Large-time asymptotics for correlated stochastic volatility models

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Outline

- Review of large-time asymptotics for the SABR and CEV-Heston models.
- The large deviation principle - motivation, application to Brownian motion and definition.
- The Donsker-Varadhan LDP for occupation measure of the Ornstein-Uhlenbeck process.
- Large-time asymptotics for correlated stochastic volatility models using the Donsker-Varadhan LDP.
- Computing the asymptotic implied volatility numerically using the Ritz method.
- Extension to the CEV-Ornstein-Uhlenbeck local-stochastic volatility model.

The standard SABR model for $\beta = 1, \rho \leq 0$

- Consider the SABR model with $\beta = 1$

$$\begin{cases} dS_t = S_t Y_t dW_t \,, \\ dY_t = \alpha Y_t dB_t \,, \end{cases} \tag{1}$$

where $dW_t dB_t = \rho dt$, $\bar{\rho} = \sqrt{1-\rho^2}$ and $\rho \in (-1, 0]$, which is necessary to ensure that (S_t) is a martingale (see [Jour04]).

- Let $T_\infty = \sigma^2 A_\infty^{(-\frac{1}{2})}$, where $\sigma = y_0/\alpha$ and $A_t^\mu = \int_0^t e^{2(\mu s + B_s)} ds$ is the well known Brownian exponential functional.
- From [Duf90] (see also [MY05]), for $\mu > 0$, $A_\infty^{-(\mu)}$ is distributed as $Z = (2\gamma_\mu)^{-1}$, where γ_μ denotes a gamma R.V. with parameter μ.
- From this we can the obtain large-time behaviour of European puts

$$\mathbb{E}(K - S_\infty)^+ \quad = \quad \mathbb{E}(P^{\mathrm{BS}}(S_0 e^{-\frac{1}{2}\rho^2 T_\infty - \frac{\rho}{\alpha} y_0}, K, 1, \bar{\rho}^2 T_\infty) \leq K \,,$$

where $P^{\mathrm{BS}}(S, K, \sigma, T$ denotes the Black-Scholes put formula with zero interest rates (see [Forde11b]). Proof obtained by conditioning on T_∞ and using that $Y_t \to 0$ a.s. as $t \to \infty$.

The CEV-Heston model

- Combining the CEV model with a CIR time-change, we can define the uncorrelated *CEV-Heston model*, governed by the following SDEs

$$\begin{cases} dS_t = S_t^{\beta}\sqrt{Y_t}dW_t^1 \,, \\ dY_t = \kappa(\theta - Y_t)dt + \sigma\sqrt{Y_t}dW_t^2 \end{cases}$$

with $dW_t^1 dW_t^2 = 0,\ Y_0 = y_0 > 0,\ \kappa > 0, \sigma > 0, \beta \in (0,1)$.

- For this model, we have the following large-time behaviour for call options

$$S_0 - \mathbb{E}(S_t - K)^+ \;=\; cK\theta^{-\frac{\gamma}{2}}\, t^{-\frac{\gamma}{2}}(1 + o(1)) \tag{2}$$

as $t \to \infty$, where $c = c(\beta, S_0) = \frac{1}{\Gamma(1+\frac{\gamma}{2})}[\frac{1}{2}(\frac{S_0^{-2\bar{\beta}}}{\bar{\beta}^2})]^{\frac{\gamma}{2}}$,
$\bar{\beta} = \beta - 1 \in (-1,0)$ and $\gamma = 1/|\bar{\beta}| > 1$.

- Proof uses saddlepoint methods (see [Forde11b]).

The Large deviation principle (LDP): motivation

- Suppose we have sequence of random variables (X_n) such that X_n is concentrated around x_0 as $n \to \infty$, and for sets A away from x_0, $\mathbb{P}(X_n \in A)$ tends to zero exponentially rapidly in n:

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(X_n \in A) = -I(A)$$

$$\text{i.e.} \quad \forall \delta > 0, \quad e^{-n(I(A)+\delta)} \leq \mathbb{P}(X_n \in A) \leq e^{-n(I(A)-\delta)}$$

  for $n = n(\delta)$ sufficiently large, and some *rate function* $I \geq 0$.

