The K-theory of $\mathbb{Z}$

Weinan Lin

Discussed with Peter May, Nick Rozenblyum and Aaron Silberstein

1 Introduction

In this topic I intend to give a description of the K-theory of $\mathbb{Z}$ and its relationship with Vandiver's Conjecture. I will start with the definition of the algebraic K-theory of a ring using Quillen's $+$-construction and then focus on the K-theory of number fields and some of their properties. As the main part of the topic, I will introduce the Motivic-to-K-theory spectral sequence and use it to compute $K_{2i-1}(\mathbb{Z})$. Then we calculate the order of $K_{4k+2}(\mathbb{Z})$ by the Main Conjecture of Iwasawa Theory. Finally assuming that Vandiver's Conjecture holds, we can give a complete description of all the K-groups of $\mathbb{Z}$.

2 The definition and properties of K-theory

First we define the K_0 and K_1 groups of an associative ring R with unit. Let $\mathrm{P}(R)$ be the commutative monoid of isomorphism classes of finitely generated projective R-modules with direct sum $\oplus$ and identity 0. We define $K_0(R)$ to be the group completion of $\mathrm{P}(R)$.

Consider the infinite general linear group $GL(R)$, which is the union of the groups $GL_n(R)$. Let $E(R)$ be the subgroup of $GL(R)$ generated by all elementary matrices. Actually $E(R) = [GL(R), GL(R)]$ is the commutator of $GL(R)$, and we define $K_1(R)$ to be the abelian group $GL(R)/E(R)$.

Now we introduce the definition of K-theory using Quillen's $+$-construction. Let $BGL(R)$ be the classifying space of $GL(R)$.

Definition 2.1 Let $BGL(R)^+$ denote any CW complex which has a distinguished map $BGL(R) \to BGL(R)^+$ such that

(1) $\pi_1 BGL(R)^+ \cong K_1(R)$, and the natural map from $GL(R) = \pi_1 BGL(R)$ to $\pi_1 BGL(R)^+$ is onto with kernel $E(R)$;

(2) $H_*(BGL(R);\mathbb{Z}) \xrightarrow{\cong} H_*(BGL(R)^+;\mathbb{Z})$.

By a theorem due to Quillen we know that $BGL(R)^+$ exists and it is unique up to homotopy. Quillen also proved that $BGL(R) \to BGL(R)^+$ is universal for maps into H-spaces, which is one of the most useful criteria, and $BGL(R)^+$ is also an infinite loop space, and extends to an Ω-spectrum. Now we define all the K-groups of R.

Definition 2.2 Write $K(R)$ for the product $K_0(R) \times BGL(R)^+$. That is, $K(R)$ is the disjoint union of copies of the connected space $BGL(R)^+$. We define $K_n(R) = \pi_n K(R)$.

We give some computational results about K-groups. Let $\mathbb{F}_q$ be a finite field with $q = p^v$ elements. Consider the trivial and standard n-dimensional representations $1_n, id_n : GL_n(\mathbb{F}_q) \to U$. Since BU is an H-space, we can obtain a map $\rho_n = B(id_n) - B(1_n) : BGL_n(\mathbb{F}_q) \to BU$. Quillen observed that ρ_n and ρ_{n+1} are compatible up to homotopy hence there is a map $\rho : BGL(\mathbb{F}_q) \to BU$ well-defined up to homotopy.

Theorem 2.3 (Quillen) *The map $BGL(\mathbb{F}_q)^+ \to BU$ induced by ρ identifies $BGL(\mathbb{F}_q)^+$ with the homotopy fiber of $\psi^q - 1$. That is, the following is a homotopy fibration:*

$$BGL(\mathbb{F}_q)^+ \xrightarrow{\rho} BU \xrightarrow{\psi^q - 1} BU.$$

Since ψ^q is multiplication by q^i on $\pi_{2i}BU = \widetilde{KU}(S^{2i})$, we immediately deduce:

Corollary 2.4 *For every finite field $\mathbb{F}_q$, and $n \geq 1$, we have*

$$K_n(\mathbb{F}_q) = \pi_n BGL(\mathbb{F}_q)^+ \cong \begin{cases} \mathbb{Z}/(q^i - 1) & n = 2i - 1, \\ 0 & n \text{ even.} \end{cases}$$

Moreover, if $\mathbb{F}_q \subset \mathbb{F}_{q'}$, then $K_n(\mathbb{F}_q) \to K_n(\mathbb{F}_{q'})$ is an injection.

