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GRAPH THEORY AND INTEGER PROGRAMMING

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It has become clear that integer programming and combinatorics are tied together much more closely than thought before. A very large part of combinatorics deals, or can be formulated as to deal, with optimization problems in discrete structures. Generally, the constraints and the objective function are linear forms of certain variables, which are restricted to integers or — mostly — to 0 and 1. Thus the combinatorial problem is translated to a linear integer programming problem. Of course, the value of such a translation depends on whether or not it provides new insight or new methods for the solution. We hope that we shall be able to show several examples where the translation to integer programming is very useful, if not essential, for the solution. We shall restrict ourselves to problems concerning graphs.

Historically, the first theorem in graph theory with integer programming flavour was the Marriage Theorem [29]. Linear programming didn't exist at that time and it took another half century until the famous book of Ford and Fulkerson appeared [17], in which one of the first applications of linear programming to graph theory was given. Another early hint of this relationship is in the paper of Gallai [21]. Now the linear programming formulation of combinatorial problems is a common approach.

In this paper we first survey some of the most important results in integer programming which have been successfully applied to graph theory and then discuss those fields of graph theory where an integer programming approach has been most effective. Due to the limited space we shall not go into some of the more advanced fields like the theory of blocking and anti-blocking polyhedra, and shall touch those fields of graph theory where the merging of combinatorial, algorithmic and discrete programming ideas has produced new disciplines, like flow theory or matching theory. We have to ignore the algorithmic aspects completely (although these may be the most important). Since our main concern is the interaction between graph theory and discrete optimization, we shall give only those proofs where this is most characteristic. On the other hand, we list many graph theoretical results which have a linear programming flavour but no explicit treatment this way so far.

1. Some preliminaries from discrete programming

Discrete (linear) programming deals with the problem of finding (say) the maximum of $c^{\mathrm{T}}x$, where c is a given vector and x is a *lattice point* (a point with integral coordinates) in a convex polyhedron described by

$$Ax \leqslant b. \tag{1}$$

Dropping the condition that x is an integer we obtain a new problem, the *linear relaxation*, which is relatively easily solvable; at any rate, a minimax formula for the optimum using the Duality Theorem can be derived.

Since the linear optimum of $c^{\mathrm{T}}x$ is always attained at a vertex (provided (1) describes a bounded polyhedron, which we assume in this discussion for sake of simplicity), if the vertices of (1) are lattice points then the linear relaxation has the same optimum as the original discrete problem. So it is extremely important to know when are the vertices of (1) lattice points. The following theorem is implicite in results of Gomory [23], Chvátal [10], Hoffman [24], Fulkerson [18], Edmonds and Giles [15].

Theorem 1.1. *The vertices of* (1) *are lattice points if and only if* $\max\{c^{\mathrm{T}}x \mid Ax \leqslant b\}$ *is an integer for every integral vector c.*

Using the Duality Theorem, this condition can be rephrased as: $\min\{b^{\mathrm{T}}y \mid A^{\mathrm{T}}y = c,\ y \geqslant 0\}$ is an integer for every integral vector c. Usually b is an integer vector and then what can be verified is that the vertex of $\{A^{\mathrm{T}}y = c,\ y \geqslant 0\}$ which minimizes $b^{\mathrm{T}}y$ is a lattice point.

Sometimes it suffices to consider a smaller set of vectors c to test in this problem [31, 32].

Theorem 1.2. *Assume that A is a* 01*-matrix. Then* $\{Ax \leqslant 1, x \geqslant 0\}$ *has integral vertices iff* $\max\{c^{\mathrm{T}}x \mid Ax \leqslant 1, x \geqslant 0\}$ *is an integer for every* 01*-vector c.*

This result has many versions, most of them discovered in order to find applications in graph theory. See Lovász [34, 35].

The most important and most widely used observation which yields polyhedra with integral vertices is due to Hoffman and Kruskal [25].

Theorem 1.3. *Assume that A is a totally unimodular matrix* (*i.e. every square submatrix has determinant* ± 1 *or* 0), *and b an integral vector. Then* $Ax \leqslant b$ *has integral vertices.*

(The proof of this assertion is very easy: any vertex is the intersection of some hyperplanes of the form $a_i^{\mathrm{T}}x = b_i$ where a_i^{T} is a row of A and b_i is the corresponding entry of b. Determine this point by solving this system of equations by Cramer's rule. The denominator is a non-zero subdeterminant of A and hence ± 1; the numerators are integers. So the solution has integral entries.)

