Weak convergence for a fast mean-reverting rough Heston model with jumps to a general class of Lévy processes

Abstract

We show weak convergence of the marginals for a fast-mean reverting/large vol-of-vol rough Heston model to a Normal Inverse Gaussian (NIG) Lévy model. This shows we can obtain a such a limit without having to impose that the true Hurst exponent H for the model is $\frac{1}{2}$ as in [AC24][1], or that $H \searrow -\frac{1}{2}$ as in [AAR25], so the result potentially has increased financial relevance. We later extend to the case when V has jumps, where we show weak convergence of the finite-dimensional distributions of the integrated variance to a deterministic time-change of the inverse of the minimum process for a more general class of Lévy processes.

1 Introduction

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The Rough Heston stochastic volatility model was introduced in [JR16], and (using C-tightness arguments) they show that the model arises naturally as a weak large-time limit of a high-frequency market microstructure model driven by two nearly unstable Hawkes process. [ER19] and [GK19] show that the characteristic function of the log stock price for Rough Heston-type models can be expressed in terms of the solution to a non-linear Volterra integral equation (VIE) (see also [EFR18],[ER18] and [BPS24],[BLP24] for extensions to jumps in V), which allows for accurate option pricing for $H \ll 1$ (and even $H = 0$) using an Adams scheme to solve the VIE numerically which avoids Monte Carlo (MC) for which traditional MC schemes are notoriously inaccurate for $H \ll 1$ (both in terms of bias and sample variance), see [AA25] for some recent developments on this issue.

V is $(H - \varepsilon)$-Hölder continuous like fBM (see e.g. Theorem 3.2 in [JR16]) and the model exhibits power-law skew for implied volatility in the small-time limit (see e.g Theorem 3.1 in [FGS21]/Corollary 3.4 in [FSV21]). See also [ALP19],[FGS21],[FS21],[Cuch22],[FG24] for further related results on rough Heston-type models. [BL24] use the sinh-acceleration method to improve accuracy in estimating the tail part of the integral for Fourier inversion for e.g. pricing call options under the rough Heston model, using the a sinh contour which goes outside the usual strip of analyticity for the mgf (but is admissible as long as it avoid poles of the characteristic function), and also discuss refinements to the usual Adams schemes for solving the rough Heston VIE.

[AC24] show that a re-scaled standard Markov Heston model with fast mean-reversion and large vol-of-vol (via a H parameter which is not the Hurst exponent) tends weakly on path space to one of three different models (either Black-Scholes, an Normal Inverse Gaussian or a Normal Lévy model, depending on whether their H parameter is $>$, $=$, or $< -\frac{1}{2}$. [AAR25] obtain a similar result without any ε parameter but instead letting $H \searrow -\frac{1}{2}$ for the so-called *hyper-rough Heston model* (see also section 5 in [FGS21] for more on this model), and exploiting Dirac-type behaviour in their Lemma 2.4 (see Appendix D here for a short summary/formal derivation of their result).

In this note, we fill in the gap between [AC24] and [AAR25], by showing that a similar result is obtained for any $H \in (0, \frac{1}{2}]$ (in particular our regime for $H = \frac{1}{2}$ corresponds to the regime in Eq 0.3 in [AC24] with their $H = -\frac{1}{2}$, $\theta = 0$ and their V_0 replaced with θ and $\lambda = 1$. In section 2, we extend the model to allow positive jumps in V^ε, and in this case (using the basic classical result for the Laplace transform of the hitting time to a lower barrier for a spectrally positive Lévy processes), we find that the limiting process is a time-change of the inverse minimum process for a more general class of Lévy processes.

1.1 Asymptotics for the terminal log stock price in the large fast mean-reversion, large vol-of-vol limit

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ throughout with filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions.

Proposition 1.1 *Consider a re-scaled rough Heston model for a log stock price process X_t^ε:*

$$\begin{aligned} dX_t^\varepsilon &= -\frac{1}{2}V_t^\varepsilon dt + \sqrt{V_t^\varepsilon}dB_t \\ V_t^\varepsilon &= V_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\frac{1}{\varepsilon}\lambda(\theta - V_s^\varepsilon)ds + \frac{1}{\varepsilon}\nu\sqrt{V_s^\varepsilon}dW_s) \end{aligned} \quad (1)$$

[1] The H parameter in [AC24] is not fixed to be $\frac{1}{2}$, but the Hurst exponent for their family of V^ε processes is of course $\frac{1}{2}$ since the models are all standard Heston for $\varepsilon > 0$.

where B and W are two Brownian motions with $dB_tdW_t = \rho dt$ with $\rho \in [-1,0]$, $\alpha \in (\frac{1}{2},1]$ and $\lambda,\nu > 0$. Then (for t fixed) X^ε_t tends weakly to X_t as $\varepsilon \to 0$, where X is a Normal Inverse Gaussian Lévy process which does not depend on $H = \alpha - \frac{1}{2}$.

