The rough Heston model

Recall the classical 1993 Heston stochastic volatility model for stock price process S, defined by the following stochastic differential equations when the interest rate $r = 0$:

$$\begin{cases} dS_t = S_t\sqrt{V_t}(\rho dW_t^2 + \sqrt{1-\rho^2}dW_t^1)\,, \\ dV_t = \kappa(\theta - V_t)dt + \nu\sqrt{V_t}dW_t^2 \end{cases}$$

where W^1, W^2 are two independent Brownian motions, and with $V_0 > 0$, $\kappa, \theta, \nu > 0$, $|\rho| \le 1$ and $2\kappa\theta > \nu^2$, which ensures that V cannot hit zero (see FM14 for proof of this result).

- We cannot compute the density of $X_t = \log S_t$ exactly. However, there is a closed-form expression for the **characteristic function** $\phi(k) = \mathbb{E}(e^{ikX_t})$ of the form

$$u(x,v,t) \quad = \quad \mathbb{E}(e^{ikX_t}|X_0 = x, V_0 = v) \quad = \quad e^{ikx+vh(t)+g(t)} \tag{1}$$

  for $k \in \mathbb{R}$, where g and h also depend on k, and we note that the exponent is **affine** in the v variable.

- To see where (1) comes, we note that from the 2d version of the Feynman-Kac formula, $u(x,v,t)$ satisfies the PDE

$$u_t \quad = \quad -\frac{1}{2}vu_x + \frac{1}{2}vu_{xx} + \kappa(\theta - v)u_v + \rho\nu v u_{xv} + \frac{1}{2}\nu^2 u_{vv}$$

  with initial condition $u(x,v,0) = e^{ikx}$. Guessing the form of u as in (1) and then equating coefficients in v and terms that do not contain v, we find that h and g must satisfy

$$h'(t) \quad = \quad \frac{1}{2}(-ik - k^2) \,+\, (\rho\nu ik - \lambda)h(t) \,+\, \frac{1}{2}\nu^2 h(t)^2 \quad , \quad g'(t) \quad = \quad \kappa\theta h(t) \tag{2}$$

  with $h(0) = 0$ and $g(0) = 0$ (Mathematica is very useful for doing these type of computations).

- We can extend this to the **Rough Heston** model for which V satisfies a **Stochastic Volterra Equation** (SVE):

$$V_t \quad = \quad V_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\kappa(\theta - V_s)ds \,+\, \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\nu\sqrt{V_s}dW_s^2 \tag{3}$$

  for $\alpha = H + \frac{1}{2}$ with $H \in (0, \frac{1}{2}]$, where Γ is the Gamma function. The Rough Heston model is the standard Heston model when $H = \frac{1}{2}$. H controls the **roughness** of the sample path of V, i.e. V_t is rougher than the standard Heston model when $H < \frac{1}{2}$ and smoother when $H > \frac{1}{2}$; more precisely $|V_t - V_s| \le \mathit{const.} \times |t-s|^{H-\varepsilon}$ for all $\varepsilon \in (0, H]$ where const. is a random constant that depends on the V path. One can show that when $H < \frac{1}{2}$, V spends zero amount of time at zero.

- For the rough Heston model, (1) generalizes to

$$\psi(k,t) \quad := \quad \mathbb{E}(e^{ikX_t}) \quad = \quad e^{ikX_0 + V_0(I^{1-\alpha}\phi)(t) + \lambda\theta(I^1\phi)(t)} \tag{4}$$

  where $(I^r f)(t) = \frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}f(s)ds$ denotes the rth **fractional integral** of a general function f, and now ϕ satisfies the **Volterra Integral Equation (VIE)**:

$$\begin{aligned} \phi(t) \quad &= \quad I^\alpha\Big(\frac{1}{2}(-ik-k^2) \,+\, (\rho\nu ik - \lambda)\phi \,+\, \frac{1}{2}\nu^2\phi^2\Big)(t) \\ &= \quad \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big(\frac{1}{2}(-ik-k^2) \,+\, (\rho\nu ik - \lambda)\phi(s) \,+\, \frac{1}{2}\nu^2\phi(s)^2\Big)ds \end{aligned} \tag{5}$$

  which implies that $h(0) = 0$ (for $H = \frac{1}{2}$ this simplifies to the Riccati ODE in (2)). Note there's no randomness in this equation, and I^1 in (4) is just the usual integration operator, since $r - 1 = 0$ and $\Gamma(1) = 1$. This VIE can be solved numerically with an **Adams scheme** as follows:

$$\phi(t_i) \quad = \quad \frac{1}{(H+\frac{1}{2})\Gamma(\alpha)}\sum_{j=1}^{i}[(t_i - s_j)^{H+\frac{1}{2}} - (t_i - s_{j-1})^{H+\frac{1}{2}}]f(\phi(s_{j-1}))$$

  which comes from integrating $(t-s)^{\alpha-1} = (t-s)^{H-\frac{1}{2}}$ over each small time step with $s_i = \frac{i}{n}dt$ and $f(w) = \frac{1}{2}(p^2 - p) + (\rho p\nu - \lambda)w + \frac{1}{2}\nu^2 w^2$ and $p = ik$.

