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Simple Control Variates Method for Options Pricing with Stochastic Volatility Model

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Abstract

This paper applies the New Method to price European options, Geometric Average Asian options and Arithmetic Average Asian options with the risk-neutral pricing formula. Then constructing the control variates by combining the stochastic volatility with the deterministic volatility. In the numerical experiment, the results illustrate that the control variates with New Method are more efficient than the Plain Monte Carlo. Following sensitivity test results also demonstrate the discipline of variance reduction rate when varying different parameter settings.

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List of Symbols and Abbreviations

S_t	Asset Price of Underlying Process
$\hat{S}_t$	Asset Price of Underlying Process in Auxiliary Process
r	The Percentage Drift of Underlying Process
σ_t	The Stochastic Volatility of Underlying Process
$\hat{\sigma}_t$	The Deterministic Volatility of Underlying Process
Y_t	The Stochastic Risk Factor
$\hat{Y}_t$	The Deterministic Risk Factor
t	Time
W_{1t}	The Wiener Process Term of Underlying Process
W_{2t}	The Wiener Process Term of Volatility Process
ρ	The Correlated Coefficient Between Underlying Process and Volatility Process
ξ	The Volatility of Volatility Process
μ	The Percentage Drift of The Hull-White Model
k	The Mean Reverting Rate of The Heston Model
θ	The Long Run Average Price Variance of The Heston Model
α	The Mean Reverting Rate of The Stein-Stein Model
β	The Long Run Average Price Variance of The Stein-Stein Model
m	The Orders of Moment
V_p	The Value of European Put Option
X_{cAO}	The Control Variates for Continuous Sample Arithmetic Average or Geometric Average Asian Option

S_0	The Initial Value of Underlying Asset Price
y_0	The Initial Value of Risk Factor
K	The Strike Price
T	The Expiry Date
GBM	Geometric Brownian Motion
MC	The Plain Monte Carlo Method
CV	The Control Variates Method
SE	Standard Error
v_0	The Analytic Solution of Control Variates Method
v_1	The Option Value of Plain Monte Carlo Method
se_1	The Standard Error of Plain Monte Carlo Method
t_1	The Time Taken of Plain Monte Carlo Method
v_2	The Option Value of Controlhttps://www.overleaf.com/project/5d347dedae6d607832346bce Variates Method
se_2	The Standard Error of Control Variates Method
t_2	The Time Taken of Control Variates Method
$\hat{R}$	The Variance Reduction Ratio of Control Variates Method

1 Introduction

Since Black and Scholes (1973) proposed the Black-Scholes formula for European options based on a series of idealised assumptions, option pricing has been a heated topic in mathematical finance. In subsequent researches, numerous pricing methods have been putting forward to broaden the restrictions of the Black-Scholes model. The Black-Scholes formula has two major shortcomings. First, the model assumes there is no dividends payment of the underlying asset before the expiry date. Considering this, Merton et al. (1973) provides the Black-Scholes model with dividends, which is the well-known Black-Scholes-Merton model. Second, the volatility of the underlying process is assumed to be constant in the Black-Scholes formula, which does not represent the practical situation in the real financial world. Many researches contribute their effort to improve the model. One direction is to build a stochastic volatility model to simulate the motion of volatility. In this case, Hull and White (1987), Heston (1993) and Stein and Stein (1991) constructed different forms of stochastic volatility models on option pricing, which are classical stochastic volatility models in the last century.

Problems emerge when the stochastic volatility models are produced. The complexity of the stochastic volatility model increases the difficulty of option valuation, which means it is very hard to derive the analytic solution for an option. Under the circumstance, numerical solutions become a wider choice than the theoretical formulas. Some classical numerical methods, such as the lattice method and finite difference method, only have limited states in simulation. The estimated results appear to be inaccurate in general situations. In contrast, the Monte Carlo method is more precise by expanding the number of state variables. Also, With the development of the derivative market, more and more exotic options are invented to fit ongoing demand. Options such as Basket option and Rainbow option involve many underlying assets. Monte-Carlo is considered as a wide-fitting pricing method to deal with them. But there is a dilemma when the number of simulation rises in the numerical experiment. The results of the Monte Carlo method often takes a long time to converge. The decrease of standard error takes far more time in simulations compared with its variance reduction speed.

The plain Monte Carlo method is easy to use but slows in convergence, so Monte Carlo method usually needs speeding up. There are some variance reduction methods, such as stratified sampling, importance sampling and control variates. This paper introduces new simple control variates methods produced by Liu, Zhao and Gu (2015) and evaluates their variance reduction efficiency in numerical experiments. The experimental results present both their advantages and disadvantages by varying values of different parameters. Their control variates method applied in the Hull-White model, the Heston model and the Stein-Stein model have good performance on variance reduction efficiency. Among numerical tests in the three models, the option values priced by the Hull-White model appears to have the highest variance reduction speed. However, their methods are only reliable under low volatility circumstance. For each stochastic volatility model, varying different parameters setting from the seed paper, this paper finds that high values of the volatility of volatility process (parameter ξ) and extreme absolute values of the correlated coefficient between volatility process and underlying process (ρ) lead to inaccurate approximations on original process and low variance reduction rate. The variance reduction rate is highly sensitive to the parameter settings of ρ and ξ.

This paper is organised as follows. Section 2 illustrates the principle of control variates method and briefly looks back some related papers. In section 3, this paper presents three classic stochastic volatility models, which is frequently cited in the upcoming chapters. Section 4 and section 5 illustrate the algorithm of the new control method and practical option pricing, which is the core methodology of the seed paper. Concluded as the derived methods of the new method, Method I and Method II are presented in section 6. Section 7 describes the structure of our code and how to implement the algorithm in programming. In section 7, a series of numerical experiments are held to evaluate the performance Liu, Zhao and Gu method on option pricing and variance reduction rate. Section 8 is the conclusion of this paper.

2 Literature Review

An obvious shortcoming of the Black-Scholes formula is that the volatility of underlying process is set as a constant value, which is too idealised in the real finance world. In order to solve the problems, one direction is to build an econometric model to capture the motion features of volatility, Engle (1982) created Generalized Autoregressive Conditional Heteroskedasticity (GARCH) model to capture the volatility clustering. Another direction is to build a stochastic volatility model to simulate the motion of volatility.

Although stochastic volatility models improve the performance of option pricing, the complexity of the stochastic volatility model also increases the difficulty of option valuation, which means it does not have a closed-form solution for an option. Under this circumstance, numerical solutions become a wider choice. Boyle (1977) first applied the Monte Carlo simulation on option pricing and soon this method is popular. In contrast to other classical numerical methods, such as the Lattice method and the Finite Difference method, which only have limited dimensions in simulation, Monte Carlo method is more accurate by expanding the number of dimensions. But it also takes more time to converge, the standard error of solution decreases much slower than the increasing times of simulations. Using variance reduction method to improve the performance of Monte Carlo is inevitable.

Boyle et al. (1997) compare several Monte Carlo variance reduction methods on Arithmetic Asian option pricing. Among the antithetic method, the moment matching method, the latin hypercube sampling, the control variates method performs the best. Other methods suffer from difficult error estimation or ordinary error reduction, but the control variates must choose carefully in each case.

Joy et al. (1996) proposed one method to improve the convergence is quasi-Monte Carlo method. In quasi-Monte Carlo method, the traditional sample sequence is replaced by low discrepancy sequences. The most important problem for quasi-Monte Carlo method is how to construct the low discrepancy sequences. In 1960, Halton (1960) introduce a method to generate low discrepancy sequences. One year later, Haselgrove Sequence (Haselgrove, 1961) is presented and then Sobol' Sequence is introduced in 1976 (Soboĺ, 1976). As low discrepancy sequences help to speed up the convergence rate, quasi-Monte Carlo method start to be a popular method in derivative pricing. For the option pricing under the variance gamma model, Avramidis and L'Ecuyer (2006) analysis the different efficient between the traditional Monte Carlo and quasi-Monte Carlo. Later, the quasi-Monte Carlo is applied for pricing Asian Basket Option by Hu and Chen (2012). These research find that, in low dimension derivative pricing problems, the performance of quasi-Monte Carlo is better than the original Monte Carlo, and the convergence rate can be speed up obviously.

Another classic way of variance reduction is control variates method, which has been used in Monte Carlo simulation long before. The idea behind the control variates method is using the information from the known property to reduce the standard error of the unknown simulation error (Lemieux, 2014). The principle is called "Using what you known".

The key point of the control variates method is to find a suitable quantities which is highly related to the original simulations and cost little time. For different options, the control variates is chosen case by case.

An ordinary control variates is founded by valuing two similar instruments simultaneously and use one's known explicit solution to control the other stand error. For example, Geometric Asian option and Arithmetic Asian option. The second way is to fix the instrument and apply different processes to price the same instrument which has an explicit solution on one of them. The seed paper adopts the latter one in the European put option pricing and combines them in the Arithmetic Asian option pricing. Kemna and Vorst (2004) used the counterpart Geometric Average Asian option to price the Arithmetic Average Asian option and get a successful application result.

There are various control variates methods developed by researches. Since Boyle (1977) first use the control variates in Monte Carlo simulation, control variates is the most widely applicable. Clewlow and Carverhill (1994) design a control variates method from the "Greek Letter" of an option, and using Delta approximation or higher order Gamma approximation to form a new option to compute control variates. The most dramatic variance reduction is attained by Delta approximation in numerical test. Broadie and Glasserman (1996) create a control variates method called re-simulation using a known simulated derivatives' information to simulate other derivatives share the same underlying. Lidebrandt (2007) compare two control variates methods on option price with an ordinary control variates on Asian option and Cliquet options. The result shows the re-simulation has the best efficiency, except one case the ordinary control variates uses a Geometric Average Asian option as control variates to price Arithmetic Asian option.

In the case of stochastic volatility process, the chosen of the constant volatility to replace the stochastic volatility is a key problem which will influence the efficiency of the control variates. Han and Lai (2010) apply a Martingale Control variates method to choose the volatility basing on the initial volatility of the process. However, this method sometimes is hard to handle as the invariant distribution function of stochastic volatility is unknown.

The seed paper, Liu et al. (2015), uses the control variates method to accelerate the Monte Carlo method. There are three kinds of control variates methods mentioned in it, including: 1) the control variates method in constant volatility form, such as Johnson and Shanno (1987), 2) the control variates method with underlying asset in first and second order moment carried out by Ma and Xu (2010), 3) the control variates method with the order moment of the stochastic volatility process proposed by Kun et al. (2013). The first one can be easily experimented but it has the lowest efficiency in variance reduction. The principle of new method is to derive a deterministic volatility process for the stochastic process, the deterministic volatility process constructed by a deterministic risk factor having the same order moment with the stochastic risk factor. By doing this, the simulation time of volatility will decrease significantly. The seed paper tests New Method in European put option and Asian call option.

3 Classic Stochastic Volatility Model

A stochastic volatility model is called stochastic because the diffusion part σ_t is not a deterministic number at each time step. The diffusion part can be used to derive a deterministic process of Y_t by moment approximation. In classic stochastic volatility model mentioned in the seed paper, risk factor Y has a square or absolute value functional form ($f(Y) = \sqrt{Y}$, $f(Y) = |Y|$). The stochastic property involved the diffusion part is the reason why some stochastic models are hard to solve. A reasonable deterministic process should be picked up with an explicit solution for the target option.

The underlying process under the risk-neutral measure is

$$\begin{aligned} dS_t &= S_t\left(rdt + \sigma_t dW_{1t}\right), \\ \sigma_t &= f\left(Y_t\right), \\ dY_t &= \alpha\left(Y_t\right)dt + \beta\left(Y_t\right)dW_{2t}, \end{aligned} \tag{1}$$

where W_{1t} and W_{2t} are standard Wiener process, and $\operatorname{cov}\left(dW_{1t}, dW_{2t}\right) = \rho dt$.

For different stochastic volatility models, we use the specific process of volatility Y_t to explore the control variates method on that. The specific volatility process Y_t is defined by the stochastic model itself.