Proof. Let $I^\alpha(f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}f(s)ds$ denote the αth-order fractional derivative of a function f for $\alpha \in (0,1]$, and (without loss of generality), we assume $X^\varepsilon_0 = 0$. Then for $p \in (0,1)$ (which will be sufficient for our purposes when we invoke the [Bill86] weak convergence result below)

$$\mathbb{E}(e^{pX^\varepsilon_t}) \quad = \quad e^{V_0 I^{1-\alpha}\phi_\varepsilon(t) + \frac{1}{\varepsilon}\lambda\theta I^1\phi_\varepsilon(t)} \tag{2}$$

where ϕ_ε is the unique solution to the non-linear Volterra integral equation (VIE):

$$\phi_\varepsilon(t) \quad = \quad \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\frac{1}{2}(p^2-p) \,+\, \frac{1}{\varepsilon}(\rho p\nu - \lambda)\phi_\varepsilon(s) \,+\, \frac{1}{\varepsilon^2}\frac{1}{2}\nu^2\phi_\varepsilon(s)^2)ds$$

(see e.g. section 4 in [ER19]). Now let $\phi_\varepsilon(t) = \varepsilon\psi(\varepsilon^q t)$. Then

$$\begin{aligned}
\varepsilon\psi(\varepsilon^q t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(\frac{1}{2}(p^2-p) \,+\, (\rho p\nu - \lambda)\psi(\varepsilon^q s) \,+\, \frac{1}{2}\nu^2\psi(\varepsilon^q s)^2)ds \\
&= \frac{1}{\Gamma(\alpha)}\int_0^{\varepsilon^q t} (t-u\varepsilon^{-q})^{\alpha-1}(\frac{1}{2}(p^2-p) \,+\, (\rho p\nu - \lambda)\psi(u) \,+\, \frac{1}{2}\nu^2\psi(u)^2)du\,\varepsilon^{-q} \\
&= \frac{\varepsilon^{-q(\alpha-1)}}{\Gamma(\alpha)}\int_0^{\varepsilon^q t} (\varepsilon^q t-u)^{\alpha-1}(\frac{1}{2}(p^2-p) \,+\, (\rho p\nu - \lambda)\psi(u) \,+\, \frac{1}{2}\nu^2\psi(u)^2)du\,\varepsilon^{-q}
\end{aligned}$$

where we set $\varepsilon^q s = u$ in the second line, so $\varepsilon^q ds = du$. Then setting $\varepsilon^q t \mapsto t$, we see that

$$\varepsilon\psi(t) \quad = \quad \frac{\varepsilon^{-q\alpha}}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}F(\psi(u))du \tag{3}$$

where $F(w) = \frac{1}{2}(p^2-p)+(\rho p\nu-\lambda)w+\frac{1}{2}\nu^2w^2$, and if now let $q = -\frac{1}{\alpha}$, the VIE is independent of ε, so $\phi_\varepsilon(t) = \varepsilon\psi(\frac{t}{\varepsilon^{\frac{1}{\alpha}}})$. Hence (from e.g. Lemma 4.4 in [FGS21])

$$\lim_{\varepsilon\to\infty}\frac{1}{\varepsilon}\phi_\varepsilon(t) \quad = \quad \lim_{t\to\infty}\psi(t) \quad = \quad \psi(\infty) \quad = \quad U_1(p) \quad = \quad \frac{1}{\nu^2}[\lambda - p\nu\rho - \sqrt{\lambda^2 - 2\lambda\rho\nu p + \nu^2 p(1-\bar{\rho}p)}] \qquad (\varepsilon \to 0)$$

for $p \in (0,1)$, where $\psi(\infty)$ is the smallest root w_* of $G(w) = \frac{1}{2}(p^2-p) + (\rho p\nu - \lambda)\nu w + \frac{1}{2}\nu^2 w^2$ (which is the stable equilibrium point since $G'(w_*) < 0$, and $\bar{\rho}^2 = 1-\rho^2$, and the convergence to $\psi(\infty)$ is rapid for $\varepsilon \ll 1$ since $\frac{1}{\varepsilon}\phi_\varepsilon(t) = \psi(\frac{t}{\varepsilon^{\frac{1}{\alpha}}})$. Then for the exponent in (2), from the bounded convergence theorem we see that

$$V_0 I^{1-\alpha}\phi_\varepsilon(t) \,+\, \frac{\lambda\theta}{\varepsilon}I^1\phi_\varepsilon(t) \quad = \quad \frac{V_0}{\Gamma(\alpha)}\int_0^t (t-s)^\alpha\phi_\varepsilon(s)ds \,+\, \lambda\theta\int_0^t \frac{1}{\varepsilon}\phi_\varepsilon(s)ds \quad \to \quad 0 \,+\, \lambda\theta U_1(p)t$$

as $\varepsilon \to 0$, and $\lambda\theta U_1(p)t$ is the log mgf of a Normal Inverse Gaussian Lévy process (see e.g. Remark 2.3 in [FJ11]). Then from e.g. the solution to Problem 30.4 on page 573 in [Bill86] (which is also used in [GK19]), X^ε_t tends weakly to the marginal law of an NIG process. ■

2 Adding jumps into V^ε

We now assume that the forward variance $\xi^\varepsilon_t(u) := \mathbb{E}(V^\varepsilon_u|\mathcal{F}^W_t)$ satisfies

$$d\xi^\varepsilon_t(u) \quad = \quad \kappa_\varepsilon(u-t)(\sigma\sqrt{V^\varepsilon_t}dW_t \,+\, d\tilde{J}^\varepsilon_t) \tag{4}$$

where $d\tilde{J}^\varepsilon_t = \int_{[0,\infty)} xN^\varepsilon(dx,dt) - V^\varepsilon_t\nu(dx)dt$ and $N^\varepsilon(dx,dt)$ is a (time-inhomogenous) Poisson random measure with (random) intensity $V^\varepsilon_t\nu(dx)dt$ [2] and ν only has positive support with $\nu(\{0\}) = 0$ and $\int_0^\infty x^2\nu(dx) < \infty$ so $\tilde{J}^\varepsilon$ has positive-only jumps and $\kappa_\varepsilon(t) = \frac{1}{\varepsilon}t^{\alpha-1}E_{\alpha,\alpha}(-\frac{\lambda}{\varepsilon}t^\alpha)$ with $\alpha \in (\frac{1}{2},1)$, where $E_{\alpha,\beta}(z)$ denotes the Mittag-Leffler function (see e.g. Appendix A.1 of [ER19] for definition). The $V_t = \xi_t(t)$ process here falls under the framework of Eq 1 in [BPS24]. Recall also that $f^{\alpha,\lambda}(t) = \lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$ is the Mittag-Leffler probability density (again see Appendix A.1 of [ER19]), and this property will be important for the convergence argument below.

