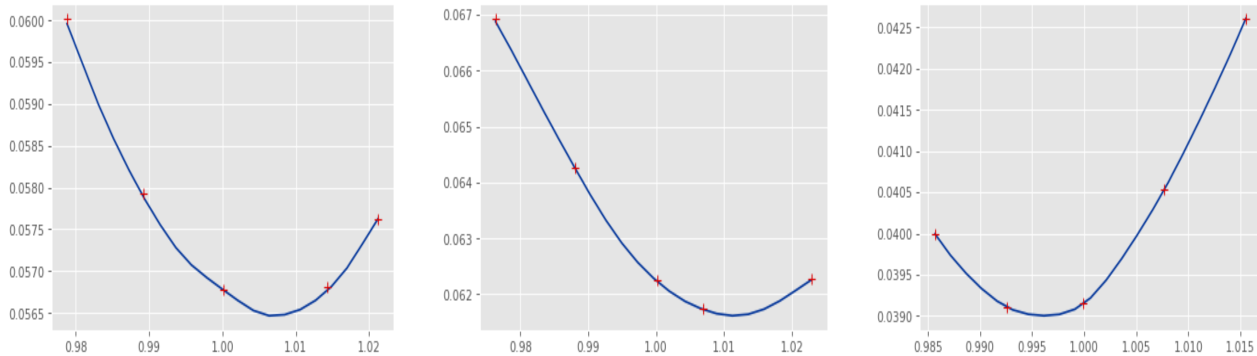

Figure 1: EUR/USD, GBP/USD and EUR/GBP smiles obtained using the finite-dimensional concave maximization scheme in ..., for which the optimal weights (with the forward as the first value) are $u = [-64.367, 15.119, 0.0, 22.66, 0.081, 26.506]$, $v = [-205.85, 31.443, 21.266, 19.126, 22.184, 35.532]$, $w = [178.197, -51.776, -52.955, -86.27, -37.829, -37.829]$.

The finite option case via maximization

Let Π denote the space of probability measures on $[0,\bar{X}]\times[0,\bar{Y}]$, and C^X, C^Y, C^Z denote random vectors with components $C_j^X = (X-K_j^X)^+$, $C_j^Y = (Y-K_j^Y)^+$, $C_j^Z = (X-K_j^Z Y)^+$ with corresponding market prices $P^{X,mkt}$, $P^{Y,mkt}$ and $P^{Z,mkt}$ (all in dollars), and let $\Pi(P^{X,mkt},P^{Y,mkt},P^{Z,mkt}) = \{\mu\in\Pi : \mathbb{E}^\mu((X-K_j^X)^+) = P_j^{X,mkt}, \mathbb{E}^\mu((Y-K_j^Y)^+) = P_j^{Y,mkt}, \mathbb{E}^\mu((X-K_j^Z Y)^+) = P_j^{Z,mkt}, j=0..n\}$ with $K_0^X=0$, $K_0^Y=0$ and $K_0^Z=0$ which correspond to the tradeable forward contracts. Then the n-option analogue of ... is

$$\begin{aligned}\inf_{\mu\in\Pi(P^X,P^Y)} H(\mu,\bar{\mu}) &= \inf_{\mu\in\Pi}\sup_{u,v,w\in\mathbb{R}^n} f(u,v,\mu) = \sup_{u,v,w\in\mathbb{R}^n}\inf_{\mu\in\Pi} f(u,v,\mu)\\ &= \sup_{u,v,w\in\mathbb{R}^n}[u.P^{mkt,X}+v.P^{mkt,Y}+w.P^{mkt,Z}-\log\mathbb{E}^{\bar{\mu}}(e^{u.C^X+v.C^Y+w.C^Z})]\end{aligned}$$

where $f(u,v,\mu) = H(\mu,\bar{\mu}) - \mathbb{E}^\mu(u.C^X+v.C^Y+w.C^Z)+u.P^{mkt,X}+v.P^{mkt,Y}+w.P^{mkt,Z}$.