- **Example**: for standard Brownian motion (W_t), for $x > 0$ we have

$$\begin{aligned} \lim_{t\to 0} t \log \mathbb{P}(W_t > x) &= -\frac{1}{2}x^2, \\ \lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(\frac{W_t}{t} > x\right) &= -\frac{1}{2}x^2 \end{aligned}$$

  In this case $I(x) = \frac{1}{2}x^2$.

The Large deviation principle (LDP): definition

Definition. A sequence of random variables (X_n) in a topological space S satisfies the LDP with a lower semicontinuous rate function $I \geq 0$ if we have the following exponential upper/lower bounds for $A \in \mathcal{B}(S)$:

$$
\begin{aligned}
-\inf_{x \in A^\circ} I(x) \quad &\leq \quad \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in A) \\
&\leq \quad \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in A) \leq -\inf_{x \in \bar{A}} I(x) \,.
\end{aligned}
$$

Definition. (X_n) is said to satisfy the *weak LDP* if

$$
\lim_{\delta \to 0} \lim_{n \to \infty} \frac{1}{n} \log \mathbb{P}(X_n \in B_\delta(x)) = -I(x) \,.
$$

- By applying the full LDP to a ball and using the lower semicontinuity of I, we can show that the LDP implies the weak LDP.

The Donsker-Varadhan LDP for the occupation measure of the Ornstein Uhlenbeck process

- Let $dY_t = -\theta Y_t dt + dW_t$ be an OU process for $\theta > 0$, and let

$$\mu_t(A) = \frac{1}{t}\int_0^t 1_A(Y_s)ds$$

denote the proportion of time that Y spends in A, for $A \in \mathcal{B}(\mathbb{R})$. $\mu_t(.)$ is a *random probability measure* on $\mathbb{R}$.

- Then from [DV76] $\mu_t(.)$ satisfies the LDP as $t \to \infty$ in the topology of weak convergence, with rate function:

$$I_B(\mu) = \frac{1}{2}\int_{-\infty}^{\infty} \psi'(y)^2 \, \mu_\infty(y)dy$$

if μ has a density (otherwise $I_B(\mu) = \infty$), where $\mu_\infty(y) = (\frac{\theta}{\pi})^{\frac{1}{2}} e^{-\theta y^2}$ is the unique stationary distribution for Y, i.e. $N(0, 1/(2\theta))$, and $\psi^2 = \frac{\mu(y)}{\mu_\infty(y)}$ is the ratio of the two densities.

- μ must integrate to 1, so $\int_{-\infty}^{\infty} \psi(y)^2 \mu_\infty(y)dy = 1$.

$\mathcal{P}(\mathbb{R})$ as a metrizable space

- Given two measures μ and ν in $\mathcal{P}(\mathbb{R})$, the **Prokhorov** metric is defined by

$$d(\mu,\nu) = \inf\{\delta > 0 : \mu(C) \leq \nu(C^\delta) + \delta,\ \nu(C) \leq \mu(C^\delta) + \delta \quad \text{for all closed } C \in \mathcal{B}(\mathcal{P}(\mathbb{R}))\}$$

  where C^δ is the δ-neighborhood of C [1].

- The metric ensures that μ and ν are close to each other only if $\mu(C)$ and $\nu(C)$ are close to each other for *all* measurable C. Under this metric, $\mathcal{P}(\mathbb{R})$ then becomes a metric space (note that $d(\mu,\nu) \leq 1$) for all μ,ν).

- The topology induced by the Prokhorov metric is equivalent to the topology induced by weak convergence of measures, so the Donsker-Varadhan LDP for μ_t also holds in the topology induced by the metric d.

[1] The set of all points which are of distance $\leq \delta$ from C.

- Donsker-Varadhan originally gave the following variational representation for $I_B(\mu)$

$$I_B(\mu) = -\inf_{u \in \mathcal{D}^+} \int_{-\infty}^{\infty} \frac{Lu}{u}\, d\mu \tag{3}$$

for each $\mu \in \mathcal{P}(\mathbb{R})$, where $L = -\alpha y \frac{d}{dy} + \frac{1}{2}\frac{d^2}{dy^2}$ is the infinitesimal generator for Y and $\mathcal{D}^+$ is the set of u in the domain $\mathcal{D}$ of L with $u > \epsilon$ for some $\epsilon > 0$. It can be shown that this expression simplifies to the previous expression for I_B.