Note we are now using σ not ν for the vol-of-vol term in (4) since ν is being used here for the Lévy density, and (in the absence of jumps) (4) is the usual equation for the forward variance under the standard rough Heston model, see e.g. [ER18] or Proposition 2.2 in [FGS21]. This model can be viewed as a generalized rough Heston model where the mean-reversion speed, vol-of-vol, and jump-intensity all scale as $\frac{1}{\varepsilon}$.

[2] see also Eq 1 and the equation below it in [BPS24], and slide 6 in [Cuch22].

Remark 2.1 The SVE

$$V_t^\varepsilon \;=\; V_0 + \int_0^t K(t-s)\frac{1}{\varepsilon}(\lambda(\theta - V_s^\varepsilon)ds + \sigma\sqrt{V_s^\varepsilon}dW_s \,+\, d\tilde{J}_s^\varepsilon) \tag{5}$$

with $K(t)=\frac{1}{\Gamma(t)}t^{\alpha-1}$ (with the same jump structure for $\tilde{J}^\varepsilon$ as in (4)) is a special case of Eq 7 in the arxiv version of [BLP24] with their $g_0 \equiv V_0$, $b_0=\frac{1}{\varepsilon}\lambda\theta$, $B=b_1=-\frac{\lambda}{\varepsilon}$, $A_0=\frac{1}{\varepsilon}\sigma^2$, and (from Remark 5 in [BLP24]), this process is equivalent to (4) , for which their $R_B=\lambda\kappa_\varepsilon$ [3], $E_B=K-R_B*K=\kappa_\varepsilon$, and $\xi_0^\varepsilon(.)=g_0-R_B*g_0+E_B*b_0=V_0-(V_0-\theta)\int_0^{.} f^{\alpha,\frac{\lambda}{\varepsilon}}(s)ds$ (which agrees with Proposition 2.1 in [FGS21]), and $\xi_0^\varepsilon(u)\to\theta$ for $u>0$ as $\varepsilon\to 0$. Conversely, if ξ_0 is independent of ε and given exogenously, then we can find a g_0 consistent with ξ_0 by solving the linear VIE $g_0-R_B*g_0+E_B*b_0=\xi_0(.)$; specifically, letting $f=\xi_0(.)-E_B*b_0$, we can re-write the VIE as $g_0-R_B*g_0=f$, which has solution $g_0=\; f-f*r$ where r is the resolvent of the 2nd kind of $-R_B$ (which will depend on ε in general). [4]

Assumption 2.1 *CHANGE/REMOVE THIS. We assume $\xi_0^\varepsilon(.)$ is uniformly bounded and continuous and $\xi_0^\varepsilon(.)$ tends pointwise to a bounded continuous function $\xi_0^0(.)$.*

Let $V_1(p)=\int_0^\infty(e^{px}-1-px)\nu(dx)$. positive root issue in App B? [5]

Assumption 2.2 *$V_1(p)<\infty$ for $p\in(-\infty,p^*)$ for some $p^*>0$.*

We now state the main result for this section:

Proposition 2.3 *The finite-dimensional distributions of $A_.^\varepsilon=\int^{(.)}V_s^\varepsilon ds$ tend weakly to those of a time-changed Lévy process $X_{g(.)}$, where $X_t=H_{-t}=\inf\{s: Z_s<-t\}$, Z is a Lévy process with Lévy triple $(-\lambda,\sigma^2,\nu)$, and $g(t)=\lambda\int_0^t\xi_0^0(u)du$.*

Remark 2.2 X is a Lévy subordinator, see e.g. Theorem 46.2 in [Sato99].

Proof. Integrating (4) from $t=0$ to u and then setting $u=t$, we see that

$$V_t^\varepsilon \;=\; \xi_0^\varepsilon(t) \,+\, \int_0^t \kappa_\varepsilon(t-s)(\sigma\sqrt{V_s^\varepsilon}dW_s + d\tilde{J}_s^\varepsilon)\,.$$

Let $f:[0,T]\to(-\infty,0)$ be a left continuous piecewise constant function with (a finite number of) discontinuities at $0<s_1<..<s_n$ [6]. Then from Appendix C (see also [BPS24]), we know that

$$M_t \;:=\; e^{\int_0^t f(T-s)V_s^\varepsilon ds + G_t} \tag{6}$$

is a local martingale if

$$G_t \;=\; \int_t^T g_\varepsilon(T-s)\xi_t^\varepsilon(s)ds \;=\; \int_0^{T-t} G(u,\psi_\varepsilon(u))\,\xi_t^\varepsilon(T-u)du$$

[7] and $\psi_\varepsilon(\tau):=\int_0^\tau \kappa_\varepsilon(\tau-r)g_\varepsilon(r)dr$, satisfies the non-linear VIE

$$\psi_\varepsilon(t) \;=\; \int_0^t \kappa_\varepsilon(t-s)G(s,\psi_\varepsilon(s))ds \tag{7}$$

with

$$G(s,w) \;=\; f(s)+\frac{1}{2}\sigma^2w^2+V_1(w)$$

and existence/uniqueness results for such VIEs are established in Theorem 13 in [BPS24] (or see Appendix A here for a short self contained proof for λ sufficiently large, using the Banach fixed point theorem). $V_t^\varepsilon\geq 0$ for all $t\geq 0$ and (from Appendix B) we know that $G(s,\psi_\varepsilon(t))\leq 0$, so

$$M_t \;\leq\; e^{\int_0^t f(T-s)V_s^\varepsilon ds} \;\leq\; 1\,.$$

[3] R_B is the resolvent of the second kind of $-KB$, see e.g. page 13 of [BLP24] for definition.