Here we list three typical models briefly and put control variates methods into further practice with these models in the following chapters.

For the Hull-White model, we have

$$\begin{aligned} \sigma_t &= \sqrt{Y_t}, \\ dY_t &= Y_t\left(\mu dt + \xi dW_{2t}\right), \end{aligned} \tag{2}$$

where μ and ξ are constant number.

In the Hull-White model, the variance function has a log-normal distribution. The Hull-White model is the simplest stochastic volatility model, but it implies that the volatility grows exponentially, which is against reality. According to empirical analysis, the volatility fluctuates with a constant number in a long time, so a mean-reverting property is more preferable.

For the Heston model, we have

$$\begin{aligned} \sigma_t &= \sqrt{Y_t}, \\ dY_t &= k\left(\theta - Y_t\right)dt + \xi\sqrt{Y_t}dW_{2t}, \end{aligned} \tag{3}$$

where k, θ and ξ are constant variables.

The Heston model has a CIR process for volatility which has the mean-reverting property. If the Feller condition $2\kappa\theta > \xi^2$ is satisfied, the process is always strictly positive.

For the Stein-Stein model, it is a mean-reverting Ornstein–Uhlenbeck process, the variance term can not less than zero in the model.

$$\begin{aligned} \sigma_t &= \left|Y_t\right|, \\ dY_t &= \alpha\left(\beta - Y_t\right)dt + \xi dW_{2t}, \end{aligned} \tag{4}$$

where α, β and ξ are constant numbers.

4 New Method

The principle of the new control variates method proposed by Liu, Zhao and Gu (2015) is to fit the stochastic risk factor by a deterministic functional form.

4.1 Introduction

The New Method (Liu et al., 2015) looks like an improved version of the Method I (Kun et al., 2013). We use $\hat{Y}_t$ to represent the deterministic volatility model process, Y_t as the stochastic risk factor. As formula (1) showed, we derive a specific function form of Y_t.

The auxiliary process is formed by choosing a deterministic function $\hat{Y}_t$ as close as possible to the stochastic risk factor Y_t. So we can choose a non-random risk factor $\hat{Y}_t$ having the same m-th order moment (m is a positive integer) related to Y_t

$$\hat{Y}_t^m = E\left[Y_t^m\right]. \tag{5}$$

By the non-random part $\hat{\sigma}_t = f(\hat{Y}_t)$, we can get the analytic solution and further complete the algorithm of control variates, the auxiliary process is

$$\begin{aligned} d\hat{S}_t &= \hat{S}_t\left(rdt + \hat{\sigma}_t dW_{1t}\right), \\ \hat{S}_0 &= s_0, \end{aligned} \tag{6}$$

where s_0 is the initial value of the underlying, W_{1t} is a standard Brownian motion and r is risk-free interest rate.

4.2 Algorithm

The algorithm of new method follows procedures mentioned below,

1) Determine the volatility function $\hat{Y}_t^m = E\left[Y_t^m\right]$. Here m is the order of moment, if the model is hard to find the closed solution for m (m is a positive integer), the moment order m can be set 1 or 2 is also fine. We use $\hat{Y}_t$ to represent estimated volatility model process (auxiliary process), Y_t as the stochastic variance process.

2) We know the functional form of $\sigma_t = f\left(Y_t\right)$, so for auxiliary process we set $\hat{\sigma}_t = f(\hat{Y}_t)$.

3) Divide the time period $[0,T]$ into n intervals with $T/n = t_{k+1} - t_k$.

4) Generate M paths with auxiliary variance function $\hat{\sigma}_t$, Using following formula to compute underlying price $\hat{S}_{t_k}$ for $\hat{S}_{t_{k+1}}$.

$$\hat{S}_{t_{k+1}} = \hat{S}_{t_k}\exp(r\Delta t - \frac{1}{2}\int_{t_k}^{t_{k+1}} \hat{\sigma}_s^2 ds + \int_{t_k}^{t_{k+1}} \hat{\sigma}_s dW_{1s}), \tag{7}$$

where W_{1s} is a Wiener process, as $\int_{t_k}^{t_{k+1}} \hat{\sigma}_s dW_{1s} \sim N\left(0, \int_{t_k}^{t_{k+1}} \sigma^2(s)ds\right)$, we generate a random number following standard normal distribution $Z_k^{1,j} \sim N(0,1)$

$$S^j(t_{k+1}) = S^j(t_k)\exp(r\Delta t - \frac{1}{2}\int_{t_k}^{t_{k+1}} \hat{\sigma}_s^2 ds + \sqrt{\int_{t_k}^{t_{k+1}} \hat{\sigma}_s^2 ds}\, Z_k^{1,j}\ ),\ \ (j = 1, \cdots, M), \tag{8}$$

where j is the path index.

5) According to the specific payoff function of the option, we get the price of target option on each path. Here δ is the discount factor, $H(S)$ is the payoff function of target option.

$$X^j = \delta H(S^j), \tag{9}$$

6) Calculate the analytic solution V to target option with the auxiliary process, the control variates d^j is set as

$$d^j = X^j - V, \tag{10}$$

7) Similarly, we can generate M paths with original variance function from S_{t_k} to $S_{t_{k+1}}$, using following formulas to compute underlying price paths.

$$S^j_{t_{k+1}} = S^j_{t_k} \exp\left\{ \left(r - \frac{\left(\sigma^j_{t_k}\right)^2}{2} \right) \Delta t + \sigma^j_{t_k} \sqrt{\triangle t} Z^{1,j}_k \right\}, \tag{11}$$

$$Y^j_{t_{k+1}} = Y^j_{t_k} \exp\left[\left(\mu - \frac{1}{2}\xi^2 \right) \Delta t + \xi \sqrt{\triangle t} Z^{2,j}_k \right], \tag{12}$$

where $Z^{1,j}_k \sim N(0,1)$, $Z^{2,j}_k \sim N(0,1)$ for formula (14) and (15), there is a correlation coefficient ρ between them. The random number $Z^{1,j}_k$ is the same random number in formula (16).

8) According to clause of target option contract, we can get the price of option on each path. Here δ is the discount factor,

$$C^j = \delta H(S^j), \tag{13}$$

9) Now we get a set of pairs $\left(c^j, d^j\right)_{j=1,\dots,M}$.

10) Using the following formula to calculate W^j on each path j

$$W^j = c^j - \beta d^j, \tag{14}$$

where $\beta = \operatorname{cov}(c,d)/\operatorname{var}(d) = \rho_{c,d} \times sd(c)/sd(d)$.

11) Calculate the average of the option price W^j on every path to get the final solution. $W = \frac{1}{M}\sum_{j=1}^{M} W^j$.

5 Example Models in the New Method

In this section, this paper provides the deterministic volatility process for the Hull-White Model, the Heston Model and the Stein Stein Model by implementing the New Method, the deterministic volatility will be deviated as a function of time t.

5.1 Hull-White Model

For the Hull-White model, first we get an m-th order moment result for stochastic process Y_t.

$$E\left[Y_t^m\right] = y_0^m \exp\left\{mt\left(\mu + \frac{1}{2}(m-1)\xi^2\right)\right\}. \tag{15}$$

Then it is easy to choose a proper m-th order to the $\hat{Y}_t$ and get deterministic volatility.

$$\hat{Y}_t = \left(E\left[Y_t^m\right]\right)^{\frac{1}{m}} = y_0 \exp\left\{t\left(\mu + \frac{1}{2}(m-1)\xi^2\right)\right\}. \tag{16}$$

Next we can get the deterministic variance function $\hat{\sigma}_t$ (y_0 is the initial value of the random factor Y_t)

$$\hat{\sigma}_t = \sqrt{\hat{Y}_t} = y_0^{\frac{1}{2}} \exp\left\{\frac{1}{2}t\left(\mu + \frac{1}{2}(m-1)\xi^2\right)\right\}. \tag{17}$$

5.2 Heston Model

For the Heston model, the closed formula solution for Y_t is hard to derive, only the expectation can be derived. This paper ignore the higher order moment of $E\left[Y_t^m\right]$ and just consider the first order moment,

$$E\left[Y_t\right] = e^{-kt}y_0 + \theta\left(1 - e^{-kt}\right). \tag{18}$$

Then we have

$$\hat{\sigma}_t = \sqrt{E\left[Y_t\right]} = \sqrt{e^{-kt}y_0 + \theta\left(1 - e^{-kt}\right)}. \tag{19}$$

5.3 Stein-Stein Model

For the Stein-Stein Model, the expectation of Y_t can be easily derived as

$$E\left[Y_t\right] = e^{-\alpha t}y_0 + \beta\left(1 - e^{-\alpha t}\right). \tag{20}$$

In the New Method, we choose $\hat{Y}_t$ to be

$$E[\hat{Y}_t] = E[Y_t], \tag{21}$$

so $\hat{\sigma}_t$ is

$$\hat{\sigma}_t = |\hat{Y}_t| = \left| e^{-\alpha t} y_0 + \beta \left(1 - e^{-\alpha t}\right) \right|. \tag{22}$$

5.4 The Analytic Solution for Options

5.4.1 European Put Option

If we replace the stochastic volatility with deterministic $\hat{Y}_t$, there is an analytic solution for European put option.

$$V_p\big|_{t=0} = Ke^{-rT} N(d_1) - s_0 N(d_1 - b), \tag{23}$$

where

$$d_1 = \frac{\ln K - a}{b},\ a = \ln s_0 + rT - \frac{1}{2}\int_0^T \hat{\sigma}_t^2 dt,\ b = \sqrt{\int_0^T \hat{\sigma}_t^2 dt}. \tag{24}$$

For different model, we can integrate the deterministic volatility $\hat{\sigma}_t$ and get the specific parameters d_1, a, b

For the Hull-White model,[1]

$$d_1 = \frac{\ln k - a}{\sqrt{b}},\ a = \ln s_0 + rT - \frac{b}{2}, \quad b = y_0 \frac{e^{ct} - 1}{c},\ c = \mu + \frac{1}{2}(m-1)\xi^2. \tag{25}$$

For the Heston model,

$$d_1 = \frac{\ln k - a}{\sqrt{b}},\ a = \ln s_0 + rT - \frac{b}{2},\ b = \theta T + \frac{1}{k}(y_0 - \theta)\left(1 - e^{-kT}\right). \tag{26}$$

For the Stein-Stein model,[2]

[1] In the seed paper, the equation "$a = \ln s_0 + rT + b$" is wrong.

[2] In the seed paper, the equation "$b = \beta^2 + 2\beta (y_0 - \beta)^2 \frac{e^{-\alpha T} - 1}{-\alpha} + (y_0 - \beta)^2 \frac{e^{-2\alpha T} - 1}{-2\alpha}$" is wrong.

$$
\begin{aligned}
&d_1 = \frac{\ln k - a}{\sqrt{b}}, \quad a = \ln s_0 + rT - \frac{b}{2}, \\
&b = \beta^2 T + 2\beta\left(y_0 - \beta\right)\frac{e^{-\alpha T} - 1}{-\alpha} + \left(y_0 - \beta\right)^2 \frac{e^{-2\alpha T} - 1}{-2\alpha}.
\end{aligned} \tag{27}
$$

Having the analytic solution on auxiliary process and Monte-Carlo method solution, we apply control variates Monte-Carlo easily.

