Recent topics in Financial Mathematics

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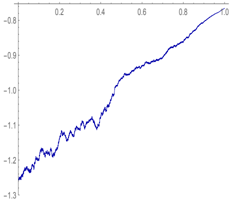
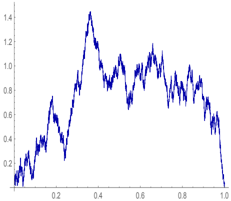
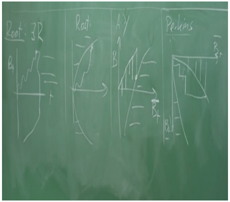






Contents

- The **Robust hedging** problem with tradeable **barrier options**.
- **Portfolio optimization** - linear and non-linear **price impact** under exponential utility.
- Portfolio optimization under **proportional transaction costs** with log utility and a jump-to-default, using **shadow prices**.
- **Fractional stochastic volatility models** - basic properties of fBM, large deviations, application to small-time asymptotics for a stochastic volatility model driven by fBM; extensions using **rough paths theory**.
- **Optimal order execution** for an **Almgren&Chriss**-type model and the Bayraktar-Ludkovski power-law **limit order book** model with stochastic liquidity.

The Skorokhod embedding problem and robust hedging

- Compute $\sup_{\tau\in\mathcal{T}(\mu)}\mathbb{E}(\Phi(B_{\tau\wedge .},\tau))$, where Φ is some path-dependent functional of B and $\mathcal{T}(\mu)$ is the set of all stopping times for which $B_\tau\sim\mu$ with $\int\mu(dx)dx=0$ and $\int|x|\mu(dx)<\infty$ and the stopped process $(B_{\tau\wedge t})_{t\geq 0}$ is uniformly integrable.
- **Azéma-Yor/Vallois**: $\Phi=1_{\bar{B}_\tau>x}(1_{L_\tau>x})$, $\tau=\inf\{t:B_t\leq K^*_\mu(\bar{B}_t)\}$ $(\tau=\inf\{t:B_t\notin(\phi^-(L_t),\phi^+(L_t))\})$, proved using excursion theory.

- **Perkins**: $\Phi=1_{\bar{B}_\tau<x}$, $\tau=\inf\{t:B_t\notin(-\gamma^+(\bar{B}_t),\gamma^-(-\underline{B}_t))\}$.
- **Root/Rost**: $\Phi=\mp(\tau-K)^+$, $\tau=\inf\{t:\mathrm{sgn}(R^\pm(B_t)-t)=\pm1\}$.
- **[BHR01]**: $\sup_{\tau_1\in\mathcal{T}(\mu_1)\leq\tau_2\in\mathcal{T}(\mu_2)}\mathbb{P}(\bar{B}_{\tau_2}>y)$, then $\tau_1=\tau_2=\tau^{AY_{\mu_1}}$ if $\xi_2(\bar{B}_{\tau_1})\geq K^*_{\mu_1}(y)(\bar{B}_{\tau_1})$, else $\tau_1=\tau^{AY_{\mu_1}}$, $\tau_2=\inf\{t:B_t\leq\xi_2(\bar{B}_t)\}$, where $\xi_2(y)=\mathrm{argmin}_{\zeta_2\leq y}[\frac{c_2(\zeta_2)}{y-\zeta_2}-1_{\zeta_2>K^*_{\mu_1}(y)}(\frac{c_1(\zeta_2)}{y-\zeta_2}-\frac{c_1(K^*_{\mu_1}(y))}{y-K^*_{\mu_1}(y)})]$.
- **[HK13]**: Minimize $\mathbb{E}(|X_{T_2}-X_{T_1}|)$ over all martingales X: $X_{T_1}\sim\mu$, $X_{T_2}\sim\nu$, $T_1<T_2$. Optimal joint law is such that $X_{T_2}|X_{T_1}$ has a trinomial distribution with points $p(X_{T_1}),X_{T_1},q(X_{T_1})$ for some functions p,q.

- In **[FK15]**, we consider: $P(\mu):=\sup_{\tau:(B_\tau,\underline{B}_\tau)\sim\mu}\mathbb{E}(\Phi)$ and $\sup_{\tau:B_\tau\sim\mu,\underline{B}_\tau\sim\nu}\mathbb{E}(\Phi)$; we adapt existing results in [GTT15] + use the **Rogers** necc+suff. condition on μ; we work under Wasserstein topology $\mathcal{W}^1$ on space of admissible μ's (instead of **peacocks** as in [GTT15]).