We have used the Sion minimax theorem to interchange the inf and sup here, since $f(u,v,\mu)$ is LSC and convex in μ (see Lemma 1.3 in [?]; recall also that $H(\mu,\bar{\mu})$ is the large deviation rate function in Sanov's theorem), and concave and continuous in (u,v), and Π is compact and convex and $\mathbb{R}^n$ is convex. This approach allows us to avoid ad hoc interpolation and extrapolation for the Z-smile (e.g. use of the SVI parametrization) although we have used SVI for the X and Y smiles for the numerics below to construct $\bar{\mu}(x,y)$. Note if μ_X and μ_Y have Gaussian tails instead of finite support, then

$$\sum_{j=1}^n u_j \le 0, \quad \sum_{j=1}^n v_j \le 0$$

Then the inner integral

$$\int_1\int_1 e^{u_i(x-K_i^X)^+ + w_i(x-K^Z y)^+}\mu_X(x)dx\mu_X(y)dy$$

is infinite for all y-values if $\sum_{j=1}^n(u_j+w_j)\le 0$.

issue with Guyon Sinkhorn scheme
since otherwise $\log\mathbb{E}^{\bar{\mu}}(e^{u.C^X+v.C^Y+w.C^Z})=\infty$ if $\bar{\mu}$ has Gaussian (or fatter) tails (since we considering the exponential of the exponential of a Gaussian in the former case), and we used this in our numerical results for the finite option case below.

Forwards. Cross options Normalization, tail condition, coord ascent scheme result? Do we need that $K=\mu\in\Pi(P^X,P^Y)$ is compact for anything

Finite option Sinkhorn scheme

Let

$$f(x,y) \quad = \quad e^{\sum_{j=1}^n w_j^X (x-K_j^X)^+ + \sum_{j=1}^n w_j^Y (y-K_j^Y)^+ + \sum_{j=1}^n w_j^Z (x-K_j^Z y)^+}$$

As $w_i^X \to -\infty$, double integral tends to

$$\int\int 1_{x \le K_i^X} e^{\sum_{j\neq i}^n w_j^X (x-K_j^X)^+ + \sum_{j=1}^n w_j^Y (y-K_j^Y)^+ + \sum_{j=1}^n w_j^Z (x-K_j^Z y)^+} dxdy$$

this is a problem if this is $> P_i^{X,mkt}$ (sometimes minimization doesnt lead to a first order condition)

$$P_i^X(w_1,...,w_n) \quad = \quad \int\int (x-K_i^X)^+ \, f(x,y)\bar{\mu}(x,y)dxdy \,.$$

The integrand is monotonically increasing in w_i and tends to $+\infty$ or 0 on $\{x : (x-K_j^X)^+ > 0\}$ as $w_i \to \pm\infty$, **non-zero contribution ignored?** so by the monotone convergence theorem $P_i \to \infty$ as $w_i \to \infty$, and (since $\bar{\mu}$ has compact support) we can also invoke the reverse Monotone convergence theorem) to say that $P_i \to 0$ tends to zero as $w_i \to -\infty$. Moreover, if we consider an arbitrary sequence $w_i^n \to w_i$, then (from the bounded convergence theorem), we see that $P_i(w_1,...,w_n)$ is continuous in w_i. Hence there is a unique w_i^* for which $P_i = P_i^{mkt}$, since $P_i^{mkt} > 0$ by assumption.

Clearly the same proof works for call options on Y as well.

Now let

$$P_i^Z(w_1,...,w_n) \quad = \quad \int\int (x-K_i y)^+ e^{\sum_{j=1}^n w_j (x-K_j)^+} \, \bar{\mu}(x,y)dxdy \,.$$

tail condition

Match forward price as well gives correct expectation but not necessarily density if we dont normalize.
avoid root finding if we use maximization. Is it still convex with awkward denominator?

