MMSE, Conditional Independence and MNIST

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1. *Bayes for continuous feature vectors.* Let $\boldsymbol{x} \in \mathbb{R}^n \sim f(\boldsymbol{x})$ be vector of continuous random variables, and let y be a discrete random variable. Show from first principles that

$$p(y|\boldsymbol{x}) = \frac{f(\boldsymbol{x}|y)p(y)}{f(\boldsymbol{x})}.$$

Hint: Start with definition of conditional probability

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A, B)}{\mathbb{P}(B)}$$

and use the definition of a multivariate pdf

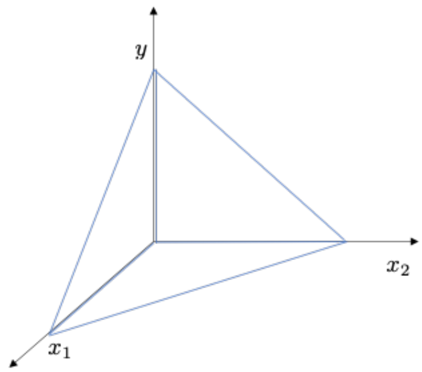
$$f(\boldsymbol{x}) = \lim_{\Delta \to 0} \frac{\mathbb{P}(x_1 < X_1 \leq x_1 + \Delta, x_2 < X_2 \leq x_2 + \Delta, \dots)}{\Delta^n}.$$

SOLUTION: Starting with

$$\begin{aligned}
p(y|\boldsymbol{x}) &= \mathbb{P}(Y = y|\boldsymbol{x}) \\
&= \lim_{\Delta \to 0} \mathbb{P}(Y = y | x_1 \leq X_1 \leq x_1 + \Delta, x_2 \leq X_2 \leq x_2 + \Delta, \dots) \\
&= \lim_{\Delta \to 0} \frac{\mathbb{P}(x_1 \leq X_1 \leq x_1 + \Delta, x_2 \leq X_2 \leq x_2 + \Delta, \dots | Y = y)\mathbb{P}(Y = y)}{\mathbb{P}(x_1 \leq X_1 \leq x_1 + \Delta, x_2 \leq X_2 \leq x_2 + \Delta, \dots)} \\
&= \lim_{\Delta \to 0} \frac{\frac{1}{\Delta^n}\mathbb{P}(x_1 \leq X_1 \leq x_1 + \Delta, x_2 \leq X_2 \leq x_2 + \Delta, \dots | Y = y)\mathbb{P}(Y = y)}{\frac{1}{\Delta^n}\mathbb{P}(x_1 \leq X_1 \leq x_1 + \Delta, x_2 \leq X_2 \leq x_2 + \Delta, \dots)} \\
&= \frac{f(\boldsymbol{x}|y)p(y)}{f(\boldsymbol{x})}
\end{aligned}$$

2. *Regression with MMSE.* Consider a feature vector $\boldsymbol{x} \in \mathbb{R}^2$ and $y \in \mathbb{R}$ with joint pdf

$$p(\boldsymbol{x}, y) = \begin{cases} 6 \text{ for } x_1, x_2, y \geq 0 \text{ and } x_1 + x_2 + y \leq 1 \\ 0 \text{ otherwise.} \end{cases}$$



You aim to find a function $f(\boldsymbol{x})$ to estimate y.

a) Find the function $f(\boldsymbol{x})$ that minimizes $E[\ell(f(\boldsymbol{x}), y)]$ under the squared error loss function.

b) What is the minimum true risk $E[\ell(f(\boldsymbol{x}), y)]$, under the squared error loss function? You can leave your answer as as integral.

SOLUTION:

a) First recall that the risk is given by $E[\ell(f(\boldsymbol{x}), y)] = E[(y - E[y|\boldsymbol{x}])^2]$. The function $f(\boldsymbol{x})$ that minimize risk under squared error is the MMSE estimate, given by $E[y|\boldsymbol{x}]$. We first find the marginal $p(y|x)$. From the picture, for a fixed x_1, x_2 with $x_1 + x_2 \leq 1$,

$$p(y|\boldsymbol{x}) = c \text{ for } y \leq 1 - (x_1 + x_2)$$

and after normalizing

$$p(y|\boldsymbol{x}) = \frac{1}{1 - (x_1 + x_2)} \text{ for } y \leq 1 - (x_1 + x_2).$$

To find $E[y|\boldsymbol{x}]$, we calculate:

$$\begin{aligned} E[y|\boldsymbol{x}] &= \int_0^{1-(x_1+x_2)} \frac{y}{1 - (x_1 + x_2)} dy \\ &= \frac{1 - (x_1 + x_2)}{2} \end{aligned}$$

b) We need to compute the value of $E[\ell(f(\boldsymbol{x}), y)] = E[(y - E[y|\boldsymbol{x}])^2]$. For a fixed $\boldsymbol{x}$, y is uniformly distributed over $0 \leq y \leq 1 - (x_1 + x_2)$, we have the minimum MSE

$$\text{var}(y|\boldsymbol{x}) = \frac{(1 - (x_1 + x_2))^2}{12}$$

since the variance of a uniform random variable over a, b is $(b-a)^2/12$. The next step is finding the expected value over $\boldsymbol{x}$, i.e.,