5.4.2 Asian Call Option

Using a deterministic square-integral volatility $\hat{\sigma}_t$ to replace the stochastic volatility σ_t, for the fixed strike continuous sampling Asian call options, the analytic formula of the geometric average form is

$$
\begin{aligned}
X_{cAO}\big|_{t=0} &= E\left[e^{-rT}\left(X_{cAO}\big|_{t=T}\right)\right] \\
&= e^{-rT} E\left[\left(e^{\frac{1}{T}\int_0^T \log S(t)dt} - K\right)^{+}\right] \\
&= e^{\frac{1}{2}\widehat{\sigma}^2 - rT + a} N\left(d_+\right) - K e^{-rT} N\left(d_-\right),
\end{aligned} \tag{28}
$$

where

$$
a = \log S_0 + \frac{1}{2} rT - \frac{1}{2T}\int_0^T \left[\int_0^t \sigma^2(s) ds\right] dt,
$$

$$
\widehat{\sigma}^2 = \lim_{n\to\infty} \frac{1}{n^2} \sum_{j=1}^{n} \left[2(n-j)+1\right] \int_0^{j\frac{T}{n}} \sigma^2(s) ds,
$$

and

$$
d_- = \frac{a - \log K}{\widehat{\sigma}}, \; d_+ = d_- + \widehat{\sigma}.
$$

For the Hull-White model, the parameter a and $\widehat{\sigma}^2$ follows

$$
a = \begin{cases} \log S_0 + \frac{1}{2} rT - \frac{1}{4}\sigma_0^2 T & \text{if } a_m = 0, \\ \log S_0 + \frac{1}{2} rT - \frac{\sigma_0^2}{2T a_m}\left[\frac{1}{a_m}\left(e^{a_m T} - 1\right) - T\right] & \text{if } a_m \neq 0, \end{cases} \tag{29}
$$

$$
\widehat{\sigma}^2 = \begin{cases} \frac{1}{3}\sigma_0^2 T & \text{if } a_m = 0, \\ \frac{2\sigma_0^2}{T^2 a_m^3}\left(e^{a_m T} - 1\right) - \frac{2\sigma_0^2}{T a_m^2} & \text{if } a_m \neq 0, \end{cases} \tag{30}
$$

where $a_m = \mu + \frac{1}{2}(m-1)\sigma^2$, $y_0 = {\sigma_0}^2$.

This experiment provides the variance reduction ratios, when X_{cAO} is applied to the control variates for the Arithmetic Average or Geometric Average Asian option with continuous sampling.

6 Comparison with Other Two Methods

In this section, the New Method will be compared with the other two methods, which are called Method I and Method II. The Method I, introduced by Du, Liu and Gu (2013), is constructed with m-th order moment of stochastic volatility. The Method II approximates the deterministic volatility from the second order moment of the underlying stock price S_t by Ma and Xu (2010).

6.1 Method I

In order to use control variates Monte Carlo pricing for Asian rate option and Asian strike option, as well as the multi-asset options pricing, Du, Liu and Gu (2013) introduced this method. The core of this method is to create a deterministic variance function, which has the same moment of the original stochastic variance. The difference between New Method and Method I is easy to find.

In Method I, we derive the Y_t from variance function $\sigma_t = f(Y_t)$, and use Y_t to choose a deterministic $\hat{Y}_t$ which shares same m-th order moment $\hat{Y}_t^m = E\left[Y_t^m\right]$. However in Method I, we directly get

$$\hat{\sigma}_t^m = E\left[\sigma_t^m\right], \tag{31}$$

and form new auxiliary process based on $\hat{\sigma}_t$ directly.

That is the reason why we called the New Method is an improved version of Method I. So the variance process $\hat{\sigma}_t$ in Method I just changed a little compared with the New Method, which means the algorithm of Method I is similar to New Method.

6.1.1 Example Models

The deterministic volatility for Hull-White Model and Heston Model combine with Method I are derived as a function of time t.

For Hull-White Model, the deterministic volatility combined with Method I is derived as

$$\hat{\sigma}_t = Y_0^{\frac{1}{2}} \exp\left\{\tfrac{1}{2}t\left(\mu + \tfrac{1}{4}(m-2)\xi^2\right)\right\}, \qquad \text{Method I} \tag{32}$$

$$\hat{\sigma}_t = \sqrt{\hat{Y}_t} = y_0^{\frac{1}{2}} \exp\left\{\tfrac{1}{2}t\left(\mu + \tfrac{1}{2}(m-1)\xi^2\right)\right\}, \qquad \text{New Method} \tag{33}$$

We can see the two diffusion processes is similar. We just need to use $\hat{\sigma}_t$ to replace the one in the new method.

For the Heston model, the seed paper derives the second order moment of σ_t^2 in the Method I and the first order moment of Y_t in the New Method respectively. Because in the Heston model, $\sigma_t^2 = Y_t$, so the variance process is same between Method I and New method. The algorithm does not need to change.

$$\hat{\sigma}_t = \sqrt{E\left[Y_t\right]} = \sqrt{e^{-kt}y_0 + \theta\left(1 - e^{-kt}\right)}. \tag{34}$$

For the Stein-Stein model, the variance function form is $\sigma_t = |Y_t|$. Method I uses the expectation of the absolute value Y_t to get $E\left[|Y_t|\right]$, the new method gets the expectation value of $E\left[Y_t\right]$ first and gets the absolute value $\hat{\sigma}_t = |\mathrm{E}[\hat{Y}_t]|$.

$$E\left[|Y_t|\right] = \tfrac{2\rho}{\sqrt{2\pi}} \exp\left\{-\tfrac{\nu^2}{2\rho^2}\right\} + \nu - 2\nu\Phi\left(-\tfrac{\nu}{\rho}\right), \qquad \text{Method I} \tag{35}$$

$$\hat{\sigma}_t = |\hat{Y}_t| = |e^{-\alpha t}y_0 + \beta(1 - e^{-\alpha t}). \qquad \text{New Method} \tag{36}$$

Here the seed paper fails to derive the analytic solution with non-random variance function $\hat{\sigma}_t$, so Method I cannot be used for the Stein-Stein model.

6.1.2 The Analytic Solution for European Put Option in Method I

Recalling the analytic solution in equation (23) and equation (24), which are

$$V_p\big|_{t=0} = Ke^{-rT}N\left(d_1\right) - s_0 N\left(d_1 - b\right),$$

where

$$d_1 = \frac{\ln K - a}{b},\ a = \ln s_0 + rT - \frac{1}{2}\int_0^T \hat{\sigma}_t^2 dt,\ b = \sqrt{\int_0^T \hat{\sigma}_t^2 dt}.$$

Due to the $\hat{\sigma}_t^2$ is defined by different models. The integration of $\int_0^T \hat{\sigma_t^2} dt$ dominants the difference among models in the Method I.

For the Hull-White model, the deterministic volatility is based on the Method I, parameters a, b , c are

$$a = lnS + rT - \frac{1}{2}b,\ b = Y_0\frac{e^{cT} - 1}{c},\ c = \mu + \frac{1}{2}(m-2)\xi^2. \tag{37}$$

For the Heston model, the parameters a , b, c are

$$a = lnS + rT - \frac{1}{2}c,\ b = \sqrt{c},\ c = \frac{Y_0 - \theta}{k}(1 - e^{-kT}) + \theta T. \tag{38}$$

However, for the Stein-Stein model, there is no analytic solution in the Method I.

6.2 Method II

Method II is proposed by Ma and Xu (2010) when pricing variance swaps by control variates Monte Carlo method. This closed form solution is derived for the approximate model with deterministic volatility, which plays a key role in their paper, and an efficient control variates technique is therefore proposed when the volatility obeys a log-normal process. By the analysis of moments for the underlying processes, the optimal volatility function in the approximate model is constructed. The numerical results demonstrate the high efficiency in this Ma and Xu's method. However, Ma and Xu (2010) only considered the first and two order moments of underlyging to choose control variates,

$$\begin{aligned} \hat{S}_t &= E\left[S_t\right], \\ \hat{S}_t{}^2 &= E\left[S_t^2\right]. \end{aligned} \tag{39}$$

Choosing $\hat{\xi}_t$ with the condition that $\hat{S}_t = E\left[S_t\right]$ and $\hat{S}_t{}^2 = E\left[S_t^2\right]$. Then the auxiliary process $\hat{S}_t$ is attained.

In seed paper extends this method to m-th order of underlying to derive the deterministic volatility. Ma and Xu (2010).

6.2.1 Example Models

Combined with Method II, the deterministic volatility for the Hull-White Model and the Heston Model are derived.

For the Hull-White model, the m-th order of the underlying stock S_t can be derived with the stochastic volatility σ_t.

$$\begin{aligned} E\left[S_t^m\right] &\approx E\left[s_0^m e^{mrt} \exp\left\{-\tfrac{m}{2}\int_0^t Y_s ds + m^2 \int_0^t Y_s ds\right\}\right], \\ E\left[\hat{S}_t^m\right] &= s_0^m e^{mrt} E\left[\exp\left\{-\tfrac{m}{2}\int_0^t \hat{\sigma}_s^2 ds + m^2 \int_0^t \hat{\sigma}_s^2 ds\right\}\right]. \end{aligned} \tag{40}$$

The extension form is constructed into $E(S_t^m) = E(\hat{S}_t^m)$, which derive the stochastic volatility with the boundary condition of stochastic factor Y_0 and time t.

$$\hat{\sigma}_t = Y_0^{\frac{1}{2}} \exp\left\{\frac{1}{2}t\left(\mu - \frac{1}{4}\xi^2\right)\right\}. \tag{41}$$

For the Heston model, the situation is similar to Method I. Considering the first order and two order moment of the underlying.

$$\begin{aligned} &E\left[S_t\right] = s_0 e^{rt}, \\ &E\left[S_t^2\right] = s_0^2 e^{2rt} E\left[\exp\left\{\textstyle\int_0^t E\left[Y_s\right]\right\}\right] ds, \\ &E[\hat{S}_t] = s_0 e^{rt}, \\ &E\left[s_0^2\right] = s_0^2 e^{2rt} \exp\left\{\textstyle\int_0^t \sigma^2(s) ds\right\}. \end{aligned} \tag{42}$$

Then we can derive $\hat{\sigma}_t = \sqrt{E\left[Y_t\right]}$ from the second order processes.

For the Stein-Stein model, the seed paper fails to apply Method II on that.

6.2.2 The Analytic Solution for European Put Option in Method II

Recalling the analytic solution in equation (23) and equation (24), which are

$$V_p\big|_{t=0} = Ke^{-rT} N\left(d_1\right) - s_0 N\left(d_1 - b\right),$$

where

$$d_1 = \frac{\ln K - a}{b},\ a = \ln s_0 + rT - \frac{1}{2}\int_0^T \hat{\sigma}_t^2 dt,\ b = \sqrt{\int_0^T \hat{\sigma}_t^2 dt}.$$

As the integration of sigma $\int_0^T \sigma^2(t)dt$ is different, the analytic formula of parameters a and b are different than the New Method.

For the Hull-White model, the deterministic volatility bases on the Method II, a and b are

$$a = lnS + rT - \frac{1}{2}b,\ b = Y_0 \frac{e^{cT} - 1}{c},\ c = \mu - \frac{1}{2}\xi^2. \tag{43}$$

For the Heston model, a and b are

$$a = lnS + rT - \frac{1}{2}c,\ b = \sqrt{c},\ c = \frac{Y_0 - \theta}{k}(1 - e^{-kT}) + \theta T. \tag{44}$$

7 Implementation in C++

In our program, we construct different the objects and make them communicate with each other base on the principle of object-oriented. This paper will reveals the high lights of our code, and explain every object and how then connected in the second part. In appendix, there is a flow chart for our structure.

7.1 Code Highlights

- We only create one GBM (Geometric Brownian Motion) process in "Valuation" object, the "process_GBM" is initialised with a deterministic volatility process in a "MonteCarlo_CV" object and a stochastic volatility process in the "MonteCarlo_plain" object respectively. The two processes should use the same random number and initialise in the two different Monte Carlo.
- The new method control variates method is implemented in the deterministic volatility class, this class store two functions, one returns the deterministic volatility to the GBM process class, the other returns the explicit formula for the option class. Besides, we add Method I and Method II in the deterministic volatility in the same way.
- In our control variates method, "MonteCarlo_plain" and "MonteCarlo_CV" share the same random number in "filling_up_path" part. In our code, there is an independent random number manager class to manage random numbers on each path. After Monte Carlo method has been executed on every path, the random number will update a new series of random number.
- By applying double dispatch, for any deterministic volatility with different process, the program easily recognises the control variates has explicit or not. If the control variates has explicit, then the explicit can be obtained, else it will return an error message and stop the program.
- For the arithmetic average Asian option, the option will create a geometric average Asian option inside, when the auxiliary process is computing payoff and explicit solution, the code returns the geometric average Asian option function to compute these value.

