The growth optimal portfolio for an exponential Ornstein-Uhlenbeck asset with proportional transaction costs

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Abstract

We compute the optimal trading strategy to maximize the long-term growth rate (i.e. logarithmic utility from terminal wealth) for an investor who trades a safe and a risky asset subject to proportional transaction costs of size λ, when the stock price is the exponential of an Ornstein-Uhlenbeck process, which extends the earlier work of Gerhold,Muhle-Karbe&Schachermayer[GMS13] for the Black-Scholes case. For this we construct a shadow price process (i.e. a fictitious price process $\tilde{S}_t$ which evolves between the bid/ask spread such that frictionless trading in $\tilde{S}$ yields the same optimal strategy as the original problem with transaction costs), and we show that the width of the no-trade region and the adjustment to the Merton (frictionless) risky fraction are both $O(\lambda^{1/3})$ as for the Black-Scholes case, except for two special stock price values for which these expansions are $O(\lambda^{\frac{1}{4}})$ and $O(\lambda^{\frac{1}{2}})$ (the latter phenomenon appears to be new and does not occur for the Black-Scholes model).

1 Introduction

Fix a càdlàg stock price process S and a riskless bank account with constant price equal to one, and assume we are subject to proportional transactions costs $\lambda \in (0, 1)$, i.e. we pay the ask price S when buying the stock, but we only receive the bid price $(1 - \lambda)S$ when selling. A standard problem is then to maximize the utility from terminal wealth over all self-financing, admissible trading strategies starting from initial capital x in the bank and zero holdings in stock. The classical way to approach

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this problem is to solve the Hamilton-Jacobi-Bellman equation. c.f. Taksar, Klass and Assaf [14], Davis&Norman[DN90], Shreve&Soner[SS94] and many others.

An alternate approach is try to construct a so-called *shadow price* process. This is a fictitious price process taking values in the bid-ask spread such that frictionless trading in the shadow price process gives the same optimal strategy as the original utility maximization problem with transaction costs. The idea behind this concept is as follows: frictionless trading at any price process $\tilde{S}$ valued in the bid-ask spread yields a higher terminal liquidation value, and hence a higher expected utility. Thus, a shadow price corresponds to the least favourable frictionless market evolving in the bid-ask spread. The optimal strategy for the shadow price only buys if the shadow price equals the ask price, and sells if the shadow price equals the bid price. If such a shadow price exists, we obtain the optimal strategy by solving a frictionless optimization problem; we can then use previous known results for frictionless problems to compute the optimal strategy. Moreover, this allows us to explain the behaviour of the agent by passing to a suitable frictionless market which implies that no qualitatively new effects arise under transaction costs that cannot be observed in frictionless financial markets.

The basic idea of the shadow price goes back to Cvitanivic&Karatzas[CK95], where they also show the following: in a Brownian setting, if the minimizer to a suitable dual problem is a local martingale, then a shadow price exists and is given by the ratio of both components of the minimizer. However, this local minimizer is thus far only guaranteed to be a supermartingale, by results of Cvitanic&Wang[CW01]. The difference between having super or local martingales becomes irrelevant if no assets can be sold short. This allowed Loewenstein[Low00] to prove the absence of shadow prices in a Brownian setting with a no short-selling condition.

[GM13] and [GGMS14] consider an investor who is permitted to trade a safe and a risky asset under the Black-Scholes model, subject to proportional transaction costs with exponential and power utility respectively. The optimal solution is formally obtained by deriving a HJB equation for the value function, and then solving a free boundary problem to solve the buy and sell boundaries. The optimal trading policy involves buying the risky asset when the position in the risky asset hits the buy boundary and selling when we hit the sell boundary (with no trading in between, hence the phrase "no-trade region", and both boundaries are constant in time), and the amount to buy/sell at each boundary is governed by the boundary local time processes of a Brownian motion with drift and two reflecting barriers. Hence the number of shares is singular with respect to the Lebesgue measure as a function of time but has no jumps, and the risky fraction is confined to stay within the no-trade region. The solution for the value function is obtained in terms of the solution to a Riccati ODE, and when the bid-ask spread $\varepsilon \ll 1$, the size of no-trade region is shown to be $O(\varepsilon^{1/3})$, centred around the Merton frictionless risky fraction which is constant. The $O(\varepsilon^{1/3})$ term can be viewed as the leading order correction to the classical Merton solution. This analysis is made rigorous using the shadow price construction.

Similar results are derived in [AGS13] for a market with *fixed* transaction costs for an investor with constant relative risk aversion; in this case the width of the no-trade region is $O(\varepsilon^{\frac{1}{4}})$ but in this setting the optimal trading strategy is not governed by a local-time type process. [KL13] extend these results to the case of a general Itô dynamics for the stock price process, using convex duality techniques.

In this article, we consider a financial market consisting of one riskless bond with a constant price equal to 1 and a risky asset. The price of the risky asset $S = (S_t)_{0\leq t<\infty}$ is given by an exponential Ornstein-Uhlenbeck process $S_t = \exp(X_t)$, where

$$dX_t \;=\; \kappa(\bar{x} - X_t)dt + \sigma dW_t$$

with positive parameters $\kappa, \bar{x}, \sigma$. By Itô's formula, S_t satisfies

$$\frac{dS_t}{S_t} \;=\; [\kappa(\bar{x} - \log(S_t)) + \frac{\sigma^2}{2}]dt + \sigma dW_t =: \mu(S_t)\, dt + \sigma dW_t. \tag{1}$$

We now model the bid-ask interval by $[(1-\lambda)S_t, S_t]$ for some $\lambda \in (0,1)$. While an investor can buy at the price S_t, he can only realize the (lower) price $(1-\lambda)S_t$ when selling a unit of stock. The holdings of bond and stock will be modeled by a self financing trading strategy starting at zero:

Definition 1.1. *Assume the investor starts with x dollars in cash ($x > 0$). Then a pair of adapted processes (φ_t^0, φ_t) is called an admissible self-financing trading strategy if both processes are predictable, have finite variation and:*

(i) The self-financing condition:

$$d\varphi_t^0 \;=\; (1-\lambda)S_t\, d\varphi_t^{1,\downarrow} \;-\; S_t\, d\varphi_t^{1,\uparrow} \tag{2}$$

for all $0 \leq t \leq T$, where $\varphi_t = \varphi_t^{1,\uparrow} - \varphi_t^{1,\downarrow}$ is the canonical decomposition of as the difference of two increasing processes, where $\varphi_t^{1,\uparrow}$, $\varphi_t^{1,\downarrow}$ are two increasing processes

(ii) The solvency condition: there exists an $M > 0$ such that the liquidation value

$$V_t(\varphi^0, \varphi) \;=\; \varphi_t^0 \,+\, \varphi_t^+(1-\lambda)S_t \,-\, \varphi_t^- S_t \;\;\geq\;\; -M \tag{3}$$

a.s., for all $0 \leq t \leq T$.

The self-financing condition in (2) ensures that no funds are added or withdrawn to the portfolio, and (3) ensures that the investor cannot owe more than M dollars at any time. The investor wishes to maximize his expected logarithmic growth rate, i.e. compute

$$\sup_{(\psi^0,\psi^1)} \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}[\log V_T(\varphi_T^0, \varphi_T)]\,. \tag{4}$$

To solve this problem, we will construct a *shadow price* process $\tilde{S}_t$, which moves inside the bid-ask spread and is such that its frictionless optimizer (i.e. without

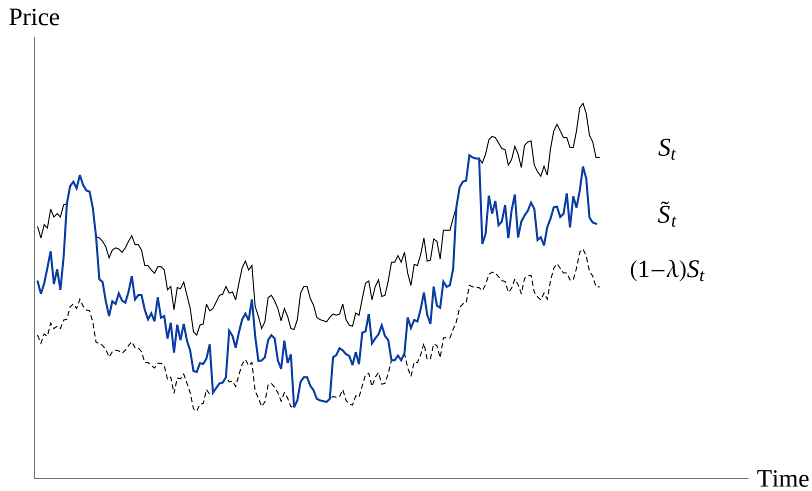


Figure 1: Here we have plotted an example of a shadow price process.

transaction costs) simultaneously solves problem (4). We remark that in the absence of frictions, V_t reduces to

$$V_t \quad = \quad \varphi_t^0 \,+\, \varphi_t S_t\,.$$

After a brief discussion on the solution to the frictionless problem for the exponential OU model in section 2, we will construct the shadow price process explicitly.

We first recall the definition of the shadow price process from Definition 1.6 in [GMS13]:

Definition 1.2. *A shadow price is a continuous semimartingale $\tilde{S}_t$ taking values in $[(1-\lambda)S_t, S_t]$, such that the log-optimal portfolio $(\varphi_t^0, \varphi_t^1)$ for the frictionless market with price process $\tilde{S}_t$ exists, has finite variation and the number of stocks φ_t only increases (resp. decreases) on the set $\{S_t = \tilde{S}_t\}$ (resp. $\{S_t = (1-\lambda)\tilde{S}_t\}$).*

Remark 1.1. *Clearly any price process $\tilde{S}_t$ with zero transaction costs which lies in $[(1-\lambda)S_t, S_t]$ leads to more favourable terms of trade than the original market with transaction costs. But a shadow price process is a particularly unfavourable model, for which it's optimal to only buy when $\tilde{S}_t = S_t$, sell when $\tilde{S}_t = (1-\lambda)S_t$ + do nothing in between.*

From Corollary 1.9 in [GMS13], we have the following proposition, which shows why the shadow price construction is so useful:

Proposition 1.2. *(Corollary 1.9 in [GMS13]). Let $\tilde{S}_t$ be a shadow price process with log optimal portfolio (φ_t^0, φ_t), with $\varphi_t^0, \varphi_t \geq 0$. Then under **non-zero** transaction costs, we have*

$$\sup_{(\psi^0,\psi)} \mathbb{E}[\log V_T((\psi^0,\psi))] \quad \geq \quad \mathbb{E}[\log V_T((\varphi^0,\varphi))] \quad \geq \quad \mathbb{E}[\log V_T((\psi^0,\psi))] \,+\, \log(1-\lambda) \qquad (5)$$

for any admissible (ψ^0, ψ).

Thus if we choose λ sufficiently small so that $|\log(1-\lambda)| < \varepsilon$ and take the sup over all (ψ^0, ψ), we see that (φ^0, φ) is an ε-optimal trading strategy for the original problem.

Corollary 1.3. *Taking the liminf as $T \to \infty$ on both sides of the second inequality in* (5) *and supping over all admissible strategies, we obtain*

$$\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}[\log V_T((\varphi^0, \varphi))] \quad = \quad \sup_{(\psi^0,\psi)} \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}[\log V_T((\psi^0, \psi))] \,.$$

Thus the optimal portfolio for the shadow price process is asymptotically optimal for the original problem under transaction costs, as $\lambda \to 0$ and/or as $T \to \infty$.

Throughout the paper, we denote by $B_\delta(a)$ the small ball centered at $a > 0$ with radius $\delta > 0$.

2 The log-optimal portfolio for the frictionless case

We first consider the case when $\lambda = 0$, i.e. zero transaction costs. Then it is well known that the Merton optimal fraction is given by

$$\theta(s) \quad = \quad \frac{\mu(s)}{\sigma(s)^2} \,,$$

and the investor's total wealth V_t evolves as

$$\begin{aligned} dV_t \quad &= \quad [(1-\theta(S_t))r + \theta(S_t)\mu(S_t)]dt \;+\; \theta(S_t)\sigma dW_t \\ &= \quad \theta(S_t)V_t dS_t/S_t \end{aligned}$$

because we are assuming the interest rate $r = 0$. Then $\varphi_t^1 = \theta(S_t)V_t/S_t$, which satisfies

$$\begin{aligned} d\varphi_t^1 \quad &= \quad \frac{\theta'(S_t)V_t}{S_t}dS_t \;+\; \frac{\theta(S_t)^2 V_t}{S_t^2}dS_t \;-\; \frac{\theta(S_t)V_t}{S_t^2}dS_t \;+\; \text{drift term} \\ &= \quad -\frac{V_t}{S_t^2}\,\Gamma(S_t)dS_t \;+\; \text{drift term} \end{aligned}$$

where $\Gamma(s) = -s\theta'(s) + \theta(s)(1-\theta(s))$.

Substituting the Merton fraction $\theta(s) = \frac{\mu(s)}{\sigma(s)^2} = [\kappa(\bar{x} - \log s) + \frac{1}{2}\sigma^2]/\sigma^2$ for our exponential OU model in (1), we find that there are two values of s for which $\Gamma(s)$ vanishes, given by

$$\begin{aligned} a^* &= \exp[\bar{x} - \frac{\sigma}{2\kappa}\sqrt{4\kappa + \sigma^2}]\,, \\ b^* &= \exp[\bar{x} + \frac{\sigma}{2\kappa}\sqrt{4\kappa + \sigma^2}]. \end{aligned}$$

3

4 Non-zero transaction costs: the shadow price process for a single excursion from the buy boundary to the sell boundary

We make the following ansatz inspired by [GMS13]: assume that $\tilde{S}_0 = S_0 = a$, and if S increases from a to b without setting a new minimum in the meantime, then we guess that $\tilde{S}_t = g(S_t)$ for $0 \leq t \leq \tau_b$, for some $g \in C^2$ and target value $b = b(a, \lambda)$, to be determined. Moreover, we make an initial trade at time zero, and we postulate that the optimal trading strategy involves no further trading until τ_b).

