A Feasibility Procedure for Unidirectional Forward-Pass Heuristics for Multi-Item Single-Level Capacitated Lot-Sizing Problems*

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Abstract

Some inconveniences of unidirectional forward-pass heuristics for multi-item single-level capacitated dynamic lot-sizing problems are pointed out. After giving the necessary and sufficient conditions for a feasible solution, a feasibility procedure is suggested to overcome these inconveniences.
Keywords: Production/Inventory: Capacitated Lot-Sizing; Algorithm: Heuristics

多产品单层能力受限批量问题的前向启发式算法的可行性过程

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摘 要: 本文讨论多产品、单层、能力受限批量问题的前向启发式算法，指出了这类算法中现有算法的一些不足之处．为了克服这些不足，我们给出了可行解存在的充分必要条件，并在此基础上提出了一个可行化算法．
关键词: 生产库存，能力受限批量问题，启发式算法

1. Introduction

The multi-item single-level capacitated lot-sizing problem (CLSP) consists of scheduling N products (or items) over a horizon of T periods. In this problem, demands are known for each product at each period and should be satisfied without backlogging. The objective is to minimise the sum of setup and inventory-holding costs over the entire planning horizon

* This research project is supported by NSFC project No. 69904007. It is also supported by the Direct Grant, Faculty of Business Administration, The Chinese University of Hong Kong.

subject to an aggregate capacity constraint in each period. Mathematically, this problem can be formulated as follows [4].

$$Min\sum_{i=1}^{N}\sum_{t=1}^{T}\{s_i Y_{it} + h_i I_{it}\} \tag{1}$$

$$s.t. \quad I_{i,t-1} + X_{it} - I_{it} = d_{it}, \qquad i = 1,\cdots,N, t = 1,\cdots,T, \tag{2}$$

$$\sum_{i=1}^{N} X_{it} \le C_t, \qquad t = 1,\cdots,T, \tag{3}$$

$$Y_{it} = \begin{cases} 0, & if \quad X_{it} = 0, \\ 1, & if \quad X_{it} > 0, \end{cases} \qquad i = 1,\cdots,N, t = 1,\cdots,T, \tag{4}$$

$$I_{it}, X_{it} \ge 0, \qquad i = 1,\cdots,N, t = 1,\cdots,T, \tag{5}$$

$$Y_{it} \in \{0,1\}, \qquad i = 1,\cdots,N, t = 1,\cdots,T, \tag{6}$$

$$I_{i0} = I_{iT} = 0, \qquad i = 1,\cdots,N, \tag{7}$$

where the given parameters are

N = the number of products,

T = the number of time periods,

d_{it} = the external demand for product i in period t ,

s_i = the setup cost for product i ,

h_i = the holding cost for unit end-of-period inventory of product i ,

C_t = the available capacity for the capacitated resource (or facility) in period t ,

and the decision variables are

X_{it} = the amount of product i produced in period t (lot-size),

Y_{it} = a binary variable indicating where production is allowed for product i in period t ,

I_{it} = the inventory of product i at the end of period t .

We assume that all the parameters in the formulation are nonnegative real numbers or integers. The objective function (1) means to minimize the total cost, which includes setup and inventory holding costs. The constraint (2) is the conservation equation for materials and the constraint (3) enforces capacity feasibility. The constraints (4) and (6) imply that the setup corresponding to a product must be paid before producing the product. The non-negative restrictions for inventory and production quantities in constraint (5) assure no backlogging occurs and the last equality (7) states that there are no initial and final inventories for any products.

From equations (2) and (7), the inventory for product i at the end of period t, i.e., the excess production for product i until period t, can be rewritten as

$$I_{it} = \sum_{\tau=1}^{t}(X_{i\tau} - d_{i\tau}), \qquad i = 1,2,\ldots,N, t = 1,2,\ldots,T. \tag{8}$$

Since a single bottleneck facility often limits the output of the entire production system, the solution to this problem is of practical interest within many industrial settings, e.g., in the application of material requirements planning (MRP) systems [2]. However, the problem is known to be NP-hard since the single item capacitated dynamic lot-sizing problem is already NP-hard[3]. Optimal solution methods have failed to solve all but very small problems within reasonable computation times. Therefore, many researchers studied the problem and a lot of heuristics were developed [2,4,5]. One of the best known heuristics is the unidirectional forward-pass algorithm proposed in [1]. However, we found that the feasibility procedure for this heuristic has some inconveniences. Actually, they may not stop at a feasible solution even though there exists one. The purpose of this paper is to present an enhancement to the heuristic to ensure feasibility.

