Schrödinger Bridge Problem

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The classical Schrödinger Bridge problem (SBP) can be formulated into a stochastic optimal control problem as follows

$$\inf_{\boldsymbol{u}\in\mathcal{U}} \mathbb{E}\left\{ \int_0^1 \frac{1}{2}\|\boldsymbol{u}(\mathbf{x},t)\|_2^2 dt \right\}, \tag{1}$$

under the constraint

$$d\mathbf{x}_t = \boldsymbol{u}(\mathbf{x},t)dt + \sqrt{2}\sigma d\boldsymbol{B}_t, \tag{2}$$

$$\mathbf{x}(t=0) \sim \rho_0(\mathbf{x}), \mathbf{x}(t=1) \sim \rho_1(\mathbf{x}), \tag{3}$$

where $\mathcal{U} := \{\boldsymbol{u} : \mathbb{R}^d \times [0,1] \to \mathbb{R}^n | \langle \boldsymbol{u}, \boldsymbol{u} \rangle < \infty\}$ is a set of control variables; $\boldsymbol{B}_t$ is the standard Brownian motion in $\mathbb{R}^n$. The expectation is taken w.r.t the joint state PDF $\rho(\mathbf{x},t)$ given initial and terminal conditions.

Rewrite SBP (1) into a fluid-dynamic formulation [CGP21], we can get that

$$\inf_{\boldsymbol{u}\in\mathcal{U},\rho} \int_0^1 \int \frac{1}{2}\|\boldsymbol{u}(\mathbf{x},t)\|_2^2 \rho(\mathbf{x},t) d\mathbf{x} dt, \tag{4}$$

subject to

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) = \sigma^2\Delta\rho, \tag{5}$$

$$\rho(\mathbf{x},0) = \rho_0(\mathbf{x}), \quad \rho(\mathbf{x},1) = \rho_1(\mathbf{x}), \tag{6}$$

where Eq.(5) is the Fokker-Plank equation or Kolmogorov's forward PDE for the corresponding controlled diffusion process (2) based on decision variables $(\rho, \boldsymbol{u}) \in \mathcal{P}(\mathbb{R}^n) \times \mathcal{U}$ and $\mathcal{P}(\mathbb{R}^n)$ is the set of probability measures on $\mathbb{R}^n$ or more generally on Polish metric spaces.

Consider the Lagrangian for Eq.(4) and introduce $\psi(\mathbf{x},t)$ as a $C^1(\mathbb{R}^n, \mathbb{R}_{>0})$ Lagrange multiplier:

$$\begin{aligned}\mathcal{L}(\rho,\boldsymbol{u},\psi) :&= \int_0^1 \int \frac{1}{2}\|\boldsymbol{u}(\mathbf{x},t)\|_2^2 \rho(\mathbf{x},t) + \psi(\mathbf{x},t)\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) - \sigma^2\Delta\rho\right) d\mathbf{x}dt,\\ &= \int_0^1 \int \left(\frac{1}{2}\|\boldsymbol{u}(\mathbf{x},t)\|_2^2 - \frac{\partial\psi}{\partial t} - \langle\nabla\psi, \boldsymbol{u}\rangle - \sigma^2\Delta\psi\rangle\right)\rho(\mathbf{x},t)d\mathbf{x}dt,\end{aligned} \tag{7}$$

where the second equation is obtained through integration by parts.