Applying Itô's formula to the shadow price process, we obtain

$$dg(S_t) \;=\; g'(S_t)dS_t + \frac{1}{2}g''(S_t)\sigma^2 S_t^2 dt\,.$$

Thus

$$dg(S_t)/g(S_t) \;=\; \hat{\mu}_t dt + \frac{1}{2}\hat{\sigma}_t dW_t$$

where $\hat{\mu}_t = [g'(S_t)S_t\mu(S_t) + \frac{1}{2}g''(S_t)\sigma^2 S_t^2]/g(S_t)$, $\hat{\sigma}_t = g'(S_t)\sigma S_t/g(S_t)$. But the optimal fraction of wealth for a log-utility maximizer without transaction costs is given by

$$\frac{\hat{\mu}(s)}{\hat{\sigma}(s)^2} \;=\; g(s)\frac{g'(s)s\mu(s) + \frac{1}{2}g''(s)\sigma^2 s^2}{g'(s)^2\sigma^2 s^2} \;=\; \frac{\varphi^1 g(s)}{\varphi^0 + \varphi^1 g(s)}\,. \tag{6}$$

Multiplying the right hand side by $\frac{a}{\varphi^0+\varphi^1 a}$ and using that $\bar{\pi} = \frac{a\varphi^1}{\varphi^0+\varphi^1 a}$ we have

$$\frac{\bar{\pi}g(s)}{\frac{a\varphi^0}{\varphi^0+\varphi^1 a} + \bar{\pi}g(s)} \;=\; \frac{\bar{\pi}g(s)}{a\,\frac{\varphi^0+a\varphi^1-a\varphi^1}{\varphi^0+\varphi^1 a} + \bar{\pi}g(s)} \;=\; \frac{\bar{\pi}g(s)}{a\,(1-\bar{\pi}) + \bar{\pi}g(s)}\,.$$

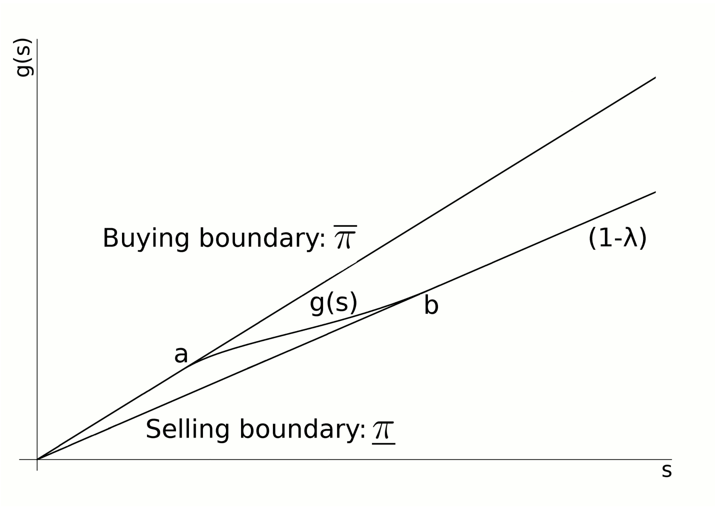
Combining with (6) yields the following ODE for g:

$$\begin{aligned}
\frac{1}{2}\sigma^2 s^2 g''(s) &= \frac{g'(s)^2\sigma^2 s^2}{g(s)}\frac{\bar{\pi}g(s)}{a\,(1-\bar{\pi}) + \bar{\pi}g(s)} - g'(s)s\mu(s) \\
&= g'(s)^2\sigma^2 s^2\frac{\bar{\pi}}{a\,(1-\bar{\pi}) + \bar{\pi}g(s)} - g'(s)s\mu(s)
\end{aligned}$$

which simplifies to

$$\begin{aligned}
g''(s) &= \frac{2\bar{\pi}g'(s)^2}{a\,(1-\bar{\pi}) + \bar{\pi}g(s)} - \frac{2g'(s)s\mu(s)}{\sigma^2 s^2} \\
&= \frac{2\bar{\pi}g'(s)^2}{a\,(1-\bar{\pi}) + \bar{\pi}g(s)} - \frac{2g'(s)\theta(s)}{s}
\end{aligned} \tag{7}$$

where $\theta(s) = \mu(s)/\sigma^2$.



To give a reasonable shadow price process, the function $g(s)$ has to pass smoothly to the bid/ask prices at the boundaries of the no-trade region $[a, b]$, i.e.

$$g(a) = a, \quad g'(a) = 1 \tag{8}$$

$$g(b) = (1-\lambda)b, \quad g'(b) = (1-\lambda). \tag{9}$$

The first two conditions just arise from the definition of the shadow price process, and the second ensure that the volatility of $\tilde{S}_t$ vanishes on both boundaries, so $\tilde{S}_t$ cannot go outside the bid-offer spread (see e.g. top of page 15 in [GM13]). Let

$$h(s) \quad := \quad \exp[\frac{\kappa}{\sigma^2}(\log(s) - \bar{x} - \frac{\sigma^2}{2\kappa})^2] \,.$$

Then straightforward calculations verify that the unique solution to (7) which also satisfies the two conditions for the ask boundary in (8) is given by

$$g(s) = g(s, a, \overline{\pi}) = a\frac{ah(a) + (1-\overline{\pi})H(a,s)}{ah(a) - \overline{\pi}H(a,s)} = a\left(1 + \frac{H(a,s)}{ah(a) - \bar{\pi}H(a,s)}\right). \tag{10}$$

where

$$H(a,s) = \int_a^s h(u)\; du \,,$$

which evaluates to

$$H(a,s) = e^{\bar{x}+\sigma^2/4\kappa}\frac{\sqrt{\pi}}{2\sqrt{\kappa}}\sigma\left(\mathrm{Erfi}[(\sqrt{\kappa}(\bar{x} - \log a))/\sigma] - \mathrm{Erfi}[(\sqrt{\kappa}(\bar{x} - \log s))/\sigma]\right) .$$

where $\mathrm{Erfi}(x) = \mathrm{Erf}(ix)/i$ is the imaginary error function.

To make the dependence on the ask price and the fraction of wealth more explicit, we will sometimes use the notation $g(s) = g(s; \overline{\pi}, a)$. It is straightforward to check that the equation (10) not only solves the ODE (7) but also the boundary conditions

(8) at the ask price a. For future reference, note that the first and second derivatives are given by

$$g'_{\overline{\pi},a}(s) = \frac{a^2 h(a)h(s)}{(ah(a) - \overline{\pi}H(a,s))^2} > 0, \tag{11}$$

$$g''_{\overline{\pi},a}(s) = \frac{a^2 h(a)(2\overline{\pi}h(s)^2 + (ah(a) - \overline{\pi}H(a,s))h'(s)}{(ah(a) - \overline{\pi}H(a,s))^3}. \tag{12}$$

The problem reduces now to finding $0 < \lambda < 1$ such that for some b the boundary conditions at the bid price (9) are satisfied as well:

$$g(b) = (1-\lambda)b, \quad g'(b) = (1-\lambda). \tag{13}$$

There are two possible scenarios: For any fixed a and λ, it is *a priori* possible that either $a < b$ or $a > b$. In the Black-Scholes case, this depends on the value of θ, a constant, but here the qualitative behavior of the shadow price process depend on the ask price a.

By plugging our function $g(s)$ into the boundary conditions at b, we obtain an implicit function connecting a, b and λ. We have

$$\begin{aligned} 0 &= g(b) - bg'(b) \\ &= a\,\frac{ah(a) + (1-\overline{\pi})H(a,b)}{ah(a) - \overline{\pi}H(a,s)} - \frac{ba^2 h(a)h(b)}{(ah(a) - \overline{\pi}H(a,b))^2}. \end{aligned}$$

From the first boundary condition that $g(b) = (1-\lambda)b$, we obtain the following formula for $\overline{\pi}$:[1]

$$\overline{\pi} = \frac{a(H(a,b) + \lambda bh(a) - bh(a) + ah(a))}{(a + \lambda b - b)H(a,b)} = \frac{a}{a + \lambda b - b} + \frac{ah(a)}{H(a,b)}, \tag{14}$$

which we can use to simplify the expression considerably to obtain

$$F(a,b,\lambda) \quad = \quad H(a,b)^2(\lambda - 1) + (a + b(\lambda-1))^2 h(a)h(b) \quad = \quad 0. \tag{15}$$

That is, the boundary conditions at the bid price are satisfied iff the function F vanishes. We now make the notion that b can lie on both sides of a precise:

Lemma 4.1. *For every value $a > 0$ and $\lambda \in (0,1)$ there are at least two values $b_\pm$ satisfying equation* (15)*, and $b_- \in (0,a)$ and $b_+ \in (a,\infty)$.*

Proof. Recall that $F(a,b,\lambda) := H(a,b)^2(\lambda-1) + (a + b(\lambda-1))^2 h(a)h(b)$ and fix $0 < \lambda < 1$ and $a > 0$. Then we have

$$F(a,a,\lambda) = a^2\lambda^2 h(a)^2 > 0. \tag{16}$$

We will show that $\lim_{s\to 0} F(a,s,\lambda)$ and $\lim_{s\to\infty} F(a,s,\lambda)$ both tend towards $-\infty$. Note that $h(s) \to \infty$ for $s \to 0$ and $s \to \infty$, so that if s becomes large or small enough, we have $h(s) > h(a)$ for all values of $a \in (0,\infty)$.

[1] Notice that for any fixed $\lambda > 0$, $\overline{\pi}(a,a,\lambda) = 1$.

Consider first the case where $s < a$. For $s > 0$ sufficiently small, we have that $a+s(\lambda-1) < a$ and $F(a,s,\lambda) < H(a,s)^2(\lambda-1)+a^2h(a)h(s)$. This can be expressed as the difference of two strictly positive functions f_+-f_-, where $f_+(s) = a^2h(a)h(s) > 0$ and $f_-(s) = H(a,s)^2(1-\lambda) > 0$. Using that $\mathrm{Erfi}(x) \sim e^{z^2}/(z\sqrt{\pi})$ as $z \to \infty$, we find that

$$\lim_{s\to 0} f_+(s)/f_-(s) \quad \sim \quad const. \times e^{-\kappa(\log s - 4\bar{x})\log s}\, s^{-(1+2\bar{x}\kappa/\sigma^2)}(\log s)^2 \to 0 \qquad (s \to 0)$$

which implies that $f_+ - f_- \to -\infty$, and since $F < f_+ - f_-$ we are done.

We now consider the case $s > a$. Then, for large s,

$$(a + s(\lambda - 1))^2 = (s(1-\lambda) - a)^2 < s^2,$$

and we have $F(s) < H(a,s)^2(\lambda-1)+s^2h(s)h(a)$, which can be expressed as the difference of two strictly positive functions $f_+ - f_-$ as before. Using the same asymptotic relation for $\mathrm{Erfi}(x)$ as above, we find that

$$\lim_{s\to 0} f_+(s)/f_-(s) \quad \sim \quad const. \times e^{-\kappa(\log s)^2}\, s^{1+2\bar{x}\kappa/\sigma^2}(\log s)^2 \to 0 \qquad (s \to 0)$$

which implies that $f_+ - f_- \to -\infty$, and since $F < f_+ - f_-$ we are done. □

As seen in Lemma 4.1, no matter how small $\lambda > 0$ is, there are multiple solutions $b = b(a,\lambda)$ to (15). However, an economically meaningful bid price should satisfy the following:

1. (**S1**) the shadow price curve given by the bid price b and the ask price a via (10) and (14) should be bounded for all s between the bid and the ask prices. Letting $D(s;a,\lambda) := ah(a)-\bar{\pi}(a,b,\lambda)H(a,s)$ be the denominator in (10). Then it is clear that $D(s;a,\lambda)$ is monotonic in s. Given that $D(a;a,\lambda) = ah(a) > 0$, we know that there is no explosion in the shadow price for all s between a and b if and only if $D(b;a,\lambda) > 0$;

2. (**S2**) The bid price converges to the corresponding ask price as the transaction cost tends to zero, i.e., $\lim_{\lambda\downarrow 0} b(a,\lambda) = a$.

Any economically meaningful bid price can lead to a meaningful shadow price curve. Below we construct the bid price function $b(\cdot,\cdot)$ satisfying all these requirements.

To this end, recall that (15) is a quadratic equation in λ. Thus, for each pair (a,b), we have two λ-values as solutions. Let $\gamma := a + b(\lambda - 1)$, which must lie in $(a-b, a)$ because the transaction cost $\lambda \in (0,1)$. More specifically, we obtain a quadratic in γ (using $\lambda - 1 = \frac{\gamma - a}{b}$):

$$G(\gamma) := \gamma^2 + \frac{H(a,b)^2}{bh(a)h(b)}\gamma - \frac{aH(a,b)^2}{bh(a)h(b)} = 0$$

with solutions

$$\gamma_\pm \quad = \quad \frac{-H(a,b)^2 \pm |H(a,b)|\sqrt{4abh(a)h(b) + H(a,b)^2}}{2bh(a)h(b)}, \qquad (17)$$

and we see that $\gamma_- < 0 < \gamma_+$. We now show that only one λ-value satisfies the no explosion condition (**S1**). Indeed, if $a \geq b$ we have $a - b > 0$, then from the requirement that $\gamma(a,b) \in (a-b, a)$, we must take $\gamma = \gamma_+(a,b)$ and therefore $\lambda = \lambda_+(a,b)$. After some algebra using (14), we have

$$D(b; a, \lambda) = ah(a) - \bar{\pi}(a, b, \lambda(a,b))H(a,b) = -\frac{aH(a,b)}{\gamma(a,b)} = \frac{aH(b,a)}{\gamma(a,b)} > 0.$$

Thus, condition (**S1**) is satisfied. On the other hand, if $b - a > 0$, then from the same calculation, we have that

$$D(b; a, \lambda_\pm) = ah(a) - \bar{\pi}(a, b, \lambda_\pm(a,b))H(a,b) = -\frac{aH(a,b)}{\gamma_\pm(a,b)} = \frac{aH(a,b)}{-\gamma_\pm(a,b)} \lessgtr 0.$$

Thus, condition (**S1**) is satisfied if and only if we take $\gamma = \gamma_-(a,b)$ and $\lambda = \lambda_-(a,b)$.

In summary, we obtain a smooth mapping from $\mathbb{R}_+ \times \mathbb{R}_+$ to $\mathbb{R}$:

$$\lambda(a,b) \quad = \quad 1 - \frac{a}{b} - \frac{1}{2}\frac{H(a,b)^2}{b^2 h(a)h(b)} - \frac{H(a,b)}{2b}\sqrt{\frac{H(a,b)^2}{b^2 h(a)^2 h(b)^2} + \frac{4a}{bh(a)h(b)}}\,. \tag{18}$$

It is easily seen that $\lambda(a,a) = 0$, for all $a > 0$. In order to construct a bid price that satisfies both (**S1**) and (**S2**), we need to construct the implicit function $b(a,\lambda)$ that satisfies $\lambda(a, b(a,\lambda_0)) = \lambda_0$ for all $\lambda_0 > 0$ sufficiently small. In particular, from the above discussion we know that, for any fix $\lambda_0 \in (0,1)$, the bid price that satisfies (**S1**) and (**S2**), must be either $b(a,\lambda_0) < a$ or $b(a,\lambda_0) > \frac{a}{1-\lambda_0}$ (since $\gamma(a, b(a,\lambda_0)) = a + (\lambda_0 - 1)b < 0$).

4.1 Asymptotics for $\lambda \ll 1$

Expanding λ in a Taylor series in b around a fixed $a > 0$ we obtain

$$\lambda(a,b) \quad = \quad \frac{\Gamma(a)}{6a^3}(b-a)^3 + O((b-a)^4)\,, \tag{19}$$

where $\Gamma(\cdot)$ is defined in Section 2. Formally, (19) defines the bid price $b(a,\lambda)$ via an asymptotic expansion. This is made rigorous in the following lemma.