Conclusion on when call price function leads to genuine density? Need $C'(0) = -1$, $C'(\infty) = 0$, condition even for finite options may not be completely trivial

Proof of Guyon convergence for finite case

Reference measure auto hits X and Y smiles

$$\int\int y\delta(\frac{x}{y}-K)\mu(x,y)dxdy \quad = \quad 1$$

When is min of convex function attained on compact set

$$e^{w_0}e^{\sum_{j=0}^{N} w_j(x-K_j)^+}$$

I think is decreasing function of w_j, similar to

$$\frac{d}{dp}\frac{e^{px}}{\mathbb{E}(e^{pX})} \quad = \quad pe^{px}\mathbb{E}(e^{-pX}) \quad \geq \quad 0$$

for $p > 0$

$$\begin{aligned} & \mathbb{E}^{\mu}((X-Y)^2) - \mathbb{E}^{\mu}(u(X)+v(Y)+Yw(Z)) \\ = \quad & \int((x-y)^2 - u(x) + v(y) + yw(x/y))\varepsilon\mu_1(x,y)dxdy \;-\; \lambda(\int \varepsilon\mu_1(x,y)dxdy) \end{aligned}$$

We dont have nice first order optimality condition

Coordinate ascent for Julien finite scheme or prove Sinkhorn convergence Fwd starter Sinkhorn convergence, sub-gradients, links to large devs theory, does value of λ matter, Julien lemma in Appendix

Let $\bar{\mu} \in \mathcal{P}(\mathbb{R}_+)$ denote the market model[1], $s^j_{b/a}$ the bid/ask prices for call options with strike K_j and $q^j_{b/a}$ the upper and low bounds available to buy/sell. Consider the minmax problem:

$$\inf_{\mu\in\mathcal{P}(\mathbb{R})} \sup_{x\in\mathbb{R}^n: q^j_b \leq x^j \leq q^j_a} [\frac{1}{\alpha}H(\mu|\bar{\mu}) + \sum_j x^j(\mathbb{E}^{\mu}((X_T-K_j)^+) - s^j_a 1_{x^j>0} - s^j_b 1_{x^j<0})] \tag{1}$$

for $\alpha > 0$, where $H(\mu,\bar{\mu}) := \mathbb{E}^{\mu}(\log\frac{d\mu}{d\bar{\mu}}) = \mathbb{E}^{\bar{\mu}}(\frac{d\mu}{d\bar{\mu}}\log\frac{d\mu}{d\bar{\mu}})$ is the entropy of μ with respect to $\bar{\mu}$. Then assuming we can interchange inf and sup, we can re-write this as

$$\frac{1}{\alpha}\sup_{x\in\mathbb{R}^n: q^j_b \leq x^j \leq q^j_a} \inf_{\mu\in\mathcal{P}(\mathbb{R})} [H(\mu|\bar{\mu}) + \alpha\sum_j x^j(\mathbb{E}^{\mu}((X_T-K_j)^+) - s^j_a 1_{x^j>0} - s^j_b 1_{x^j<0})]\,.$$

The inner inf and the optimal μ can be computed explicitly as described on pg 8 in [?] (or see short proof at end of document), so we can further re-write the sup inf as

$$\sup_{x\in\mathbb{R}^n: q^j_b \leq x^j \leq q^j_a} (-\log\mathbb{E}^{\bar{\mu}}(e^{-\alpha\sum_j x^j((X_T-K_j)^+ - s^j_a 1_{x^j>0} - s^j_b 1_{x^j<0})})) \tag{2}$$

where the optimal μ (for each $x = (x^1, .., x^j)$) is

$$\mu(dz) \quad = \quad \frac{e^{-\alpha\sum_j x^j(z-K_j)^+}}{\mathbb{E}^{\bar{\mu}}(e^{-\alpha\sum_j x^j(X_T-K_j)^+}}\,\bar{\mu}(dz)\,.$$

We can re-write (1) as

$$\inf_{\mu\in\mathcal{P}(\mathbb{R})}[H(\mu|\bar{\mu}) + \sum_j q^j_a(\mathbb{E}^{\mu}((X_T-K_j)^+) - s^j_a)^+ + \sum_j |q^j_b|\,(s^j_b - \mathbb{E}^{\mu}((X_T-K_j)^+))^+])$$

i.e. we minimize entropy over models which fall within the bid-offer spread, and models which don't (but these model incurs an additional finite penalty for each option which falls outside). In [?],[?] we just minimize entropy over models which perfectly price all options and there is no bid-offer spread i.e. there is an infinite penalty for non-calibrated models. For the simple setup in the project (assuming all options have the same maturity), as long as the options don't violate basic butterfly arbitrage, we can always find a model i.e. a distribution for X which prices all the options exactly (usual Breeden-Liztenberger type argument).