[4] A general linear VIE of the form $x(\tau)+(k*x)(\tau)=f(\tau)$ has solution $x(\tau)=f(\tau)-(r*f)(\tau)$ where r is the **resolvent** of the 2nd kind of k, which is the unique function r which satisfies $r+r*k=k$ (the resolvent exists if k is locally integrable). To see this, we substitute $x(\tau)$ into the VIE to get $x\,+\,k*x=x+k*(f-r*f)=x+(k-k*r)*f=x+r*f=f$.

[5] an instructive example to keep in mind here is the case when V_1 is the cgf for a one-sided tempered stable (CGMY) process with $\nu(dx)=\frac{Ce^{-Mx}}{x^{1+Y}}1_{x>0}$ for $C,M>0$ and $Y\in(0,2)\setminus\{1\}$ for which $V_1(p)=C(M(M-p)^Y+M^Y(-M+pY))\Gamma(-Y))/M$

[6] the strict negativity here will be needed in Appendix B.

[7] the final expression in just follows from setting $u=T-s$ in the left integral; then $s=T-u$ and $ds=-du$

Thus M is a bounded local martingale, and hence a martingale [8], and (since $G_T=0$) we see that

$$M_t \;=\; \mathbb{E}(M_T|\mathcal{F}^W_t) \;=\; \mathbb{E}(e^{\int_0^T f(T-s)V^\varepsilon_s ds}|\mathcal{F}^W_t)$$

and in particular, at $t=0$ we have

$$\mathbb{E}(e^{\int_0^T f(T-s)V^\varepsilon_s ds}) \;=\; e^{\int_0^T G(s,\psi_\varepsilon(s))\xi^\varepsilon_0(T-s)ds}\,. \tag{8}$$

Lemma 2.4 *We have the uniform bounds $\Phi_0^-(t)\le\psi_\varepsilon(t)\le 0$ for all $t\ge 0$, where $\Phi_0^-(t)$ is the negative root of $G(t,.)$ (defined below Eq (7)).*

Proof. See Appendix B. ■

Using the same notation as Appendix A in [ER19], we also note that $\int_t^\infty \lambda\kappa_\varepsilon(s)ds = 1-F^{\alpha,\lambda/\varepsilon}(t)=\int_t^\infty \frac{\lambda}{\varepsilon}s^{\alpha-1}E_{\alpha,\alpha}(-\frac{\lambda}{\varepsilon}s^\alpha)ds$, where $F^{\alpha,\lambda}(t)=\int_0^t f^{\alpha,\lambda}(s)ds$. Letting $u=\frac{s}{\varepsilon^{\frac{1}{\alpha}}}$, we can re-write $1-F^{\alpha,\lambda/\varepsilon}(t)$ as

$$\int_t^\infty \lambda\kappa_\varepsilon(s)ds \;=\; \varepsilon^{\frac{1}{\alpha}}\int_{\frac{t}{\varepsilon^{\frac{1}{\alpha}}}} \frac{\lambda}{\varepsilon}\varepsilon^{\frac{\alpha-1}{\alpha}}u^{\alpha-1}E_{\alpha,\alpha}(-\lambda u^\alpha)du \;=\; \int_{\frac{t}{\varepsilon^{\frac{1}{\alpha}}}}^\infty \lambda u^{\alpha-1}E_{\alpha,\alpha}(-\lambda u^\alpha)ds \;=\; 1-F^{\alpha,\lambda}(\tfrac{t}{\varepsilon^{\frac{1}{\alpha}}}) \;\sim\; \frac{1}{\lambda\Gamma(1-\alpha)}\varepsilon t^{-\alpha} \tag{9}$$

as $\varepsilon\to 0$ (see e.g. Appendix A in [ER19] for the asymptotic relation here on the right, although we will not need this relation).

Add in arguments from Convergence.pdf here if correct

Hence $\psi_\varepsilon(t)$ tends to the (unique) negative solution of

$$f(t)\;-\;\lambda\psi_0(t)\;+\;\frac{1}{2}\sigma^2\psi_0(t)^2\;+\;V_1(\psi_0(t)) \;=\; 0 \tag{10}$$

for $t>0$ at the continuity points of f (since Appendix B rules out the positive solution). Note we do not have convergence at $t=0$ since $\psi_\varepsilon(0)=0$, but $\psi_0(0)\neq 0$ because $f(0)<0$).

Hence we also have convergence for $G(s,\psi_\varepsilon(s))$ Lebesgue a.e. (since we have convergence at the continuity points of f away from zero) in (8), so (from the bounded convergence theorem), we have

$$\lim_{\varepsilon\to 0}\mathbb{E}(e^{\int_0^T f(T-s)V^\varepsilon_s ds}) \;=\; \lim_{\varepsilon\to 0} e^{\int_0^T G(s,\psi_\varepsilon(s))\xi^\varepsilon_0(T-s)ds} \;=\; e^{\int_0^T G(s,\psi_0(s))\xi^0_0(T-s)ds} \;=\; e^{\lambda\int_0^T \psi_0(s)\xi^0_0(T-s)ds} \tag{11}$$

since ψ_ε converges to ψ_0 Lebesgue a.e. on $(0,T]$ (and ξ^ε_0 converges pointwise to ξ^0_0 by assumption). We now characterize the process which has (11) as its characteristic function.

