The Riemann-Liouville field as a $H \to 0$ limit- sub, critical and super critical GMC, decompositions and explicit spectral expansions

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Abstract

Building on [NR18], we consider a re-scaled Riemann-Liouville (RL) process $Z_t^H = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$, and using Lévy's continuity theorem for random fields we show that Z^H tends weakly to an almost log-correlated Gaussian field Z as $H \to 0$. Away from zero, this field differs from a standard Bacry-Muzy field by an a.s. Hölder continuous Gaussian process, and we show that $\xi_\gamma^H(dt) = e^{\gamma Z_t^H - \frac{1}{2}\gamma^2 \mathrm{Var}(Z_t^H)} dt$ tends to a Gaussian multiplicative chaos (GMC) random measure ξ_γ for $\gamma \in (0,1)$ as $H \to 0$. We also show convergence in law for ξ_γ^H as $H \to 0$ for $\gamma \in [0, \sqrt{2})$ using tightness arguments, and the super-critical phase $\gamma > \sqrt{2}$ using an additional independent stable subordinator. For the critical case $\gamma = \sqrt{2}$, we construct a GMC by decomposing Z into the sum of a restriction of a whole-plane massive free field from quantum field theory plus a continuous Gaussian process, for which a critical GMC can then be constructed by known results. For the sub-critical regime, ξ_γ is non-atomic and we show that ξ_γ is locally multifractal away from zero. We also compute a spectral expansion for the RL process and field, for which all terms can be explicitly computed (unlike the standard Karhunen-Loève expansion) by identifying the Cameron-Martin space for the field Z, and this expansion gives exceptional accuracy in sampling $\xi_\gamma^H([0,T])$ for $H \ll 1$ and $H = 0$, unlike traditional Monte Carlo methods (Cholesky, Riemann sum etc.) which are useless in this regime, and we discuss VIX option pricing in the $H \to 0$ limit for the rough Bergomi model.[1]

1 Introduction

Originally pioneered by Kahane[Kah85], Gaussian multiplicative chaos (GMC) is a random measure on a domain of $\mathbb{R}^d$ that can be formally written as

$$M_\gamma(dx) \quad = \quad e^{\gamma X_x - \frac{1}{2}\gamma^2 \mathbb{E}(X_x^2)} dx \tag{1}$$

where X is a Gaussian field with zero mean and covariance $K(x,y) := \mathbb{E}(X_x X_y) = \log^+ \frac{1}{|y-x|} + g(x,y)$ for some bounded continuous function g. X is not defined pointwise because there is a singularity in its covariance, rather X is a random tempered distribution, i.e. an element of the dual of the Schwartz space $\mathcal{S}$ under the locally convex topology induced by the Schwartz space semi-norms. For this reason, making rigorous sense of (1) requires a regularizing sequence X^ϵ of Gaussian processes (with the singularity removed), (see e.g. [BBM13] and [BM03] for a description of such a regularization in 1d based on integrating a Gaussian white noise over truncated triangular region, which we summarize in Section 2.3 here, or page 17 in [RV10] and section 3.4 in [Sha16] for a general method in $\mathbb{R}^d$ using a convolution to smooth X). In most of the literature on GMC, the choice of X^ϵ is a martingale in ϵ, from which we can then easily verify that $M_\gamma^\epsilon(A) = \int_A e^{\gamma X_x^\epsilon - \frac{1}{2}\gamma^2 \mathrm{Var}(X_x^\epsilon)} dx$ is a martingale, and then obtain a.s. convergence of $M_\gamma^\epsilon(A)$ using the martingale convergence to a random variable $M_\gamma(A)$ with $\mathbb{E}(M_\gamma(A)) = \mathrm{Leb}(A)$, and with a bit more work we can verify that $M_\gamma(.)$ defines a random measure (see the end of Section 4 on page 18 in [RV10]).

If $\gamma^2 < 2d$, $M_\gamma^\epsilon(dx) = e^{\gamma X_x^\epsilon - \frac{1}{2}\gamma^2 \mathbb{E}((X_x^\epsilon)^2)} dx$ tends weakly to a multifractal random measure M_γ with full support a.s. which satisfies the local multifractality property

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}(M_\gamma([x, x+\delta]^d)^q)}{\log \delta}) \quad = \quad \zeta(q) \tag{2}$$

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for $q \in (1, q^*)$ (see Proposition 3.7 in [RV10]), where $\zeta(q^*) = 1$ [2] and

$$\zeta(q) \;=\; dq - \frac{1}{2}\gamma^2(q^2 - q)$$

so $q^* = \frac{2}{\gamma^2}$ for $d = 1$, and $\mathbb{E}(M_\gamma([0,t])^q) = \infty$ if $q > q^*$, see Theorem 2.13 in [RV14] and Lemma 3 in [BM03]). Moreover, we can show that the support of M_γ are the so-called *γ-thick points* of X, i.e. points x such that $\lim_{\epsilon\to 0} \frac{X^\epsilon(x)}{\log\frac{1}{\epsilon}} = \gamma$ (see e.g. section 2 in [Aru17], [Ber17b] and page 7 in [RV16] for more on this), and for $g \equiv 0$, explicit expressions are known for the Mellin transform of the law of $M_\gamma([0,1])$ (see e.g. [Ost09], [Ost13], [Ost18] and [RZ18]), which show that $\log M_\gamma([0,1])$ has an infinitely divisible law.

M_γ is the zero measure for $\gamma^2 = 2d$ and $\gamma^2 > 2d$; in these cases a different re-normalization is required to obtain a non-trivial limit. Specifically, for $\gamma^2 = 2d$, we obtain a non-trivial limit by considering $\sqrt{\log\frac{1}{\epsilon}} \cdot M^\epsilon_{\gamma=2}$ as $\epsilon \to 0$ or the "derivative measure" $\frac{d}{d\gamma} e^{\gamma X^\epsilon - \frac{1}{2}\gamma^2 \mathrm{Var}(X_\epsilon)}|_{\gamma=\sqrt{2d}}$. [DRSV14] show that both these objects tend weakly to the same measure μ' as $\epsilon \to 0$, and for the case when the underlying field is the GFF in 2d Aru et al.[APS19] have shown that $\frac{M_\gamma}{2-\gamma} \to 2\mu'$ in probability as γ tends to the critical value of 2, and the critical γ-value is particularly important in Liouville quantum gravity (again see [DRSV14] and Powell[Pow20] for further discussion).

In the sub-critical case, using a limiting argument it can be shown that M_γ satisfies

$$\mathbb{E}\Big(\int_D F(X,z) M_\gamma(dz)\Big) \;=\; \mathbb{E}\Big(\int_D F(X + \gamma^2 K(z,.), z) dz\Big) \tag{3}$$

for any measurable function F and any interval D, which comes from the Cameron-Martin theorem for Gaussian measures and the notion of *rooted measures* and the disintegration theorem (see section 2.1 in [Aru17] and [FS20] for details on where this equation comes from). (3) can be taken as the definition of GMC, and it uniquely determines M_γ as a measurable function of X, and hence also uniquely fix its law.

GMC also has a natural and important application in Liouville Quantum Field Theory; LQFT is a 2d model of random surfaces, which (formally) we can view as a random metric in the context of quantum gravity, where we weight the classical free field action with an interaction term given by the exponential of a GMC and can be viewed as a toy model to understand in quantum gravity how the interaction with matter influences the geometry of space-time.

[HN20] consider a re-scaled modification of a fractional Brownian field B^H of the form $X^H(x) = c_H(B^H(x) - \int B^H(u)\psi(u,x)du)$ for some family of normalizing functions ψ, and show the covariance of this process tends to that of an almost log-correlated Gaussian field (LGF) in the limit as $H \to 0$. Theorem 2.4 in [HN20] states that the sequence of GMCs M^H_γ of of X^H converges in probability as $H \to 0$ for all $\gamma \le \gamma^*(d)$ for some $\gamma^*(d)$ which is known to be $> \sqrt{\frac{7}{4}d}$ but may not be less than the usual critical value of $\sqrt{2d}$. Their proof uses a new notion of good points adapted from [Ber17b] and they show that for H small enough M^H_γ is arbitrarily small outside the set of good points.

Continuing in the same vein as [NR18], we consider a re-scaled Riemann-Liouville process $Z^H_t = \int_0^t (t-s)^{H-\frac{1}{2}} dW_s$ in the $H \to 0$ limit (we say "re-scaled" here because we do not have the usual $\sqrt{2H}$ prefactor in front of the integral). Using Lévy's continuity theorem for tempered distributions, we show that Z^H tends weakly to an almost log-correlated Gaussian field Z (which we refer to as the *Riemann-Liouville field*) as $H \to 0$, which is a random tempered distribution, i.e. a random element of the dual of the Schwartz space $\mathcal{S}$. From Theorem A in [JSW19], we know this field differs from a standard Bacry-Muzy field by a Hölder continuous Gaussian process, and we show that $\xi^H_\gamma(dt) = e^{\gamma Z^H_t - \frac{1}{2}\gamma^2 \mathrm{Var}(Z^H_t)} dt$ tends to a Gaussian multiplicative chaos (GMC) random measure ξ_γ for $\gamma \in (0,1)$ as $H \searrow 0$. Unlike standard constructions of GMC, our approximating sequence Z^H_t is not a martingale so we cannot appeal to the martingale convergence theorem.

We later address the more difficult "L^1-regime" where $\gamma \in [1, \sqrt{2})$ using standard tightness/weak convergence arguments and comparing ξ^H_γ to the sequence of GMCs ξ^H_φ constructed in section 3 in [FFGS20] using a Gaussian white noise integrated over curved regions in the upper half plane under the Haar measure. We also construct a candidate GMC for the super-critical phase $\gamma > \sqrt{2}$, using an independent stable subordinator time-changed by our Riemann-Liouville GMC (similar to section 3 in [BJRV14]) to construct an atomic GMC with the correct (locally) multifractal exponent for γ-values greater than $\sqrt{2}$, which is closely related to the non-standard branch of gravity in conformal field theory, and for the critical value $\gamma = \sqrt{2}$, we construct a critical GMC by decompose Z into the sum of a massive free field from quantum field theory plus a continuous Gaussian process with a $\star$-invariant kernel, for which a critical GMC can be constructed by known results. For the sub-critical regime, ξ_γ is non-atomic and locally multifractal away from zero. We also compute a spectral expansion for the RL process and field, for which all terms can be explicitly computed

[2] see Lemma 3 in [BM03] to see why the critical q value is q^*

(unlike the standard Karhunen-Loève expansion) by identifying the Cameron-Martin space for Z, and when we use the first n terms of this expansion and n time points to approximate $\xi_\gamma^H([0,T])$ with n large (e.g. $n=1000$) this expansion gives exceptional accuracy in sampling $\xi_\gamma^H([0,T])$ for $H \ll 1$ and $H=0$ where traditional Monte Carlo methods (Cholesky, Riemann sum etc) becomes useless, and we discuss VIX option pricing in the $H \to 0$ limit for the standard rough Bergomi model.

We expect our ξ_γ measure (obtained by letting $H \to 0$) to have the same limiting law away from zero as a GMC μ obtained by applying the usual convolution mollification method in e.g. Eq 1.5 in [Ber17] to our limiting field Z, since (modulo some technicalities which we do not go into here) we expect ξ_φ (see above) and μ to satisfy the same "master eq" in (3) which uniquely fixes the law of the GMC, and ξ_γ has the same law as ξ_γ. This being the case, ξ_γ will then be supported on the so-called *thick points* of the field Z (using the definition and result discussed in section 2 in [Ber17]). Note this definition of thick points is different to the definition of good/bad points in [HN20] which is specifically designed to obtain stronger (i.e. non weak) convergence results for their GMC in the limit as $H \to 0$, but requires a much lengthier argument since they do not use a Sandwich lemma as we do to compare their GMC to a Bacry-Muzy multifractal random measure (MRM).

These results have a natural application to the popular Rough Bergomi stochastic volatility model, since ξ_γ^H is the quadratic variation of the log stock price for this model and values of H as low as .03 have been reported in empirical studies of this model (see e.g. Fukasawa et al.[FTW19]). Using our Riemann-Liouville GMC and Jacod's stable convergence theorem, the companion article [FFGS20] proves the surprising result that the martingale component X_t of the log stock price for the Rough Bergomi model tends weakly to $B_{\xi_\gamma([0,t])}$ as $H \to 0$ where B is a Brownian motion independent of everything else, which means the smile for the rBergomi model with $\rho \le 0$ is symmetric in the $H \to 0$ limit for $\gamma \in (0,1)$, and we find that $\mathbb{E}(X_t^3)$ decays exponentially fast or blows up exponentially fast depending on whether γ is less than or greater than a critical $\gamma \approx 1.61711$, and we can also define a $H=0$ model with non-zero skew for which $X_t/\sqrt{t}$ tends weakly to a non-Gaussian random variable X_1 with non-zero skewness as $t \to 0$. In [FFGS20] we also define a generalized family of non-Gaussian rough volatility models, with an associated non-Gaussian multiplicative chaos in the $H \to 0$ limit, using a similar construction to [BM03] with an independently scattered infinitely divisible random measure, and we show that the resulting GMC in the $H \to 0$ limit is locally multifractal for integer moments with a non-quadratic multifractal exponent.