Now we give a result about the rank of K_n over number fields.

Theorem 2.5 (Borel) *Let F be a number field, and A a central simple F-algebra. Then for $n \geq 2$ we have $K_n(A) \otimes \mathbb{Q} \cong K_n(F) \otimes \mathbb{Q}$ and*

$$\text{rank } K_n(A) \otimes \mathbb{Q} = \begin{cases} r_2, & n \equiv 3 \pmod 4, \\ r_1 + r_2, & n \equiv 1 \pmod 4, \\ 0, & \text{else.} \end{cases} \tag{1}$$

In particular, the $K_n(A)$ are torsion for all even $n \geq 2$.

Suppose A is a finite dimensional semisimple algebra over $\mathbb{Q}$, such as number fields, and that R is a subring of A which is finite generated over $\mathbb{Z}$ and has $R \otimes \mathbb{Q} = A$. Borel also proved that $K_n(R) \otimes \mathbb{Q} \cong K_n(A) \otimes \mathbb{Q}$ for all $n \geq 2$. Thus (1) also holds for the integer rings $\mathcal{O}_F$ for number fields F for $n \geq 2$. For example, if $A = F = \mathbb{Q}$, $R = \mathbb{Z}$ and $n \geq 2$, we have

$$\text{rank } K_n(\mathbb{Z}) \otimes \mathbb{Q} = \begin{cases} 1 & n \equiv 1 \pmod 4, \\ 0, & \text{else.} \end{cases}$$

We want to know when $K_n(R)$ is finitely generated. Here is another theorem due to Quillen ([Q2]):

Theorem 2.6 (Quillen) *Let R be either an integrally closed subring of a number field F, finite over $\mathbb{Z}$, or else the coordinate ring of a smooth affine curve over a finite field. Then $K_n(R)$ is a finitely generated group for all n.*

It is often useful to consider K-theory with finite coefficients. Consider the Moore space $P^m(\mathbb{Z}/l)$ which is formed from the sphere S^{m-1} by attaching an m-cell via a degree l map. If $m \geq 2$, We define the mod l homotopy "group" $\pi_m(X;\mathbb{Z}/l)$ of a based topological space X to be the pointed set $[P^m(\mathbb{Z}/l), X]$ of based homotopy maps. Actually it is a group for any $m > 2$, but not a group when $m = 2$ for general X. But if X is an infinite loop space, we can ignore this restriction on m.

Definition 2.7 The mod l K-groups of R are defined to be the abelian groups:

$$K_m(R;\mathbb{Z}/l) = \pi_m(K(R);\mathbb{Z}/l).$$

Similar to the ordinary homology theory in topology, we have the following universal coefficient theorem for K-theory

Theorem 2.8 *There is a short exact sequence*

$$0 \to K_m(R) \otimes \mathbb{Z}/l \to K_m(R;\mathbb{Z}/l) \to {}_lK_{m-1}(R) \to 0$$

for every $m \in \mathbb{Z}$ and l. Here ${}_lK_{m-1}(R)$ is the subgroup of all elements $a \in R$ such that $la = 0$. It is split exact unless $l \equiv 2$ mod 4. The splitting is not natural in R.