2. Inconveniences of an existing feasibility procedure

A unidirectional forward-pass algorithm begins by determining the lot sizes for earlier periods before moving to study the later periods. In such an algorithm, one way to avoid infeasibilities or backtracking is to use the feasibility constraints to put lower bounds and upper bounds on production lot sizes. A possible lower bound for cumulative production of all products until any period t is proposed in [1] and we restate it using our notation as follows.

Lemma 1[1] In any period t (t=1,2,...,T),

$$\sum_{i=1}^{N}\sum_{\tau=1}^{t}(X_{i\tau}-d_{i\tau})\geq R_t, \tag{9}$$

or equivalently

$$\sum_{i=1}^{N} I_{it}\geq R_t, \tag{10}$$

where

$$R_t=\begin{cases} Max(\sum_{i=1}^{N} d_{i,t+1}+R_{t+1}-C_{t+1},0), & for \quad t=1,2,\cdots,T-1, \\ 0, \quad for \quad t=T. \end{cases} \tag{11}$$

It is obvious that R_t is the required minimum excess aggregate production for any period t, i.e., an amount of production that will not be used before period t+1, but on the other hand needs to be produced not later than period t. In order to make clear how to disaggregate this value into the lot sizes of the N products, another variable, L_{it}, is defined in [1] as the portion of I_{it} that can be used to meet the R_t requirement.

Theorem 1[1] $$L_{it}=\begin{cases} Min\{R_t,d_{i,t+1}+L_{i,t+1}\}, & for \quad t<T, \\ 0, & for \quad t=T. \end{cases} \tag{12}$$

Based on L_{it}, a possible upper bound for X_{it} is presented in [1] as follows.

Theorem 2[1] In a forward-pass algorithm, during the lot-sizing of any period t, while X_{it} is being reviewed for increases, for any i=1,2,...,N the amount to be pulled into production for product i can not exceed δ_{it} as given below:

$$\delta_{it} = Min\{\delta_{it}^1, \delta_{it}^2\}, \tag{13}$$

where

$$\delta_{it}^1 = \sum_{\tau=1}^{T} d_{i\tau} - \sum_{\tau=1}^{t} X_{i\tau}, \tag{14}$$

$$\delta_{it}^2 = C_t - \sum_{k=1}^{N} X_{kt} - \omega_{it}, \tag{15}$$

$$\omega_{it} = Max\left(\omega_{it}^1 - \omega_{it}^2, 0\right), \tag{16}$$

$$\omega_{it}^1 = Max\left(R_t - \sum_{k=1}^{N} Min\left[\sum_{\tau=1}^{t} (X_{k\tau} - d_{k\tau}), L_{kt} \right], 0 \right), \tag{17}$$

$$\omega_{it}^2 = Max\left(L_{it} - \sum_{\tau=1}^{t} (X_{i\tau} - d_{i\tau}), 0 \right). \tag{18}$$

According to the arguments in [1], ω_{it}^1 is the portion of R_t yet to be filled (note that it only depends on period t but does not depend on product i) and ω_{it}^2 is the portion of possible further production of product i in period t which can be used to meet R_t. Therefore ω_{it} denotes how much of ω_{it}^1 cannot be filled by further production of product i in period t, and δ_{it}^2 is the portion of remaining capacity in period t which can be used to further produce for product i. Furthermore, δ_{it}^1 demotes the portion of total requirement of product i which still needs to be produced until period t, thus the minimum of δ_{it}^1 and δ_{it} is an upper bound for further production of product i in period t.

The above arguments seem reasonable. However, according to the definition of L_{it} ("the portion of I_{it} that can be used to meet the R_t requirement"), it is a natural conclusion that $L_{it} \le I_{it}$ and the value of L_{it} should be dependent on the value of I_{it}, which contradicts Theorem 1. Therefore the value of L_{it} calculated from equation (12) is not what is defined in the paper. Actually, the value of L_{it} calculated from equation (12) is the possible maximum portion of I_{it} that can be used to meet the R_t requirement. That's to say, at most L_{it} can be used to meet R_t at any cases, and it may be possible that what can be used to meet R_t is actually less than L_{it} even if $I_{it} \ge L_{it}$. For this misunderstanding exists, the upper bound calculated in Theorem 2 is also misleading when it is used to guide a forward-pass lot-sizing algorithm. Ignoring the objective of cost reduction, the feasibility procedure proposed by [1] for the unidirectional forward-pass algorithm can be simplified as following:

Algorithm 1

Step 0. Calculate R_t and L_{it} for all i and t. Set X_{it}=0 for all i and t. Set t=0.