- Usual problem: compute **minimal superhedging cost** for an **path-dependent option** payoff $\Phi(X.)$ on an (unspecified) continuous martingale stock price process X, when we have observed tradeable European call options at all strikes K with a single fixed maturity T at $t = 0$, and we can dynamically trade X. We adapt this problem to compute minimal superhedging cost $D(\mu)$ when we can also trade **barrier options**, and we prove that $D(\mu) = P(\mu)$.
- Recall the **DDS time-change** result: any continuous martingale X can be written as time-changed Brownian motion $X_t = B_{\langle X \rangle_t}$ for some Brownian motion B and $B_t = X_{A_t}$ where $A_t = \inf\{s : \langle X \rangle_s > t\}$.
- The **duality result** shows that $P(\mu)$ equals

$$\sup_{\mathbb{P} \in \mathcal{M}_\mu} \mathbb{E}_{\mathbb{P}}[\Phi(X)]$$

$$= \inf_{\lambda \in \Lambda} \{ \int \lambda(x,y) d\mu(x,y) | \exists \gamma : \int_0^T \gamma_t dX_t + \lambda(X_T, \underline{X}_T) \geq \Phi(X)\, a.s. \forall \mathbb{P} \in \mathcal{M}$$

$\mathcal{M}_\mu$ is the set of all $\mathbb{P}$: X is a cts martingale and $(X_T, \underline{X}_T) \sim \mu$, where μ is the measure implied by observed barrier option prices. Use **Prokhorov**'s thm and **bi-conjugate** theorem applied to $P(\mu)$ under $\mathcal{W}^1$ to prove $P(\mu) = \inf_{\lambda \in \Lambda} \sup_\tau \mathbb{E}[\Phi - \lambda(B_\tau, \underline{B}_\tau) + \mu(\lambda)]$. Then **Doob-Meyer** to the associated **Snell envelope**, **MRT** to construct γ_t.

Portfolio optimization with transaction costs - linear and non-linear price impact

- Consider a market with a safe asset earning zero interest and a risky asset whose best quoted price S_t satisfies GBM:

$$dS_t = S_t(\mu dt + \sigma dW_t)\,.$$

- Unlike a frictionless market, trades in the risky asset are not realized at the best quote S_t, but rather at a less favourable price which effectively penalizes the trader for making large trades in a short period of time. More precisely, we assume that the average price for trading is

$$\tilde{S}_t := S_t(1 + \lambda S_t \dot{\theta}_t)$$

where θ_t is the number of shares held at time t, which we assume is differentiable in t. Let C_t denote the cash position, which must evolve as

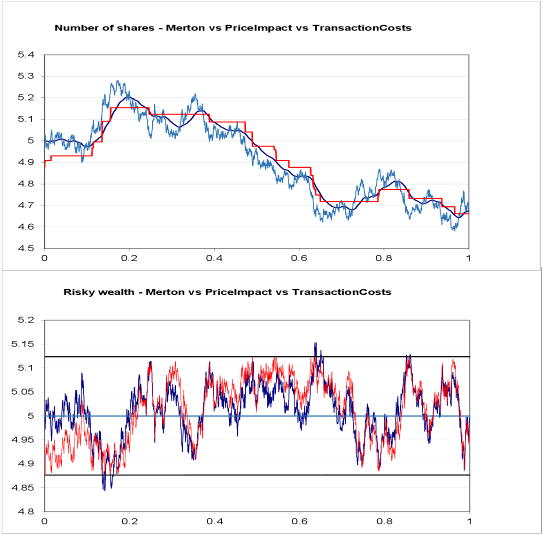
$$dC_t = -\tilde{S}_t d\theta_t\,.$$

- We now let $u_t = \dot{\theta}_t S_t$, $X_t = \theta_t S_t + C_t$ denote the total wealth, and $Y_t = \theta_t S_t$ the **risky wealth**.

- We assume the investor has **exponential utility** and a long-time horizon, and thus trades to maximize $\liminf_{T\to\infty} -\frac{1}{\alpha T}\log \mathbb{E}(e^{-\alpha X_T})$.
- This is a **stochastic control** problem - we are looking for the optimal u_t process. From standard stochastic control arguments, for any **admissible control** u_t, $V(t, X_t, Y_t)$ is a **supermartingale** and is a martingale for the **optimal control** $\hat{u}_t$, and applying Ito's lemma to $V(t, X_t, Y_t)$ +setting drift=0, $V(t,x,y)$ satisfies a **HJB eq**. We then substitute the ansatz $V(t,x,y) = -e^{-\alpha x}e^{\alpha\beta t}e^{\alpha\int_0^y q(\zeta)d\zeta}$.
- In [FWZ15], we find that the **optimal trading policy** is

$$\hat{u}_t \;=\; \hat{u}(Y_t) \;\sim\; -\frac{\alpha\sigma^2(Y_t-\bar{Y})}{\sigma\sqrt{2\alpha}}\frac{1}{\sqrt{\lambda}} \qquad (\lambda\to 0)$$

