HOMOLOGICAL ALGEBRA: THE LONG EXACT SEQUENCE

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Abstract. The focus of this paper will be to cover the Long Exact Sequence in Homological Algebra. In order to do this we need to understand basic concepts like abelian groups, rings, kernels, etcetera. We also need to cover concepts of category theory like morphisms of objects and the relationships between them. Having this knowledge we are then able to grasp chain complexes where we can then head on to short exact sequences where we will look over some diagram lemma's that will lead us to our understanding of the long exact sequence.

1. Introduction

Homology, in mathematics, is a way of relating algebraic objects, such as abelian groups, to other objects such as a group or topological space. A circle has a one-dimensional hole, while a disk has none, whereas a sphere has a two-dimensional hole. These shapes more specifically are manifolds. (Manifolds are topological spaces that locally resembles Euclidean space near each point, but globally may not. For example, one-dimensional manifolds include lines and circles, while two-dimensional manifolds include spheres, tori, and Klein bottles. Each manifold has cycles, classified by dimensions of the manifold, which can be thought of as objects to split and put back together, that can have in different orientations.)

1.1. Basics.

In this chapter, we begin by going over fundamental definitions that are needed to explain homological algebra.

Definition 1.1. A **group** is a set A together with a binary operation $\cdot\colon A \times A \to A$ such that:

- *Closure*: For all a and b in A, $a \cdot b$ is also in A.
- *Associativity*: For all a, and b and c in A, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
- *Identity*: There exists an element e in A, such that for all elements a in A, $e \cdot a = a \cdot e = a$.
- *Inverse*: For each a in A, there exists b in A, such that $a \cdot b = b \cdot a = e$, the identity.

Example 1.2. The group of integers $\mathbb{Z}$ under addition, denoted $(\mathbb{Z}, +)$, is a group while $(\mathbb{Z}, \cdot)$, with the operation of multiplication instead of addition, do not form a group. For $(\mathbb{Z}, \cdot)$, the reason is that the closure, associativity and identity axioms are satisfied, but inverses do not exist. For instance, $a = 2$ is an integer, but the only solution to the equation $a \cdot b = 1$ in this case is $b = 1/2$, which is a rational number, but not an integer. Therefore, not every element of $\mathbb{Z}$ has a (multiplicative) inverse.

Definition 1.3. An **abelian** group is a set A with binary operator $\cdot\colon A \times A \to A$ such that all previously shown axioms for a group hold, along with:

- *Commutativity*: For all a and b in A, $a \cdot b = b \cdot a$.

Example 1.4. $(\mathbb{Z}, +)$, is an abelian group that satisfies the axioms accordingly: associativity by addition, additive identity by zero, additive inverse, $-n$, of every n, and commutativity since $k + n = n + k$ for any two integers k and n.

Definition 1.5. A **ring** (abelian group with additional qualifiers) is a set R with binary operators $+\colon R \times R \to R$ and $\cdot\colon R \times R \to R$ such that:

(1) R is an abelian group under addition:
   - *Commutativity*: For all a and b in R, $a + b = b + a$.
   - *Associativity*: For all a, and b and c in R, $(a + b) + c = a + (b + c)$.
   - *Identity*: There exists an element 0 in R, such that for all elements a in R, $0 + a = a + 0 = a$.
   - *Inverse*: For each a in R, there exists $-a$ in R, such that $a + (-a) = (-a) + a = 0$, the identity.

(2) R is associative under multiplication and has a multiplicative identity:
   - *Associativity*: For all a, and b and c in R, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
   - *Identity*: There exists an element 1 in R, such that for all elements a in R, $1 \cdot a = a \cdot 1 = a$.

(3) Multiplication, with respect to addition, is distributive (from both sides):
   - *Left side*: For all a, and b and c in R, $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.
   - *Right side*: For all a, and b and c in R, $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$.