Now assume that l is a prime number. We consider the l-adic completion of a spectrum $\mathbf{E}$, which is the homotopy limit over v of the spectra $\mathbf{E} \wedge \mathbf{P}^\infty(\mathbb{Z}/l^v)$. We let $\pi_n(E;\mathbb{Z}_l)$ denote the homotopy groups of this spectrum; If $\mathbf{E} = \mathbf{K}(R)$ we write $K_n(R;\mathbb{Z}_l)$ for $\pi_n(\mathbf{E}(R);\mathbb{Z}_l)$. There is an extension

$$0 \to \varprojlim{}^1 \pi_{n+1}(\mathbf{E};\mathbb{Z}/l^v) \to \pi_n(\mathbf{E};\mathbb{Z}_l) \to \varprojlim \pi_n(\mathbf{E};\mathbb{Z}/l^v) \to 0.$$

Where the $\varprojlim{}^1$ term vanishes if the homotopy groups $\pi_{n+1}(\mathbf{E};\mathbb{Z}/l^v)$ are finite. By Theorem 2.6 and Theorem 2.8 we can prove that

$$K_{2i}(R)_{l-\mathrm{tors}} \cong K_{2i}(R;\mathbb{Z}_l) \tag{2}$$

for every ring R in Theorem 2.6. Here we use $A_{l-\mathrm{tors}}$ to denote the l-torsion subgroup of a finitely generated abelian group A.

On the other hand, we consider $P^m(\mathbb{Z}/l^\infty) = \varinjlim P^m(\mathbb{Z}/l^v)$. We have that $\pi_m(X;\mathbb{Z}/l^\infty) := [P^m(\mathbb{Z}/l^\infty), X]$ is the direct limit of $\pi_m(X;\mathbb{Z}/l^v)$. There is a universal coefficient sequence for $m \geq 3$:

$$0 \to (\pi_m X) \otimes \mathbb{Z}/l^\infty \to \pi_m(X;\mathbb{Z}/l^\infty) \xrightarrow{\partial} (\pi_{m-1}X)_{l-\mathrm{tors}} \to 0.$$

We write $K_m(R;\mathbb{Z}/l^\infty)$ for $\pi_m(K(R);\mathbb{Z}/l^\infty)$. By Theorem 2.5, Theorem 2.6 and Theorem 2.8 we can deduce that

$$K_{2i-1}(R)_{l-\mathrm{tors}} \cong K_{2i}(R;\mathbb{Z}/l^\infty)$$

for every ring R in Theorem 2.6.

Finally we introduce a theorem about the K-groups of Dedekind domains which is a consequence of the localization theorem in K-theory:

Theorem 2.9 *Let R be a Dedekind domain whose field of fractions F is a global field. Then $K_n(R) \cong K_n(F)$ for all odd $n \geq 3$; for even $n \geq 2$ the localization sequence breaks up into exact sequence:*

$$0 \to K_n(R) \to K_n(F) \to \bigoplus_{\mathfrak{p}} K_{n-1}(R/\mathfrak{p}) \to 0.$$

3 The e-invariant of a field

Before we introduce the e-invariant we give the characterization of K-groups of algebraically close fields due to Suslin [Su].

Theorem 3.1 *Let F be an algebraically closed field of characteristic $p > 0$. Then*

$$K_n(F) = \begin{cases} \mathbb{Q}/\mathbb{Z}[1/p] \oplus \textit{uniquely divisible group}, & n = 2i-1 \\ \textit{uniquely divisible group} & n > 0 \textit{ even} \end{cases}$$

If $char(F) = 0$, then

$$K_n(F) = \begin{cases} \mathbb{Q}/\mathbb{Z} \oplus \textit{uniquely divisible group}, & n = 2i-1 \\ \textit{uniquely divisible group} & n > 0 \textit{ even} \end{cases}$$

Let $\mu = \mu(F)$ denote the roots of unity of F, and write $\mu(i)$ for the abelian group μ, made into a $\mathrm{Aut}(F)$-module by letting $g \in \mathrm{Aut}(F)$ act as $\zeta \mapsto g^i(\zeta)$. We know that as an abelian group μ is isomorphic to either $\mathbb{Q}/\mathbb{Z}$ or $\mathbb{Q}/\mathbb{Z}[1/p]$, according to the characteristic of F. Actually we have following conclusion

Proposition 3.2 *If F is algebraically closed and $i > 0$, the torsion submodule of $K_{2i-1}(F)$ is isomorphic to $\mu(i)$ as an $\mathrm{Aut}(F)$-module.*

Now we define the e-invariant of a field F.