where $\bar{Y}=\mu/(\alpha\sigma^2)$ is the **frictionless target**, i.e. the optimal Y-value when $\lambda=0$ (which is constant), and $\beta(\lambda)=\frac{\mu^2}{2\alpha\sigma^2}-c_1\sqrt{\lambda}+o(\sqrt{\lambda})$. We expect $\hat{u}(y)$ to blow up as $\lambda\to 0$, because in the frictionless case $Y_t=\theta_t S_t=\bar{Y}$ so $\theta_t=\frac{\bar{Y}}{S_t}$, which clearly is not differentiable in t a.s.
- We later extend to the **non-linear price impact** : $\tilde{S}_t = S_t(1+\lambda|S_t\dot{\theta}_t|^{\gamma}\mathrm{sgn}(\dot{\theta}))$. $c_1(\gamma)$ is now determined (numerically) as the unique value for which the solution $s(w)$ to the non-linear ODE: $-c+w^2-|s(w)|^{1+\frac{1}{\gamma}}+s'(w)=0$ satisfies $s(w)\sim|w|^{\frac{2\gamma}{1+\gamma}}$ as $|w|\to\infty$.


Number of shares - Merton vs PriceImpact vs TransactionCosts
5.4
5.3
5.2
5.1
5
4.9
4.8
4.7
4.6
4.5
0
0.2
0.4
0.6
0.8
1
Risky wealth - Merton vs PriceImpact vs TransactionCosts
5.2
5.15
5.1
5.05
5
4.95
4.9
4.85
4.8
0
0.2
0.4
0.6
0.8
1

In the upper graph , we have plotted a Monte Carlo simulation of how the **number of shares** $\theta_t = Y_t/S_t$ evolves using the (asymptotically) optimal trading strategy $\hat{u}(y) = -\frac{\sqrt{\alpha}\,\sigma(y-\bar{Y})}{\sqrt{2}}\frac{1}{\sqrt{\lambda}}$ under linear price impact (dark blue) against the evolution of θ_t in the frictionless Merton setting (light blue) and the optimal θ_t process under the [GMK15] model (red) with **proportional transaction costs** but no price impact. We see that the price impact curve **smoothly tracks** the **frictionless Merton portfolio** and the transaction costs curve is **a.s. piecewise constant** because of the **no-trade region**. Here the parameters are $\mu = .05$, $\sigma = .1$, $\lambda = .0.00001$, $\alpha = 1$, the size of the proportional transaction costs is $\varepsilon = .0001$ and the time horizon here is $t = 1$ year. The lower graph shows the corresponding evolution of the **risky wealth** under all three models.

Transaction costs with jumps

- First consider a (frictionless) stock price process which evolves as

$$dS_t = S_{t-}[\mu_t dt + \sigma_t dW_t - dJ_t] \quad (1)$$

where J is a standard Poisson process with intensity λ_J and μ_t, σ_t are progressively measurable process with $\int_0^t \mu_s^2 ds, \int_0^t \sigma_s^2 ds < \infty$.

- Let ϕ_t denote the amount of stock. Then $V_t = V_0 + \int_0^t \phi_u dS_u$ is our total wealth at time t, and we wish to maximize $\mathbb{E}[\log V_T]$.
- From the change of variable formula we can show that

$$\mathbb{E}(V_T) = V_0 + \mathbb{E}[\int_0^T e^{-\lambda_J t}(\pi_{t-}\mu_t - \frac{1}{2}\pi_{t-}^2\sigma_t^2 + \lambda_J \log(1-\pi_{t-}))\, dt]\,.$$

- Differentiating with respect to π, the optimal π_{t-}^* satisfies

$$\mu_t - \pi_{t-}^*\sigma_t^2 - \frac{\lambda_J}{1-\pi_{t-}^*} = 0 \quad (2)$$

and for $\mu_t \in (0,1)$ the unique $\pi_{t-}^* = \frac{1}{2}[\theta_t + 1 - \sqrt{(1-\theta_t)^2 + 4\bar{\lambda}_t^J}]$ where $\theta_t = \mu_t/\sigma_t^2$ and $(\bar{\lambda}_J)_t = \lambda_J/\sigma_t^2$.

- (See Appendix for primer on **shadow prices**). With transaction costs, we now make the following ansatz for the shadow price: assume that $\tilde{S}_0 = S_0 = 1$, and if S increases from 1 to $\bar{s}$ without setting a new minimum, then we guess that $\tilde{S}_t = g(S_t)$ for $0 \le t \le \tau_{\bar{s}}$, for some $g \in C^2$ and target value $\bar{s}$, to be determined.
- Make an initial trade at $t = 0$, + postulate that the optimal trading strategy involves no further trading until $\tau_{\bar{s}}$ (i.e. a **no-trade region**).
- Applying Itô's formula to $\tilde{S}_t$, we obtain

$$\begin{aligned} d\tilde{S}_t &= dg(S_{t-}) = g'(S_{t-})dS_t + \frac{1}{2}g''(S_{t-})\sigma^2 S_{t-}^2 dt - g(S_{t-})dJ_t \\ \Rightarrow d\tilde{S}_t &= \tilde{S}_t[\tilde{\mu}(S_t)dt + \frac{1}{2}\tilde{\sigma}(S_t)dW_t - dJ_t] \end{aligned}$$

where $\tilde{\mu}(s) = [g'(s)s\mu + \frac{1}{2}g''(s)\sigma^2 s^2]/g(s)$, $\tilde{\sigma}(s) = g'(s)\sigma s/g(s)$. Hence we see that that $\tilde{S}_t$ follows a process of the form in (1). Setting $c = \varphi_t^0/\varphi_t$ we see that $\pi_{t-} = \frac{\varphi g(S_t)}{\varphi^0 + \varphi g(S_t)} = \frac{1}{1+\frac{c}{g(S_t)}}$