Lemma 4.2. *For any fixed $a \notin \{a^*, b^*\}$, there exist $\delta_a, \delta'_a > 0$, and a C^∞-mapping $\varphi : B_{\delta_a}(a) \times B_{\delta'_a}(0) \to \mathbb{R}_+$, such that for every $x \in B_{\delta_a}(a)$, $y \in B_{\delta_a}(a)$ and $\varepsilon \in B_{\delta'_a}(0)$, we have $\lim_{\varepsilon\to 0}\varphi(x,\varepsilon) = x$, and*

$$\varepsilon^3 = \lambda(x,y) \qquad \text{iff} \qquad y = \varphi(x,\varepsilon).$$

*Thus, the bid price $b(a',\lambda) := \varphi(a', \lambda^{\frac{1}{3}})$ satisfies both (**S1**) and (**S2**), and is well-defined over $B_{\delta_a}(a) \times B_{(\delta'_a)^3}(0)$. Moreover, the sign of $b(a',\lambda) - a'$ is the sign of $\Gamma(a)$.*

Proof. Consider the function

$$\psi(x,y) = \begin{cases} \lambda(x,y)/(y-x)^3, & x \neq y \\ \Gamma(x)/6x^3, & x = y \end{cases}, \qquad \forall x, y > 0.$$

Then it is easily seen that $\psi(\cdot,\cdot)$ is a C^∞-mapping. If $a \notin \{a^*, b^*\}$, then we can find a $\delta_a > 0$ sufficiently small such that $\psi(x,y) \neq 0$ for all $(x,y) \in B_{\delta_a}(a) \times B_{\delta_a}(a)$. For a sufficiently small $\delta'_a > 0$, we consider the equation $\varepsilon^3 = \lambda(x,y)$ for $(x,y,\varepsilon) \in B_{\delta_a}(a) \times B_{\delta_a}(a) \times B_{\delta'_a}(0)$, which is equivalent to

$$\psi^{\frac{1}{3}}(x,y)(y-x) - \varepsilon = 0.$$

By the classical implicit function theorem in a neighborhood of $(a,a,0)$, we know that there exists a C^∞-function as stated in the lemma, and the sign of $\varphi(a',\varepsilon) - a'$ agrees with the sign of $\psi^{\frac{1}{3}}(a,a) = \sqrt[3]{\Gamma(a)/6a^3}$. □

From the previous lemma and (19) we obtain that

$$b(a,\lambda) \;\; = \;\; a + a(\frac{6}{\Gamma(a)})^{\frac{1}{3}}\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}).$$

Remark 4.3. *From Lemma 4.2, we know that the bid price that satisfies (***S1***) and (***S2***) has the following property: if the ask price $a \notin [a^*, b^*]$, the bid price $b(a,\lambda) < a$ for λ sufficiently small; conversely, if the ask price $a \in (a^*, b^*)$, the bid price $b(a,\lambda) > a$ for λ sufficiently small.*

Lemma 4.2 constructs the ask-bid price mapping $b(\cdot,\cdot)$ at each ask price a different from a^*, b^*; and the transaction cost level that is allowed for $b(a,\lambda)$ to be well-defined may depend on the ask price a. Below we "glue" these mappings together to obtain a global smooth mapping $b : (\mathbb{R}_+ \backslash \{a^*, b^*\}) \times [0,\delta) \ni (a,\lambda) \mapsto b(a,\lambda) \in \mathbb{R}_+$, for some small $\delta > 0$. We shall first consider all ask price $a < a^*$ or $a \gg b^*$.

Lemma 4.4. *For any $\lambda \in (0,1)$ and $a \leq a^*$, there exists a unique solution $b = b(a,\lambda)$ to $F(a,b,\lambda) = 0$ in $(0,a)$. Moreover, $b(a,\lambda)$ is differentiable in a, λ and* $\lim_{\lambda \downarrow 0} b(a,\lambda) = a$.

Proof. See Lemma A.1. □

Lemma 4.5. *For any $\lambda \in [0, \frac{1}{2})$ and $a > b^*$ sufficiently large, there exists a unique solution $b = b(a,\lambda)$ to $F(a,b,\lambda) = 0$ over (b^*, a). Moreover, $b(a,\lambda)$ is differentiable in a, λ and that* $\lim_{\lambda \downarrow 0} b(a,\lambda) = a$.

Proof. See Lemma A.2. □

Remark 4.6. ***For*** $\lambda \in (0,1)$ ***fixed, there may not be an economically meaningful*** $b(a;\lambda)$ ***which satisfies*** (15)***. We will see in section 5 that this is the case if and only if*** a ***lies in some interval*** $(a_1(\lambda), a_2(\lambda))$***.***

The above formula gives the small-λ expansions of the bid price for any given ask price that is not equal to a^* or b^*. Going back to the function (14) for $\overline{\pi}$, we can now express the fractions of wealth invested in stock, when the stock price S is equal to the ask price a, as a function of λ. Since $\overline{\pi}$ is the ratio of two C^∞-functions, we can plug in our series expansion of $b(a, \lambda)$ to obtain a small λ series expansion of $\overline{\pi}$ in a neighborhood around 0:

$$\overline{\pi}(a, \lambda) \equiv \overline{\pi}(a, b(a, \lambda), \lambda) = \theta(a) - (\frac{3}{4}\Gamma(a)^2)^{\frac{1}{3}}\lambda^{\frac{1}{3}} + O_a(\lambda^{\frac{2}{3}}). \tag{20}$$

To obtain an expression for $\underline{\pi}$ use the relation

$$\varphi_t^0 + \varphi_t^1 a_t = V_t$$

and the definition of $\overline{\pi}$ to obtain

$$\underline{\pi} \quad = \quad \frac{\varphi^1 b}{\varphi^0 + \varphi^1 b} \quad = \quad \frac{\frac{\varphi^1 a}{\varphi^0 + a\varphi^1}\, b}{\frac{\varphi^0 a}{\varphi^0 + a\varphi^1} + \frac{\varphi^1 a}{\varphi^0 + a\varphi^1} b} \quad = \quad \frac{\overline{\pi} b}{a + \overline{\pi}(b - a)} \tag{21}$$

(where we have also used that φ^0 and φ^1 only change by a infinitesimal amount - **we only prove this later!**). Plugging the expansion for b into this formula yields
Plugging the expansion for b into this formula yields

$$\underline{\pi} \quad = \quad \theta(b) \; + \; (\frac{3}{4}\;\Gamma(b)^2\lambda)^{\frac{1}{3}} \; + \; O(\lambda^{\frac{2}{3}}).$$

The above discussion also holds when we switch from the buying boundary (when the stock price is equal to an ask price a) to the selling boundary (when the stock price is equal to a bid price b). More specifically, fix any $b \notin \{a^*, b^*\}$, then for all sufficiently small $\lambda > 0$, there is a unique economically meaningful ask price, denoted by $a(b, \lambda)$, such that $\lambda(a(b, \lambda), b) = \lambda$ and that

$$a(b, \lambda) \quad = \quad b \; - \; b(\frac{6}{\Gamma(b)})^{\frac{1}{3}}\lambda^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}}).$$

$$\underline{\pi}(b, \lambda) \equiv \underline{\pi}(a(b, \lambda), b, \lambda) = \theta(b) + \left(\frac{3}{4}\Gamma(b)^2\right)^{\frac{1}{3}}\lambda^{\frac{1}{3}} + O_b(\lambda^{\frac{2}{3}}).$$

4.2 The special case $a = a^*$

As previously discussed, the asymptotic behaviour is different at the points $a = a^*$ and $b = b^*$. These are the points where Γ vanishes, which means we have to consider higher order terms when inverting the series expansion (19). For $a = a^*$, mathematically we have two solutions to (15) with corresponding series expansions:

$$b_{\pm}(a^*, \lambda) \quad = \quad a^* \; \pm \; a^*\sqrt{2}\,(\frac{3\sigma^4}{\kappa\sigma\sqrt{(4\kappa + \sigma^2)}}\lambda)^{\frac{1}{4}} \; + \; O(\sqrt{\lambda})\,,$$

However, we shall see that b_- is the only physically meaningful solution at $a = a^*$ for $\lambda \ll 1$, but there will be two physically relevant solutions at a special value $a = a_0(\lambda)$ which tends to a^* as $\lambda \to 0$ (see section 5.2).

Similarly at $b = b^*$ we have

$$a_\pm \;=\; b \pm b\sqrt{2}\,(\frac{3\sigma^4}{\kappa\sigma\sqrt{(4\kappa+\sigma^2)}}\lambda)^{\frac{1}{4}} + O(\sqrt{\lambda}).$$

and

$$\begin{aligned}\bar{\pi} &= \theta(a) - \frac{\sqrt{\kappa}\,(4\kappa+\sigma^2)^{\frac{1}{4}}}{\sqrt{3}\,\sigma^{\frac{3}{2}}}\lambda^{\frac{1}{2}} + o(\sqrt{\lambda})\,,\\ \underline{\pi} &= \theta(b) + \frac{\sqrt{\kappa}\,(4\kappa+\sigma^2)^{\frac{1}{4}}}{\sqrt{3}\,\sigma^{\frac{3}{2}}}\lambda^{\frac{1}{2}} + o(\sqrt{\lambda})\,.\end{aligned}$$

In the case of the two points a^* and b^* we have to modify Lemma 4.2.

Lemma 4.7. *For $a = a^*$ there exist $\delta, \delta' > 0$ and C^∞-mappings $\varphi_\pm : B_\delta(a) \times B_{\delta'}(0) \to \mathbb{R}^+$, such that for every $x \in B_\delta(a)$, $y \in B_\delta(a)$ and $\varepsilon \in B_{\delta'}(0)$ we have $\lim_{\varepsilon\to 0}\varphi_\pm(x,\varepsilon) = x$ and*

$$\varepsilon^4 = \lambda(x,y) \qquad \text{iff} \qquad y = \varphi_-(x,\varepsilon) \text{ or } y = \varphi_+(x,\varepsilon).$$

Proof. Fix $(x,y,\varepsilon) \in B_\delta(a) \times B_\delta(a) \times B_{\delta'}(0)$ such that $\varepsilon^4 = \lambda(x,y)$. Then

$$\lambda(x,y) = \psi(x,y)(x-y)^4,$$

for some C^∞ function ψ, since the first three derivatives of λ with respect to the second derivative vanish at the point (a,a). Since the fourth derivative gives

$$\frac{\partial^4\lambda}{\partial y^4}(a,a) \;=\; 4!\psi(a,a) \;=\; 4!\,\frac{\kappa\sqrt{4\kappa+\sigma^2}}{12a^4\sigma^3} \;>\; 0\,,$$

we can write

$$\varepsilon \;=\; \psi(x,y)^{\frac{1}{4}}(y-x). \tag{22}$$

By the classical implicit function theorem in a neighborhood of $(a,a,0)$, we know that there exists two C^∞-functions as stated in the lemma. $\square$

Remark 4.8. *Note that*

$$\frac{\partial\varphi_\pm}{\partial\varepsilon}(a^*,0) = \pm\frac{1}{\left(\frac{1}{4!}\frac{\partial^4\lambda}{\partial y^4(a^*,a^*)}\right)^{\frac{1}{4}}}$$

and the functions $\varphi_\pm$ are therefore distinct.

5 What happens when a physically meaningful $b(\lambda)$ or $a(\lambda)$ do not exist

Throughout this section, we will denote the shadow price function by $\widetilde{S}_t = g(s; a, \overline{\pi})$, to make the dependence on the ask price and the fraction of wealth invested in stocks held at that ask price clear. If not otherwise indicated, derivatives are taken with respect to the stock price s. We begin constructing the shadow price in a region surrounding the point a_0:

Theorem 5.1. *For any $e^{\frac{\sigma^2}{2\kappa}} b^* > a_2 > b^*$, there exists a unique value $\overline{\pi} \in \mathbb{R}$ and a unique ask price $a_1 = a_1(a_2) \in (a^*, b^*)$ such that g satisfies the boundary conditions at this price, i.e. $g(a_1; a_2, \overline{\pi}) = a_1$ and $g'(a_1; a_2, \overline{\pi}) = 1$. Moreover, the mapping $a_2 :\mapsto a_1(a_2)$ is smooth, monotonically decreasing, $\lim_{a_2 \downarrow b^*} a_1(a_2) = b^*$ and $\lim_{a_2 \downarrow b^*} a_1'(a_2) = -1$.*

Proof. For any fixed $s \neq a_2$, from (10) we know that, $g(s; a_2, \overline{\pi}) = s$ if and only if $\overline{\pi} = p(s; a_2)$, where

$$p(s; a_2) := a_2 \frac{(a_2 - s)h(a_2) + H(a_2, s)}{(a_2 - s)H(a_2, s)}. \tag{23}$$

Furthermore, it is straightforward to verify that

$$\left.\frac{\partial g(s; a_2, \overline{\pi})}{\partial s}\right|_{\overline{\pi} = p(s; a_2)} - 1 = \frac{-H^2(s, a_2) + (a_2 - s)^2 h(a_2) h(s)}{H^2(s, a_2)^2} =: f(s; a_2).$$

Clearly, a solution to $f(s; a_2) = 0$ over $(0, a_2)$ is the same as a solution to $k(s; a_2, 0) = 0$ over the same domain, where

$$k(s; a_2, 0) := -H(s, a_2) + (a_2 - s)\sqrt{h(a_2)h(s)}, \quad \forall s \leq a_2.$$

The remaining of the proof now follows from Lemma A.3. □

We have constructed a "special region" around the price b_0, at which the investor only buys stocks until the stock price either drops below a_1 or rises above a_2, but only locally in a neighborhood around b_0. This result can be extended, however:

5.1 The special point $b_0(\lambda)$ - geometric definition and small-λ expansion

We now wish to find the special value of $a_2 = a_2(\lambda)$ in Theorem 5.1 such that the g curve joining a_2 to $a_1(a_2) = a_1(\lambda)$ just touches the buying boundary, at some point $b = b_0(\lambda)$, to be determined (see figure 2). More specifically, we want series expansions for $a_2(\lambda), a_1(\lambda)$ and $b_0(\lambda)$.

To this end, let $a_2 = b^* + \delta$ for some $\delta > 0$ small. Then setting $a_1 = b^* - c_1\delta$, we find that the leading order term in

$$g(s; a_1, \underline{\pi})|_{s=a_2, \underline{\pi}=p(a_2,a_1)} - s$$

vanishes if and only if $c_1 = 1$; thus we see that (at leading order for $\delta \ll 1$) the g curve returns to the buying boundary equidistant from b^* but on the opposite side.