2 The Riemann-Liouville process and its Gaussian multiplicative chaos as $H \to 0$

2.1 Weak convergence of the Riemann-Liouville process Z^H to the Riemann-Liouville field Z

We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $(\mathcal{F}_t)_{t\ge 0}$ throughout, which satisfies the usual conditions. In this section we consider a re-scaled Riemann-Liouville process in the limit as $H \to 0$; To this end, let $(W_t)_{t\ge 0}$ denote a standard Brownian motion and consider the following family of re-scaled Riemann-Liouville processes:

$$Z_t^H \quad = \quad \int_0^t (t-s)^{H-\frac{1}{2}} dW_s \tag{4}$$

for $H \in (0, \frac{1}{2})$, for which

$$R_H(s,t) \quad := \quad \mathbb{E}(Z_s^H Z_t^H) \quad = \quad \int_0^{s\wedge t} (s-u)^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}} du\,.$$

The integrand here is dominated by

$$h(u,s,t) \quad = \quad ((s-u)^{-\frac{1}{2}} \vee 1)\cdot((t-u)^{-\frac{1}{2}} \vee 1) \tag{5}$$

which is integrable for $s<t$, so using the dominated convergence theorem, we find that

$$R_H(s,t) \;\to\; R(s,t) \quad := \quad \int_0^{s\wedge t} (s-u)^{-\frac{1}{2}}(t-u)^{-\frac{1}{2}} du$$

for $s \ne t$ as $H \to 0$ and $R_H(s,t) \to \infty$ for $s=t>0$. We note also that $R(0,0)=\lim_{n\to\infty}\int_0^0 n ds = 0$ (from the definition of Lebesgue integration) and we also note that $R_H(0,0)=0$ so $\lim_{H\to 0} R_H(0,0)=R(0,0)=0$. We can evaluate this integral to obtain

$$R(s,t) \;:=\; 2\tanh^{-1}(\frac{\sqrt{s}}{\sqrt{t}}) \;=\; \log\frac{1+\frac{\sqrt{s}}{\sqrt{t}}}{1-\frac{\sqrt{s}}{\sqrt{t}}} \;=\; \log\frac{\sqrt{t}+\sqrt{s}}{\sqrt{t}-\sqrt{s}} \;=\; \log\frac{(\sqrt{t}+\sqrt{s})^2}{t-s} \;=\; \log\frac{1}{t-s} \;+\; g(s,t) \tag{6}$$

for $0 < s < t$, where

$$g(s,t) \quad = \quad 2\log(\sqrt{s}+\sqrt{t}) \tag{7}$$

and note that $R(s,t) \geq 0$ for all $s,t \geq 0$. $\int_{[0,T]^2} R_H(s,t)dsdt \leq 2\int_{[0,T]^2}\int_0^t((s-u)^{-\frac{1}{2}}\vee 1)\cdot((t-u)^{-\frac{1}{2}}\vee 1)dudsdt < \infty$, so from the dominated convergence theorem, we have

$$\lim_{H\to 0}\int_{[0,T]^2}\phi_1(s)\phi_2(t)R_H(s,t)dsdt \quad = \quad \int_{[0,T]^2}\phi_1(s)\phi_2(t)R(s,t)dsdt \tag{8}$$

for any $\phi_1,\phi_2 \in \mathcal{S}$, where $\mathcal{S}$ denotes the Schwartz space. Similarly, for any sequence $\phi_k \in \mathcal{S}$ with $\|\phi_k\|_{m,j} \to 0$ for all $m,j \in \mathbb{N}_0^n$ for any $n \in \mathbb{N}$ (i.e. under the Schwartz space semi-norm defined in Eq 1 in e.g. [BDW18])

$$\lim_{k\to\infty}\int_{[0,T]^2}\phi_k(s)\phi_k(t)R(s,t)dsdt \quad = \quad 0 \tag{9}$$

since $\mu(A) = \int_A R(s,t)dsdt$ is a bounded non-negative measure (since $\int_0^T\int_0^t R(s,t)dsdt = \int_0^T 2tdt = T^2 < \infty$), and the convergence here implies in particular that ϕ_k tends to zero pointwise, so we can use the bounded convergence theorem. Thus if we define

$$\begin{aligned} \mathcal{L}_{Z^H}(f) &:= \mathbb{E}(e^{i(f,Z^H)}) \quad = \quad e^{-\frac{1}{2}\int_{[0,T]^2} f(s)f(t)R_H(s,t)dsdt} \\ \mathcal{L}(f) &:= e^{-\frac{1}{2}\int_{[0,T]^2} f(s)f(t)R(s,t)dsdt} \end{aligned}$$

for $f \in \mathcal{S}$, and note at the moment that we do not have a process or field as a subscript in $\mathcal{L}(f)$ since we have not yet shown that this is the characteristic functional of a random field. Then from (8) and (9) and Lévy's continuity theorem for generalized random fields in the space of tempered distributions (see Theorem 2.3 and Corollary 2.4 in [BDW18]), we see that $\mathcal{L}_{Z^H}(f)$ tends to $\mathcal{L}_Z(f)$ pointwise and $\mathcal{L}(.)$ is continuous at zero, then there exists a generalized random field Z (i.e. a random *tempered distribution*, see Appendix A for details) such that $\mathcal{L}_Z = \mathcal{L}$ and Z^H tends to Z in distribution with respect to the strong and weak topology (see page 2 in [BDW18] for definition).

Remark 2.1 Based on the right hand side of (6), we can say that Z is an *almost log-correlated Gaussian field* (LGF), see Appendix A for definitions and background on this.

Remark 2.2 One can refine this result to show that the restriction of Z^H to $[0,T]$ converges in probability in the fractional Sobolev space $\mathcal{H}^{-\frac{1}{2}-\epsilon}(\mathbb{R}) \subset \mathcal{S}'$ (see e.g. [JSW18] for definition) for any $\epsilon > 0$, but we will not need this result.

2.1.1 Decomposing Z

Since $g(s,t)$ is smooth away from $(0,0)$, from Theorem A in [JSW19], we know that Z differs from the standard Bacry-Muzy field on $(0,T]$ with covariance $\log\frac{1}{|t-s|}$ by some Gaussian process G_t which is a.s. Hölder continuous on $(0,T]$.

2.1.2 Transforming Z to a stationary field

$Y_t = Z_{e^t}$ has covariance $R_Y(s,t) = 2\tanh^{-1}(e^{-\frac{1}{2}|t-s|})$, and hence is a strictly stationary field. $R_Y(s,t)$ can be re-written as

$$R_Y(s,t) \quad = \quad \log\frac{1}{|t-s|} + h(|t-s|) \tag{10}$$

for some function h which is continuous on $\mathbb{R}$, and hence bounded on any finite interval, so Y is a strictly stationary almost log correlated Gaussian field (this is a special case of the *Lamperti transformation*, which gives a simple one-to-one mapping between any H-self-similar process and a strictly stationary process, in our case $H = 0$).

2.2 Constructing a Gaussian multiplicative chaos from Z^H as $H \to 0$

Using similar notation to [NR18], we now define the family of random measures :

$$\xi_\gamma^H(dt) \quad := \quad e^{\gamma Z_t^H - \frac{1}{2}\gamma^2\mathrm{Var}(Z_t^H)}dt\,.$$

Theorem 2.1 *Let $H_n \searrow 0$. Then for any $A \in \mathcal{B}([0,T])$ and $\gamma \in (0,1)$, $\xi_\gamma^{H_n}(A)$ tends to some non-negative random variable $\xi_{\gamma,A}$ in L^2 (and hence also converges in probability), $\xi_\gamma([0,T])$ is a non-trivial random variable (i.e. has finite non-zero variance), and there exists a random measure ξ_γ on $[0,T]$ such that $\xi_\gamma(A) = \xi_{\gamma,A}$ a.s. for all $A \in \mathcal{B}([0,T])$. ξ_γ is the GMC associated with the family of process Z^H as $H \to 0$.*

Proof. We wish to show that $\mathbb{E}((\xi_\gamma^{H_n}[0,T] - \xi_\gamma^{H_m}[0,T]))^2 \to 0$, i.e. that $\xi_\gamma^{H_n}[0,T]$ is a Cauchy sequence in L^2. To this end, we first note that

$$\begin{aligned}
\mathbb{E}(\xi_\gamma^{H_n}([0,T])\xi_\gamma^{H_m}([0,T])) &= \mathbb{E}(\int_{[0,T]^2} e^{\gamma^2(Z_t^{H_n}+Z_s^{H_m})-\frac{1}{2}\gamma^2\mathbb{E}((Z_t^{H_n})^2)-\frac{1}{2}\gamma^2\mathbb{E}((Z_s^{H_m})^2)}ds\,dt) \\
&= \int_{[0,T]^2} \mathbb{E}(e^{\gamma^2(Z_t^{H_n}+Z_s^{H_m})-\frac{1}{2}\gamma^2\mathbb{E}((Z_t^{H_n})^2-\frac{1}{2}\gamma^2\mathbb{E}((Z_s^{H_m})^2})ds\,dt \\
&= \int_{[0,T]^2} e^{\frac{1}{2}\gamma^2 R_{H_n}(t,t)+\frac{1}{2}\gamma^2 R_{H_m}(s,s)+\gamma^2\mathbb{E}(Z_t^{H_n}Z_s^{H_m})-\frac{1}{2}\gamma^2 R_{H_n}(t,t)-\frac{1}{2}\gamma^2 R_{H_m}(s,s)}ds\,dt \\
&= \int_{[0,T]^2} e^{\gamma^2\mathbb{E}(Z_t^{H_n}Z_s^{H_m})}ds\,dt\,.
\end{aligned}$$

The integrand here is bounded by $e^{\gamma^2\int_0^{s\wedge t}h(u,s,t)du}$ (where $h(u,s,t)$ is defined in (5)) and is integrable on $[0,T]^2$, and $\mathbb{E}(Z_t^{H_n}Z_s^{H_m}) = \int_0^s (t-u)^{H_n-\frac{1}{2}}(s-u)^{H_m-\frac{1}{2}}du \to R(s,t)$ Lebesgue a.e. on $[0,T]^2$ as $n,m\to\infty$, so from the dominated convergence theorem we see that

$$\begin{aligned}
\mathbb{E}(\xi_\gamma^{H_n}([0,T])\xi_\gamma^{H_m}([0,T])) &\to \int_{[0,T]^2} e^{\gamma^2 R(s,t)}ds\,dt \qquad (n,m\to\infty) \\
&= 2\int_{[0,T]}\int_{[0,t]} e^{\gamma^2 R(s,t)}ds\,dt \\
&= 2\int_{[0,T]}\int_{[0,t]} (\frac{\sqrt{t}+\sqrt{s}}{\sqrt{t}-\sqrt{s}})^{\gamma^2}dsdt \\
&= 2\int_{[0,T]} t\int_{[0,1]} (\frac{\sqrt{t}+\sqrt{tu}}{\sqrt{t}-\sqrt{tu}})^{\gamma^2}du\,dt \\
&= 2\int_{[0,T]} t\int_{[0,1]} (\frac{1+\sqrt{u}}{1-\sqrt{u}})^{\gamma^2}du\,dt \;=\; 2\int_0^T t a_\gamma dt \;=\; a_\gamma T^2 \;<\; \infty \qquad (11)
\end{aligned}$$

for $\gamma \in (0,1)$, where

$$a_\gamma \;:=\; \int_{[0,1]}(\frac{1+\sqrt{u}}{1-\sqrt{u}})^{\gamma^2}du \;=\; \frac{2\cdot {}_2F_1(2,-\gamma^2,3-\gamma^2,-1)}{(1-\gamma)(1+\gamma)(2-\gamma^2)} \qquad (12)$$

where ${}_2F_1(z)$ is the hypergeometric function, and using that $1-\sqrt{u}\sim\frac{1}{2}(1-u)$ as $u\to 1$, we can easily verify that $a_\gamma\to\infty$ as $\gamma\uparrow 1$. Hence

$$\mathbb{E}((\xi_\gamma^{H_n}([0,T]) - \xi_\gamma^{H_m}([0,T]))^2) \;=\; \mathbb{E}(\xi_\gamma^{H_n}([0,T])^2) \;-\; 2\mathbb{E}(\xi_\gamma^{H_n}([0,T])\xi_\gamma^{H_m}([0,T])) \;+\; \mathbb{E}(\xi_\gamma^{H_m}([0,T])^2) \;\to\; 0$$

so $\xi_\gamma^{H_n}([0,T])$ converges in $L^2(\Omega,\mathcal{F},\mathbb{P})$ to some a.s. non-negative random variable $\xi_{\gamma,[0,T]}$, and hence also converges in probability. Similarly, for any $A\in\mathcal{B}([0,T])$, we can trivially modify the argument above to show that

$$\mathbb{E}(\xi_\gamma^{H_n}(A)\xi_\gamma^{H_m}(A)) \;\to\; \int_A\int_A e^{\gamma^2 R(s,t)}ds\,dt \;\leq\; a_\gamma T^2 \;<\; \infty$$

so $\xi_\gamma^H(A)$ tends to some random variable $\xi_{\gamma,A}$ in L^2, and hence in probability.

