Schrödinger Bridge Problem

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1 Classical Schrödinger Bridge (SBP) problem

The Schrödinger Bridge problem (SBP) have obtained tremendous attention in deep generative models and financial mathematics. SBP is closely connected to stochastic optimal control and entropic optimal transport. Today, we study the formulation of SBP as a stochastic optimal control problem

$$\inf_{\boldsymbol{u}\in\mathcal{U}} \mathbb{E}\bigg\{\int_0^1 \frac{1}{2}\|\boldsymbol{u}(\mathbf{x},t)\|_2^2 dt\bigg\}, \tag{1}$$

under the constraint

$$\begin{aligned} d\mathbf{x}_t &= \boldsymbol{u}(\mathbf{x},t)dt + \sqrt{2}\sigma d\boldsymbol{B}_t, \\ \mathbf{x}(t=0) &\sim \rho_0(\mathbf{x}), \mathbf{x}(t=1) \sim \rho_1(\mathbf{x}), \end{aligned} \tag{2}$$

where $\mathcal{U}$ is a set of control variables $\boldsymbol{u}: \mathbb{R}^d \times [0,1] \to \mathbb{R}^n$; the state-space is $\mathbb{R}^d$ by default and it also applies to compact spaces; $\boldsymbol{B}_t$ is the standard Brownian motion in $\mathbb{R}^n$. The expectation is taken w.r.t the joint state PDF $\rho(\mathbf{x},t)$ given initial and terminal conditions.

Rewrite SBP (1) into a fluid-dynamic formulation [CGP21], we can get that

$$\inf_{\boldsymbol{u}\in\mathcal{U},\rho} \int_0^1 \int \frac{1}{2}\|\boldsymbol{u}(\mathbf{x},t)\|_2^2 \rho(\mathbf{x},t) d\mathbf{x} dt, \tag{3}$$

subject to

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) &= \sigma^2\Delta\rho, \\ \rho(\mathbf{x},0) = \rho_0(\mathbf{x}), &\quad \rho(\mathbf{x},1) = \rho_1(\mathbf{x}), \end{aligned} \tag{4}$$

where Eq.(4) is the Fokker-Plank equation or Kolmogorov's forward PDE for the corresponding controlled diffusion process (2) based on decision variables $(\rho, \boldsymbol{u}) \in \mathcal{P}(\mathbb{R}^n) \times \mathcal{U}$ and $\mathcal{P}(\mathbb{R}^n)$ is the set of probability measures on $\mathbb{R}^n$ or more generally on Polish metric spaces.

Consider the Lagrangian for Eq.(3) and introduce $\boldsymbol{\phi}(\mathbf{x},t)$ as a $C^1(\mathbb{R}^n, \mathbb{R}_{>0})$ Lagrange multiplier:

$$\begin{aligned} \mathcal{L}(\rho,\boldsymbol{u},\boldsymbol{\phi}) :&= \int_0^1 \int \frac{1}{2}\|\boldsymbol{u}(\mathbf{x},t)\|_2^2 \rho(\mathbf{x},t) + \boldsymbol{\phi}(\mathbf{x},t)\bigg(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{u}) - \sigma^2\Delta\rho\bigg) d\mathbf{x}dt, \\ &= \int_0^1 \int \bigg(\frac{1}{2}\|\boldsymbol{u}(\mathbf{x},t)\|_2^2 \; - \frac{\partial\boldsymbol{\phi}}{\partial t} - \langle\nabla\boldsymbol{\phi}, \boldsymbol{u}\rangle - \sigma^2\Delta\boldsymbol{\phi}\rangle\bigg)\rho(\mathbf{x},t)d\mathbf{x}dt, \end{aligned} \tag{5}$$

where the second equation is obtained through integration by parts.

Minimizing with respect to $\boldsymbol{u}$, we get the optimal control $\boldsymbol{u}^\star$ as follows

$$\boldsymbol{u}^\star(\mathbf{x},t) = \nabla\boldsymbol{\phi}(\mathbf{x},t). \tag{6}$$

As such, the value function at time $t \in [0,1]$ follows that

$$\mathbb{E}\bigg\{\int_t^1 \frac{1}{2}\|\boldsymbol{u}(\mathbf{x},s)\|_2^2 - \log\boldsymbol{\phi}(\mathbf{x},s)ds\bigg\}.$$

Plugging Eq.(6) into Eq.(5), we have

$$\mathcal{L}(\rho, \boldsymbol{u}, \boldsymbol{\phi}) = \int_0^1 \int \bigg(-\frac{\partial \boldsymbol{\phi}}{\partial t} - \frac{1}{2}\|\nabla \boldsymbol{\phi}(\mathbf{x}, t)\|_2^2 - \sigma^2 \Delta \boldsymbol{\phi} \bigg) \rho(\mathbf{x}, t) d\mathbf{x} dt. \tag{7}$$