Now let $\varepsilon = a_1(a_2) - b^* - \delta = o(\delta)$ denote the remainder term. We want to derive the leading order term for ε in terms of δ. To this end, let

$$U(a_1, a_2) = \frac{\partial g(s; a_1, \underline{\pi})}{\partial s}|_{s=a_2, \underline{\pi}=p(a_2,a_1)} = \frac{(a_1 - a_2)^2 h(a_1) h(a_2)}{H(a_1, a_2)^2}.$$

Then it is straightforward to verify (using e.g. Mathematica or L'Hôpital's rule) that

$$U(b^* - \delta, b^* + \delta) - 1 = A_1\delta^4 + O(\delta^5) \tag{24}$$

where $A_1 = \frac{2\kappa(-2\kappa+5\sigma\sqrt{4\kappa+\sigma^2})}{15(b^*\sigma)^4} < 0$. By Theorem 5.1, we know that $U(b^* - \delta, b^* + \delta + \varepsilon) = 1$, so the effect of the error term ε is to remove the residue $A_1\delta^4 + O(\delta^5)$. Since U is analytic, we consider its Taylor expansion:

$$U(b^* - \delta + \varepsilon, b^* + \delta) = U(b^* - \delta, b^* + \delta) + U_2(b^* - \delta, b^* + \delta)\varepsilon + \frac{1}{2}U_{22}(b^* - \delta, b^* + \delta)\varepsilon^2 + o(\varepsilon^2)$$

where $U_2(x, y) = \frac{\partial}{\partial y}U(x, y), U_{22}(x, y) = \frac{\partial^2}{\partial y^2}U(x, y)$ (recall that $\varepsilon = a_1(a_2) - b^* - \delta$). Notice that (24) gives the leading term. For the second term, we have

$$U_2(b^* - \delta, b^* + \delta) \quad = \quad U(b^* - \delta, b^* + \delta)\left(-\frac{1}{\delta} + \frac{h'(b^* + \delta)}{h(b^* + \delta)} + \frac{2h(b^* - \delta)}{H(b^* - \delta, b^* + \delta)}\right).$$

Using e.g. Mathematica, we find that

$$\frac{1}{\delta} + \frac{h'(b^* - \delta)}{h(b^* - \delta)} - \frac{2h(b^* + \delta)}{H(b^* - \delta, b^* + \delta)} = B_1\delta^2 + O(\delta^3).$$

where $B_1 = -\frac{2\kappa\sqrt{4\kappa+\sigma^2}}{3(b^*\sigma)^3}$. It follows that, as $\delta \downarrow 0$,

$$U_2(b^* - \delta, b^* + \delta) = (1 + A_1\delta^4 + O(\delta^5))(B_1\delta^2 + O(\delta^3)) = B_1\delta^2 + O(\delta^3)$$

Similarly, using Mathematica we have

$$U_{22}(b^* - \delta, b^* + \delta) = O(\delta)$$

Combining all the above, we have that

$$\begin{aligned} U(b^* - \delta, b^* + \delta + \varepsilon) - 1 \quad &= \quad A_1\delta^4 + O(\delta^5) + [1 + A_1\delta^4 + O(\delta^5)][B_1\delta^2 + O(\delta^3)]\varepsilon + \frac{1}{2}O(\delta)\varepsilon^2 + o(\varepsilon^2) \\ &= \quad A_1\delta^4 + B_1\delta^2\varepsilon + o(\delta^4) + \frac{1}{2}O(\delta)\varepsilon^2 + o(\varepsilon^2). \end{aligned}$$

In order to remove the leading term δ^4, we need to have

$$\varepsilon \;=\; -\frac{A_1}{B_1}\delta^2 + o(\delta^2) \;=\; d_1\delta^2 + o(\delta^2)$$

where $d_1 = \frac{1}{5b^*\sigma}\frac{-2\kappa+5\sigma\sqrt{4\kappa+\sigma^2}}{\sqrt{4\kappa+\sigma^2}}$. Thus from the definition of ε above, we have

$$a_1(a_2) \;=\; b^* \,-\, \delta \,+\, d_1\delta^2 \,+\, o(\delta^2)\,.$$

If we now let $s = (1-\alpha)(b^*+\delta) + \alpha(b^* - \delta + d_1\delta^2)$ (so $s = b^* + \delta$ when $\alpha = 0$ and $s = b^* - \delta + d_1\delta^2$ when $\alpha = 1$) , we find that

$$\begin{aligned} g(s; a_2, \underline{\pi})|_{s=a_2,\,\underline{\pi}=p(a_2,a_1)} - s \;&=\; -A\alpha^2(\alpha-1)^2\delta^4 \\ &\quad+\; B\alpha^2(\alpha-1)^2(2\alpha-3)\delta^5 + O(\delta^6) \end{aligned}$$

where $A = \frac{4\kappa\sqrt{4\kappa+\sigma^2}}{3(b^*\sigma)^3}$, $B = \frac{4\kappa(2\kappa-5\sigma\sqrt{4\kappa+\sigma^2})}{15(b^*\sigma)^4}$. We now have to solve for α and λ so that this curve just touches the sell boundary. Imposing the smooth pasting condition:

$$\frac{d}{d\alpha}[-A\alpha^2(\alpha-1)^2\delta^4 \,+\, B\alpha^2(\alpha-1)^2(2\alpha-3)\delta^5 + \lambda s] \;=\; 0$$

we find that the (unique) critical λ is given by

$$\lambda \;=\; \textbf{to do ..}$$

We now guess that the critical α value is given by $\alpha = \frac{1}{2} + c_1\delta + O(\delta^2)$ for some c_1. Substituting this into the value-matching condition:

$$-A\alpha^2(\alpha-1)^2\delta^4 \,+\, B\alpha^2(\alpha-1)^2(2\alpha-3)\delta^5 - \lambda s \;=\; 0$$

we find that $-A + (B + 8Ac_1)b_0 = 0$ or

$$c_1 \;=\; -\frac{-A + b^*B}{8Ab^*} \;=\; \frac{-\kappa + 5\sigma\sqrt{4\kappa+\sigma^2}}{20b^*\sigma\sqrt{4\kappa+\sigma^2}}\,.$$

Then we have

$$\lambda \;=\; \frac{\kappa\sqrt{4\kappa+\sigma^2}}{12(b^*)^4\sigma^3}\delta^4 + o(\delta^4)\,.$$

Inverting this expansion, we see that the critical δ value is given by

$$\delta \;=\; (\frac{12(b^*)^4\sigma^3}{\kappa\sqrt{4\kappa+\sigma^2}})^{\frac{1}{4}}\lambda^{\frac{1}{4}} + o(\lambda^{\frac{1}{4}})\,.$$

Thus we have

$$\begin{aligned} a_1(\lambda) \;&=\; b^* \,-\, \frac{\sqrt{2}\cdot 3^{\frac{1}{4}}b^*\sigma^{\frac{3}{4}}}{\kappa^{\frac{1}{4}}(4\kappa+\sigma^2)^{\frac{1}{8}}}\lambda^{\frac{1}{4}} \,+\, o(\lambda^{\frac{1}{4}})\,, \\ a_2(\lambda) \;&=\; b^* \,-\, \frac{\sqrt{2}\cdot 3^{\frac{1}{4}}b^*\sigma^{\frac{3}{4}}}{\kappa^{\frac{1}{4}}(4\kappa+\sigma^2)^{\frac{1}{8}}}\lambda^{\frac{1}{4}} \,+\, o(\lambda^{\frac{1}{4}})\,. \end{aligned}$$

Plugging these expansions for α and δ into the expression for s above, we obtain the following expansions

$$b_0(\lambda) \;=\; b^* - \frac{b^*\sqrt{3\kappa\sigma}}{5(4\kappa+\sigma^2)^{\frac{3}{4}}}\lambda^{\frac{1}{2}} + o(\lambda^{\frac{1}{2}})$$

We conclude the following:

There is a special region $(a_1(\lambda), a_2(\lambda))$ such that for a values inside this interval, there is no corresponding $b(a;\lambda)$ value.

5.2 The special point $b_0(\lambda)$ - financial meaning and alternate derivation for the small-λ expansion

We now consider an asymptotic regime where λ and $|b-b^*|$ both simultaneously go to zero. More specifically, we set $\lambda = \varepsilon^4$ and $b = b^* + x\varepsilon^2$. But from Lemma 4.6 we know that $\varphi_\pm$ are C^∞, so

$$a_\pm(a,\lambda) \;=\; \varphi_\pm(b^* + x\varepsilon^2, \varepsilon) \;=\; b + b\beta_1^\pm\varepsilon + b\beta_2^\pm(x)\varepsilon^2 + O(\varepsilon^3)$$

where $\varepsilon = \lambda^{\frac{1}{4}}$, and the coefficients $\beta_1^\pm$, $\beta_2^\pm(x)$ are to be determined. Plugging this expansion into (15), we find that

$$\begin{aligned}
\beta_1^\pm &= \pm\frac{\sqrt{2}\cdot 3^{\frac{1}{4}}\sigma^{\frac{3}{4}}}{\kappa^{\frac{1}{4}}(4\kappa+\sigma^2)^{\frac{1}{8}}}, \\
\beta_2^\pm(x) &= -\frac{x}{2b^*} + \frac{\sqrt{3}\,(-3\kappa\sqrt{\sigma} + 10\sqrt{4\kappa\sigma^3+\sigma^5})}{10\sqrt{\kappa}(4\kappa+\sigma^2)^{\frac{3}{4}}}.
\end{aligned}$$

Substituting these expansions into the expression for $\underline{\pi}$ in (21), we find that

$$\underline{\pi}_\pm(b,\lambda) \;=\; \theta(b) + c_1^\pm\varepsilon^2 + c_2^\pm\varepsilon^3,$$

and

$$\begin{aligned}
c_1^\pm &= -\frac{\sqrt{\kappa}\,(4\kappa+\sigma^2)^{\frac{1}{4}}}{\sqrt{3}\,\sigma^{\frac{3}{2}}}, \\
c_2^+ &= \ldots \\
c_2^- &= \ldots.
\end{aligned}$$

and $\beta_1^\pm, c_1^\pm$ are independent of x.

We now wish to find the x value for which $\underline{\pi}_+ = \underline{\pi}_-$ - why? because this will yield a special b-value on the sell boundary for which there is no trading required if $(S_t, \tilde{S}_t) = (b, (1-\lambda)b)$, because whether S goes up or down, the two $\underline{\pi}$ values are equal. But this is equivalent to solving $c_2^+ = c_2^-$, for which we find the critical value to be

$$x^* = \frac{a^*\sqrt{3\kappa\sigma}}{5(4\kappa+\sigma^2)^{\frac{3}{4}}}$$

which corresponds to an a-value given by

$$b^* - x^*\lambda^{\frac{1}{2}} + o(\lambda^{\frac{1}{2}})$$

and plugging $x = x^*$ into $a_\pm$, we find that

$$a_\pm(\lambda) = b^* \pm \frac{\sqrt{2}\cdot 3^{\frac{1}{4}}\, b^*\sigma^{\frac{3}{4}}}{\kappa^{\frac{1}{4}}(4\kappa+\sigma^2)^{\frac{1}{8}}}\lambda^{\frac{1}{4}} + \ldots .$$

But we see that these expansions agree with the expansions for $b_0(\lambda)$ and $a_1(\lambda), a_2(\lambda)$ in the previous subsection. Thus (at leading order) $b_0(\lambda)$ is a special point at which there is no trading required if $(S_t, \tilde{S}_t) = (b_0(\lambda), b_0(\lambda))$.

Similarly, we have the following counterpart of Theorem 5.1 around a^*:

Theorem 5.2. *Let $e^{-\frac{\sigma^2}{2\kappa}}a^* < b_1 < a^*$ and $\lambda \in (0,1)$. Then there exists a unique value $\overline{\pi} \in \mathbb{R}$ and a unique bid price $b_2 = b_2(b_1) \in (a^*, b^*)$ such that g satisfies the boundary conditions at this price, i.e. $g(b_2; b_1, \underline{\pi}) = (1-\lambda)b_2$ and $g'(b_2; b_1, \underline{\pi}) = 1-\lambda$. Moreover, the mapping $b_1 :\mapsto b_2(b_1)$ is smooth, monotonically decreasing, $\lim_{b_1\uparrow a^*} b_2(b_1) = a^*$ and $\lim_{b_1\uparrow a^*} b_2'(b_1) = -1$.*

Proof. The proof is omitted for the sake of brevity □

5.3 The special point $a_0(\lambda)$ - geometric definition and small-λ expansion

We now wish to find the special value of $b_1 = b_1(\lambda)$ in Theorem 5.2 such that the g curve joining b_1 to $b_2(b_1) = b_2(\lambda)$ just touches the buying boundary, at some point $a = a_0(\lambda)$, to be determined (see figure 2). More specifically, we want series expansions for $b_1(\lambda), b_2(\lambda)$ and $a_0(\lambda)$.

To this end, let $b_1 = a^* - \delta$ for some $\delta > 0$ small. Then setting $b_2 = a^* + c_1\delta$, we find that the leading order term in

$$g(s; b_1, \underline{\pi})|_{s=b_2,\, \underline{\pi}=p(b_1,b_2)} - (1-\lambda)s$$

vanishes if and only if $c_1 = 1$; thus we see that (at leading order for $\delta \ll 1$) the g curve returns to the selling boundary equidistant from a^* but on the opposite side.

Now let $\varepsilon = b_2(b_1) - a^* - \delta = o(\delta)$ denote the remainder term. We want to derive the leading order term for ε in terms of δ. To this end, let

$$U(b_1,b_2) = \frac{1}{1-\lambda}\frac{\partial g(s;b_1,\underline{\pi})}{\partial s}|_{s=b_2,\underline{\pi}=p(b_1,b_2)} = \frac{(b_1-b_2)^2 h(b_1)h(b_2)}{H(b_1,b_2)^2}\,.$$

Then it is straightforward to verify (using e.g. Mathematica or L'Hôpital's rule) that

$$U(a^*-\delta, a^*+\delta) - 1 = A_1\delta^4 + O(\delta^5) \tag{25}$$

where $A_1 = -\frac{2\kappa(2\kappa+5\sigma\sqrt{4\kappa+\sigma^2})}{15(a^*\sigma)^4} < 0$. By Theorem 5.2, we know that $U(a^*-\delta, a^*+\delta+\varepsilon) = 1$, so the effect of the error term ε is to remove the residue $A_1\delta^4 + O(\delta^5)$. Since U is analytic, we consider its Taylor expansion:

$$U(a^*-\delta, a^*+\delta+\varepsilon) = U(a^*-\delta, a^*+\delta) + U_2(a^*-\delta, a^*+\delta)\varepsilon + \frac{1}{2}U_{22}(a^*-\delta, a^*+\delta)\varepsilon^2 + o(\varepsilon^2)$$

where $U_2(x,y) = \frac{\partial}{\partial y}U(x,y), U_{22}(x,y) = \frac{\partial^2}{\partial y^2}U(x,y)$ (recall that $\varepsilon = b_2(b_1) - a^* - \delta$). Notice that (25) gives the leading term. For the second term, we have

$$U_2(a^*-\delta, a^*+\delta) \;=\; U(a^*-\delta, a^*+\delta)\left(\frac{1}{\delta} + \frac{h'(a^*+\delta)}{h(a^*+\delta)} - \frac{2h(a^*+\delta)}{H(a^*-\delta, a^*+\delta)}\right).$$

Using Mathematica, we find that

$$\frac{1}{\delta} + \frac{h'(a^*+\delta)}{h(a^*+\delta)} - \frac{2h(a^*+\delta)}{H(a^*-\delta, a^*+\delta)} = B_1\delta^2 + O(\delta^3).$$

where $B_1 = \frac{2\kappa\sqrt{4\kappa+\sigma^2}}{3(a^*\sigma)^3}$. It follows that, as $\delta \downarrow 0$,

$$U_2(a^*-\delta, a^*+\delta) = (1 + A_1\delta^4 + O(\delta^5))(B_1\delta^2 + O(\delta^3)) = B_1\delta^2 + O(\delta^3)$$

Similarly, using Mathematica we have

$$U_{22}(a^*-\delta, a^*+\delta) = O(\delta)$$

Combining all the above, we have that

$$\begin{aligned} U(a^*-\delta, a^*+\delta+\varepsilon) - 1 \;&=\; A_1\delta^4 + O(\delta^5) + [1 + A_1\delta^4 + O(\delta^5)][B_1\delta^2 + O(\delta^3)]\varepsilon + \frac{1}{2}O(\delta)\varepsilon^2 + o(\varepsilon^2) \\ &=\; A_1\delta^4 + B_1\delta^2\varepsilon + o(\delta^4) + \frac{1}{2}O(\delta)\varepsilon^2 + o(\varepsilon^2)\,. \end{aligned}$$

In order to remove the leading term δ^4, we need to have

$$\varepsilon - \frac{A_1}{B_1}\delta^2 + o(\delta^2) \;=\; d_1\delta^2 + o(\delta^2)$$

where $d_1 = \frac{1}{5a^*\sigma}\frac{2\kappa+5\sigma\sqrt{4\kappa+\sigma^2}}{\sqrt{4\kappa+\sigma^2}}$. Thus from the definition of ε above, we have

$$b_2(b_1) \;=\; a^* + \delta + d_1\delta^2 + o(\delta^2)\,.$$

Now let $s = b_1 + \alpha(b_2 - b_1) = a^* - \delta + \alpha(2\delta + \varepsilon) = a^* + (2\alpha - 1)\delta + \alpha d_1\delta^2 + o(\delta^2)$ for $\alpha \in [0, 1]$.