- The expression for $I_B(\mu)$ in (3) is useful, because we can establish the following lemma which characterizes the tail behaviour of the measures inside a level set of I_B:

Lemma. If $I_B(\mu) \le c$ and $k \in (0, \alpha)$, we have

$$\int_{-\infty}^{\infty} y^2 \mu(dy) \quad \le \quad \frac{c+k}{2k(\alpha - k)}\,. \tag{4}$$

We prove this just by using a test function $u = e^{ky^2}$ in (3). This lemma is key to the whole talk and accompanying paper.

Large time asymptotics for a general correlated stoc vol model

- Consider an correlated stochastic volatility model for a log stock price process $X_t = \log S_t$ defined by the SDEs

$$\begin{cases} dX_t = -\frac{1}{2}\sigma(Y_t)^2 dt + \sigma(Y_t) dW_t^1 \,, \\ dY_t = -\alpha Y_t dt + dW_t^2 \end{cases} \tag{5}$$

$dW_1 dW_2 = \rho dt$, $\rho \in (-1, 0]$.

- We set $f(y) = \sigma^2(y)$ and we assume that $\sigma : \mathbb{R} \mapsto (0, \infty)$ is continuous and satisfies the sublinear growth condition

$$\sigma(y) \;\leq\; A(1 + |y|^p)$$

for some $A > 0, p \in (0, 1)$.

- We take $S_0 = 1$ (i.e. $x_0 = 0$) without loss of generality, because $X_t - x_0$ is independent of x_0 as the SDEs have no dependence on x.

The uncorrelated case

- **Theorem**. For the OU model defined in (5) with $\rho = 0$, $(X_t/t, \mu_t)$ satisfies the LDP as $t \to \infty$ with good rate function

$$I_0(x, \mu) = \frac{(x-c)^2}{2F(\mu)} + I_B(\mu)\,, \tag{6}$$

where $c = c(\mu) = -\frac{1}{2}F(\mu)$ and $F(\mu) = \int_{-\infty}^{\infty} f(y)\mu(y)dy$.
- Heuristically, the proof of the *weak LDP* here is just based on a simple conditioning argument:

$$\begin{aligned}\mathbb{P}(\frac{X_t}{t} \in B_\delta(x), \mu_t \in B_\delta(\mu)) &= \mathbb{P}(\frac{X_t}{t} \in B_\delta(x) | \mu_t \in B_\delta(\mu))\mathbb{P}(\mu_t \in B_\delta(\mu)) \\ &\approx e^{-\frac{(x-c(\mu))^2}{2F(\mu)}t}\, e^{-I_B(\mu)t}\,,\end{aligned}$$

where we first use the Donsker-Varadhan weak LDP for μ_t to estimate $\mathbb{P}(\mu_t \in B_\delta(\mu))$, and we then condition on $\mu_t \in B_\delta(\mu)$, i.e. μ_t falling inside a ball of radius δ in the Prokhorov metric, so we expect $F(\mu_t)$ to be "close" to $a = F(\mu)$.
- Finally we use that $\frac{1}{t}[-\frac{1}{2}at + W_{at})$ satisfies the LDP as $t \to \infty$ with rate function $\frac{(x+\frac{1}{2}a)^2}{2a}$, and then establish *exponential tightness*.

The gory details

- The main technical obstacle for the proof is that the functional $F(\mu)$ is not continuous because σ is not bounded, but we show that the probability that $F(\mu_t)$ is not close to $F(\mu)$ is a higher order term than we are interested in, so we can ignore it for our purposes.

- More precisely, we subdivide $B_\delta(\mu)$ into two sets - $\{\mu_1 : I_B(\mu_1) \leq c_2\}$ and $\mu_1 : I_B(\mu_1) \geq c_2$; then the μ_1's in the first set have bounded second moment (using (4) which can be used to show that $F(\mu_1)$ is close to $F(\mu)$ for δ sufficiently small, and the probability that μ_t falls in the second set is $\approx e^{-c_2 t}$ by definition of the LDP, which is a higher order term than we care about when c_2 is chosen to be greater than or equal to $I(x, \mu)$.