- Combining this with (2) we obtain the **non-linear ODE**:

$$\tilde{\mu}(s) - \frac{1}{1+\frac{c}{g(s)}}\tilde{\sigma}(s)^2 - \frac{\lambda_J}{1-\frac{1}{1+\frac{c}{g(s)}}} = 0\,.$$

- Setting $\pi^*(s) = \frac{g(s)}{g(s)+c} = \frac{1}{1+c/g(s)}$, then

$$\frac{1}{2}\sigma^2 s^2 (\pi^*)''(s) + s\mu\,(\pi^*)'(s) - \lambda_J \pi^*(s) = 0.$$

which is an Euler ODE, so $\pi^*(s) = As^{\alpha^+} + Bs^{\alpha^-}$ for some $\alpha^{\pm}$.
- The **true shadow price** process for all t is then given by

$$\tilde{S}_t \quad = \quad m_t g(\frac{S_t}{m_t})$$

with m_t defined as in [GMS13].
- (φ^0, φ) is self-financing strategy, and satisfies

$$d\varphi_t^0 \quad = \quad -\tilde{S}_t d\varphi_t \quad = \quad -m_t d\varphi_t$$

an such that $\frac{\varphi_t^0}{\varphi_t} = cm_t$, so we see that we buy when m_t decreases, as required for the definition of a shadow price process. Thus for $t < \tau_{\bar{s}}$

$$d\varphi_t^0 \quad = \quad cm_t d\varphi_t + c\varphi_t dm_t \quad = \quad -cd\varphi_t^0 + \frac{\varphi_t^0}{m_t} dm_t$$

so $\frac{d\varphi_t^0}{\varphi_t^0} = \frac{1}{1+c}\frac{dm_t}{m_t}$ (we can perform similar analysis on the **sell boundary** $\tilde{S} = (1-\lambda)S$).

Asymptotics and the implied welfare

- Using the **implicit function theorem** we can show that

$$
\begin{aligned}
\bar{s}(\lambda) &= 1 + \frac{1}{A^{\frac{1}{3}}}\lambda^{\frac{1}{3}} + o(\lambda^{\frac{1}{3}}), \\
c(\lambda) &= \bar{c} + \frac{\phi'(1)}{A^{\frac{1}{3}}}\lambda^{\frac{1}{3}} + o(\lambda^{\frac{1}{3}})
\end{aligned}
$$

for some A which is easily computed in terms of the parameters.

- **Welfare**: Expected long term log utility of wealth is

$$
\delta := \lim_{T\to\infty} \mathbb{E}(\log V_T) = V_0 + \lim_{T\to\infty} \mathbb{E}[\int_0^T e^{-\lambda_J t}\,\Upsilon(\frac{S_t}{m_t})\,dt]
$$

where

$$
\Upsilon(y) := \pi^*(y)\tilde{\mu}(y) - \frac{1}{2}\pi^*(y)^2\tilde{\sigma}(y)^2 + \lambda_J \log[1-\pi^*(y))] .
$$

- $\log \frac{S_t}{m_t}$ is identical in law to a geometric Brownian motion $dY_t = Y_t \sigma dB_t$ with two **reflecting** barriers at 1 and $\bar{s} = \bar{s}(\lambda)$. Thus

$$
\delta = V_0 + \lim_{T\to\infty} \mathbb{E}[\int_0^T e^{-\lambda_J t}\Upsilon(Y_t)\,dt] = V_0 + \frac{1}{\lambda_J}\mathbb{E}[\Upsilon(Y_\tau)] .
$$

Numerics

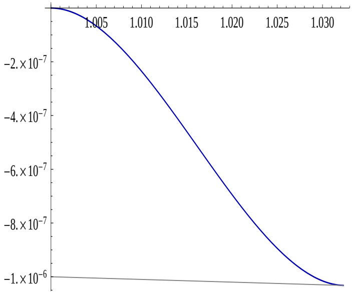


Figure: We set $\mu = .01$, $\sigma = .2$, $\lambda = .000001$, $\lambda_J = .0001$, we find (numerically) that the correct c-value is $c = 3.102951$ and $\bar{s} = 1.03231$. Here we have plotted $g(s) - s$ and $-\lambda s$, and we see that the two curves are tangential at $\bar{s}$.