- We use the **Lewis Fourier inversion** formula to get the price of a call option with strike $K = e^x$:

$$\mathbb{E}(\max(S_T - K, 0)) \quad = \quad e^{-rT}\mathbb{E}((e^{X_T} - e^x)^+) \quad = \quad e^{-rT}(1 - e^{\frac{1}{2}x}\int_{-\infty}^{\infty}\frac{e^{-iux}}{u^2+\frac{1}{4}}\psi(u-\frac{1}{2}i, T)du)\,.$$

where $x = \log\frac{K}{F_0}$ and $F_0 = e^{(r-q)t}$ is the initial forward price. So in short, we have to solve Eq (5) to get ϕ, then compute $1-\alpha$th order fractional integral of ϕ to get $\psi(u, T)$, then implement the inverse Fourier transform to obtain the call price.

- Alternatively, we can simulate V using Monte Carlo with an **Euler**-type scheme as follows:

$$V_{(i+1)\Delta t} \quad = \quad V_0 + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{i-1}((i-j)\Delta t)^{H-\frac{1}{2}}(\lambda(\theta - V_{j\Delta})\Delta t + \sqrt{V_{j\Delta}}\,\Delta W_j) \tag{6}$$

where ΔW_j is a sequence of i.i.d. $N(0, \Delta t)$ random variables. We can easily implement (6) in Matlab as:

```
for j=1:N
    t=j*dt;
    V(j)=V0;
    for k=1:j-1
        s=k*dt;
        V(j) =max(V(j)+c1*(t-s)^(H-.5)*(lambda*(theta-V(k))*dt+sqrt(V(k))*dW(k),0);
  end
end
```

or similarly in **Python**. Note the code requires an additional inner loop which the usual Euler scheme does not need, so the Monte Carlo for rough models like is slower, i.e. $O(N^2)$ not $O(N)$, which is an important issue in practice. For small H-values below e.g. .05, typically a large amount of simulations and time steps are required to get decent accuracy, so it's preferable to run code on a GPU.

1 More background on the Rough Heston model

- For the general case $\alpha \neq 1$, the two integrands in (3) contain $(t-s)^{\alpha-1}$ terms (and thus depend on t), so this is not the integrated form of a standard SDE). The Rough Heston model is more realistic than Black-Scholes because volatility is now stochastic and **mean-reverting**, the model has fat tails i.e. $\mathbb{E}(e^{pX_t}) = \infty$ for some $p = p^*(t) < \infty$ sufficiently large (unlike the Black-Scholes model for which $\mathbb{E}(e^{pX_t}) = e^{\frac{1}{2}\sigma^2(p^2-p)t} < \infty$ for all $p \in \mathbb{R}$ when $r = 0$), and the Rough Heston model is more consistent with observed behaviour of traded European option prices, particularly at small maturities.

- Taking expectations of (3) and using that the expectation of the stochastic integral term is zero, we see that

$$\begin{aligned}
\mathbb{E}(V_t) &= V_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda(\theta - \mathbb{E}(V_s))ds + \frac{1}{\Gamma(\alpha)}\mathbb{E}(\int_0^t (t-s)^{\alpha-1}\nu\sqrt{V_s}dB_s) \\
&= V_0 + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\theta - \mathbb{E}(V_s))ds. \qquad (7)
\end{aligned}$$

- Now let $\xi_t(u) = \mathbb{E}(V_u|\mathcal{F}_t)$. From the tower propery in FM01, we can easily verify that any process of the form $\mathbb{E}(X|\mathcal{F}_t)$ is a martingale, if X is random variable with $\mathbb{E}(|X|) < \infty$. Thus for our particular example here, $\xi_t(u)$ is a martingale in t with respect to $\mathcal{F}_t^B$, and

$$\begin{aligned}
\xi_t(u) &= V_0 + \frac{1}{\Gamma(\alpha)}\int_0^u (u-s)^{\alpha-1}\lambda(\theta - \mathbb{E}(V_s|\mathcal{F}_t))ds + \frac{1}{\Gamma(\alpha)}\mathbb{E}((\int_0^t + \int_t^u)(u-s)^{\alpha-1}\nu\sqrt{V_s}dB_s) \\
&= V_0 + \frac{1}{\Gamma(\alpha)}\int_0^u (u-s)^{\alpha-1}\lambda(\theta - \mathbb{E}(V_s|\mathcal{F}_t))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (u-s)^{\alpha-1}\nu\sqrt{V_s}dB_s
\end{aligned}$$

  where we have used that $\int_0^t (u-s)^{\alpha-1}\nu\sqrt{V_s}dB_s$ is $\mathcal{F}_t$-measurable.

