An explicit Karhunen-Loève theorem for the Riemann-Liouville process, and accurate Monte Carlo for the Rough Bergomi model for H small

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Abstract

We compute a new Karhunen-Loève (K-L) expansion for the Riemann-Liouville process as an eigenfunction expansion in its Cameron-Martin space with random coefficients, for which all terms can be explicitly computed (unlike the standard Karhunen-Loève expansion which requires the eigenfunctions and eigenvalues of the covariance operator of the process which are not known explicitly). When applied to the Rough Bergomi volatility model, the new expansion gives exceptional accuracy in sampling the integrated variance for small H-values where traditional Monte Carlo methods (Cholesky, Riemann sum etc) break down.

1 Introduction

Following section 4.3 in [Gia15], we first briefly recall the classical Karhunen-Loève theorem. Let $(X_t)_{t\in[a,b]}$ be a centred continuous-parameter real-valued process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is second order (i.e. $\mathbb{E}(X_t^2) < \infty$ for all $t \in [a, b]$) with continuous covariance function $K_X(s, t)$. Let

$$Z_k = \int_a^b X_t e_k(t)$$

where $\{e_k\}_{k=1}^{\infty}$ are the eigenfunctions of the Hilbert-Schmidt integral operator on $L^2([a; b])$ given by $(Af)(t) = \int_a^b K_X(s,t)f(s)ds$, which is an orthonormal basis for the space spanned by the eigenfunctions corresponding to the non-zero eigenvalues of A. Then $\mathbb{E}(Z_j Z_k) = \lambda_k \delta_{jk}$ for all j, k, $\mathbb{E}(Z_j) = 0$ for all j, and the series

$$\sum_{n=1}^{\infty} Z_k e_k(t)$$

converges to X_t in mean square, uniformly for $t \in [a, b]$. This expansion is often said to be *bi-orthogonal*, since the random coefficients Z_k are orthogonal in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ and the eigenfunctions are orthogonal in $L^2([a, b])$. If X is Gaussian, then the Z_k's are independent Gaussians.

2 An explicit Karhunen-Loève expansion for the Riemann-Liouville process

The K-L expansion of standard Brownian motion is given by

$$W_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, \phi_n(t) Z_n$$

is a standard Brownian motion on $[0, 1]$ (see page 50 in [Gia15]), where Z_n is a sequence of i.i.d. standard Normals, $\lambda_n = \frac{4}{(2n-1)^2\pi^2}$, $\phi_n(t) = \sqrt{2}\sin((n - \frac{1}{2})\pi t)$. λ_n and ϕ_n are the eigenvalues and eigenfunctions of the Hilbert-Schmidt covariance operator $R_{\frac{1}{2}}\theta : L^2([0, 1]) \to L^2([0, 1])$ given by $R_{\frac{1}{2}}\theta(t) = \int_0^1 R_{\frac{1}{2}}(s,t)\theta(s)ds = \int_0^1 (s \wedge t)\theta(s)ds$, and ϕ_n form a basis of $L^2([0, 1])$.

We now consider a Riemann-Liouville process

$$Y_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$$

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where $H \in (0,1)$ and W is a standard Brownian motion, then $R_H(s,t) := \mathbb{E}(Y_s Y_t) = \int_0^{s\wedge t}(s-u)^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}}du$. We define the operator $K_H : L^2 \to C[0,1]$ as $K_H f(t) = \int_0^t (t-s)^{H-\frac{1}{2}} f(s)ds$ for $H \in [0,\frac{1}{2})$, and set

$$X_t^n \quad = \quad \sum_{k=1}^{n} \sqrt{\lambda_k}\, K_H \phi_k'(t) Z_n\,. \tag{1}$$

Then the covariance function of X^n is

$$R_{H,n}(s,t) \quad = \quad \sum_{k=1}^{n} \lambda_k K_H \phi_k'(s) K_H \phi_k'(s) \tag{2}$$

and we define the Hilbert space $\mathcal{H} = K_H L^2([0,1])$ with inner product $\langle F, G\rangle_{\mathcal{H}} = \langle f, g\rangle_{L^2([0,1])}$ if $F = K_H f$ and $G = K_H g$.

We will require the following variant of Mercer's theorem:

Proposition 2.1 *For $H \in (0,\frac{1}{2})$*

$$R_{H,n}(s,t) \quad = \quad \sum_{k=1}^{n} \lambda_k K_H \phi_k'(s) K_H \phi_k'(t) \quad \to \quad R_H(s,t) \tag{3}$$

in $L^2([0,1]^2)$.