Definition 3.3 *Let F be a field, with separable closure $\bar{F}_{sep}$ and Galois group $G = \mathrm{Gal}(\bar{F}_{sep}/F)$. Since $K_*(F) \to K_*(\bar{F}_{sep})$ is a homomorphism of G-modules with G acting trivially on $K_n(F)$, it follows that there is a natural map*

$$e: \; K_{2i-1}(F)_{\mathrm{tors}} \to K_{2i-1}(\bar{F}_{sep})^G_{\mathrm{tors}} \cong \mu(i)^G.$$

We call e the e-invariant.

If $\mu(i)^G$ is a finite group it is cyclic, and we write $w_i(F)$ for its order, so that $\mu(i)^G \cong \mathbb{Z}/w_i(F)$. If l is a prime, we write $w_i^{(l)}(F)$ for the order of the l-primary subgroup $\mu_{(l)}(i)^G$ of $\mu(i)^G$. We have the following formulas to compute $w_i^{(l)}(F)$.

Proposition 3.4 *Fix an odd prime l, and let F be a field of characteristic $\neq l$. Let $a \leq \infty$ be maximal such that $F(\zeta_l)$ contains a primitive l^ath root of unity and set $r = [F(\zeta_l) : F]$. If $i = cl^b$, where $l \nmid c$, then the numbers $w_i^{(l)}(F)$ are l^{a+b} if $r|i$ and 1 otherwise.*

Proposition 3.5 ($l = 2$) *Let F be a field of characteristic $\neq 2$. Let a be maximal such that $F(\sqrt{-1})$ contains a primitive 2^ath root of unity. If $i = c2^b$, where $2 \nmid c$, then the 2-primary numbers $w_i^{(2)}(F)$ are given by:*

(a) If $\sqrt{-1} \in F$ then $w_i^{(2)}(F) = 2^{a+b}$.

(b) If $\sqrt{-1} \notin F$ and i is odd then $w_i^{(2)}(F) = 2$.

(c) If $\sqrt{-1} \notin F$, F is exceptional and i is even then $w_i^{(2)}(F) = 2^{a+b}$.

(d) If $\sqrt{-1} \notin F$, F is non-exceptional and i is even then $w_i^{(2)}(F) = 2^{a+b-1}$.

As a special case, we have

Proposition 3.6 *If i is odd, $w_i(\mathbb{Q}) = 2$. If $i = 2k$ is even then $w_{2k}(\mathbb{Q}) = w_{2k}$ is the denominator of $B_{2k}/4k = (-1)^k c_k/w_{2k}$. The prime l divides $w_i(\mathbb{Q})$ exactly when $(l-1)$ divides i.*

Remark 3.7 The complex Adams e-invariant for stable homotopy is a map from π_{2i-1}^s to $\mathbb{Z}/w_i$. Quillen observed in [Q1] that the Adams e-invariant is the composition $\pi_{2i-1}^s \to K_{2i-1}(\mathbb{Q}) \xrightarrow{e} \mathbb{Z}/w_i(Q)$

In most cases we can think of the e-invariant as detecting a direct summand of the K-groups $K_{2i-1}(F)$ because of the following theorem:

Harris-Segal Theorem 3.8 *Let F be a field with $1/l \in F$; if $l = 2$ we also suppose that F is non-exceptional. Then each $K_{2i-1}(F)$ has a direct summand isomorphic to $\mathbb{Z}/w_i^{(l)}(F)$, detected by e-invariant. If F is the field of fractions of an integrally closed domain R then $K_{2i-1}(R)$ also has a direct summand isomorphic to $\mathbb{Z}/w_i^{(l)}(F)$, detected by the e-invariant.*

Remark 3.9 If F is exceptional such as $\mathbb{Q}$, $\mathbb{Q}(\sqrt{-7})$, there is a cyclic summand of $K_{2i-1}(F)$ whose order is either $w_i(F)$, $2w_i(F)$ or $w_i(F)/2$. We also call these Harris-Segal summands.

4 Milnor K-theory of fields and Galois Cohomology

Fix a field F, and consider the tensor algebra of the group $F^\times$,

$$T(F^\times) = \mathbb{Z} \oplus F^\times \oplus (F^\times \otimes F^\times) \oplus (F^\times \otimes F^\times \otimes F^\times) \oplus \cdots .$$

We write $l(x)$ for the element of degree one of $T(F^\times)$ corresponding to $x \in F^\times$.

