Portfolio optimization for an exponential Ornstein-Uhlenbeck model with proportional transaction costs

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Outline of talk

- The exponential Ornstein-Uhlenbeck model with **proportional transaction costs**.
- **Admissible self-financing trading strategies**.
- **Shadow price** processes - definition and why we use them.
- Explicit construction of the shadow price process for the exponential OU model.
- Asymptotics for the **no-trade region** and the **risky fraction** when the transaction cost is small. Results extend the work of [GMS13], who deal with the Black-Scholes case, and show new phenomena.
- The **verification argument**, and links to **excursion theory**.
- Brief discussion on **duality**.

The modelling framework

- We work on some $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and consider a financial market with one riskless bond with a constant price equal to 1 (i.e. zero interest rates) and a risky asset S_t. We assume that S_t is given by the exponential of an **Ornstein-Uhlenbeck** process $S_t = e^{X_t}$, where

$$dX_t \quad = \quad \kappa(\overline{x} - X_t)dt + \sigma dW_t$$

with $\kappa, \overline{x}, \sigma > 0$. By Itô's formula, S_t satisfies

$$dS_t/S_t \quad = \quad [\kappa(\overline{x} - \log(S_t)) + \frac{\sigma^2}{2}]dt + \sigma dW_t =: \mu(S_t)dt + \sigma dW_t \,.$$

- We now model the bid-ask interval by $[(1-\lambda)S_t, S_t]$ for some $\lambda \in (0,1)$. The investor pays S_t for each share bought, but only receives $(1-\lambda)S_t$ for each share sold. Let (φ_t^0, φ_t) denote our holding in the riskless and risky asset at time t.
- Investor wishes to maximize his expected **long-term growth rate**:

$$\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}[\log V_T(\varphi_T^0, \varphi_T)]$$

where $V_t((\varphi_t^0, \varphi_t)) = \varphi_t^0 + \varphi_t^+(1-\lambda)S_t - \varphi_t^- S_t$ is the **liquidation value** of the portfolio at time t (investor has **log utility**).

Admissible trading strategies

Assume the investor starts with x dollars in cash ($x > 0$). Then a pair of adapted processes (φ_t^0, φ_t) is called an **admissible self-financing trading strategy** if both processes are predictable, have finite variation and:

(i) The **self-financing condition**:

$$d\varphi_t^0 \quad = \quad (1-\lambda)S_t \, d\varphi_t^{\downarrow} \; - \; S_t \, d\varphi_t^{\uparrow} \tag{1}$$

for all $0 \leq t \leq T$. φ_t has F.V. so $\varphi_t = \varphi_t^{\uparrow} - \varphi_t^{\downarrow}$, where $\varphi_t^{\uparrow}$, $\varphi_t^{\downarrow}$ are two increasing processes

(ii) The **solvency condition**: there exists an $M > 0$ such that the liquidation value

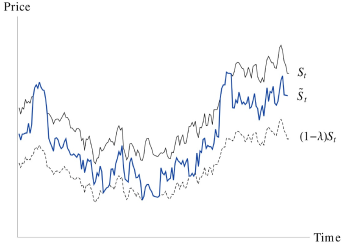
$$V_t(\varphi^0, \varphi) \quad = \quad \varphi_t^0 + \varphi_t^+(1-\lambda)S_t \; - \; \varphi_t^- S_t \quad \geq \quad -M \tag{2}$$

a.s., for all $0 \leq t \leq T$.

- The self-financing condition in (1) ensures that no funds are added or withdrawn to the portfolio, and (2) ensures that the investor **cannot owe** more than M dollars at any time.

Definition of a shadow price

- **Definition**. A **shadow price** is a continuous semimartingale $\tilde{S}_t \in [(1-\lambda)S_t, S_t]$, such that the optimal trading strategy (φ_t^0, φ_t) for a fictitious market with price process $\tilde{S}_t$ and zero transaction costs exists, has finite variation and the number of stocks φ_t only increases when $\tilde{S}_t = S_t$ and decreases when $\tilde{S}_t = (1-\lambda)S_t$.



- Clearly any price process $\tilde{S}_t$ with zero transaction costs which lies in $[(1-\lambda)S_t, S_t]$ leads to **more favourable** terms of trade than the original market with transaction costs. But a shadow price process is a particularly unfavourable model, for which it's optimal to only **buy** when $\tilde{S}_t = S_t$, **sell** when $\tilde{S}_t = (1-\lambda)S_t$ + do nothing in between.