Corollary (X_t/t) satisfies the LDP as $t \to \infty$ with a good rate function given by

$$I_0(x) \;=\; \inf_{\mu \in \mathcal{P}(\mathbb{R})} \left[\frac{(x - c(\mu))^2}{2F(\mu)} + I_B(\mu)\right] \;\leq\; \frac{(x + \frac{1}{2}\bar{\sigma}^2)^2}{\bar{\sigma}^2}. \qquad (7)$$

where $\bar{\sigma}^2 = \int_{-\infty}^{\infty} \sigma(y)^2 \mu_\infty(y) dy$.

Proof. The LDP just follows from the *contraction principle* from large deviations theory, and the bound comes from setting $\mu = \mu_\infty$, for which $F(\mu_\infty) = \bar{\sigma}^2$ and $I_B(\mu_\infty) = 0$.

The correlated case

Theorem. For the correlated OU model in (5) with $\rho \in (-1, 0]$, (X_t/t) satisfies the large deviation principle as $t \to \infty$ with good rate function

$$I(x) = \inf_{\mu \in \mathcal{P}(\mathbb{R})} \left[\frac{(x-m)^2}{2\nu^2} + I_B(\mu)\right] \tag{8}$$

where

$$m = m(\mu) = -\frac{1}{2}F(\mu) + \rho G(\mu) \,,$$

$\nu^2 = \nu(\mu)^2 = \bar{\rho}^2 F(\mu)$, $G(\mu) = \int_{-\infty}^{\infty} b(y)\mu(y)dy$ and $b(y) = \alpha y \sigma(y) - \frac{1}{2}\sigma'(y)$.

Remark. This theorem includes the case $\rho = 0$, and $I(x)$ reduces to $I_0(x)$ for $\rho = 0$.

- Integrating (5), we see that

$$X_t = -\frac{1}{2}\int_0^t \sigma(Y_s)^2 ds + \int_0^t \sigma(Y_s)(\rho dW_s^2 + \bar{\rho} dW_s^1)\,.$$

- If we let $V(y) = \int_{y_0}^{y} \sigma(u)du$, then

$$\begin{aligned} dV(Y_t) &= \sigma(Y_t)dY_t + \frac{1}{2}\sigma'(Y_t)d\langle Y\rangle_t \\ &= \sigma(Y_t)(-\alpha Y_t dt + dW_t^2) + \frac{1}{2}\sigma'(Y_t)dt\,, \end{aligned} \tag{9}$$

  which we can integrate and re-arrange as follows

$$\int_0^t \sigma(Y_s)dW_s^2 = V(Y_t) + \int_0^t b(Y_s)ds\,,$$

  where $b(y) = \alpha y \sigma(y) - \frac{1}{2}\sigma'(y)$.

- Thus we can re-write X_t as

$$\begin{aligned} \frac{1}{t}X_t &= -\frac{1}{2}F(\mu_t) + \rho[G(\mu_t) + \frac{1}{t}V(Y_t)] + \bar{\rho}W^1_{tF(\mu_t)}\,, \\ &= m(\mu_t) + \bar{\rho}W^1_{tF(\mu_t)} + \frac{\rho}{t}V(Y_t)\,. \end{aligned} \tag{10}$$

- We can show that $\mathbb{P}(\frac{1}{t}|V(Y_t)| > v)$ is a higher order term than we wish to retain as $t \to \infty$; more precisely that

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(|\frac{1}{t} V(Y_t)| > v) = -\infty\,. \tag{11}$$

Corollary. The infimum of $I(x)$ is attained uniquely at

$$x^* \; = m(\mu_\infty) \; = \; -\frac{1}{2}\bar{\sigma}^2 + \rho\bar{b}$$

where $\bar{b} = \int_{-\infty}^{\infty} b(y)\mu_\infty(y)dy$.

Proof. For $\mu = \mu_\infty$, $I(x^*, \mu_\infty) \; = \; \frac{(x^* - m(\mu_\infty))^2}{2\nu(\mu_\infty)^2} = 0$ and $x = x^*$ is the only minimizer of $\frac{(x^* - m(\mu))^2}{2\nu(\mu)^2}$ when $\mu = \mu_\infty$. But for any other values of μ, $I_B(\mu) > 0$ so $I(x, \mu) > 0$; thus, (x^*, μ^*) is the only pair that makes $I(x, \mu)$ vanish.