Step 1. Set t=t+1. If $t>T$, then a feasible solution is got, stop. Otherwise set

$$X_{it} = Max\left(\sum_{\tau=1}^{t} d_{i\tau} - \sum_{\tau=1}^{t-1} X_{i\tau}, 0 \right) \text{ for all } i. \tag{19}$$

Table 1. An Example

		t=1	t=2	t=3
D_{it}	i=1	0	0	5
	i=2	0	0	7
	i=3	0	12	0
C_t		10	10	10
R_t		4	2	0
L_{it}	i=1	2	2	0
	i=2	2	2	0
	i=3	4	0	0
X_{it} using Algorithm 1	i=1	5	0	
	i=2	5	0	
	i=3	0	?	
X_{it} using Algorithm 2	i=1	5	0	0
	i=2	0	0	7
	i=3	2	10	0

Set i=0.

Step 2. If

$$R_t \le \sum_{k=1}^{N} Min\left[\sum_{\tau=1}^{t}(X_{k\tau}-d_{k\tau}), L_{kt}\right], \tag{20}$$

then go to Step 1. Otherwise set i=i+1.

Step 3. If i>N, then no feasible solution exists, stop. Otherwise calculate δ_{it} as given in (13) and further calculate

$$q = Max\left[Min\left(\delta_{it}, L_{it}-\sum_{\tau=1}^{t}(X_{i\tau}-d_{i\tau})\right), 0\right]. \tag{21}$$

Set X_{it}= X_{it}+q and go to Step 2.

If we use it to guide a feasibility procedure, we may not reach a feasible solution even if there exists one for the original problem. A three-item three-period problem is given in Table 1. For this example, the above procedure will set X_{11}= X_{21}=5, X_{31}=0 for the lot sizes in the first period. However, in the second period, the above procedure will set X_{12}= X_{22}=0, X_{32}=12>C_2, thus the resulted schedule is infeasible.

3. Exact lower and upper bounds for production

In order to really avoid infeasibilities in a unidirectional forward-pass algorithm, we must first present the exact lower bounds and upper bounds on production lot sizes.

Theorem 3 In a unidirectional forward-pass algorithm, during the lot sizing of any period t, while X_{it} are being reviewed for increase, for any i=1,2,...,N the amount to be pulled into production for product i can not be less than l_{it} as given below:

$$l_{it} = \underset{t \le \tau \le T}{Max}(\alpha_{i\tau}, 0), \tag{22}$$

where

$$\alpha_{i\tau} = Max\left[\sum_{j=1}^{\tau} d_{ij} - \sum_{j=1}^{t} X_{ij}, 0\right] - \sum_{j=t+1}^{\tau} C_j . \tag{23}$$

Proof In the equation (23), $Max\left[\sum_{j=1}^{\tau} d_{ij} - \sum_{j=1}^{t} X_{ij}, 0\right]$ is the amount which must be produced for product i from periods t to τ. However, $\sum_{j=t+1}^{\tau} C_j$ is the cumulated capacity available from periods t+1 to τ. Thus, $\alpha_{i\tau}$ is the amount which must be produced for product i in period t in order to ensure feasibility for product i till period τ. Therefore, l_{it} is the minimum amount which must be pulled into production for product i in periods t.

Theorem 4 In a unidirectional forward-pass algorithm, during the lot sizing of any period t, while X_{it} is being reviewed for increase, for any i=1,2,...,N the amount to be pulled into production for product i can not exceed u_{it} as given below:

$$u_{it} = Min\{u_{it}^1, u_{it}^2\}, \tag{24}$$

where

$$u_{it}^1 = \sum_{\tau=1}^{T} d_{i\tau} - \sum_{\tau=1}^{t} X_{i\tau}, \tag{25}$$

$$u_{it}^2 = C_t - \sum_{k=1}^{N} X_{kt} - \underset{t \le \tau \le T}{Max}\left\{\sum_{k \ne i} Max\left[\sum_{j=1}^{\tau} d_{kj} - \sum_{j=1}^{t} X_{kj}, 0\right] - \sum_{j=t+1}^{\tau} C_j\right\}. \tag{26}$$

Furthermore, if the increase in X_{it} is below u_{it} as given in (24), then there exists a feasible solution for periods from t to T.