Fractional Brownian motion and stoc vol models

Recall that a zero-mean real-valued **Gaussian process** $(Z_t)_{t\geq 0}$ is a stochastic process such that on any finite subset $\{t_1,..,t_n\}\subset\mathbb{R}$, $(Z_{t_1},...,Z_{t_n})$ has a multivariate normal distribution with mean zero. The law of a Gaussian process is entirely determined by its **covariance function** $R(s,t)=\mathbb{E}(Z_s Z_t)$. A zero-mean Gaussian process B^H_t is called standard **fractional Brownian motion** (fBM) with Hurst parameter $H\in(0,1)$ if

$$R(s,t) \;=\; \mathbb{E}(B^H_t B^H_s) \;=\; \frac{1}{2}(|t|^{2H}+|s|^{2H}-|t-s|^{2H})\,.$$

- fBM is continuous a.s. and H-**self-similar**, i.e. $(B_{at})_{t\geq 0}\overset{(\mathrm{d})}{=}a^H(B_t)_{t\geq 0}$ (i.e. they have the same finite dimensional distributions). For $H\neq\frac{1}{2}$, B^H does not have independent increments;
- $B^H_t-B^H_s\sim N(0,|t-s|^{2H})$; thus B^H has **stationary increments**.
- B_k-B_{k-1} and $B_{k+n}-B_{k+n-1}$ are positively correlated if $H\in(\frac{1}{2},1)$ and negatively correlated if $H\in(0,\frac{1}{2})$. Thus B^H is **persistent** when $H>\frac{1}{2}$ and anti-persistent when $H<\frac{1}{2}$.
- Sample paths of B^H are α-**Hölder-continuous**, for all $\alpha\in(0,H)$.
- B^H_t is neither a Markov process nor a semimartingale (see [Nual96]).

- There is a **Volterra**-type representation of fBM on the interval $[0,t]$:

$$B_t^H = \int_0^t K_H(s,t)dB_s \,.$$

- $\sqrt{\varepsilon}B^H$ satisfies the LDP on $C_0[0,1]$ as $\varepsilon \to 0$ with speed $\frac{1}{\varepsilon}$ and rate function $\Lambda(f) = \frac{1}{2}\int_0^1 h(s)^2 ds$ if $f(t) = \int_0^t K_H(s,t)h(s)ds \ \forall t \in [0,1]$ and some $h \in L^2[0,1]$, and $\Lambda(f) = \infty$ otherwise.
- The space of functions with $\Lambda(f) < \infty$ is known as the **reproducing kernel Hilbert space** $\mathcal{H}$ of fBM, with $\langle f_1, f_2\rangle_{\mathcal{H}} = \langle h_1, h_2\rangle_{L^2[0,1]}$.
- We have used this to compute small-t asymptotics for the model

$$\begin{cases} dS_t = S_t\sigma(Y_t)(\sqrt{1-\rho^2}dW_t + \rho dB_t)\,, \\ dY_t \;=\; dB_t^H \end{cases} \tag{3}$$

- In [FZ15] we show that $t^{H-\frac{1}{2}}\log S_t$ satisfies the LDP as $t \to 0$ with speed $\frac{1}{t^{2H}}$ and rate $I(x) = \inf_{f\in H_1}\left[\frac{(x-\rho G(f))^2}{2\bar{\rho}^2 F(\mathbf{K}_H f')} + \frac{1}{2}\|f\|^2_{H_1}\right]$, where $F(f) = \int_0^1 \sigma(f(s))^2 ds$, $G(f) = \int_0^1 \sigma((\mathbf{K}_H f')(s))f'(s)ds$, $(\mathbf{K}_H f)(t) = \int_0^t K_H(s,t)f(s)ds$, $\sigma(.)$ can be unbounded but must satisfy a linear growth condition which still allows for moment explosions for S_t.

Generalizing the model using rough paths theory

- **Rough paths theory** is concerned with differential equations of type

$$dY_t = \sum_{i=1}^{d} V^i(Y_t) dX^i_t$$

where the driving signals X^i_t may be rougher than standard Brownian motion (rough in the sense of **Hölder continuity**) e.g. fBM.

- Eqs of this type can be solved **pathwise** using rough paths theory. **Lyons** proved that the **Itô map** which takes the signal X to the solution Y is continuous in an appropriate **rough path topology**, which for $H > \frac{1}{3}$ requires keeping track of the higher order iterated integrals of X:

$$X^i_{s,t} = X^i_t - X^i_s \quad , \quad \mathbb{X}^{ij}_{s,t} = \int_s^t \int_s^r dX^i_u dX^j_r \,.$$

- We let $\mathbf{x}_{s,t} = (1, X_{s,t}, \mathbb{X}_{s,t})$. We can then define a norm and metric on the space in which $\mathbf{x}$ lives, and Lyon's Universal limit theorem says thats the Itô map is continuous under the topology associated with this norm, which makes proving LDPs much simpler, using the contraction principle.