We also know that $\mathbb{E}(\xi_\gamma^{H_n}([0,T])) = T$ for all n and we have already established L^2-convergence for $\xi_\gamma^{H_n}(A)$ as $n\to\infty$ which implies L^1 convergence, so $\mathbb{E}(\xi_{\gamma,[0,T]}) = T$, which further implies that $\mathbb{P}(\xi_{\gamma,[0,T]}>0)>0$ and

$$\mathbb{E}(\xi_{\gamma,[0,T]}^2) \;=\; a_\gamma T^2$$

so in particular ξ_γ is not multifractal at zero, since the power is 2 and not $\zeta(2)$. The L^2-convergence also means that $\xi_\gamma^H[0,T]\to\xi_{\gamma,[0,T]}$ in L^q as $H\to 0$ for all $q\in[1,2]$ which implies that

$$\lim_{H\to 0}\mathbb{E}(\xi_\gamma^H([0,T])^q) \;=\; \mathbb{E}(\xi_{\gamma,[0,T]}^q)\,. \qquad (13)$$

Given that $\mathbb{E}(\xi_{\gamma,[0,T]}) = T$ and $\mathrm{Var}(\xi_{\gamma,[0,T]}) = \int_{[0,T]^2} e^{\gamma^2 R(s,t)}ds\,dt - T^2 > 0$ since $a_\gamma > 1$ for $\gamma\in(0,1)$, we see that $\xi_{\gamma,[0,T]}$ is a non-trivial random variable.

For $A, B \in \mathcal{B}([0,T])$ disjoint, $\xi^H_{\gamma,A\cup B} = \xi^H_{\gamma,A} + \xi^H_{\gamma,B}$ a.s. since ξ^H_γ is a measure, and we know that both sides tend to $\xi_{\gamma,A\cup B}$ and $\xi_{\gamma,A} + \xi_{\gamma,B}$ in probability. But by a standard result, if $X_n \xrightarrow{p} X$ and $X_n \xrightarrow{p} Y$, then $X = Y$ a.s., hence

$$\xi_{\gamma,A\cup B} \quad = \quad \xi_{\gamma,A} + \xi_{\gamma,B} \tag{14}$$

a.s.

Similarly for any sequence $A_n \downarrow \emptyset$ with $A_n \in \mathcal{B}([0,T])$, $\mathbb{E}(\xi_{\gamma,A_n}) = \text{Leb}(A_n)$, so by Markov's inequality $\mathbb{P}(\xi_\gamma(A_n) > \delta) \leq \frac{\text{Leb}(A_n)}{\delta}$, so $\xi_\gamma(A_n)$ tends to zero in probability, and from (14), we know that $\xi_\gamma(A_n)$ is decreasing, and hence also tends to some random variable Y a.s. (and hence also in probability). Thus by the same standard result discussed above, $Y = 0$ a.s. Thus by Theorem 9.1.XV in [DV07] (see also the end of Section 4 on page 18 in [RV10]), there exists a random measure ξ_γ on $[0,T]$ such that $\xi_\gamma(A) = \xi_{\gamma,A}$ a.s. for all $A \in \mathcal{B}([0,T])$. ■

Remark 2.3 If we replace the definition of Z^H with the usual Riemann-Liouville process $Z^H_t = \sqrt{2H}\int_0^t (t-s)^{H-\frac{1}{2}} dW_s$, then adapting the arguments above, we see that

$$\mathbb{E}((\int_A e^{\gamma^2 Z^H_t - \frac{1}{2}\gamma^2 \text{Var}(Z^H_t)} dt)^2) \quad \to \quad \text{Leb}(A)^2$$

as $H \to 0$, for all $A \in \mathcal{B}([0,T])$. But we know that the first moment of $\int_A e^{\gamma^2 Z^H_t - \frac{1}{2}\gamma^2 \text{Var}(Z^H_t)} dt$ is $\text{Leb}(A)$ as well, hence $\int_A e^{\gamma^2 Z^H_t - \frac{1}{2}\gamma^2 \text{Var}(Z^H_t)} dt \to \text{Leb}(A)$ in L^2.

Corollary 2.2 *For $c \in (0,1]$, $(W_c, \xi_\gamma([0,c])) \sim (\sqrt{c}\, W_1, c\xi_\gamma[0,1])$, so in particular, $\xi_\gamma([0,(.)])$ is a self-similar process, and hence $\xi_\gamma([0,c])$ is monofractal at zero, i.e.*

$$\mathbb{E}(\xi_\gamma([0,c])^q) \quad = \quad c^q\, \mathbb{E}(\xi_\gamma([0,1])^q)\,.$$

Proof. See Appendix A. ■

2.3 Construction and properties of the usual Bacry-Muzy multifractal random measure (MRM) via Gaussian white noise on triangles

In this subsection we briefly describe the family of (stationary) Gaussian process used in [BM03]; the Bacry-Muzy multifractal random measure (MRM) is then the GMC associated with this family of processes as the l parameter tends to zero. Define $\omega_l(t)$ as in Eq 7 in [BBM13] with $\lambda = 1$ and $T = 1$, and set $\bar{\omega}_l(t) := \omega_l(t) - \mathbb{E}(\omega_l(t))$, so

$$\bar{\omega}_l(t) \quad = \quad \int_{(u,s)\in\mathcal{A}_l(t)} dW(u,s)$$

where $dW(u,s)$ is two-dimensional Gaussian white noise with variance $s^{-2}duds$, and $\mathcal{A}_l(t) = \{(u,s) : |u-t| \leq (\frac{1}{2}s) \wedge T, s \geq l\}$ is the cone-like region defined in Eq 11 in [BM03] (for the special case when $f(l) = f^{(e)}(t)$ in their notation, see Eqs 12 and 15 in [BM03]). Then

$$K^T_l(s,t) \quad := \quad \mathbb{E}(\bar{\omega}_l(t)\bar{\omega}_l(s)) \quad = \quad \begin{cases} \log\frac{T}{\tau} & l \leq \tau \leq T \\ \log\frac{T}{l} + 1 - \frac{\tau}{l} & \tau \leq l \\ 0 & \tau > T \end{cases} \tag{15}$$

where $\tau = |t-s|$, and one can easily verify that $K^T_l(s,t) \leq \log\frac{T}{\tau}$ (see Eq 25 in [BM03]). From a picture, we also see that $\mathbb{E}(\bar{\omega}_l(t)\,\bar{\omega}_{l'}(s)) = K_l(s,t)$ for $l > l'$ (i.e. the answer does not depend on l'), and $K^T_l(s,t) \nearrow \log\frac{T}{|t-s|}$ as $l \to 0$. We now define the measure

$$M^{T,l}_\gamma(dt) \quad = \quad e^{\gamma\bar{\omega}_l(t) - \frac{1}{2}\gamma^2\text{Var}(\bar{\omega}_l(t))} dt \tag{16}$$

and we use $M^l_\gamma(dt)$ as shorthand for $M^{1,l}_\gamma(dt)$. One can easily verify that $M^l_\gamma(A)$ is a backwards martingale with respect to the filtration $\mathcal{F}_l := \sigma(W(A,B) : A \subset \mathbb{R}^+, B \subseteq [l,\infty])$ (see e.g. subsection 5.1 in [BM03] and page 17 in [RV10]) and $\sup_l \mathbb{E}(M^l_\gamma(A)^q) < \infty$ (Lemma 3 i) in [BM03]), so from the martingale convergence theorem, $M^{T,l}_\gamma(A)$ converges to $M^T_\gamma(A)$ in L^q for $q \in (1,q^*)$, and from the reverse triangle inequality this implies that

$$\lim_{l\to 0} \mathbb{E}((M^{T,l}_\gamma(A))^q) \quad = \quad \mathbb{E}((M^T_\gamma(A))^q) \tag{17}$$

and M^T is perfectly multifractal, i.e. $\mathbb{E}(|M^T_\gamma([0,t])|^q) = c_{q,T}\, t^{\zeta(q)}$ (see e.g. Lemma 4 in [BM03]) for some finite constant $c_{q,T} > 0$, depending only on q and T. For integer $q \geq 1$, we also note that

$$\begin{aligned}\mathbb{E}(M^T_\gamma(A)^q) &= \int_A \ldots \int_A e^{\gamma^2 \sum_{1\leq i<j\leq q} \log \frac{T}{|u_i-u_j|}} du_i \ldots du_q \\ &= \int_A \ldots \int_A e^{\gamma^2 q(q-1)\log T + \sum_{1\leq i<j\leq q} \log \frac{1}{|u_i-u_j|}} du_i \ldots du_q \;=\; T^{\gamma^2 q(q-1)} \mathbb{E}(M_\gamma(A)^q) \qquad (18)\end{aligned}$$

so we see that

$$c_{q,T} \;=\; c_q T^{\gamma^2 q(q-1)} \qquad (19)$$

where $c_q = c_{q,1}$, and this also holds for non-integer q (see e.g. Theorem 3.16 in [Koz06]).

3 ξ_γ for the full sub-critical range $\gamma \in (0, \sqrt{2})$

In this section we extend the definition of the GMC ξ_γ to the full range $\gamma \in (0, \sqrt{2})$.

3.1 The Sandwich lemma

We will make use of the following standard result:

Theorem 3.1 (Kahane's Inequality) *(see e.g. Appendix of [RV10]). Let I be a bounded subinterval of $\mathbb{R}$ and $(X(u))_{u\in I}$, $(Y(u))_{u\in I}$ be two centred continuous Gaussian processes with $\mathbb{E}[X(u)X(u')] \leq \mathbb{E}[Y(u)Y(u')]$ for all u, u'. Then, for all convex functions $F : \mathbb{R} \to \mathbb{R}$, we have:*

$$\mathbb{E}[F(\int_I e^{X(u)-\frac{1}{2}\mathbb{E}(X(u)^2)} du)] \;\leq\; \mathbb{E}[F(\int_I e^{Y(u)-\frac{1}{2}\mathbb{E}(Y(u)^2)} du)]\,.$$

for $0 < \tau < \tau + \delta < 1$ and some $l^*(H) > 0$ and $l_*(H,\tau) > 0$, and the upper bound will actually hold for $\tau = 0$ as well.

Lemma 3.2 *(The Sandwich lemma). For $0 < \tau < \tau+\delta < 1$ and $\tau \leq s \leq t \leq t+\delta$ we can sandwich $R_H(s,t)$ as follows:*

$$K^{4\tau}_{l_*(H,\tau)}(k) \;\leq\; R_H(s,t) \;\leq\; K^4_{l^*(H)}(k) \qquad (20)$$

for $k = |t-s|$ for $0 < s < t < 1$ and some $l_*(H), l^*(H) > 0$ which both tend to zero as $H \to 0$. Note the upper bound trivially holds for $s = 0$ as well, since $R_H(0,k) = 0$ and $K^T_l(k) \geq 0$. We also remind the reader that if $0 = s < t$, $R(s,t) = 0$ not $\log \frac{1}{t-0} + g(0,t) = \infty$.

Remark 3.1 The lower bound of the Sandwich lemma will only be used to prove the local multifractality of ξ_γ, and can be skipped for everything else in the article.

