Stat243: Homework 7

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1 Problem 1

We compare the standard error $\frac{s}{\sqrt{m}}$ with effect size to see if it correctly characterizes the uncertainty of the estimated regression coefficient.

2 Problem 2

$$A = \Gamma\Lambda\Gamma^T$$
$$Az = \Gamma\Lambda(\Gamma^T z)$$
$$Az^T(Az) = z^T A^T Az = z^T \Gamma\Lambda\Gamma^T\Gamma\Lambda\Gamma^T z = z^T\Gamma\Lambda\Lambda\Gamma^T z$$
$$y = \Gamma^T z$$
$$= y^T\Lambda^2 y$$
$$||y||_2{}^2 = \sum(\Gamma_{i1}z_1 + ... + \Gamma_{in}z_n)^2 = \sum \Gamma_{i1}{}^2 z_1{}^2 + ... + \Gamma_{in}z_n{}^2 + 2\sum \Gamma_{ik}\Gamma_{il}z_i z_l$$
$$k \neq l, ||z||_2 = 1$$
$$||y||_2{}^2 = z_1{}^2 + ... + z_n{}^2 = 1$$
$$y^T\Gamma^2 y = \sum y_i{}^2\Gamma_i{}^2$$
$$||A||_2 = sup\sqrt{\sum y_i{}^2\Lambda_i{}^2}$$
$$max\Gamma_{i..n}{}^2 = \Gamma_k{}^2$$
$$||A||_2 = ||\Gamma_k||$$

Therefore, the norm A is the largest of the absolute value of the eigenvalues of A for symmetric A.

3 Problem 3

3.1 3A

$$
\begin{aligned}
X &= USV^T \\
X^T &= VSU^T \\
X^TX = VSU^TUSV^T &= VS^2V^T \\
X^TXV &= VS^2 \\
V^T(X^TX)V = (XV)^TXV = \|XV\|_2{}^2 &\geqslant 0
\end{aligned}
$$

Therefore, X^TX is a positive definite matrix, so the SVD for X^TX is equal to eigendecomposition. VS^2V^T is the SVD of X^TX, based on the definition of eigendecomposition, columns of V are eigenvectors of X^TX. Also, columns of V are right singular vectors of X. In conclusion, the right singular vectors of X are eigenvectors of X^TX. Since S^2 's diagnol elements are eigenvalues of X^TX, S's diagnol elements are singular values of X, the eigenvalues of X^TX are squares of the singular values of X.

3.2 3B

$$
\begin{aligned}
Zv &= \lambda v \\
(\sum +cI)v &= \lambda v \\
\sum v &= (\lambda - c)v
\end{aligned}
$$

so $\lambda - c$ is the eigenvalue of $\sum$. The eigenvalues of Z can be computed by adding constant c to the eigenvalues of $\sum$

4 Problem 4

4.1 Problem 4b

```
Efficient_Solve <- function(X,Y,A,b){
 C = crossprod(X)
 d = crossprod(X,Y)
 tmp1 = crossprod(solve(c,t(A)),t(A))
 tmp2 = crossprod(solve(c, t(A)), d)
 beta_hat = solve(c,d) + solve(c,crossprod(A,solve(tmp1, -tmp2+b)))
 beta_hat
}
```

5 Problem 5

5.1 5A

Instrument variables have an important property: The matrix of correlation between variables in X and the variables in Z is of maximum possible rank, meaning that we want X to retain only the correlation between X and Z. However, since X and Z are sparse matrixes, resulting X is also sparse, which would fail the expectation of stage 1 which is supposed to geenrate a matrix containing all the correlation items(fitted values) for the second stage.

5.2 5B

Stage 1: Regress each variable in X on Z to obtain a matrix with fitted values.
Stage 2: Regress y on X
Key:Use Pz as an intermediate

$$Z(Z^T Z)^{-1} Z^T X = P_z X$$

$$\beta = (\hat{X}^T \hat{X})^{-1} X^T y = (X^T {P_z}^T P_z X)^{-1} X^T {P_z}^T y = (X^T P_z X) X^T P_z y$$

$$P_z^T P_z = [Z(Z^T Z)^{-1} Z^T]^T = Pz$$

Pseudocode:
Compute $P_z$using spam package
plug P_z into β equation
return β