If we now let $s = a^* + (2\alpha - 1)\delta + \alpha d_1\delta^2$ (so $s = a^* - \delta$ when $\alpha = 0$ and $s = a^* + \delta + d_1\delta^2$ when $\alpha = 1$) , we find that

$$\begin{aligned} g(s; b_1, \underline{\pi})|_{s=b_2,\, \underline{\pi}=p(b_1,b_2)} - (1-\lambda)s &= A(1-\lambda)\alpha^2(\alpha-1)^2\delta^4 \\ &- B(1-\lambda)\alpha^2(\alpha-1)^2(2\alpha-3)\delta^5 + O(\delta^6) \end{aligned}$$

where $A = \frac{4\kappa\sqrt{4\kappa+\sigma^2}}{3(a^*\sigma)^3}$, $B = \frac{4\kappa(2\kappa+5\sigma\sqrt{4\kappa+\sigma^2})}{15(a^*\sigma)^4}$. We now have to solve for α and λ so that this curve just touches the buy boundary. Imposing the smooth pasting condition:

$$\frac{d}{d\alpha}[A(1-\lambda)\alpha^2(\alpha-1)^2\delta^4 \,-\, B(1-\lambda)\alpha^2(\alpha-1)^2(2\alpha-3)\delta^5 - \lambda s] \;=\; 0$$

we find that the (unique) critical λ is given by

$$\lambda^* \;=\; \frac{-2A\alpha(1-3\alpha+2\alpha^2)\delta^4 + 2B\alpha(-3+12\alpha-14\alpha^2+5\alpha^3)\delta^5}{(-2\delta - 2A\alpha(1-3\alpha+2\alpha^2)\delta^4 + 2B\alpha(-3+12\alpha-14\alpha^2+5\alpha^3)\delta^5 - \frac{\delta^2(2\kappa+5\sigma\sqrt{4\kappa+\sigma^2})}{5a^*\sigma\sqrt{4\kappa+\sigma^2})}}$$

We now guess that the critical α value is given by $\alpha = \frac{1}{2} + c_1\delta + O(\delta^2)$ for some c_1. Substituting this into the value-matching condition:

$$A(1-\lambda)\alpha^2(\alpha-1)^2\delta^4 \,-\, B(1-\lambda)\alpha^2(\alpha-1)^2(2\alpha-3)\delta^5 - \lambda s \;=\; 0$$

we find that $A + (B + 8Ac_1)a_0 = 0$ or

$$c_1 \;=\; -\frac{A + a^*B}{8Aa^*} \;=\; -\frac{\kappa + 5\sigma\sqrt{4\kappa+\sigma^2}}{20a^*\sigma\sqrt{4\kappa+\sigma^2}}\,.$$

Then we have

$$\lambda \;=\; \frac{\kappa\sqrt{4\kappa+\sigma^2}}{12(a^*)^4\sigma^3}\delta^4 \,+\, o(\delta^4)\,. \tag{26}$$

Inverting (25), we see that the critical δ value is given by

$$\delta^* \;=\; (\frac{12(a^*)^4\sigma^3}{\kappa\sqrt{4\kappa+\sigma^2}})^{\frac{1}{4}}\lambda^{\frac{1}{4}} + h.o.t.$$

Thus we have

$$\begin{aligned} b_1(\lambda) &= a^* - \frac{\sqrt{2}\cdot 3^{\frac{1}{4}}a^*\sigma^{\frac{3}{4}}}{\kappa^{\frac{1}{4}}(4\kappa+\sigma^2)^{\frac{1}{8}}}\lambda^{\frac{1}{4}} \,+\, o(\lambda^{\frac{1}{4}})\,, \\ b_2(\lambda) &= a^* - \frac{\sqrt{2}\cdot 3^{\frac{1}{4}}a^*\sigma^{\frac{3}{4}}}{\kappa^{\frac{1}{4}}(4\kappa+\sigma^2)^{\frac{1}{8}}}\lambda^{\frac{1}{4}} \,+\, o(\lambda^{\frac{1}{4}})\,. \end{aligned}$$

Plugging these expansions for α and δ into $s = a^* + (2\alpha-1)\delta + \alpha d_1\delta^2$, we obtain the following expansions

$$a_0(\lambda) \;=\; a^* + \frac{a^*\sqrt{3\kappa\sigma}}{5(4\kappa+\sigma^2)^{\frac{3}{4}}}\lambda^{\frac{1}{2}} \,+\, o(\lambda^{\frac{1}{2}})$$

We conclude the following:

There is a special region $(b_1(\lambda), b_2(\lambda))$ such that for b values inside this interval, there is no corresponding $a(b;\lambda)$ value.

5.4 The special point $a_0(\lambda)$ - financial meaning and alternate derivation for the small-λ expansion

We now consider an asymptotic regime where λ and $|a-a^*|$ both simultaneously go to zero. More specifically, we set $\lambda=\varepsilon^4$ and $a=a^*+x\varepsilon^2$. But from Lemma 4.6 we know that $\varphi_\pm$ are C^∞, so

$$b_\pm(a,\lambda) \quad=\quad \varphi_\pm(a^*+x\varepsilon^2,\varepsilon) \quad=\quad a+a\beta_1^\pm\varepsilon+a\beta_2^\pm(x)\varepsilon^2 \;+\; O(\varepsilon^3)$$

where $\varepsilon=\lambda^{\frac{1}{4}}$, and the coefficients $\beta_1^\pm$, $\beta_2^\pm(x)$ are to be determined. Plugging this expansion into (15), we find that

$$\begin{aligned}\beta_1^\pm &= \pm\frac{\sqrt{2}\cdot 3^{\frac{1}{4}}\sigma^{\frac{3}{4}}}{\kappa^{\frac{1}{4}}(4\kappa+\sigma^2)^{\frac{1}{8}}}\,,\\ \beta_2^\pm(x) &= -\frac{x}{2a^*}+\frac{\sqrt{3\sigma}\,(3\kappa+10\sigma\sqrt{4\kappa+\sigma^2})}{10\sqrt{\kappa}(4\kappa+\sigma^2)^{\frac{3}{4}}}\,.\end{aligned}$$

Substituting these expansions into the expression for $\bar{\pi}$ in (14), we find that

$$\pi_\pm(a,\lambda) \quad=\quad \theta(a)+c_1^\pm\varepsilon^2+c_2^\pm\varepsilon^3\,,$$

and

$$\begin{aligned}c_1^\pm &= -\frac{\sqrt{\kappa}\,(4\kappa+\sigma^2)^{\frac{1}{4}}}{\sqrt{3}\,\sigma^{\frac{3}{2}}}\,,\\ c_2^+ &= \frac{3\sqrt{2}\,\kappa^{\frac{5}{4}}\sqrt{\sigma}-\frac{5}{a^*}\sqrt{6}\,x\kappa^{\frac{3}{4}}(4\kappa+\sigma^2)^{\frac{3}{4}}}{15\cdot 3^{\frac{1}{4}}\sigma^{\frac{9}{4}}(4\kappa+\sigma^2)^{\frac{3}{8}}}\\ c_2^- &= \frac{\sqrt{2}\,\kappa^{\frac{3}{4}}[-\sqrt{3\kappa\sigma}+\frac{5}{a^*}x(4\kappa+\sigma^2)^{\frac{3}{4}}]}{5\cdot 3^{\frac{3}{4}}\sigma^{\frac{9}{4}}(4\kappa+\sigma^2)^{\frac{3}{8}}}\,.\end{aligned}$$

and $\beta_1^\pm, c_1^\pm$ are independent of x.

We now wish to find the x value for which $\pi_+=\pi_-$ - why? because this will yield a special a-value on the buy boundary for which there is no trading required if $(S_t,\tilde{S}_t)=(a,a)$, because whether S goes up or down, the two π values are equal. But this is equivalent to solving $c_2^+=c_2^-$, for which we find the critical value to be

$$x^* \quad=\quad \frac{a^*\sqrt{3\kappa\sigma}}{5(4\kappa+\sigma^2)^{\frac{3}{4}}}$$

which corresponds to an a-value given by

$$a^*+x^*\lambda^{\frac{1}{2}}\;+o(\lambda^{\frac{1}{2}})$$

and plugging $x=x^*$ into $b_\pm$, we find that

$$b_\pm(\lambda) \quad=\quad a^*\pm\frac{\sqrt{2}\cdot 3^{\frac{1}{4}}a^*\sigma^{\frac{3}{4}}}{\kappa^{\frac{1}{4}}(4\kappa+\sigma^2)^{\frac{1}{8}}}\lambda^{\frac{1}{4}}\;+\dots\,.$$

But we see that these expansions agree with the expansions for $a_0(\lambda)$ and $b_1(\lambda), b_2(\lambda)$ in the previous subsection. Thus (at leading order) $a_0(\lambda)$ is a special point at which there is no trading required if $(S_t, \tilde{S}_t) = (a_0(\lambda), a_0(\lambda))$.

We have now fixed the two special regions and the corresponding level of transaction costs. We still need to show that outside of these regions the boundary conditions are satisfied as well.

Theorem 5.3. *For all real numbers $\underline{a}, \bar{a} > 0$ such that $[a_0(0), b_0(0)] \subset [\underline{a}, \bar{a}] \subset (0, \infty)$, there exists a $\bar{\lambda} > 0$ such that for all $\lambda \in (0, \bar{\lambda})$, there is a globally defined $b(a, \lambda)$, $b : [\underline{a}, \bar{a}] \times [0, \bar{\lambda}] \to \mathbb{R}^+$.*

Proof. We have already showed that for every $a \in \mathbb{R}^+ \backslash \{a^*, b^*\}$ there exist some $\delta_a, \delta_b, \delta_\varepsilon > 0$ and a C^∞-function $\varphi : B_{\delta_a}(a) \times B_{\delta_\varepsilon}(a) \to B_{\delta_b}(a)$ such that

$$\begin{aligned} \{(x, y,\varepsilon)\big|\lambda(x, y) = \varepsilon^3\} \cap B_{\delta_a}(a) \times B_{\delta_b}(a) \times B_{\delta_\varepsilon}(0) = \\ =\{(x, \varphi(x, \varepsilon), \varepsilon)\big|(x, \varepsilon) \in B_{\delta_a}(a) \times B_{\delta_\varepsilon}(0). \end{aligned}$$

Similarly we showed that in the case $a \in \{a^*, b^*\}$ there exist two C^∞-functions $\varphi_\pm : B_{\delta_a}(a) \times B_{\delta_\varepsilon}(0)$ such that

$$\begin{aligned} \{(x, y,\varepsilon)\big|\lambda(x, y) = \varepsilon^4\} \cap B_{\delta_a}(a) \times B_{\delta_b}(a) \times B_{\delta_\varepsilon}(0) = \\ =\{(x, \varphi_-(x, \varepsilon), \varepsilon)\big|(x, \varepsilon) \in B_{\delta_a}(a) \times B_{\delta_\varepsilon}(0) \\ \cup \{(x, \varphi_+(x, \varepsilon), \varepsilon)\big|(x, \varepsilon) \in B_{\delta_a}(a) \times B_{\delta_\varepsilon}(0). \end{aligned}$$

It is clear that

$$[a_0(\lambda), a_1(\lambda)] \quad \subseteq \quad \bigcup_{a \in [a_0(\lambda), a_1(\lambda)]} B_{\delta_a}(a)$$

is an open covering of the interval $[a_0(\lambda), a_1(\lambda)]$ and there even exists a finite subcovering, i.e. $N \in \mathbb{N}$ and $\tilde{a}_1, \ldots, \tilde{a}_N \in [a_0(\lambda), a_1(\lambda)]$ such that

$$[a_0(\lambda), a_1(\lambda)] \quad \subseteq \quad \bigcup_{n=1}^{N} B_{\delta_{\tilde{a}_n}}(\tilde{a}_n)$$

and there is a $\varphi_n(x, \varepsilon)$ function associated with each n. We now set

$$b(a, \lambda) \quad = \quad \begin{cases} \varphi_n(a, \lambda^{\frac{1}{3}}), & \text{if } a \in B_{\delta_{\tilde{a}_n}}(\tilde{a}_n)\,,\, \tilde{a}_n \notin \{a^*, b^*\}, \\ \varphi_\pm(a, \lambda^{\frac{1}{4}}), & \text{if } a \in B_{\delta_{\tilde{a}_n}}(\tilde{a}_n)\,,\, \tilde{a}_n \in \{a^*, b^*\}, . \end{cases}$$

If e.g. $a \in B_{\delta_{a^*}}(a^*) \cap B_{\delta_{\tilde{a}_n}}(\tilde{a}_n)$ with $\tilde{a}_n \notin \{a^*, b^*\}$, then we have that

$$\lambda(a, \varphi_n(a, \lambda^{\frac{1}{3}})) \quad = \quad \lambda(a, \varphi_\pm(a, \lambda^{\frac{1}{4}})) \quad = \quad \lambda\,.$$

(mild abuse of notation here).

Since φ is unique on every ball we can piece together a C^∞-function $\tilde{\varphi} : [a_0(\lambda), a_1(\lambda)] \times [0, \tilde{\lambda}] \to \mathbb{R}^+$, where

$$0 < \bar{\lambda} := \begin{cases} \min\limits_n \{\delta_\varepsilon(\tilde{a}_n)^{\frac{1}{3}}\}, & \text{if } \tilde{a}_n \notin \{a^*, b^*\}, \\ \min\limits_n \{\delta_\varepsilon(\tilde{a}_n)^{\frac{1}{4}}\}, & \text{if } \tilde{a}_n \in \{a^*, b^*\}. \end{cases}$$

Deal with left side and right side ... □

Thus, for λ sufficiently small, we see that for every ask price in $[\underline{a}, \bar{a}] \setminus (a_1(\bar{\lambda}), a_2(\bar{\lambda}))$ we can find a bid price such that the boundary conditions are satisfied. It remains to show that the solution $g(s)$ remains in the bid-ask interval:

Lemma 5.4. *Fix $a \in [a_0, a_2]$ and $b(\lambda)$ as before. Then there exists $\varepsilon > 0$ such that for all $\lambda < \varepsilon$ we have*

$$g(s; a, b(a, \lambda)) \in [(1-\lambda)s, s],$$

for all $s \in [a, b(a, \lambda)]$.

