Probabilistic Graphical Models

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1. Consider a joint distribution over a black and white 3×3 image with pixels $x_1, \ldots, x_9$ as shown below:

x_1	x_4	x_7
x_2	x_5	x_8
x_3	x_6	x_9

Note that $\boldsymbol{x}$ can take on one of 2^9 values in $\{0,1\}^9$. Without further assumption, we would need to estimate 2^9 parameters to estimate $p(\boldsymbol{x})$.

a) Recall that by repeated application of the product rule of probability, a joint distribution $p(\boldsymbol{x})$ can be written as:

$$p(\boldsymbol{x}) = p(x_1) \prod_{i=2}^{n} p(x_i | x_1, \ldots x_{i-1}) \tag{1}$$

Write the expression for the joint distribution $p(\boldsymbol{x})$, $\boldsymbol{x} = [X_1, X_2, \ldots, X_9]^T$ using the factorization in expression (1).

b) Is the factorization unique?

c) Assume that each pixel, conditioned on the values of the four pixels immediately to the left/right/above/below, is independent of all the other pixels in the image. This *conditional independence* assumption implies, for example

$$p(x_5 | x_1, x_2, x_3, x_4, x_6, x_7, x_8, x_9) = p(x_5 | x_2, x_4, x_6, x_8).$$

Assume that for a pixel on the edge of the image, for example, x_3,

$$p(x_3 | x_1, x_2, x_4, x_5, x_6, x_7, x_8, x_9) = p(x_3 | x_2, x_6).$$

Simplify your expression from **(a)** by incorporating the conditional independence assumption.

d) Draw a directed graphical model (i.e, a DAG) that matches the decomposition and conditional independence structure found above.

e) How many parameters do you need to estimate to estimate $p(\boldsymbol{x})$ with the conditional independence assumption?