Issue with Wasserstein topology, rate of convergence, theoretical background on linprog, coordinate ascent scheme for finite case, problem with closedness for Touzi argument, Sinkhorn integrals give the iff condition for the problem to be admissible (recall doesnt require densities), only real issue is the additional assumption of finite entropy?

$Y = \Pi$ is space of probability measures on $\mathbb{R}_+\times\mathbb{R}_+$, $\mathcal{X}$ is $\mathcal{C}^1$. From the Sion minimax theorem

$$\begin{aligned} \inf_{\mu\in\Pi(P^X,P^Y)} H(\mu,\bar{\mu}) \quad &= \quad \inf_{\mu\in\Pi}\sup_{u,v\in\mathbb{R}^n}[H(\mu,\bar{\mu}) + \mathbb{E}^{\mu}(u.C^X + v.C^Y) - P^X(u) - P^Y(v)] \\ &= \quad \sup_{u,v\in\mathbb{R}^n}\inf_{\mu\in\Pi}[H(\mu,\bar{\mu}) + \mathbb{E}^{\mu}(u.C^X + v.C^Y) - P^X(u) - P^Y(v)] \\ &= \quad \sup_{u,v\in\mathbb{R}^n}(H^*(u,v) - P^X(u) - P^Y(v)) \end{aligned}$$

[1] where $\mathcal{P}(\mathbb{R}_+)$ denotes the collection of all probability measures on $\mathbb{R}_+$

$f(u, v, \mu)$ is LSC and convex in μ (since $H(\mu, \bar{\mu})$ is the large deviation rate function in Sanov's theorem), and concave and continuous in (u, v), and Π is compact and convex and $\mathbb{R}^n$ is convex.

Do we need that $K = \mu \in \Pi(P^X, P^Y)$ is compact Need to give clear definition of $\Pi(P^X, P^Y)$ as set of models that fall within bid-offer spread. With constrains on u, v, set is still convex

Let $C^X = (X, (X-K_1)^+, ..., (X-K_n)^+$. Assume $\bar{\mu}$ satisfies

$$\mathbb{E}^{\bar{\mu}}\left(\frac{e^{|\underline{u}|.C^X+|\underline{v}|.C^Y}}{\mathbb{E}^{\bar{\mu}}(e^{-u.C^X-v.C^Y})}(X,Y)\right) = (c_{X,\bar{\mu}}, c_{Y,\bar{\mu}}) < \infty$$

In the minmax argument below, we will need the following probability measure

$$\mu^*(A) = \mathbb{E}^{\bar{\mu}}\left(\frac{e^{-u.C^X-v.C^Y}}{\mathbb{E}^{\bar{\mu}}(e^{-u.C^X-v.C^Y})}1_A\right)$$

for which we see that

$$\mathbb{E}^{\mu^*}((X,Y)) = \mathbb{E}^{\bar{\mu}}\left(\frac{e^{-u.C^X-v.C^Y}}{\mathbb{E}^{\bar{\mu}}(e^{-u.C^X-v.C^Y})}(X,Y)\right) \leq \mathbb{E}^{\bar{\mu}}\left(\frac{e^{|\underline{u}|.C^X+|\underline{v}|.C^Y}}{\mathbb{E}^{\bar{\mu}}(e^{-u.C^X-v.C^Y})}(X,Y)\right) \leq (c_{X,\bar{\mu}}, c_{Y,\bar{\mu}}) \vee 1 < \infty$$

and we let $\mathcal{P}_1$ denote the collection of probability measures on $\mathbb{R}_+ \times \mathbb{R}_+$ which satisfy this condition. Then $\Pi(P^X, P^Y) \subset \mathcal{P}_1$ since all the probability measures in $\Pi(P^X, P^Y)$ have expectation $(X_0, Y_0) = (1, 1)$.