Our code is classified as auxiliary part and valuation part, the auxiliary part consists "Input" object, "Output" object, "Environment_stuff", "Factory_stuff" and "Library". The valuation part contains ”Monte_Carlo” object, ”Option” object, ”GBM” object, ”Stochastic|Volatility” object, ”Deterministic_Volatility” object.

7.2 Code Architecture

For “IO_stuff”, first, we construct configuration manager which include environment manager and monitor. For the environment, it is set to control the ways to input the data to the program and the ways to output the result. For the monitor, it is set to monitor the program running statement. Second, we construct input stuff, output stuff and parameter. For input stuff, it provides 3 approaches to receive the data, which are input data from console, input data from code and input data from ’txt’ file. For the output stuff, output results to console and output results to file can be chosen. Third, the whole structure of parameter is designed to store all parameter that will be used in other classes, and the other classes can receive these data by using the “IO_management” to obtain any data they want.

For “Factory”, it is a creation mode, which defines a creation object interface, and can instantiate any subclass. In this project, we can use factory to choose different type of Monte Carlo method, stochastic volatility, deterministic volatility and option to register in the program. Then, pass them through all the program to compute some values and get the values.

For Library, we contain RegressionOLS, lib_val, rv_library and utility. The RegressionOLS to do the ordinary least squares regression for control variates. The lib_val is used to transform the data type or do some data type recognition. rv_library contains some function formula that can be applied to the option price computation. Some control command can obtain from the utility such as print command.

“Application Wrapper” is to initialise valuation class, Monitor class, Output class and StopWatch class, and start to monitor the whole program, call the Valuation to obtain values from other objects, then set up the output.

The whole valuation part starts with a Valuation object and end with an “Output” object.

“Valuation” object involves Option base object, GBM process object and Monte Carlo Manager class which has two derived class CV (Control Variates) Monte Carlo and plain Monte Carlo. The valuation object initialises a GBM process class and run a “MonteCarlo_manager”.

The “GBM_process” is composed of two volatility processes, one is stochastic volatility class initialised in “valuation” object, the other one is deterministic volatility process created and initialised in the “MonteCarlo_CV” object.

The stochastic volatility class is required to return a generated stochastic volatility value to GBM process, The deterministic volatility class is demanded to return a deterministic volatility value to GBM process and return the explicit solution corresponding to the present deterministic volatility process.

There are three different stochastic volatility processes, the “Heston_process”, the “Stein-Stein _process”, the “Hull-White_process”.

The eight deterministic volatility processes include three processes which combined New Method and three above stochastic volatility processes and other three processes combined the Method I with the three stochastic volatility, the other two processes are Method II combined with the Heston, the Hull-White process, the Method II can not be applied in the Stein-Stein process.

The “Valuation” object has two Monte Carlo pricing methods, plain Monte Carlo and CV Monte Carlo. The input file will decide which one to run in program.

The “MonteCarlo_Manager” contains a “Randomnumber_Manager” object and create a “MonteCarlo _Base” object, which can be inherited by plain Monte Carlo and CV Monte Carlo. For our control variates method, the plain Monte Carlo and CV Monte Carlo must share the same random number that’s why we create “MonteCarlo_Manager” here. With the manager object, the two different Monte Carlo methods are able to run independently, which means that even an option has no control variates method still cam use plain Monte Carlo pricing. In its run function, the “MonteCarlo_Manager” will initialise the “MonteCarlo_Base” with “Randomnumber_Manager”, option and GBM process and run the “MonteCarlo_Base”.

The plain Monte Carlo does the simulation with setting GBM process to fill up the underlying path and create an accumulator object to store each path payoff.

The “Monte Carlo CV” creates a “Monte Carlo plain” in its constructor and initialises it with own random number manager object. This “Monte Carlo CV” has been initialised with an option base and a “process GBM” which only has a stochastic volatility process. When a “Monte Carlo CV” is executing, it will create a deterministic volatility process and put it into the “process GBM”.

In its run function, the “Monte Carlo CV” will firstly calculate the explicit solution with deterministic volatility process. Secondly, the “Monte Carlo CV” will do a similar simulation work with “Monte Carlo plain”, filling up an underlying path and compute the option payoff. The difference is that when “Monte Carlo CV” run one path with deterministic volatility, it will also call owned plain Monte Carlo run one path with stochastic volatility at the same time, the "Monte Carlo CV" then transfer the explicit solution and deterministic volatility simulation payoff to "Control variates" to compute control variates. In the end, the plain Monte Carlo payoff and control variates will be stored in the "Accumulator_CV".

On each path, a "Monte Carlo CV" has an accumulator to store the payoff with deterministic volatility process. The deterministic volatility process payoff is not presented in the result table, we still keep it to check whether the control variates is right.

Apart from the plain accumulator, the Monte Carlo CV class has another powerful accumulator "Accumulator_CV". During computing each path, the control variates and plain Monte Carlo payoff will be produced and save in it. The final option value will be calculated in "Accumulator_CV" by Ordinary Least Square regression. The CV average and the correlation coefficient ρ will be printed at the console to check the correctness.

The Computation of control variates consists two parts, the Monte Carlo simulation solution and the explicit solution, control variates is the difference of the two value on each path. How to generate simulation solution has been explained before, the explicit solution is computed in option base. We use double dispatch to catch the proper explicit formula, the explicit solution will be stored as a private member in the Monte Carlo CV class.

There is an additional function in Monte Carlo CV class running an independent Monte Carlo plain just for the time consuming t_1, the seed paper neglects the time consuming between the Monte Carlo plain and Monte Carlo CV, which defines variance reduction ratio as

$$\hat{R} = \frac{se_1^2}{se_2^2}, \tag{45}$$

In our report, we define a new variance reduction ratio as

$$\hat{R} = \frac{t_1 se_1^2}{t_2 se_2^2}. \tag{46}$$

8 Numerical Experiment

In this section, this paper follows the routine of Liu et al. (2015), giving the numerical experiment of New Method to demonstrate the efficiency of the variance reduction in pricing European put options, European call options and Arithmetic Asian call options.

Deterministic volatility dominates the discrepancy among Method I, Method II, New Method. The slight changes of the deterministic volatility formulas do not influence the discipline of their performance when varying parameter values. So this paper only present the numerical experiment of the new method, including reproducing the results of the seed paper and sensitivity test for extreme parameter values.

In order to quantify how efficient the methods are, the variance reduction ratio (the variance by the plain Monte Carlo method divided by the control variates Monte Carlo method), cited by Liu et al. (2015), is used to illustrate the acceleration of new methods. However, this evaluation is inaccurate. Liu et al. followed the work of Ma and Xu (2010). They ignored the time taken in computing deterministic volatility by control variates methods. The computational time of plain MC method is not closed to the time of control variates method. So they contribute to the calculation of variance reduction ratio.

In the following subsection, this paper defines several symbols to represent distinct methods and outcomes."MC" is used for representing the plain Monte Carlo results."CV" is used for the relative control variates ones. v_0 is the analytic solution in New Method. v_1 is the option price of "MC". v_2 is the option price of "CV". se_1 is the standard error of ordinary Monte Carlo method. se_2 is the standard error of control variates Monte Carlo method. $\hat{R}$ is the variance reduction ratio, calculated as,

$$\hat{R} = \frac{t_1 se_1^2}{t_2 se_2^2}. \tag{47}$$

where t_1 is the time taken of the plain Monte Carlo method, t_2 is the time taken of the control variates Monte Carlo method.

In the seed paper, Liu et al. set distinct values for the same or distinct parameters in each model. They are presented in the following tables before the experiments. The simulation paths are 100,000, the number of simulation points in each path is 50. Through the experiment, we find that separate parameters influence the variance reduction efficiency of control variates in different ways.

8.1 European Put Options

The experiment for European put option is generated by using New Method in the Hull and White process and the Heston process. Each table concludes the output of option value, standard error and computational time of both "MC" and "CV" method. Varying by different parameter values, the numerical results reveal some discipline of Liu et al.'s methods.

8.1.1 Hull-White Model

option parameters		process parameters						
K	T	r	S_0	y_0	μ	ξ	ρ	m
40	1	0.05	[34, 50]	0.02	0.02	[0.01, 0.25]	[-1, 1]	[-50, 50]

Table 1: Parameter Settings of Hull-White Model

S_0		34	36	40	44	46	50
v_0		4.7025	3.2994	1.3707	0.4635	0.2507	0.0646
MC	v_1	4.69	3.27	1.387	0.461	0.252	0.065
	se_1	(0.0121)	(0.0108)	(0.0076)	(0.0043)	(0.0031)	(0.0015)
	t_1	[1.469]	[1.469]	[1.469]	[1.485]	[1.461]	[1.470]
CV	v_2	4.6373	3.2763	1.3991	0.4635	0.2577	0.0669
	se_2	(0.0004)	(0.0004)	(0.0004)	(0.0003)	(0.0003)	(0.0002)
	t_2	[2.361]	[2.345]	[2.360]	[2.376]	[2.350]	[2.329]
$\hat{R}$		581.7	488.4	252.8	103.4	73.01	34.86

Table 2: Variance Reduction by New Method with different S_0
(European Put Option, Hull-White Model, Setting $\rho = 0$, $m = 1$, $\sigma = 0.1$)

The results in Table 2 show that, in the Hull-White model, New Method presents great variance reduction efficiency in pricing European put options while varying S_0.[3]. For European put options, greater original prices (S_0) lead to smaller variance reduction ratios. This does not means the decline of variance reduction efficiency of New Method when S_0 increases. It is just because of the property of put options. Considering the payoff function of European put options, the Monte Carlo Method produces more worthless options on the expiry date when the initial underlying prices increase. This property declines standard errors of the plain Monte Carlo Method when S_0 values become larger by including more and more zero in the calculation formula.

[3]Liu et al. (2015) use wrong value of parameters in the seed paper and they do not provide the value of ξ and T in the parameter settings. These lead to the error in their table II-IV.

ρ		-1	-0.6	-0.1	0	0.1	0.6	1
v_0		1.3707	1.3707	1.3707	1.3707	1.3707	1.3707	1.3707
MC	v_1	1.382	1.372	1.363	1.374	1.361	1.365	1.355
	se_1	(0.0077)	(0.0076)	(0.0075)	(0.0075)	(0.0074)	(0.0074)	(0.0072)
	t_1	[1.453]	[4.087]	[2.437]	[1.437]	[1.485]	[1.482]	[1.486]
CV	v_2	1.3796	1.3788	1.3679	1.3896	1.3695	1.3755	1.3456
	se_2	(0.0002)	(0.0003)	(0.0004)	(0.0004)	(0.0004)	(0.0003)	(0.0002)
	t_2	[2.398]	[2.605]	[2.437]	[2.375]	[2.376]	[2.422]	[2.359]
$\hat{R}$		801.6	332.4	244.1	237.4	244.9	319.4	815.2

Table 3: Variance Reduction by New Method with Different ρ
(European Put Option, Hull-White Model, Setting $S_0 = 40$, $\sigma = 0.1$, $m = 1$)

ξ		0.01	0.05	0.1	0.15	0.2	0.25
v_0		1.3706	1.3713	1.373	1.3765	1.3811	1.3870
MC	v_1	1.376	1.361	1.380	1.379	1.370	1.354
	se_1	(0.0075)	(0.0075)	(0.0075)	(0.0075)	(0.0076)	(0.0075)
	t_1	[1.453]	[1.468]	[1.469]	[1.435]	[1.500]	[1.485]
CV	v_2	1.3876	1.3696	1.3859	1.3853	1.3695	1.3584
	se_2	(4×10^{-5})	(0.0002)	(0.0038)	(0.0006)	(0.0008)	(0.0009)
	t_2	[2.344]	[2.39]	[2.389]	[2.391]	[2.375]	[2.375]
$\hat{R}$		25388	961.8	249.1	108.1	61.65	38.77