Proof. At this point we refer the reader to Appendix B for the definition and basic properties of the function $G_H(k)$. Then choosing $l^* = l^*(H)$ such that $G_H(0) = \frac{(\tau+\delta)^{2H}}{2H} \leq \frac{1}{2H} = \log(\frac{4}{l^*})$, we find that

$$l^*(H) \;=\; 4\, e^{-\frac{1}{2H}} \downarrow 0 \quad \text{as} \quad H \to 0$$

(B-2) implies that $G_H(k) \leq \log \frac{4}{k}$, and for $k \in [l^*, 4]$, $K^4_{l^*}(k) = \log \frac{4}{k}$, so in this case $G_H(k) \leq K^4_{l^*}(k)$. For $k \in (0, l^*)$, $K^4_{l^*}(k) = \log(\frac{4}{l^*}) + 1 - \frac{k}{l^*} > \log \frac{4}{l^*} \geq G_H(0) > G_H(k)$. Hence for both cases, we have established the upper bound

$$G_H(k) \;=\; R_H(\tau+\delta-k, \tau+\delta) \;\leq\; K^4_{l^*(H)}(k)\,.$$

Note that this holds even if $\tau = 0$. From Appendix B, we recall that

$$R_H(s, k+s) \;=\; \int_0^s (u(k+u))^{H-\frac{1}{2}} du$$

and if we restrict attention to $A_\delta := \{(s,t) : t-s = k, (s,t) \in [\tau, \tau+\delta]\}$ for $0 < \tau < \tau+\delta < 1$ with $k \in [0,\delta]$, then $R_H(s,t)$ is maximized at $s = \tau+\delta-k$ and minimized at $s = \tau$ (see Figure 2). Thus

$$R_H(s,t) \;\leq\; G_H(k) \;\leq\; K^4_{l^*(H)}(k) \qquad (21)$$

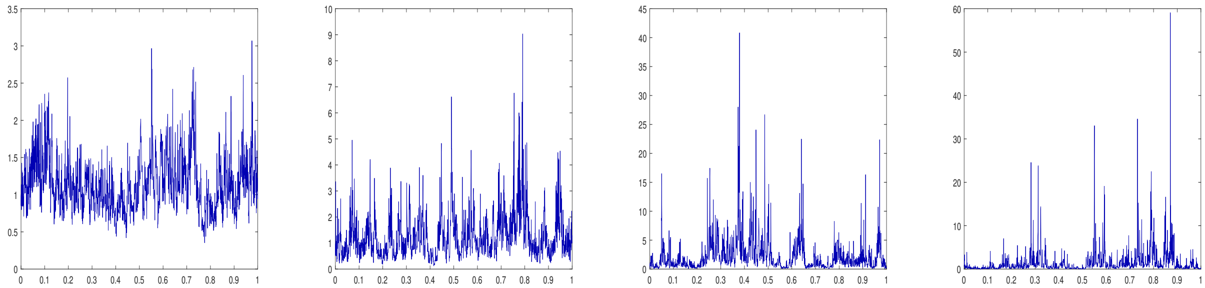

Figure 1: Here we see simulations of ξ_γ using our spectral expansion in (36) with the basis in (31) for (from left to right) $\gamma = 0.125,\ 0.25,\ 0.375$ and 0.5 with $n = 1000$ eigenfunctions, 1000 time points, $H = 0$ and we have used Gauss-Legendre quadrature. For this range of γ-values, the first four raw sample moments are in very close agreement with the theoretical values for $H = 0$, see below for tabulated values.

for $(s,t) \in [\tau, \tau+\delta]^2$ where $k = |t-s|$.

To obtain the lower bound we now assume $\tau \in (0,1)$, and recall from Appendix B that $F_H(k) := R_H(\tau, \tau+k)$. The following facts will be used: $F_H'(k) \downarrow -\infty$ as $k \downarrow 0$, $F_H(k) \uparrow F_0(k)$ uniformly on intervals not containing the origin (this is Dini's theorem), and that $F_H(k)$ is convex.

From the start of the lower bound part of Appendix B, we know that $F_0(k) > \log(\frac{4\tau}{k})$ but we also know that $F_H(0) < \infty$ but clearly $\log(\frac{4\tau}{k}) \to \infty$ as $k \downarrow 0$, so from the aforementioned uniform convergence, we see that for $H > 0$ sufficiently small there exists a $k^* = k^*(H,\tau) > 0$ such that

$$F_H(k^*) \quad = \quad \log \frac{4\tau}{k^*} \tag{22}$$

with

$$F_H(k) \;\geq\; \log \frac{4\tau}{k} \quad \text{for} \quad k \in [k^*, 4\tau] \quad , \quad F_H(k) \;\leq\; \log \frac{4\tau}{k} \quad \text{for} \quad k \leq k^* \tag{23}$$

(at least in a sufficiently small neighborhood of k^*). Now set $l_* = l_*(H,\tau)$ such that

$$|F_H'(k^*)| \quad = \quad \frac{1}{l_*}\,.$$

Such an l_* will be in $[\tau, \tau+\delta]$ for sufficiently small H. For such a choice we must have $l_* \geq k^*$, since

$$\frac{1}{k^*} \quad = \quad |\frac{d}{dk} \log \frac{4\tau}{k}|_{k=k^*}| \quad > \quad |F_H'(k^*)|$$

where the final inequality follows from (23) and (22) and the convexity of $F_H(k)$ (see Figure 2). Then $K_{l_*}^{4\tau}(k) \leq F_H(k)$ for $k \in (0, l^*)$ and hence in $(0, k^*)$. We also see that $l_* \downarrow 0$ as $H \downarrow 0$, since $F_H'(k^*(H)) \to -\infty$ as $H \to 0$. Thus we have shown that

$$G_H(k) \quad = \quad R_H(\tau+\delta-k, \tau+\delta) \quad \leq \quad K_{l^*(H)}^4(k)$$

and

$$K_{l_*(H,t)}^{4\tau}(k) \quad \leq \quad F_H(k) \quad = \quad R_H(\tau, \tau+k)$$

for $k \in [0, 4t]$. From Appendix B, we recall that

$$R_H(s, k+s) \quad = \quad \int_0^s (u(k+u))^{H-\frac{1}{2}} du$$

and if we restrict attention to $A_\delta := \{(s,t) : t-s = k, (s,t) \in [\tau, \tau+\delta]\}$ for $0 < \tau < \tau+\delta < 1$ with $k \in [0,\delta]$, then $R_H(s,t)$ is maximized at $s = \tau+\delta-k$ and minimized at $s = \tau$. Thus

$$K_{l_*(H,\tau)}^{4\tau}(k) \quad \leq \quad F_H(k) \quad \leq \quad R_H(s,t) \quad \leq \quad G_H(k) \quad \leq \quad K_{l^*(H)}^4(k) \tag{24}$$

for $(s,t) \in [\tau, \tau+\delta]^2$ where $k = |t-s|$. ■

3.2 Existence of a limiting law for ξ_γ for $\gamma \in (0,\sqrt{2})$

We begin this subsection by briefly recalling the setup in section 3 in [FFGS20] for the easier Gaussian case: Let P be an independently scattered infinitely divisible random measure (see [BM03] for details) with

$$\mathbb{E}(e^{iqP(A)}) = e^{\varphi(q)\mu(A)}$$

for $q \in \mathbb{R}$ where $\mu(du,dw) = \frac{1}{w^2}dwdu$ denotes the *Haar measure*. Here we restrict attention to the special case where $\varphi(q) = \frac{1}{2}\gamma^2 q^2$, in which case $P(du,dw)$ is just γ times a Gaussian white noise with variance $\frac{1}{w^2}dudw$ (similar to to Section 2.3). Let $A_t^H := \{0 \le u \le t,\, w \ge g_H(u,t)\}$ for a family of functions which satisfy the following condition:

Condition 1 *$g_H(.,t) \ge 0$ with $g_H(u,t)$ increasing in t and H.*

We now define the process

$$\omega_t^H = P(A_t^H)$$

for $t \ge 0$ with filtration $\mathcal{F}_H := \sigma(P(A \times B) : B \subseteq [H,\infty], A, B \in \mathcal{B}(\mathbb{R}))$ (compare to a similar filtration on page 17 in [RV10]), which is a Gaussian process since $\varphi(q)$ is the characteristic function of a Gaussian, with covariance

$$\mathbb{E}(\omega_s^H \omega_t^H) = \int_0^s \int_{g_H(u,t)}^\infty \frac{1}{w^2} dwdu = \int_0^s \frac{1}{g_H(u,t)} du$$

for $0 \le s \le t$, and differentiating with respect to s, we see that if g satisfies $\frac{1}{g_H(s,t)} = R_s^H(s,t)$ then (for H fixed) the Gaussian process ω^H has the same covariance as our process Z^H, and the explicit formula for g_H is given in [FFGS20] (see also first two plots in Figure 3 in [FFGS20]), and in Appendix C in [FFGS20] we verify that Condition 1 above is satisfied. For $H = 0$ we have $g_0(s,t) = \frac{\sqrt{s}\,(t-s)}{\sqrt{t}}$.

For $H_2 < H_1$, $\omega_t^{H_2} - \omega_t^{H_1} = P(A_t^{H_2} \setminus A_t^{H_1})$ and $\omega_t^H = P(A_t^H)$ are independent for any $H \ge H_1$, so ω_t^H is an $\mathcal{F}_H$-backwards martingale, and from this one can easily verify that $\xi_\varphi^H(I)$ is also an $\mathcal{F}_H$ backward martingale for any Borel set I.

We now let ξ_φ^H denote the GMC of $\gamma\omega^H$ on $[0,1]$.

Theorem 3.3 *For any $q \in (1,q^*)$ and any interval $I \subseteq [0,1]$, $\xi_\varphi^H(I)$ tends to some non-negative random variable $\xi_{\varphi,I}$ as $H \to 0$ a.s. and in L^q, and $\mathbb{E}(\xi_\varphi^H(I)^q) \to \mathbb{E}(\xi_{\varphi,I}^q)$.*

Proof. From the upper bound in the Sandwich Lemma, we have the following inequality for $0 < s < t < 1$:

$$R_H(s,t) \le K_{l^*(H)}^\theta(s,t)$$

where $\theta = 4 \cdot \sup(I)$ and $K_l^T(s,t)$ is the covariance of the model in [BM03], and $l^*(H) \downarrow 0$ as $H \downarrow 0$. Then from Kahane's inequality we have that

$$\mathbb{E}(\xi_\varphi^H(I)^q) \le \mathbb{E}(M_{l^*(H)}^\theta(I)^q) \tag{25}$$

where M_l^T is defined as in Section 2.3. Moreover, from Lemma 3 in [BM03] we know that

$$\sup_{l>0} \mathbb{E}(M_l^\theta(I)^q) < \infty$$

for $q \in [1,q^*)$, so we have the uniform bound $\sup_{H>0} \mathbb{E}(\xi_\varphi^H(I)^q) < \infty$.

From above we know that $\xi_\varphi^H(I)$ is a $\mathcal{F}^H$ backwards martingale. Then (by Doob's martingale convergence theorem for continuous martingales) $\xi_\varphi^H(I)$ tends to some random variable which we call $\xi_{\varphi,I}$ as $H \to 0$ a.s. and in L^q for $q \in [1,q^*)$. Moreover, from the reverse triangle inequality, the aforementioned L^q-convergence implies that

$$\mathbb{E}((\xi_\varphi(I)^H)^q) \to \mathbb{E}(\xi_{\varphi,I}^q) \tag{26}$$

as $H \to 0$, for integer $q \in [1,q^*)$. ■

Theorem 3.4 *The laws of $\xi_\gamma^H([0,.)$ on $C_0([0,1])$ converge weakly to the law of an increasing process on $C_0([0,1])$ which induces a non-atomic measure ξ_γ on $[0,T]$ with $\mathbb{E}(\xi_\gamma(A)) = \mathrm{Leb}(A)$.*

Remark 3.2 In a previous version of this article, we stated a slightly stronger result involving L^1-convergence using the Shamov approximation theorem (Theorem 25 in [Sha16]) via generalized randomized shifts which generalize deterministic Cameron-Martin shifts, but in practice we are really just interested in simulating ξ^H for some single small H-value, and seeing whether the law of ξ^H is close to some limiting law (or put another way, we do not get to observe Z^H for a range of H-values for a single W in real life).

Proof. Note that although

$$\mathbb{E}(\omega_s^H \omega_t^H) \quad = \quad \mathbb{E}(Z_s^H Z_t^H)$$

this does not imply that $\mathbb{E}(\omega_s^H \omega_t^{H_2}) = \mathbb{E}(Z_s^H Z_t^{H_2})$ for $H \neq H_2$. However, crucially, ξ_φ^H has the same law as our original ξ_γ^H measure for all $H > 0$, and the non-decreasing process $\xi_\varphi^H([0,(.))$ and $\xi_\gamma^H([0,(.))$ have the same finite-dimensional distributions, so it suffices to prove weak convergence in law of the sequence $\xi_\varphi^H([0,(.))$. Thus from the a.s. convergence in Theorem 3.3 and the bounded convergence theorem, we see that for n distinct time values $t_1, ...t_n \in [0,1]$ and $u_1, ..u_n \in \mathbb{R}$

$$\lim_{H \to 0} \mathbb{E}(e^{\sum_{k=1}^n \; iu_k \xi_\varphi^H([0,t_k))}) \quad = \quad \mathbb{E}(e^{\sum_{k=1}^n \xi_{\gamma,[0,t_k]}})\,.$$

So we have convergence of the finite-dimensional distributions of the process $\xi_\gamma^H([0,.]))$. Moreover, from the upper bound for the Sandwich lemma, for $0 < s < t < 1$ we have

$$\mathbb{E}(\xi_\gamma^H([s,t])^q) \quad \leq \quad \mathbb{E}((M_\gamma^{4,l^*(H)}([s,t]))^q) \;\; \nearrow \;\; \mathbb{E}((M_\gamma^4([s,t]))^q) \quad = \quad c_{q,4}|t-s|^{\zeta(q)}\,.$$

Moreover, $\zeta(q) = 1 + (1 - \frac{1}{2}\gamma^2)(q-1) + O((q-1)^2)$, and hence $\zeta(q) > 1$ for $q > 1$ sufficiently small for $\gamma \in (0, \sqrt{2})$. Hence by Problem 2.4.11 in [KS91] (or Theorem 1.8 in [RY99]) with $X_t^m := \xi_\gamma^H([0,t])$ and $m = 1/H$, the probability measures $\mathbb{Q}^H = \mathbb{P} \circ (X^m)^{-1}$ induced by the sequence of processes $\xi_\gamma^H([0,.])$ on $C_0([0,1])$ are tight under the usual sup norm topology. Thus by Proposition 2.4.15 in [KS91] (see also Theorem B.1.3 in [FH05] and page 1 in [BM16]), the sequence $\mathbb{Q}^H$ converges weakly to a measure $\mathbb{Q}$ on $C_0([0,1])$. Moreover, since

$$\xi_\varphi^H([0,s]) \quad \leq \quad \xi_\varphi^H([0,t])$$

for $0 < s < t$, and we have a.s. convergence of both sides, so $\xi_\varphi([0,s])) \leq \xi_\varphi([0,t]))$ and hence $\mathbb{Q}$ is the law of a non-decreasing continuous process, which induces a measure on $[0,1]$ which we call ξ_γ, with no atoms. We already also know from the previous Theorem that $\mathbb{E}(\xi_{\gamma,A}) = \text{Leb}(A)$, so $\mathbb{E}(\xi_\gamma(A)) = \text{Leb}(A)$. ■

3.2.1 Local multifractality

Proposition 3.5 *For $\gamma \in (0, \sqrt{2})$, ξ_γ has the following locally multifractal behaviour away from zero:*

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}(\xi_\gamma([t,t+\delta])^q)}{\log \delta} \quad = \quad \zeta(q) \tag{27}$$

for $t \in (0,1)$ and $q \in (0, q^*)$.