Example 1.6. The set 2-by-2 matrices with real number entries written

$$\mathcal{M}_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a, b, c, d \in \mathbb{R} \right\}$$

is a ring. With operations addition and multiplication, this set satisfies the axioms above with the matrix, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, being the multiplicative identity. If, for instance, we have $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $AB = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; This ring is noncommutative.

Definition 1.7. A **left *R*-module** M (or ${}_RM$) consists of an abelian group $(M, +)$ and an binary operation $\cdot\colon R \times M \to M$ such that for all r, s in R and x, y in M, we have:

- $r \cdot (x + y) = r \cdot x + r \cdot y$,

- $(r+s)\cdot x = r\cdot x + s\cdot x$,
- $(rs)\cdot x = r\cdot(s\cdot x)$, and
- $1_R \cdot x = x$, where 1_R is the multiplicative identity in R.

A **right R-module** M (or M_R) behaves similarly, except the ring acts on the right, i.e., $\cdot\colon M\times R \to M$, where r and s are written on the right-hand side of x and y.

In this paper, a module, usually denoted M, refers to a finitely generated left R-module over a ring as above. Given a module M over a ring R we write $M \xrightarrow{x} M$, where $x\in R$, for the R-module homomorphism $\varphi\colon M \longrightarrow M$ defined by $\varphi(x) = xm$ for all $m\in M$.

Example 1.8. If R is a ring, we can define the ring, R^x, which has the same underlying set and the same addition operation, but the opposite multiplication as follows: if $ab = c$ in R, then $ba = c$ in R^x. Any left R-module M can then be seen to be a right module over R^x, and any right module over R can be considered a left module over R^x.

Definition 1.9. Let G and H be groups and f be a group homomorphism from G to H. Then the **kernel** of f is

$$\ker f = \{g\in G \mid f(g) = e_H\},$$

where e_H is the identity element of H. Similarly for rings, $\ker f = \{r\in R \mid f(r) = 0_S\}$ where 0_S is the zero element.

Example 1.10. In the category of topological spaces, if $f\colon X\to Y$ is a continuous map, then the preimage of K, is a subspace of X. The inclusion map of K into X ($g\colon K\to X \mid f(k) = k$ for all $k\in K$) is the categorical kernel of f.

Definition 1.11. Given $f\colon X\to Y$ as a function from set X to set Y, and if x is a member of X, then $f(x) = y$ is the **image** of x under f. The image of a subset $A\subseteq X$ under f is the subset $f(A)\subseteq Y$ denoted as:

$$\operatorname{im} f = f(A) = \{y\in Y \mid y = f(x), \text{for some } x\in A\}.$$

Example 1.12. Let $f\colon \mathbb{R}\to\mathbb{R}$ defined by $f(x) = x^2$. The image of $\{-1,2\}$ under f is $f(\{-1,2\}) = \{1,4\}$, and the image of f is $\mathbb{R}^+$ while the preimage of $\{1,4\}$ under f is $f^{-1}(\{1,4\}) = \{-2,-1,1,2\}$. The preimage of set $N = \{n\in\mathbb{R} \mid n<0\}$ under f is the empty set, because the negative numbers do not have square roots in the set of real numbers.

Definition 1.13. A morphism $f\colon X\to Y$ in a category is an **isomorphism** if it admits a two-sided inverse, meaning that there is another morphism $g\colon Y\to X$ in that category such that $g\circ f = 1_X$ and $f\circ g = 1_Y$, where 1_X and 1_Y are the identity morphisms of X and Y, respectively [2].