- We prove a similar small-time LDP for a general model where the stock price S_t satisfies

$$\begin{aligned} dS_t &= S_t[\sigma(Y_t)dW_t + \eta_1 dB^1_t], \\ dY_t &= V^1(Y_t)dB^{H_1}_t + V^2(Y_t)dB^{H_2}_t \end{aligned} \tag{4}$$

where $B^{H_1}_t = \int_0^t K_{H_1}(s,t)dB^1_s$, $B^{H_2}_t = \int_0^t K_{H_2}(s,t)dB^2_s$ and W, B^1, B^2 are 3 independent standard Brownian motions and $\frac{1}{4} < H_1 < \frac{1}{2} < H_2 < 1$, under suitable regularity conditions on the coefficients.

- To construct the solution to (4), we let D_n be a sequence of partitions of $[0,T]$ with mesh size $\to 0$. Let $\pi_V(0, y_0; x)$ denote the solution to the controlled ODE $dY_t = V^1(Y_t)dX^1_t + V^2(Y_t)dX^2_t$ when X^1_t, X^2_t have finite variation. Then the random sequence of ODE solutions $\pi_V(0, y_0; x^{D_n})$ (where x^{D_n} is the **piecewise linear approximation** to x) is Cauchy in probability under the uniform topology and its unique limit point is a $C([0,T],\mathbb{R})$-valued random-variable which does not depend on the choice of sequence D_n, and is identified as the random **RDE solution** $(Y)_{t\geq 0}$.

Numerical results: the small-maturity implied volatility smile

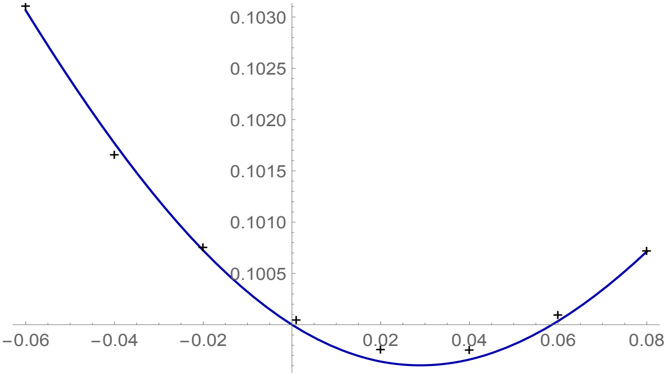


Figure: Here we have plotted the small-maturity implied volatility smile for the model in (3) for $\rho = -0.1$, $\sigma(y) = 1 + .05\tanh(y)$, $H = 0.25$ and $t = .002$. verses the values obtained by Monte Carlo using the Willard conditioning method with 500,000 simulations and 100 time steps.

Optimal liqidation for the Gatheral-Schied problem with stochastic liquidity

- We consider an extension of an **Almgren&Chriss**-type model of transient and permanent price impact, where the price that we pay for a stock is

$$\tilde{S}_t = S_t + \eta_t \dot{x}_t + \gamma(x_t - x_0)$$

where $dS_t = S_t \sigma dW_t$, $d\eta_t = \eta_t \alpha dB_t$ and $dW_t dB_t = \rho dt$, and x_t is the number of shares held at time t which we assume is differentiable in t.

- The η term corresponds to the temporary (**transient**) price impact, which effectively penalizes a trader for making a large change in his stock position in a small time period, but goes away as soon as he stops trading. The γ term captures the effect of **permanent** price impact, whereby a large number of buy (sell) orders permanently moves the underlying stock price up (down), and S is known as the *unaffected* stock price.

- Assume $x_0 = X > 0$, and we implement a **liquidation strategy** x_t over $[0, T]$ so $x_T = 0$. Then it's well known that the cost of doing this is:

$$\begin{aligned} R_T = \int_0^T \tilde{S}_t \dot{x}_t dt &= \int_0^T S_t \dot{x}_t dt + \int_0^T (\eta_t \dot{x}_t + \gamma(x_t - x_0))\dot{x}_t dt \\ &= S_T x_T - S_0 x_0 - \int_0^T x_t dS_t + \int_0^T (\eta_t \dot{x}_t^2 + \gamma(x_t - x_0)\dot{x}_t) dt \\ &= -S_0 x_0 - \int_0^T x_t dS_t + \int_0^T \eta_t \dot{x}_t^2 + \frac{1}{2}\gamma(x_T^2 - x_0^2) + \gamma x_0^2 \\ &= -S_0 X - \int_0^T x_t dS_t + \frac{1}{2}\gamma X^2 + \int_0^T \eta_t \dot{x}_t^2 dt\,. \end{aligned}$$

Thus $\mathbb{E}(R_T) = -S_0 X + \frac{1}{2}\gamma X^2 + \mathbb{E}(\int_0^T \eta_t \dot{x}_t^2 dt)$ and if $\eta_t = 0$ for all t, we see that $\mathbb{E}(R_T)$ is independent of the trading strategy x.