Proof. It suffices to show that $\tilde{g}(a, b; s) := g(a, b; s)/s$ is decreasing in s on the interval $[a, b]$, since this implies the assertion. Note that $\tilde{g}$ is continuously differentiable with respect to s and define

$$f(a, b; s) := \frac{\partial \tilde{g}}{\partial s}(a, b; s).$$

Then f is C^∞ and it is straightforward to check that $f(a, a; a) = \frac{\partial f}{\partial s}(a, a; a) = 0$ and $\frac{\partial^2 f}{\partial s^2}(a, a; a) = \frac{2h(a)h''(a) - 3h'(a)^2}{2ah(a)^2} \neq 0$. As shown in [Morse28], there exist open neighborhoods $B_{\delta_a}(a), B_{\delta_b}(a)$ and $B_{\delta_s}(a)$ as well as two C^∞-functions $\varphi_\pm : B_{\delta_a}(a) \times B_{\delta_b}(a) \to B_{\delta_s}(a)$ such that $f(x, y; s) = 0$ for every triple $(x, y, s) \in B_{\delta_a}(a) \times B_{\delta_b}(a) \times B_{\delta_s}(a)$ iff $s = \varphi_-(x, y)$ or $s = \varphi_+(x, y)$.

Since we also have $f(a, b; a) = f(a, b; b) = 0$, it follows from the continuity of f that either $f(x, y; s) > 0$ or $f(x, y; s) < 0$, depending on the sign of the second derivative. □

For every ask price a we have thus found a bid price b which is connected by the function g, that is, g solves our original ODE and satisfies the boundary conditions at a and b. It remains to show the it is indeed a shadow price process remaining inside the bid-ask interval:

Theorem 5.5. *For every pair (a, b) as in Theorem* **??** *we have $g(s) \in ((1-\lambda)s, s)$ for all $s \in (b, s)$.*

Proof. We prove this by contradiction. For this, we assume that there exists $s' \in (b, a)$ with $g(s') > s'$. We have $h(b) > h(s') > h(a) > 0$, as well as $H(a, s') < 0$ and further that g is concave at a and convex at b, which can be checked by plugging into (12). By continuity of g, this implies that there are two values $s'_{1,2} \in (b, a)$ such that $g(s'_{1,2}) = s'_{1,2}$. This in turn implies

$$H(a, s'_{1,2}) = f_1(s'_{1,2}),$$

where

$$f_1(s) := \frac{h(a)(a-s)\big(a + b(\lambda - 1)\big)H(a, b)}{h(a)(a-s)(a + b(\lambda-1)) - (b(\lambda-1) + s)H(a, b)}.$$

But this impossible since $H(a, a) = f_1(a) = 0$ and both functions are concave. Indeed, we have

$$H_{ss}(a, s) = h(s)\frac{2\kappa\left(-\frac{\sigma^2}{2\kappa} + \log(s) - \bar{x}\right)}{s\sigma^2} < 0,$$

since $s < b_0$ and

$$f_1''(s) = \frac{2h(a)(a+b(\lambda-1))^2 H(a,b)^2\big(H(a,b)+h(a)(a+b(\lambda-1))\big)}{(h(a)(a-s)(a+b(\lambda-1))-(b(\lambda-1)+s)H(a,b))^3}$$
$$= \frac{2h(a)(a-b(1-\lambda))^2 H(a,b)^2 f_2(a,b)}{f_3(a,b,s)^3}.$$

In the last expression, the numerator is negative, since

$$\begin{aligned} f_2(a,b) &= (a-b(1-\lambda))h(a) + H(a,b) \\ &= (a-b(1-\lambda))\left(h(a) - \sqrt{\frac{h(a)h(b)}{1-\lambda}}\right) \\ &< (a-b(1-\lambda))h(a)\left(1-\frac{1}{\sqrt{1-\lambda}}\right) < 0. \end{aligned}$$

In the second equality we have used the fact that a and b are corresponding bid and ask prices, and we therefore have $F(a,b,\lambda)=0$ in (15). To see that the denominator is positive, observe that the function

$$f_3(a,b,s) := h(a)(a-s)(a+b(\lambda-1)) - (b(\lambda-1)+s)H(a,b)$$

is positive at the point $s=b$ and its derivative is given by

$$\frac{\partial f_3(a,b,s)}{\partial s} = (a+b(\lambda-1))\left(\frac{\sqrt{h(a)h(b)}}{\sqrt{1-\lambda}} - h(a)\right) > 0.$$

Since both functions are concave and conicide at $s=a$, they can have at most one intersection in the interval (b,a), which contradicts the assumption.

The proof that there doesn't exist a value s such that $g(s) < (1-\lambda)s$ is completely analogous. □

Exactly as above we can prove that the shadow price process stays inside the bid-ask interval for big enough stock prices.

6 Construction of the a_t process and the verification argument

Before we get into the details of the construction of the a_t process, we have to determine the initial value of the shadow price process. At time zero, we assume the investor comes to the market with portfolio $(\varphi^0_-, \varphi^1_-)$. The investor then has to adjust this fraction instantaneously to the optimal fraction $(\varphi^0_0, \varphi^1_0)$. If he has to buy stocks to attain the optimal allocation, then $d\varphi^1_0 > 0$, and we start on the buying boundary, i.e. $\tilde{S}_0 = S_0$. Conversely, if he has to sell stock to reach the optimal allocation, then $d\varphi^1_0 < 0$, and we start on the seling boundary $\tilde{S}_0 = S_0$.

We now have to specify how a_t and b_t evolve over time (recall that a and b were essentially fixed in section 4 where we only consider one excursion from the buy boundary to the sell boundary).

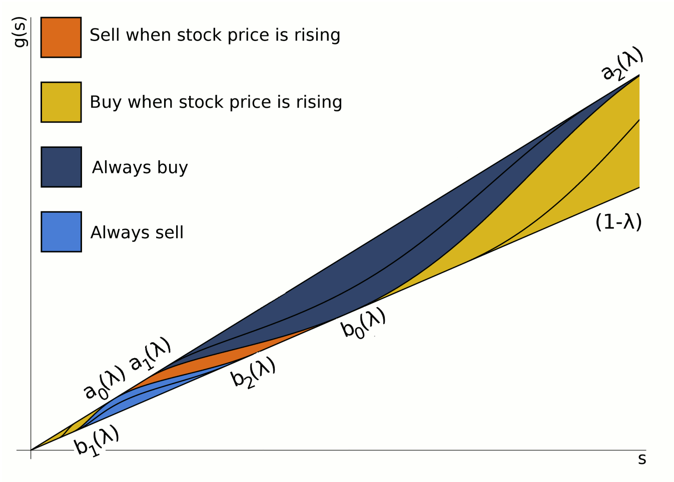


Figure 2: Here we have plotted various shadow price curves $g(S;a,b)$ as a function of s, for $\lambda = .3$ and $\kappa = 3, \sigma = 1, \bar{x} = 1$, for which we find that $a_0(\lambda) = 1.3914, b_0(\lambda) = 5.31052$ and, and $a_1(\lambda) = 1.95159, a_2(\lambda) = 10.5602$ and $b_1(\lambda) = 0.699707, b_2(\lambda) = 3.78616$.

6.1 The case when $\tilde{S}_0 = S_0$

From the construction of $g(S;a,\pi)$ and Figure 2, we see that there is really only one way to do this: if there is a valid g curve to follow (i.e. which stays within the bid/offer spread), then $\tilde{S}$ follows this curve, otherwise we need a new a or b value depending on which boundary we are on, which (locally) will be given by a minimum or maximum process of the underlying stock price, depending on whether the target b (resp. a) value on the opposite boundary is greater than or less than a (resp b). Specifically, we set $\rho_0 = 0$ and for $n \geq 1$ let

$$
\begin{aligned}
a_t &= \begin{cases} \min(a_0(\lambda), \max_{\rho_{n-1} \leq u \leq t} S_u) & S_{\rho_{n-1}} \in (0, a_0(\lambda)), \\ \max(a_0(\lambda), \min_{\rho_{n-1} \leq u \leq t} S_u) & S_{\rho_{n-1}} \in [a_0(\lambda), b^*], \\ \max_{\rho_{n-1} \leq u \leq t} S_u & S_{\rho_{n-1}} \in (b^*, \infty), \end{cases} \\
b_t &= b(a_t, \lambda)
\end{aligned}
$$

for $t \in [\rho_{n-1}, \sigma_n)$, and $\sigma_n = \inf\{t \geq \rho_{n-1} : S_t = b(a_t, \lambda)$ if $a_t \notin (a_1(\lambda), a_2(\lambda))$ or $S_t = (a^-)^{-1}(a_t)$ if $a_t \in (a_1(\lambda), b^*]$ or $S_t = a^-(a_t)$ if $a_t \in (b^*, a_2(\lambda))\}$[2] (see Theorem 5.1 for the definition of $a^-(.)$). Similarly, we define

$$
\begin{aligned}
b_t &= \begin{cases} \min_{\sigma_n \leq u \leq \rho_n} S_u & S_{\sigma_n} \in (0, a^*), \\ \min(b_0(\lambda), \max_{\sigma_n \leq u \leq \rho_n} S_u) & S_{\sigma_n} \in (a^*, b_0(\lambda), \\ \max(b_0(\lambda), \min_{\sigma_n \leq u \leq \rho_n} S_u) & S_{\sigma_n} \in (b_0(\lambda), \infty), \end{cases} \\
a_t &= a(b_t, \lambda)
\end{aligned}
$$

for $t \in [\sigma_n, \rho_n)$, and $\rho_n = \inf\{t \geq \sigma_n : S_t = a(b_t, \lambda)$ if $b_t \notin (b_1(\lambda), b_2(\lambda))$ or $S_t = b^+(b_t)$ if $b_t \in (b_1(\lambda), a^*]$ or $S_t = (b^+)^{-1}(b_t)$ if $b_t \in (b_1(\lambda), a^*)\}$.[3] The shadow price

[2] Here it is understood that there will be two values for $b(a_t, \lambda)$ if $a_t = a_0(\lambda)$ and in this case $\sigma_n$is the first hitting time to either b-value.

[3] Similar to before, there will be two target a-values if $b_t = b_0(\lambda)$.

process is then defined to be

$$\tilde{S}_t = g(S_t; a_t, b_t).$$

a_t has a.s. finite variation for $t \in [\rho_{n-1}, \sigma_n)$ and for $t \in [\sigma_n, \rho_n)$ because locally it it just a continuous function of the maximum or minimum process of a exponential Ornstein-Uhlenbeck process (which has a.s. finite variation). Moreover, a_t is a.s. continuous during excursions *and* at the stopping times σ_n because $a(b(a,\lambda),\lambda) = a$, and similarly for b_t.

Remark 6.1. *Note that the length of the special interval* $(a_1(\lambda), a_3(\lambda))$ *is* $O(\lambda^{\frac{1}{4}})$.

6.2 The case when $\tilde{S}_0 = S_0(1-\lambda)$

In this case, we essentially just swap σ and ρ: we now set $\sigma_0 = 0$ and for $n \geq 1$ let

$$\begin{aligned} b_t &= \begin{cases} \min_{\sigma_n \leq u \leq \rho_n} S_u & S_{\sigma_{n-1}} \in (0, a^*), \\ \min(b_0(\lambda), \max_{\sigma_n \leq u \leq \rho_n} S_u) & S_{\sigma_{n-1}} \in (a^*, b_0(\lambda), \\ \max(b_0(\lambda), \min_{\sigma_n \leq u \leq \rho_n} S_u) & S_{\sigma_{n-1}} \in (b_0(\lambda), \infty), \end{cases} \\ a_t &= a(b_t, \lambda) \end{aligned}$$

for $t \in [\sigma_{n-1}, \rho_n)$, and $\rho_n = \inf\{t \geq \sigma_{n-1} : S_t = a(b_t, \lambda)$ if $b_t \notin (b_1(\lambda), b_2(\lambda))$ or $S_t = b^+(b_t)$ if $b_t \in (b_1(\lambda), b_2(\lambda))\}$[4] and we set

$$\begin{aligned} a_t &= \begin{cases} \min(a_0(\lambda), \max_{\rho_{n-1} \leq u \leq t} S_u) & S_{\rho_{n-1}} \in (0, a_0(\lambda)), \\ \max(a_0(\lambda), \min_{\rho_{n-1} \leq u \leq t} S_u) & S_{\rho_{n-1}} \in [a_0(\lambda), b^*], \\ \max_{\rho_{n-1} \leq u \leq t} S_u & S_{\rho_{n-1}} \in (b^*, \infty), \end{cases} \\ b_t &= b(a_t, \lambda) \end{aligned}$$

for $t \in [\rho_n, \sigma_n)$, and $\sigma_n = \inf\{t \geq \rho_n : S_t = b(a_t, \lambda)$ if $a_t \notin (a_1(\lambda), a_2(\lambda))$ or $S_t = (a^-)^{-1}(a_t)$ if $a_t \in (a_1(\lambda), b^*]$ or $S_t = a^-(a_t)$ if $a_t \in (b^*, a_2(\lambda))\}$[5]. Similarly, we define The shadow price process is then defined to be

$$\tilde{S}_t = g(S_t; a_t, b_t). \tag{27}$$

a_t has a.s. finite variation for $t \in [\rho_{n-1}, \sigma_n)$ and for $t \in [\sigma_n, \rho_n)$ because locally it it just a continuous function of the maximum or minimum process of a exponential Ornstein-Uhlenbeck process (which has a.s. finite variation). Moreover, a_t is a.s. continuous during excursions *and* at the stopping times σ_n because $a(b(a,\lambda),\lambda) = a$, and similarly for b_t.

[4] Similar to before, there will be two target a-values if $b_t = b_0(\lambda)$

[5] Here it is understood that there will be two values for $b(a_t, \lambda)$ if $a_t = a_0(\lambda)$ and in this case $\sigma_n$is the first hitting time to either b-value.