Definition 1.14. An **epimorphism** is a morphism $f\colon X\to Y$ that is right-cancellative, that is, for all morphisms $g_1,\, g_2\colon Y\to Z$,

$$g_1\circ f = g_2\circ f \quad\implies\quad g_1 = g_2.$$

Similarly, an **monomorphism** is a morphism $f\colon X \to Y$ that is left-cancellative, that is, for all morphisms $g_1,\, g_2\colon Z \to X$,

$$f \circ g_1 = f \circ g_2 \quad \implies \quad g_1 = g_2.$$

2. Homological Algebra

Homological Algebra is a branch of mathematics that studies homology on a more general algebraic stage. It studies the partitions of space and geometric components (lines, triangles, etc) and the interrelationships between these components. Homology, originating from the topological branch of mathematics along with abstract algebra, became prevalent around the closing of the 19th century. It was developed through investigations mostly by Henri Poincaré and David Hilbert. Homology is closely related to category theory, the study of mathematical structures using objects and arrows to show their relations. Mostly, homological algebra is the study of functors (the mappings or morphisms between categories) and their usefulness to lead us to chain complexes (algebraic structures showing the relationships between objects in a topological space).

2.1. **Category Theory.** In Category theory, a category consists of objects (sets) that are linked by arrows (functions). These categories have identity maps and associative operations on their arrows. In general, a category is used to describe the mathematical pieces and their relationships. Category theory purposefully tries to generalize all of mathematics through these said objects and arrows. This is nice because practically all of mathematics can be placed into categories which shows the similarities between branches. Examples of categories are **Set**: sets and their functions, **Ring**: rings and ring homomorphisms, and **Top**: topological spaces and their continuous maps.

Definition 2.1. A **category** C consists of two classes, a class $obj(C)$ of objects and a class $hom(C)$ of morphisms (see following definition of functor).

Example 2.2. The class of all sets, together with all functions between sets, where composition is the usual function composition, forms a large category, Set. It is the most basic and the most commonly used category in mathematics. The category Rel, on the other hand, consists of all sets with binary relations as morphisms. A few concrete categories are:

- **Ring** (as above) - objects: rings | morphisms: ring homomorphisms
- **Grp** - objects: groups | morphisms: group homomorphisms
- **R-mod** - objects: R-modules | morphisms: R-module homomorphisms.

Definition 2.3. A **functor** consists of two objects, a *source* and a *target*. Commonly, objects are sets where the functor is a map (geometrically, an arrow or line) from one object to another, with *domain* as the source and *codomain* as the target.

As shown in [5], a functor from set A to set B is a mapping that does the following:

(1) Associates to each object X in A an object $F(X)$ in B, and
(2) Associates to each morphism $f\colon X \to Y$ in A a morphism $F(f)\colon F(X) \to F(Y)$ in B, such that
  - for every object X in A, $F(1_X) = 1_{F(X)}$, and
  - for all morphisms $f\colon X \to Y$ and $g\colon X \to Y$ in A, $F(g \circ f) = F(f) \circ F(g)$.

Example 2.4. Consider

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "g\circ f"'] & Y \arrow[d, "g"] \\ & Z, \end{tikzcd}$$

where $hom(X, Y)$ would be the collection of morphisms from X to Y, and $g \circ f$ is the zero morphism from X to Z (kernel of g, an object X together with morphism f).

2.2. **Chain Complexes.** A chain complex is a way of representing a relationship through algebraic structures. Specifically, in homological algebra, they represent no underlying space and therefore are studied through axioms.

Definition 2.5. A **chain complex**

$$M_*\colon \quad \cdots \longrightarrow M_{n+1} \xrightarrow{f_n} M_n \xrightarrow{f_{n-1}} M_{n-1} \longrightarrow \cdots$$

is a sequence of modules M_n (for any n), connected by homomorphisms $f_n\colon M_{n+1} \to M_n$, such that two consecutive morphisms is zero, i.e., $f_n \circ f_{n-1} = 0$. [1]

Example 2.6. An example of a complex of the $\mathbb{Z}$-modules over a ring $\mathbb{Z}$ is as follows:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2\cdot} \mathbb{Z} \longrightarrow \frac{\mathbb{Z}}{2\mathbb{Z}} \longrightarrow 0,$$

where the map from $\mathbb{Z}$ to $\mathbb{Z}$ is multiplication by 2, and the map from $\mathbb{Z}$ to $\mathbb{Z}/2\mathbb{Z}$ is given by reducing integers, modulo 2.

