7. Lévy models

7.1 Definition of a Lévy process

A Lévy process is a generalization of the jump diffusion process $X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} \xi_i$ discussed in the previous lecture, ai. X is said to be a Lévy process if:

- $X_0 = 0$;
- X has independent increments;
- $X_t - X_s$ has the same distribution as X_{t-s} for any $0 < s < t$;
- X can only jump at random times.

Remark 0.1 Examples of Lévy processes: standard Brownian motion W_t, the Poisson process N_t, the sum $X_t = W_t + N_t$, the general jump diffusion $X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} \xi_i$ from the previous lecture with general jump size distribution $\mu(dx)$ and $(H_b)_{b\geq 0}$ (the first hitting time process for standard Brownian motion) where now b is the time variable.

7.2 A Levy process

A Lévy process has an associated Lévy measure $\nu(dx)$ which is such that for any n disjoint sets $A_1, A_1, ..., A_n$ in $\mathcal{B}(\mathbb{R})$, the number of jumps which fall in $A_1, ..., A_n$ over $[0, t]$ is a vector of n independent Poisson random variables with parameters $\nu(A_1)t, ..., \nu(A_n)t$. When X is just a jump diffusion then $\nu(dx) = \lambda\mu(x)$, where $\mu(x)$ is the jump size distribution and λ is the arrival rate for the Poisson process. For some models e.g. CGMY, $\nu(0, x] = \infty$ for any $x > 0$; this means there is an infinite number of positive jumps almost surely, and this is known as an **infinite activity** Lévy process.

Theorem 0.1 *Lévy-Khintchine representation). Let X be a Lévy process with characteristic triple (γ, σ, ν). Then*

$$\mathbb{E}(e^{iuX_t}) = \exp\left[t(i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux1_{|x|\leq 1})\nu(dx)\right] \tag{1}$$

Notice that there is one additional term in the integral which is not there when X is a jump diffusion. We will not prove this here, but note that if $\nu(\mathbb{R}) = +\infty$, this term is needed to ensure that the integral here is finite.

7.3 Examples of Lévy processes

- The double exponential **Kou model** (as previously discussed) is a jump diffusion model for which the Lévy measure is just a multiple λ times a probability measure, given by a two-sided exponential distribution, so

$$\nu(x) = \lambda[p\lambda_+ e^{-\lambda_+ x}1_{x>0} + (1-p)\lambda_- e^{\lambda_- x}1_{x<0}] .$$

  This is a **finite activity** model, because $\nu(x)$ is just a multiple of a probability density, so $\nu(\mathbb{R}) < \infty$.

- The CGMY (Carr-Geman-Madan-Yor) model has a Lévy density of the form

$$\nu(x) = \frac{Ce^{-Mx}}{x^{1+Y}}\mathbf{1}_{\{x>0\}} + \frac{Ce^{Gx}}{|x|^{1+Y}}\mathbf{1}_{\{x<0\}}$$

  for $C, G, M > 0$ and $Y \in (0, 2)$, so we the see the jump rate tends to infinity as the jump size tends to zero. This is what we call an **infinite activity** model, because $\nu(x)$ is not a multiple of a pdf.

- Its characteristic function is given as

$$\phi_t(u) = \mathbb{E}(e^{iuX_t}) = \exp\left[t\,C\Gamma(-Y)\left\{(M-iu)^Y + (G+iu)^Y - M^Y - G^Y\right\} + ibut\right],$$

  for $Y \neq 1$ and some constant b which controls the drift, and from this we can compute the critical moments p_+, p_- (see homework ...)

- To compute the density of $p_t(x)$, we use the inverse Fourier transform as before.