For a Lévy process Z with Lévy triple $(-\lambda,\sigma^2,\nu)$ where ν is defined as above, $e^{uZ_t-\Lambda(u)t}$ is an $\mathcal{F}^Z$-martingale for $u\le 0$, where $\Lambda(u)=-\lambda u+\frac{1}{2}\sigma^2u^2+V_1(u)$. Hence from the optional stopping theorem (OST)

$$\mathbb{E}(e^{-qH_a}) \;=\; e^{-\Phi(q)a} \;=\; e^{\Phi(q)|a|}$$

for $q>0$ and $a<0$, where $H_a:=\inf\{t\ge 0: Z_t<a\}$ and $\Phi(q)<0$ is the smallest root of $\Lambda(u)=q$ (since we have a *lower* barrier here), and by standard theory on hitting times for Lévy processes (since $-\lambda+V_1'(0)=-\lambda<0$) $H_a<\infty$ a.s. for $a<0$ (see e.g. Eq 2.5 in [KKR13] for details on this, and the limiting argument used to prove this using the dominated convergence theorem [9]). Now let $X_t=H_{-t}$ (not the same X process as section 1) which is a Lévy subordinator. Then from the i.i.d. property for Lévy processes, for any left continuous piecewise constant function f and differentiable function g, we have

$$\mathbb{E}(e^{\int_0^T f(T-s)d(X_{g(s)})}) \;=\; e^{\int_0^T \Phi(-f(T-s))g'(s)ds} \;=\; e^{\int_0^T \Phi(-f(s))g'(T-s)ds} \tag{12}$$

(see also Lemma 15.1 in [CT04], which is used in [AAR25]), (recall $f<0$ by assumption). But $\Phi(-f(.))$ is $\psi_0(.)$ from Eq (11). Hence for $0<t_1<..<t_n$, choosing $f(s)=(u_1+u_2+...+u_n)1_{0\le s\le s_1}+(u_2+...+u_n)1_{s_1<s\le s_2}+...+u_n1_{t_{n-1}<s\le s_{n-1}}$ with $u_1,u_2,...<0$, we see that

$$\int_0^T f(s)V^\varepsilon_s ds \;=\; u_1A^\varepsilon_{s_1}+...+u_nA^\varepsilon_{s_n} \quad,\quad \int_0^T f(s)dX_{g(s)} \;=\; u_1X_{g(s_1)}+...+u_nX_{g(s_n)}$$

where $A^\varepsilon_t=\int_0^t V^\varepsilon_s ds$, so (again by Problem 30.4 in [Bill86]) we see that the finite-dimensional distributions of $A^\varepsilon_\cdot=\int_0^{(.)}V^\varepsilon_s ds$ converge weakly to those of the time-changed Lévy process $X_{g(t)}$ with $g'(t)=\lambda\xi^0_0(t)$ and $g(0)=0$. ■

[8] see also related discussion below Eq 2.24 in [GK19].

[9] in [KKR13] they take the largest root since they are considering an upper barrier

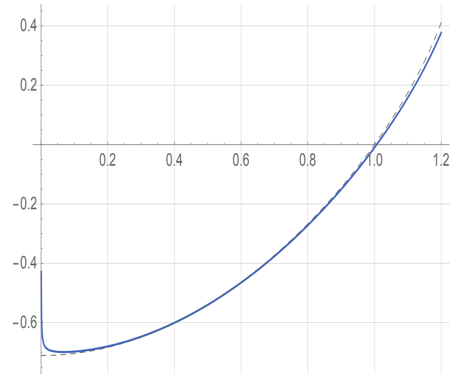

Figure 1: Here we have plotted ψ_ε in Remark 2.2 (in blue) and ψ_0 (grey dashed), using an Adams scheme with 2000 time steps with $\varepsilon = .01$, $H = 0.2$, $\nu = .4$, $\lambda = 1$, $T = 1$, $f(s) = -\cos(\frac{1}{2}\pi s)$ and $\nu(x) = \frac{Ce^{-Mx}}{x^{1+Y}} 1_{x>0}$ for $C = 1$, $M = 3$ and $Y = 1.5$, and we see convergence to ψ_0 away from zero (see e.g. [BL24] for details on refinements to Adams schemes).

Remark 2.3 For the process V^ε in (5), if we instead set $M_t = e^{\int_0^t f(T-s)V_s^\varepsilon ds + G_t}$ with $G_t = \int_t^T g(T-s) g_t^\varepsilon(s) ds$ and we define the "adjusted forward process" $g_t^\varepsilon(s)$ as in Eq 4 in [BPS24] as

$$g_t^\varepsilon(s) \quad = \quad V_0 + \int_0^t K(s-r)\frac{1}{\varepsilon}(\lambda(\theta - V_r^\varepsilon)dr + \sigma\sqrt{V_r^\varepsilon} dW_r + d\tilde{J}_r^\varepsilon)$$

for $s \geq t$, so $g_t^\varepsilon(t) = V_t^\varepsilon$ (with $K(t) = \frac{1}{\Gamma(\alpha)} t^\alpha$), then following the same arguments as Appendix C we can check that ψ_ε satisfies $\varepsilon\psi_\varepsilon(\tau) = \int_0^\tau K(\tau - s)(f(s) - \lambda\psi_\varepsilon(s) + \frac{1}{2}\sigma^2\psi_\varepsilon(s)^2 + V_1(\psi_\varepsilon))ds$ and the limiting solution agrees with ψ_0 in the proof of Proposition 2.2 (we test this numerically below).

