Rough stochastic volatility with a vol history and the law of the iterated logarithm - Laplace's method on path space

VIX, OU process for Ben Arous, Large-time, Ben Arous small noise multifractal using RKHS. $\log J \leq \delta\varepsilon^{-\theta}$ implies that $J \leq e^{-\delta\varepsilon^{-\theta}}$

Assume interest rates are zero and consider a log stock price process $X_t = \log S_t$ which satisfies

$$\begin{cases} dX_t = -\frac{1}{2}\sigma(Y_t)^2 dt + \sigma(Y_t)(\rho dB_t + \bar{\rho} dW_t)\,, \\ Y_t \;=\; B_t^H \\ B_t^H = c_H \int_{-\infty}^{\infty} [(t-s)_+^{H-\frac{1}{2}} - (-s)_+^{H-\frac{1}{2}}] dB_s \end{cases}$$

for $H \in (0, \frac{1}{2})$ with $\sigma \in C_b^2$, where $c_H = [\int_0^\infty (1+s)^{H-\frac{1}{2}} - s^{H-\frac{1}{2}} + \frac{1}{2H}]^{\frac{1}{2}}$ (which ensures that $\mathbb{E}((B_t^H)^2) = t^{2H}$, where W, B are two independent Brownian motions, $\bar{\rho} = \sqrt{1-\rho^2}$. B^H has the conditional decomposition:

$$\begin{aligned} B_t^H &= c_H \int_{-\infty}^0 [(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}] dB_s \,+\, c_H \int_0^t (t-s)^{H-\frac{1}{2}} dB_s \\ &= \mathbb{E}(B_t^H \,|\, \mathcal{F}_0^B) \,+\, c_H \int_0^t (t-s)^{H-\frac{1}{2}} dB_s \;=\; \zeta_t + \hat{B}_t \end{aligned}$$

where $\zeta_t := \mathbb{E}(B_t^H \,|\, \mathcal{F}_0^B) = c_H \int_{-\infty}^0 [(t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}] dB_s$ is a Fredholm Gaussian process, and $\hat{B}_t = c_H \int_0^t (t-s)^{H-\frac{1}{2}} dB_s$ is a Riemann-Liouville process. From the self-similarity of $\hat{B}$, we know that $X_t \sim X_1^\varepsilon$, where

$$dX_t^\varepsilon \;=\; -\frac{1}{2}\varepsilon^2 \sigma(\zeta_{\varepsilon^2 t} + \hat{\varepsilon}\hat{B}_t)^2 dt + \varepsilon\sigma(\zeta_{\varepsilon^2 t} + \hat{\varepsilon}\hat{B}_t)(\rho dB_t + \bar{\rho} dW_t)$$

where $\hat{\varepsilon} = \varepsilon^{2H}$. It turns out that ζ is also H-self-similar, but we will not need this here since we will be conditioning on the history $(B_t^H)_{t \leq 0}$.

Laplace asymptotics

Following [BFGHS17], we let $H_0^1 = \{f \in C_b[0,1] : \int_0^1 \dot{f}(s)^2 ds < \infty\,,\, f(0) = 0\}$ and

$$I(x) \;=\; \inf_{h,f \in H_0^1} \{\frac{1}{2}\int_0^1 \dot{h}^2 dt \,+\, \frac{1}{2}\int_0^1 \dot{f}^2 dt \,:\, \int_0^1 \sigma(\hat{f})(\bar{\rho} dh + \rho df) = x\}$$

where $\hat{f} = K\dot{f}$ with $(Kf)(t) = \int_0^t (t-s)^{H-\frac{1}{2}} \dot{f}(s) ds$, for $f \in H_0^1$. Moreover, the inf in $I(x)$ is attained and is unique for x sufficiently small (see Lemma 5.10 in [BFGHS17]).

From here on, $f, \hat{f}$, h will refer to $f^x, \hat{f}^x$ and h^x.