6.3 The verification argument

In Theorem 5.3 and Lemma **??**, we have verified that b is a differentiable function of a for all $a > 0$. We also note that

$$\begin{aligned}
\bar{\pi}_a &= \frac{H(a,b)(b(\lambda-1)H(a,b) + a(a+b(\lambda-1))^2 h'(a)) - (a+b(-1+\lambda))^2 h(a)(-H(a,b) - ah(a))}{(a+b(-1+\lambda))^2 H(a,b)^2} \\
\bar{\pi}_b &= \frac{a[(-1+\lambda)H(a,b)^2 + (a+b(-1+\lambda))^2 h(a)h(b)]}{(a+b(-1+\lambda))^2 H(a,b)^2}
\end{aligned}$$

and both expressions can only blow up if $b = \frac{a}{1-\lambda}$ **(is this ever a problem when $b > a$?** From this we obtain that

$$\frac{d}{da}\bar{\pi}(a, b(a,\lambda); \lambda) \;=\; \bar{\pi}_a(a, b(a,\lambda); \lambda) \,+\, \bar{\pi}_b(a, b(a,\lambda); \lambda)\frac{db}{da}\,.$$

We know that $b(a, \lambda)$ and $\bar{\pi}(a, b(a, \lambda); \lambda)$ are both differentiable functions of a. From Ito's formula (and using that $S_t = a_t$ on the growth set of a_t), we now obtain that

$$d\tilde{S}_t \;=\; g_S(S_t, a_t, b_t)dS_t \,+\, \mu(a_t, a_t)da_t \,+\, \frac{1}{2}g_{SS}(S_t, a_t, b_t)d\langle S_t, S_t\rangle$$

where $\mu(S, a) = \frac{d}{da}g(S_t, a, b(a, \lambda))$. But

$$\frac{d}{da}g(S; a, b(a,\lambda)) \;=\; g_a(S, a, b(a,\lambda)) \,+\, g_b(S, a, b(a,\lambda))\frac{db}{da}$$

and we can easily verify that $g_a(a, a, b) = g_b(a, a, b) = 0$, so the da_t term in (28) vanishes. Now consider a self-financing trading strategy $(\varphi_t^0, \varphi_t^1)$ such that

$$\frac{\varphi_t^1 a_t}{\varphi_t^0 + \varphi_t^1 a_t} \;=\; \frac{a_t}{\frac{\varphi_t^0}{\varphi_t^1} + a_t} \;=\; \frac{1}{\frac{Z_t}{a_t} + 1} \;=\; \bar{\pi}(a_t, b(a_t,\lambda), \lambda) \;:=\; \bar{\pi}_t$$

where $Z_t = \frac{\varphi_t^0}{\varphi_t^1}$ depends only on a_t. Re-arranging, we see that $Z_t = \chi(a_t)$, where $\chi(a) = a(\frac{1}{\bar{\pi}(a,b(a,\lambda),\lambda)} - 1)$. Then we have that

$$\begin{aligned}
dZ_t \;=\; d[a_t(\frac{1}{\bar{\pi}_t} - 1)] &\;=\; (\frac{1}{\bar{\pi}_t} - 1)da_t \,+\, a_t \cdot -\frac{\pi'(a_t)}{\bar{\pi}_t^2}da_t \\
&\;=\; \frac{1}{\bar{\pi}_t^2}\bar{\Gamma}(a_t)da_t
\end{aligned} \tag{28}$$

when $\varphi_t^1 \neq 0$, where $\bar{\pi}(a) = \bar{\pi}(a, b(a,\lambda), \lambda)$ and $\bar{\Gamma}(a) = -a\bar{\pi}'(a) + \bar{\pi}(a)(1 - \bar{\pi}(a))$. From (28)we easily find that

$$dY_t \;=\; -\frac{Y_t^2}{\bar{\pi}_t^2}\bar{\Gamma}(a_t)da_t$$

when $\varphi_t^0 \neq 0$, where $Y_t = 1/Z_t = \frac{\varphi_t^1}{\varphi_t^0}$. Using the expansion in (27), we find that

$$\lim_{a\uparrow a_0(\lambda)}\bar{\Gamma}(a) \;=\; -\lim_{a\downarrow a_0(\lambda)}\bar{\Gamma}(a) \;=\; \frac{\sqrt{2}\,\kappa^{\frac{3}{4}}(4\kappa+\sigma^2)^{\frac{3}{8}}}{3^{\frac{3}{4}}\sigma^{\frac{9}{4}}}\lambda^{\frac{1}{4}}\,,$$

so for $\lambda \ll 1$ ($\lambda > 0$), $\bar{\Gamma}(a)$ has a discontinuity at $a_0(\lambda)$.

Deal with the other point where $\bar{\Gamma}$ flips sign.

$\bar{\pi}'$ not defined at $a_0(\lambda)$?

Z_t changes if and only if φ_t^1 and φ_t^0 change, because $(\varphi_t^0, \varphi_t^1)$ is self-financing, and from (28), we see that trading only occurs when a_t changes, i.e. if we are on the buying or selling boundary of the bid/ask cone. But from the construction of a_t we know that on $\{\tilde{S}_t = S_t\}$ (i.e. on the selling boundary) we have

- $da_t > 0$ for $a_t \in (0, a_0(\lambda)$.
- $da_t = 0$ when $a_t = a_0(\lambda)$, so $d\varphi_t^1 = 0$ when $a_t = a_0(\lambda)$.
- $da_t < 0$ for $a_t \in (a_0(\lambda), a_1(\lambda))$
- $da_t > 0$ for $a_t \in (a_2(\lambda), \infty)$.

However, it turns out that we always have $dZ_t \leq 0$ on $\{\tilde{S}_t = S_t\}$, which is required for $\tilde{S}_t$ to be a true shadow price process, because $\bar{\Gamma}(a)$ also flips sign at $a = a_0(\lambda)$ (see subsection 5.4) for details).

If we now compute the risky fraction

$$\frac{\varphi_t^1 \tilde{S}_t}{\varphi_t^0 + \varphi_t^1 \tilde{S}_t}$$

then using that a_t is constant when $\tilde{S}$ is away from the bid/ask cone, we find that this agrees with the Merton fraction for the conjectured shadow price process, iff $g(S; a, b)$ satisfies the ODE in (7).

From this we obtain

$$\begin{aligned} d\varphi_t^1 &= \frac{1}{Z_t} d\varphi_t^0 - \frac{\varphi_t^0}{Z_t^2} dZ_t \\ &= -\frac{\varphi_t^1}{\varphi_t^0} a_t d\varphi_t^1 - \frac{\varphi_t^1}{Z_t} \frac{1}{\bar{\pi}_t^2} \bar{\Gamma}(a_t) da_t \\ &= -\frac{a_t}{\chi(a_t)} d\varphi_t^1 - \frac{\varphi_t^1}{Z_t} \frac{1}{\bar{\pi}_t^2} \bar{\Gamma}(a_t) da_t \,. \end{aligned} \tag{29}$$

Re-arranging, we see that

$$d\varphi_t^1 = -\frac{1}{1 + \frac{a_t}{\chi(a_t)}} \frac{\varphi_t^1}{\bar{\pi}_t^2} \bar{\Gamma}(a_t) da_t \,.$$

but the answer should be ...

$$\frac{d\varphi_t^1}{da_t} = \varphi_t^1 \left(\frac{\bar{\pi}' a_t - \bar{\pi}(1 - \bar{\pi})}{\bar{\pi} a_t} \right) . \tag{30}$$

$$
\begin{aligned}
\overline{\pi}' da_t = d\overline{\pi} = d\left(\frac{\varphi_t^1 a_t}{\varphi_t^0 + \varphi_t^1 a_t}\right) \\
&= \frac{1}{(\varphi_t^0 + \varphi_t^1)^2}\left(\varphi_t^0\varphi_t^1 da_t + a_t(\varphi_t^0 d\varphi_t^1 - \varphi_t^1 d\varphi_t^0)\right) \\
&= \frac{\overline{\pi}}{a_t}(1-\overline{\pi})da_t + \frac{\overline{\pi}}{\varphi_t^1} d\varphi_t^1.
\end{aligned}
$$

which leads to

$$
d\varphi_t^1 \quad = \quad \varphi_t^1 \frac{a_t\bar{\pi}'(a_t) - \bar{\pi}_t(1-\bar{\pi}_t)}{\bar{\pi}_t a_t} dt \quad = \quad -\frac{\varphi_t^1}{\bar{\pi}_t a_t}\,\bar{\Gamma}(a_t;\lambda)dt
$$

where $\bar{\pi}'(a) = \frac{d}{da}\bar{\pi}(a, b(a,\lambda),\lambda)$.

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A Appendix A

For $\lambda \in [0,1)$, and $s, a \in (0,\infty)$, we consider function

$$k(s;a,\lambda) = -\sqrt{1-\lambda}\ H(s,a) + (a + s(\lambda-1))\sqrt{h(a)h(s)}. \tag{31}$$

We list the following calculation about the derivative $k_s(s;a,\lambda)$ for future reference: for $s \le a$,

$$\begin{aligned} k_s(s;a,\lambda) =& \sqrt{1-\lambda}\ h(s) - (1-\lambda)\sqrt{h(a)h(s)} + (a+s(\lambda-1))\frac{1}{2}\sqrt{\frac{h(a)}{h(s)}}h'(s) \\ =& h(s)\left(\sqrt{1-\lambda} - (1-\lambda)\sqrt{\frac{h(a)}{h(s)}} - (a+s(\lambda-1))\frac{1}{2}\sqrt{\frac{h(a)}{h(s)}}f(s)\right) \end{aligned} \tag{32}$$

where

$$f(s) = -\frac{h'(s)}{h(s)} = \frac{2\kappa}{\sigma^2}(\bar{x} + \frac{\sigma^2}{2\kappa} - \log s)\frac{1}{s} = \frac{2}{s}\theta(s). \tag{33}$$

Further, define $w(s;a) := \frac{k_s(s;a,\lambda)}{h(s)}$. Then we have that

$$
\begin{aligned}
w_s(s;a,\lambda) =& -\frac{1}{2}\sqrt{\frac{h(a)}{h(s)}}\frac{h'(s)}{h(s)}\Big(-(1-\lambda)-(a+s(\lambda-1))\frac{f(s)}{2}\Big)+\frac{(1-\lambda)}{2}\sqrt{\frac{h(a)}{h(s)}}f(s)\\
&-(a+s(\lambda-1))\frac{1}{2}\sqrt{\frac{h(a)}{h(s)}}f'(s)\\
=&\frac{1}{2}\sqrt{\frac{h(a)}{h(s)}}\Big((\lambda-1)f(s)-(a+s(\lambda-1))\frac{f^2(s)}{2}+(1-\lambda)f(s)-(a+s(\lambda-1))f'(s)\Big)\\
=&-\frac{1}{2}\sqrt{\frac{h(a)}{h(s)}}(a+s(\lambda-1))\Big(\frac{f^2(s)}{2}+f'(s)\Big)\\
=&\sqrt{\frac{h(a)}{h(s)}}(a+s(\lambda-1))\frac{\Gamma(s)}{s^2},
\end{aligned}
\tag{34}
$$

where the last equality follows from (33).

Similarly, for $s > \frac{a}{1-\lambda}$, we have

$$
k_s(s;a,\lambda) = h(s)\Big(\sqrt{1-\lambda}-(1-\lambda)\sqrt{\frac{h(a)}{h(s)}}-(a+s(\lambda-1))\sqrt{\frac{h(a)}{h(s)}}\frac{\theta(s)}{s}\Big), \tag{35}
$$

$$
w_s(s;a,\lambda) = -\frac{1}{s^2}\sqrt{\frac{h(a)}{h(s)}}(s(1-\lambda)-a)\Gamma(s), \tag{36}
$$

where $w(s;a,\lambda) := \frac{k_s(s;a,\lambda)}{h(s)}$.

Below we prove a uniqueness result that the bid price $b = b(a)$ that is less than the ask price a. More specifically, we donate by $s_0 := \exp(-\frac{\sqrt{1-\lambda}-(1-\lambda)}{\lambda\kappa/\sigma^2}+\bar{x}+\frac{\sigma^2}{2\kappa}) \in (e^{\bar{x}}, e^{\bar{x}+\frac{\sigma^2}{2\kappa}}) = (\sqrt{a^*b^*}, b^*)$. It can be easily verified that s_0 is the unique solution to the equation

$$
\sqrt{1-\lambda}-(1-\lambda)-\lambda\theta(s) = 0.
$$

Lemma A.1. *For any $\lambda \in (0,1)$ and $a \le s_0$, there exists a unique solution to $k(s;a,\lambda) = 0$ over $(0,a)$. Denoting this solution by $b = b(a)$, then $b = b(a)$ is differentiable.*

Proof. The existence of one solution to the equation $k(s;a,\lambda) = 0$ over $(0,a)$ is ensured by Lemma 4.1. Below we prove the uniqueness. To this end, we notice from (32) that

$$
w(a;a,\lambda) \quad = \quad \sqrt{1-\lambda}-(1-\lambda)-\lambda\theta(a) \quad \le \quad 0, \qquad \forall a \le s_0. \tag{37}
$$

On the other hand, from (34), for all fixed $a \le s_0 < b^*$, by considering the two cases $a < a^*$ and $a \ge a^*$ separately and recalling that $\Gamma(s)$ switches sign at $s = a^*$, we see that the function $w(\cdot;a,\lambda)$ is strictly decreasing over $(0, a^*\wedge a)$ and strictly increasing over $(a^*\wedge a, a)$. Thus, there exists at most one root $s = \hat{s}$ to $w(s,a,\lambda) = 0$ over $(0,a)$.

Equivalently, there exists at most one root to $k_s(s;a,\lambda)=0$ over $(0,a)$. We claim that there is at most one solution to $k(s;a,\lambda)=0$ over $(0,a)$. Indeed, if there were more than one solution, then since $\lim_{s\downarrow 0} k(s;a,\lambda)<0$ (from the proof of Lemma 4.1) and $k(a;a,\lambda)=\lambda a h(a)>0$, there must be at least three distinct solutions to $k(s;a,\lambda)=0$ over $(0,a)$. But if this were so, the mean value theorem would imply that there are at least two distinct solutions to $k_s(s;a,\lambda)=-\infty$ over $(0,a)$. This is a contradiction, hence we have the uniqueness of $b=b(a)<a$. Moreover, given that $\lim_{s\downarrow 0} k(s;a,\lambda)<0$, and $k(.;a,\lambda)$ has a unique turning point at some $\hat{s}\in(0,a)$ which is not an inflection point, and $k(a;a,\lambda)>0$, we must have that $k(.;a,\lambda)$ has a unique root $b(a)\in(0,\hat{s})$.

To prove that $b(a)$ is differentiable in a, we first show that $k_s(b;a,\lambda)>0$. Indeed, by construction we know that $k_s(b;a,\lambda)\geq 0$. Suppose that $k_s(b;a,\lambda)=0$, then from the previous discussion we know that $k_s(s;a,\lambda)\neq 0$ for all $s\in(b,a)$. Further, recall that $k_s(a;a,\lambda)=w(a;a,\lambda)h(a)\leq 0$, hence we must have $k_s(s;a,\lambda)<0$ for all $s\in(b,a)$. However, this will imply that $\lambda a h(a)=k(a;a,\lambda)<k(b;a,\lambda)=0$, a contradiction. We thus have $k_s(b;a,\lambda)>0$ for all $a\leq s_0$. Thus for any $\bar{a}\in(0,s_0)$, $k(b(a),a,\lambda)=0$ and $k_s(b(a),a,\lambda)\neq 0$, so the implicit function theorem implies that there exists a differentiable function $b=b(a;\bar{a})$ in a small neighborhood of $\bar{a}$, and that

$$\frac{db(a;\bar{a})}{da}=-\frac{k_a(b;a,\lambda)}{k_s(b;a,\lambda)}.$$

This implies the differentiability of $b=b(a)$ over $(0,s_0)$, as the uniqueness we proved earlier ensures that $b(a)=b(a,\bar{a})$, for all $\bar{a}\in(0,s_0)$. □

Lemma A.2. *For any $\lambda\in[0,\frac{1}{2})$ and $a>b^*$ sufficiently large, there exists a unique solution to $k(s;a,\lambda)=0$ over (b^*,a). Denoting this solution by $b=b(a,\lambda)$, then $b(a,\lambda)$ is differentiable in a,λ and that $b(a,0+)=a$.*

Proof. The argument is similar as the one in Lemma A.1. We know that for any fixed $a>b^*\geq s_0(\lambda)=\exp(-\frac{\sqrt{1-\lambda}-(1-\lambda)}{\lambda\kappa/\sigma^2}+\bar{x}+\frac{\sigma^2}{2\kappa})=\exp(\bar{x}+\frac{\sigma^2}{2\kappa}\frac{1-\sqrt{1-\lambda}}{1+\sqrt{1-\lambda}})$, we have that $w(a;a,\lambda)=\sqrt{1-\lambda}-(1-\lambda)-\lambda\theta(a)\geq 0$. Furthermore, we have $w_s(s;a,\lambda)<0$ for all $s\in[b^*,a)$, thus $w(s;a,\lambda)>0$ for all $s\in[b^*,a)$. Thus, the function $k(\cdot;a,\lambda)$ is strictly increasing over $[b^*,a)$. Recall that $k(a;a,\lambda)=\lambda a h(a)\geq 0$, there is a unique solution to $k(s;a,\lambda)=0$ over $[b^*,a]$ if and only if $k(b^*;a,\lambda)<0$. The latter inequality is equivalent to

$$-\sqrt{1-\lambda}H(b^*,a)+(a+b^*(\lambda-1))\sqrt{h(a)h(b^*)}<0. \tag{38}$$

However, notice that for all $\lambda\in[0,\frac{1}{2}]$, we have

$$\begin{aligned}&\frac{-\sqrt{1-\lambda}H(b^*,a)+(a+b^*(\lambda-1))\sqrt{h(a)h(b^*)}}{H(b^*,a)}\\ &\leq -\frac{1}{2}+\frac{a\sqrt{h(a)h(b^*)}}{H(b^*,a)}\\ &=-\frac{1}{2}+\frac{a\sqrt{h(a)h(b^*)}}{H(b^*,a)}\to-\frac{1}{2},\end{aligned}$$

as $a \to \infty$ by L'Hospital's rule. Thus, (38) holds and the claim about the uniqueness of $b(a, \lambda)$ is proved. The differentiability of $b(a, \lambda)$ in a and λ follows from the implicit function theorem and the fact that $k_s(b; a, \lambda) > 0$ for $b = b(a, \lambda)$.