Why do we use shadow prices?

- **Proposition** (Corollary 1.9 in Schachermayer et al.[GMS13]). Let $\tilde{S}_t$ be a shadow price process whose optimal trading strategy (for zero transaction costs) is given by (φ^0_t, φ_t), with $\varphi^0_t, \varphi_t \geq 0$. Then under **non-zero** transaction costs, we have

$$\begin{aligned}\sup_{(\psi^0,\psi)} \mathbb{E}[\log V_T((\psi^0,\psi))] &\geq \mathbb{E}[\log V_T((\varphi^0,\varphi))] \\ &\geq \mathbb{E}[\log V_T((\psi^0,\psi))] + \log(1-\lambda)\end{aligned}$$

  for *any* admissible (ψ^0,ψ). Thus if we choose λ suff small so that $|\log(1-\lambda)| < \varepsilon$ and take the sup over all (ψ^0,ψ), we see that (φ^0,φ) is an ε-**optimal trading strategy** for the original problem.
- Or take liminf as $T \to \infty$ + sup over all admissible strategies, we obtain

$$\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}[\log V_T((\varphi^0,\varphi))] = \sup_{(\psi^0,\psi)} \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}[\log V_T((\psi^0,\psi))] .$$

  Thus the optimal portfolio for the shadow price process is **asymptotically optimal** for the original problem under transaction costs, as $\lambda \to 0$ and/or as $T \to \infty$.

The optimal portfolio for the frictionless case

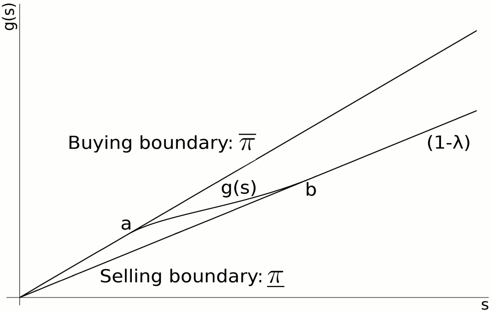
- First consider the case when $\lambda = 0$, i.e. zero transaction costs, and assume S_t follows a general Itô process of the form $dS_t = S_t(\mu_t dt + \sigma_t dW_t)$ with zero interest rates.
- For the frictionless case, we are looking to maximize:

$$
\begin{aligned}
\mathbb{E}[\log V_T] &= \mathbb{E}[\log[x + \int_0^T \phi_t dS_t] \\
&= \mathbb{E}[\log x + \int_0^T \frac{\phi_t S_t}{x + \int_0^t \phi_t dS_t} \frac{dS_t}{S_t} - \frac{1}{2}\int_0^T \frac{\phi_t^2 S_t^2 \sigma_t^2 \, dt}{(x + \int_0^t \phi_t dS_t)^2} dt] \\
&= \mathbb{E}[\log x + \int_0^T (\pi_t \mu_t - \frac{1}{2}\pi_t^2 \sigma_t^2 dt)] \,.
\end{aligned}
$$

- Maximizing the integrand over all π_t, we obtain that $\hat{\pi}_t = \frac{\mu_t}{\sigma_t^2}$, which is known as the **Merton fraction**. For the Black-Scholes case, $\hat{\pi}_t = \hat{\pi} = \mu/\sigma^2$ is constant, but in general $\hat{\pi}_t$ has infinite variation and so will φ_t (unlike the case $\lambda > 0$).
- For the BS case, $dV_t = \hat{\pi} V_t dS_t/S_t = V_t(\hat{\pi}\mu dt + \hat{\pi}\sigma dW_t)$ is GBM, and so is $\phi_t = \hat{\pi} V_t/S_t$.

Construction of the shadow price for the exp OU model for a single excursion from the buy bndry to the sell bndry

Ansatz: if S increases from a to b without setting a new minimum in the meantime, then we guess that $\tilde{S}_t = g(S_t)$ for $0 \leq t \leq \tau_b$, for some $g \in C^2$ and target value $b = b(a, \lambda)$ to be determined. In general, $\tilde{S}_t = g(S_t; a_t, b(a_t, \lambda))$ where $a_t = \min_{0 \leq u \leq t} S_u$ up to time τ_b. For $t \geq \tau_b$, we set $b_t = \max_{\tau_b \leq u \leq t} S_u$, and then $\tilde{S}_t = g(S_t; a(b_t, \lambda), b_t)$ until $\tilde{S}$ returns to the buy boundary (possibly along a new g curve), and so on..