Assume now that the polyhedron P described by (1) does *not* have integral vertices. Then the polyhedron $\hat{P}$ which is the convex hull of lattice points in P is properly contained in P. If we can write up the system of linear inequalities characterizing $\hat{P}$ we are through with most difficult part of our task again. A general scheme to obtain these inequalities is the following. Assume that $a^{\mathrm{T}}x = b$ is a hyperplane which touches P at a vertex and, moreover, a^{T} is integral, but b is not. Then $a^{\mathrm{T}}x \leqslant [b]$ is an inequality which eliminates a piece of P but certainly does not cut off anything from $\hat{P}$. Inequalities obtained by this construction are called *cuts*. It follows from the famous algorithm of Gomory [23], and was proved independently and more directly by Chvátal [10] that

Theorem 1.4. *If P is a (bounded) polytope then adding appropriate cuts repeatedly we obtain the convex hull $\hat{P}$ of lattice points in P in a finite number of steps.*

The Gomory algorithm also specifies how to choose these cuts, but in general it is very complicated. In some cases, e.g. for matchings, the appropriate cuts can be described in a very nice way [13].

Some notation from graph theory

$V(G)$: set of vertices,
$E(G)$: set of edges,
$\bar{G}$: complement of G,
$\nu(G)$: maximum number of disjoint edges (matching number),
$\alpha(G)$: maximum number of independent points (stability number),
$\tau(G)$: minimum number of points covering all edges,
$\rho(G)$: minimum number of edges covering all points,
$\chi(G)$: chromatic number,
$\chi'(G)$: chromatic index,
$\omega(G)$: maximum number of pairwise adjacent points (clique number).

2. Flow theory

We begin with a classical application of discrete optimization ideas to graph theory (or vice versa).

Let G be a directed graph with two specified vertices a, b, and a "capacity" $c(x, y) \geqslant 0$ asigned to every edge (x, y). A *flow* is a function f that assignes an "intensity" $f(x, y) \geqslant 0$ to each edge such that Kirchoff's law is satisfied, i.e.

$$\sum_x f(x, p) = \sum_x f(p, x) \tag{1}$$

for every point $p \neq a, b$. The amount

$$v(f) = \sum_x f(a, x) - \sum_x f(x, a) \tag{2}$$

is called the *value* of the flow. The flow is *feasible* if

$$0 \leq f(x, y) \leq c(x, y) \tag{3}$$

for all (x, y). The task is to find the maximum possible value of a feasible flow.

It is immediately seen that what we have is a linear programming problem with constraints (1) and (3) and objective function (2). There is even no integrality constraint involved. Let us consider the dual:

$$\text{minimize} \quad \sum_{x,y} g(x, y)c(x, y)$$

subject to

$$g(x, y) \geq h(x) - h(y),$$
$$g(x, y) \geq 0,$$
$$h(a) = 1, \qquad h(b) = 0.$$

Now it is very easy to see that the matrix of this l.p. problem is totally unimodular, so looking for an optimal solution we may assume that the variables $h(x)$, $g(x, y)$ are integers. Also trivially $0 \leq h(x) \leq 1$ and $g(x, y) = \max(0, h(x) - h(y))$. Thus an optimal solution is defined by specifying a set $S \subset V(G)$ such that $a \in S$, $b \notin S$, and setting

$$h(x) = \begin{cases} 1 & \text{if } x \in S, \\ 0 & \text{if } \notin S, \end{cases}$$

$$g(x, y) = \begin{cases} 1 & \text{if } x \in S, y \notin S, \\ 0 & \text{otherwise} \end{cases}$$

and so the minimum value of flow is

$$\min_{a \in S \subseteq V(G) - b} \sum_{\substack{x \in S \\ y \notin S}} c(x, y).$$

This result is the famous *max-flow-min-cut theorem.* Since the original problem also has a totally unimodular matrix, whenever the capacities are integral then one of the optimum flows is integral.

The max-flow-min-cut theorem, with the supplement given, implies a variety of important-graph-theoretical results, such as Menger's Theorem (see e.g. [7]), but has applications to seemingly distant combinatorial problems such as the construction of certain resolvable designs [4].

We cannot go into the numerous interesting and extremely important extensions of flow theory, but refer to Hu [27], Rothschild and Whinston [51], Hoffman [24] and the monograph by Hu [26].

3. Further minimax theorems

The following minimax result was conjectured by Robertson and Younger, and proved in 1973 by Lucchesi and Younger. See also Lovász [34, 35].

We say that a set S determines a *directed cut* D if $S \neq \emptyset$, $S \neq V(G)$ and no edge connects $V(G)-S$ to S. Then D is defined as the set of edges connecting S to $V(G)-S$.

Theorem 3.1. *Let G be a digraph. Then the maximum number of edge-disjoint directed cuts in G equals to the minimum number of edges covering all directed cuts.*

If we wrote up the two linear programs whose integral optima were the two numbers in question, we could not apply any of the basic discrete programming methods directly. So we have to do some combinatorial preparation.