Proof. Applying Kahane's inequality and Sandwich Lemma for $q \in (1, q^*)$ we have

$$\mathbb{E}[(M_\gamma^{4\tau, l_*(H,\tau)}([\tau,\tau+\delta]))^q] \quad \leq \quad \mathbb{E}[(\xi_\gamma^H([\tau,\tau+\delta]))^q] \quad \leq \quad \mathbb{E}[(M_\gamma^{4,l^*(H)}([\tau,\tau+\delta]))^q] \tag{28}$$

where $M_\gamma^{T,l}$ is defined as in Section 2.3. Using the L^q convergence of $M_\gamma^{T,l}(A)$ in (17) and (26), we see that

$$\mathbb{E}[(M_\gamma^{4\tau}([\tau,\tau+\delta]))^q] \quad \leq \quad \mathbb{E}[(\xi_\gamma([\tau,\tau+\delta]))^q] \quad \leq \quad \mathbb{E}[(M_\gamma^4([\tau,\tau+\delta]))^q]\,.$$

Then using the multifractality property of M_γ^T we see that:

$$c_{q,4\tau}\delta^{\zeta(q)} \quad = \quad c_{q,1}(4\tau)^{\gamma^2 q(q-1)}\delta^{\zeta(q)} \quad \leq \quad \mathbb{E}[(\xi_\gamma([\tau,\tau+\delta]))^q] \quad \leq \quad c_{q,4}\delta^{\zeta(q)} \quad = \quad c_{q,1}4^{\gamma^2 q(q-1)}\delta^{\zeta(q)}$$

where we have used (19) in the final line. Taking the logarithm of the above inequality, dividing by $\log\delta$ and taking limits yields the local multifractality property for ξ_γ (recall that we are assuming that $\tau > 0$ here). ■

4 Critical and super critical GMCs for the Riemann-Liouville field

4.1 The supercritical case $\gamma > \sqrt{2}$

Following Remark 5 in [BJRV14], we now construct an atomic GMC by considering a Radon measure N_{ξ_γ} whose law conditioned on ξ_γ is a Poisson random measure on $\mathbb{R} \times \mathbb{R}_+$ with intensity $\frac{\xi_\gamma(dt)dx}{x^{1+\alpha}}$ for $\alpha \in (0,1)$. Then $\bar{M}(A) = \int_A \int_{\mathbb{R}_+} x N_{\xi_\gamma}(dx, dt)$ satisfies

$$\mathbb{E}(e^{-u\bar{M}(A)}) = \mathbb{E}(e^{-\int_A \int_{\mathbb{R}_+}(e^{-xu}-1)\frac{dx}{x^{1+\alpha}}\xi_\gamma(dt)}) = \mathbb{E}(e^{-\frac{\Gamma(1-\alpha)}{\alpha}u^\alpha \xi_\gamma(A)})$$

and in particular $A_t = \bar{M}([0,t])$ conditioned on ξ_γ is just an additive process, and A_t has the same law (on path space) as $L_{\xi_\gamma([0,t])}$, where L is a stable subordinator independent of ξ_γ with $\mathbb{E}(e^{-uL_t}) = e^{-\frac{\Gamma(1-\alpha)}{\alpha}u^\alpha t})$.

Using the identity $x^\beta = \frac{\beta}{\Gamma(1-\beta)}\int_0^\infty (1-e^{-xz})\frac{dz}{z^{1+\beta}}$ twice (this is Eq 31 in [BJRV14]), we can then easily mimic the proof of Proposition 6 in [BJRV14] to show that

$$\mathbb{E}(\bar{M}(A)^q) = c'_q \mathbb{E}(\xi_\gamma(A)^{q/\alpha}) \tag{29}$$

where $c'_q = \frac{\Gamma(1-q/\alpha)\Gamma(1-\alpha)^{q/\alpha}}{\Gamma(1-q)\alpha^{q/\alpha}}$. Then setting $\gamma\bar{\gamma} = 2$ so $\bar{\gamma} > \sqrt{2}$, and $\alpha = \frac{1}{2}\gamma^2$, we find that $\zeta(q/\alpha) = \zeta(q)_{\gamma=\bar{\gamma}}$ (see page 13 in [BJRV14]). Setting $A = [t, t+\delta]$ and taking the logarithm of (29), dividing by $\log\delta$ and taking the limit as $\delta \to 0$ and using the local multifractality property of ξ_γ, we see that

$$\lim_{\delta\to 0} \frac{\log \mathbb{E}(\bar{M}([t,t+\delta])^q)}{\log \delta} = \zeta(q)|_{\gamma=\bar{\gamma}}$$

for $q \in (0, \alpha = \frac{1}{2}\gamma^2 = \frac{2}{\bar{\gamma}^2} = q^*|_{\gamma=\bar{\gamma}})$. Thus we see that $\bar{M}$ is a natural candidate for a GMC with γ-value equal to $\bar{\gamma} > \sqrt{2}$.

4.2 The critical case $\gamma = \sqrt{2}$ - decomposing ξ_γ

Recall the stationary RL process $Z^{H,\theta}$ introduced in Appendix C. Then we see that

$$Z_t^H - Z_t^{H,\theta} = \int_0^t (1-e^{-\theta(t-u)})(t-u)^{H-\frac{1}{2}}dW_u + \int_{-\infty}^0 e^{-\theta(t-u)}(t-u)^{H-\frac{1}{2}}dW_u$$

is a well defined Gaussian process for $t > 0$ for $H > 0$ and $H = 0$, and the two processes on the right hand side are independent. Let $R_t := Z_t - Z_t^{\theta,0}$, where $Z_t := Z_t^0$. Then from Eq 5.3 and the final eq in the proof of Theorem 5.3 in [JSW19] with $d = 1$

$$\xi_\gamma(dt) = e^{\frac{1}{2}\gamma^2(\tilde{g}_\theta(0)-g(t,t))}e^{\gamma R_t}\nu_{\gamma,\theta}(dt) \tag{30}$$

for $t > 0$, where $\tilde{g}_\theta$ is defined in (B-3), $g(s,t)$ is defined as in (7) and $\nu_{\gamma,\theta}$ is the GMC of $Z^{0,\theta}$, and the critical GMC of Z looks like

$$e^{\frac{1}{2}\gamma^2(\tilde{g}_\theta(0)-g(t,t))}e^{\gamma R_t}\nu_{\sqrt{2},\theta}(dt)|_{\gamma=\sqrt{2}}$$

where $\nu_{\sqrt{2},\theta}$ is the *critical* GMC of $Z^{0,\theta}$ defined as in Eq 5.3 in [JSW19] as the weak*-limit of

$$(\log\frac{1}{\epsilon})^{\frac{1}{2}}\, e^{\sqrt{2}\,Z_\epsilon^{0,\theta}(t)-\mathbb{E}(Z_\epsilon^{0,\theta}(t)^2)}dt$$

as $\epsilon \to 0$, where $Z_\epsilon^{0,\theta}(t) = (\psi_\epsilon * Z^{0,\theta})(t)$ is a standard mollification of $Z^{0,\theta}$.

5 Explicit spectral expansions for Z^H for $H \geq 0$

Following section 4.3 in [Gia15], we first briefly recall the classical Karhunen-Loève theorem. Let $(X_t)_{t\in[a,b]}$ be a centred continuous-parameter real-valued process defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is second order (i.e. $\mathbb{E}(X_t^2) < \infty$ for all $t \in [a,b]$) with continuous covariance function $K_X(s,t)$. Let

$$Z_k = \int_a^b X_t e_k(t)$$

where $\{e_k\}_{k=1}^{\infty}$ are the eigenfunctions of the Hilbert-Schmidt integral operator on $L^2([a;b])$ given by $(Af)(t) = \int_a^b K_X(s,t)f(s)ds$, which is an orthonormal basis for the space spanned by the eigenfunctions corresponding to the non-zero eigenvalues of A. Then $\mathbb{E}(Z_jZ_k) = \lambda_k\delta_{jk}$ for all j,k, $\mathbb{E}(Z_j) = 0$ for all j, and the series

$$\sum_{n=1}^{\infty} Z_k e_k(t)$$

converges to X_t in mean square, uniformly for $t \in [a,b]$. This expansion is often said to be *bi-orthogonal*, since the random coefficients Z_k are orthogonal in $L^2(\Omega,\mathcal{F},\mathbb{P})$ and the eigenfunctions are orthogonal in $L^2([a,b])$. If X is Gaussian, then the Z_k's are independent Gaussians.

The K-L expansion of standard Brownian motion on $[0,1]$ is given by

$$W_t = \sum_{n=1}^{\infty} \sqrt{\lambda_n}\,\phi_n(t)Z_n$$

(see page 50 in [Gia15]), where Z_n is a sequence of i.i.d. standard Normals, and

$$\lambda_n = \frac{4}{(2n-1)^2\pi^2} \quad , \quad \phi_n(t) = \sqrt{2}\sin((n-\frac{1}{2})\pi t)\,. \tag{31}$$

λ_n and ϕ_n are the eigenvalues and eigenfunctions of the Hilbert-Schmidt covariance operator $R_{\frac{1}{2}} : L^2([0,1]) \to L^2([0,1])$ given by $R_{\frac{1}{2}}\theta(t) = \int_0^1 R_{\frac{1}{2}}(s,t)\theta(s)ds = \int_0^1 (s\wedge t)\theta(s)ds$,and and ϕ_n forms an orthonormal basis of $L^2([0,1])$ and $\sqrt{\lambda}_n\phi'_n$ forms an orthonormal basis of $L^2([0,1])$.

5.0.1 Application to the Riemann-Liouville process

We now recall our re-scaled Riemann-Liouville process from section 2:

$$Z_t^H = \int_0^t (t-s)^{H-\frac{1}{2}}dW_s$$

with $H \in (0,1)$ for which $R_H(s,t) := \mathbb{E}(Z_s^H Z_t^H) = \int_0^{s\wedge t}(s-u)^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}}du$. We define the operator $K_H : L^2 \to C[0,1]$ as $K_Hf(t) = \int_0^t (t-s)^{H-\frac{1}{2}}f(s)ds$ for $H \in [0,\frac{1}{2})$, and set

$$X_t^n = \sum_{k=1}^{n} \sqrt{\lambda_k}\, K_H\phi'_k(t)Z_n\,.$$

Recall that $\sqrt{\lambda}_n\phi'_n$ forms an orthonormal basis of $L^2([0,1])$, and we see that the covariance function of X^n is

$$R_{H,n}(s,t) = \sum_{k=1}^{n} \lambda_k K_H\phi'_k(s)K_H\phi'_k(t)\,.$$

The aim of the next few subsections is to show that (32) converges to Z_t^H in appropriate sense for $H > 0$ and for $H = 0$. To do this, we first have to give some background on the Cameron-Martin and Reproducing Kernel Hilbert spaces associated with Gaussian fields.