Finally, as $\lambda \downarrow 0$, we have that $k(a; a, \lambda) \to 0$. Thus the uniqueness of $b(a, \lambda)$ over $[b^*, a]$ implies that $b(a, \lambda) \to a$ as $\lambda \downarrow 0$. □

Lemma A.3. *For any $e^{\frac{\sigma^2}{2\kappa}} b^* > a_2 > b^*$, we have exactly one root to $k(s; a_2, 0) = 0$ over (a^*, a_2). Denoting this root by $a_1(a_2)$, then $a_1(a_2) \in (a^*, b^*)$, $a_1(\cdot)$ is smooth, monotonically decreasing, $\lim_{a_2 \downarrow b^*} a_1(a_2) = b^*$ and $\lim_{a_2 \downarrow b^*} a_1'(a_2) = -1$..*

Proof. For any $a_2 > b^*$, we know from (34) that $w_s(s; a, 0) < 0$ for all $s \in (b^*, a)$, so the function $w(\cdot; a, 0)$ is strictly decreasing over $[b^*, a)$. It follows that $w(s; a, 0) > w(a; a, 0) = 0$ for all $s \in (b^*, a)$, and hence $k_s(s; a, 0) > 0$, i.e. $k(\cdot; a, 0)$ is strictly increasing over $s \in [b^*, a)$. But the latter implies that $k(s; a, 0) < 0$ for all $s \in [b^*, a)$. To prove the existence the root to to $k(s; a_2, 0) = 0$ over (a^*, b^*), we notice that for any $a_2 \in (a^*, b^*)$, $a' := \exp(2(\bar{x} + \frac{\sigma^2}{2\kappa}))/a_2 \in (a^*, e^{\bar{x} + \frac{\sigma^2}{2\kappa}}) \subset (a^*, b^*)$ and that $h(a') = h(a_2)$. Since $H(a', a_2)$ is the integral of the function h, which is convex, we have

$$H(a', a_2) < (a_2 - a')h(a_2),$$

which is equivalent to $k(a'; a_2, 0) > 0$. It follows that there must be a solution to $k(s; a_2, \lambda) = 0$ over $(a', b^*) \subset (a^*, b^*)$.

To prove the uniqueness of the root, we use (34) to conclude that $w(\cdot; a, 0)$ is monotonically decreasing over (a^*, b^*), and hence there is at most one root to $w(s; a, 0) = 0$ over this interval. Using the argument as above, we know that the latter implies the uniqueness of the root to $k(s; a, 0) = 0$ over (a^*, b^*).

Denoting this root by a_1, then from $k(a_1; a_2, 0) = 0 > k(a_1-; a_2, 0)$ we know that $k_s(a_1; a_2, 0) < 0$. On the other hand, notice that $k_a(a_1; a_2, 0)/h(a_2) = -w(a_2; a_1, 0)$. From

$$-\frac{\partial}{\partial a} w(a; a_1, 0) = \frac{1}{a^2} \sqrt{\frac{h(a_1)}{h(a)}} (a - a_1)\Gamma(a), \forall a \geq a_1 \in (a^*, b^*),$$

we know that the function $-w(\cdot; a_1, 0)$ is increasing over (a_1, b^*) and then decreasing over (b^*, ∞). Given that $-w(a_1; a_1, 0) = 0$ and $\lim_{a \to \infty}[-w(a; a_1, 0)] = -1 < 0$, we have $-w(a; a_1, 0) > 0$ for all $a \in (a_1, b^*]$, and there is exactly one solution to $-w(a; a_1, 0) = 0$ over (b^*, ∞). Denoting this unique root by a'', then for all $a \in (a_1, a'')$, we have $-w(a; a_1, 0) > 0$ and hence $k_a(a_1; a, 0) < 0$. Because $k(a_1; a_1, 0) = 0$, we have that $k(a_1; a, 0) < 0$ for all $a \in (a_1, a'')$, which implies that $a_2 > a''$. For all $a \in (a'', \infty)$, we have $-w(a; a_1, 0) < 0$ and hence $k_a(a_1; a, 0) > 0$. In particular, $k_a(a_1; a_2, 0) < 0$. By the implicit function theorem, we have $a_1(\cdot)$ is differentiable and

$$\frac{da_1(a_2)}{da_2} = -\frac{k_a(a_1(a_2); a_2, 0)}{k_s(a_1(a_2); a_2, 0)} < 0.$$

Finally, using similar argument as above, we can show that $k_s(s; b^*, 0) < 0$ for all $s \in (a^*, b^*)$ and thus, $k(s; b^*, 0) > 0$ for all $s \in (a^*, b^*)$. This implies that $\lim_{a_2 \downarrow b^*} a_1(a_2) = b^*$, in particular, $\lim_{a_2 \downarrow b^*} |a_2 - a_1(a_2)| = 0$ and $\lim_{a_2 \downarrow b^*} a_1'(a_2) = -\lim_{a_2 \downarrow b^*} \frac{k_a(a_1(a_2); a_2, 0)}{k_s(a_1(a_2); a_2, 0)} = -\lim_{a_2 \downarrow b^*} \frac{w(a_1(a_2); a_2, 0)}{w(a_2; a_1(a_2), 0)}$. By L'Hospital's rule, we have

$$\lim_{s \to a} \frac{w(s; a, 0)}{w(a; s, 0)} = 1, \forall a, s > 0, a \neq s.$$

Thus, $\lim_{a_2 \downarrow b^*} a_1'(a_2) = -1$. $\square$

B Approximate behaviour of the shadow price curves for $\lambda \ll 1$

B.1 The behaviour of the shadow price function starting at the buy boundary for $\lambda \ll 1$

B.1.1 The general case $a \neq a^*$

Consider any $a \neq a^*$. Performing a Taylor series expansion of $g(s, a, \overline{\pi}) - s$ around $(s = a, \pi = \theta(a))$ to third order, we find that

$$g(s, a, \overline{\pi}) - s \;\; = \;\; \frac{1}{3!} g_{sss}(s-a)^3 \; + \; \frac{1}{2!} g_{ss\bar{\pi}}(s-a)^2(\bar{\pi} - \theta(a)) \; + \; h.o.t.$$

$$g(s, a, \overline{\pi}) - s \;\; = \;\; \frac{1}{3!} \psi^{(sss)}(s)(s-a)^3 \; + \; \frac{1}{2!} \psi^{(ss\bar{\pi})}(s-a)^2(\bar{\pi} - \theta(a)) \; + \; h.o.t.$$

where the partial derivatives are evaluated at $(a, a, \theta(a))$, and all other partial derivatives vanish. Setting $\varepsilon = \lambda^3$ using (22) we know that

$$b \;\; = \;\; a\,[1 \; + \; (\frac{6}{\Gamma(a)})^{\frac{1}{3}} \varepsilon + O(\varepsilon^2)]. \tag{B-1}$$

Now let $\alpha \in [0, 1]$ and set

$$s \;\; = \;\; a + \alpha(b - a)\,. \tag{B-2}$$

Substituting (B-1) into (B-2), and then substituting s and the expansion $\bar{\pi} = \theta(a) - (\frac{3}{4}\Gamma(a)^2)^{\frac{1}{3}}\varepsilon + O(\varepsilon^2)$ (from (20)) into (B-1), we obtain

$$g(s, a, \overline{\pi}) - s \;\; = \;\; a\alpha^2(2\alpha - 3)\varepsilon^3 \; + \; h.o.t.$$

Thus all g curves are essentially the same for λ sufficiently small, and just characterized by the cubic polynomial

$$P(\alpha) \;\; = \;\; \alpha^2(2\alpha - 3)$$

in the dimensionless parameter α, and (surprisingly) is independent of all parameters (numerics confirm this). From elementary calculations we find that $P(0) = 0$, $P(1) = -1$ and $P'(0) = P'(1) = 0$ (see Figure 2).

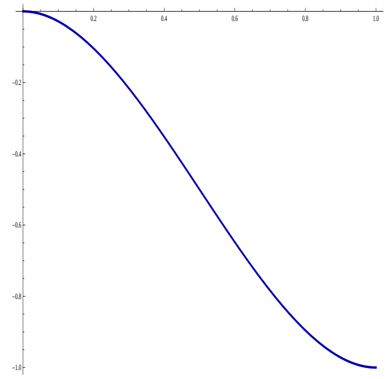

Figure 3: Here we have plotted $P(\alpha)$.

B.1.2 Proving that curves don't cross

Transforming back from α to S, we find that

$$g(s,a,\overline{\pi}) - s \;\; = \;\; -\frac{1}{a}\Big(\frac{3\Gamma(a)^2}{4}\Big)^{\frac{1}{3}}\varepsilon(s-a)^2 \,+\, h.o.t.$$

We can view $g(s,a,\overline{\pi})-s$ as the original g curve rotated by 45 degrees clockwise. We now wish to prove that g curves don't cross each other for λ suff small. Differentiating (B-3) with respect to a, we find that From this we find that

$$\begin{aligned}\frac{\partial}{\partial a}[g(s,a,\overline{\pi}) - s] \;\; &= \;\; \varepsilon(s-a)\,\frac{3(a+s)\Gamma(a) - 2a(s-a)\Gamma'(a)}{6^{\frac{2}{3}}a^2\Gamma(a)^{\frac{1}{3}}} \,+\, h.o.t. \\ &= \;\; [\frac{3}{4}\Gamma(a)]^{\frac{1}{3}}\,\frac{2}{a}(s-a) > 0 \end{aligned} \tag{B-3}$$

which has sign equal to sign of $\Gamma(a)$ for $a \in (a_0(0), a_1(0))$ (when $s > a$ which is relevant in this region). Thus curves cannot cross for λ suff small.

B.1.3 The special case $a = a^*$

Now let $a = a^*$. Performing a Taylor series expansion of $g(s,a,\overline{\pi}) - s$ around $(a,a,\theta(a))$ to fourth order, we find that

$$g(s,a,\overline{\pi}) - s \;\; = \;\; \frac{1}{4!}g_{ssss}(s-a)^4 \,+\, \frac{1}{2!}g_{ss\overline{\pi}}(s-a)^2(\overline{\pi} - \theta(a)) \,+\, h.o.t. \tag{B-4}$$

where the partial derivatives are evaluated at $(a,a,\theta(a))$, and all other partial derivatives vanish. Set $\varepsilon = \lambda^4$. Then we know that

$$b \;\; = \;\; a \pm a\sqrt{2}\,[\frac{3\sigma^4}{\kappa\theta\sqrt{4\kappa+\sigma^2}}]^{\frac{1}{2}}\,\varepsilon + O(\varepsilon^2)]. \tag{B-5}$$

Now let $\alpha \in [-1,1]$ and set

$$s \;\; = \;\; a + \alpha(b-a)\,. \tag{B-6}$$

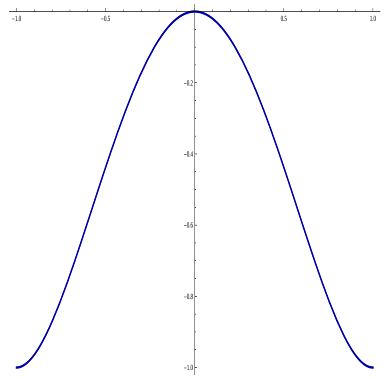

Figure 4: Here we have plotted $Q(\alpha)$.

Substituting (B-5) into (B-6), and then substituting s and our expansion $\bar{\pi} = \theta(a) - \sqrt{\frac{\kappa\theta\sqrt{4\kappa+\sigma^2}}{3\sigma^4}}\,\varepsilon^2 + O(\varepsilon^3)$ into (B-4), we obtain

$$g(s, a, \overline{\pi}) - s \;\; = \;\; a\alpha^2(\alpha^2 - 2)\varepsilon^4 \; + \; h.o.t.$$

Thus in this special case, the g curve is just parametrized by the quartic

$$Q(\alpha) \;\; = \;\; \alpha^2(\alpha^2 - 2)$$

in the dimensionless parameter α, and (surprisingly) is independent of all parameters (numerics confirm this). From elementary calculations we find that $Q(0) = 0$, $Q(1) = -1$ and $Q'(0) = Q'''(0) = Q'(1) = 0$ (see Figure 3).

B.2 The behaviour of the shadow price function starting at the sell boundary for $\lambda \ll 1$

B.2.1 The case $b \neq b^*$

Consider any $b \neq b^*$. Performing a Taylor series expansion of $g(s, b, \overline{\pi}) - s$ around $(b, b, \theta(b))$ to third order, we find that

$$g(s, b, \underline{\pi}) - s \;\; = \;\; \frac{1}{3!} g_{sss}(s-b)^3 \; + \; \frac{1}{2!} g_{ss\underline{\pi}}(s-b)^2(\underline{\pi} - \theta(b)) \; + \; h.o.t. \qquad \text{(B-7)}$$

where the partial derivatives are evaluated at $(a, a, \theta(a))$, and all other partial derivatives vanish. Set $\varepsilon = \lambda^3$. Then we know that

$$a \;\; = \;\; b\,[1 \; - \; (\frac{6}{\Gamma(b)})^{\frac{1}{3}}\varepsilon + O(\varepsilon^2)]. \qquad \text{(B-8)}$$

Now let $\alpha \in [0, 1]$ and set

$$s \;\; = \;\; b + \alpha(a - b)\,. \qquad \text{(B-9)}$$

Substituting (B-8) into (B-9), and then substituting s and our expansion $\underline{\pi} = \theta(a) + \left(\frac{3}{4}\,\Gamma(b)^2\lambda\right)^{\frac{1}{3}} + O(\lambda^{\frac{2}{3}})$ into (B-7), we obtain

$$g(s, b, \underline{\pi}) - s \quad = \quad (1 - \lambda)b\alpha^2(2\alpha - 3)\varepsilon^3 \, + \, h.o.t.$$

Thus all the g curves are essentially the same for λ sufficiently small, characterized by the cubic polynomial:

$$P(\alpha) \quad = \quad \alpha^2(2\alpha - 3)$$

in the dimensionless parameter α, is independent of all parameters. From elementary calculations we find that $P(0) = 0$, $P(1) = -1$ and $P'(0) = P'(1) = 0$ (see graph below).