We say that the directed cuts D_1 determined by S_1 and D_2 determined S_2 are *crossing* if $S_1 \cap S_2 \neq \emptyset$, $S_1 \cup S_2 \neq V(G)$, $S_1 \nsubseteq S_2$ and $S_2 \nsubseteq S_1$. In this case $S_1 \cap S_2$ and $S_1 \cup S_2$ also determine directed cuts, which will be denoted by $D_1 \wedge D_2$ and $D_1 \vee D_2$.

Consider now the directed-cut-packing program. We have a variable x_D assigned to each directed cut D and

$$\text{minimize } \sum_D x_D \tag{1}$$

subject to

$$x_D \geqslant 0 \qquad (\forall D),$$

$$\sum_{D \ni e} x_D \leqslant c(e) \quad (\forall e \in E(G)), \tag{2}$$

where the $c(e)$ are arbitrary integers.

Let D_1, D_2 be the two crossing cuts with $0 < x_{D_1} \leqslant x_{D_2}$. Define

$$x'_D = \begin{cases} 0 & \text{if } D = D_1, \\ x_{D_2} - x_{D_1} & \text{if } D = D_2, \\ x_{D_1 \wedge D_2} + x_{D_1} & \text{if } D = D_1 \wedge D_2, \\ x_{D_1 \vee D_2} + x_{D_1} & \text{if } D = D_1 \vee D_2, \\ x_D & \text{otherwise.} \end{cases}$$

Then

$$x'_D \geqslant 0 \qquad (\forall D),$$

$$\sum_{D \ni e} x'_D = \sum_{D \ni e} x_D \quad (\forall e),$$

and

$$\sum_D x'_D = \sum_D x_D.$$

So if (x_D) is an optimum solution of (1)–(2) then so is (x'_D). Repeating this procedure it is easy to see that we never get to a cycle and in a finite number of steps we end up with an optimum solution (x_D) of (1)–(2) such that the set

$$\mathcal{D} = \{D : x_D > 0\}$$

consists of pairwise non-crossing directed cuts. So (1)–(2) is replaced by the program

$$\text{maximize} \sum_{D \in \mathcal{D}} x_D \tag{3}$$

subject to

$$x_D \geqslant 0 \qquad (\forall D \in \mathcal{D}),$$

$$\sum_{D \ni e} x_D \leqslant c(e) \quad (\forall e \in E(G)), \tag{4}$$

where $\mathcal{D}$ is a family of non-crossing directed cuts. But it is well known that the matrix of (4) is totally unimodular and so, by the Hoffman–Kruskal Theorem, the maximum of $\sum x_D$ is attained at integral values of x_D. In particular if $c(e) = 1$ then this maximum is the maximum number of edge-disjoint directed cuts.

Consider now the dual of (1)–(2); we have a variable y_e for each edge e and want to

$$\text{minimize} \sum_{e} c(e) y_e \tag{5}$$

subject to

$$y_e \geqslant 0 \qquad (\forall e),$$

$$\sum_{e \in D} y_e \geqslant 1 \quad (\forall D).$$

We know by the above that the minimum of $\sum_e c(e) y_e$ is an integer for all integral $c(e)$. Hence by Theorem (1.2), the polytope (6) has integral vertices and hence the minimum is attained for an integral choice of the y_i's. Again if $c(e) = 1$ then this minimum is the minimum number of edges covering all directed cuts.

Generalizations of the Lucchesi–Younger Theorem were given by Edmonds and Giles [15]. The "Chinese Postman Theorem" of Edmonds and Johnson [16] can be viewed as an analogue for undirected graphs (cf. [36]).

The packing of other kinds of subgraphs has also been a favourite problem in graph theory. Very little is known about packing circuits or directed circuits; it is an old problem of Gallai [22] whether the ration between the (discrete) optima of the directed circuit packing problem and its dual remains bounded.

Some results are known about the problem of packing circuits through a given point [34, 35]. This is strongly related to the problem of packing paths with both endpoints in a specified subset. This problem was considered by Nash–Williams [41], Gallai [22], Lovász [30, 34, 35], Mader (1977), Karzanov-Lomonosov (1977), and Seymour (1977).

The problems of packing spanning trees in a graph [56], covering a graph by spanning trees [42], packing spanning arborescences rooted at a given point [14], or rooted cuts [20] are solved.

It is my impression that the common nature of these results is not yet completely understood, especially in terms of discrete programming.

4. Matching and point packing

The following are four important optimization problems in graph theory.

(a) *matching*: given a graph G, determine the maximum number $\nu(G)$ of disjoint edges.

(b) *point-packing*: determine the maximum number $\alpha(G)$ of independent points.

(c) *point-covering*: determine the minimum number $\tau(G)$ of points covering all edges.

(d) *edge-covering*: determine the minimum number $\rho(G)$ of edges covering all points. (Here we assume that there are no isolated points.)