Proposition 0.1 *(Modified pricing formula) For x sufficiently small that the inf in $I(x)$ is attained, we have*

$$c(\hat{x}, t) \;:=\; \mathbb{E}((e^{X_t} - e^{\hat{x}})^+) \;=\; e^{-I(x)/\hat{\varepsilon}^2} e^{\frac{\varepsilon}{\hat{\varepsilon}}x} \mathbb{E}((e^{\frac{\varepsilon}{\hat{\varepsilon}}(\hat{\varepsilon}g_1 + \int_0^1 \sigma'(\hat{f})\zeta_{\varepsilon^2(.)} d(\bar{\rho}h + \rho f) + \hat{\varepsilon}^2 \tilde{R}_2^\varepsilon)} - 1)^+ e^{-I'(x) g_1/\hat{\varepsilon}}) \tag{1}$$

Proof. As in section 6 of [BFGHS17]), we define $\hat{X}_t = \frac{\hat{\varepsilon}}{\varepsilon} X_t^\varepsilon$ so

$$d\hat{X}_t^\varepsilon \;=\; -\frac{1}{2}\varepsilon\hat{\varepsilon}\sigma(\zeta_{\varepsilon^2 t} + \hat{\varepsilon}\hat{B}_t)^2 dt + \hat{\varepsilon}\sigma(\zeta_{\varepsilon^2 t} + \hat{\varepsilon}\hat{B}_t)(\rho dB_t + \bar{\rho} dW_t)\,.$$

Now consider the usual Cameron-Martin change of measure $\hat{\varepsilon}(W, B) \mapsto \hat{\varepsilon}(W, B) + (h, f)$, for which the associated Girsanov density is

$$G_\varepsilon \;=\; \exp[-\frac{1}{\hat{\varepsilon}}\int_0^1 \dot{h}_s dW_s \,-\, \frac{1}{\hat{\varepsilon}}\int_0^1 \dot{f}_s dB_s - \frac{1}{2\hat{\varepsilon}}\int_0^1 (\dot{h}^2 + \dot{f}^2) ds]$$

Under the new measure $\hat{X}_1^\varepsilon$ becomes $\tilde{Z}_1^\varepsilon$, where

$$\begin{aligned} \tilde{Z}_1^\varepsilon &= \int_0^1 \sigma(\hat{f} + \hat{\varepsilon}\hat{B} + \zeta_{\varepsilon^2 s})[\hat{\varepsilon} d(\bar{\rho} dW + \rho dB) + d(\bar{\rho}h + \rho f)] \,-\, \frac{1}{2}\varepsilon\hat{\varepsilon}\int_0^1 \sigma(\hat{\varepsilon}\hat{B}_s + \hat{f} + \zeta_{\varepsilon^2 s})^2 dt \\ &= \int_0^1 \sigma(\hat{f}) d(\bar{\rho}h + \rho f) \,+\, \hat{\varepsilon}\int_0^1 \sigma(\hat{f})(d(\bar{\rho} dW + \rho dB)) \,+\, \hat{\varepsilon}\int_0^1 \sigma'(\hat{f})\hat{B} d(\bar{\rho}h + \rho f) \\ &+\; \sigma'(\hat{f})\zeta_{\varepsilon^2 s} d(\bar{\rho}h + \rho f) \,+\, \hat{\varepsilon}^2 \tilde{R}_2^\varepsilon \\ &= \int_0^1 \sigma(\hat{f}) d(\bar{\rho}h + \rho f) \,+\, \hat{\varepsilon} g_1 \,+\, \sigma'(\hat{f})\zeta_{\varepsilon^2 s} d(\bar{\rho}h + \rho f) \,+\, \hat{\varepsilon}^2 \tilde{R}_2^\varepsilon \end{aligned} \tag{2}$$

where g_1 is defined as in [BFGHS17] (for the case when $K(s,t) = c_H(t-s)^{H-\frac{1}{2}}$), but $\tilde{R}_2^\varepsilon$ is slightly different to R_2^ε in [BFGHS17] since it includes additional $O(\zeta_{\varepsilon^2 t}^2)$ terms. Then $c(\hat{x},t)$ in Lemma 6.3 in [BFGHS17] is now replaced by

$$e^{-I(x)/\hat{\varepsilon}^2} e^{\frac{\varepsilon}{\hat{\varepsilon}}x} \mathbb{E}((e^{\frac{\varepsilon}{\hat{\varepsilon}}(\hat{\varepsilon}g_1 + \int_0^1 \sigma'(\hat{f})\zeta_{\varepsilon^2(.)}d(\bar{\rho}h+\rho f)+\hat{\varepsilon}^2\tilde{R}_2^\varepsilon)} - 1)^+ e^{-I'(x)g_1/\hat{\varepsilon}})$$

