Densities and Expectation

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1. *Unfair dice, convolution.*

a) Consider two independent dice $X \sim p(x)$ and $Y \sim q(y)$ where

$$p(x) = \begin{cases} p_1 & \text{if } x = 1 \\ \vdots & \\ p_6 & \text{if } x = 6 \\ 0 & \text{otherwise} \end{cases}$$

and

$$q(y) = \begin{cases} q_1 & \text{if } y = 1 \\ \vdots & \\ q_6 & \text{if } y = 6 \\ 0 & \text{otherwise.} \end{cases}$$

Find an expression for the pmf of $X + Y$. It may be helpful to specify the ways in which $X + Y = i$ for $i = 2, 3, \ldots, 12$.

b) Next consider the more general case. Let X and Y be integer valued independent random variables with pmfs given by p_X and p_Y, and define the random variable $Z = X + Y$. Show that

$$p_Z(z) = \sum_i p_X(x_i) p_Y(z - x_i).$$

2. In homework 1, you wrote a few lines of code to find the minimum of n i.i.d. samples of a uniform random variable. Here, you will address the same problem analytically. More precisely, let

$$X_i \overset{i.i.d.}{\sim} U[0, 1],$$

and define

$$Y = \min_{i=1,\ldots,n} X_i.$$

a) Find an expression for the pdf of Y. Your answer should depend on n.

b) Find an expression for $E[Y]$.

c) Does this agree with the plot you made previously?

3. *Expectation Basics.* Let $\mathbb{P}(X = 1) = 1/2$, $\mathbb{P}(X = 2) = 1/4$, and $\mathbb{P}(X = 3) = 1/4$.

a) Compute $E[X]$.

b) Compute $E[g(X)]$ if $g(x) = x^2$.

c) The variance of an random variable is defined as $\text{Var}(X) = E[(X - E[X])^2]$. Compute $\text{Var}(X)$.

d) Write an expression for $E[-\log(g(X))]$ in terms of the function $g(X)$.

e) One particular function is $g(X) = \mathbb{P}(X)$, where $\mathbb{P}(X)$ is defined above. Write an expression for $E[-\log_2(\mathbb{P}(X))]$. This quantity is the *entropy* of X.

4. *Bi-variate Random Variables.* The joint PMF $p(x, y)$ is defined as follows:

$$\mathbb{P}(X = 1, Y = 1) = 1/8$$
$$\mathbb{P}(X = 1, Y = 2) = 1/8$$
$$\mathbb{P}(X = 2, Y = 1) = 1/2$$
$$\mathbb{P}(X = 2, Y = 2) = 1/4$$

a) Are X and Y independent? Why or why not?

b) For any two random variables X and Y, the *covariance* is defined as $\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$. Compute $\text{Cov}(X, Y)$.

c) Define a new random variable $Z = X + Y$. Specify the pmf of z $p(z)$.

d) Write a function (in code) that generates X and Y at random as specified by the joint PMF. Your function should take no input arguments, and should return $X, Y \in \{1, 2\}^2$ with probability specified above.

5. *Total Expectation.* Let X and Y be discrete random variables. Conditional expectation is defined as $E_X[X|Y] = \sum_x x \; \mathbb{P}(X = x|Y)$. Use this definition to show that

$$E[X] = E_Y[E_X[X|Y]].$$

6. *MAP classification and 1-d discriminant analysis.* Let $X \in \mathbb{R}$ represent a feature, and $Y = 0$ or $Y = 1$ the class label. The distribution of X depends on the label:

$$X|Y = 0 \sim \mathcal{N}(0, 1)$$

$$X|Y = 1 \sim \mathcal{N}(4, 1)$$

where $\mathcal{N}(\mu, \sigma^2)$ is the Gaussian density: $p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$. Let the prior probability of the classes be $p(y = 0) = 3/4$ and $p(y = 1) = 1/4$.

a) Use a computer to create a plot with both pdfs $p(x|y = 0)$ and $p(x|y = 1)$ on the same axis.

b) Use Bayes and total probability to find an expression for the posterior $p(y|x)$.

c) Use a computer to evaluate $p(y = 0|x = 2)$ using your expression above. What is $p(y = 0|x = 2)$?

d) Recall that maximum a posteriori (MAP) classification rule predicts the label y as follows:

$$\widehat{y} = \arg\max_y p(y|x).$$

Use *maximum a posteriori* to design a classification rule that will predict if $Y = 0$ or $Y = 1$ given $X = x$.

e) What is the *true risk* of your MAP classifier? Use a computer to find a numerical answer.