5.1 The Cameron-Martin space of a log-correlated Gaussian field

Let $C(x,y) = \log\frac{1}{|y-x|} + g(x,y)$ for some function g which is continuous and bounded away from $(0,0)$. If the associated bi-linear operator $C(\phi,\psi) := \int\int C(x,y)\phi_1(x)\phi_2(y)dxdy$ is positive definite (i.e. $C(\phi,\phi) \geq 0$ for all $\phi \in \mathcal{S}$) and continuous at zero (i.e. under the Schwartz space semi-norm defined in Eq 1 in e.g. [BDW18]) then the Minlos-Bochner theorem implies that C is the covariance of a centred Gaussian measure μ on the space $\mathcal{S}'$ of *tempered distributions* which is the dual of the Schwartz space $\mathcal{S}$ (see e.g. Theorem 2.1 in [BDW18] or page 8 in Janson[Jan97]). If $X \sim \mu$, then (due to the log term in (5.1)) we say that X is an *almost log-correlated Gaussian field*, which has the following covariance structure:

$$\mathbb{E}(X(\phi_1)X(\phi_2)) = \int\int C(x,y)\phi_1(x)\phi_2(y)dxdy\,. \tag{32}$$

$\mathcal{S}$ is a Montel space and thus is reflexive, i.e. $(\mathcal{S}')'$ is isomorphic to $\mathcal{S}$ using the canonical embedding of S into its bi-dual $(S')'$.

We can complete $\mathcal{S}$ using the inner product $\langle \phi_1, \phi_2 \rangle_{R^\mu} := \int\int C(x,y)\phi_1(x)\phi_2(y)dxdy$ to form the Hilbert Space R^μ (see e.g. page 44 in Bogachev[Bog91]). If we define H_μ to be the dual of R^μ i.e. the space of linear bounded functionals on R^μ with norm

$$\|h\|_{H_\mu} \quad := \quad \sup\{h(l) : l \in R^\mu,\, C(l,l) \leq 1\}$$

then H_μ is known as the *Cameron-Martin space* of X (again see page 44 in [Bog91]).

We now recall the standard Riesz Representation theorem:

Theorem 5.1 *Given a Hilbert space H, the map $\phi : H \to H^*$ defined by:*

$$\phi(h) \quad = \quad \langle h, . \rangle$$

is an isometric isomorphism.

Any element of H_μ is a bounded linear operator on the Hilbert space R^μ, so (by Riesz) for any $h \in H_\mu$ there exists a unique $g \in R^\mu$ such that

$$h(f) \quad = \quad \langle f, g \rangle_{R^\mu} \quad = \quad \mathbb{E}(X(f)X(g)) \quad = \quad (Cg)(f) \qquad \forall f \in R^\mu\,.$$

h and Cg are both bounded linear functions on R_μ that agree on every element of R_μ so $h = Cg$, so the Riesz map $\phi : H_\mu \to R^\mu$ here is given explicitly by $\phi(h) = C^{-1}h$.

5.2 Characterizing H_μ when $C = AA^*$

Suppose we have some Hilbert space E and we can factorise the covariance as $C = AA^*$

$$A :\; E \to H_\mu \quad , \quad A^* :\; R^\mu \to E$$

where A and A^* are bounded linear operators and A^* is the adjoint of each other in the sense that $(A\phi)(f) = \langle A^* f, \phi \rangle_E$ for all $\phi \in E$ and all $f \in R^\mu$ i.e. $A^* f = f(A(.))$. Then we have the following:

Proposition 5.2 *H_μ is equal to the Hilbert space $H = AE$ with inner product $\langle Af, Ag \rangle_H := \langle f, g \rangle_E$ for all $f, g \in E$.*

Before proceeding with the proof, we recall a definition: given a set $\mathcal{X}$, a *Reproducing Kernel Hilbert Space* H (RKHS) is a Hilbert space of functions on $\mathcal{X}$ admitting a reproducing kernel, i.e. there exists a positive definite function $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that:
(i) $K(p, .) \in H$ for all $p \in \mathcal{X}$.
(ii) $f(p) = \langle K(p, .), f \rangle_H$ for all $f \in H$ and $p \in \mathcal{X}$.

Proof. One proof can be found on pages 107-108 of [Bog91]. Alternatively, note that for all $f \in E$ and $g \in R^\mu$:

$$(Af)(g) \quad := \quad \langle A^* g, f \rangle_E \quad = \quad \langle AA^* g, Af \rangle_H \quad = \quad \langle Cg, Af \rangle_H\,. \tag{33}$$

But this is the statement that H is an RKHS on R^μ with kernel C. Such spaces are uniquely characterised by their kernel (this is the Aronszajn-Moore theorem) and so all that remains is to show that H_μ is also a RKHS with the same kernel. To this end, we note that any $h \in H_\mu$ can be written as $C\phi$ for some $\phi \in R^\mu$, so $\forall g \in R^\mu$ we have

$$h(g) \quad = \quad \langle C\phi, g \rangle \quad = \quad \langle C\phi, Cg \rangle_{H_\mu} \quad = \quad \langle Cg, h \rangle_{H_\mu}$$

which is the reproducing condition. ■

Our particular case of interest is of course for the Riemann-Liouville process (resp. field) where $(Af)(t) := \int_0^t (t-s)^{H-\frac{1}{2}} f(s)ds$ for $H \in [0,1)$ which is a bounded linear map on $E = L^2([0,1])$ (see Theorem 2.6 in Samko et al.[SKM93] for a proof of this) and $H_\mu = AL^2([0,1])$. Moreover, using the Stone-Weierstrauss argument in Appendix A of [FZ17], we can also verify that H_μ is dense in L^2

As another simple example, we can trivially write $C = C\iota$ where ι is the identity mapping on R^μ. Let $E = R^\mu$, $A = C$ and $A^* = \iota$ we see the adjoint condition is simply $(C\phi)(f) = \langle \phi, f \rangle_{\mathbb{R}^\mu}$. By Proposition 1.2 we recover the result $H_\mu = CR^\mu$.

5.3 Karhunen-Loève type expansions

The Hilbert space structure allows us to expand our field in a basis. We have established a linear isomorphism $H_\mu \leftrightarrow R^\mu$ which is also an isometry. Thus if we have an O.N. basis of H_μ then there is a corresponding O.N. basis $\{A_k = C^{-1}e_k\}$ of R^μ. Then for all $\phi \in \mathcal{S}$ we have

$$\phi \quad = \quad \sum_{k=1}^{\infty} c_n^{\phi} A_k$$

where $c_k^\phi = \langle \phi, A_k \rangle_{R^\mu}$ and convergence is of course in the R^μ norm i.e. $(C(\phi - \phi_n), \phi - \phi_n) \to 0$, where $\phi_n = \sum_{k=1}^{n} \langle \phi, A_k \rangle A_k$ denotes the nth partial sum, or equivalently

$$X(\phi) \quad = \quad \sum_{k=1}^{\infty} c_k^{\phi} X(A_k)$$

in the sense that

$$\|X(\phi) - X^n(\phi)\|^2 \quad = \quad \langle C(\phi - \phi_n), \phi - \phi_n \rangle \quad \to \quad 0 \tag{34}$$

as $n \to \infty$, where $X^n = \sum_{k=1}^{n} c_k^\phi X(A_k)$. Using the isometry and the reproducing property in H we see that

$$c_k^\phi \quad = \quad \langle C\phi, A_k \rangle \quad = \quad \langle C\phi, CA_k \rangle_{H_\mu} \quad = \quad \langle C\phi, e_k \rangle_{H_\mu} \quad = \quad e_k(\phi)$$

by the reproducing property. In this sense we say that $X = \sum_{k=1}^{\infty} X(A_k) e_k$ and we can re-write the convergence in (34) as

$$\|X(\phi) - X^n(\phi)\|^2 \quad = \quad \mathbb{E}([(X, \phi) - (X^n, \phi)]^2) \quad \to \quad 0$$

where $X^n := \sum_{k=1}^{n} X(A_k) e_k$.

We also have that

$$\mathbb{E}(X(A_j)X(A_k)) \quad = \quad \langle CA_j, A_k \rangle \quad = \quad \langle A_j, A_k \rangle_{R^\mu} \quad = \quad \langle CA_j, CA_k \rangle_{H_\mu} \quad = \quad 1_{j=k}\,. \tag{35}$$

and we know that each $Z_k := X(A_k)$ is Gaussian so they must be i.i.d. standard Normals, so we can re-write our expansion as

$$X(\phi) \quad = \quad \sum_{k=1}^{\infty} e_k(\phi)\, Z_k\,.$$

Moreover, $X^n(\phi) = \sum_{k=1}^{n} e_k(\phi)\, Z_k$ is a discrete-time L^2-martingale, so by the martingale convergence theorem $X_n(\phi)$ converges a.s. to $X(\phi)$, and hence $X_n \to X$ a.s. in the weak topology on $\mathcal{S}$ (and the strong topology, see page 2 in [BDW18]).

5.3.1 Digression for the case $H > 0$

The Riemann-Liouville process Z^H for $H > 0$ lives in $C_0([0,1]) \subset \mathcal{S}'$, and setting $R_H = C$ we see that $(R\delta_u)(t) = R(u,t)$ where δ_u denotes a dirac mass at u, and for $H > 0$ we see that $\delta_u \in R^\mu$ since $(A^*\delta_u)(s) = \int_s^1 (t-s)^{H-\frac{1}{2}} \delta_u(t) dt = (u-s)^{H-\frac{1}{2}}$ is in L^2, and $C\delta_u = AA^*\delta_u \in H_\mu$, which is not the case when $H = 0$. Note this is allowed since R^μ is a closure of $\mathcal{S}$ in the larger space $\mathcal{S}'$ of generalized functions which includes dirac delta functions, and setting $\phi(t) = 1_t$ this means that for $H > 0$

$$X_t \quad = \quad \sum_{k=1}^{\infty} e_k(t)\, Z_k$$

where now convergence is in $L^2(\mathbb{P})$.

5.4 Choice of basis and explicit computation of terms

The operator C is a Hilbert-Schmidt, compact, linear and self-adjoint operator on L^2 so by the spectral theorem we can form the O.N. basis (e_n) of eigenfunctions of C which is known as the *Karhunen-Loève basis*, which is frequently used in theoretical proofs but is not so useful in practice aside from a few special cases (e.g. Brownian motion and the Brownian bridge) since typically these eigenfunctions cannot be computed explicitly.

For this reason we appeal to Proposition 2.2 instead: let $(\tilde{e}_k)$ be an O.N. basis of $L^2([0,1])$. A is injective (due to the positive definiteness of C), so $e_k = A\tilde{e}_k$ is an O.N. basis of $AL^2([0,1)$. Then in this basis we have

$$X = \sum_{k=1}^{\infty} A\tilde{e}_k Z_k \tag{36}$$

and the $Z_k = X(A_k)$'s are i.i.d. Normals.

If we use the O.N. basis of $L^2([0,1])$ given by $\sqrt{\lambda}_n \phi_k(t)$ from (31), then we can compute $K_H\phi'_k(t)$ explicitly as

$$\begin{aligned} K_H\phi'_k(t) &= \frac{\sqrt{2}}{1+2H}(2n-1)\pi t^{\frac{1}{2}+H}{}_1F_1\!\left(\frac{3}{4}+\frac{1}{2}H, \frac{5}{4}+\frac{1}{2}H, -\frac{1}{16}(2n-1)^2\pi^2t^2\right) \qquad (H>0) \\ K'_H\phi_k(t) &= \sqrt{2}\sqrt{2n-1}\,\pi\left[\cos\!\left(\frac{1}{2}(2n-1)\pi t\right)\mathrm{FresnelC}\!\left(\sqrt{(2n-1)t}\right) + \mathrm{FresnelS}\!\left(\sqrt{(2n-1)t}\right)\sin\!\left[\frac{1}{2}(2n-1)\pi t\right]\right] \quad (H=0) \end{aligned} \tag{37}$$

where ${}_pF_q$ is the generalized hypergeometric function[3], $\mathrm{FresnelC}(z) = \int_0^z \cos(\frac{1}{2}\pi t^2)dt$, $\mathrm{FresnelS}(z) = \int_0^z \sin(\frac{1}{2}\pi t^2)dt$. No such explicit formulae are known for the standard K-L expansion for the RL process which requires knowledge of the eigenvalues of eigenfunctions of the covariance operator (see Gulisashvili et al.[GVZ19] for asymptotic results in this direction).

5.5 Using the spectral expansion to sample the GMC mass $\xi_\gamma^H([0,T])$ for $H \ll 1$ and $H = 0$

From the well known Selberg formula we have that

$$\mathbb{E}(\xi_\gamma^H([0,T])^q) = \int_{[0,T]} \ldots \int_{[0,T]} e^{\gamma^2 \sum_{1\le i<j\le q} R_H(u_i,u_j)} du_i \ldots du_q \tag{38}$$

for $q > 0$. In the following table we have tabulated the first four raw moments $\xi_\gamma([0,1]))$ for (using (38)) and their corresponding estimates using Monte Carlo simulation with the K-L expansion in (32) using (37) (which we denote by $\hat{\mu}_n$), with $n = 1000$ eigenfunctions, 1000 time steps and 1 million simulations for both cases, and Gaussian quadrature for the numerical integration, and we find that the exact and MC answers are in very close agreement (and similarly we get very close agreement for small positive H values, e.g. $H = .0001$, for which the numbers are almost identical to those in the table). If we perform the same computations for e.g. $H = .0001$ using a traditional Cholesky decomposition with a simple Riemann sum, then we get nonsensical answers, which we have not tabulated here (we comment more on this bad behaviour in the next subsection). Based on these results, our KL expansion clearly useful for pricing variance options under the rough Bergomi model with H small or zero (i.e. options on $\xi_\gamma^H([0,T])$); our expansion does not appear to work so well for the (driftless) rough Bergomi model with non-zero correlation defined in Eq 5 in [FFGS20] in estimating the third moment of the driftless log stock price: $\mathbb{E}((\tilde{X}_T^H)^3)$ (when compared to the analytical expression for this quantity given in Eq 7 in Proposition 2.3 in [FFGS20]).