Add cross-rate, $P_X(u)$ could be more general convex function to incorporate LOB, but note that $\bar{q}$ is part of LOB structure, use your Autumn 22 approach. Then

$$\begin{aligned}
&\inf_{\mu\in\mathcal{P}(\mathbb{R}_+\times\mathbb{R}_+)} \sup_{\underline{u}\leq u\leq\bar{u},\underline{v}\leq u\leq\bar{v}} [H(\mu,\bar{\mu}) + \mathbb{E}^{\mu}(u.C^X + v.C^Y) - P^X(u) - P^Y(v)] \\
= &\inf_{\mu\in\mathcal{P}_1} \sup_{\underline{u}\leq u\leq\bar{u},\underline{v}\leq u\leq\bar{v}} [H(\mu,\bar{\mu}) + \mathbb{E}^{\mu}(u.C^X + v.C^Y) - P^X(u) - P^Y(v)] \\
= &\sup_{\underline{u}\leq u\leq\bar{u},\underline{v}\leq u\leq\bar{v}} \inf_{\mu\in\Pi(P^X,P^Y)} [H(\mu,\bar{\mu}) + \mathbb{E}^{\mu}(u.C^X + v.C^Y) - P^X(u) - P^Y(v)] \\
= &\sup_{\underline{u}\leq u\leq\bar{u},\underline{v}\leq u\leq\bar{v}} [-\log \mathbb{E}^{\bar{\mu}}(e^{-u.C^X-v.C^Y}) - P^X(u) - P^Y(v)]
\end{aligned}$$

does 2nd line hold? (Cite improved Guyon proof).

by assumption.

$$\mathbb{E}^{\mu^*}(X) \leq \mathbb{E}^{\bar{\mu}}\left(\frac{e^{|\underline{u}|.C^X+|\underline{v}|.C^Y}}{\mathbb{E}^{\bar{\mu}}(e^{-u.C^X-v.C^Y})}X\right) = c_{\bar{\mu}} < \infty$$

Then

$$\mu^*(X > K) \leq \frac{c_{\bar{\mu}}}{K}$$

Switch from $\mathcal{W}^1$ to weak, can weaken condition on $\bar{\mu}$

Hence we want $\mathbb{E}^{\bar{\mu}}(e^{|\underline{u}|.C^X+|\underline{v}|.C^Y}) < \infty$. $H(\mu,\bar{\mu}) = \mathbb{E}^{\mu}(\log\frac{d\mu}{d\bar{\mu}}) = \mathbb{E}^{\mu}(-u.C^X - v.C^Y) - ....$ Does finite wealth help?

Need $\sum u_i \geq \delta$

if $\mathbb{E}^{\bar{\mu}}(e^{-u.C^X-v.C^Y}) < \infty$ (which can always be true of $\bar{\mu}$ has thin enough tails) seemingly only useful for Chebychev bound if $\sum u_i > 0$, in which it's saying that μ^* is v well behaved (because $\bar{\mu}$ is v well behaved), what about effect of two assets here?

Need **artificial contract** to enforce X, Y bounded, what about if $\bar{\mu}$ is real-world measure

Chung result: require

$$\begin{aligned}
&\inf_{\mu\in\mathcal{P}} \sup_{u\in K} [H(\mu,\bar{\mu}) + \mathbb{E}^{\mu}(u.C^X + v.C^Y) - P^X(u) - P^Y(v)] \\
\leq &\inf_{\mu\in H^c} \sup_{u\in K} [H(\mu,\bar{\mu}) + \mathbb{E}^{\mu}(u.C^X + v.C^Y) - P^X(u) - P^Y(v)]
\end{aligned}$$

We'd need all $\mu \in \Pi(P^X, P^Y)$ to satisfy this c_1 conditon (v unlikely since this is only finite call options)

cross-rate! for some $c_1 < \infty$, hence we can restrict attention μ in this set, which we hope is weakly compact, check Guyon proof

u, v portfolios have linear growth here, stock price needs finite exp moments (which is thinner than BS)