In all cases, we may consider a weighting of the elements (points or edges) and may want to determine the optimum total weight of the sets in consideration.

The problems above are not independent. The complement of a point-cover is an independent set of points and vice versa, therefore

$$\alpha(G)+\tau(G)=|V(G)|.$$

The following assertion is less trivial:

$$\gamma(G)+\rho(G)=|V(G)|.$$

These formulas are called *Gallai's identities.* They justify that we shall restrict ourselves to problems (a) and (b).

Even these problems are not independent: the matching problem is the point-packing problem for the line-graph. But the matching problem is much easier in the sense that much more can be said about it.

If we want to use linear programming we have to introduce the following notation. Let $V(G)=\{v_1,\ldots,v_n\}$, $E(G)=\{e_1,\ldots,e_m\}$. Let $A=(a_{ij})_{i=1}^{n}{}_{j=1}^{m}$ be the point-edge incidence matrix of G, i.e.

$$a_{ij}=\begin{cases}1 & \text{if } v_i\in e_j,\\ 0 & \text{otherwise}\end{cases}.$$

Let $x_1,\ldots,x_n$, $y_1,\ldots,y_m$ be variables. We shall identify a subset $S\subset V(G)$ with its characteristic vector (x_i), where

$$x_i=\begin{cases}1 & \text{if } v_i\in S,\\ 0 & \text{otherwise,}\end{cases}$$

and similarly for subsets of edges.

Let M(G) and PP(C) denote the convex hulls of all matchings and point-packings, respectively. These are clearly polytopes. If we could write up the systems of linear inequalities which describe these polytopes, in other words if we could find their facets, then we could find the optima of problems (a) and (b), at least in a sense. In fact, maximizing—say—the form $\sum_{j=1}^{m} y_j$ over the polytope M(G) we get the maximum number of independent edges, since there always exists an optimum feasible solution which is a vertex of M(G), i.e. matching.

So let us try to find—first—valid inequalities for these polytopes and then see how close we are to the solution. Clearly every matching $y=(y_j)_{j=1}^{m}$ satisfies

$$\begin{aligned} Ay &\leqslant 1, \\ y &\geqslant 0, \end{aligned} \tag{1}$$

or, written out in detail,

$$\begin{aligned} \sum_{e_j \ni v_i} y_j &\leqslant 1 \qquad (i=1,\ldots,n), \\ y_j &\geqslant 0 \qquad (j=1,\ldots,m). \end{aligned} \tag{1'}$$

Similarly, every independent set $x=(x_i)_{i=1}^{n}$ of points satisfies

$$\begin{aligned} xA &\leqslant 1, \\ x &\geqslant 0. \end{aligned} \tag{2}$$

In general, the polytopes (1), (2) properly contain the polytopes M(G), PP(G). Let e.g. G be a triangle, then the polytope described by (1) contains the vertex $(\frac{1}{2},\frac{1}{2},\frac{1}{2})^{\mathrm{T}}$ which is not a convex combination of matchings (Fig. 1).

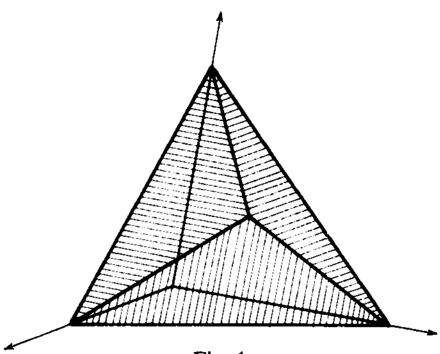

Fig. 1.

However, each *lattice point* satisfying (1) is a matching; therefore it is natural to call the points in (1) and (2) *fractional matchings* and *fractional point packings*, resp. Denote the polytopes (1) and (2) accordingly by FM(G) and FPP (G).

The case of bipartite graphs is very simple and well-known:

Theorem 4.1. *The following are equivalent:*
(i) *G is bipartite;*
(ii) *its point-edge incidence matrix A is totally unimodular;*
(iii) $\mathrm{FM}(G)$ *has integral vertices,* i.e. $\mathrm{FM}(G)=\mathrm{M}(G)$;
(iv) $\mathrm{FPP}(G)$ *has integral vertices,* i.e. $\mathrm{FPP}(G)=\mathrm{PP}(G)$.

Theorem 4.1 has important consequences for bipartite graphs. $\nu(G)$ is the integral optimum of $1\cdot x$, subject to (1). Since $\mathrm{FM}(G)$ has integral vertices, this is the same as the real optimum. The dual problem gives $\tau(G)$. So the Duality Theorem implies

König's Theorem 4.2. *In a bipartite graph $\nu(G)=\tau(G)$.*

If we want to proceed to non-bipartite graphs, we face two questions:
(a) describe the vertices of the polytopes $\mathrm{FM}(G)$, $\mathrm{FPP}(G)$.
(b) describe the facets of $\mathrm{M}(G)$, $\mathrm{PP}(G)$.
The first problem is easier, and completely solved [1, 2, 43].

