

Chapter 7 First-order Circuits

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7.1 Introduction

- In this chapter, we examine two types of simple circuits: a circuit comprising a resistor and a capacitor and a circuit comprising a resistor and an inductor. These are called *RC* and *RL* circuits, respectively.

- RC and RL circuits are characterized by first-order differential equations. Hence, the circuits are collectively called *first-order* circuits.
- There are two ways to excite the circuits. The first way is by energy initially stored in the capacitive or inductive element. The second way is by independent sources.

Overview

EXCITATION METHODS	By energy initially stored in the capacitive or inductive element	By independent sources
<i>RC</i>	7.2 The Source-Free <i>RC</i> Circuit $v_c(0) = V_0$	7.5 An <i>RC</i> Circuit with Step Input
<i>RL</i>	7.3 The Source-Free <i>RL</i> Circuit $i_L(0) = I_0$	7.6 An <i>RL</i> Circuit with Step Input

7.2 The Source-Free RC Circuit

- A source-free RC circuit occurs when its dc source is suddenly disconnected. The energy already stored in the capacitor is released to the resistor.

Consider the circuit in Fig. 7.1. The capacitor is initially charged, we assume that at time $t = 0$, the initial voltage is

$$v(0) = V_0$$

with the corresponding value of energy stored as

$$w_C(0) = \frac{1}{2} C V_0^2$$

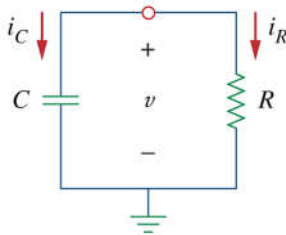


Figure 7.1 A source-free RC circuit.

Our objective is to determine the circuit response, say, the capacitor voltage v .

$$i_C + i_R = 0$$

$$i_C = C \frac{dv}{dt}, i_R = \frac{v}{R}$$

$$C \frac{dv}{dt} + \frac{v}{R} = 0$$

or

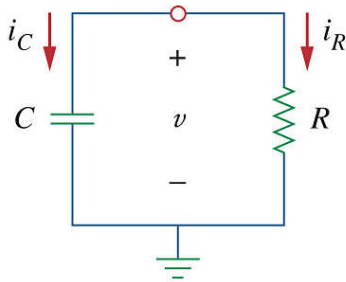


Figure 7.1 A source-free RC circuit.

$$\frac{dv}{dt} + \frac{v}{RC} = 0$$

This is a first-order differential equation.

$$r + \frac{1}{RC} = 0$$

$$r = -\frac{1}{RC}$$

$$v = Ae^{rt} = Ae^{-\frac{t}{RC}}$$

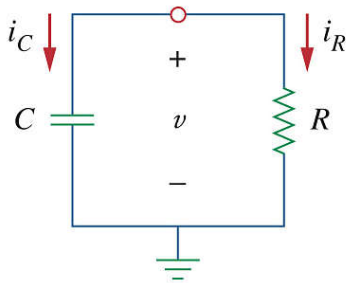


Figure 7.1 A source-free
 RC circuit.

$$v(0) = A = V_0$$

$$v = V_0 e^{-\frac{t}{RC}}$$

Since the response is due to V_0 (i.e., the *initial state* of the circuit) and not due to some external voltage or current source (i.e., the input to the circuit), it is called the zero - input response of the circuit.

= source free

The response of the source-free RC circuit is illustrated in Fig. 7.2. Note that at $t = 0$, $v = V_0$. As t increases, v decreases toward zero. The rapidity with which v decreases is expressed in terms of the *time constant*, denoted by τ .

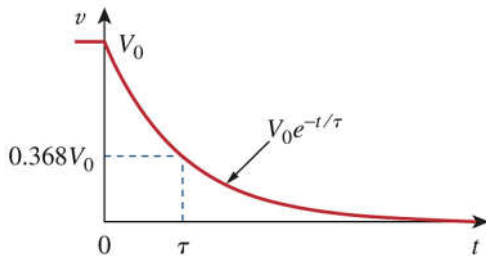


Figure 7.2 The voltage response of the source-free RC circuit.