3. Exact Sequences

An exact sequence is either a finite or infinite sequence of objects and morphisms (or arrows), between them such that the image of one morphism is the kernel of the next one. They can be of different sizes leading to the terms long and short exact sequences.

3.1. **Short Exact Sequences.** A short exact sequence is a special case of exact sequences, length of five, with maps $f_1\colon A_1 \to A_2$ and $f_2\colon A_2 \to A_3$ (f_1 being injective and f_2 being surjective), beginning and ending with 0, the zero module.

Definition 3.1. A **short exact sequence** is a sequence (length five) of groups and their homomorphisms in the form

$$0 \longrightarrow A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \longrightarrow 0,$$

with $\mathrm{im}(f_1) = \ker(f_2)$.

Example 3.2. Looking back at Example 2.6 we see that it is an exact sequence, since:

- The image of the map $0 \to \mathbb{Z}$ is $[0]$, and the kernel of multiplication by 2 is also $[0]$. Therefore, the sequence is exact at the first $\mathbb{Z}$.
- The image of multiplication by 2 is $2\mathbb{Z}$, and the kernel of reducing modulo 2 is also $2\mathbb{Z}$. Therefore, the sequence is exact at the second Z.
- The image of reducing modulo 2 is all of $\mathbb{Z}/2\mathbb{Z}$, and the kernel of the zero map is also all of $\mathbb{Z}/2\mathbb{Z}$. Therefore, the sequence is exact at the position $\mathbb{Z}/2\mathbb{Z}$.

Lemma 3.3 (Short Five Lemma)**.** Consider the following diagram of two short exact sequences. If g_1 and g_3 are isomorphisms, then so is g_2.

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "f_1"] \arrow[d, "g_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "g_2"] & A_3 \arrow[r] \arrow[d, "g_3"] & 0 \\
0 \arrow[r] & A_1' \arrow[r, "f_1'"'] & A_2' \arrow[r, "f_2'"'] & A_3' \arrow[r] & 0.
\end{tikzcd}$$

Proof. Suppose $u \in \ker(g_2)$. Commutativity of the diagram tells us that

$$g_3(f_2(u)) = f_2'(g_2(u)) = g'(0) = 0,$$

and since g_3 is an isomorphism, $f_2(u) = 0$. Exactness at B means that $u \in \ker(f_2)$ implying that $u \in \mathrm{im}(f_1)$, so there exists $a \in A$ such that $f_1(a) = u$. Commutativity of the diagram now gives

$$f_1'(g_1(a)) = g_2(f_1(a)) = g_2(u) = 0,$$

meaning $g_1(a) = 0$ (by injectivity of f_1' using exactness at A_1') and so $a = 0$ (since g_1 is an isomorphism), so that $u = f_1(0) = 0$. Hence $u \in \ker(g_2)$ implies that $u = 0$, so g_2 is injective.

Next, consider $a_2' \in A_2'$. Then $f_2'(a_2') \in A_3'$, so ${g_3}^{-1}(f_2'(a_2')) \in A_3$, and by surjectivity of f_2 there exists $a_2 \in A_2$ such that

$$f_2(a_2) = {g_3}^{-1}(f_2'(a_2')).$$

By commutativity of the diagram,

$$f_2'(g_2(a_2)) = g_3(f_2(a_2)) = g_3({g_3}^{-1}(f_2'(a_2'))) = f_2'(a_2'),$$

so $f_2'(a_2' - g_2(a_2)) = 0$. This means that $f_2' - g_2(a_2) \in \ker(f_2') = \operatorname{im}(f_2')$ (by exactness at A_2'), so there exists a_1' in A_1' with $f_1'(a_1') = a_2' - g_2(a_2)$. Since g_1 is an isomorphism, there exists $a_1 \in A_1$ such that $g_1(a_1) = a_1'$, and

$$a_2' - g_2(a_2) = f_1'(g_1(a_1)) = g_2(f_1(a_1))$$

by commutativity of the diagram. Hence

$$a_2' = g_2(a_2) + g_2(f_1(a_1)) = g_2(a_2 + f_1(a_1)),$$

so $a_2' \in A_2'$ implies that $a_2' \in \operatorname{im}(g_2)$, showing that g_2 is surjective and therefore completing the proof of the Short Five Lemma, see [1]. $\square$