- If $\lambda = 0$, the 1st integral term on the right hand side is zero, and we can re-write the 2nd term in the differential form:

$$d\xi_t(u) = \kappa(u-t)\sqrt{V_t}dB_t \qquad (8)$$

  where $\kappa(u-t) = \frac{\nu}{\Gamma(\alpha)}(u-t)^{\alpha-1}$.

- For general $\lambda \neq 0$, we can show that

$$d\xi_t(u) = \kappa(u-t)\sqrt{V_t}dB_t \qquad (9)$$

  for some (more complicated) function κ for which we can compute a series expansion. Note that $\xi_t(t) = \mathbb{E}(V_t|\mathcal{F}_t) = \xi_t(t)$.

- $\xi_t(u)$ (considered as a function of u, for a fixed t) is known as the **forward variance curve** at time t, which moves up and down and tilts as time evolves, since (depending on κ) some parts of the curve are responsive than others to changes in W. $\xi_t(u)$ satisfies the **Markov property** in itself, since we only need to know $V_t = \xi_t(t)$ to be able to compute $d\xi_t(u)$. Note that V is not Markov in itself (see (12) below to see why).

- We can also consider other models which are not Rough Heston but for which (9) is still satisfied, and models of this form are known as **affine forward variance models**. We can integrate this relation to obtain

$$\xi_t(u) = \xi_0(u) + \int_0^t \kappa(u-s)\sqrt{V}_s dB_s \qquad (10)$$

  so in particular

$$V_t = \xi_t(t) = \xi_0(t) + \int_0^t \kappa(t-s)\sqrt{V}_s dB_s \qquad (11)$$

  which generalizes the Rough Heston model.

- Note we can either specify the dynamics for V (as we do for Rough Heston, in which case $\xi_0(u)$ is obtained by solving (7)) and $\kappa(u-t)$ can be computed as in Homework 2 q1, or much easier when $\lambda = 0$), in which case $\kappa(u-t) = \frac{1}{\Gamma(\alpha)}\nu(u-t)^{-\alpha}$), *or* we can specify the initial variance curve $\xi_0(t)$ and $\kappa(t-u)$ exogenously, and V_t is then given by (11).

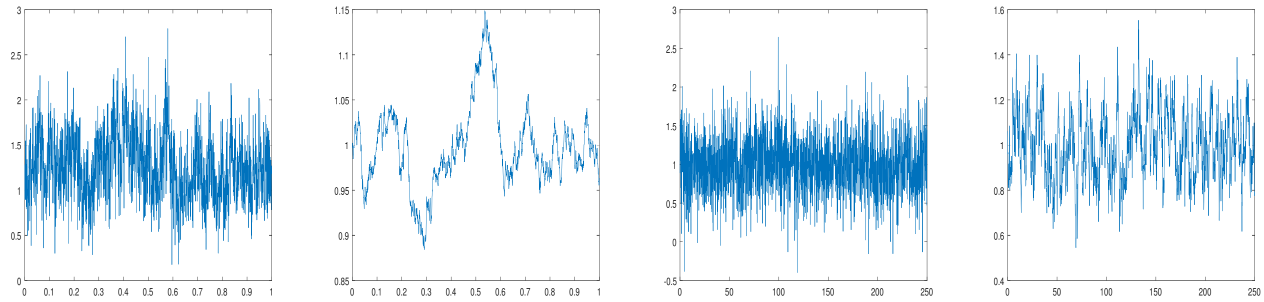

Figure 1: Here we have plotted a Monte Carlo simulation of V_t for the rough Heston model with $\alpha = .55$ (i.e. $H = .05$) (first and third plots), i.e. α close to the lower limit of 0.5, and $\alpha = 1$ (second and final plot), with $\lambda = 1$, $\theta = V_0 = 1$ and $\nu = .4$. One can see that V becomes rougher as α becomes smaller. Note the time horizon is much larger for the 3rd and 4th plots.

- Note that for general κ, (11) is no longer the Rough Heston model, but rather a more general *affine variance curve model*. Note that

$$V_t - V_s \quad = \quad \xi_0(t) \,-\, \xi_0(s) \,+\, \int_0^t \kappa(t-r)\sqrt{V}_r dB_r \,-\, \int_0^s \kappa(s-r)\sqrt{V}_r dB_r$$

  so V is not Markov in itself, since the right hand side depends on V going back to time zero, not just over $[s,t]$. F

- Compare this to the **Rough Bergomi** model

$$d\xi_t(u) \quad = \quad \eta\xi_t(u)dB_t$$

  or the **standard Bergomi** model:

$$d\xi_t(u) \quad = \quad \eta e^{-\lambda(T-u)}\xi_t(u)dB_t$$

  where we typically extract the initial variance curve $\xi_0(t)$ from the market prices of variance swaps which pay $\int_0^t V_s ds$, since $\xi_0(t) = \frac{d}{dt}\mathbb{E}(\int_0^t V_s ds)$.