Proof The necessary and sufficient condition for a feasible solution to exist for periods from t to T is that, for any period $\tau \ge t$, the cumulated net requirement for all the products from periods t to τ does not exceed the cumulated remaining capacity from periods t to τ. That's to say, the condition is, for all $\tau = t, t+1, \ldots, T$,

$$\sum_{k=1}^{N} Max\left[\sum_{j=1}^{\tau} d_{kj} - \sum_{j=1}^{t} X_{kj}, 0\right] \le C_t - \sum_{k=1}^{N} X_{kt} + \sum_{j=t+1}^{\tau} C_j . \tag{27}$$

According to the logic of the unidirectional forward-pass algorithm, we can assume this condition holds before X_{it} is being reviewed for increase. While X_{it} is being reviewed for increase by an amount of Δ_{it}, we only need to consider the impact of the increase on the inequality (27). For any given τ, if $\sum_{j=1}^{\tau} d_{ij} - \sum_{j=1}^{t} X_{ij} - \Delta_{it} \geq 0$, then both sides of (27) will be reduced by an amount of Δ_{it} when X_{it} is increased by Δ_{it}, thus the inequality (27) still holds. However, if $\sum_{j=1}^{\tau} d_{kj} - \sum_{j=1}^{t} X_{kj} - \Delta_{it} < 0$, then only the right side of (27) will be reduced by Δ_{it}, thus the inequality (27) becomes

$$\sum_{k \neq i} Max\left[\sum_{j=1}^{\tau} d_{kj} - \sum_{j=1}^{t} X_{kj}, 0\right] \leq C_t - \sum_{k=1}^{N} X_{kt} - \Delta_{it} + \sum_{j=t+1}^{\tau} C_j, \tag{28}$$

or

$$\Delta_{it} \leq C_t - \sum_{k=1}^{N} X_{kt} - \left\{\sum_{k \neq i} Max\left[\sum_{j=1}^{\tau} d_{kj} - \sum_{j=1}^{t} X_{kj}, 0\right] - \sum_{j=t+1}^{\tau} C_j\right\}. \tag{29}$$

Recalling the formula (26), and noticing that the condition (29) must hold for all $\tau = t$, $t+1,\ldots,T$, the condition (29) is equivalent to

$$\Delta_{it} \leq u_{it}^2. \tag{30}$$

On the other hand, from the definition (25), u_{it}^1 is the remaining unmet demand for product i. In order to make (7) hold, Δ_{it} can not exceed u_{it}^1. Thus, u_{it} as given in (24) is an exact upper bound for Δ_{it}, the possible further production of product i in period t. This completes the proof for Theorem 4.

Theorem 5 In a unidirectional forward-pass algorithm, during the lot sizing of any period t, while X_{it} are determined for all i=1,2,...,N, the necessary and sufficient condition for a feasible solution to exit for periods from t+1 to T is

$$\beta_\tau \geq 0, \qquad \text{for all } \tau = t+1, t+2,\ldots,T, \tag{31}$$

where

$$\beta_\tau = \sum_{j=t+1}^{\tau} C_j - \sum_{k=1}^{N} Max\left(\sum_{j=1}^{\tau} d_{kj} - \sum_{j=1}^{t} X_{kj}, 0\right). \tag{32}$$

Proof In equation (32), $Max\left(\sum_{j=1}^{\tau} d_{kj} - \sum_{j=1}^{t} X_{kj}, 0\right)$ is the cumulated net requirement for product k from period t+1 to τ, after X_{it} are determined for all i=1,2,...,N in period t. Therefore, β_τ is the cumulated capacity from periods t+1 to τ, reduced by the cumulated net requirement for all products from periods t+1 to τ. Thus we know that the necessary and sufficient condition for a feasible solution to exist is that (31) holds.

4 Feasibility procedure

Ignoring the objective of cost reduction, a feasibility procedure can be suggested based on the exact lower and upper bounds given in Section 3.

Algorithm 2

Step 0. Set X_{it}=0 for all i and t. Set t=0.

Step 1. Set t=t+1. If t>T, then a feasible solution is got, stop. Otherwise set $X_{it} = l_{it}$ for all i, as defined in (22). Set i=0.

Step 2. Compute β_τ for all τ = t+1, t+2,…,T. Check whether (31) holds or not. If yes, then go to Step 1. Otherwise set i=i+1.

Step 3. If i>N, then no feasible solution exists, stop. Otherwise calculate u_{it} as given in (24) and set X_{it}= X_{it}+u_{it}. Go to Step 2.

If we use this algorithm to guide a feasibility procedure, we will definitely reach a feasible solution if there exists one for the original problem. The three-item three-period problem given in Section 2 is resolved using Algorithm 2 and is shown in the last three rows in Table 1. We also programmed the algorithm and incorporated it into some well-known capacitated lot-sizing heuristics to evaluate the performance of the feasibility procedure. The numerical results from a large set of test problems show that the algorithm performs well in reasonable computing times.

5. Summary

This study presents a feasibility procedure to the unidirectional forward-pass algorithms for CLSP. Furthermore, the exact lower and upper bounds provided in this paper can be incorporated into other capacitated lot-sizing heuristics to guide their feasibility procedures.

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