The next result is the Five Lemma, that can be thought of a combination of two other theorems, the **Four Lemmas**, which are dual to each other. The Five Lemma is a stronger statement of the Short Five Lemma. It states that(using the following diagram), if the rows are exact, g_1 and g_2 are isomorphisms, g_0 is an epimorphism, and g_4 is a monomorphism, then g_2 is also an isomorphism.

Lemma 3.4 (The Five Lemma)**.** Given the commutative diagram of abelian groups with exact rows, the following hold:

(1) If g_0 is surjective, and g_1 and g_3 are injective, then g_2 is also injective;
(2) If g_4 is injective, and g_1 and g_2 are surjective, then g_3 is also surjective.
(3) If g_0, g_1, g_3, and g_4 are all bijective, then using (1) and (2), so is g_2.

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, "g_0"] & A_1 \arrow[r, "f_1"] \arrow[d, "g_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "g_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "g_3"] & A_4 \arrow[d, "g_4"] \\
A_0' \arrow[r, "f_0'"'] & A_1' \arrow[r, "f_1'"'] & A_2' \arrow[r, "f_2'"'] & A_3' \arrow[r, "f_3'"'] & A_4'
\end{tikzcd}$$

Proof. The Five Lemma follows immediately from combining the following two Four Lemmas, see [3]. $\square$

Lemma 3.5 (Four Lemma I)**.** If the rows in the commutative diagram are exact and g_1 and g_3 are epimorphisms and g_3 is monomorphism, then g_2 is also an epimorphism.

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "g_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "g_2"] & A_3 \arrow[r, "f_3"] \arrow[d, "g_3"] & A_4 \arrow[d, "g_4"] \\
A_1' \arrow[r, "f_1'"'] & A_2' \arrow[r, "f_2'"'] & A_3' \arrow[r, "f_3'"'] & A_4'
\end{tikzcd}$$

Proof. Assume that g_1 and g_3 are surjective, and g_4 injective. Let a_2' be an element of A_2'. Since g_3 is surjective, there exists an element a_3' in A_3' with $g_3(a_3') = f_2'(a_2')$. By commutativity of the diagram, $f_3'(g_3(a_3')) = g_4(j(a_3'))$. Since $\operatorname{im}(f_2') = \ker(f_3')$ by exactness,

$$0 = f_3'(f_2'(a_2')) = f_3'(g_3(a_3')) = g_4(f_3(a_3')).$$

Since g_4 is injective, $f_3(a_3') = 0$, so a_3' is in $\ker(f_3) = \operatorname{im}(f_2)$. Therefore, there exists a_2 in A_2 with $f_2(a_2) = a_3'$. Then

$$f_2'(g_2(a_2)) = g_3(f_2(a_2)) = f_2'(a_2').$$

Since f_2' is a homomorphism, it follows that $f_2'(a_2' - g_2(a_2)) = 0$. By exactness, $a_2' - g_2(a_2)$ is in the image of f_1', so there exists a_1' in A_1' with $f_1'(a_1') = a_2' - g_2(a_2')$. Since g_1 is surjective, we can find a_1 in A_1 such that $a_1' = g_1(a_1)$. By commutativity,

$$g_2(g(a_1)) = f_1'(g_1(a_1')) = a_2' - g_2(a_2).$$

Since g_2 is a homomorphism,

$$g_2(f_1(a_1) + a_2) = g_2(f_1(a_1)) + g_2(a_2) = a_2' - g_2(a_2) + g_2(a_2) = a_2'.$$

Therefore, g_2 is surjective. $\square$

Lemma 3.6 (Four Lemma II)**.** If the rows in the commutative diagram are exact and g_1 and g_3 are monomorphisms and g_0 is a epimorphism, then g_2 is also an monomorphism.