Table 4: Variance Reduction by New Method with Different ξ
(European Put Option, Hull-White Model, Setting $S_0 = 40$, $m = 2$, $\rho = 0$)

m		-50	-10	-1	0	1	10	50
v_0		1.3453	1.3424	1.3655	1.3681	1.3707	1.3943	1.3554
MC	v_1	1.367	1.371	1.361	1.372	1.371	1.373	1.376
	se_1	(0.0075)	(0.0075)	(0.0075)	(0.0075)	(0.0075)	(0.0075)	(0.0075)
	t_1	[1.500]	[1.454]	[1.469]	[1.469]	[1.469]	[1.453]	[1.453]
CV	v_2	1.3734	1.3798	1.3701	1.3799	1.3695	1.3779	1.3833
	se_2	(0.0005)	(0.0004)	(0.0004)	(0.0004)	(0.0004)	(0.0004)	(0.0005)
	t_2	[2.437]	[2.389]	[2.404]	[2.403]	[2.373]	[2.391]	[2.375]
$\hat{R}$		120.1	223.8	242.1	256.2	254.4	241.4	123.8

Table 5: Variance Reduction by New Method with Different m
(European Put Option, Hull-White Model, Setting $S_0 = 40$, $\sigma = 0.1$, $\rho = 0$)

In table 3, when the absolute values of ρ grow, the efficiency ratios raise. The high correlated coefficient (ρ) intensifies the spreading of stock price correlated with the stochastic volatility, which makes the Monte Carlo method harder to be predicted in a deterministic way. According to result from Table 4, the variance reduction ratio ($\hat{R}$) rapidly declines when ξ increase. Variance reduction effect has a significant difference when $\sigma = 0.01$ or 0.25, but the "CV" option values remain precise. This discipline appears similarly in Table 5. When the absolute values of m become very large, the efficiency ratios have obvious declines. According to the analytic solution for European put options in the New Method (Equation 25), the orders of moment (m) not only influence the efficiency ratios, but also produce biased results in option pricing by deterministic volatility when m is equal to -50 or 50. This prompts us that New Method in Liu et al. (2015) 's paper may have restriction in parameter settings. In extreme circumstance, New Method may lose its efficiency on variance reduction.

8.1.2 Heston model

option parameters		process parameters						
K	T	r	S_0	y_0	k	θ	ξ	ρ
100	100	0	[90, 100]	0.01	[1,14]	0.01	[0.01, 0.25]	[-1, 1]

Table 6: Parameter Settings of Heston Model

S_0		80	90	100	110	120
v_0		20.001	10.201	2.82	0.305	0.0121
MC	v_1	20.00	10.22	2.80	0.318	0.0167
	se_1	(0.0178)	(0.0187)	(0.0126)	(0.0043)	(0.0009)
	t_1	[3.611]	[3.744]	[3.64]	[3.604]	[3.591]
CV	v_2	20.019	10.263	2.889	0.329	0.0179
	se_2	(0.0033)	(0.0033)	(0.0028)	(0.0019)	(0.0006)
	t_2	[4.709]	[4.643]	[4.556]	[4.512]	[4.507]
$\hat{R}$		23.41	25.98	16.29	4.293	1.532

Table 7: Variance Reduction by New Method with Different S_0
(European Put Option, Heston Model, Setting $\rho = 0$, $\sigma = 0.1$, $T = 0.5$, $k = 2$)

T		0.25	0.5	0.75	1	1.5
v_0		1.995	2.820	3.454	3.988	4.843
MC	v_1	1.970	2.80	3.40	3.93	4.81
	se_1	(0.0090)	(0.0125)	(0.0153)	(0.0175)	(0.0211)
	t_1	[4.325]	[4.282]	[4.218]	[4.216]	[4.243]
CV	v_2	1.985	2.830	3.453	3.962	4.822
	se_2	(0.0016)	(0.0028)	(0.0037)	(0.0044)	(0.0055)
	t_2	[5.211]	[5.333]	[5.361]	[5.172]	[5.203]
$\hat{R}$		24.04	16.21	13.97	12.69	11.49

Table 8: Variance Reduction by New Method with Different T
(European Put Option, Heston Model, Setting $S_0 = 100$, $\rho = 0$, $\sigma = 0.1$, $k = 2$)

k		1	2	4	8	14
v_0		2.820	2.820	2.820	2.820	2.820
MC	v_1	2.84	2.83	2.80	2.81	2.82
	se_1	(0.0126)	(0.0126)	(0.0126)	(0.0126)	(0.0125)
	t_1	[4.232]	[4.360]	[4.200]	[4.362]	[4.297]
CV	v_2	2.853	2.826	2.817	2.819	2.833
	se_2	(0.0032)	(0.0028)	(0.0023)	(0.0018)	(0.0013)
	t_2	[5.180]	[5.325]	[5.144]	[5.290]	[5.198]
$\hat{R}$		12.04	16.28	24.00	42.03	67.08

Table 9: Variance Reduction by New Method with Different k
(European Put Option, Heston Model, Setting $S_0 = 100$, $\rho = 0$, $\sigma = 0.1$, $T = 0.5$)

The results in Table 7-9 also illustrate that, in the Heston model, New Method presents good variance reduction efficiency, but it is not good as the performance in the Hull-White model. For European put options, the efficiency ratios decline when the value of initial underlying prices (S_0) or expiry time (T) rises. This is similar with the pattern in Hull-White model. The efficiency ratios increase when mean reverting rate (k) goes up. The performance of New Method is quite well within the time periods of one and half years. When the original underlying price equals to 120, The efficiency ratio ($\hat{R}$) is approached to 1. The standard error of "MC" and "CV" is closed to each other. New Method almost loses its effect on variance reduction.

ρ		-1	-0.6	-0.1	0	0.1	0.6	1
v_0		2.820	2.820	2.820	2.820	2.820	2.820	2.820
MC	v_1	2.73	2.78	2.77	2.78	2.81	2.81	2.81
	se_1	(0.0141)	(0.0136)	(0.0127)	(0.0126)	(0.0124)	(0.0116)	(0.0107)
	t_1	[4.186]	[4.258]	[4.118]	[4.314]	[4.345]	[4.309]	[4.272]
CV	v_2	2.763	2.795	2.776	2.756	2.819	2.822	2.831
	se_2	(0.0020)	(0.0025)	(0.0028)	(0.0028)	(0.0028)	(0.0025)	(0.0020)
	t_2	[5.152]	[5.371]	[5.158]	[5.139]	[5.425]	[5.299]	[5.207]
$\hat{R}$		37.71	22.80	16.96	16.45	15.40	16.23	23.16

Table 10: Variance Reduction by New Method with Different ρ
(European Put Option, Heston Model, Setting $S_0 = 100$, $\sigma = 0.1$. $T = 0.5$, $k = 2$)

ξ		0.01	0.05	0.1	0.15	0.2	0.25
v_0		2.8204	2.820	2.820	2.820	2.820	2.820
MC	v_1	2.81	2.81	2.80	2.76	2.72	2.67
	se_1	(0.0126)	(0.0125)	(0.0126)	(0.0126)	(0.0126)	(0.0128)
	t_1	[4.147]	[4.333]	[4.161]	[4.276]	[4.354]	[4.292]
CV	v_2	2.8156	2.822	2.819	2.789	2.729	2.668
	se_2	(0.0003)	(0.0014)	(0.0028)	(0.0041)	(0.0054)	(0.0065)
	t_2	[5.421]	[5.312]	[5.184]	[5.215]	[5.255]	[5.386]
$\hat{R}$		1542	63.80	16.34	7.479	4.368	3.090

Table 11: Variance Reduction by New Method with Different ξ
(European Put Option, Heston Model, Setting $S_0 = 100$, $\rho = 0$, $T = 0.5$, $k = 2$)

Table 10-11 give us interesting results about New Method. Even though variance reduction ratio still increases when the absolute value of ρ approaching to 1, the value of v_0 is no longer closely approached to the value of v_1. Similarly, under the condition of $\rho = 0$, the efficiency ratios have a rapid shrinking while the fluctuation of volatility is going bigger. When ξ is 0.25, $\hat{R}$ becomes 3.090 and there is a distinct difference between v_0 and v_1, the analytic solution is no longer valid to replace the outcome of the plain Monte Carlo simulation and New Method has a low effect on variance reduction.

Based on the observed results from Table 3, 4, 5, 10 and 11, the variance reduction ratio ($\hat{R}$) shows highly sensitive to the parameter settings of ρ, ξ and m for European put option in the Hull-White and ρ and ξ for the Heston model. Following the pattern in the seed paper, in the further sections of this paper, the efficiency ratio of European call is tested by the Stein-Stein model and the efficiency ratio of Asian call option is tested by the Hull-White model. By varying ξ in a range of [0.01, 0.50], a sensitivity test is held to detect the feasible parameter interval for each model.

8.2 European Call Option

The Stein-Stein model is used to price European call option in this paper. In the seed paper, Liu et al. does not mention anything about European call option, but actually their results in Table X-XIV are produced by pricing European call options in the Stein-Stein model.[4]

8.2.1 Stein-Stein model

option parameters		process parameters						
K	T	r	S_0	y_0	α	β	ξ	ρ
[90, 100]	[1/12,1]	0.095	100	0.2	[4, 100]	[0.1, 0.35]	[0.01, 0.25]	[-1, 1]

Table 12: Parameter Settings of Stein-Stein Model

K		90	95	100	105	110
v_0		15.107	11.332	8.133	5.576	3.653
MC	v_1	15.12	11.31	8.19	5.60	3.70
	se_1	(0.0410)	(0.0376)	(0.0336)	(0.0286)	(0.0238)
	t_1	[3.029]	[3.110]	[3.118]	[3.061]	[3.088]
CV	v_2	15.133	11.319	8.185	5.622	3.718
	se_2	(0.0062)	(0.0060)	(0.0056)	(0.0054)	(0.0051)
	t_2	[4.139]	[4.038]	[4.082]	[4.061]	[4.077]
$\hat{R}$		32.39	30.80	27.16	20.99	16.60

Table 13: Variance Reduction by New Method with Different k
(European Call Option, Stein-Stein Model,
Setting α=4, β=0.2, $S_0 = 100$, $\rho = 0$, $\sigma = 0.1$, $T = 0.5$)

[4]Liu et al. carelessly set $y_0 = 0.1$, which makes us impossible to reproduce their results. The correct setting of y_0 is 0.2. They also forgot to present the outcomes by varying β, even though there is a parameter interval of β in the seed paper.

T		1/12	0.25	0.5	0.75	1
v_0		2.710	5.239	8.153	10.657	12.975
MC	v_1	2.70	5.28	8.18	10.71	13.05
	se_1	(0.0119)	(0.0223)	(0.0336)	(0.0429)	(0.0513)
	t_1	[3.163]	[3.061]	[3.132]	[2.989]	[3.177]
CV	v_2	2.713	5.275	8.169	10.719	13.069
	se_2	(0.0012)	(0.0033)	(0.0057)	(0.0076)	(0.0093)
	t_2	[4.248]	[4.045]	[4.149]	[4.058]	[4.123]
$\hat{R}$		68.83	30.80	26.63	23.45	23.63

Table 14: Variance Reduction by New Method with Different T
(European Call Option, Stein-Stein Model,
Setting α=4, β=0.2, $S_0 = 100$, $K = 100$, $\rho = 0$, $\sigma = 0.1$)

α		4	8	14	20	100
v_0		8.133	8.133	8.133	8.133	8.133
MC	v_1	8.14	8.18	8.14	8.17	8.12
	se_1	(0.0335)	(0.0334)	(0.0333)	(0.0333)	(0.0332)
	t_1	[3.186]	[3.061]	[3.078]	[3.124]	[2.732]
CV	v_2	8.149	8.175	8.156	8.179	8.129
	se_2	(0.0056)	(0.0043)	(0.0034)	(0.0029)	(0.0013)
	t_2	[4.191]	[4.082]	[3.992]	[4.073]	[3.658]
$\hat{R}$		26.84	44.69	73.88	102.2	471.6

Table 15: Variance Reduction by New Method with Different α
(European Call Option, Stein-Stein Model,
Setting T=0.5, β=0.2, $S_0 = 100$, $K = 100$, $\rho = 0$, $\sigma = 0.1$)

β		0.1	0.15	0.2	0.25	0.3	0.35
v_0		6.740	7.413	8.133	8.884	9.655	10.441
MC	v_1	6.80	7.47	8.17	8.95	9.60	10.42
	se_1	(0.0253)	(0.0293)	(0.0335)	(0.0379)	(0.0424)	(0.0475)
	t_1	[3.022]	[3.126]	[3.107]	[3.043]	[3.077]	[3.002]
CV	v_2	6.816	7.485	8.164	8.945	9.612	10.429
	se_2	(0.0057)	(0.0057)	(0.0057)	(0.0058)	(0.0058)	(0.0059)
	t_2	[4.009]	[4.231]	[3.976]	[3.997]	[4.006]	[3.994]
$\hat{R}$		15.05	19.90	26.99	33.15	41.67	48.72

Table 16: Variance Reduction by New Method with Different β
(European Call Option, Stein-Stein Model,
Setting α=4, T=0.5, $S_0 = 100$, $K = 100$, $\rho = 0$, $\sigma = 0.1$)

Table 13-16 also present a good variance reduction of New Method with Stein-Stein model. The variance reduction ratio decreases when exercise price(K) or expiry time (T) increases. The increasing of mean reverting rate (α) or the mean value (β) leads to the rise of efficiency ratio. In Table 13, the original price (S_0) is set to 100. According to the payoff function of European Call option, the simulation produces more zero payoff value when K is higher. So the changing of $\hat{R}$ in Table 13 is only because the mathematical property of call option. Particularly, the mean reverting rate has a great influence on efficiency ratio. It increases quickly when α value varying as shown in Table 15.