References

[AAR25] Abi Jaber, E., E.Attal and M.Rosenbaum, "From Hyper Roughness to Jumps as $H \to -\frac{1}{2}$", preprint, 2025

[AA25] Abi Jaber, E. and E.Attal, "Simulating integrated Volterra square-root processes and Volterra Heston models via Inverse Gaussian", preprint, 2025

[AC24] Abi Jaber, E., and N. De Carvalho, "Reconciling rough volatility with jumps", *SIAM Journal on Financial Mathematics*, 15(3), 785-823, 2024

[ALP19] Abi Jaber, E., M.Larsson, and S.Pulido, "Affine Volterra processes", Annals of Applied Prob-ability, Volume 29, Number 5 (2019), 3155-3200

[AE19] Abi Jaber, O. El Euch, "Multifactor approximation of rough volatility models", *SIAM Journal on Financial Mathematics*, 10(2):309–349, 2019

[BER25] Ballotta, L., Eberlein, E. and Rayée, G., "The term structure of implied correlations between S&P and VIX markets", *Frontiers of Mathematical Finance*, 5, pp. 73- 93.

[Bill86] Billingsley, P., "Probability and measure", John Wiley&Sons, 2nd edition edition, 1986.

[BL24] Boyarchenko, S. and S.Levendorskii, "Correct implied volatility shapes and reliable pricing in the rough Heston model", preprint, 2024

[BLP24] Bondi, A., G.Livieri and S.Pulido, "Affine Volterra processes with jumps", *Stochastic Processes and their Applications*, Volume 168, February 2024, pgs 1042-64

[BPS24] Bondi, A., S.Pulido and S.Scotti, "The rough Hawkes Heston stochastic volatility model", *Mathematical Finance*, Volume 34, Issue 4, Oct 2024, pages 1197-1241

[CT04] Cont, R. and P. Tankov, "Financial modelling with Jump Processes", Chapman&Hall, 2004.

[Cuch22] Cuchiero, C., "Modelling rough covariance processes", talk, February 2022

[DZ98] Dembo, A. and O.Zeitouni, "Large deviations techniques and applications", Jones and Bartlet publishers, Boston, 1998

[ER19] El Euch, O. and M.Rosenbaum, "The characteristic function of Rough Heston models", *Mathematical Finance*, 29(1), 3-38, 2019.

[ER18] El Euch, O. and M.Rosenbaum, "Perfect hedging in rough Heston models", *Annals of Applied Probability*, 28(6):3813–3856, 2018

[EFR18] El Euch, O., M.Fukasawa, and M.Rosenbaum, "The microstructural foundations of leverage effect and rough volatility", *Finance&Stochastics*, 12 (6), p. 241-280, 2018.

[FGS21] Forde, M., S.Gerhold and B.Smith, "Small-time, large-time and $H \to 0$ asymptotics for the rough Heston model, *Mathematical Finance*, 31(1), 203-241, 2021

[FG24] Friz, P. and J.Gatheral, "Computing the SSR", preprint

[FSV21] Forde, M., B.Smith and L.Viitasaari, "Rough volatility and CGMY jumps with a finite history and the Rough Heston model - small-time asymptotics in the $k\sqrt{t}$ regime", with B.Smith and L.Viitasaari, *Quantitative Finance*, 21(4), 541-563, 21(4), 2021.

[FGS21II] Forde, M., S.Gerhold and B.Smith,"Small-time VIX smile and the stationary distribution for the Rough Heston model", with S.Gerhold and B.Smith

[FS21] Forde, M and B.Smith, "Rough Heston with jumps - joint calibration to SPX/VIX level and skew as $T \to 0$, and issues with the quadratic rough Heston model"

[FJ11] Forde, M. and A.Jacquier, "The large-maturity smile for the Heston model", *Finance and Stochastics*, 15(4):755–780, 2011

[GGP19] Gerhold, S., C.Gerstenecker and A.Pinter, "Moment Explosions In The Rough Heston Model", *Decisions in Economics and Finance*, 42(2), pp. 575-608, 2019

[GK19] Gatheral, J. and M.Keller-Ressel, "Affine forward variance models", *Finance and Stochastics*, Volume 23, pages 501-533, 2019.

[JR16] Jaisson, T. and M.Rosenbaum, "Rough fractional diffusions as scaling limits of nearly unstable heavy tailed Hawkes processes", *Annals of Applied Probability*, 26(5), 2860-2882, 2016.

[JR20] P.Jusselin and M.Rosenbaum, "No-arbitrage implies power-law market impact and rough volatility", *Mathematical Finance*, 30(4):1309–1336, 2020.

[KKR13] Kuznetsov, A., A.Kyprianou and V.Rivero, "The theory of scale functions for spectrally negative Lévy processes", review article

[Sato99] Sato,K., "Lévy Processes and Infinitely Divisible Distributions", Cambridge University Press: Cambridge, UK, 1999.