$$E_{\boldsymbol{x}}[\text{var}(y|\boldsymbol{x})].$$

To compute $p(\boldsymbol{x})$, we integrate out y, and have

$$p(\boldsymbol{x}) = 6(1-(x_1+x_2)) \text{ for } x_1+x_2 \leq 1.$$

$$\begin{aligned} E_{\boldsymbol{x}}[\text{var}(y|\boldsymbol{x})] &= \int_{\boldsymbol{x}\in\Omega} p(\boldsymbol{x})\text{var}(y|\boldsymbol{x})d\boldsymbol{x} \\ &= \int_0^1 \int_0^{1-x_2} \frac{(1-(x_1+x_2))^3}{2} dx_1 dx_2 \\ &= \frac{1}{40} \end{aligned}$$

3. *Bayes Nets on MNIST*. Note that Activity 12 will be helpful for completing this problem. Recall the MAP classification rule:

$$\widehat{y} = \arg\min_y p(\boldsymbol{x}|y)p(y)$$

By repeated application of the product rule of probability, a distribution $p(\boldsymbol{x}|y)$ can be written as:

$$p(\boldsymbol{x}|y) = p(x_1|y)p(x_2|x_1, y)p(x_3|x_1, x_2, y)p(x_4|x_1, x_2, x_3, y)\ldots p(x_n|x_1, \ldots, x_{n-1}, y).$$

Last time we used *Naïve Bayes* to estimate $p(\boldsymbol{x}|y)$. Recall that the class $y \in \{0, 1, \ldots, 9\}$ represents the true digit, while $\boldsymbol{x} \in \{0,1\}^{784}$ is the 28×28 black and white image that we'd like to classify. Naïve Bayes make the often poor assumption that $p(\boldsymbol{x}|y) \approx \prod_{i=1}^n p(x_i|y)$.

In this problem we will make a compromise to model relationships *between* pixels. More specifically, we will estimate $p(\boldsymbol{x}|y)$ by making a *conditional independence assumption*: we will assume that the probability of a pixel, conditioned on the values of the pixels to the left and above, it is independent of all the other pixels with a lower index in the image.

a) Imagine the pixels are enumerated so that the x_1 is the pixel in the upper left corner of the image, x_2 is the pixel below x_1 (in the first column and second row) and so on. Simplify the expression

$$p(x_{34}|x_1, x_2, \ldots, x_{33})$$

if the pixels are conditionally independent given their neighbors (the pixel immediately to the left and above).

b) Find an expression for $p(\boldsymbol{x}|y)$ given the conditional independence assumptions. You can ignore any discrepancy for edges cases (when x_i corresponds to a pixel in the first column or bottom row).

c) How many parameters do we need to estimate? How does this compare to the Naive Bayes case or the general case?

d) We can estimate $p(x_i = 0|x_j = 0, x_k = 0, y = 9)$ (for example) empirically by counting the number of times pixel i is equal to zero when $x_j = 0, x_k = 0$, vs. the number of times it is equal to 1 when $x_j = 0, x_k = 0$ among the images that are labeled $y = 9$:

$$p(x_i = 0|x_j = 0, x_k = 0, y = 9) =$$

$$\frac{1 + \text{count of examples with pixel } i \text{ equal to 0 where } x_j = 0, x_k = 0 \text{ from class 9}}{2 + \text{count of examples where } x_j = 0, x_k = 0 \text{ from class 9}}.$$

The 1 in the numerator and 2 in the denominator is called 'Laplace Smoothing', and deals with cases where there are no corresponding examples by assigning them an uncommitted value of 1/2.

Write code that estimates $p(x_i = 0|x_j, x_k, y)$ and $p(x_i = 1|x_j, x_k, y)$ by counting the occurrences. Note that for each pixel, you will have to consider 80 cases: 2 values for the pixel itself $x_i \in \{0, 1\}$, 4 cases for the conditionals, $(x_j, x_k) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, times 10 cases for $y = 0, 1, \ldots, 9$. Store your estimates of $p(x_i|y)$ in a $2 \times 10 \times 4 \times 28 \times 28$ array.

e) Write code that computes log likelihood of a new image for each class $y = 1, \ldots, 9$. Recall that the log likelihood is given by $\log(p(\boldsymbol{x}|y))$ when provided with a test image $\boldsymbol{x}$.

f) Use maximum likelihood and your estimate of the log likelihood to classify the test images. What is the classification error rate of the maximum likelihood classifier on the $10k$ test images?

SOLUTION:

a) $p(x_{34}|x_1, x_2, \ldots, x_{34}) = p(x_{34}|x_6, x_{33})$

b) $p(\boldsymbol{x}|y) = \prod_{i=1}^{728} p(x_i|x_{i-1}, x_{i-28})$

c) For each pixel, there are four possible states for the two neighbors. Each possible state corresponds to one parameter that we must estimate. If we don't adjust for the edge cases or the parent pixel, this means we have approximately 4×784 parameters. It is four times the Naive Bayes case, but much smaller than the general case.

d) See the included code.