γ	μ_1	μ_2	μ_3	μ_4	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\mu}_3$	$\hat{\mu}_4$
0.05	1	1.00502	1.01513	1.0305	1	1.005	1.0151	1.0305
0.1	1	1.02028	1.06216	1.12836	0.99999	1.0202	1.062	1.1281
0.15	1	1.04644	1.14633	1.31508	0.99999	1.0464	1.1462	1.3148
0.2	1	1.08466	1.27764	1.63646	0.99999	1.0845	1.2771	1.6351
0.25	1	1.13669	1.47323	2.1843	0.99999	1.1367	1.4734	2.1852
0.3	1	1.20512	1.76191	3.14796	0.99985	1.2042	1.7582	3.1336
0.35	1	1.29359	2.19289	4.94361	0.99975	1.2919	2.1849	4.9066
0.4	1	1.40729	2.85324	8.56902	1.0001	1.4078	2.8651	8.9761
0.45	1	1.5537	3.90481	16.6981	1	1.5532	3.8953	16.461
0.5	1	1.74375	5.66824	37.5977	1.0001	1.744	5.6369	35.0148

The following table performs the same computations as above but for $H = .03$ (empirical values as low as this are reported in Fukasawa et al.[FTW19]), and in the final column we compare against a traditional Cholesky scheme using Simpson's rule (also with 1000 time steps and 1 million simulations), and we see that our K-L expansion method outperforms the latter to an increasingly greater extent as γ increases. In the second column, we have computed the usual vol-of-variance parameter $\eta = \frac{\gamma}{\sqrt{2H}}$ corresponding to each choice of γ. Matlab was unable to compute a positive-definite 1000 point Cholesky decomposition when we tried using Gaussian quadrature instead of Simpson's rule (the former of course has non-equidistant time points), and also for $H = .05$, see final table below).

[3]using Mathematica's definition

γ	$\eta = \frac{\gamma}{\sqrt{2H}}$	μ_1	μ_2	μ_3	μ_4	$\hat{\mu}_1^H$	$\hat{\mu}_2^H$	$\hat{\mu}_3^H$	$\hat{\mu}_4^H$	$\hat{\mu}_1^{H,\text{chol}}$	$\hat{\mu}_2^{H,\text{chol}}$	$\hat{\mu}_3^{H,\text{chol}}$	$\hat{\mu}_4^{H,\text{chol}}$
0.05	0.204124	1	1.0043	1.01306	1.0263	1	1.0043	1.013	1.0262	1	1.0043	1.0131	1.0264
0.1	0.408248	1	1.01749	1.05345	1.10988	0.99999	1.0175	1.0534	1.1098	1	1.0176	1.0538	1.1107
0.15	0.612372	1	1.03993	1.125	1.26441	1	1.04	1.1253	1.267	1	1.0402	1.1259	1.2684
0.2	0.816497	1	1.0725	1.23489	1.52805	0.99997	1.0725	1.2349	1.5281	1.0001	1.0735	1.238	1.5353
0.25	1.02062	1	1.11644	1.39505	1.95613	1.0002	1.1174	1.3977	1.963	0.9999	1.1173	1.399	1.9678
0.30	1.22474	1	1.17353	1.62473	2.66892	0.99977	1.1727	1.6226	2.6646	0.99999	1.176	1.6352	2.7041
0.35	1.42887	1	1.24619	1.95509	3.90479	1.0002	1.2477	1.9635	3.951	0.99987	1.2505	1.976	3.9871
0.40	1.63299	1	1.3378	2.43775	6.17752	0.99988	1.3371	2.4352	6.1555	0.9998	1.3481	2.5016	6.5463
0.45	1.83712	1	1.45292	3.16099	10.6697	0.99971	1.4505	3.1351	10.2878	0.99937	1.4714	3.2987	11.6325
0.50	2.04124	1	1.59791	4.28211	20.4214	0.99966	1.5947	4.2561	20.1089	1.0003	1.6488	4.8197	28.5117

5.6 Poor convergence and sampling error in simulating ξ_γ^H using traditional Monte Carlo methods for $H \ll 1$

If we use a standard Riemann sum (or rectangle rule) approximation $X_t^n = X_{\frac{1}{n}[nt]}$ for X (note we are overloading notation here; X^n is not the K-L expansion in (32) here), then X^n is still a Gaussian process and its covariance function $R_H^n(s,t)$ is piecewise constant. Then the 2nd moment of the GMC $\xi_\gamma^{H,n}$ associated with X^n is

$$\begin{aligned} \hat{\mu}_2^2 &= \mathbb{E}((\xi_\gamma^{H,n}[0,T]^2) = \int_{[0,T]^2} e^{\gamma^2 R_H^n(s,t)} ds dt \\ &= (\Delta t)^2 (2 \sum_{i=0}^{n-1} \sum_{j=0}^{i-1} e^{\gamma^2 R_H(i\Delta t, j\Delta t)} + \sum_{i=0}^{n-1} e^{\gamma^2 R_H((i-1)\Delta t, (i-1)\Delta t)}) \end{aligned} \tag{39}$$

where $\Delta t = \frac{T}{n}$. At least for the case $T \le 1$, then from the monotone convergence theorem, $\mathbb{E}((\xi_\gamma^{H,n}[0,T]^2)$ tends to the true value $\mathbb{E}((\xi_\gamma^H[0,T]^2)$, but the convergence in H is very slow when H is small.

To give some specific numerical examples, for $\gamma = .01$ and using 10,000 time steps (for which the Cholesky decomposition would take weeks to compute in Matlab) with $H = .0001$, and using (39) we find that the Riemann sum Monte Carlo estimate for $\text{Var}(\xi_\gamma^H([0,1])^2)$, i.e. $\hat{\sigma}^2 = \hat{\mu}_2^2 - 1$ is $\hat{\sigma}^2 = 0.000264642$ when the true value for $\text{Var}(\xi_\gamma^H([0,1])^2) = 0.000199772$ (i.e. a 32% error) so the Monte Carlo estimate is wildly biased even for a very low γ value (and for comparison the true $\text{Var}(\xi_\gamma([0,T])^2) = 0.000199871$), and this doesn't even account for how many simulations would be required in practice to get close to the theoretical value of $\hat{\sigma}^2$. More disturbingly, if we change γ to 0.1, the rectangle rule estimate is 5.12841×10^{18} when (from the first table above) we know the true answer is $1.02028 - 1$ (contrast this behaviour with the startling convergence we see in the tables above using the K-L method).

6 VIX options for the rough Bergomi model in the $H \to 0$ limit

For $0 \le s \le t$, we can decompose $V_t^H := e^{\gamma \int_0^t (t-u)^{H-\frac{1}{2}} dB_u - \frac{1}{2} \cdot \frac{\gamma^2}{2H} t^{2H}}$ as

$$V_t^H = e^{\gamma \int_0^t (t-u)^{H-\frac{1}{2}} dB_u - \frac{1}{2} \cdot \frac{\gamma^2}{2H} t^{2H}} = e^{\gamma \int_0^s (t-u)^{H-\frac{1}{2}} dB_u - \frac{\gamma^2}{4H}[t^{2H} - (t-s)^{2H}]} e^{\gamma \int_s^t (t-u)^{H-\frac{1}{2}} dB_u - \frac{\gamma^2}{4H}(t-s)^{2H}}.$$

Thus

$$\xi_s(t) = e^{\gamma \int_0^s (t-u)^{H-\frac{1}{2}} dB_u - \frac{\gamma^2}{4H}[t^{2H} - (t-s)^{2H}]} = e^{\gamma Z_s^{H,t} - \frac{1}{2}\gamma^2 \text{Var}(Z_s^{H,t})}$$

where $Z_s^{H,t} = \int_0^s (t-u)^{H-\frac{1}{2}} dW_u$. $\text{Var}(Z_s^{H,t}) \to \frac{1}{2} \log \frac{t}{t-s}$ as $H \to 0$ which blows up at $s = t$, which is to be expected since $\xi_s(s) = V_s^{H=0}$ which is not defined, but $Z_s^{H,t}$ is a well defined Gaussian process for $t > s$ and hence amenable to standard Monte Carlo methods. VIX options are written on $(\int_T^{T+\Delta} \xi_T(u) du)^{\frac{1}{2}}$, and using that

$$R_s(t_1, t_2) := \lim_{H \to 0} \mathbb{E}(Z_s^{H,t_1} Z_s^{H,t_2}) = 2 \log(\frac{\sqrt{t_1} + \sqrt{t_2}}{\sqrt{t_1 - s} + \sqrt{t_2 - s}})$$

for $t_1, t_2 \ge s \ge 0$ and that $\mathbb{E}(\xi_s(t)) = 1$, and we can easily verify (e.g. in Mathematica) that

$$\int_s^1 \int_s^1 e^{\gamma^2 R_s(t_1, t_2)} dt_1 dt_2 < \infty$$

for all $\gamma > 0$, and setting $s = T$, we can show (as before) that $\int_T^{T+\Delta} \xi_T(u)du$ tends to some non-trivial random variable Υ in L^2 as $H \to 0$, with $\mathrm{Var}(\Upsilon) = \int_T^{T+\Delta} \int_T^{T+\Delta} e^{\gamma R_T(t_1,t_2)} dt_1 dt_2 - 1 > 0$, so although volatility itself is not well defined in the $H \to 0$ limit, the value of the VIX at time T, i.e. $(\int_T^{T+\Delta} \xi_T(u)du)^{\frac{1}{2}}$ can be perfectly well defined as an L^2-limit.

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A Log-correlated Gaussian fields

The material in this Appendix is a brief summary of the material on log-correlated Gaussian fields in [DRSV17]. A log-correlated Gaussian field X on $\mathbb{R}^d$ is a random distribution on $\mathbb{R}^d$ with a certain Gaussian structure; X_t is not a random function on $\mathbb{R}^d$ since X_t does not exist pointwise, but $X(f)=\langle X,f\rangle$ is well defined for any element f of the Schwartz space $\mathcal{S}=\{\phi\in C^\infty : p_{\alpha,\beta}=\|\phi\|_{\alpha,\beta}:=\sup_x|x^\alpha D^\beta\phi|<\infty,\,\forall\,\alpha,\beta\}$ (the set of $f:\mathbb{R}\to\mathbb{R}$ whose derivatives of all orders exist and decay faster than any polynomial at infinity). X is a Gaussian process indexed by elements of $\mathcal{S}$; X is Gaussian here means that $\langle X,f_1\rangle,...,\langle X,f_n\rangle$ for $f_i\in\mathcal{S}$ has a multivariate Normal distribution, and X has covariance

$$\mathbb{E}(\langle X,\phi_1\rangle\langle X,\phi_2\rangle) \quad=\quad \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\log\frac{1}{|x-y|}\,\phi_1(y)\phi_2(x)dy\,dx$$

for all $\phi_1,\phi_2\in\mathcal{S}$. X is a (random) tempered distribution, i.e. an element of the dual space $\mathcal{S}'$ of S under the locally convex topology generated by the seminorms $p_{\alpha,\beta}$, which has a neighborhood base of convex sets the form $\{f\in\mathcal{S}:p_{\alpha,\beta}(f-g)<\epsilon,\,\forall\alpha,\beta\}$; an element of $\mathcal{S}'$ is a continuous linear functional on $\mathcal{S}$ under this topology, i.e. h is continuous iff $h(\phi_n)\to 0$ whenever $p_{\alpha,\beta}(\phi_n)\to 0$, and any $h\in\mathcal{S}'$ can be written as $h=\sum_{|\alpha+\beta|\le k}x^\beta D^\alpha u_{\alpha\beta}$ where $u_{\alpha\beta}\in C_b$ and differentiation is defined via integration by parts.