Definition 4.1 *The graded ring $K_*^M(F)$ is defined to be the quotient of $T(F^\times)$ by the ideal generated by the homogeneous elements $l(x) \otimes l(1-x)$ with $x \neq 0, 1$. The Milnor K-group $K_n^M(F)$ is defined to be the subgroup of elements of degree n. We shall write $\{x_1, \dots, x_n\}$ for the image of $l(x_1) \otimes \cdots \otimes l(x_n)$ in $K_n^M(F)$.*

That is, $K_n^M(F)$ is presented as the group generated by symbols $\{x_1, \dots, x_n\}$ subject to two defining relations: $\{x_1, \dots, x_n\}$ is multiplicative in each x_i and equals zero if $x_i + x_{i+1} = 1$ for some i.

Let F_{sep} denote the separable closure of a field F, and let $G = G_F$ denote the Galois group $\text{Gal}(F_{\text{sep}}/F)$. The family of subgroups $G_E = \text{Gal}(F_{\text{sep}}/E)$, as E runs over all finite extension of F, forms a basis for a topology of G. A G-module M is called discrete if the multiplication $G \times M \to M$ is continuous.

For example, the abelian group $G_m = F_{\text{sep}}^\times$ of units of F_{sep} is a discrete module, as is the subgroup μ_n of all n-th roots of unity.

The G-invariant subgroup M^G of a discrete G_F-module M defines a left exact functor on the category of discrete G_F-modules. The Galois cohomology groups $H^i_{et}(F; M)$ are defined to be its right derived functors. In particular, $H^0_{et}(F; M)$ is just M^G.

Kummer Theory 4.2 *We have $H^0_{et}(F, G_m) = F^\times$, $H^1_{et}(F, G_m) = 0$ and $H^2_{et}(F, G_m) = Br(F)$. If n is prime to char(F), consider the exact sequence of discrete modules*

$$1 \to \mu_n \to G_m \xrightarrow{n} G_m \to 1.$$

The corresponding cohomology sequence is called the Kummer sequence:

$$1 \to \mu_n(F) \to F^\times \xrightarrow{n} F^\times \to H^1_{et}(F; \mu_n) \to 1$$
$$1 \to H^2_{et}(F; \mu_n) \to Br(F) \xrightarrow{n} Br(F)$$

This yields isomorphisms $H^1_{et}(F; \mu_n) \cong F^\times / F^{\times n}$ and $H^2_{et}(F; \mu_n) \cong {}_nBr(F)$. If $\mu_n \subset F^\times$, this yields a natural isomorphism $H^2_{et}(F; \mu_n^{\otimes 2}) \cong {}_nBr(F) \otimes \mu_n(F)$.

5 Motivic-to-K-theory spectral sequence

There is a powerful spectral sequence which we heavily rely on in this topic to compute the K-groups of $\mathbb{Z}$.

Theorem 5.1 *For any coefficient group A and any smooth scheme X over a field k, there is a spectral sequence, natural in X and A:*

$$E_2^{p,q} = H^{p-q}(X, A(-q)) \Rightarrow K_{-p-q}(X; A).$$

Here is the fundamental structure theorem for motivic cohomology with finite coefficients, due to Rost and Voevodsky.

Norm Residue Theorem 5.2 (Rost-Voevodsky) *If k is a field containing $1/m$, the natural map induces isomorphisms*

$$H^n(k, \mathbb{Z}/m(i)) \cong \begin{cases} H^n_{et}(k, \mu_m^{\otimes i}) & n \leq i, \\ 0 & n > i. \end{cases}$$

If X is a smooth scheme over k, the natural map $H^n(X, \mathbb{Z}/m(i)) \to H^n_{et}(X, \mu_m^{\otimes i})$ is an isomorphism for $n \leq i$. For $n > i$, the map identifies $H^n(X, \mathbb{Z}/m(i))$ with the Zariski hypercohomology on X of the truncated direct image complex $\tau^{\leq i} Ra_*(\mu_m^{\otimes i})$.