ρ		-1	-0.6	-0.1	0	0.1	0.6	1
v_0		8.133	8.133	8.133	8.133	8.133	8.133	8.133
MC	v_1	8.33	8.30	8.17	8.15	8.16	8.10	8.03
	se_1	(0.0299)	(0.0314)	(0.0332)	(0.0335)	(0.0335)	(0.0357)	(0.0370)
	t_1	[2.997]	[3.054]	[3.030]	[3.060]	[3.091]	[3.100]	[3.007]
CV	v_2	8.323	8.311	8.175	8.143	8.171	8.120	8.035
	se_2	(0.0048)	(0.0053)	(0.0056)	(0.0056)	(0.0057)	(0.0054)	(0.0050)
	t_2	[4.043]	[4.238]	[4.284]	[4.247]	[4.237]	[4.28]	[4.383]
$\hat{R}$		28.723	25.54	25.81	25.72	26.03	31.82	38.01

Table 17: Variance Reduction by New Method with Different ρ
(European Call Option, Stein-Stein Model,
Setting α=4, β=0.2, $S_0 = 100$, $K = 100$, $\sigma = 0.1$, $T = 0.5$)

ξ		0.01	0.05	0.1	0.15	0.2	0.25
v_0		8.1331	8.133	8.133	8.133	8.133	8.133
MC	v_1	8.18	8.13	8.13	8.19	8.29	8.37
	se_1	(0.0335)	(0.0331)	(0.0336)	(0.0338)	(0.0349)	(0.0356)
	t_1	[3.029]	[3.022]	[2.927]	[2.946]	[3.12]	[3.057]
CV	v_2	8.1756	8.138	8.125	8.198	8.285	8.367
	se_2	(0.0006)	(0.0028)	(0.0057)	(0.0086)	(0.0117)	(0.0146)
	t_2	[4.092]	[4.213]	[4.213]	[3.859]	[4.213]	[4.226]
$\hat{R}$		2526	105.8	26.48	11.84	7.047	4.602

Table 18: Variance Reduction by New Method with Different ξ
(European Call Option, Stein-Stein Model,
setting α=4, β=0.2, $S_0 = 100$, $K = 100$, $\rho = 0$, $T = 0.5$)

The results from Table 17-18, in the Stein-Stein model, the relative coefficient (ρ) and fluctuation of volatility (ξ) have higher influence on of variance reduction ratio ($\hat{R}$) than in the Hull-White model and the Heston model. For European call option, the $\hat{R}$ increases when $|\rho|$ grows and the $\hat{R}$ decreases when ξ increases. According to the calculation formula of volatility process by risk factor ($\sigma_t = |Y_t|$), the change of ρ and ξ leads to more severe diffusion effect than in the other two models. In table 17, when the absolute value of ρ equals to 1, the analytic formula presents us abnormal value of options, which is not closed to the plain Monte Carlo value. In table 18, high volatility also leads to inaccurate option pricing of v_0. When ξ is equal to 0.25 in Table 18, the $\hat{R}$ becomes 4.602, which means that the variance reduction effect is limited in a relatively high volatility circumstances and analytic option value becomes meaningless.

8.3 Asian Call Option

In the following experiment of Asian call option pricing, this paper compares the numerical results of both Geometric Average Asian call option (V_{GAO}) and Arithmetic Average Asian call option (V_{AAO}) priced by the Hull-White model. First we separately compute the geometric average solution and the arithmetic average solution by varying m values (setting $k = 100$), similarly with Table XV in the seed paper. Then we present the efficiency ratio outcomes to compare their performance to the control variates method by varying both K and m values. The analytic solution of Asian call option is based on continuous sampling geometric average. In the seed paper, the number of points on the underlying path is 50, it is easy to find that the results of plain Monte Carlo method frequently shaking because of small sampling. So this paper expand the number of points on the underlying path (N) to 100. This makes both the plain Monte Carlo method and the control variates method produced more precise outcome.

option parameters		process parameters						
K	T	r	S_0	y_0	μ	ξ	ρ	m
[90, 110]	1	0.05	100	0.15^2	0.05	0.01	0	[-25, 50], $1-\frac{2\mu}{\sigma^2}$

Table 19: Parameter Settings of Asian Call Option Priced by Hull-White Model ($m = 1 - \frac{2\mu}{\sigma^2} = -999$)

m		-25	-1	0	1	2	50	-999
v_0		4.57156	4.55152	4.57154	4.57158	4.57154	4.57149	4.5713
MC	v_1	4.60	4.58	4.61	4.60	4.57	4.58	4.61
	se_1	(0.0189)	(0.0188)	(0.0189)	(0.0189)	(0.0188)	(0.0188)	(0.0189)
	t_1	[3.629]	[3.615]	[3.592]	[3.607]	[3.658]	[3.634]	[3.640]
CV	v_2	4.61536	4.58334	4.61554	4.61255	4.58335	4.57559	4.6196
	se_2	(5×10^{-5})	(5×10^{-5})	(5×10^{-5})	(5×10^{-5})	(5×10^{-5})	(5×10^{-5})	(0.0001)
	t_2	[6.064]	[6.043]	[5.923]	[6.026]	[6.225]	[6.030]	[6.050]
$\hat{R}$		79872	81136	77211	79692	81018	80708	14413

Table 20: Variance Reduction by New Method with Different m
(Asian Call Option, Geometric Average V_{GAO}, Setting $K = 100$)

m		-25	-1	0	1	2	50	-999
v_0		4.7052	4.7018	4.7047	4.7703	4.7424	4.7527	4.7053
MC	v_1	4.74	4.74	4.72	4.75	4.72	4.74	4.73
	se_1	(0.0192)	(0.0192)	(0.0193)	(0.0193)	(0.0193)	(0.0193)	(0.0193)
	t_1	[3.246]	[3.238]	[3.188]	[3.155]	[3.266]	[3.280]	[3.179]
CV	v_2	4.7566	4.7478	4.7198	4.7664	4.7283	4.7334	4.7345
	se_2	(0.0004)	(0.0004)	(0.0004)	(0.0004)	(0.0004)	(0.0004)	(0.0005)
	t_2	[5.713]	[5.633]	[5.634]	[5.519]	[5.798]	[5.675]	[5.560]
$\hat{R}$		1179	1230	1237	1236	1255	1246	864.9

Table 21: Variance Reduction by New Method with Different m
(Asian Call Option, Arithmetic Average V_{AAO}, Setting $K = 100$)

The results from Table 20 and Table 21 demonstrate that when the orders of moment m varying from -25 to 50, the option value of both Geometric Average Asian call options and Arithmetic Average Asian call options calculated by the plain Monte Carlo method is similar with the value from the control variates method. The efficiency ratio ($\hat{R}$) changes slightly. When the orders of moment (m) become -999, although the option value from the plain Monte Carlo method and the control variates method are similar, as for Asian options, different order moments (m) do not influence the pricing process, the efficiency ratios ($\hat{R}$) decrease a lot for both geometric average Asian call options and arithmetic average Asian call options. Based on the equation (29) and equation (30), the parameter a_m is equal to 0 when $m = -999$, whose calculation formula is different from other orders of moment. Comparing the efficiency ratio $\hat{R}$, it is obvious to find that the correction effect for Geometric Average Asian call options is better than that for Arithmetic Average Asian call options.

K \ m	-25	-1	0	1	2	50	-999
	V_{GAO}						
90	110637	102637	100672	100689	100798	101658	15725
100	79872	81136	77211	79692	81018	80708	14413
110	29375	29926	28493	30066	29826	29858	7053
	V_{AAO}						
90	1538	1559	1568	1519	1584	1565	1230
100	1179	1230	1237	1236	1255	1246	864.9
110	391.8	406.3	403.4	407.1	395.6	418.5	293.8

Table 22: The Efficiency Ratio by Using New Method for V_{GAO} and V_{AAO}

As for the Table 22, for the same strike price (K), when the orders of moment (m) is between -25 to 50, the efficiency ratio ($\hat{R}$) just changes a bit, but when the order moment becomes -999, the efficiency ratio declines a lot. For the same orders of moment, when the strike price increases, the efficiency ratio for Geometric Average Asian call options and Arithmetic Average Asian call options declines. This phenomenon is caused by the increase of strike price, which leads to more worthless values of the call options.

8.4 Sensitivity Test

In the previous subsection of numerical experiment, the outcome of New Method from the seed paper, Liu et al. (2015), has high sensitive on the setting of parameters correlated with the correlated coefficient (ρ) and the volatility of volatility process (ξ). They have significant influence on option pricing by the analytic formula and variance reduction ratio. In several conditions, New Method produces extremely low $\hat{R}$, which represent the invalid of variance reduction. The plain Monte Carlo results are highly distinct from the analytic ones. These outcomes are also presented in the seed paper on Table II, VI, VII, XI, XII, but they did not discuss their limitation of modelling.

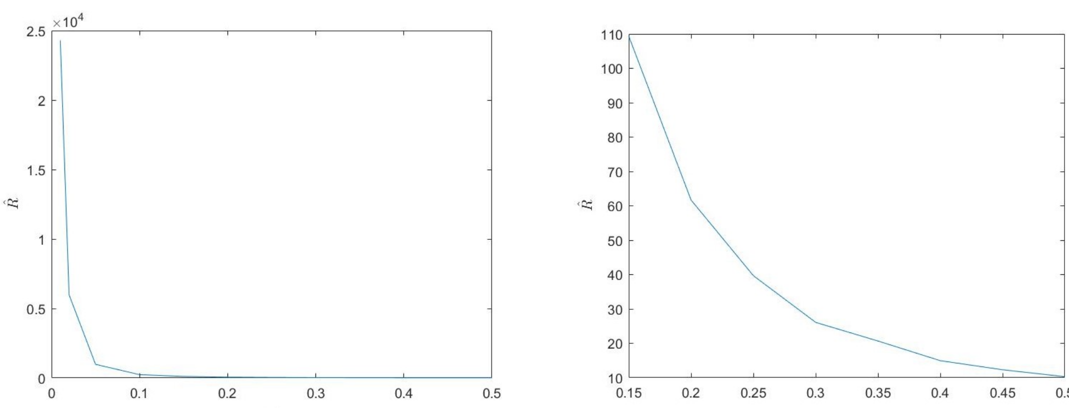
In order to experimenting the performance of Liu et al. (2015).'s method in extreme parameter settings condition, a series of sensitivity test is held to reproducing the ξ tables in separated models by fixing ρ value to 1 and varying ξ value in [0.01.0.5]. The European put options are priced by the Hull-White model and the Heston model. The European call options are priced by the Stein-Stein model. The Asian Call options are priced by the Hull-White model.