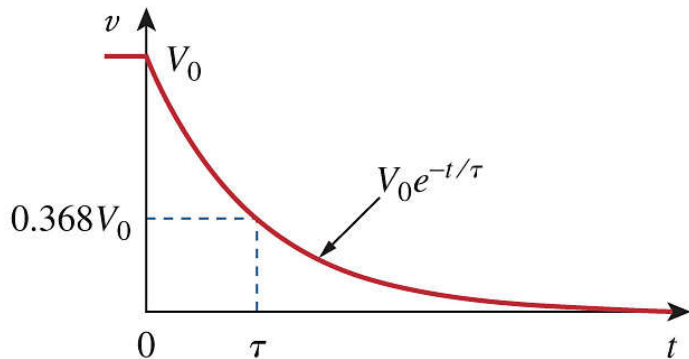


Figure 7.2 The voltage response of the source-free RC circuit.

The *time constant* of a circuit is the time required for the response to decay to a factor of $1/e$ or 36.8 % of its initial value.

This implies that at $t = \tau$,

$$v = V_0 e^{-\frac{t}{RC}} = V_0 e^{-1}$$

or

$$\tau = RC$$

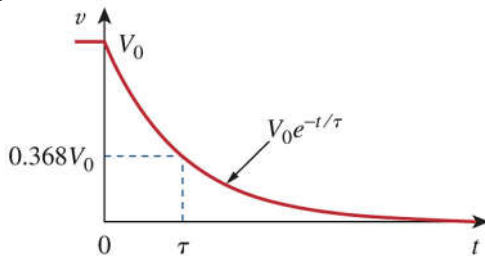
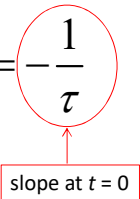


Figure 7.2 The voltage response of the source-free RC circuit.

In terms of the time constant,

$$v = V_0 e^{-t/\tau}$$

The time constant may be viewed from another perspective. Evaluating the derivative of v / V_0 at $t = 0$, we obtain

$$\left. \frac{d}{dt} \left(\frac{v}{V_0} \right) \right|_{t=0} = -\frac{1}{\tau} e^{-t/\tau} \bigg|_{t=0} = -\frac{1}{\tau}$$


Thus, the time constant is the initial rate of decay. To find τ from the response curve, draw the tangent to the curve at $t = 0$, as shown in Fig. 7.3. The tangent intercepts with the time axis at $t = \tau$.

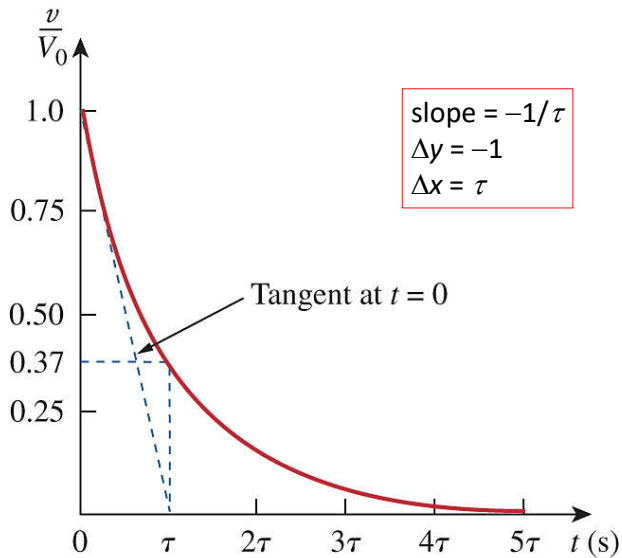


Figure 7.3 Graphical determination of the time constant from the response curve.

Five values of $v / V_0 = e^{-t/\tau}$ are shown in Table 7.1. It is evident from Table 7.1 that v is less than 1% of V_0 after 5τ . Thus, it is customary to assume that the capacitor is fully discharged after five time constants.

TABLE 7.1 Values of $v / V_0 = e^{-t/\tau}$

t	τ	2τ	3τ	4τ	5τ
v / V_0	0.36788	0.13534	0.04979	0.01832	0.00674

In other words, it takes 5τ for the circuit to reach its *final state* or *steady state* when no changes take place with time.

The smaller the time constant, the more rapidly the voltage decreases, that is, the faster the response. This is illustrated in Fig. 7.4. A circuit with a small time constant gives a fast response in that it reaches the steady state quickly due to quick dissipation of energy stored.

