Logistic Regression

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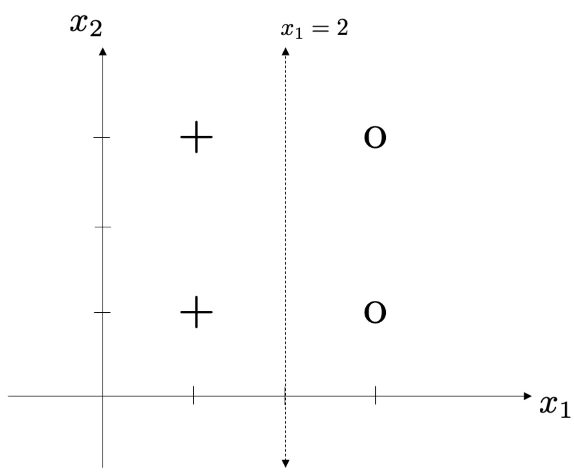
1. Consider the four data points shown in the below. The data points at $(1,1)$ and $(1,3)$ belong to the class labeled $y = 1$, while the data points at $(3,1)$ and $(3,3)$ belong to the class labeled $y = -1$.



a) A decision boundary at $x_1 = 2$ can be expressed as the set of points that satisfy $\boldsymbol{x}^T\boldsymbol{w} = 0$, where

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} \text{ and } \boldsymbol{w} = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \end{bmatrix}$$

Find w_0, w_1, w_2 so that the set of points that satisfy $\boldsymbol{x}^T\boldsymbol{w} = 0$ correspond to the vertical line at $x_1 = 2$. Note that your answer will not be unique. Express your answer for w_1 in terms of w_0.

b) The starting point for motivating logistic regression is to assume that $y \in \{-1, 1\}$ is a Bernoulli random variable with bias that depends on $\boldsymbol{x}$:

$$p(y = 1|\boldsymbol{x}) = \frac{1}{1 + e^{-\boldsymbol{x}^T\boldsymbol{w}}}$$

and

$$p(y = -1|\boldsymbol{x}) = \frac{1}{1 + e^{\boldsymbol{x}^T\boldsymbol{w}}}$$

Find an expression for the likelihood $L(\boldsymbol{w}) = p(y_1, y_2, y_3, y_4)$ given the data in the figure, and assuming *i)* independence and *ii)* that $\boldsymbol{w}$ is restricted to correspond to the result from (a).

c) Find an expression for the negative log-likelihood.

d) Recall that we are usually interested in maximizing the likelihood $L(\boldsymbol{w})$, or equivalently minimizing the negative log-likelihood. Does the likelihood have a maximum value in this case?

e) Sketch or describe the estimate of $p(y = 1|\boldsymbol{x})$ for a large w_0 over the x_1, x_2 plane.

SOLUTION:

a) Since the set of points is a vertical line, there is no dependence on x_2 and $w_2 = 0$. Since $x_1 = 2$, we have $w_0 + 2w_1 = 0$, and conclude

$$\boldsymbol{w} = \begin{bmatrix} w_0 \\ -\frac{w_0}{2} \\ 0 \end{bmatrix}.$$

b) From the independence assumption, we have $p(y_1, y_2, y_3, y_4) = p(y_1)p(y_2)p(y_3)p(y_4)$. Next note that for all four points, $y_i \boldsymbol{x}_i^T \boldsymbol{w} = \frac{w_0}{2}$. Hence,

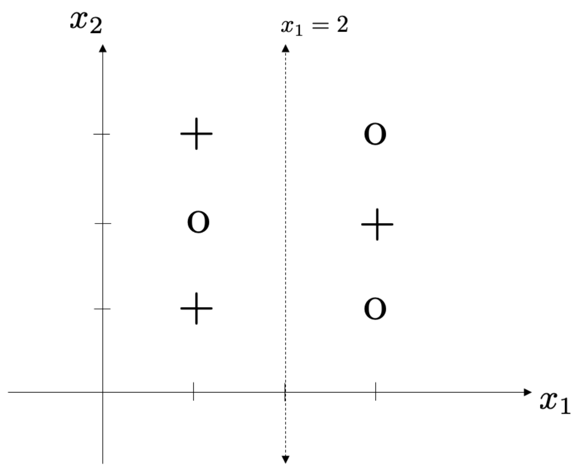
$$p(y_1, y_2, y_3, y_4) = \left(\frac{1}{1 + e^{-w_0/2}} \right)^4$$

c)

d) No. The function continues to approach 1 as w_0 grows, so it has a supremum, but not a maximum.

e) The surface becomes closer and closer to a step function centered at the decision boundary of $x_1 = 2$.

2. You collect two more labeled data points, which are show on the plot below. The points are at $(1, 2)$ and $(3, 2)$.



a) As before, find an expression for the likelihood $L(\boldsymbol{w})$, under the restriction that the decision boundary is a vertical line at $x_1 = 2$.

b) Maximize the likelihood function to specify $\boldsymbol{w}$ (note: you may need to use a computer to find an explicit answer, alternatively, you may leave your answer in a simple, but not explicit, analytic form).

c) Sketch or describe the estimate of $p(y = 1|\boldsymbol{x})$ over the x_1, x_2 plane, and describe the effect of changing w_0 on the surface. How does this compare to your result from problem 1?

SOLUTION:

a) As before, for the points we have $y_i \boldsymbol{x}_i^T \boldsymbol{w} = \frac{w_0}{2}$. For the additional two points, we have $y_i \boldsymbol{x}_i^T \boldsymbol{w} = -\frac{w_0}{2}$ Hence,

$$p(y_1, y_2, y_3, y_4) = \left(\frac{1}{1 + e^{-w_0/2}}\right)^4 \left(\frac{1}{1 + e^{w_0/2}}\right)^2$$

b) To minimize the function, we take the log, and then differentiate and set to zero. After some algebra attempting to solve for w_0, we arrive at the expression $1 = e^{w_0/2} - 2e^{-w_0/2}$.

c) The surface is no longer a step function at the decision boundary. This is to accommodate for the two new points which cannot be correctly classified.