8.4.1 European Put Options

ξ		0.01	0.02	0.05	0.1	0.15	0.2
v_0		1.37068	1.37076	1.3713	1.3733	1.3765	1.3811
MC	v_1	1.367	1.358	1.373	1.368	1.378	1.363
	se_1	(0.0075)	(0.0075)	(0.0075)	(0.0075)	(0.0075)	(0.0075)
	t_1	[1.453]	[1.453]	[1.453]	[1.453]	[1.453]	[1.469]
CV	v_2	1.36995	1.35568	1.3748	1.3795	1.3644	1.3659
	se_2	(3.70×10^{-5})	(7.50×10^{-5})	(0.00019)	(0.00038)	(0.00057)	(0.00076)
	t_2	[2.344]	[2.372]	[2.333]	[2.367]	[2.339]	[2.332]
$\hat{R}$		24294	5978	979.2	238.1	109.4	61.7

ξ		0.25	0.3	0.35	0.4	0.45	0.5
v_0		1.3870	1.394	$\boxed{1.403}$	$\boxed{1.413}$	$\boxed{1.425}$	$\boxed{1.438}$
MC	v_1	1.387	1.365	$\boxed{1.348}$	$\boxed{1.348}$	$\boxed{1.353}$	$\boxed{1.352}$
	se_1	(0.0075)	(0.0075)	(0.0075)	(0.0075)	(0.0075)	(0.0075)
	t_1	[1.438]	[1.453]	[1.502]	[1.466]	[1.469]	[1.500]
CV	v_2	1.3956	1.375	1.363	1.356	1.358	1.366
	se_2	(0.00093)	(0.0011)	(0.0013)	(0.00149)	(0.00168)	(0.00185)
	t_2	[2.343]	[2.484]	[2.360]	[2.469]	[2.359]	[2.404]
$\hat{R}$		39.68	26.09	20.67	14.95	12.31	10.29

Table 23: Variance Reduction by New Method with Different ξ
(European Put Options, Hull-White model, Setting $S_0 = 40$, $m = 2$, $\rho = 1$)



$\sigma \in [0.01, 0.5]$ $\sigma \in [0.15, 0.5]$

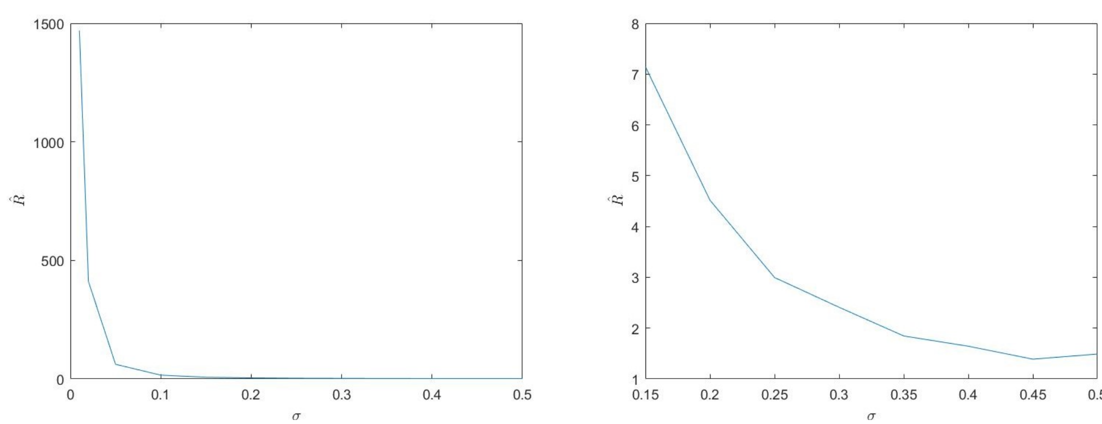
Figure 1: Variance Reduction by New Method with Different ξ
(European Put Options, Hull-White model, setting $S_0 = 40$, $m = 2$, $\rho = 1$)

For the Hull-White model, the abnormal outputs of v_0 are shown in the text boxes. In Table 23, starting from ξ greater equal to 0.35, v_0 and v_1 are no longer closed to each other since the one after the decimal point. The options pricing by plain Monte Carlo method stay stable when varying ξ values. Figure 1 reveals that the efficiency ratio ($\hat{R}$) is diving when ξ is varied from 0.01 to 0.1, then it presents a relatively andante declining rate in $\xi \in [0.1, 0.5]$.

ξ		0.01	0.02	0.05	0.1	0.15	0.2
v_0		2.8204	2.8204	2.820	2.820	2.820	2.820
MC	v_1	2.83	2.80	2.83	2.79	2.78	2.73
	se_1	(0.0126)	(0.0125)	(0.0126)	(0.0126)	(0.0126)	(0.0128)
	t_1	[3.618]	[3.685]	[3.744]	[3.884]	[3.920]	[4.065]
CV	v_2	2.8216	2.8089	2.853	2.785	2.746	2.739
	se_2	(0.0003)	(0.0006)	(0.0014)	(0.0028)	(0.0041)	(0.0054)
	t_2	[4.881]	[4.744]	[4.860]	[4.983]	[5.127]	[5.054]
$\hat{R}$		1469	410.9	61.71	15.97	7.147	4.518

ξ		0.25	0.3	0.35	0.4	0.45	0.5
v_0		$\boxed{2.820}$	$\boxed{2.820}$	$\boxed{2.820}$	$\boxed{2.820}$	$\boxed{2.820}$	$\boxed{2.82}$
MC	v_1	$\boxed{2.66}$	$\boxed{2.62}$	$\boxed{2.55}$	$\boxed{2.50}$	$\boxed{2.44}$	$\boxed{2.40}$
	se_1	(0.0126)	(0.0125)	(0.0129)	(0.0130)	(0.0131)	(0.0132)
	t_1	[3.945]	[4.037]	[3.908]	[3.937]	[3.828]	[3.939]
CV	v_2	2.648	2.646	2.559	2.516	2.427	2.48
	se_2	(0.0064)	(0.0074)	(0.0083)	(0.0089)	(0.0095)	(0.0100)
	t_2	[5.178]	[5.036]	[5.185]	[5.093]	[5.205]	[4.610]
$\hat{R}$		2.996	2.410	1.846	1.644	1.389	1.489

Table 24: Variance Reduction by New Method with Different ξ
(European Put Option, Heston Model, Setting $S_0 = 100$, $\rho = 1$, $T = 0.5$)



$\sigma \in [0.01, 0.5]$ $\sigma \in [0.15, 0.5]$

Figure 2: Variance Reduction by New Method with Different ξ
(European Put Option, Heston Model, setting $S_0 = 100$, $\rho = 1$, $T = 0.5$)

For the Heston model, the analytic solution of option priced by the control variates method and the variance reduction ratios are more sensitive through the changing of ξ values. The values of v_0 are inaccurate when ξ is higher than 0.25 in Table 24. In Figure 2, although the rate of decline is slower in than the Hull-White model, the control variates method appears to be useless when $\sigma = 0.5$. The value of $\hat{R}$ is 1.489, $se_1 = 0.0132$ and $se_2 = 0.0100$. The control variates method loses its effect in the extremely high volatility condition because the standard error of two methods are too closed.

8.4.2 European Call Options

ξ		0.01	0.02	0.05	0.1	0.15	0.2
v_0		8.1331	8.133	8.133	8.133	8.133	$\boxed{8.13}$
MC	v_1	8.12	8.14	8.17	8.17	8.19	$\boxed{8.20}$
	se_1	(0.0333)	(0.0331)	(0.0332)	(0.0336)	(0.0340)	(0.0346)
	t_1	[2.730]	[2.863]	[2.931]	[2.949]	[2.915]	[2.877]
CV	v_2	8.1285	8.136	8.159	8.195	8.185	8.25
	se_2	(0.0006)	(0.0011)	(0.0028)	(0.0056)	(0.0084)	(0.0111)
	t_2	[3.871]	[3.978]	[3.998]	[3.922]	[3.869]	[3.858]
$\hat{R}$		2468	615.1	100.6	27.22	12.35	7.187

ξ		0.25	0.3	0.35	0.4	0.45	0.5
v_0		$\boxed{8.13}$	$\boxed{8.13}$	$\boxed{8.13}$	$\boxed{8.13}$	$\boxed{8.13}$	$\boxed{8.13}$
MC	v_1	$\boxed{8.37}$	$\boxed{8.49}$	$\boxed{8.56}$	$\boxed{8.67}$	$\boxed{8.79}$	$\boxed{8.83}$
	se_1	(0.0356)	(0.0369)	(0.0379)	(0.0394)	(0.0406)	(0.0424)
	t_1	[2.860]	[2.899]	[2.946]	[2.955]	[2.956]	[2.944]
CV	v_2	8.38	8.46	8.56	8.65	8.74	8.86
	se_2	(0.0141)	(0.0170)	(0.0195)	(0.0224)	(0.0247)	(0.0278)
	t_2	[3.875]	[4.062]	[4.110]	[4.108]	[4.113]	[3.655]
$\hat{R}$		4.742	3.357	2.709	2.225	1.941	1.879

Table 25: Variance Reduction by New Method with Different ξ
(European Call Option, Stein-Stein Model, Setting $S_0 = 100$, $\rho = 1$, $T = 0.5$)

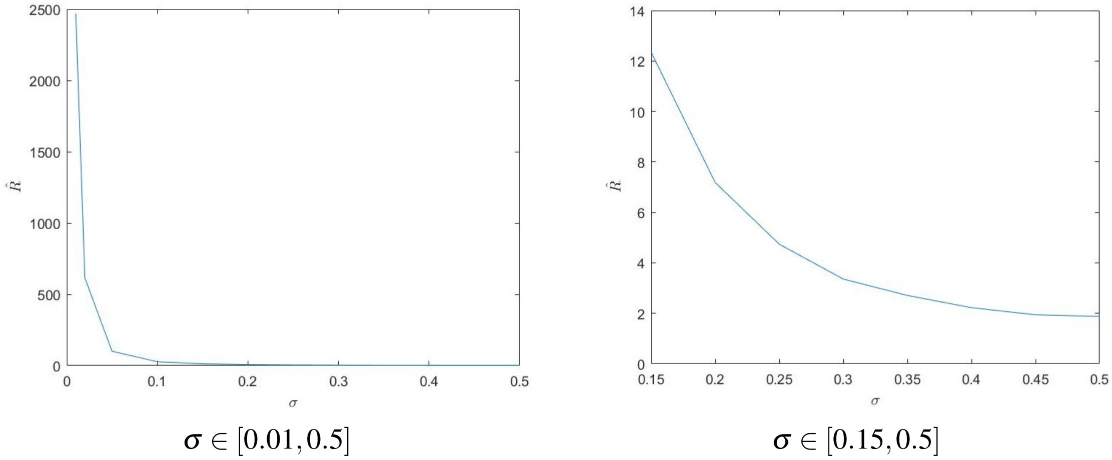


Figure 3: Variance Reduction by New Method with Different ξ
(European Call Option, Stein-Stein Model, Setting $S_0 = 100$, $\rho = 1$, $T = 0.5$)

For the Stein-Stein Model, Table 25 shows that, under the parameter settings of the seed paper, the Stein-Stein model presents worst analytic solution performance and variance reduction ratio when ξ value is relatively high. In Figure 3, the efficiency ratio goes down as the similar routine in the Hull-White model and the Heston model. The declining rate is slightly lower than in the other two models.