Explicit construction of the shadow price contd.

Assume that $\tilde{S}_0 = S_0 = a$ and $\tilde{S}_t = g(S_t)$ during an excursion from $S = a$ to $S = b$ (we postulate that no trading occurs until S hits b, we will then show how to choose $b = b(a, \lambda)$).

- From the drawing we see that: $g(a) = a$ and $g(b) = (1-\lambda)b$.
- **Smooth-pasting condition**: $g'(a) = 1$, $g'(b) = 1-\lambda$ - this ensures that the **volatility** of $\tilde{S}_t$ vanishes on both boundaries (see below).
- $(1-\lambda)s \le g(s) \le s$ for all $s \in [a, b]$.
- Applying Itô's formula to the shadow price process, we obtain

  $$d\tilde{S}_t \;=\; dg(S_t) \;=\; g'(S_t)dS_t \;+\; \frac{1}{2}g''(S_t)\sigma^2 S_t^2 dt$$

  or $dg(S_t)/g(S_t) = \hat{\mu}_t dt \;+\; \frac{1}{2}\hat{\sigma}_t dW_t$, where $\hat{\mu}_t = [g'(S_t)S_t\mu(S_t) + \frac{1}{2}g''(S_t)\sigma^2 S_t^2]/g(S_t)$, $\hat{\sigma}_t = g'(S_t)\sigma S_t/g(S_t)$.
- But the optimal risky fraction for an investor who maximizes log-utility (with zero transaction costs) is given by

$$\frac{\hat{\mu}(s)}{\hat{\sigma}(s)^2} \;=\; \frac{(g'(s)s\mu(s) + \frac{1}{2}g''(s)\sigma^2 s^2)/g(s)}{(g'(s)^2\sigma^2 s^2)/g(s)^2} \;=\; \frac{\varphi g(s)}{\varphi^0 + \varphi g(s)}. \quad (3)$$

ODE for the shadow price

- Multiplying the numerator and denominator of the right hand side of (3) by $\frac{a}{\varphi^0+\varphi a}$ and setting $\bar{\pi} = \frac{a\varphi}{\varphi^0+\varphi a}$ we have

$$\frac{\bar{\pi} g(s)}{\frac{a\varphi^0}{\varphi^0+\varphi a} + \bar{\pi} g(s)} = \frac{\bar{\pi} g(s)}{a\,\frac{\varphi^0+a\varphi-a\varphi}{\varphi^0+\varphi a} + \bar{\pi} g(s)} = \frac{\bar{\pi} g(s)}{a\,(1-\bar{\pi}) + \bar{\pi} g(s)}.$$

- Combining with (3) yields the following ODE for g:

$$\begin{aligned}\frac{1}{2}\sigma^2 s^2 g''(s) &= \frac{g'(s)^2\sigma^2 s^2}{g(s)} \frac{\bar{\pi} g(s)}{a\,(1-\bar{\pi}) + \bar{\pi} g(s)} - g'(s) s\mu(s) \\ &= g'(s)^2\sigma^2 s^2 \frac{\bar{\pi}}{a\,(1-\bar{\pi}) + \bar{\pi} g(s)} - g'(s) s\mu(s)\end{aligned}$$

which simplifies to

$$\begin{aligned}g''(s) &= \frac{2\bar{\pi} g'(s)^2}{a\,(1-\bar{\pi}) + \bar{\pi} g(s)} - \frac{2g'(s)s\mu(s)}{\sigma^2 s^2} \\ &= \frac{2\bar{\pi} g'(s)^2}{a\,(1-\bar{\pi}) + \bar{\pi} g(s)} - \frac{2g'(s)\theta(s)}{s} \qquad (4)\end{aligned}$$

where $\theta(s) = \mu(s)/\sigma^2$.

