①

a) Let $Z = X+Y$

$$p(z) = \begin{cases} p_1 q_1 & , z=2 \\ p_1 q + p_2 q_1 & , z=3 \\ p_1 q_3 + p_2 q_2 + p_3 q_1 & z=4 \\ p_1 q_4 + p_2 q_3 + p_3 q_2 + p_4 q_1 & , z=5 \\ p_1 q_5 + p_2 q_4 + p_3 q_3 + p_4 q_2 + p_5 q_1 & , z=6 \\ p_1 q_6 + p_2 q_5 + p_3 q_4 + p_4 q_3 + p_5 q_2 + p_6 q_1 & , z=7 \\ p_2 q_6 + p_3 q_5 + p_4 q_4 + p_5 q_3 + p_6 q_2 & , z=8 \\ p_3 q_6 + p_4 q_5 + p_5 q_4 + p_6 q_3 & , z=9 \\ p_4 q_6 + p_5 q_5 + p_6 q_4 & , z=10 \\ p_5 q_6 + p_6 \cdot q_5 & , z=11 \\ p_6 q_6 & , z=12 \end{cases}$$

b) $\because Z = X+Y$

$\therefore P(z) = P(x+y)$

$\because$ X, Y independent

$\therefore P(z) = P_x(a) \cdot P_Y(b) \quad \forall a+b = z$.

$\because$ Law of total probability

$\therefore P_Z(z) = \sum_i P_x(x_i) P_Y(z - x_i)$ ☒

②

a) $$p(y) = \begin{cases} 0, & \text{if } y \notin [0,1] \\ 1, & \text{if } y = \min(x_1, \dots, x_i) \text{ where } x_i \overset{iid}{\sim} U[0,1] \ \forall i \le n. \end{cases}$$

b) $E[Y] = \min\{x_i \overset{iid}{\sim} U[0,1] \ \forall i \le n\} = \frac{1}{n}$.

c) Yes

③

a) $E[X] = 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{4} = \frac{1}{2} + \frac{1}{2} + \frac{3}{4} = \frac{7}{4}$

b) $E[g(x)] = 1 \times \frac{1}{2} + 4 \times \frac{1}{4} + 9 \times \frac{1}{4} = \frac{1}{2} + 1 + \frac{9}{4} = \frac{15}{4}$

c) $Var(x) = E[(1-\frac{7}{4})^2 + (2-\frac{7}{4})^2 + (3-\frac{7}{4})^2] = (\frac{3}{4})^2 + (\frac{1}{4})^2 + (\frac{5}{4})^2$

$= \frac{9}{16} + \frac{1}{16} + \frac{25}{16} = \frac{35}{16}$

d) suppose X is a discrete random variable

$E[-\log(g(x))] = \sum_{x \in X} -\log(g(x)) = -\sum_{x \in X} \log(g(x))$

e) $E[-\log_2(P(x))] = 1 \times 1 + 2 \times 2 + 3 \times 2 = 1 + 4 + 6 = 11$

④ a) $P(X=1) = \frac{1}{4}$ $P(X=2) = \frac{3}{4}$

$P(Y=1) = \frac{5}{8}$ $P(Y=2) = \frac{3}{8}$

since $P(X=1) \cdot P(Y=1) = \frac{3}{16} \neq \frac{1}{8} = P(X=1, Y=1)$

X and Y are not independent.

b) $Cov(X,Y) = E[XY] - E(X)E(Y)$

$E[XY] = \frac{1}{8} + 2 \times \frac{1}{8} + 2 \times \frac{1}{2} + 4 \times \frac{1}{4} = \frac{1}{8} + \frac{1}{4} + 1 + 1 = \frac{1+2+16}{8} = \frac{19}{8}$

$E[X] = \frac{1}{4} + \frac{3}{2} = \frac{7}{4}$, $E[Y] = \frac{5}{8} + \frac{3}{4} = \frac{11}{8}$

$\therefore Cov(X,Y) = \frac{19}{8} - \frac{7}{4} \times \frac{11}{8} = -\frac{1}{32}$

c) $p(z) = \begin{cases} \frac{1}{8}, & z = 2 \\ \frac{5}{8}, & z = 3 \\ \frac{1}{4}, & z = 4 \end{cases}$

d) see other pdf

⑤ $RHS = E_Y[E_X[X|Y]] = E_Y[\sum_x x P(X=x|Y)]$

$= E_Y[x_1 P(X=x_1|Y)] + \cdots + E_Y[x_n P(X=x_n|Y)]$

$= x_1 E_Y[P(X=x_1|Y)] + \cdots + x_n E_Y[P(X=x_n|Y)]$

$= x_1 \cdot P(x_1) + \cdots + x_n \cdot P(x_n)$

$= \sum_x x \cdot P(x) = E[X]$ ∎

⑥ a) see script

b) $P(y|x) = \frac{P(x|y)P(y)}{P(x)}$

$P(y=1|x) = \frac{P(x|y=1)P(y=1)}{P(y=1)P(x|y=1) + P(y=2)P(x|y=2)} = \frac{N(4,1) \cdot \frac{1}{4}}{\frac{1}{4} \cdot N(4,1) + \frac{3}{4} \cdot N(0,1)}$

$P(y=0|x) = \frac{\frac{3}{4} \cdot N(0,1)}{\frac{1}{4} \cdot N(4,1) + \frac{3}{4} N(0,1)}$

c) see script

d) see script

f) see script