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, "g_0"] & A_1 \arrow[r, "f_1"] \arrow[d, "g_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "g_2"] & A_3 \arrow[d, "g_3"] \\
A_0' \arrow[r, "f_0'"'] & A_1' \arrow[r, "f_1'"'] & A_2' \arrow[r, "f_2'"'] & A_3'
\end{tikzcd}$$

Proof. Assume that g_1 and g_3 are injective, and g_0 surjective. Let a_2 in A_2 be such that $g_2(a_2) = 0$. $f_2'(g_2(a_2))$ is then 0. By commutativity, $g_3(f_2(a_2)) = 0$. Since g_3 is injective, $f_2(a_2) = 0$. By exactness, there is an element a_1 of A_1 such that $f_1(a_1) = a_2$. By commutativity,

$$f_1'(g_1(a_1)) = g_2(f_1(a_1)) = g_2(a_2) = 0.$$

By exactness, there is then an element a_0' of A_0' such that $f_0'(a_0') = g_1(a_1)$. Since g_0 is surjective, there is a_0 in A_0 such that $g_0(a_0) = a_0'$. By commutativity,

$$g_1(f_0(a_0)) = f_0'(g_0(a_0)) = g_1(a_1).$$

Since g_1 is injective, $f_0(a_0) = a_1$. So $a_2 = f_1(f_0(a_0))$. Since the composition of f_1 and f_0 is trivial, $a_2 = 0$. Therefore, g_2 is injective. $\square$

Lemma 3.7. The following shows a commutative diagram of abelian groups with exact rows, also known as the *Snake Lemma*:

$$\begin{tikzcd}
 & A_1 \arrow[r, "f_1"] \arrow[d, "g_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "g_2"] & A_3 \arrow[r] \arrow[d, "g_3"] & 0 \\
0 \arrow[r] & A_1' \arrow[r, "f_1'"'] & A_2' \arrow[r, "f_2'"'] & A_3' &
\end{tikzcd}$$

If $f_1\colon A_1 \to A_2$ is injective and $f_2'\colon A_2' \to A_3'$ is surjective, then with ingenuity we can see that this brings about an exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \ker(f_1) \arrow[r] & \ker(g_1) \arrow[r] & \ker(g_2) \arrow[r] & \ker(g_3) \arrow[dllll, dotted, "S"] & \\
\operatorname{coker}(g_1) \arrow[r] & \operatorname{coker}(g_2) \arrow[r] & \operatorname{coker}(g_3) \arrow[r] & \operatorname{coker}(f_2') \arrow[r] & 0. &
\end{tikzcd}$$

This is better understood by looking at the following, where $\xrightarrow{S}$ is the map connecting $\ker\, g_3$ and $\operatorname{coker}\, g_1$ (giving rise to the name), called a **connecting homomorphism**:

$$\begin{tikzcd}
0 \arrow[r] & \ker\ f_1 \arrow[r] & \ker\ g_1 \arrow[r] \arrow[d] & \ker\ g_2 \arrow[r] \arrow[d] & \ker\ g_3 \arrow[d] \arrow[dddll, dotted, "S"] & & \\
 & & A_1 \arrow[r, "f_1"] \arrow[d, "g_1"] & A_2 \arrow[r, "f_2"] \arrow[d, "g_2"] & A_3 \arrow[r] \arrow[d, "g_3"] & 0 & \\
 & 0 \arrow[r] & A_1' \arrow[r, "f_1'"] \arrow[d] & A_2' \arrow[r, "f_2'"] \arrow[d] & A_3' \arrow[d] & & \\
 & & \operatorname{coker}\ g_1 \arrow[r] & \operatorname{coker}\ g_2 \arrow[r] & \operatorname{coker}\ g_3 \arrow[r] & \operatorname{coker}\ f_2' \arrow[r] & 0.
\end{tikzcd}$$

The connecting homomorphism is an exact sequence relating the kernels and cokernels of g_1, g_2 and g_3. The maps between the kernels and the maps between the cokernels are known by the diagram's commutativity. The exactness of the two sequences follows from the exactness of the rows of the original diagram. The important remark of the lemma is that a connecting homomorphism S exists which completes the exact sequence.