Solution for the shadow price

- General solution to ODE in (4) with $g(a) = a, g'(a) = 1$:

$$g(s) \;=\; g(s; a, \overline{\pi}) \;=\; a\,\frac{ah(a) + (1-\overline{\pi})H(a,s)}{ah(a) - \overline{\pi}H(a,s)}$$

  where $H(a,s) = \int_a^s h(u)du,\ h(s) = \exp[\frac{\kappa}{\sigma^2}(\log(s) - \overline{x} - \frac{\sigma^2}{2\kappa})^2]$.
- Plugging this into $g(b) = (1-\lambda)b, g'(b) = 1-\lambda$ we obtain:

$$\overline{\pi} \;=\; \frac{a(H(a,b) + \lambda bh(a) - bh(a) + ah(a))}{(a + \lambda b - b)H(a,b)} \tag{5}$$

  and $F(a,b,\lambda) = H(a,b)^2(\lambda - 1) + (a + b(\lambda - 1))^2 h(a)h(b) = 0$.
- Solving the quadratic for λ we find the physically meaningful solution is given by

$$\lambda(a,b) = 1 - \frac{a}{b} - \frac{1}{2}\frac{H(a,b)^2}{b^2h(a)h(b)} - \frac{H(a,b)}{2b}\sqrt{\frac{H(a,b)^2}{b^2h(a)^2h(b)^2} + \frac{4a}{bh(a)h(b)}}.$$

- For λ fixed, eq has 2 solNs $b = b_{1/2}(a,\lambda)$ with $b_1 < a < b_2$. To choose the physically meaningful solution, it turns out there is a critical $a = a_0(\lambda)$ such that $b = b_2(a,\lambda)$ for $a \geq a_0$ and $b = b_1(a,\lambda)$ for $a < a_0$.

Asymptotics

- Since λ is smooth, we can expand it in a Taylor series around the point $b = a$:

$$\lambda(a,b) = \frac{\Gamma(a)}{6a^3}(b-a)^3 + O((b-a)^4), \tag{6}$$

where $\Gamma(s) = \theta(s)(1-\theta(s)) - \theta'(s)s$.

- Inverting (6), for $a \notin \{a^*, b^*\}$, we obtain

$$b(a,\lambda) \;=\; a \;+\; a(\frac{6}{\Gamma(a)})^{\frac{1}{3}}\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}})].$$

- For the risky fraction $\bar{\pi}$, plugging this expansion into (5) we get:

$$\bar{\pi} = \theta(a) - (\frac{3}{4}\,\Gamma(a)^2)^{\frac{1}{3}}\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}).$$

- For $a = a_0(\lambda)$ there are two b-values, and

$$\begin{aligned} b_{1,2}(a,\lambda) \;&=\; a^* \pm a^*\sqrt{2}\,(\frac{3\sigma^4}{\kappa\sigma\sqrt{(4\kappa+\sigma^2)}})^{1/4}\lambda^{\frac{1}{4}} \;+\; O(\sqrt{\lambda})\,, \\ \bar{\pi} \;&=\; \theta(a) - \frac{\sqrt{\kappa}\,(4\kappa+\sigma^2)^{\frac{1}{4}}}{\sqrt{3}\,\sigma^{\frac{3}{2}}}\lambda^{\frac{1}{2}} \;+\; O(\sqrt{\lambda})\,, \end{aligned}$$

Remarks

- The special value $a_0(\lambda)$ does show up for the Black-Scholes model where we do not see the $\lambda^{\frac{1}{4}}$ asymptotic behaviour.
- In general $\tilde{S}_t = g(S_t; a_t, \bar{\pi}(a_t, b(a_t, \lambda), \lambda)$ for some continuous process a_t with finite variation, which only increases/decreases when $\tilde{S}_t$ is on boundaries of the bid-ask cone.
- If $S_0 = a$, then before S hits $b(a, \lambda)$ or $a_0(\lambda)$, $a_t = \min_{0 \leq u \leq t} S_u$: in words, every time S_t sets a new minimum, we need a new a-value for $g(.)$, but when S_t makes an **excursion** away from its **minimum process**, $da_t = 0$, and $\tilde{S}_t$ just follows the g curve from left to right.
- For $t \geq \tau_b$, we set $b_t = \max_{\tau_b \leq u \leq t} S_u$, and then $\tilde{S}_t = g(S_t; a(b_t, \lambda), b_t)$ until $\tilde{S}$ returns to the buy boundary or b_t hits the critical value $b_0(\lambda)$.
- The g curves change direction to the left of $a_0(\lambda)$ and to the right of $b_0(\lambda)$. At these critical values, there are two valid g curves (not one).