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From pg 220 (see also pages 216, and 214 for the definition of their modulus of uniform continuity) in [Lif95], we know that for all $\varepsilon > 0$,

$$|B_t^H| \quad \leq \quad (c_1+\delta)|\log t|^{\frac{1}{2}} t^H$$

for $t = t(\delta)$ sufficiently small, for some (non-random) constant c_1. Thus we see that

$$|\zeta_t| \quad = \quad |\mathbb{E}(B_t^H|\mathcal{F}_0)| \quad \leq \quad \mathbb{E}(|B_t^H||\mathcal{F}_0) \quad \leq \quad (c_1+\delta)|\log t|^{\frac{1}{2}} t^H \,.$$

This will be needed in the next lemma. **check 36 rule, proper LOIL would sharpen your result**

Remark 0.1 From Theorem 3.1 in [TX07], we also know that $\liminf_{t\to 0} \frac{\sup_{u\in[0,t]}|B_u^H|}{t^H/(\log\log\frac{1}{t})^H} = c_3 > 0$ which implies that $|\zeta_t| \geq (c_3 - \varepsilon)\frac{t^H}{(\log\log\frac{1}{t})^H}$, which in particular shows that W_t^H cannot stay constant over any time interval, but we will not using this result here.

$\frac{\varepsilon}{\hat{\varepsilon}}\varepsilon^{2H}\sqrt{2\log\varepsilon} = \varepsilon\sqrt{2\log\varepsilon}$ moderate deviations regime

Lemma 0.2 *Let $a, b, d, \alpha, \beta, \gamma$ be real constants, a, α strictly positive. Let $\psi(z) = (e^z - 1)^+$, and $Z \sim N(0,1)$. Then*

$$\mathbb{E}(\exp(-a\varepsilon^{-\alpha}Z)\,\psi(b\varepsilon Z + d\varepsilon(\log\varepsilon)^{\frac{1}{2}})) \quad = \quad \frac{1}{\sqrt{2\pi}}\varepsilon^{-\frac{d^2}{2b^2}} e^{\frac{ad}{b}\varepsilon^{-\alpha}\sqrt{\log\varepsilon}} \frac{b^3\varepsilon^{1+2\alpha}}{a^2}(1 + 2\frac{d}{ba}\varepsilon^\alpha\sqrt{\log\varepsilon} + O(\varepsilon^2))$$

Proof. This is a minor variant of Lemma 2.6 in the working paper [FGP18]. Setting $y = \frac{x - d\sqrt{\log\varepsilon}}{b}$, we can re-write the expectation as $\bar{c}\,I$, where

$$I \quad = \quad (\int_0^\delta + \int_\delta^\infty)\, e^{-\frac{ax}{b\varepsilon^\alpha}}\,\psi(\varepsilon x) e^{-\frac{x^2}{2b^2} + \frac{x}{b^2}d\sqrt{\log\varepsilon}} \frac{dx}{b\sqrt{2\pi}}$$

where $\bar{c} = e^{-\frac{d^2}{2b^2}\log\varepsilon} e^{\frac{ad}{b}\varepsilon^{-\alpha}\sqrt{\log\varepsilon}}$. The second integral gives a higher order contribution than we care about here, and for $x > 0$ ... ■

Lemma 0.3

$$\log c(\hat{x},t) \quad = \quad -\frac{I(x)}{t^{2H}} + O(\frac{|\log t|^{\frac{1}{2}}}{t^H}) \tag{3}$$

a.s. as $t \to 0$. t_0 issue

Proof. The first order correction due to ζ in (1) is

$$\int_0^1 \sigma'(\hat{f})\zeta_{\varepsilon^2(.)}d(\bar{\rho}h + \rho f) \quad \leq \quad \|\sigma'\| \sup_{s\in[0,1]} |\zeta_{\varepsilon^2 s}| \cdot (\bar{\rho}I(x) + \rho I(x)) \quad = \quad \|\sigma'\| \sup_{s\in[0,1]} |\zeta_{\varepsilon^2 s}| \cdot I(x) \tag{4}$$