Remark 5.3 Let k be a field. The edge map $K_{2i}(k; \mathbb{Z}/m) \to H^0_{et}(k, \mu_m^{\otimes i})$ is the e-invariant, and the other edge map $E_2^{0,-n} = H^n(k, \mathbb{Z}(n)) \cong K_n^M(k) \to K_n(k)$ is the natural map from Milnor K-theory into K-theory.

Corollary 5.4 (Block-Kato conjecture) *If k is a field containing $1/m$, the Galois symbols $K_i^M(k)/m \to H^i_{et}(k, \mu_m^{\otimes i})$ are isomorphisms for all i. They induce a ring isomorphism:*

$$\oplus K_i^M(k)/m \cong \oplus H^i(k, \mathbb{Z}(i))/m \cong \oplus H^i(k, \mathbb{Z}/m(i)) \cong \oplus H^i_{et}(k, \mu_m^{\otimes i}).$$

When $\mathcal{O}_S$ is a ring of integers in a number field F, the mod l cohomology dimension of $\mathcal{O}_S$ is 2 if l is an odd prime. This forces the motivic spectral sequence $E_2^{p,q}$ to degenerate completely. This is also true when $l = 2$ and F is total imaginary ($r_1 = 0$). Using the spectral sequence and results in the first section now we can compute the l-torsion subgroups of K-groups of $\mathbb{Z}$ in odd degrees.

Theorem 5.5 *Let F be a number field, and let $\mathcal{O}_S$ be a ring of integers in F. Fix a prime l; if $l = 2$ we suppose F totally imaginary. Then for all $n \geq 2$:*

$$K_n(\mathcal{O}_S)_{l-tors} \cong \begin{cases} H^2_{et}(\mathcal{O}_S[1/l]; \mathbb{Z}_l(i+1)) & \text{for } n = 2i, \\ \mathbb{Z}/w_i^{(l)}(F) & \text{for } n = 2i - 1. \end{cases}$$

The computation of the 2-torsion part of the K-groups of real number fields needs much more work. After careful examination of the real embeddings:

$$\alpha_S^n(i) : H^n(\mathcal{O}_S; \mathbb{Z}/2^\infty(i)) \to \bigoplus^{r_1} H^n(\mathbb{R}; \mathbb{Z}/2^\infty(i)),$$

we are able to get the following data:

Theorem 5.6 *Let F be a real number field, and let $\mathcal{O}_S$ be a ring of S-integers in F containing $\mathcal{O}_F[\frac{1}{2}]$. Then for all $n \geq 0$:*

$$K_n(\mathcal{O}_S; \mathbb{Z}/2^\infty) \cong \begin{cases} \mathbb{Z}/w_{4k}^{(2)}(F) & n = 8k \\ \mathbb{Z}/2 & n = 8k+2 \\ \mathbb{Z}/2w_{4k+2}^{(2)}(F) \oplus (\mathbb{Z}/2)^{r_1 - 1} & n = 8k+4 \\ 0 & n = 8k+6 \end{cases}$$

Combining Theorem 5.5 and Theorem 5.6 we have

Theorem 5.7 *Let $\mathcal{O}_S$ be a ring of S-integers in a number field F. Then for each odd $n \geq 3$, the group $K_n(\mathcal{O}_S) \cong K_n(F)$ is given by:*

a) If F is totally imaginary. $K_n(F) \cong \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i(F)$;

b) If F has $r_1 > 0$ real embeddings then, setting $i = (n+1)/2$,

$$K_n(F) \cong \begin{cases} \mathbb{Z}^{r_1+r_2} \oplus \mathbb{Z}/w_i(F) & n \equiv 1 \pmod 8 \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/2w_i(F) \oplus (\mathbb{Z}/2)^{r_1-1} & n \equiv 3 \pmod 8 \\ \mathbb{Z}^{r_1+r_2} \oplus \mathbb{Z}/\frac{1}{2}w_i(F) & n \equiv 5 \pmod 8 \\ \mathbb{Z}^{r_2} \oplus \mathbb{Z}/w_i(F) & n \equiv 7 \pmod 8 \end{cases}$$

6 Main Conjecture of Iwasawa Theory

The following theorem relates the orders of K-groups to the orders of étale cohomology groups, conjectured by Lichtenbaum in [Li] up to a factor of 2^{r_1}.

