①

a) we know that solution for $Xw=y$ is

$$w = \frac{1}{\sqrt{2}} \begin{bmatrix} 1+\frac{1}{\gamma} \\ 1-\frac{1}{\gamma} \end{bmatrix}$$

$$\|w\|_2^2 = \frac{1}{2} (1+\tfrac{1}{\gamma})^2 (1-\tfrac{1}{\gamma})^2$$

when $\gamma = 0.1$, $w = \frac{1}{\sqrt{2}} \begin{bmatrix} 11 \\ -9 \end{bmatrix}$ $\|w\|_2^2 = \frac{1}{\sqrt{2}} \sqrt{11^2 + 9^2} \approx 10.05$

when $\gamma = 10^{-8}$ $w = \frac{1}{\sqrt{2}} \begin{bmatrix} 1+10^8 \\ 1-10^8 \end{bmatrix}$ $\|w\|_2^2 = \frac{1}{\sqrt{2}} \sqrt{(1+10^8)^2 + (1-10^8)^2} \approx 1 \times 10^8$

b) $w_e = V \begin{bmatrix} 1 & 0 \\ 0 & 1/\gamma \end{bmatrix} U^T \begin{bmatrix} e \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$= \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\gamma} \end{bmatrix} \cdot \cancel{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} e \\ 0 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} e \\ e \end{bmatrix}$$

$$= \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & \frac{1}{\gamma} \\ 1 & -\frac{1}{\gamma} \end{bmatrix} \begin{bmatrix} e \\ e \end{bmatrix} = \frac{e}{2\sqrt{2}} \begin{bmatrix} 1+\frac{1}{\gamma} \\ 1-\frac{1}{\gamma} \end{bmatrix}$$

Take $\varepsilon = 0.01$ and $\gamma = 0.1$, $\|w_e\|_2^2 \approx 0.0502$

$\gamma = 10^{-8}$ $\|w_e\|_2^2 \approx 5 \times 10^5$.

$\frac{\sqrt{2}}{2}$

c) In this case

$$w = V \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U^T y = \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} y$$

$$= \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} y$$

$$= \frac{1}{2\sqrt{2}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1+\varepsilon \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2+\varepsilon \\ 2+\varepsilon \end{bmatrix}$$

Taking $\varepsilon = 0.01$ and compute $\|w_e\|_2^2$

$\|w_e\|_2 = \left\| \frac{1}{2\sqrt{2}} \begin{bmatrix} e \\ e \end{bmatrix} \right\|_2 = 0.005$

$\|w_e\|_2^2$ is well-contained comparing to (b).