Hence

$$\int_0^1 \sigma'(\hat{f})\zeta_{\varepsilon^2 t}d(\bar{\rho}h + \rho f) \quad \leq \quad \|\sigma'\|(c_1+\delta)I(x)(\log\varepsilon^2)^{\frac{1}{2}}\varepsilon^{2H} \quad = \quad C_{H,\sigma}I(x)(\log\varepsilon)^{\frac{1}{2}}\varepsilon^{2H} \tag{5}$$

where $C_{H,\sigma} := (c_1+\delta)/\sqrt{2}$. Now setting $\gamma = 1$, $\alpha = 2H$, $a = I'(x)\sigma_x$, $b = \sigma_x$ where $\sigma_x = \mathrm{Var}(g_1)^{\frac{1}{2}} = O(1)$ in the previous lemma, and using (5) and the modified pricing formula, we see that

$$c_- \quad \leq \quad \liminf_{t\to 0} c(\hat{x},t) \quad \leq \quad \limsup_{t\to 0} c(\hat{x},t) \quad \leq \quad c_+ \quad a.s. \tag{6}$$

where

$$c_\pm \quad = \quad \frac{\sigma_x}{I'(x)^2\sqrt{2\pi}}\, t^{\frac{1}{2}+2H-\frac{C_{H,\sigma}^2 I(x)^2}{4\sigma_x^2}}\, e^{-\frac{I(x)}{t^{2H}} \pm \frac{|I'(x)I(x)C_{H,\sigma}|}{\sqrt{2}} \frac{|\log t|^{\frac{1}{2}}}{t^H} + t^{1-2H}x}\, (1 \pm \sqrt{2}\sigma_x^2| \frac{C_{H,\sigma}I(x)}{I'(x)}|t^H|\log t|^{\frac{1}{2}})$$

Lemma 0.4 *Let $A = \{|\hat{\varepsilon}B + \zeta_{\varepsilon^2 t}|_{\infty;[0,1]} \leq \kappa\}$. Then*

$$\mathbb{P}(A^c) \quad \leq \quad e^{-c\kappa^2/\varepsilon^2}$$

for κ sufficiently large??, and some constant $c > 0$.

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modified g_1, can we view history term as e.g. $\hat{\varepsilon}Z_\varepsilon$, and then apply saddlepoint formula in Pig

Lemma 0.5

$$\mathbb{P}(\{|R_1^{\varepsilon,x}| > r\} \cap A) \quad \leq \quad c_1 e^{-c_2 r}$$

for all $c_1, c_2 > 0$.

Proof. The proof for the no-history case appears in the unpublished note by Friz&Bayer; we make minor modifications to their proof so as to include the $\xi_{\varepsilon^2 t}$ term. Now define

$$\begin{aligned} M^\varepsilon &:= \hat{\varepsilon} \int_0^1 (\sigma(\hat{f} + \hat{\varepsilon}\hat{B} + \zeta_{\varepsilon^2 t}) - \sigma(f)) d(\bar{\rho}dW + \rho dB) \\ N^\varepsilon &:= \int_0^1 (\sigma(\hat{f} + \hat{\varepsilon}\hat{B} + \zeta_{\varepsilon^2 t}) - \sigma(f) - \sigma'(f)(\hat{\varepsilon}\hat{B} + \zeta_{\varepsilon^2 t}))d(\bar{\rho}h + \rho f) - \frac{1}{2}\varepsilon\hat{\varepsilon}\int_0^1 \sigma(\hat{\varepsilon}\hat{B}_t + \hat{f} + \zeta_{\varepsilon^2 t})^2 dt \end{aligned}$$

Then $\hat{\varepsilon}^2 \tilde{R}_2^\varepsilon = M^\varepsilon + N^\varepsilon$. Let $\tau^{\varepsilon,\kappa} = \inf\{t : |\varepsilon B_t + \zeta_{\varepsilon^2 t}| > \kappa\}$. Then the stopped process $\tilde{M}_t^\varepsilon := M^\varepsilon_{t \wedge \tau^{\varepsilon,\kappa}}$ is still a local martingale and if A occurs, $\tilde{M}^\varepsilon = \tilde{M}^\varepsilon$ on the interval $[0,1]$. Moreover, using Taylor's remainder theorem, we see that

$$\langle M^{\kappa,\varepsilon}\rangle_1 \quad = \quad \int_0^1 \hat{\varepsilon}^2(\sigma(\hat{f} + \hat{\varepsilon}\hat{B} + \zeta_{\varepsilon^2 t}) - \sigma(f))^2 dt \quad \leq \quad \hat{\varepsilon}^2 \|\sigma'\|^2 \int_0^1 (\hat{\varepsilon}\hat{B}_t + \zeta_{\varepsilon^2 t})^2 dt \quad \leq \quad \hat{\varepsilon}^2 \|\sigma'\|^2 \kappa^2$$