- Now recall the **Gatheral-Schied**[GS11] optimal liquidation problem:

$$V(T, S, \eta, X) = \inf_{v \in \mathcal{V}} \mathbb{E}\Big(\int_0^T (\lambda x_t S_t + \eta_t v_t^2) dt \,|\, S_0 = S, \eta_0 = \eta, x_0 = X \Big) \quad (1)$$

where $v_t = \dot{x}_t$ and $\mathcal{V}$ is the space of all progressively measurable processes v for which $x_T = X + \int_0^T v_s ds = 0$ with certain integrability conditions.

- The expectation in (1) (aside from the constants in (5)) is the expected cost of liquidation plus an additional λ-term to penalize the trader for holding large positions. The **HJB equation** for this problem is

$$V_T = \frac{1}{2}\sigma^2 S^2 V_{SS} + \rho\sigma\alpha S\eta V_{S\eta} + \frac{1}{2}\alpha^2\eta^2 V_{\eta\eta} + \lambda SX + \inf_{v\in\mathbb{R}}[\eta v^2 + vV_X] \quad (5$$

with $\lim_{T\to 0} V(T,S,\eta,X) = 0$ if $X=0$ and $+\infty$ otherwise (i.e. infinite penalty if $x_T \neq 0$) and we see that $v^* = -\frac{1}{2}V_X^*/\eta$.

- Solving the HJB eq, and using a verification argument to verify optimality, we find that the unique optimal trade execution strategy attaining the infimum is

$$x_t^* = \frac{T-t}{T}[X - \frac{\lambda T}{4}\int_0^t \frac{S_s}{\eta_s} ds]$$

which implies that $\dot{x}_t^* = -\frac{x_t^*}{T-t} - \frac{\lambda S_t(T-t)}{4\eta_t}$ and the value function is

$$V^*(T,S,\eta,X) = \frac{\eta X^2}{T} + \frac{\lambda STX}{2} + \frac{\lambda^2 S^2[2 - 2e^{T\theta} + T\theta(2+T\theta)]}{16\eta\theta^3}$$

where $\theta = \sigma^2 - 2\alpha\rho\sigma + \alpha^2$.

Asymptotics

- We have the asymptotic formulae:

$$
\begin{aligned}
V(T,S,\eta,X) &= \frac{X^2\eta}{T} + \frac{1}{2}SX\lambda T - \frac{S^2\lambda^2}{48\eta}T^3 - \frac{S^2\lambda^2\theta}{192\eta}T^4 + O(T^5) \\
V(T,S,\eta,X) &= V(T,S,\eta,X)|_{\alpha=0} \\
&+ \frac{S^2\lambda^2\rho\,(6+4T\sigma^2+T^2\sigma^4+2e^{T\sigma^2}(T\sigma^2-3))}{8\eta\sigma^7}\alpha + O(\alpha^2)\,.
\end{aligned}
$$

The **correction term** in the last equation is the leading order correction to the expected liquidation cost $\mathbb{E}(R_T)$ due to stochastic price impact i.e. α, which it turns out has the same sign as ρ. From the penultimate equation, we see that the effect of α is not felt until $O(T^4)$, i.e. very close to the terminal time. See left plot on title page for **Monte Carlo** simulation of $\dot{x}_t^*$.
- $\dot{x}_t$ satisfies a linear coupled **FBSDE**, using a similar convex analysis argument to the constrained problem in [BSV15] by setting the **Gateaux derivative** of the functional to be minimized to zero.
- Can extend to include stochastic permanent price impact or stochastic interest rates, and $\dot{x}_t^*$ remains unchanged if S,η are martingale diffusions (i.e. the optimal liquidation strategy is **robust** in some sense).

A power-law limit order book model

- Following BL14, assume that the unaffected price process S_t is an (unspecified) martingale and a trader places $N_t \in \mathbb{N}$ **ask orders** in the LOB at price $S_t + \delta_t$ at time t, where he is free to choose $\delta > 0$ as the stochastic control.
- BL14 assume that the number of shares held $(N_t)_{t \geq 0}$ is equal to N_0 minus a Poisson counting process with **controlled intensity** $\Lambda(\delta_t) = \lambda \delta_t^{-\alpha}$ for $\alpha > 1$, so trades arrive randomly one at a time and the trader keeps selling until all the stock is sold, so his cash position evolves as $dC_t = -(S_t + \delta_t) dN_t$ until $N_t = 0$.
- Let T denote time to maturity and n represent the current stock holding. Then BL14 seek to maximized expected revenue, for which the value function $V(n, T)$ satisfies the (discrete space) HJB eq

$$V_T = \sup_{\delta > 0} \frac{\lambda}{\delta^\alpha} [V(n-1, T) - V(n, T) + \delta]$$

with boundary conditions:

1. $V(n, 0) = 0$ for all n (number of shares) - no more revenue is generated when time is up.
2. $V(0, T) = 0$ for all T - no more revenue after all shares sold.