The share measure

- Let $\mathbb{P}^*(A) = \frac{1}{S_0}\mathbb{E}(S_t 1_A) = \mathbb{E}(e^{X_t} 1_A)$ for $A \in \mathcal{F}_t$ be the *Share measure*.
- For the correlated OU model with $\rho \in (-1, 0]$, (X_t/t) satisfies the LDP under $\mathbb{P}^*$ with rate function

$$I^*(x) \;=\; I(x) - x\,.$$

- The infimum of $I^*(x)$ is attained at

$$x_+ \;=\; (\frac{1}{2} - \rho^2)F(\mu) + \rho G(\mu)\,.$$

- We have the following large-t behaviour for the distribution function

$$\begin{aligned}
\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}^*(X_t > xt) &= \Lambda^*(x) \qquad (x > x_+)\,, \\
\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}^*(X_t < xt) &= \Lambda^*(x) \qquad (x < x_+)\,,
\end{aligned}$$

where $\Lambda^*(x) = \inf_{y>x} I^*(y)$ if $x \geq x_+$ and $\Lambda^*(x) = \inf_{y<x} I^*(y)$ if $x \leq x_+$

Implied volatility

- Using the same proofs as in [FJ09] for the Heston model, we have the following large-time, large-strike behaviour for the implied volatility $\hat{\sigma}_t(x)$ of a put/call option with strike $S_0 e^{xt}$

$$
\begin{aligned}
&\hat{\sigma}_\infty(x) \\
&= \lim_{t\to\infty} \hat{\sigma}_t^2(x) \quad = \quad \begin{cases} 2(2\Lambda^*(x) + x - 2\sqrt{\Lambda^*(x)^2 + \Lambda^*(x)x}) & (x \notin (x^*, x_+ \\ 2(2\Lambda^*(x) + x + 2\sqrt{\Lambda^*(x)^2 + \Lambda^*(x)x}) & (x \in (x^*, x_+ \end{cases} \\
&\lim_{t\to\infty} \hat{\sigma}_t^2(x) \quad = \quad 8\Lambda(0)\,.
\end{aligned}
$$

Numerical results

- Recall that the rate function for $\rho = 0$ is

$$
\begin{aligned}
I_0(x) &= \inf_{\mu \in \mathcal{P}(\mathbb{R})} \Big[\frac{(x + \frac{1}{2}F(\mu))^2}{2F(\mu)} + I_B(\mu) \Big] \\
&= \inf_{\psi \in L^2(\mu_\infty):\|\psi\|_2 = 1} \Big[\frac{(x + \frac{1}{2}\bar{F}(\psi))^2}{2\bar{F}(\psi)} + \frac{1}{2}\int_{-\infty}^{\infty} \psi'(y)^2\, \mu_\infty(y)dy \Big]
\end{aligned}
$$

  where $\bar{F}(\psi) = \int_{-\infty}^{\infty} f(y)\psi(y)^2\, \mu_\infty(y)dy$, i.e. subject to the constraint $\int_{-\infty}^{\infty} \psi(y)^2\, \mu_\infty(y)dy = 1$.

- $\frac{(x+\frac{1}{2}a)^2}{2a}$ is convex in a and $F(\mu)$ is linear, so the composition $\frac{(x+\frac{1}{2}F(\mu))^2}{2F(\mu)}$ is convex in μ and $I_B(\mu)$ is known to be convex, so the first line is the inf of a convex function over the convex set $\mathcal{P}(\mathbb{R})$.

- We can re-write this optimization problem as

$$
\begin{aligned}
I_0(x) &= \inf_{a\in\mathbb{R}^+} \inf_{\mu\,:\,F(\mu)=a} [\frac{(x+a)^2}{2a} + I_B(\mu)] \\
&= \inf_{a\in\mathbb{R}^+} \left[\frac{(x+a)^2}{2a} + \inf_{\mu\in\mathcal{P}(\mathbb{R}):F(\mu)=a} I_B(\mu)\right]. \qquad (12)
\end{aligned}
$$

- Then the inner inf is the inf of the convex function $I_B(\mu)$ over the convex set $\{\mu : F(\mu) = a\}$. The outer inf is then just a line search.