on the event A. Similarly

$$\begin{aligned} |N_1^\varepsilon| \quad &\leq \quad \sup_{t \in [0,1]} (\hat{\varepsilon}\hat{B} + \zeta_{\varepsilon^2 t})^2 \cdot \|\sigma''\| \, (\bar{\rho}|h|_{\mathrm{BV};[0,1]} + \rho|f|_{\mathrm{BV};[0,1]}) \, + \, \frac{1}{2}\varepsilon\hat{\varepsilon} \, \|\sigma\|^2 \\ &\leq \quad \kappa^2 \cdot \|\sigma''\| \, (\bar{\rho}|h|_{\mathrm{BV};[0,1]} + \rho|f|_{\mathrm{BV};[0,1]}) \, + \, \frac{1}{2}\varepsilon\hat{\varepsilon} \, \|\sigma\|^2 \end{aligned}$$

on A. Using that $|h|_{\mathrm{BV};[0,1]} = |\dot{h}|_{L^1;[0,1]} \leq |\dot{h}|_{L^2;[0,1]} = ||h||_{H_0^1;[0,1]} \leq I(x)$ (and similarly for f), we see that

$$|N_1^\varepsilon| \quad \leq \quad \kappa^2 \, \|\sigma''\| \cdot I(x) \, + \, \frac{1}{2}\varepsilon\hat{\varepsilon}\|\sigma\|^2 \, .$$

For a Itô process X of the form $dX_t = \sigma_t dW_t$ with σ_t bounded, we know that

$$\mathbb{E}(e^{\frac{1}{2}pX_t}) \quad = \quad \mathbb{E}(e^{\frac{1}{2}pX_t - \frac{1}{4}p^2\langle X\rangle_t + \frac{1}{4}p^2\langle X\rangle_t}) \quad \leq \quad \mathbb{E}(e^{pX_t - \frac{1}{2}p^2\langle X\rangle_t})^{\frac{1}{2}} \mathbb{E}(e^{\frac{1}{2}p^2\langle X\rangle_t})^{\frac{1}{2}} \quad \leq \quad (e^{\frac{1}{2}p^2\|\sigma\|^2})^{\frac{1}{2}} \quad < \quad \infty$$

for all $p > 0$. Thus we see that $\langle M^{\kappa,\varepsilon}\rangle_1 = O(\hat{\varepsilon}^2\|\sigma'\|^2\kappa^2)$ and $|N_1^\varepsilon| = O(1)$.

$$\mathbb{E}(1_{|R_1^{\varepsilon,x}|>r} \, 1_A) \quad = \quad \mathbb{E}(1_{|\tilde{R}_1^{\varepsilon,x}|>r} \, 1_A) \quad \leq \quad \mathbb{E}(1_{|\tilde{R}_1^{\varepsilon,x}|>r}) \quad \leq \quad c_1 e^{-c_2 r} \, .$$

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Have to be careful if we don't condition at $t = 0$ since for $t \neq 0$, $\hat{B}$ is no longer self similar, but can probably make a time shift to circumvent this issue.

For the fractional Stein-Stein model with $\sigma(y) = y$, the $O(\zeta^2_{\varepsilon^2 t}))$ term in (2) is

$$\hat{\varepsilon}\int_0^1 \zeta_{\varepsilon^2 t} d(\bar{\rho}W_t + \rho dB_t) - \frac{1}{2}\varepsilon\hat{\varepsilon}\int_0^1 \sigma(\hat{\varepsilon}\bar{B}_t + \hat{f}_t + \zeta_{\varepsilon^2 t})^2 dt \quad = \quad \hat{\varepsilon}\int_0^1 \zeta_{\varepsilon^2 t} d(\bar{\rho}W_t + \rho dB_t) + O(\varepsilon\hat{\varepsilon})$$

since σ is bounded. Moreover, conditioned on $\mathcal{F}_0^B$, $\hat{\varepsilon}\int_0^1 \zeta_{\varepsilon^2 t} d(\bar{\rho}W_t + \rho dB_t) \sim N(0, \hat{\varepsilon}^2 \int_0^1 \zeta^2_{\varepsilon^2 t} dt)$. $\hat{x} = kt^{\frac{1}{2} - H + \beta}$.

References

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