Theorem 4.3. *Let $C_1,\ldots,C_p$ be disjoint odd circuits and M a matching, vertex-disjoint from the C_i's in the graph G. Set*

$$x_1=\begin{cases}\frac{1}{2} & \text{if } e_i\in C_1\cup\cdots\cup C_p,\\ 1 & \text{if } e_i\in M,\\ 0 & \text{otherwise.}\end{cases}$$

Then (x_i) is a vertex of $\mathrm{FM}(G)$ *and all vertices of* $\mathrm{FM}(G)$ *arise this way.*

Theorem 4.4. *Let G be a graph and A, B, C a partition of $V(G)$ such that (a) A is independent, (b) B contains all neighbors of A and (c) no component of the subgraph induced by C is bipartite. Then*

$$x_i=\begin{cases}1 & \text{if } v_i\in A,\\ 0 & \text{if } v_i\in B,\\ \frac{1}{2} & \text{if } v_i\in C,\end{cases}$$

defines a vertex of $\mathrm{FPP}(G)$, *and every vertex of* $\mathrm{FPP}(G)$ *arises this way.*

These results have a variety of graph theoretical applications. We only mention a few. Tutte (1953) calls a set of edges a *q-factor*, if it forms vertex-disjoint circuits and edges, and covers all points.

Theorem 4.5. *A graph has a q-factor iff every independent set A of points has at least $|A|$ neighbors.*

The following version of this result was observed by J. Edmonds; it follows by the same kind of argument. A 2-*matching* is a collection of edges such that any point belongs to at most two of them. A 2-*cover* is a collection of points such that every edge contains at least two of them.

Theorem 4.6. *The maximum number of edges in a 2-matching equals to the minimum number of points in a 2-cover.*

Now we turn to the question of describing the facets of M(G). This was answered by Edmonds [13]:

Theorem 4.7. M(G) *is described by the following set of inequalities:*

(I) $x_j \geqslant 0 \qquad (j = 1, \dots, m)$,

(II) $\sum_{e_j \ni v_i} x_j \leqslant 1 \qquad (i = 1, \dots, n)$,

(III) $\sum_{e_j \subseteq S} x_j \leqslant \frac{|S|-1}{2} \qquad (S \subseteq V(G), |S| \text{ odd})$.

Moreover, for every linear objective function with integral coefficients the dual problem has an integral optimum solution.

So to obtain M(G) from FM(G) only the cuts (III) are needed; not even all of them, because Edmonds and Pulleyblank [49] proved:

Theorem 4.8. $\sum_{e_j \subseteq S} x_j \leqslant (|S|-1)/2$ *is a facet of* M(G) *iff* S *spans a* 2-*connected graph* G_S *such that* $G_S - v$ *has a perfect matching for every* $v \in S$.

The last two theorems can be derived from an analysis of the Edmonds Matching Algorithm. Let us sketch here a short direct proof of the first part of Theorem (3.9).

Let

$$\sum_{i=1}^{m} a_i x_i \leqslant b \tag{3}$$

be a facet of M(G); we want to show that it is of one of the forms (I), (II) or (III).

Case 1. Assume that there is an i_0 such that $a_{i_0} < 0$. Then every matching (x_i) which gives equality in (3) must satisfy $x_{i_0} = 0$. Consequently, (3) must be identical with the facet $x_{i_0} \geqslant 0$.

Case 2. Assume that there is a point v_j such that every matching with equality in (3) contains an edge adjacent to v_j. Then every such matching satisfies $\sum_{e_i \ni v_j} x_i = 1$ and, consequently, (3) must be identical with the facet $\sum_{e_i \ni v_j} x_i \leqslant 1$.

Case 3. Assume that $a_i \geqslant 0$ for all i and that for each point j, there is a matching avoiding v_j and giving equality in (3). Let G' be the graph formed by those edges e_i for which $a_i > 0$. It is easily seen that G' is connected. Next one shows that no matching which gives equality in (3) misses two points of G.

This can be seen as follows. Assume that M_1 is a matching missing both points $x, y \in V(G')$. We use induction on the distance of x and y in G'. Trivially, x and y cannot be adjacent in G', so we may choose a point z on the shortest path connecting x to y. Let M_2 be a matching which gives equality in (3) and misses z. By the induction hypothesis, M_1 covers z and M_2 covers both of x and y. Consider the connected components of $M_1 \cup M_2$ containing x and y, respectively. These are paths which may be identical, but at least one of them, P say, misses z. Set

$$M' = M_1 - (M_1 \cap P) \cup (M_2 \cap P)$$

and

$$M'' = M_2 - (M_2 \cap P) \cup (M_1 \cap P)$$

Then M' and M'' are matchings; furthermore, they are extreme matchings because

$$2b \geq \sum_{e_i \in M'} a_i x_i + \sum_{e_i \in M''} a_i x_i = \sum_{e_i \in M_1} a_i x_i + \sum_{e_i \in M_2} a_i x_i = 2b.$$

But now M'' is an extremal matching which misses z and one of x and y, contrary to the induction hypothesis.