Theorem 6.1 *Let F be a totally real number field, with r_1 real embeddings, and let $\mathcal{O}_S$ be a ring of integers in F. Then for all even $i > 0$*

$$2^{r_1} \cdot \frac{|K_{2i-2}(\mathcal{O}_S)|}{|K_{2i-1}(\mathcal{O}_S)|} = \frac{\prod_l |H^2_{et}(\mathcal{O}_S[1/l]; \mathbb{Z}_l(i))|}{\prod_l |H^2_{et}(\mathcal{O}_S[1/l]; \mathbb{Z}_l(i))|}.$$

There is a deep result of Wiles[Wi, Thm 1.6], which is often called "Main Conjecture" of Iwasawa Theory.

Theorem 6.2 (Wiles) *Let F be a totally real number field. If l is odd and $\mathcal{O}_S = \mathcal{O}_F[1/l]$, then for all even integers $2k > 0$ there is a rational number u_k, prime to l, such that*

$$\zeta_F(1-2k) = u_k \frac{|H^2_{et}(\mathcal{O}_S, \mathbb{Z}_l(2k))|}{|H^2_{et}(\mathcal{O}_S, \mathbb{Z}_l(2k))|}.$$

Using these two theorems, we can prove the following result conjectured by Lichtenbaum in [Li, 2.4-2.6], which was only stated up to powers of 2.

Theorem 6.3 *If F is totally real, and* $\mathrm{Gal}(F/\mathbb{Q})$ *is abelian, then for all $k \geq 1$:*

$$\zeta_F(1-2k) = (-1)^{kr_1} 2^{r_1} \frac{|K_{4k-2}(\mathcal{O}_F)|}{|K_{4k-1}(\mathcal{O}_F)|}.$$

7 The K-theory of $\mathbb{Z}$

Recall that c_k is numerator of $B_{2k}/4k = (-1)^k c_k / w_{2k}$. By Theorem 5.7 and Theorem 6.3, we get

Theorem 7.1 *For $n \neq 0 \pmod 4$ and $n > 1$, we have*

(1) *If $n = 8k+1$, $K_n(\mathbb{Z}) \cong K_n(\mathbb{Q}) \cong \mathbb{Z} \oplus \mathbb{Z}/2$;*

(2) *If $n = 8k+2$, $|K_n(\mathbb{Z})| = 2c_{2k+1}$;*

(3) *If $n = 8k+3$, $K_n(\mathbb{Z}) \cong K_n(\mathbb{Q}) \cong \mathbb{Z}/2w_{4k+2}$;*

(4) *If $n = 8k+5$, $K_n(\mathbb{Z}) \cong K_n(\mathbb{Q}) \cong \mathbb{Z}$;*

(5) *If $n = 8k+6$, $|K_n(\mathbb{Z})| = c_{2k+2}$;*

(6) *If $n = 8k+7$, $K_n(\mathbb{Z}) \cong K_n(\mathbb{Q}) \cong \mathbb{Z}/w_{4k+4}$.*

Remark 7.2 The group $K_{4k-2}(\mathbb{Z})$ is cyclic (of order c_k or $2c_k$) for all $k \leq 5000$. To see this, we observe that $K_{4k-2}(\mathbb{Z})_{l-\text{tors}}$ is cyclic if l^2 does not divide c_k, and in this range only seven of the c_k are not square-free. The numerator c_k is divisible by l^2 only for the following pairs (k, l): (114, 103), (142, 37), (457, 59), (717, 271), (1646, 67), (2884, 101) and (3151, 157). In each of these cases we can use the usual transfer argument to show that $K_{4k-2}(\mathbb{Z})/l$ is either 0 or $\mathbb{Z}/l$. Since c_k is divisible by l^2 but not l^3, $K_{4k-2}(\mathbb{Z})_{l-\text{tors}} \cong \mathbb{Z}/l^2$.