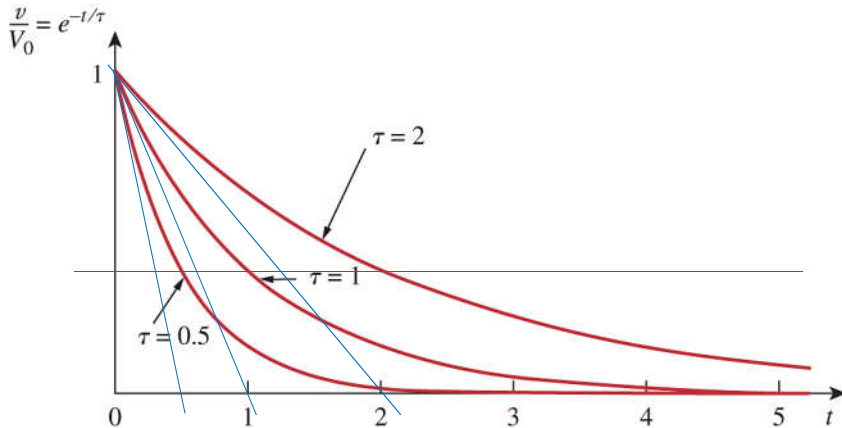


Figure 7.4 Plot of v/V_0 for various values of the time constant.

With the voltage $v = V_0 e^{-t/\tau}$, we can find the resistor current

$$i_R = \frac{v}{R} = \frac{V_0}{R} e^{-t/\tau}$$

The power dissipated in the resistor is

$$p = vi_R = \frac{V_0^2}{R} e^{-2t/\tau}$$

The energy absorbed by the resistor up to time t is

$$\begin{aligned}w_R &= \int_0^t \left(\frac{V_0^2}{R} e^{-2t/\tau} \right) dt = \frac{V_0^2}{R} \frac{e^{-2t/\tau}}{-2/\tau} \Big|_0^t \\&= \frac{1}{2} C V_0^2 (1 - e^{-2t/\tau})\end{aligned}$$

Notice that as $t \rightarrow \infty$, $w_R \rightarrow \frac{1}{2} C V_0^2 = w_C(0)$, the energy initially stored in the capacitor.

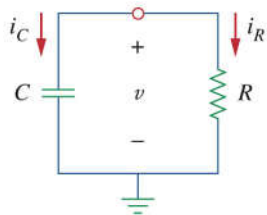
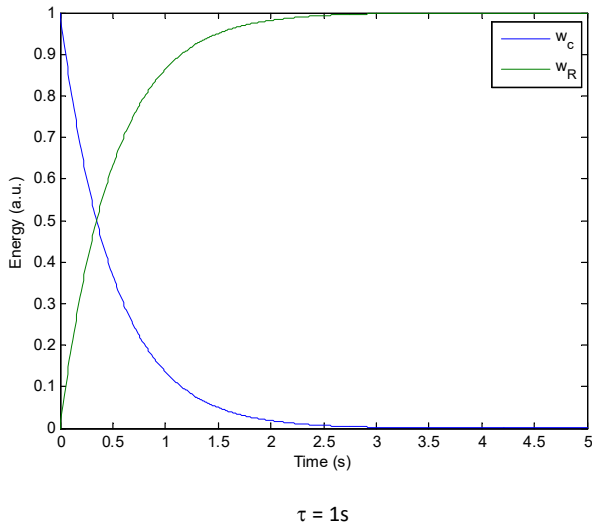


Figure 7.1 A source-free RC circuit.

In summary, the key to working with a source-free RC circuit is to find

1. The initial voltage $v(0) = V_0$ across the capacitor.
2. The time constant $\tau = RC$, where R is often the equivalent resistance at the terminals of C .

With these two items, we obtain the response as the capacitor voltage $v_C = v = v(0)e^{-t/\tau}$. Once v_C is first obtained, other variables (i_C, v_R, i_R) can be obtained.

Practice Problem 7.1 Refer to the circuit in Fig. 7.7. Let $v_C(0) = 45$ V. Determine v_C , v_x , and i_o for $t \geq 0$.

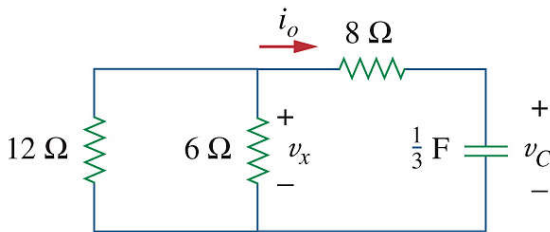


Figure 7.7 An RC circuit.

Solution :

The equivalent resistance seen at the terminals of the capacitor is

$$R_{eq} = 8 + 12 \parallel 6 = 12 \text{ } (\Omega)$$

The time constant is

$$\tau = R_{eq} C = 12 \times \frac{1}{3} = 4 \text{ (s)}$$

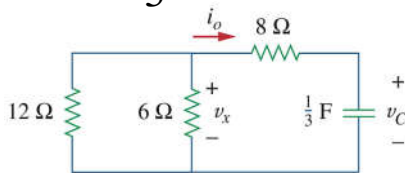


Figure 7.7 An RC circuit.