Now it follows that if $V(G') = S$ then $|S|$ is odd and every matching with equality in (3) satisfies

$$\sum_{e_i \subseteq S} x_i = \frac{|S|-1}{2}.$$

Consequently, facet (3) is of the form (III).

Edmonds' theorem has various applications in graph theory. First, let us remark that Tutte's famous condition for the existence of a 1-factor can be derived from it. Let us show another one:

The *chromatic index* of a graph is defined as the minimum number of colors necessary to color the edges so that adjacent edges have different colors. This number is clearly at least as large as the maximum degree and quite often larger. The following result is due to Edmonds [13].

Theorem 4.9. *Let G be a k-regular k-edge-connected graph with an even number of points. Then there exists a number t such that if we replace every edge by t parallel edges the resulting graph G has chromatic index kt.*

Proof. *Consider the point* $(1/k, \ldots, 1/k)^{\mathrm{T}}$. Straightforward computation shows that it satisfies all 3 types of inequalities in Edmonds' theorem, and hence it belongs to M(G). Therefore it is a convex combination of vertices of M(G):

$$\left(\frac{1}{k},\ldots,\frac{1}{k}\right)^{\mathrm{T}}=\sum_{n=1}^{N}\alpha_i F_1,\qquad \sum_{i=1}^{N}\alpha_i=1,\quad \alpha_i\geqslant 0.$$

where the F_i are matchings. Clearly one may assume that every α_i is rational. Let P be a common denominator of the numbers $k\alpha_i$ and $k\alpha_i=P_i/P$. Then

$$(P,\ldots,P)^{\mathrm{T}}=\sum P_iF_i,\qquad \sum_{i=1}^{N}P_i=kp.$$

So if we multiply edge e_i by P then the resulting graph can be decomposed into the union of $\sum P_i=kp$ matchings.

Remark. The least value of P here is not known. A conjecture of Fulkerson states $P\leqslant 2$ for $k=3$. Seymour [52] proved that if $k=3$ then P can be chosen a power of 2.

A second application in which the full strength of Edmonds' theorem can be used is the Chinese Postman Problem (Edmonds and Johnson [16]). A postman must traverse every street of a town and return to the post office. He wants to minimize the total length of his walk. In other words: Given a connected graph G with a "length" c_i assigned to every edge. Find a closed walk which contains every edge at least once and has minimum length. Note that if the graph is Eulerian then any Eulerian trail is such a walk. This motivates the following reformulation: Find a collection $\mathcal{T}$ of edges of G with minimum total length such that the number of edges of $\mathcal{T}$ incident with a point x of G has the same parity as the degree of x.

Let us divide the set of points of odd degree in pairs (x_1, y_i) $(i=1,\ldots,t)$ and take a path P_i connecting x_i and y_i. Then $\mathcal{T}=\sum_{i=1}^{t}P_i$ is a collection with the desired property. It is easy to see that, conversely, the optimal $\mathcal{T}$ arises this way. The total length of edges in $\mathcal{T}$ is equal to the sum of "lengths" of the P_i. So the problem is transformed to the following: let $d(x,y)$ denote the minimal length of (x,y)-paths in G. Partition the points of G with odd degree into pairs $(x_1,y_1),\ldots,(x_t,y_t)$ such that $\sum_{i=1}^{t}d(x_i,y_i)$ is minimal.

But this is the problem of finding a maximum-weight matching of the complete graph on the set of points of G with odd degree, and can be solved by the methods of matching theory.

Let an *odd cut* be defined as the set of edges connecting some $S\subseteq V(G)$ to $V(G)-S$, provided it has odd cardinality.

Theorem 4.10. *The minimum number of edges whose doubling results in an Eulerian graph equals half of the maximum number of odd cuts, containing each edge at most twice.*

The point-packing polytope has a much more difficult structure. As remarked before, $PP(G) = FPP(G)$ if the graph G is bipartite. If G is not bipartite then $PP(G)$ must have facets of more complicated form than $FPP(G)$. The following class of facets is immediately seen:

$$\sum_{i \in C} x_i \leq 1, \tag{1}$$

where C is an arbitrary clique [18]. The class of graphs for which these are all is wider than bipartite graphs, and is very interesting from the graph-theoretical point of view.

A graph G is called *perfect* if every induced subgraph G' of G satisfies $\chi(G') = \omega(G')$. This notion has been introduced by Berge [6]. There are many examples of perfect graphs. The most important is probably the following:

Let $V(G)$ be a partially ordered set and connect two points iff they are comparable.