B Definition and properties of $F_H(k)$ and $G_H(k)$ for the Sandwich lemma

We first note that

$$R_H(s,t) \quad=\quad \int_0^{s\wedge t}(s-u)^{H-\frac{1}{2}}(t-u)^{H-\frac{1}{2}}du \quad=\quad \int_0^s u^{H-\frac{1}{2}}(t-s+u)^{H-\frac{1}{2}}du$$

for $0\le s\le t$. Note that the integrand is non-negative. Going forward we will denote $k=t-s$. We restrict $R_H(s,t)$ to the line $\{t-s=k\}$ and the square $[\tau,\tau+\delta]^2$ with $\delta\in(0,1-\tau)$, i.e. $R_H(s,k+s)=\int_0^s(u(k+u))^{H-\frac{1}{2}}du$. This expression is maximized at $s=\tau+\delta-k$ and minimized at $s=\tau$ for constant k (see Figure 2), and this remains true if $\tau=0$. In this case, the maximum is as before but the minimum is simply zero for all H and all k. Define

$$G_H(k) \quad:=\quad R_H(\tau+\delta-k,\tau+\delta) \tag{B-1}$$

We now establish some basic properties of $G_H(k)$. By the above:

$$G_H(k) \quad=\quad \int_0^{\tau+\delta-k}(u(k+u))^{H-\frac{1}{2}}du\,.$$

Taking the derivative with respect to k and using the Leibniz rule:

$$G_H'(k) \quad=\quad -(\tau+\delta-k)^{H-\frac{1}{2}}(\tau+\delta)^{H-\frac{1}{2}}\;+\;(H-\frac{1}{2})\int_0^{\tau+\delta-k}u^{H-\frac{1}{2}}(k+u)^{H-\frac{3}{2}}du \quad<\quad 0$$

so $G_H(k)$ is decreasing in k. The integral term in the previous equation explodes as $k\downarrow 0$:

$$\int_0^{\tau+\delta-k}u^{H-\frac{1}{2}}(k+u)^{H-\frac{3}{2}}du \quad\ge\quad \int_0^{\tau+\delta-k}(k+u)^{2H-2}du \quad=\quad \frac{(\tau+\delta)^{2H-1}}{2H-1}-\frac{k^{2H-1}}{2H-1}\quad\uparrow\quad\infty\,.$$

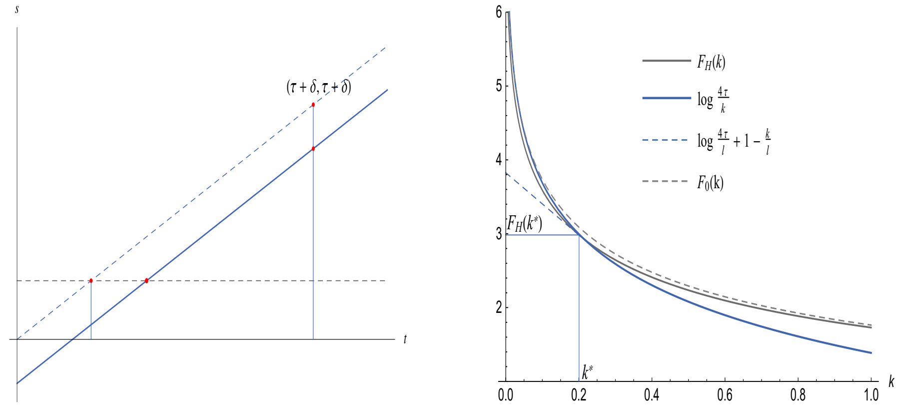

Figure 2: Left plot: $R(s,t)$ is maximized at $s = \tau + \delta - k$, and minimized at $s = \tau$. On the right we have plotted the various quantities appearing in the lower bound part of the proof of the Sandwich Lemma with $H = .01$, $\tau = .95$. We solve for k^* as a numerical root finding exercise and find that $k^* = 0.200634$ and $l_* = 0.237032$ (and $l^* = 1.48803 \times 10^{-43}$). The main point to notice is that the blue dashed line is tangential to the grey line at $k = k^*$, and the blue line has steeper slope than the grey line at this point.

Hence $G_H(k)$ has a cusp at $k = 0$. Letting $H \to 0$ in (B-1), we see that

$$\begin{aligned} G_H(k) \quad &\uparrow \quad G_0(k) \;=\; \log \frac{1}{k} + 2\log(\sqrt{\tau+\delta-k} + \sqrt{\tau+\delta}) \qquad (H \to 0) \\ &\leq \quad g(k) \;:=\; \log \frac{1}{k} + \log(4(\tau+\delta)) \end{aligned}$$

with equality at $k = 0$ in the sense that both sides of the inequality are infinite. Thus

$$G_H(k) \;\leq\; G_0(k) \;\leq\; g(k) \;\leq\; \log \frac{4}{k}\,. \tag{B-2}$$

$F_H(k)$ is also a decreasing convex function whose derivative explodes at $k = 0$. The proof is the same as for $G_H(k)$ but briefer since the integral has no k dependence in the upper limit meaning the Leibniz rule merely involves taking the partial derivative inside the integral.

$F_H(k) = R_H(\tau, \tau+k)$ increases pointwise as $H \downarrow 0$ to $F_0(k) = 2\tanh^{-1}(\sqrt{\frac{\tau}{\tau+k}}) = \log \frac{1}{k} + 2\log(\sqrt{\tau} + \sqrt{\tau+k})$. The second term is minimized at $k = 0$, so we define:

$$f(k) := \log \frac{4\tau}{k}$$

and note that $f(k) < F_0(k)$.

C The stationary Riemann-Liouville process and the massive free field

If we replace the RL process Z^H in section 2.2 with the *stationary* fractional Langevin process:

$$Z_t^{H,\theta} \;=\; \int_{-\infty}^{t} e^{-\theta(t-s)}(t-s)^{H-\frac{1}{2}}dW_s$$

for $H \in (0, \frac{1}{2})$, then for $t \geq 0$

$$R_{H,\theta}(t) \;:=\; \mathbb{E}(Z_0^{H,\theta} Z_t^{H,\theta}) \;=\; \int_{-\infty}^{t} e^{-\theta(t-u)}(t-u)^{H-\frac{1}{2}}e^{-\theta(-u)}(-u)^{H-\frac{1}{2}}du \;=\; \frac{2^{-H}\sqrt{\pi}}{\Gamma(\frac{1}{2}-H)}\,(\frac{t}{\theta})^H\,\mathrm{BesselK}(H, t\theta)\sec(\pi H)\,.$$

The integrand here is dominated by $e^{-\theta(t-u)}e^{-\theta(-u)}\max((t-u)^{-\frac{1}{2}}, 1)\max((-u)^{-\frac{1}{2}}, 1)$ which is integrable, so using the dominated convergence theorem, we find that

$$\begin{aligned} R_{H,\theta}(t) \;\to\; R_\theta(t) \;&:=\; \int_{-\infty}^{0} e^{-\theta(t-u)}(t-u)^{-\frac{1}{2}}e^{-\theta(-u)}(-u)^{-\frac{1}{2}}du \\ &=\; \mathrm{BesselK}(0, t\theta) \;=\; \log \frac{L}{|t|} + g_\theta(t) \;=\; \log \frac{1}{|t|} + \tilde{g}_\theta(t) \qquad (H \to 0) \end{aligned} \tag{B-3}$$

where $L=\frac{1}{\theta}$, $\tilde{g}_\theta(t)=\log L+g_\theta(t)$ and $g_\theta(t)$ is continuous and bounded on any compact set (unlike the $g(s,t)$ function in the previous subsection) and $g_\theta(t)\to\infty$ as $t\to\infty$. We also have $R_{H,\theta}(0)=4^{-H}\theta^{-2H}\Gamma(2H)\sim\frac{1}{2H}$ as $H\to 0$. Moreover $\int_{[0,T]^2}R_{H,\theta}(s,t)dsdt<\infty$, so from the dominated convergence theorem, we have

$$\lim_{H\to 0}\int_{[0,T]^2}\phi_1(s)\phi_2(t)R_{H,\theta}(|t-s|)dsdt \;=\; \int_{[0,T]^2}\phi_1(s)\phi_2(t)R_\theta(|t-s|)dsdt$$

for any $\phi_1,\phi_2\in\mathcal{S}$. Thus using similar argument to Section 2.2, we can easily show that $Z^{H,\theta}$ tends weakly to $Z^{0,\theta}$ as $H\to 0$, where $Z^{0,\theta}$ is a stationary Gaussian field on $[0,T]$ with covariance

$$\mathbb{E}(\langle Z^{0,\theta},\phi_1\rangle\langle Z^{0,\theta},\phi_2\rangle) \;=\; \int_{[0,T]^2}R_\theta(|t-s|)\,\phi_1(s)\phi_2(t)ds\,dt$$

for $\phi_1,\phi_2\in\mathcal{S}$.

BesselK$(0,|t-s|\theta)$ can be re-written in terms of the Green's function of the transition density of a two-dimensional Brownian motion $\hat{B}$ with killing rate $\frac{1}{2}m^2$:

$$G_m(x,y) \;=\; \int_0^\infty e^{-\frac{1}{2}m^2u-|x-y|^2/2u}du \;=\; \mathrm{BesselK}(0,mr) \;=\; R_m(|x-y|)$$

for $x,y\in\mathbb{R}^2$, so for us $m=\theta$ ($G_m(x,y)$ is the expected local time of $\hat{B}$ at y over $[0,\infty)$, when $\hat{B}_0=x$). We can re-write this as $G_m(x,y)=\int_1^\infty\frac{k_m(u(x-y))}{u}du$ where $k_m(z)=\frac{1}{2}\int_0^\infty e^{-\frac{1}{2}\frac{m^2}{v}z^2-\frac{1}{2}v}dv$. A Gaussian field with covariance function BesselK$(0,|t-s|\theta)$ on $\mathbb{R}$ is the restriction of the whole-plane *massive free field* from quantum field theory on $\mathbb{R}^2$ to the real line (see e.g. section 6.2 in [DRSV14] and section 2.3 in [MRV16], and section 4.1 in [JSW19]), and G_m is known as a $\star$-invariant kernel.

D Proof of Corollary 2.2

For $c\in(0,1)$ we have

$$\begin{aligned}(W_c,\xi^H_\gamma([0,c])) &= (W_c,\int_{[0,c]}e^{\gamma Z^H_u-\frac{1}{2}\gamma^2\mathbb{E}((Z^H_u)^2)}du) = (W_c,c\int_0^1 e^{\gamma Z^H_{cu}-\frac{1}{2}\gamma^2\mathbb{E}((Z^H_{cu})^2)}du)\\ &\sim (\sqrt{c}\,W_1,c\int_{[0,1]}e^{\gamma c^H Z^H_u-\frac{1}{2}\gamma^2c^{2H}\mathbb{E}((Z^H_u)^2)}du)\\ &= (\sqrt{c}\,W_1,c\,\xi^H_{\gamma c^H}([0,1])) \qquad\qquad \text{(B-4)}\end{aligned}$$

where we have used that $(W_c,c^{2H}Z^H)\sim(\sqrt{c}W_1,c^{2H}Z^H)$. From the proof of Theorem 2.1 we know the left hand side tends to $(W_c,\xi_\gamma([0,c])$ in L^2 (and hence also in law), and hence the right hand side also tends to $(W_c,\xi_\gamma([0,c])$ in law. Moreover, for any sequence $H_n\downarrow 0$ we have

$$\begin{aligned}\mathbb{E}(\xi^{H_n}_{c^{H_n}\gamma}([0,1])\xi^{H_m}_{c^{H_m}\gamma}([0,1])) &= \int_{[0,1]}\int_{[0,1]}e^{\gamma^2\mathbb{E}(c^{H_n}Z^{H_n}_t\cdot c^{H_m}Z^{H_m}_s)}ds\,dt\\ &= \int_{[0,1]}\int_{[0,1]}e^{\gamma^2c^{H_n}c^{H_m}\mathbb{E}(Z^{H_n}_tZ^{H_m}_s)}ds\,dt \;\to\; \int_{[0,1]}\int_{[0,1]}e^{\gamma^2R(s,t)}ds\,dt\end{aligned}$$

and

$$\mathbb{E}((\xi^H_\gamma([0,1])-\xi^H_{c^H\gamma}([0,1]))^2) \;=\; \mathbb{E}(\xi^H_\gamma([0,1])^2)\;-\;2\mathbb{E}(\xi^H_\gamma([0,1])\xi^H_{c^H\gamma}([0,1]))\;+\;\mathbb{E}(\xi^H_{c^H\gamma}([0,1])^2) \;\to\; 0$$

and

$$\mathbb{E}((\xi_\gamma([0,1])-\xi^H_{c^H\gamma}([0,1]))^2) \;\le\; \mathbb{E}((\xi_\gamma([0,1])-\xi^H_\gamma([0,1]))^2)+\mathbb{E}((\xi^H_\gamma([0,1])-\xi^H_{c^H\gamma}([0,1]))^2)$$

so $\xi^H_{c^H\gamma}([0,1])$ also tends to $\xi_\gamma([0,1])$ in L^2 (and hence also in law). Thus for any $f\in C_b$

$$\lim_{H\to 0}\mathbb{E}(f(W_c,\xi^H_\gamma([0,c])) \;=\; \mathbb{E}(f(W_c,\xi_\gamma([0,c])) \;=\; \lim_{H\to 0}\mathbb{E}(f(\sqrt{c}W_1,c\xi^H_{c^H\gamma}([0,1])) \;=\; \mathbb{E}(f(\sqrt{c}W_1,\,c\,\xi_\gamma([0,1]))\,.$$