Proof. $\phi_k'(t)$ is a multiple of $\cos((k-\frac{1}{2})\pi s)$ and $(\cos((k-\frac{1}{2})\pi s))_{k=1}^{\infty}$ also forms an orthonormal basis of $L^2([0,1])$ and from this we can easily verify that $\sqrt{\lambda}_k(\phi_k')$ forms an orthonormal basis of $L^2([0,1])$, so $\sqrt{\lambda}_k(K_H\phi_k')_{k=1}^{\infty}$ forms an orthonormal basis of $\mathcal{H}$.

Moreover, for $s < t$

$$R_H(s,t) \quad = \quad \int_0^s (s-r)^{H-\frac{1}{2}}(t-r)^{H-\frac{1}{2}}dr \quad = \quad K_H(t-(.))^{H-\frac{1}{2}}(s)$$

and $(t-(.))^{H-\frac{1}{2}} \in L^2([0,1])$, so $R_H(.,t) \in \mathcal{H}$ since $2(H-\frac{1}{2}) > -1$. We now verify that the infinite series on the left hand side of (3) is the Fourier series of $R_H(.,t)$ in $\mathcal{H}^2$ in this basis.

To this end, let $g(s,t) = (t-s)^{H-\frac{1}{2}} 1_{s<t}$ and $G(s,t) = \int_0^s (t-r)^{H-\frac{1}{2}} 1_{r<t}dr$. Then we see that

$$(K_H g(.,t))(s) \quad = \quad \int_0^s (s-r)^{H-\frac{1}{2}}(t-r)^{H-\frac{1}{2}} 1_{r<t}dr \quad = \quad \int_0^{s\wedge t} (s-r)^{H-\frac{1}{2}}(t-r)^{H-\frac{1}{2}}dr \quad = \quad R_H(s,t)\,.$$

Thus

$$\begin{aligned}\langle R_H(.,t), K_H\phi_n'\rangle_{\mathcal{H}} \quad &= \quad \langle g(.,t), \phi_n'\rangle_{L^2([0,1])} \quad = \quad \int_0^1 (t-s)^{H-\frac{1}{2}} 1_{s<t}\, \phi_n'(s)ds \\ &= \quad \int_0^t (t-s)^{H-\frac{1}{2}}\, \phi_n'(s)ds \quad = \quad K_H\phi_n'(t)\,.\end{aligned}$$

Thus the kth Fourier coefficient of $R_H(.,t)$ in the aforementioned basis is

$$\langle R_H(.,t), \sqrt{\lambda_k} K_H \phi_n'\rangle_{\mathcal{H}} \quad = \quad \sqrt{\lambda_k} K_H \phi_n'(t)\,.$$

Thus $R_{H,n}(.,t)$ tends to $R_H(.,.)$ in $\mathcal{H}$, which implies that $R_{H,n}(.,t)$ tends to $R_H(.,t)$ in $L^2([0,1])$. Then

$$\|R_H - R_{H,n}\|_{L^2([0,1]^2)} \quad = \quad \int_0^1 \int_0^1 (R_H(s,t) - R_{H,n}(s,t))^2 dsdt\,.$$

The result then follows by combining the $L^2([0,1])$-convergence of $R_{H,n}(.,t)$ to $R_H(.,.)$ and the bounded convergence theorem, since

$$R_{H,n}(s,t) \quad = \quad \mathbb{E}(X_s^n X_t^n) \quad \leq \quad \mathbb{E}((X_s^n)^2)^{\frac{1}{2}}\mathbb{E}((X_t^n)^2)^{\frac{1}{2}} \quad \leq \quad \mathbb{E}((X_s)^2)^{\frac{1}{2}}\mathbb{E}((X_t)^2)^{\frac{1}{2}} \quad = \quad s^H t^H \quad \leq \quad 1\,.$$

■

Theorem 2.2 *X_t^n tends to*

$$X_t \quad = \quad \sum_{n=1}^{\infty} \sqrt{\lambda_n}\, K_H \phi_n'(t)\, Z_n \tag{4}$$

in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ for t fixed and $X^n \to X \sim Y$ in $L^2(\Omega \times [0,1])$.

Remark 2.1 It should be possible to adapt this result for other Gaussian Volterra processes, e.g. fractional Brownian motion, and non-Gaussian processes, but for the sake of brevity we do not pursue this here.

