Assignment #1 Textbook Problems

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2. A free electron has wave function of the form:

$$\Psi(z,t) = Ae^{ikz}e^{i\omega t}$$

calculating ω:

$$\omega = \frac{E}{\hbar}$$

$$= \frac{3eV}{\hbar}$$

calculating k:

$$k = \frac{2\pi}{\lambda}$$

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{m_e 10^5} = 7.274\times10^{-9}$$

$$k = \frac{2\pi}{7.274\times10^{-9}} = 8.64\times10^{8}\ m^{-1}$$

putting it altogether:

$$\boxed{\Psi(z,t) = Ae^{i(8.64\times10^{8})z}e^{i\left(\frac{3eV}{\hbar}\right)t}}$$

To calculate the PE:

$$E = E_{PE} + E_{KE}$$
$$= V + E_{KE}$$

isolating and solving for V:

$$V = E - E_{KE}$$
$$= 3eV - \tfrac{1}{2} m_e v^2$$
$$= 3eV - \tfrac{1}{2} m_e (10^5)^2$$
$$= \boxed{2.97 \quad eV}$$

Verifying with Schrodinger time-dependent:

$$i\hbar \frac{\partial \Psi}{\partial t} = \left(\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V \right) \Psi$$
$$V\Psi = i\hbar \dot{\Psi} + \frac{\hbar^2}{2m} \Psi''$$

assuming Ψ is seperable: $\Psi = f(z)\, g(t)$

$$V f(z) g(t) = i\hbar f(z) g'(t) + \frac{\hbar^2}{2m} f''(z) g(t)$$

$$V = i\hbar \frac{g'(t)}{g(t)} + \frac{\hbar^2}{2m} \frac{f''(z)}{f(z)}$$

from earlier in the question, we know $f(z)$ and $g(t)$ so:

$$f(z) = e^{ikz} \qquad g(t) = e^{-i\omega t}$$
$$f'(z) = ike^{ikz} \qquad g'(t) = -i\omega e^{-i\omega t}$$
$$f''(z) = -k^2 e^{ikz}$$

$$V = i\hbar(-i\omega) + \frac{\hbar^2}{2m}(-k^2)$$
$$= 3eV + \frac{\hbar^2}{2m_e}\left(-(8.64\times10^8)^2\right)$$
$$= \boxed{2.97\ eV}$$

3. We begin by simply analyzing the equation in question:

$$V = \lambda f$$
$$= \frac{h}{p}\frac{\omega}{2\pi} = \frac{h}{p}\frac{E}{h} = \frac{E}{p} = \frac{\frac{1}{2}mv^2}{mv} = \frac{v}{2}$$

this equation gives phase velocity which differs from the velocity of the electron. Therefore the equation does not hold for electrons. A more accurate approach would involve using

$$E = \frac{1}{2}mv^2$$

4. We can approximate this problem to be an electron in an infinite well.

Energy for an electron in an infinite well for a given state n is:

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2 m_e L^2}$$

solving for n:

$$n = \sqrt{\frac{E_n \, 2 m_e L^2}{\hbar^2 \pi^2}}$$

$$= \sqrt{\frac{(5eV)\, 2 m_e (10)^2}{\hbar^2 \pi^2}}$$

$$= \boxed{3.64 \times 10^{10}}$$

approximate energy diff:

$$E_{211} - E_{111} = \frac{\hbar^2 \pi^2}{2 m_e (10)^2} \left[(2^2 + 1^2 + 1^2) - (1^2 + 1^2 + 1^2) \right]$$

$$= \frac{\hbar^2 \pi^2}{2 m_e (100)} (3)$$

$$= \boxed{1.8 \times 10^{-39} \text{ J}}$$

5. Exact same procedure as #4, only difference being $L = 10^{-9}$

$$n = \sqrt{\frac{(5eV)\, 2 m_e (10^{-9})^2}{\hbar^2 \pi^2}}$$

$$= \boxed{3.65}$$

$$E_{211} - E_{111} = \frac{\hbar^2 n^2}{2 m_e 10^{-18}} (3)$$
$$= \boxed{1.8 \times 10^{-19} \text{ J}}$$

6. Discretization occurs when the particle is confined to a space equal to or smaller then its wavelength. So we must calculate λ

$$E = \tfrac{1}{2} m v^2 \rightarrow v = \sqrt{\frac{2E}{m}}$$

$$\lambda = \frac{h}{mv} = \frac{h}{m}\sqrt{\frac{m}{2E}}$$

$$\lambda_e = \frac{h}{m_e}\sqrt{\frac{m_e}{2(2.5\text{eV})}} = \boxed{0.776 \text{ nm}}$$

$$\lambda_p = \frac{h}{m_p}\sqrt{\frac{m_p}{2(2.5\text{eV})}} = \boxed{0.018 \text{ nm}}$$

The electron must be confined to a space of λ_e or smaller, while the proton must be confined to λ_p or smaller to observe energy discretization