Minimizing with respect to $\boldsymbol{u}$, we get the optimal control $\boldsymbol{u}^*$ as follows

$$\boldsymbol{u}^*(\mathbf{x},t) = \nabla\psi(\mathbf{x},t). \tag{8}$$

Plugging Eq.(8) into Eq.(7), we have

$$\mathcal{L}(\rho,\boldsymbol{u},\psi) = \int_0^1 \int \left(- \frac{\partial\psi}{\partial t} - \frac{1}{2}\|\nabla\psi(\mathbf{x},t)\|_2^2 - \sigma^2\Delta\psi\rangle\right)\rho(\mathbf{x},t)d\mathbf{x}dt. \tag{9}$$

To achieve the minimum control cost, we hope $\mathcal{L}(\rho, \boldsymbol{u}, \psi) = 0$, which is the dynamic programming equation. In what follows, we choose

$$\frac{\partial \psi}{\partial t} + \sigma^2 \Delta \psi = -\frac{1}{2} \|\nabla \psi(\mathbf{x}, t)\|_2^2. \tag{10}$$

Given the optimal control variable u^*, the above PDE is known as the *Hamilton–Jacobi–Bellman* (HJB) PDE. Since the HJB PDE is non-linear due to the presence of $\frac{1}{2}\|\nabla \psi(\mathbf{x}, t)\|_2^2$, we convert it into a linear PDE through the logarithmic or Cole-Hopf transformation given by

$$\begin{aligned} \varphi(\mathbf{x}, t) &= \exp\left(\frac{\psi(\mathbf{x}, t)}{2\sigma^2}\right) \\ \hat{\varphi}(\mathbf{x}, t) &= \rho^*(\mathbf{x}, t) \exp\left(-\frac{\psi(\mathbf{x}, t)}{2\sigma^2}\right). \end{aligned} \tag{11}$$

We now can easily verify that the transformed variables $(\varphi, \hat{\varphi})$ solve the *Schrödinger system*

$$\begin{aligned} &\frac{\partial \varphi}{\partial t} + \sigma^2 \Delta \psi = 0 \\ &\frac{\partial \hat{\varphi}}{\partial t} - \sigma^2 \Delta \hat{\varphi} = 0, \end{aligned} \tag{12}$$

under the constraint that

$$\begin{aligned} \varphi(\mathbf{x}, t = 0)\hat{\varphi}(\mathbf{x}, t = 0) &= \rho_0(\mathbf{x}) \\ \varphi(\mathbf{x}, t = 1)\hat{\varphi}(\mathbf{x}, t = 1) &= \rho_1(\mathbf{x}). \end{aligned}$$

In what follows, the solution of Eq.(12) can be written into the *Schrödinger equations*

$$\varphi(\mathbf{x}, t) = \int K(t, \mathbf{x}, 1, \mathbf{y}) \hat{\varphi}(\mathbf{y}, t = 1) d\mathbf{y}, \quad t \leq 1 \tag{13}$$

$$\hat{\varphi}(\mathbf{x}, t) = \int K(0, \mathbf{y}, t, \mathbf{x}) \hat{\varphi}(\mathbf{y}, t = 0) d\mathbf{y}, \quad t \geq 0, \tag{14}$$

where $K(t, \mathbf{x}, s, \mathbf{y}) := (4\pi\sigma^2(t-s))^{-n/2} \exp\big(-\frac{\|\mathbf{x}-\mathbf{y}\|_2^2}{4\sigma^2(t-s)}\big)$ is the Markov kernel associated with the pure diffusion process $d\mathbf{x}_t = \sqrt{2}\sigma dB_t$.

Now, based on Eq.(11), the optimal decision variables $(\rho, \boldsymbol{u})$ can be obtained as follows

$$\begin{aligned} \rho^*(\mathbf{x}, t) &= \varphi(\mathbf{x}, t)\hat{\varphi}(\mathbf{x}, t) \\ \boldsymbol{u}^*(\mathbf{x}, t) &= 2\sigma^2 \nabla \log \varphi(\mathbf{x}, t). \end{aligned}$$

To compute the iterates efficiently, one may resort to the Sinkhorn algorithm with DNN approximation [CLT22] to solve it.

References

[CGP21] Yongxin Chen, Tryphon T. Georgiou, and Michele Pavon. Stochastic Control Liaisons: Richard Sinkhorn Meets Gaspard Monge on a Schrödinger Bridge. *SIAM Review*, 63(2):249–313, 2021.

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