MAP and 1-d Gaussian discriminant analysis

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1. MAP classification and 1-d discriminant analysis. Let $X \in \mathbb{R}$ represent a feature, and $Y = 0$ or $Y = 1$ the class label. The distribution of X depends on the label:

$$X|Y = 0 \sim \mathcal{N}(0, 1)$$

$$X|Y = 1 \sim \mathcal{N}(3, 1)$$

where $\mathcal{N}(\mu, \sigma^2)$ is the Gaussian density: $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Let the prior probability of the classes be $p(y = 0) = 3/4$ and $p(y = 1) = 1/4$.

a) Use a computer to create a plot with both pdfs $f(x|y = 0)$ and $f(x|y = 1)$ on the same axis.

b) Use Bayes and total probability to find an expression for the posterior $p(y|x)$.

c) What is $p(y = 0|x = 1.5)$? What is $p(y = 1|x = 1.5)$? (use a computer)

d) What is the approximate value of x such that $p(y = 0|x) = p(y = 1|x)$? (use a computer)

e) Recall that maximum a posteriori (MAP) classification rule predicts the label y as follows:

$$\widehat{y} = \arg\max_y p(y|x).$$

Argue first why this is equivalent to

$$\widehat{y} = \arg\max_y f(x|y)p(y)$$

and second, why this is equivalent to

$$\widehat{y} = \arg\max_y \log\left(f(x|y)p(y)\right).$$

f) Design a classification rule that will predict y given x. Your answer should be of the form: if $x \geq c$, then $\widehat{y} = 1$, otherwise $\widehat{y} = 0$.