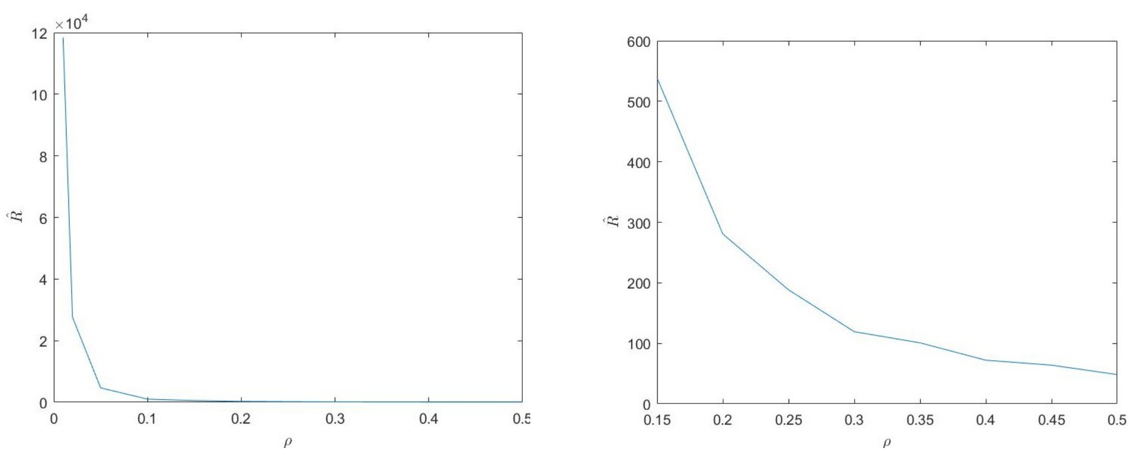
In the Heston and the Stein-Stein model, Liu, Zhao and Gu ignored the stochastic effect of volatility process and considered it as a pure mean reverting process when they derive the equation for the deterministic volatility. Their approximation is reliable when ξ is low, for example when ξ less equal to 0.15 in Table 25. Because of the ignorance to the fluctuation of volatility for the control variates process in the Heston model and the Stein-Stein model, the analytic solution of the control variates method (v_0) never changed when varying ξ. This is unreasonable in option pricing. The approximation also leads to a consequence that the performance of "CV" is worse than in the Hull-White model. According to the equation (9), the value of ξ is considered into the calculation of deterministic volatility unless m equals to 1. In conclude, the first and second order moment approximation for deriving deterministic volatility in the Heston model and the Stein-Stein model is only applicable in low ξ circumstance. The accuracy of analytic formula and the efficiency of variance reduction in the numerical experiments are significantly correlated with the parameter settings related to the fluctuation of volatility process.

8.4.3 Asian Call Options

ξ		0.01	0.02	0.05	0.1	0.15	0.2
v_0		4.57155	4.57161	4.5720	4.5733	4.5755	4.5785
MC	v_1	4.61	4.65	4.62	4.64	4.65	4.60
	se_1	(0.0190)	(0.0193)	(0.0192)	(0.0195)	(0.0196)	(0.0197)
	t_1	[1.641]	[1.625]	[1.625]	[1.641]	[1.688]	[1.625]
CV	v_2	4.62653	4.63568	4.6456	4.6489	4.6765	4.6245
	se_2	(4.30×10^{-5})	(9.01×10^{-5})	(0.0002)	(0.0004)	(0.0007)	(0.0009)
	t_2	[2.687]	[2.689]	[2.708]	[2.852]	[2.717]	[2.828]
$\hat{R}$		118392	27614	4666	1010	538.8	280.7

ξ		0.25	0.3	0.35	0.4	0.45	0.5
v_0		4.585	4.587	4.593	4.550	4.507	4.516
MC	v_1	4.58	4.53	4.58	4.52	4.55	4.52
	se_1	(0.0199)	(0.0202)	(0.0203)	(0.0205)	(0.0205)	(0.0206)
	t_1	[1.643]	[1.625]	[1.763]	[1.681]	[1.724]	[1.641]
CV	v_2	4.567	4.539	4.558	4.543	4.565	4.548
	se_2	(0.0011)	(0.0014)	(0.0016)	(0.0018)	(0.0020)	(0.0023)
	t_2	[2.688]	[2.939]	[2.837]	[2.868]	[2.761]	[2.704]
$\hat{R}$		188.4	119.2	100.8	72.12	63.89	48.32

Table 26: Variance Reduction by New Method with Different ξ
(Geometric Average Asian Call Options, Hull-White model,
Setting $S_0 = 100$, $m = 2$, $\rho = 1$)



$\sigma \in [0.01, 0.5]$ $\sigma \in [0.15, 0.5]$

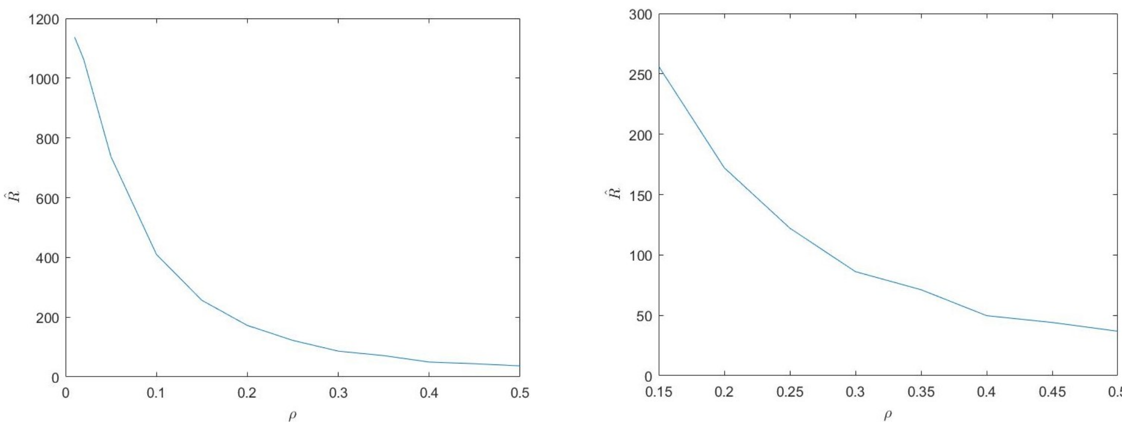
Figure 4: Variance Reduction by New Method with Different ξ
(Geometric Average Asian Call Options, Hull-White model,
Setting $S_0 = 100$, $m = 2$, $\rho = 1$)

Table 26 demonstrates that for the Geometric Average Asian options, the efficiency ratios decline when ξ grows. Figure 4 also shows the similar declining discipline with European options, but they still have some difference. Based on the Hull-White model, the extreme values of ξ no longer cause distinct discrepancy between the plain Monte Carlo options value and the values of the analytic formula. The obvious gap between v_0 and v_1 does not exist when ξ changes from 0.01 to 0.5. The decline rate in the interval [0.15, 0.5] is slower than the one in when pricing European put options.

ξ		0.01	0.02	0.05	0.1	0.15	0.2
v_0		4.5716	4.5716	4.5720	4.5733	4.5755	4.579
MC	v_1	4.59	4.51	4.60	4.58	4.54	4.57
	se_1	(0.0196)	(0.0198)	(0.0198)	(0.0200)	(0.0203)	(0.0204)
	t_1	[1.450]	[1.483]	[1.469]	[1.469]	[1.596]	[1.593]
CV	v_2	4.5865	4.5035	4.6433	4.589	4.5215	4.584
	se_2	(0.0004)	(0.0005)	(0.0006)	(0.0008)	(0.0009)	(0.0012)
	t_2	[2.547]	[2.547]	[2.563]	[2.531]	[2.630]	[2.751]
$\hat{R}$		1137	1063	737.1	410.1	256.3	172.2

ξ		0.25	0.3	0.35	0.4	0.45	0.5
v_0		4.582	4.587	4.593	4.600	4.607	4.616
MC	v_1	4.55	4.55	4.59	4.53	4.56	4.58
	se_1	(0.0205)	(0.2081)	(0.0209)	(0.0212)	(0.0214)	(0.0214)
	t_1	[1.532]	[1.479]	[1.522]	[1.487]	[1.578]	[1.516]
CV	v_2	4.546	4.533	4.599	4.553	4.526	4.575
	se_2	(0.0014)	(0.0017)	(0.0019)	(0.0022)	(0.0025)	(0.0027)
	t_2	[2.591]	[2.672]	[2.563]	[2.732]	[2.655]	[2.598]
$\hat{R}$		122.2	86.18	71.21	49.71	44.08	36.82

Table 27: Variance Reduction by New Method with Different ξ
(Arithmetic Average Asian Call Options, Hull-White model,
setting $S_0 = 100$, $m = 2$, $\rho = 1$)



$\sigma \in [0.01, 0.5]$ $\sigma \in [0.15, 0.5]$

Figure 5: Variance Reduction by New Method with Different ξ
(Geometric Average Asian Call Options, Hull-White model,
Setting $S_0 = 100$, $m = 2$, $\rho = 1$)

The results in Table 27 illustrates that for the Arithmetic Average Asian call options, the efficiency ratios decline when ξ grows. Figure 4 also demonstrates the similar decline pattern with European options, but they still have some difference. The efficiency ratio of Arithmetic Average Asian call options are lower than the corresponding Geometric Average Asian call options. Figure 5 shows that the decline rate of $\hat{R}$ is more smooth than the geometric average Asian call options. The variance reduction effect of the control variates method is less effective than in the Geometric Average Asian call options. The standard error of “CV” option values is higher than the Geometric Average Asian call options.

Previous experiment illustrates that New Method has a strict restriction on the parameter settings. Its performance becomes worse in a high volatility market. The numerical experiment finds that the situations of extremely low variance reduction rate are accompanied with the abnormal value of analytic solution. The analytic solution and auxiliary option values, presented in Equation 14, are correction terms in the control variates method. The significant discrepancy between the option values priced by the deterministic volatility (the analytic solution and auxiliary process) and the stochastic volatility (the plain Monte Carlo) seriously decrease their correlation. So the standard error of “CV” is mainly determined by the original “MC” values, because the value of β in Equation 14 is relatively low. So the se_1 is close to se_2, which concludes the inefficient of variance reduction. This means that in the real financial market, the underlying assets should be carefully selected when these methods are used. It is also unreliable in a volatile time period.

9 Conclusions

This paper reproduces New Method proposed by Liu, Zhao and Gu (2015). Liu et al. used a deterministic volatility to replace the stochastic volatility which has the same order moment risk factor. Their method performed a good variance reduction effect on the plain Monte Carlo method in the Hull-White model, the Heston model and the Stein-Stein model. However, the numerical experiment also reveals that their methods are only reliable under the low volatility circumstance. The sensitivity test based on European put options and Asian call options illustrates that New Method loses the efficiency in the high fluctuation volatility process. Method I and Method II also demonstrate similar discipline with New Method.

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A Program Input Parameters Setting

S0	The underlying asset price at time 0
X	The strike price
r	The risk-free interest rate
sigma	The volatility of volatility process
Rho	The Correlated Coefficient Between Underlying Process and Volatility Process
N	The samples points of a path
M	The number of paths
T	The expiry date of the option
Y0	The initial non-random factor
HWSigma	The sigma of Hull-White process
HWM	The orders moment of Hull-White process
Miu	The drift term of Hull-White volatility process
K	The rate at which v_t reverts to θ of Heston process
Theta	The long variance of Heston process
Alpha	The rate at which v_t reverts to β of Stein-Stein process
Beta	The long variance of Stein-Stein process
DeterministicVolatilityBase	"NMHW" is new method for the HuLL-White model, "NMHeston" is the new method for Heston model, "NMStein" is the new method for Stein-Stein model, "MIHW" is method I for the Hull-White model, "MIHeston" is the method I for Heston model, "MIStein" is the method I for Stein-Stein model, "MIIHW" is the method II for Hull-White model, "MIIHeston" is the method II for the Heston model
StochasticVolatilityBase	"HW" for Hull-White, Heston for Heston, Stein for Stein-Stein
OptionBase	"p" for Europut option,c for Eurocall option, "asian_ari_c" for Arithmetic Asian call option, "asian_ari_p" for Arithmetic Asian put option,

	“asian_geo_c” for Geometric Asian call option,
	“asian_geo_p” for Geometric Asian put option
MonteCarloBase	“plain” for Plain Monte Carlo,
	“cv” for Control variates Monte Carlop

B Code Structure Chart

C Code Listing