Let G be a graph and let H_x $(x \in V(G))$ be a vertex disjoint graphs. Consider $\bigcup H_x$ and add all edges between H_x and H_y if $(x, y) \in E(G)$. We shall say that the resulting graph is obtained by substituting the graphs H_x for the points of G. The following theorem uses this construction [31, 32]; its proof is quite easy and elementary.

Theorem 4.11. *Substituting perfect graphs for the points of perfect graphs we obtain a perfect graph.*

The next result provides further examples [31, 32]:

Theorem 4.12. *The complement of a perfect graph is perfect.*

The result was conjectured by Berge [6]. Its strong connection to linear programming was discovered by Fulkerson [18, 19], who proved it in a slightly weaker form.

We prove (3.13) together with the following theorem [11, 18, 19],

Theorem 4.13. *G is perfect iff the polytope*

$$\begin{aligned} &x_i \geq 0 \qquad (i = 1, \dots, n), \\ &\sum_{i \in C} x_i \leq 1 \qquad (\forall \text{cliques } C \text{ in } G) \end{aligned} \tag{5}$$

has integral vertices.

We shall show that G is perfect $\Rightarrow$ (5) has integral vertices $\Rightarrow \bar{G}$ is perfect. The cycle closes by interchanging the role of G and $\bar{G}$.

I. Assume first that G is perfect. Let x be a vertex of (5). Clearly x is rational and so for some integer $k > 0$, $kx = (p_1, \dots, p_n)$ is a lattice point. Substitute a complete p_i-graph for v_i ($i=1,\dots,n$). Let C' be any clique in the resulting graph G' and let C be its "projection" to G. Then

$$|C'|\le \sum_{i\in C}|P_i| = k\sum_{i\in C} x_i \le k.$$

Since G' is perfect by 3.12, $\chi(G')=\omega(G')\le k$. Let $\{A_1',\dots,A_k'\}$ be a k-coloration of G', let A_i be the "projection" of A_i' to G and a_i the corresponding vector. Then

$$\frac{1}{k}\sum_{i=1}^{n} a_i = x.$$

Since x is a vertex, this implies that $x=a_1=\cdots=a_k$, i.e. x is a lattice point.

II. Assume now that (5) has integral vertices. Note that the induced subgraphs of G have similar property, since the corresponding polytopes are faces of (5).

The face $1\cdot x\le\alpha(G)$ of (5) is contained in several facets, at least one of which is of the form $\sum_{i\in C}x_i\le 1$. This means combinatorially that the clique C intersects every maximum independent set, i.e. $\alpha(G-C)<\alpha(G)$. Repeating the argument we get $\alpha(G)$ cliques which cover all points of G. This proves that $\chi(\bar G)=\alpha(G)=\omega(\bar G)$. Since this follows for the induced subgraphs of G similarly, we conclude that G is perfect.

It is natural to study those graphs which are not perfect but all of their induced subgraphs are. Let us call these graphs *critically imperfect*. The famous Strong Perfect Graph Conjecture of Berge asserts that every critically imperfect graph is a chordless odd circuit of length ≥ 5 or the complement of such a circuit. It is known [32] that

Theorem 4.14. *A critically imperfect graph G has $\alpha(G)\omega(G)+1$ points.*

Padberg [45, 46] proved, among many interesting structural results on these graphs, that

Theorem 4.15. *If G is a critically imperfect graph then the polytope* (5) *has precisely one non-integral vertex* (*the point $1/\omega(G)\cdot 1$*) *and this is contained in precisely n facets.*

For further important results on perfectness see Chvátal [11], Meyniel [39], Parthasaraty and Ravindra [48].

As the Berge conjecture suggests, further classes of facets of $\mathrm{PP}(G)$ can be derived from chordless odd circuits C in G or $\bar G$. Assume first that G is itself an odd circuit. Then

$$\sum_{i=1}^{n} x_i \le \frac{n-1}{2} \tag{6}$$

is a facet. Similarly, if G is the complement of an odd circuit then

$$\sum_{i=1}^{n} x_i \leq 2 \tag{7}$$

is a facet. Now if $C \neq G$ then (6) yields a facet of PP(G); in fact, there is a general "lifting" procedure which derives a facet of PP(G) from a facet of PP(G') where G' is an induced subgraph of G. Geometrically, this can be described as follows. It suffices to consider the case $G' = G - v_n$. Let F be a facet of PP(G').

Since PP(G') = PP(G) $\cap \{x_n = 0\}$, F is an $(n-2)$-face of PP(G) and hence, F is the intersection of two facets of PP(G). One of these is clearly $x_n = 0$, the other is the *lift* of F. For example, lifting the facet $x_1 + \cdots + x_5 \leq 2$ of the pentagon, we obtain the facet $x_1 + \cdots + x_5 + 2x_6 \leq 2$ of the 5-wheel (Fig. 2).