Proof. (3) implies that X_t has finite second moment equal to $R_H(t,t) = \frac{t^{2H}}{2H}$, so $X_t < \infty$ a.s. and we know that $R_{H,n}(t,t) \nearrow R_H(t,t)$, so

$$R_H(t,t) - R_{H,n}(t,t) = \mathbb{E}((X_t - X_t^n)^2) \to 0$$

so X_t^n tends to X_t in $L^2(\Omega, \mathcal{F}, \mathbb{P})$, for t fixed. From Fubini, we also see that

$$\mathbb{E}(\int_0^1 (X_t - X_t^n)^2 dt) = \int_0^1 \mathbb{E}((X_t - X_t^n)^2) dt$$

We have just shown that $\lim_{n\to\infty} \mathbb{E}((X_t - X_t^n)^2)$, and $\mathbb{E}((X_t - X_t^n)^2) = \mathbb{E}((X_t)^2 - 2X_t X_t^n + \mathbb{E}((X_t^n)^2) \leq \mathbb{E}((X_t)^2 + 2\mathbb{E}(|X_t||X_t^n|) + \mathbb{E}((X_t^n)^2) \leq 4\mathbb{E}((X_t)^2) < \infty$, where we have used Cauchy-Schwarz for the final inequality. Thus, from the bounded convergence theorem $X^n \to X$ in $L^2(\Omega \times [0,1])$ as well. ■

Remark 2.2 Note we have only considered the K-L expansion on the interval $[0,1]$ here, but our results our easily adapted to an arbitrary finite interval $[0,T]$.

Remark 2.3 The practical power of our expansion in (4) is that we can compute $K'_H \phi_k(t)$ explicitly as

$$K'_H \phi_k(t) = \frac{\sqrt{2}}{1+2H}(2n-1)\pi t^{\frac{1}{2}+H} {}_1F_1(\frac{3}{4} + \frac{1}{2}H, \frac{5}{4} + \frac{1}{2}H, -\frac{1}{16}(2n-1)^2\pi^2 t^2) \tag{5}$$

where ${}_pF_q$ is the generalized hypergeometric function[1], in contrast to the usual K-L expansion for the RL process which requires knowledge of the eigenvalues of eigenfunctions of the covariance operator (see e.g. [GVZ15] for asymptotic results in this direction). Computation of ${}_1F_1$ is expensive but (for a fixed H value) this is just a one-off overhead so they can be precomputed before the Monte Carlo simulations begin (and computation of these functions is massively sped up using vectorization in Matlab).

3 Application to the Rough Bergomi model

Let $\xi^H(dt) = e^{\gamma X_t - \frac{1}{2}\gamma^2 \mathbb{E}(X_t^2)} dt$. Then $\xi^H([0,T])$ is the quadratic variation of the log stock price (also known as the "integrated variance") for the popular Rough Bergomi volatility model. Then from the well known Seiberg formula (see also page 7 in [FFGS19] for the 2d case) we have that

$$\mathbb{E}(\xi_\gamma^H([0,T])^q) = \int_{[0,T]} \dots \int_{[0,T]} e^{\gamma^2 \sum_{1\leq i<j\leq q} R_H(u_i,u_j)} du_i \dots du_q \tag{6}$$

for $q > 0$. In the following table we have tabulated the first four raw moments $\xi_\gamma^H([0,1]))$ for (using (6)) and their corresponding estimates using Monte Carlo simulation with the K-L expansion in (4) for $H = .0001$ using (5) (which we denote by $\hat{\mu}_n^H$), with $n = 1000$ eigenfunctions, 1000 time steps and 1 million simulations for both cases, and Gaussian quadrature for the numerical integration. If we perform the same computations for $H = .0001$ using a traditional Cholesky decomposition with a simple Riemann sum, then we get nonsensical answers, which we have not tabulated here (we comment more on this bad behaviour in the next subsection).