- We can use the **Ritz** method described in Gelfand&Fomin[GF00] to numerically solve this problem in terms of ψ, by considering a $\psi = \alpha_1\varphi_1 + ...\alpha_n\varphi_n$, where $\varphi_1, \varphi_2, ...$ are the eigenfunctions for the Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mu_\infty)$ of square integrable functions with respect to $\mu_\infty(y)$, which can be computed in closed form (see section 6.2.1 in Linetsky[Lin04]):

$$\begin{aligned} \lambda_n &= \alpha n \quad , \quad \varphi_n(y) = \mathcal{N}_n H_n(\xi) \\ \xi &= \sqrt{\alpha} y \quad , \quad \mathcal{N}_n^2 = \frac{\sqrt{\alpha}}{2^{n+1} n! \sqrt{\pi}} \end{aligned} \tag{13}$$

  where $H_n(x)$ denotes the nth Hermite polynomial.

- We then optimize over $(\alpha_1, ...\alpha_n)$ ...

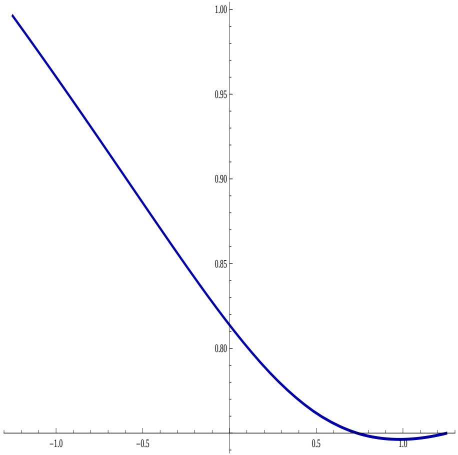

Figure : Here we have plotted the asymptotic implied volatility $\hat{\sigma}_\infty(x)$ for the correlated Ornstein-Uhlenbeck model with $\rho = -0.4, \alpha = 1$ and $\sigma(y) = \sqrt{\log(1 + e^y)}$ using the Ritz method with the NMinimize command in Mathematica and $n = 6$. In this case $x^* = -0.376131$ and $x_+ = 0.302909$. Note that $\sigma(y) \sim \sqrt{y}$ as $y \to \infty$.

x	$I(x)$	$\hat{\sigma}_{\infty}(x)$
-1.25	0.28630719	0.99617456
-1	0.15752900	0.96023119
-0.75	0.06147721	0.92330372
-.50	0.00738347	0.88583712
-0.37614149	0	0.86734248
0.00	0.08278530	0.81380735
0.25	0.25265779	0.78376673
0.50	0.53794508	0.76176957
0.75	0.94584025	0.74953993
1.00	1.46732473	0.74630916
1.25	2.08458000	0.74989235

Here is the data corresponding to the graph on the previous slide.

The CEV-OU model

- Combining the CEV model with the uncorrelated OU process, we have the following hybrid local-stochastic volatility model

$$\begin{cases} dS_t = S_t^{\beta} f(Y_t) dW_t^1 \,, \\ dY_t = -\alpha Y_t dt + dW_t^2 \end{cases}$$

  with $dW_t^1 dW_t^2 = 0$, $Y_0 = y_0 > 0$ and f satisfying the same conditions as before.

- **Proposition** (S_t/t^{γ}) satisfies the LDP on $[0,\infty)$ as $t \to \infty$ with a good rate function given by

$$I_{\mathrm{CEV/OU}}(K) = \inf_{\mu \in \mathcal{P}(\mathbb{R})} [F(\mu) I_{\mathrm{CEV}}(\frac{K}{F(\mu)^{\gamma}}) + I_B(\mu)] \;\leq\; \bar{\sigma}^2 I_{\mathrm{CEV}}(\frac{K}{\bar{\sigma}^{2\gamma}})$$

  where $I_{\mathrm{CEV}}(K) = \frac{K^{2|\bar{\beta}|}}{2\bar{\beta}^2}$ is the rate function for the pure CEV model.

- We call also derive similar result for a **Lévy model** time-changed by the OU process: let (X_t) be a Lévy process and Y be an OU process independent of X. Then the (re-scaled) time-changed Lévy process $\frac{1}{t} X_{\int_0^t f(Y_s) ds}$ satisfies an LDP.

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