Appendix

A Existence and uniqueness for the VIE in (7)

Let $G(s,w)=f(s)+\frac{1}{2}\sigma^2w^2+V_1(w)$ with $f<0$, and set $G_-=G\wedge 0$. Define the operator $(AF)(t)=\int_0^t\kappa_\varepsilon(t-s)G_-(s,F(s))ds$ acting on F in the Banach space $C([0,T])$ with the usual sup norm. Now define the sequence of Picard iterates $\psi^\varepsilon_{n+1}(t)=(A\psi^\varepsilon_n)(t)$ with $\psi^\varepsilon_0\equiv 0$. $G_-(s,.)$ is Lipschitz, so

$$|G_-(s,w')-G_-(s,w)| \quad\leq\quad K_1|w'-w|$$

for some constant $K_1>0$ which is independent of ε. Then

$$|A\phi(t)-A\psi(t)| \quad\leq\quad \int_0^t\kappa_\varepsilon(t-s)|G_-(s,\phi(s))-G_-(s,\psi(s))|ds \quad\leq\quad \frac{K_1}{\lambda}\int_0^t\lambda\kappa_\varepsilon(s)|\phi(t-s)-\psi(t-s)|ds \quad\leq\quad \frac{K_1}{\lambda}\|\phi-\psi\|$$

so A is a contraction when $\lambda>K_1$. From the Banach fixed point theorem, $\psi^\varepsilon_n(t)$ converges to some ψ^ε in $C([0,T])$ in the sup norm for any $T>0$, and ψ^ε is the unique solution of $\psi^\varepsilon=A\psi^\varepsilon$. Finally, assume (for a contradiction) that $G(\psi_\varepsilon(t))>0$ for some $t\in[0,T]$. For the case with no jumps, we can easily compute the smallest admissible K_1 as $\sigma\sqrt{-2\bar{f}}$ where $\bar{f}=\sup_{0\leq t\leq T}f(t)<0$.

But Lemma 2.4 also holds with G replaced by G_- (the final term in Eq (B-1) just becomes zero), so the solution of $\psi_\varepsilon=A\psi_\varepsilon$ that we have just constructed satisfies $G(t,\psi_\varepsilon(t))\leq 0$, and hence ψ satisfies with same VIE with G_- changed back to G.

B Proof of Lemma 2.4

Recall the VIE in (7): $\psi_\varepsilon(t)=\int_0^t\kappa_\varepsilon(t-s)G(s,\psi_\varepsilon(s))ds$ with $G(s,w)=f(s)+\frac{1}{2}\sigma^2w^2+V_1(w)$. Let $\Phi^0_\pm(s)$ denote the roots of $G(s,.)=0$ which are strictly positive and negative since $f(0)<0$ (the zero superscript is here to distinguish this from the Φ function in section 2.1 which includes a λ term). If $G(s,.)$ has no positive root because V_1 blows up, then we let $\Phi^0_+(s)=\inf\{w:G(s,w)>0\}$, which (from Assumption 2.2) is also strictly positive.

Since $\psi_\varepsilon(0)=0$ and $\psi_\varepsilon(.)$ is continuous so $\psi_\varepsilon(t)$ is close to zero for t small, and $f(s)$ is constant for $0\leq s\leq s_1$ and $f<0$, drawing a picture of $G(s,.)$ we see that $\Phi^0_-(t)<\psi_\varepsilon(t)<\Phi^0_+(t)$ for t in some non-zero interval $(0,t_*)$ (since $\Phi^0_\pm(t)$ are strictly positive and negative), so (since $G(\psi_\varepsilon(t))<0$ on $(0,t^*)$), from the definition of the VIE for ψ_ε (and the positivity of κ_ε) we see that $\psi_\varepsilon(t)<0$ on $(0,t_*)$.

Now let $\delta>0$, and (for a contradiction) let t_1 denote the first time ψ_ε hits $\Phi^0_- -\delta$, and $t_0<t_1$ denote the first time ψ_ε hits $\Phi^0_-(t_1)$. Then

$$\begin{aligned}\psi_\varepsilon(t_1) &= \int_0^{t_0}\kappa_\varepsilon(t_1-s)G(s,\psi_\varepsilon(s))ds+\int_{t_0}^{t_1}\kappa_\varepsilon(t_1-s)G(s,\psi_\varepsilon(s))ds\\ &= \psi_\varepsilon(t_0)+\int_0^{t_0}(\kappa_\varepsilon(t_1-s)-\kappa_\varepsilon(t_0-s))G(s,\psi_\varepsilon(s))ds+\int_{t_0}^{t_1}\kappa_\varepsilon(t_1-s)G(s,\psi_\varepsilon(s))ds\,. \qquad \text{(B-1)}\end{aligned}$$

Then the first integral on the right is non negative (since $\kappa_\varepsilon(.)$ is decreasing) and the second integral is non-negative because $G(s,\psi_\varepsilon(s))>0$ on $[t_0,t_1]$), but $\psi_\varepsilon(t_1)-\psi_\varepsilon(t_0)=\delta<0$, a contradiction. Thus ψ_ε cannot reach $\Phi^0_- -\delta$, and since $\delta>0$ is arbitrary, we conclude that $\psi_\varepsilon(t)\geq\Phi^0_-(t)$ for all $t\geq 0$, and note that $\Phi^0_-(.)\leq\Phi_-(.)$ since $-\lambda u\geq 0$ for $u\leq 0$.

Looking again at the right side of the VIE for ψ_ε, this also implies that ψ_ε could only go positive if ψ_ε first exceeds Φ_+, but $\Phi_+>0$, so this is clearly impossible.

