CS/ECE/ME532 Assignment 3

1. *Polynomial fitting.* Suppose we observe pairs of points (a_i, b_i), $i = 1, \ldots, m$ representing measurements from a scientific experiment. The variables a_i are the experimental conditions and the b_i correspond to the measured response in each condition. We fit a degree $d < m$ polynomial to these data. In other words, we want to find the coefficients of a degree p polynomial $w(a)$ so that $w(a_i) \approx b_i$ for $i = 1, 2, \ldots, m$.

a) Suppose $w(a)$ is a degree p polynomial. Write the general expression for $w(a_i) = b_i$.

b) Express the $i = 1, \ldots, m$ equations as a system in matrix form $\boldsymbol{Ax} = \boldsymbol{d}$ while defining $\boldsymbol{A}$ and $\boldsymbol{d}$. What is the form/structure of $\boldsymbol{A}$ in terms of the given a_i?

c) Write a script to find the least-squares model fit to the $m = 30$ data points in `polydata.mat`. Plot the points and the polynomial fits for $p = 1, 2, 3$.

2. Least Squares Approximation of Matrices.

a) Derive the solution to least-squares problem $\min_{\boldsymbol{w}} \|\boldsymbol{x} - \boldsymbol{Tw}\|_2^2$ when $\boldsymbol{T}$ is an n-by-r matrix of orthonormal columns. Your solution should not involve a matrix inverse.

b) Let $\boldsymbol{X} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_p \end{bmatrix}$ be an n-by-p matrix. Use the least-squares problems $\min_{\boldsymbol{w}_i} \|\boldsymbol{x}_i - \boldsymbol{T}\boldsymbol{w}_i\|_2^2$ to find $\boldsymbol{W} = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_p \end{bmatrix}$ in the approximation $\boldsymbol{X} \approx \boldsymbol{TW}$. Your solution should express $\boldsymbol{W}$ as a function of $\boldsymbol{T}$ and $\boldsymbol{X}$.

3. We return to the movies rating problem of Activity 5. The ratings on a scale of 1-10 are:

Movie	Jake	Jennifer	Julia	Justin	Jackson	James	Jasmine
Star Trek	4	7	2	8	7	4	2
Pride and Prejudice	9	3	5	6	10	5	5
The Martian	4	8	3	7	6	4	1
Sense and Sensibility	9	2	6	5	9	5	4
Star Wars: Empire Strikes	4	9	2	8	7	4	1

A matrix $\boldsymbol{X}$ containing this data is available in the file `movie.mat` and the csv file `movie.csv`. Our goal is to approximate $\boldsymbol{X}$ using r "tastes", the columns of $\boldsymbol{T}$, that is, $\boldsymbol{X} \approx \boldsymbol{TW}$ where $\boldsymbol{T}$ is 5-by-r. You will use a Gram-Schmidt orthogonalization code to find a set of tastes that approximate the ratings. A script that implements Gram-Schmidt orthogonalization is available.

Define a 5-by-r taste matrix $\boldsymbol{T} = \begin{bmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 & \cdots & \boldsymbol{t}_r \end{bmatrix}$ with orthonormal columns and the r-by-7 weight matrix

$$\boldsymbol{W} = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{17} \\ w_{21} & w_{22} & \cdots & w_{27} \\ & & \vdots & \\ w_{r1} & w_{r2} & \cdots & w_{r7} \end{bmatrix}$$

a) In Activity 5 you found the baseline (average) rating for each friend by requiring the first basis vector in the taste matrix to be

$$\boldsymbol{t}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

You have noticed by now that the first vector in the Gram-Schmidt procedure is a scaled version of the first vector of the matrix, so you decide to define an augmented matrix $\tilde{\boldsymbol{X}} = \begin{bmatrix} \boldsymbol{1} & \boldsymbol{X} \end{bmatrix}$ where $\boldsymbol{1}$ is a column vector containing five unity entries. Apply a Gram-Schmidt orthogonalization code to $\tilde{\boldsymbol{X}}$ to find a set of orthonormal basis vectors. Is the first basis vector you obtain equal to $\boldsymbol{t}_1$?

b) Use your solution to the preceding problem in this homework assignment to find the rank-1 approximation of $\boldsymbol{X}$ using only $\boldsymbol{t}_1$. That is, find $\boldsymbol{W}$ so that $\boldsymbol{X} \approx \boldsymbol{t}_1\boldsymbol{W}$. Use $\boldsymbol{W}$ to compute $\boldsymbol{t}_1\boldsymbol{W}$. This gives you each friend's baseline ratings. Also compute the residual error $\boldsymbol{X} - \boldsymbol{t}_1\boldsymbol{W}$.

c) Now find a rank-2 approximation using $\boldsymbol{T} = \begin{bmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 \end{bmatrix}$. That is, find $\boldsymbol{W}$ so that $\boldsymbol{X} \approx \boldsymbol{TW}$. Use $\boldsymbol{W}$ to compute $\boldsymbol{TW}$. This gives you a rank-2 approximation to the ratings. Also compute the residual error $\boldsymbol{X} - \boldsymbol{TW}$. How does $\boldsymbol{t}_2$ relate to the distinction between sci-fi and and romance movie preferences?

d) Now find a rank-3 approximation using $\boldsymbol{T} = \begin{bmatrix} \boldsymbol{t}_1 & \boldsymbol{t}_2 & \boldsymbol{t}_3 \end{bmatrix}$. That is, find $\boldsymbol{W}$ so that $\boldsymbol{X} \approx \boldsymbol{TW}$. Use $\boldsymbol{W}$ to compute $\boldsymbol{TW}$. This gives you a rank-3 approximation to the ratings. Also compute the residual error $\boldsymbol{X} - \boldsymbol{TW}$. Qualitatively discuss the effect of increasing the rank of the approximation on the residual error.

e) Suppose you interchange the order of Jake and Jennifer so that Jennifer's ratings are in the first column of $\boldsymbol{X}$ and Jake's ratings are in the second column. Does the rank-2 approximation change? Why or why not? Does the rank-3 approximation change? Why or why not?

4. Let $\boldsymbol{Q} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

a) Is $\boldsymbol{Q} \succ 0$?

b) Sketch the surface $y = \boldsymbol{x}^T \boldsymbol{Q} \boldsymbol{x}$ where $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. If you find 3-D sketching too difficult, you may draw a contour map with labeled contours.

5. Suppose $\mathbf{P} \succ 0$ and $\mathbf{Q} \succ 0$ are (symmetric) positive definite $n \times n$ matrices. Prove that $\mathbf{QPQ} \succ 0$.