Remarks

- We can show that the optimal number of shares φ_t evolves as

$$\frac{d\varphi_t}{\varphi_t} = -\frac{\bar{\Gamma}(a_t;\lambda)}{\bar{\pi}(a_t, b(a_t,\lambda),\lambda)}\,\frac{da_t}{a_t} \tag{7}$$

  where $\bar{\Gamma}(a;\lambda) = -a\bar{\pi}'(a) + \bar{\pi}(a)(1-\bar{\pi}(a))$, and here $\bar{\pi}(a)$ is shorthand for $\bar{\pi}(a, b(a,\lambda),\lambda)$.

- Integrating (7) we get $\log \frac{\varphi_t}{\varphi_0} = F(a_t) := -\int_{a_0}^{a_t} \frac{\bar{\Gamma}(u;\lambda)}{\bar{\pi}(u,b(u,\lambda),\lambda)u}du$.

- For $\tilde{S}_t$ to be a genuine shadow price, we have to verify that $d\varphi_t \geq 0$ when $S_t = \tilde{S}_t$, and $d\varphi_t \leq 0$ when $S_t = (1-\lambda)\tilde{S}_t$ (this is the so-called **verification argument**).

Numerics

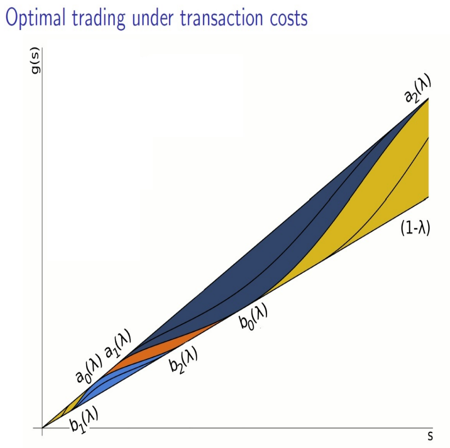


Figure: Here we have plotted various shadow price curves $g(S; a, b)$ as a function of s, for $\lambda = .3$ and $\kappa = 3, \sigma = 1, \bar{x} = 1$, for which we find that $a_0(\lambda) = 1.3914$, $b_0(\lambda) = 5.31052$ and, and $a_1(\lambda) = 1.95159$, $a_2(\lambda) = 10.5602$ and $b_1(\lambda) = 0.699707$, $b_2(\lambda) = 3.78616$.

Duality

- Let $\tilde{S}_t \in [(1-\lambda)S_t, S_t]$ and Z^0_t denote the density process of an **ELMM** $\mathbb{Q}$ for $\tilde{S}_t$, and let $Z^1_t = Z^0_t \tilde{S}_t$.
- Let $V(y) = \sup_{x>0}[U(x) - xy]$ denote the **Fenchel-Legendre** transform of U, where U is a (concave) utility function.
- Then (as before) trading the shadow price process is more favourable than trading the real risky asset, so for any admissible trading strategy (φ^0, φ) we have

$$\begin{aligned}\mathbb{E}[U(V_T(\varphi^0,\varphi)] \leq \mathbb{E}[U(x+\varphi_t \cdot d\tilde{S}_t)] &\leq \mathbb{E}[V(yZ^0_T) + (x+\varphi_t \cdot \tilde{S}_t)yZ^0_T)\\ &\leq \mathbb{E}[V(yZ^0_T] + xy\end{aligned}$$

because $\mathbb{E}(Z^0_T) = 1$ and $\tilde{S}_t$ is a local martingale under $\mathbb{Q}$.
- Taking sups and infs, we see that

$$\sup_{(\varphi^0,\varphi)} \mathbb{E}[U(V_T(\varphi^0,\varphi)] \quad \leq \quad \inf_{(Z^0,Z^1,y)} \mathbb{E}[V(yZ^0_T)] + xy\,.$$

- If the dual optimizers $(\hat{Z}^0, \hat{Z}^1, y)$ exist then we have equality, and $U'[V_T(\varphi^0,\varphi)] = \hat{y}\hat{Z}^0_T$. For $U(x) = \log x$, we have $V(y) = -\log y - 1$.

References

- [GMS13] Gerhold, S., J.Muhle-Karbe and W.Schachermayer, "The Dual Optimizer for the Growth-Optimal Portfolio under Transaction Costs", *Finance and Stochastics*, Vol. 17 (2013), No. 2, pp. 325-354.
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