Add cross rate option, tail constraint for finiteness, check RN derivatives in Guyon proof, $G(u,v)=-\infty$ for bad (u,v) anyway

Now add constraints on u,v, add back cross-rate options

We now prove the analogue of Theorem 4.2 i) in [GTT16] for our problem:

Proposition 0.1 *We have the duality relation:*

$$P(\mu_X,\mu_Y,\mu_Z) \;=\; \sup_{u,v,w\in\mathcal{C}^1}\inf_{\mu\in\mathcal{P}(\mathbb{R}_+\times\mathbb{R}_+)}\left(H(\mu,\bar\mu)-\mathbb{E}^{\mu}(u(X)+v(Y)+Yw(Z))+\mathbb{E}^{\mu_X}(u(X))+\mathbb{E}^{\mu_Y}(v(Y))+\mathbb{E}^{\mu_Z}(w(Z))\right). \quad (3)$$

Proof. See Appendix B. ■

Finally, we know that the inner inf in (3) can be computed explicitly as described on pg 8 in [**?**], so we can re-write (3) as the concave maximization problem:

$$P(\mu_X,\mu_Y,\mu_Z) \;=\; \sup_{u,v,w\in\mathcal{C}^1} G(u,v,w) \quad (4)$$

where

$$G(u,v,w) \;=\; \mathbb{E}^{\mu_X}(u(X))+\mathbb{E}^{\mu_Y}(v(Y))+\mathbb{E}^{\mu_Z}(w(Z))-\log\mathbb{E}^{\bar\mu}(e^{u(X)+v(Y)+Yw(X/Y)}) \quad (5)$$

$$\begin{aligned}
\mu_X(x) &= e^{u(x)}\int_0^\infty e^{v(y)+yw(\frac{x}{y})}\bar\mu(x,y)dy\\
\mu_Y(y) &= e^{v(y)}\int_0^\infty e^{u(x)+yw(\frac{x}{y})}\bar\mu(x,y)dx\\
\mu_Z(K) &= \int_0^\infty\int_0^\infty y\delta(\frac{x}{y}-K)e^{u(x)+v(y)+yw(\frac{x}{y})}\bar\mu(x,y)dydx\\
&= \int_0^\infty\int_0^\infty \frac{x}{z}\delta_K(dz)e^{u(x)+v(\frac{x}{z})+\frac{x}{z}w(z)}\frac{x}{z^2}\bar\mu(x,\frac{x}{z})dzdx\\
&= \int_0^\infty \frac{x}{K}e^{u(x)+v(\frac{x}{K})+\frac{x}{K}w(K)}\frac{x}{K^2}\bar\mu(x,\frac{x}{K})dx
\end{aligned} \quad (6)$$

Use MON to prove existence of root, tail problem (cant use puts cos still need forwards, same problem with $\mathcal{C}^1$ in your current paper?)

$$\begin{aligned}
c_X^{mkt}(K_i) &= \int_0^\infty\int_0^\infty (x-K_i)^+e^{\sum_i u_i(X-K_i^X)^+ +\sum_i v_i(Y-K_i^Y)^+ +\sum_i w_i(X-YK_i^Z)^+}\bar\mu(x,y)dydx\\
\mu_Y(y) &= e^{v(y)}\int_0^\infty e^{u(x)+yw(\frac{x}{y})}\bar\mu(x,y)dx\\
\mu_Z(K) &= \int_0^\infty\int_0^\infty y\delta(\frac{x}{y}-K)e^{u(x)+v(y)+yw(\frac{x}{y})}\bar\mu(x,y)dydx\\
&= \int_0^\infty\int_0^\infty \frac{x}{z}\delta_K(dz)e^{u(x)+v(\frac{x}{z})+\frac{x}{z}w(z)}\frac{x}{z^2}\bar\mu(x,\frac{x}{z})dzdx\\
&= \int_0^\infty \frac{x}{K}e^{u(x)+v(\frac{x}{K})+\frac{x}{K}w(K)}\frac{x}{K^2}\bar\mu(x,\frac{x}{K})dx
\end{aligned} \quad (7)$$