VIX modelling under a GARCH(1,1) model

Let

$$
\begin{aligned}
R_t \quad := \quad \log\frac{S_t}{S_{t-1}} \quad &= \quad \mu_{t-1} + \sqrt{V_{t-1}}\,Z_t \\
\text{where} \quad V_t \quad &= \quad \omega + \alpha V_{t-1}Z_t^2 + \beta V_{t-1} \\
&= \quad V_{t-1} + (1-\beta)(\bar{\omega} - V_{t-1}) + \alpha V_{t-1}Z_t^2
\end{aligned}
\tag{1}
$$

for $t \in \mathbb{Z}$, $\omega, \alpha, \beta > 0$, where $\bar{\omega} = \frac{\omega}{1-\beta}$, and $(Z_t)_{t\in\mathbb{Z}}$ is a sequence of i.i.d. random variables with $\mathbb{E}(Z_t) = 0$ and $\mathbb{E}(Z_t^2) = 1$ (we call $(Z_t)_{t\geq 0}$ the residuals for the model). Then

$$
\mathbb{E}(\frac{S_t}{S_{t-1}}|V_{t-1}) \quad = \quad e^{\mu_{t-1}}\mathbb{E}(e^{\sqrt{V_{t-1}}\,Z_1})\,.
$$

Let $\mathcal{F}_t = \sigma(Z_1, .., Z_t)$, then for S to be an $\mathcal{F}_t$-martingale, we require that $\mu_{t-1} = -\log\mathbb{E}(e^{\sqrt{V_{t-1}}\,Z_t}) = -\log\mathbb{E}(e^{\sqrt{V_{t-1}}\,Z_1})$, which we denote by $\varphi(V_{t-1})$. Note if the Z_t's are i.i.d. $N(0,1)$ then $\mu_{t-1} = -\frac{1}{2}V_{t-1}$, and we will need to make this assumption for the VIX modelling below. We also note that

$$
\mathbb{E}(V_t|\mathcal{F}_{t-1}) \quad = \quad \omega + (\alpha+\beta)V_{t-1}
$$

and

$$
\begin{aligned}
\mathbb{E}(V_t|\mathcal{F}_{t-2}) \quad = \quad \mathbb{E}(\mathbb{E}(V_t|\mathcal{F}_{t-1})|\mathcal{F}_{t-2}) \quad &= \quad \omega + (\alpha+\beta)\mathbb{E}(V_{t-1}|\mathcal{F}_{t-2}) \\
&= \quad \omega + (\alpha+\beta)(\omega + (\alpha+\beta)V_{t-2})
\end{aligned}
$$

and repeating this argument we see that

$$
\mathbb{E}(V_t|\mathcal{F}_{t-n}) \quad = \quad \omega(1 + (\alpha+\beta) + ... + (\alpha+\beta)^{n-1}) + (\alpha+\beta)^n V_{t-n}\,.
$$

Letting $r = \alpha + \beta$, we can re-write this expression as

$$
\mathbb{E}(V_t|\mathcal{F}_{t-n}) \quad = \quad \omega\frac{1-r^n}{1-r} + (\alpha+\beta)^n V_{t-n}\,.
$$

If we now assume that $Z_t \sim N(0,1)$ so $\mu_{t-1} = -\frac{1}{2}V_{t-1}$ (see above), then

$$
\mathrm{VIX}_t^2 \quad = \quad -\frac{2}{\Delta}\mathbb{E}(\log\frac{S_{t+\Delta}}{S_t}|\mathcal{F}_t) \quad = \quad \frac{1}{\Delta}\sum_{u=t+1}^{t+\Delta}\mathbb{E}(V_u|\mathcal{F}_t) \quad = \quad \frac{\omega}{\Delta}\sum_{n=1}^{\Delta}\frac{1-r^n}{1-r} + \frac{1}{\Delta}\sum_{n=1}^{\Delta}(\alpha+\beta)^n V_t \quad = \quad aV_t + b
$$

for some constants $a = a$ and b which depend only on $(\alpha, \beta, \omega, \Delta)$, where $\Delta = 30$ days in practice, so VIX_t^2 is an affine function of V_t itself. Note if $\beta = 1$ and $\bar{\omega} = 0$, then $\omega = 0$ as well, and

$$
\mathbb{E}(V_t|\mathcal{F}_{t-n}) \quad = \quad (\alpha+1)^n V_{t-n}\,.
$$

Stationarity condition

Taking unconditional expectations of (1) and assuming V is stationary, we see that

$$
\mathbb{E}(V_t) \quad = \quad \omega + \alpha\sigma^2\mathbb{E}(V_t) + \beta\mathbb{E}(V_t)
$$

and

$$
\mathbb{E}(R_t^2) \quad = \quad \mathbb{E}(\mathbb{E}(R_t^2|V_t)) \quad = \quad \mathbb{E}(V_t)\,.
$$

Re-arranging, we see that

$$
\mathbb{E}(V_t) \quad = \quad \frac{\omega}{1-\alpha-\beta}\,.
$$

Since V_t cannot be negative, we see that $\alpha + \beta < 1$ is the **stationarity condition** for V.