There is a relation between π_n^s the stable homotopy groups of spheres and $K_n(\mathbb{Z})$. The natural maps $\pi_n^s \to K_n(Z)$ capture most of the Harris-Segal summands. When n is $8k+1$ or $8k+7$, the Harris-Segal summand of $K_n(\mathbb{Z})$ is isomorphic to the subgroup $J(\pi_n O)$ of π_n^s. When $n = 8k+3$, the subgroups $J(\pi_n O) \cong \mathbb{Z}/w_{4k+2}$ of π_n^s is contained in the Harris-Segal summand $\mathbb{Z}/2w_{4k+2}$ of $K_n(\mathbb{Z})$.

It is interesting that the statement $K_{4i}(\mathbb{Z}) = 0$ for all i is equivalent to the following conjecture in number theory:

Vandiver's Conjecture 7.3 *If l is an irregular prime then $\mathrm{Pic}(\mathbb{Z}[\zeta_l + \zeta_l^{-l}])$ has no l-torsion. Equivalently, the natural representation of $G = \mathrm{Gal}(\mathbb{Q}(\zeta_l)/\mathbb{Q})$ on $\mathrm{Pic}(\mathbb{Z}[\zeta_l])/l$ is a sum of G-modules $\mu_l^{\otimes i}$ with i odd.*

This means that complex conjugation c acts as multiplication by -1 on the l-torsion subgroup of $\mathrm{Pic}(\mathbb{Z}[\zeta_l])/l$, because c is the unique element of G of order 2.

Using Kummer's theory, we can prove the following equivalences:

Theorem 7.4 (Kurihara) *Let l be an irregular prime number. Then the following are equivalent for every integer k between 1 and $\frac{l-1}{2}$:*

(1) $\mathrm{Pic}(\mathbb{Z}[\zeta])/l^{[-2k]} = 0$.

(2) *$K_{4k}(\mathbb{Z})$ has no l-torsion.*

(3) *$K_{2a(l-1)+4k}(\mathbb{Z})$ has no l-torsion for all $a \geq 0$.*

(4) $H^2_{et}(\mathbb{Z}[1/l], \mu_l^{\otimes 2k+1}) = 0$.

In particular, Vandiver's conjecture for l is equivalent to the assertion that $K_{4k}(\mathbb{Z})$ has no l-torsion for all $k < \frac{l-1}{2}$, and implies that $K_{4k}(\mathbb{Z})$ has no l-torsion for all k.

Theorem 7.5 *If Vandiver's conjecture holds, then the groups $K_n(\mathbb{Z})$ are given by the following table for all $n \geq 2$. Here k is the integer part of $1 + \frac{n}{4}$.*

n (mod8)	1	2	3	4	5	6	7	8
$K_n(\mathbb{Z})$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/2c_k$	$\mathbb{Z}/2w_{2k}$	0	$\mathbb{Z}$	$\mathbb{Z}/c_k$	$\mathbb{Z}/w_{2k}$	0

References

[We] Charles A Weibel, *The K-book : an introduction to algebraic k-theory*, Graduate Studies in Math. vol. 145, AMS, 2013.

[Q1] D. Quillen, *Letter from Quillen to Milnor on $\mathfrak{J}(\pi_i(O) \to \pi_i^s \to K_i\mathbb{Z})$*, pp. 182-188 in Lecture Notes in Math. 551 Springer Verlag, 1976.

[Q2] D. Quillen, *Finite generation of the groups K_i of rings of algebraic integers*, pp. 195-214 in Lecture Notes in Math. 341, Springer-Verlag, 1973.

[Su] A. Suslin, *On the K-theory of algebraically closed fields*, Invent. Math. 73 (1983), 241-245.

[MP] J. P. May and K. Ponto, *More concise algebraic topology: localization, completion, and model categories*, 2012.

[Li] S. Lichtenbaum, *Values of zeta functions, étale cohomology, and algebraic K-theory*, pp. 489-501 in Lecture Notes in Math. 342, Springer Verlag, 1973.

[Wi] A. Wiles, *The Iwasawa conjecture for totally real fields*, Annals of Math. 131 (1990), 493-540.