The fluid limit

- We can consider the *fluid limit* where the share increment is now equal to $\Delta \ll 1$ and the Poisson rate is re-scaled by $1/\Delta$. The value function is now for $x \in \{0, \Delta, 2\Delta, ...\}$, for which the HJB equation is

$$V_T^{\Delta} = \sup_{\delta > 0} \frac{\lambda}{\delta^{\alpha}\Delta}[V^{\Delta}(x-\Delta, T) - V^{\Delta}(x, T) + \delta\Delta].$$

- As $\Delta \to 0$, we have

$$V_T = \sup_{\delta > 0}\{\lambda\delta^{-\alpha}(\delta - V_X)\}$$

which now implies that we now have a continuous stream of trades so that $dx_t = -\lambda\delta_t^{-\alpha}dt$, where x_t is the number of shares at time t.
- We extend this setup by assuming that λ is stochastic: $d\lambda_t = \beta\lambda_t dZ_t$ where Z is a standard BM independent of S and N, and this is the model that we work with, for which the HJB eq is now

$$V_T = \frac{1}{2}\beta^2\lambda^2 V_{\lambda\lambda} + \sup_{\delta > 0}\{\lambda\delta^{-\alpha}(\delta - V_X)\}.$$

Solution

- Using a separable ansatz of the form

$$V(X,T,\lambda) \;=\; X^{\frac{\alpha-1}{\alpha}}(\lambda g(T))^{\frac{1}{\alpha}}$$

yields that $g(T) = \frac{1}{\kappa}(1 - e^{-\kappa T})$, where $\kappa = \frac{(\alpha-1)\beta^2}{2\alpha}$.

- Thus we obtain the optimal spread δ_t^* in closed form as

$$\delta_t^* \;=\; \Big(\frac{\lambda_t}{\kappa x_t^*}(1 - e^{-\kappa(T-t)})\Big)^{1/\alpha}.$$

- The optimal number of shares evolves as:

$$dx_t^* \;=\; -\lambda_t \delta_t^{-\alpha} dt \;=\; -\frac{\kappa x_t^*}{1 - e^{-\kappa(T-t)}} dt$$

i.e. we trade in a way that ensures that x_t^* is **deterministic**, but δ_t^* is not and varies stochastically with λ_t.

- Note the special case where $\beta = 0$ the above reduces to

$$dx_t^* \;=\; -\frac{x_t^*}{T-t} dt$$

with solution $x_t^* = \frac{T-t}{T} X$.

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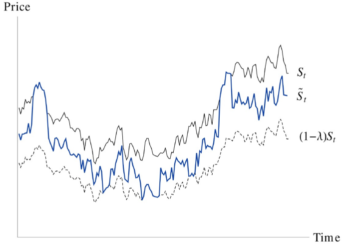
Definition of a shadow price

- **Definition**. A **shadow price** is a semimartingale $\tilde{S}_t \in [(1-\lambda)S_t, S_t]$, such that the optimal trading strategy (φ_t^0, φ_t) for a fictitious market with price process $\tilde{S}_t$ and zero transaction costs exists, has finite variation and the number of stocks φ_t only increases when $\tilde{S}_t = S_t$ and decreases when $\tilde{S}_t = (1-\lambda)S_t$.



- Clearly any price process $\tilde{S}_t$ with zero transaction costs which lies in $[(1-\lambda)S_t, S_t]$ leads to **more favourable** terms of trade than the original market with transaction costs. But a shadow price process is a particularly unfavourable model, for which it's optimal to only **buy** when $\tilde{S}_t = S_t$, **sell** when $\tilde{S}_t = (1-\lambda)S_t$ + do nothing in between.

Why do we use shadow prices?

- **Proposition** (Corollary 1.9 in Schachermayer et al.[GMS13]). Let $\tilde{S}_t$ be a shadow price process whose optimal trading strategy (for zero transaction costs) is given by (φ^0_t, φ_t), with $\varphi^0_t, \varphi_t \geq 0$. Then under **non-zero** transaction costs, we have

$$\begin{aligned} \sup_{(\psi^0,\psi)} \mathbb{E}[\log V_T((\psi^0,\psi))] &\geq \mathbb{E}[\log V_T((\varphi^0,\varphi))] \\ &\geq \mathbb{E}[\log V_T((\psi^0,\psi))] + \log(1-\lambda) \end{aligned}$$

  for *any* admissible (ψ^0,ψ). Thus if we choose λ suff small so that $|\log(1-\lambda)| < \varepsilon$ and take the sup over all (ψ^0,ψ), we see that (φ^0,φ) is an ε-**optimal trading strategy** for the original problem.
- Or take liminf as $T \to \infty$ + sup over all admissible strategies, we obtain

$$\liminf_{T\to\infty} \frac{1}{T}\mathbb{E}[\log V_T((\varphi^0,\varphi))] = \sup_{(\psi^0,\psi)} \liminf_{T\to\infty} \frac{1}{T}\mathbb{E}[\log V_T((\psi^0,\psi))] \,.$$

  Thus the optimal portfolio for the shadow price process is **asymptotically optimal** for the original problem under transaction costs, as $\lambda \to 0$ and/or as $T \to \infty$.