To achieve the minimum control cost, we hope $\mathcal{L}(\rho, \boldsymbol{u}, \boldsymbol{\phi}) = 0$, which is the dynamic programming equation. In what follows, we choose

$$\frac{\partial \boldsymbol{\phi}}{\partial t} + \sigma^2 \Delta \boldsymbol{\phi} = -\frac{1}{2}\|\nabla \boldsymbol{\phi}(\mathbf{x}, t)\|_2^2. \tag{8}$$

Given the optimal control variable $u^\star$, the above PDE is known as the *Hamilton–Jacobi–Bellman* (HJB) PDE. Since the HJB PDE is non-linear due to the presence of $\frac{1}{2}\|\nabla \boldsymbol{\phi}(\mathbf{x}, t)\|_2^2$, we convert it into a linear PDE through the logarithmic or Cole-Hopf transformation given by

$$\begin{aligned} \boldsymbol{\psi}(\mathbf{x}, t) &= \exp\bigg(\frac{\boldsymbol{\phi}(\mathbf{x}, t)}{2\sigma^2}\bigg) \\ \boldsymbol{\varphi}(\mathbf{x}, t) &= \rho^\star(\mathbf{x}, t) \exp\bigg(-\frac{\boldsymbol{\phi}(\mathbf{x}, t)}{2\sigma^2}\bigg). \end{aligned} \tag{9}$$

We now can easily verify that the transformed variables $(\boldsymbol{\psi}, \boldsymbol{\varphi})$ solve the *Schrödinger system*

$$\begin{aligned} \frac{\partial \boldsymbol{\psi}}{\partial t} + \sigma^2 \Delta \boldsymbol{\psi} = 0 \\ \frac{\partial \boldsymbol{\varphi}}{\partial t} - \sigma^2 \Delta \boldsymbol{\varphi} = 0, \end{aligned} \tag{10}$$

under the constraint that

$$\begin{aligned} \boldsymbol{\psi}(\mathbf{x}, t=0)\boldsymbol{\varphi}(\mathbf{x}, t=0) = \rho_0(\mathbf{x}) \\ \boldsymbol{\psi}(\mathbf{x}, t=1)\boldsymbol{\varphi}(\mathbf{x}, t=1) = \rho_1(\mathbf{x}). \end{aligned} \tag{11}$$

In what follows, the solution of Eq.(10) can be written into the *Schrödinger equations*

$$\begin{aligned} \boldsymbol{\psi}(\mathbf{x}, t) &= \int K(t, \mathbf{x}, 1, \mathbf{y}) \boldsymbol{\varphi}(\mathbf{y}, t=1) d\mathbf{y}, \quad t \le 1 \\ \boldsymbol{\varphi}(\mathbf{x}, t) &= \int K(0, \mathbf{y}, t, \mathbf{x}) \boldsymbol{\varphi}(\mathbf{y}, t=0) d\mathbf{y}, \quad t \ge 0, \end{aligned} \tag{12}$$

where $K(t, \mathbf{x}, s, \mathbf{y}) := (4\pi\sigma^2(t-s))^{-n/2} \exp\big(-\frac{\|\mathbf{x}-\mathbf{y}\|_2^2}{4\sigma^2(t-s)}\big)$ is the Markov kernel associated with the pure diffusion process $d\mathbf{x}_t = \sqrt{2}\sigma dB_t$.

Now, based on Eq.(9), the optimal decision variables $(\rho, \boldsymbol{u})$ can be obtained as follows

$$\begin{aligned} \rho^\star(\mathbf{x}, t) &= \boldsymbol{\psi}(\mathbf{x}, t)\boldsymbol{\varphi}(\mathbf{x}, t) \\ \boldsymbol{u}^\star(\mathbf{x}, t) &= 2\sigma^2 \nabla \log \boldsymbol{\psi}(\mathbf{x}, t). \end{aligned}$$

To obtain the minimizer of the SBP, we resort to solving a variant of *Schrödinger equations* by combiniong Eq.(11) and Eq.(12) as follows

$$\begin{aligned} \rho_0(\mathbf{x}) &= \varphi(\mathbf{x}) \int_{\mathbb{R}^n} K(0, \mathbf{x}, 1, \mathbf{y}) \psi(\mathbf{y}) d\mathbf{y} \\ \rho_1(\mathbf{y}) &= \psi(\mathbf{y}) \int_{\mathbb{R}^n} K(0, \mathbf{x}, 1, \mathbf{y}) \varphi(\mathbf{x}) d\mathbf{x}, \end{aligned} \tag{13}$$

where $\varphi(\cdot)$ and $\psi(\cdot)$ are defined as follows for notation simplicity

$$\varphi(\cdot) = \boldsymbol{\varphi}(\cdot, t=0) \;\; \text{and} \;\; \psi(\cdot) = \boldsymbol{\psi}(\cdot, t=1). \tag{14}$$

References

[CGP21] Yongxin Chen, Tryphon T. Georgiou, and Michele Pavon. Stochastic Control Liaisons: Richard Sinkhorn Meets Gaspard Monge on a Schrödinger Bridge. *SIAM Review*, 63(2):249–313, 2021.