3.2. **The Long Exact Sequence.** The long exact sequence is an exact sequence with countably infinitely many terms.

Theorem 3.8. *Let*

$$0 \longrightarrow A_* \xrightarrow{f} B_* \xrightarrow{g} C_* \longrightarrow 0$$

be a short exact sequence of chain complexes. Then

$$\cdots \longrightarrow H_{n+1}(C_*) \xrightarrow{\partial} H_n(A_*) \xrightarrow{f_*} H_n(B_*) \xrightarrow{g_*} H_n(C_*) \xrightarrow{\partial} H_{n-1}(A_*) \longrightarrow \ldots$$

is a ***long exact sequence*** *where* $\operatorname{im}(f_{i+1}) = \ker(f_i)$. *Written another way,*

$$\begin{tikzcd}
\ldots \arrow[r] & H_n(A_*) \arrow[r, "H_n(f)"] & H_n(B_*) \arrow[r, "H_n(g)"] & H_n(C_*) \arrow[dll, dotted, "S_n"'] & \\
& H_{n-1}(A_*) \arrow[r, "H_{n-1}(f)"'] & H_{n-1}(B_*) \arrow[r, "H_{n-1}(g)"'] & H_{n-1}(C_*) \arrow[r] & \ldots
\end{tikzcd}$$

is exact.

Proof. From the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A_n \arrow[r, "f_n"] \arrow[d, "\partial_n"] & B_n \arrow[r, "g_n"] \arrow[d, "\partial_n"] & C_n \arrow[r] \arrow[d, "\partial_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f_{n-1}"'] & B_{n-1} \arrow[r, "g_{n-1}"'] & C_{n-1} \arrow[r] & 0
\end{tikzcd}$$

with exact rows and the Snake Lemma, we get an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & Z_n(A_*) \arrow[r, "f_n"] & Z_n(B_*) \arrow[r, "g_n"] & Z_n(C_*) \arrow[dll, dotted, "\partial"'] & \\
& A_{n-1}/B_{n-1}(A_*) \arrow[r, "f_{n-1}"'] & B_{n-1}/B_{n-1}(B_*) \arrow[r, "g_{n-1}"'] & C_{n-1}/B_{n-1}(C_*) \arrow[r] & 0.
\end{tikzcd}$$

Thus, we have a commutative diagram

$$\begin{tikzcd}
& A_n \arrow[r, "f_1"] \arrow[d, "g_1"] & B_n \arrow[r, "f_2"] \arrow[d, "g_2"] & C_n \arrow[r] \arrow[d, "g_3"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f_1'"'] & B_{n-1} \arrow[r, "f_2'"'] & C_{n-1} &
\end{tikzcd}$$

with exact rows. Applying the snake lemma again and we get an exact sequence

$$H_n(A_*) \xrightarrow{f_*} H_n(B_*) \xrightarrow{g_*} H_n(C_*) \xrightarrow{\partial} H_{n-1}(A_*) \xrightarrow{f_*} H_{n-1}(B_*) \xrightarrow{g_*} H_{n-1}(C_*).$$

□

Example 3.9. The Mayer-Vietoris sequence is a long exact sequence relating the singular homology groups (with coefficient group the integers $\mathbb{Z}$) of the spaces X, A, B, and the intersection $A \cap B$, where A and B are subspaces that together cover X.

There is an unreduced and a reduced version. The reduced version in [4] is as follows and ends as:

$$\cdots \longrightarrow H_0(A \cap B) \xrightarrow{i_*, j_*} H_0(A) \oplus H_0(B) \xrightarrow{k_* - l_*} H_0(X) \longrightarrow 0.$$

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