C Proof of the VIE for $G(.,\psi_\varepsilon(.))$

We set $\varepsilon=1$ in this appendix and drop the ε superscripts to ease notation since the arguments will be exactly the same for general $\varepsilon>0$, and recall $G_t=\int_t^T g(T-s)\xi_t(s)ds$, $d\xi_t(u)=\kappa(u-t)(\sigma\sqrt{V_t}dW_t+d\tilde{J}_t)$ and $M_t=e^{\int_0^t f(T-s)V_s ds+G_t}$. Then

$$\begin{aligned} dG_t &= -g(T-t)V_t dt+\int_t^T g(T-s)\kappa(s-t)(\sigma\sqrt{V_t}dW_t+d\tilde{J}_t)ds \\ &= -g(T-t)V_t dt+(\int_t^T g(T-s)\kappa(s-t)ds)(\sigma\sqrt{V_t}dW_t+d\tilde{J}_t)\,. \end{aligned}$$

Then from the change-of-variable formula we see that

$$\begin{aligned} dM_t &= M_{t_-}(f(T-t)V_t dt+dG_t+\frac{1}{2}d\langle G\rangle_t) \\ &= M_{t_-}(f(T-t)V_t dt-g(T-t)V_t+\frac{1}{2}\sigma^2(\int_t^T g(T-s)\kappa(s-t)ds)^2 V_t dt+(e^{(\int_t^T g(T-s)\kappa(s-t)ds)d\tilde{J}_t}-1)+(...)dW_t^1 \end{aligned}$$

so (taking out a common factor of V_t) we see that M_t is a local martingale if g satisfies the VIE:

$$g(T-t) = f(T-t)+\frac{1}{2}\sigma^2(\int_t^T g(T-s)\kappa(s-t)ds)^2+(\int_0^\infty e^{(\int_t^T g(T-s)\kappa(s-t)ds)x}-1)\nu(dx)\,.$$

Setting $\tau=T-t$ and $r=T-s$ (so $s=T-r$ and $s-t=T-r-(T-\tau)=\tau-r$), we can re-write $\int_{T-\tau}^T g(T-s)\kappa(s-t)ds=\int_0^\tau \kappa(\tau-r)g(r)dr$, so

$$g(\tau) = f(\tau)+\frac{1}{2}\sigma^2(\int_0^\tau \kappa(\tau-r)g(r)dr)^2+V_1(\int_0^\tau \kappa(\tau-r)g(r)dr)$$

or setting $\psi(\tau)=\int_0^\tau \kappa(\tau-r)g(r)dr$, we can re-cast this in terms of ψ as

$$\psi(\tau) = \int_0^\tau \kappa(\tau-s)(f(s)+\frac{1}{2}\sigma^2\psi(s)^2+V_1(\psi(s)))ds\,.$$

and recall the exponent for the characteristic function in (6) is

$$\int_0^t f(T-s)V_s ds+\int_t^T g(T-s)\xi_t(s)ds\,.$$

D Brief summary of pgs 5-6 in [AAR25]

$$e^{iuA_t^\varepsilon} = e^{iuV_0 t+iu\int_0^t \kappa_\varepsilon(t-s)W_{A_s^\varepsilon}ds}$$

Consider a family of hyper-rough Heston models[10] (with zero mean-reversion for simplicity) for which the quadratic variation of the log stock price satisfies

$$\langle\log S^n\rangle_t = X_t^n = V_0 t+(H_n+\frac{1}{2})\sigma\int_0^t (t-s)^{H_n-\frac{1}{2}}W_{X_s^n}ds$$

for $H_n\in(-\frac{1}{2},1)$. From Lemma 2.4 in [AAR25] [11], we formally expect that

$$\lim_{H_n\searrow-\frac{1}{2}}(H_n+\frac{1}{2})\sigma\int_0^t (t-s)^{H_n-\frac{1}{2}}W_{X_s^n}ds = \sigma W_{X_t}\,,$$

where X is the weak limit of X^n, so we expect X to satisfy

$$X_t = V_0 t+\sigma W_{X_t}\,. \qquad \text{(A-1)}$$

Now let $Y_t=-t+\sigma W_t$, and set $X_t=H_{-V_0 t}$, where $H_b=\inf\{t:Y_t=b\}$. Then setting $t\mapsto X_t$ we see that

$$-V_0 t = -X_t+\sigma W_{X_t}$$

[10] see [JR20] and section 5 in [FGS21] for more on this model

[11] this lemma is particularly easy to check when f is a polynomial

which we can re-arrange to re-produce (A-1). Hence (using the notation/setup in Lemma 2.3 in [AAR25], $c=-V_0$, $b=\sigma$ and $a=-1$, i.e. we have an Inverse Gaussian Lévy process with parameters $(V_0,\frac{V_0^2}{\sigma^2})$.

To analyze this process with VIEs, using that $\frac{1}{\Gamma(\alpha)}=\alpha+O(\alpha^2)$ as $\alpha\to 0$ (i.e. as $H\to-\frac{1}{2}$), we see that the usual rough Heston VIE (with $\rho=0$) takes the form

$$\begin{aligned}\phi(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(-\frac{1}{2}(u^2+iu)\,+\,\frac{1}{2}\sigma^2\alpha\Gamma(\alpha)\phi(s)^2)ds\\ &= (1+O(\alpha))\alpha\int_0^t (t-s)^{\alpha-1}(-\frac{1}{2}(u^2+iu)\,+\,\frac{1}{2}\sigma^2(1+O(\alpha))\phi(s)^2)ds \;\;\to\;\; -\frac{1}{2}(u^2+iu)\,+\,\frac{1}{2}\sigma^2\phi(t)^2\end{aligned}$$

as $\alpha\to 0$ (again using Lemma 2.4 in [AAR25], which is just an algebraic equation for ϕ. If we ignore the linear term in u for simplicity (i.e. ignore the drift of the log stock price), then the (relevant) solution to this equation is $\phi(t)=\frac{1}{\sigma^2}(1-\sqrt{1+\sigma^2u^2})$, i.e. the smaller root as in the proof of Proposition 2.1.