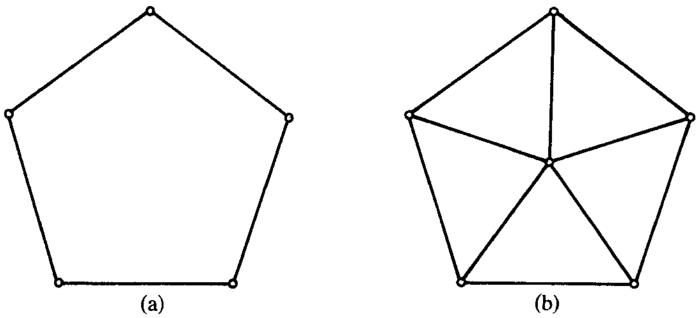

Fig. 2.

So in a sense it suffices to study those facets which are *essential* in the sense that they do not arise by lifting. This procedure is due to Padberg [47] and Nemhauser and Trotter [43].

An interesting class of facets has been described by Chvátal [11]. Call an edge e of a graph G *α-critical* if $\alpha(G-e) > \alpha(G)$.

Theorem 4.16. *Let G be a graph whose α-critical edges form a spanning connected graph. Then*

$$\sum_{i=1}^{n} x_i \leq \alpha(G)$$

is a facet of PP(G).

Proof. It is trivial that this is a valid inequality. To show that it is a facet it suffices to prove that if

$$a_i x_i \leq b \tag{8}$$

is a facet of PP(G) such that every maximum matching satisfies it with equality then $a_1 = \cdots = a_n$. Assume indirectly $a_1 \neq a_2$ (say). We may assume that v_1 and v_2

are adjacent. Let T be an independent $(\alpha+1)$-set in G-(v_1, v_2). Obviously $v_1, v_2 \in T$. Let $T - v_i$ have characteristic vector $x^{(i)}$ $(i = 1, 2)$. Then

$$1 \cdot x^{(1)} - 1 \cdot x^{(2)} = a_2 - a_1 \neq 0,$$

so it is impossible that both $x^{(1)}$ and $x^{(2)}$ give equality in (S).

These facets may and may not be essential. The structure of graphs to which theorem (3.17) applies can be very complicated, e.g. all connected α-critical graphs are among them. This fact already indicates that a general description of the facets of PP(G) in the spirit of Edmonds' theorem is hopeless.

However, there may be important classes of graphs for which the facets of PP(G) can be described. A very recent result of Minty [40] seems to point out such a class. Minty has a polynomial-bounded algorithm to determine $\alpha(G)$ provided G contains no 3-star (claw) as an induced subgraph. Since all line-graphs are claw-free this result considerably extends Edmond's algorithm for finding maximum matchings. One would expect that, analogously, Edmonds' description of the matching polytope (which is equivalent to a description of the facets of PP(G) if G is a line-graph) could be extended to a description of the facets of claw-free graphs. It is interesting that there is another result which points out the importance of claw-free graphs when independence number is investigated: this is the theorem of Parthasarathy and Ravindra [48] stating that the strong perfect graph conjecture is true for claw-free graphs.

We close this topic with the following interesting result [43] which underlines the importance of considering FPP together with PP:

Theorem 4.17. *Let x_0 be a vertex of* FPP *which maximizes* $1 \cdot x$. *Then there is a vertex x_1 of* PP *which maximizes* $1 \cdot x$ *(i.e. a maximum independent set) such that x_1 coincides with x_0 in all coordinates which are integral in x_0.*

Concluding remarks

1. One further field where linear programming has been applied in graph theory is the problem of Hamilton circuits. This is related to the well-known Travelling Salesman problem. We only refer here to the papers of Chvátal [10], Bellmore and Nemhauser [5], Christofides [9].

2. The chromatic theory of graphs is a field of possible further development. One can define the chromatic number as the discrete optimum of a linear program, and investigate the corresponding linear optimum. The two are equal for perfect graphs, but very little is known about their relationship in general. See e.g. M. Rosenfeld [50], Stahl [53], Lovász [36] and Chen and Hilton [8].

3. It is often an interesting question when are two vertices of a combinatorially defined polytope neighboring. This is solved for M(G) and PP(G) [11], and some

other classes (Hausman and Korte 1976). This points out how little is known about *faces* of these polyhedra even if the *facets* are known.

4. If there is no hope to fully describe the facets of a polytope (e.g. of PP(G)) it is still a possibility to find some kind of classification. This hope is supported by the fact that α-critical graphs, which provide a large variety of facets by (3.17), do have an interesting classification [33]. It might be the case that some classes of facets cut down FPP(G) to a polytope "not much larger" than PP(G).

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