6. Jump models

In this chapter, we enrich the Black-Scholes model by adding jumps to the model, so the stock price sample path is no longer continuous.

- To begin with, we let

$$X_t = W_t + N_t$$

  where W_t is standard Brownian motion, and N_t is an independent Poisson process with parameter $\lambda > 0$, which means that

$$\mathbb{P}(N_t = n) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

  i.e. N_t is a Poisson random variable with parameter λt (rather than the usual λ).

- Recall that a Poisson process has the property that the times between jumps are i.i.d. exponential $\mathrm{Exp}(\lambda)$ random variables and the probability of a jump in a small time interval Δt tends to $\lambda \Delta t$ as $\Delta t \to 0$.

- We now wish to compute $\mathbb{P}(X_t > x)$. We first note that we can re-write the event $\{X_t > x\}$ as follows:

$$\{X_t > x\} = \cup_{n=0}^{\infty} \{X_t > x\} \cap \{N_t = n\}\,.$$

  But the (infinite) union on the right hand side is a union of *disjoint* events. Thus the probability of the union is equal to the sum of the individual probabilities:

$$\begin{aligned}
\mathbb{P}(X_t > x) &= \sum_{n=0}^{\infty} \mathbb{P}(X_t > x, N_t = n) \\
&= \sum_{n=0}^{\infty} \mathbb{P}(X_t > x \,|\, N_t = n)\, \mathbb{P}(N_t = n)
\end{aligned}$$

  where we have used the usual rule of conditional probability that $\mathbb{P}(A \cap B) = \mathbb{P}(A \,|\, B)\mathbb{P}(B)$ in the last line.

- We can now compute the distribution function of X_t by *conditioning* on the independent N_t process first:

$$\begin{aligned}
\mathbb{P}(X_t > x) &= \sum_{n=0}^{\infty} \mathbb{P}(X_t > x \,|\, N_t = n)\, \mathbb{P}(N_t = n) \\
&= \sum_{n=0}^{\infty} \mathbb{P}(W_t + n > x \,|\, N_t = n)\, \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= \sum_{n=0}^{\infty} \mathbb{P}(W_t + n > x)\, \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= \sum_{n=0}^{\infty} \mathbb{P}(\frac{W_t}{\sqrt{t}} > \frac{x-n}{\sqrt{t}})\, \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= \sum_{n=0}^{\infty} \Phi^c(\frac{x-n}{\sqrt{t}})\, \frac{(\lambda t)^n e^{-\lambda t}}{n!}\,.
\end{aligned}$$

- X_t has a Brownian motion component, and a jump component, and is a simple example of a **Lévy process**.

6.1 The Merton jump diffusion model

- Now let $S_t = e^{X_t}$ denote the stock price process, where

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} \xi_i$$

where W_t is standard Brownian motion, the ξ_i's are independent $N(\alpha, \delta^2)$ random variables i.e. the jumps are now of **random** size, and N_t is a Poisson process with parameter $\lambda > 0$. We assume that N_t, W_t and all the ξ_i's are independent of one another.

- The sum $\sum_{i=1}^{N_t} \xi_i$ is known as a compound Poisson process. Note that if $\xi_i \equiv 1$, then $\sum_{i=1}^{N_t} \xi_i = N_t$ and we are back to the simple model discussed in the previous section.

- As before, we compute the distribution function of X_t by *conditioning* on the independent N_t process first:

$$\begin{aligned}\mathbb{P}(X_t > x) &= \sum_{n=0}^{\infty} \mathbb{P}(X_t > x \,|\, N_t = n)\, \mathbb{P}(N_t = n) \\ &= \sum_{n=0}^{\infty} \mathbb{P}(\mu t + \sigma W_t + \sum_{i=1}^{n} \xi_i > x)\, \frac{(\lambda t)^n e^{-\lambda t}}{n!}\end{aligned}$$