γ	μ_1	μ_2	μ_3	μ_4	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\mu}_3$	$\hat{\mu}_4$
0.05	1	1.00502	1.01513	1.0305	1	1.005	1.0151	1.0305
0.1	1	1.02028	1.06216	1.12836	0.99999	1.0202	1.062	1.1281
0.15	1	1.04644	1.14633	1.31508	0.99999	1.0464	1.1462	1.3148
0.2	1	1.08466	1.27764	1.63646	0.99999	1.0845	1.2771	1.6351
0.25	1	1.13669	1.47323	2.1843	0.99999	1.1367	1.4734	2.1852
0.3	1	1.20512	1.76191	3.14796	0.99985	1.2042	1.7582	3.1336
0.35	1	1.29359	2.19289	4.94361	0.99975	1.2919	2.1849	4.9066
0.4	1	1.40729	2.85324	8.56902	1.0001	1.4078	2.8651	8.9761
0.45	1	1.5537	3.90481	16.6981	1	1.5532	3.8953	16.461
0.5	1	1.74375	5.66824	37.5977	1.0001	1.744	5.6369	35.0148

The following table performs the same computations as above but for $H = .03$ (empirical values as low as this are reported in Fukasawa et al.[FTW19]), and in the final column we compare against a traditional Cholesky scheme using Simpson's rule (also with 1000 time steps and 1 million simulations), and we see that our K-L expansion method outperforms the latter to an increasingly greater extent as γ increases. In the second column, we have computed the usual

[1] using Mathematica's definition

vol-of-variance parameter $\eta = \frac{\gamma}{\sqrt{2H}}$ corresponding to each choice of γ. Matlab was unable to compute a positive-definite 1000 point Cholesky decomposition when we tried using Gaussian quadrature instead of Simpson's rule (the former of course has non equidistant time points), and also for $H = .05$, see final table below). For pricing call options under the rBergomi model, there is no reference to compare a Monte Carlo estimator for the call price against, since we don't have a explicit formula for calls like we do for integer moments of $\xi^H([0,T])$, and we don't have an explicit form for the characteristic function of the log stock price as for Rough Heston, so we advise great caution in pricing calls under rBergomi using traditional MC methods for H small (see next subsection for more discussion on this).

γ	$\eta = \frac{\gamma}{\sqrt{2H}}$	μ_1	μ_2	μ_3	μ_4	$\hat{\mu}_1^H$	$\hat{\mu}_2^H$	$\hat{\mu}_3^H$	$\hat{\mu}_4^H$	$\hat{\mu}_1^{H,\text{chol}}$	$\hat{\mu}_2^{H,\text{chol}}$	$\hat{\mu}_3^{H,\text{chol}}$	$\hat{\mu}_4^{H,\text{chol}}$
0.05	0.204124	1	1.0043	1.01306	1.0263	1	1.0043	1.013	1.0262	1	1.0043	1.0131	1.0264
0.1	0.408248	1	1.01749	1.05345	1.10988	0.99999	1.0175	1.0534	1.1098	1	1.0176	1.0538	1.1107
0.15	0.612372	1	1.03993	1.125	1.26441	1	1.04	1.1253	1.267	1	1.0402	1.1259	1.2684
0.2	0.816497	1	1.0725	1.23489	1.52805	0.99997	1.0725	1.2349	1.5281	1.0001	1.0735	1.238	1.5353
0.25	1.02062	1	1.11644	1.39505	1.95613	1.0002	1.1174	1.3977	1.963	0.9999	1.1173	1.399	1.9678
0.30	1.22474	1	1.17353	1.62473	2.66892	0.99977	1.1727	1.6226	2.6646	0.99999	1.176	1.6352	2.7041
0.35	1.42887	1	1.24619	1.95509	3.90479	1.0002	1.2477	1.9635	3.951	0.99987	1.2505	1.976	3.9871
0.40	1.63299	1	1.3378	2.43775	6.17752	0.99988	1.3371	2.4352	6.1555	0.9998	1.3481	2.5016	6.5463
0.45	1.83712	1	1.45292	3.16099	10.6697	0.99971	1.4505	3.1351	10.2878	0.99937	1.4714	3.2987	11.6325
0.50	2.04124	1	1.59791	4.28211	20.4214	0.99966	1.5947	4.2561	20.1089	1.0003	1.6488	4.8197	28.5117

The final table here performs the same computations for $H = .05$ [2]:

γ	$\eta = \frac{\gamma}{\sqrt{2H}}$	μ_1	μ_2	μ_3	$\hat{\mu}_1^H$	$\hat{\mu}_2^H$	$\hat{\mu}_3^H$	$\hat{\mu}_1^{H,\text{chol}}$	$\hat{\mu}_2^{H,\text{chol}}$	$\hat{\mu}_3^{H,\text{chol}}$
0.05	0.158114	1	1.00395	1.01189	1	1.0039	1.0119	0.99999	1.0039	1.0119
0.1	0.316228	1	1.01591	1.04855	1	1.016	1.0487	1	1.016	1.0488
0.15	0.474342	1	1.03628	1.11314	0.99999	1.0362	1.113	1	1.0365	1.1137
0.2	0.632456	1	1.06573	1.2115	1	1.0658	1.2118	0.99999	1.0659	1.2122
0.25	0.790569	1	1.10527	1.35317	1	1.1054	1.3533	0.99993	1.1054	1.3536
0.3	0.948683	1	1.15631	1.55321	1	1.1564	1.5534	1.0002	1.158	1.5594
0.35	1.1068	1	1.22078	1.83538	1.0001	1.221	1.8361	0.9999	1.2216	1.8392
0.4	1.26491	1	1.30126	2.2377	0.99986	1.3006	2.2351	1.0002	1.3041	2.2491
0.45	1.42302	1	1.40123	2.82265	0.99991	1.4007	2.8226	1.0002	1.4059	2.8495
0.5	1.58114	1	1.52537	3.69624	1	1.5254	3.6893	0.99966	1.5291	3.73

3.0.1 Poor convergence and sampling error in simulating ξ_γ^H using traditional Monte Carlo methods for $H \ll 1$

If we use a standard Riemann sum (or rectangle rule) approximation $X^n = X_{\frac{1}{n}[n(.)]}$ for X, then X^n is still a Gaussian process and its covariance function $R_H^n(s,t)$ is piecewise constant. Then the 2nd moment of the GMC $\xi_\gamma^{H,n}$ associated with X^n is

$$\begin{aligned}\hat{\mu}_2^2 &= \mathbb{E}((\xi_\gamma^{H,n}[0,T]^2) &= \int_{[0,T]^2} e^{\gamma^2 R_H^n(s,t)} ds dt \\ & &= (\Delta t)^2 (2\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} e^{\gamma^2 R_H(i\Delta t, j\Delta t)} + \sum_{i=0}^{n-1} e^{\gamma^2 R_H((i-1)\Delta t,(i-1)\Delta t)}) \qquad (7)\end{aligned}$$

where $\Delta t = \frac{T}{n}$. At least for the case $T \le 1$, then from the monotone convergence theorem, $\mathbb{E}((\xi_\gamma^{H,n}[0,T]^2)$ tends to the true value $\mathbb{E}((\xi_\gamma^H[0,T]^2)$, but the convergence in H is very slow when H is small.

To give some specific numerical examples, for $\gamma = .01$ and using 10,000 time steps (for which the Cholesky decomposition would take weeks to compute in Matlab) with $H = .0001$, and using (7) we find that the Riemann sum Monte Carlo estimate for $\text{Var}(\xi_\gamma^H([0,1])^2)$, i.e. $\hat{\sigma}^2 = \hat{\mu}_2^2 - 1$ is $\hat{\sigma}^2 = 0.000264642$ when the true value for $\text{Var}(\xi_\gamma^H([0,1])^2) = 0.000199772$ (i.e. a 32% error) so the Monte Carlo estimate is wildly biased even for a very low γ value (and for comparison the true $\text{Var}(\xi_\gamma([0,T])^2) = 0.000199871$), and this doesn't even account for how many simulations would be required in practice to get close to the theoretical value of $\hat{\sigma}^2$. More disturbingly, if we change γ to 0.1, the rectangle rule estimate is 5.12841×10^{18} when (from the first table above) we know the true answer is $1.02028 - 1$ (contrast this behaviour with the startling convergence we see in the tables above using the K-L method).

[2] Mathematica had some issues in computing the exact 4th moments μ_4 (for which the formula is a 4-dimensional integral) so we have omitted these values here)

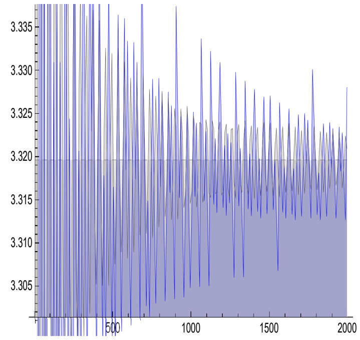

Figure 1: Here we have plotted $R_{0,n}(s,t)$ (see (2) for definition) in grey, and $R(\frac{1}{n}[sn],\frac{1}{n}[tn])$ (blue) as a function of n for the irrational time values $s=1/\pi$ and $t=1/e$, and the dashed line is the limiting value $R(s,t)$

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