8. Probability distribution of a particle is given by:

$$P = \int_0^L \psi^* \psi \, dx$$

To find the particle in the range
$L/2 \leq x \leq L$:

$$P = \int_{L/2}^{L} \psi^* \psi \, dx$$

$$= \frac{2}{L} \int_{L/2}^{L} \sin\left(\frac{n\pi x}{L}\right) e^{+iE_n t/\hbar} \sin\left(\frac{n\pi x}{L}\right) e^{-iE_n t/\hbar} dx$$

$$= \frac{2}{L} \int_{L/2}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \int_{L/2}^{L} \sin^2(u) \, du \left(\frac{L}{n\pi}\right)$$

$$= \frac{2}{n\pi} \int_{u_1}^{u_2} \sin^2(u) \, du$$

$$= \frac{2}{n\pi} \left[\frac{u}{2} - \frac{\cos(u)\sin(u)}{2}\right]_{u_1}^{u_2}$$

$$= \frac{2}{n\pi} \left[\frac{n\pi x}{2L} - \frac{\sin\left(\frac{2n\pi x}{L}\right)}{2}\right]_{L/2}^{L}$$

$$= \frac{2}{n\pi} \left[\left(\frac{n\pi}{2}\right) - \left(\frac{n\pi}{4}\right)\right]$$

$$= \frac{2}{n\pi} \left(\frac{n\pi}{4}\right)$$

$$= \boxed{\frac{1}{2}}$$

9. We know the energy for such an electron can be written as:

$$E_n = \frac{\hbar^2 \pi^2 n^2}{2 m_e L^2}$$

$$E_2 = \frac{\hbar^2 \pi^2 4}{2 m_e (2.3 \times 10^{-9})^2}$$

$$= \boxed{0.284 \text{ eV}}$$

to calculate velocity:

$$v = \sqrt{\frac{2E}{m_e}}$$

$$= \sqrt{\frac{2(0.284 \text{eV})}{m_e}}$$

$$= \boxed{316060 \text{ m/s}}$$

11a. A normalized wave function ψ obeys the following equation:

$$\int_0^L |\psi|^2 \, dx = 1$$

plugging $\psi(x)$ into the above equation we can solve for A:

$$\int_0^L A^2 (x(L-x))^2 \, dx = 1$$

$$A^2 \int_0^L x^2 (L^2 - 2xL + x^2) \, dx = 1$$

$$A^2 \int^L L^2 x^2 - 2x^3 L + x^4 \, dx = 1$$

$$A^2\int_0^L x^2(L^2-2xL+x^2)\,dx = 1$$

$$A^2\int_0^L L^2x^2-2x^3L+x^4\,dx = 1$$

$$A^2\left[\frac{L^2x^3}{3}-\frac{x^4L}{2}+\frac{x^5}{5}\right]_0^L = 1$$

$$A^2\left[\frac{L^5}{3}-\frac{L^5}{2}+\frac{L^5}{5}\right] = 1$$

$$A^2\,\frac{L^5}{30} = 1$$

$$A = \boxed{\sqrt{30/L^5}}$$

11b. $$\sqrt{\frac{30}{L^5}}\,(x\,(L-x)) = \sum_{n=1}^{\infty} c_n\psi_n(x)$$

We know that the eigenstates of an infinite well are orthogonal to eachother such that:

$$\langle\psi_m|\psi_n\rangle = \delta_{mn}$$

We can use the above identity to determine c_n:

$$\langle\psi_m|\sum_{n=1}^{\infty}c_n\psi_n\rangle = \sum_{n=1}^{\infty}c_n\langle\psi_m|\psi_n\rangle$$

$$= c_m$$

From the question, we know we can write $\psi(x)$ in terms of c_n, ψ_n such that:

$$c_n = \int_0^L \psi_n \psi(x)\, dx$$

$$= \sqrt{\frac{2}{L}} \sqrt{\frac{30}{L^5}} \int_0^L \sin\left(\frac{n\pi x}{L}\right) (x(L-x))\, dx$$

$$= -\sqrt{\frac{2}{L}} \sqrt{\frac{30}{L^5}}\, L^3 \, \frac{2\cos(n\pi) - 2}{n^3 \pi^3}$$

$$= -\sqrt{60}\, \frac{L^3}{L^3}\, \frac{2(\cos(n\pi) - 1)}{n^3 \pi^3}$$

$$= \boxed{4\sqrt{15}\, \frac{1 - \cos(n\pi)}{n^3 \pi^3}}$$

11c. The probability E_n is just the coefficient c_n's magnitude:

$$P(E_n) = |c_n|^2$$

$$= \left| 4\sqrt{15} \left(\frac{1 - \cos(n\pi)}{n^3 \pi^3} \right) \right|^2$$

$$= 240 \left(\frac{(1 - \cos(n\pi))^2}{n^6 \pi^6} \right)$$

$$P(E_1) = \frac{240\,(+2)^2}{1^6 \pi^6} = \boxed{0.999}$$

$$P(E_2) = 240\,(0) = \boxed{0}$$

$$P(E_3) = \frac{240\,(2)^2}{3^6 \pi^6} = \boxed{0.00137}$$

$$P(E_4) = 240\,(0) = \boxed{0}$$

$$P(E_5) = \frac{240\,(2)^2}{5^6 \pi^6} = \boxed{6.39 \times 10^{-5}}$$

$$P(E_6) = 240\,(0) = \boxed{0}$$

It makes sense that only odd Energy eigenstates are possible since $\int |A(x(L-x))|^2 dx$ is an odd function.