- But $\mu t + \sigma W_t + \sum_{i=1}^{n} \xi_i$ is just a constant (i.e. μt) plus a sum of $n+1$ independent Normal random variables. But a sum of independent Normal random variables is still a Normal random variable, with mean and variance given by the sum of the individual means and variances. So in this case

$$\mu t + \sigma W_t + \sum_{i=1}^{n} \xi_i \sim N(\mu t + n\alpha, \sigma^2 t + n\delta^2)\,. \tag{1}$$

- Thus we have

$$\mathbb{P}(X_t > x) = e^{-\lambda t} \sum_{n=0}^{\infty} \Phi^c(\frac{x - \mu t - n\alpha}{\sqrt{\sigma^2 t + n\delta^2}})\, \frac{(\lambda t)^n}{n!}$$

This series converges rapidly and be easily computed in VBA (see the Excel sheet JumpDiffusionModel.xls).

- Differentiating this expression with respect to x, and multiplying by -1, we obtain the density of X_t (we omit the details for the same of brevity).

6.2 A general jump diffusion model

Now consider a more general jump diffusion model where $X_t = \mu t + \sigma W_t + Y_t$, where $Y_t = \sum_{i=1}^{N_t} \xi_i$ and the ξ_i's are independent and identically distributed (i.i.d) random variables with density $\mu(x)$ and N_t is a Poisson process with parameter $\lambda > 0$, and W, N and the ξ_i are all independent of each other. Using independence we have

$$\mathbb{E}(e^{iuX_t}) = \mathbb{E}(e^{iu(\mu t + \sigma W_t)})\, \mathbb{E}(e^{iuY_t}) = \exp\left[i\mu u t - \frac{1}{2}\sigma^2 u^2 t\right] \mathbb{E}(e^{iuY_t}) \tag{2}$$

where $i = \sqrt{-1}$. Here we have used that $\mathbb{E}(e^{pZ}) = e^{\mu_1 p + \frac{1}{2}\sigma_1^2 p^2}$ if $Z \sim N(\mu_1, \sigma_1^2)$; here $Z = \mu t + \sigma W_t$ so $\mu_1 = \mu t$, $\sigma_1^2 = \sigma^2 t$ and $p = iu$.

$$\begin{aligned}\mathbb{E}(e^{iuY_t}) = \sum_{n=0}^{\infty} \mathbb{E}(e^{iuY_t} \,|\, N_t = n)\, \mathbb{P}(N_t = n) &= \sum_{n=0}^{\infty} \mathbb{E}(e^{iu(\xi_1 + \ldots \xi_n)})\, \mathbb{P}(N_t = n) \\ &= \sum_{n=0}^{\infty} \mathbb{E}(e^{iu\xi_1})\mathbb{E}(e^{iu\xi_2}) \ldots \mathbb{E}(e^{iu\xi_n})\, \mathbb{P}(N_t = n) \\ &= \sum_{n=0}^{\infty} \mathbb{E}(e^{iu\xi_1})^n\, \mathbb{P}(N_t = n)\end{aligned}$$

where we have used that $\xi_1, \xi_2, ...$ are i.i.d. Now let $\phi(u) = \mathbb{E}(e^{iu\xi_1})$. Then we have

$$\begin{aligned}
\mathbb{E}(e^{iuY_t}) &= \sum_{n=0}^{\infty} \phi(u)^n \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= \exp(-\lambda t + \phi(u)\lambda t) \\
&= \exp(-\lambda t + \lambda t \int e^{iux}\mu(x)dx) \\
&= \exp[\lambda t \int_{-\infty}^{\infty} (e^{iux} - 1)\mu(x)dx]
\end{aligned}$$

using that $\int \mu(x)dx = 1$ because $\mu(x)$ is a density. In general, if we know the characteristic function $\phi(k)$ of a random variable, if its density exists, we can compute its density using an **inverse Fourier transform**:

$$p(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx}\phi(k)dk\,.$$