The capacitor voltage is

$$v_C = v_C(0)e^{-t/\tau} = 45e^{-t/4} = 45e^{-0.25t} \text{ (V)}$$

$$i_o = C \frac{dv_C}{dt} = \frac{1}{3} \times \frac{d}{dt} (45e^{-0.25t})$$

$$= 15 \times (-0.25e^{-t/4}) = -3.75e^{-t/4} \text{ (A)}$$

$$v_x = 8i_o + v_C = 8 \times (-3.75e^{-0.25t}) + 45e^{-0.25t}$$

$$= 15e^{-0.25t} \text{ (V)}$$

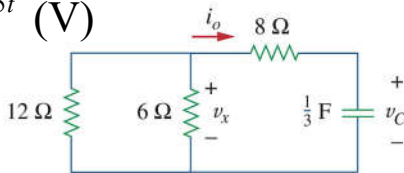


Figure 7.7 An RC circuit.

Practice Problem 7.12 If the switch in Fig. 7.10 opens at $t = 0$, find $v(t)$ for $t \geq 0$ and $w_C(0)$.

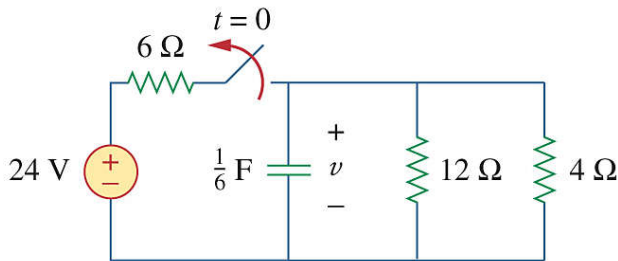


Figure 7.10

Solution :

When $t \leq 0$, the capacitor voltage

$$v(t) = 24 \times \frac{12 \parallel 4}{6 + 12 \parallel 4} = 8 \text{ (V)}$$

Hence, $v(0) = 8 \text{ V}$.

When $t \geq 0$, the circuit becomes a source-free RC circuit with

$$\tau = R_{eq} C = (12 \parallel 4) \times \frac{1}{6} = \frac{1}{2} \text{ (s)}$$

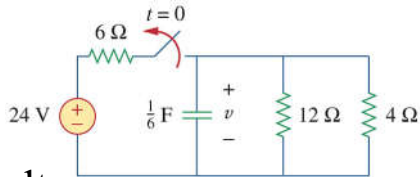


Figure 7.10

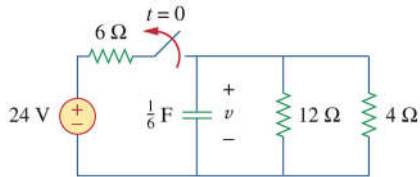


Figure 7.10

Therefore,

$$v(t) = v(0)e^{-t/\tau} = 8e^{-t/(1/2)} = 8e^{-2t} \text{ (V)}$$

$$w_C(0) = \frac{1}{2} C (v(0))^2 = \frac{1}{2} \times \frac{1}{6} \times 8^2$$

$$= \frac{16}{3} \approx 5.33 \text{ (J)}$$

7.3 The Source-Free RL Circuit

Consider the circuit in Fig. 7.11. Our goal is to determine the circuit response, which we assume to be the current i through the inductor. The inductor has an initial current

$$i(0) = I_0$$

with the corresponding energy

$$w_L(0) = \frac{1}{2}LI_0^2$$

$$v_L + v_R = 0$$

$$v_L = L \frac{di}{dt}, v_R = iR$$

$$L \frac{di}{dt} + iR = 0$$

or

$$\frac{di}{dt} + \frac{R}{L}i = 0$$

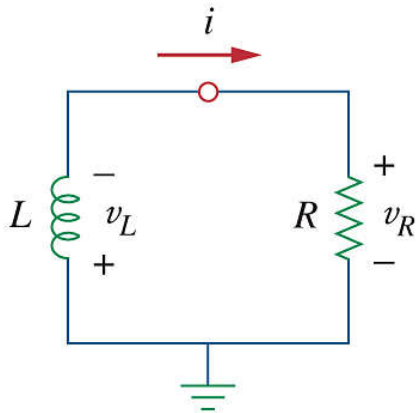


Figure 7.11 A source-free RL circuit.

$$r + \frac{R}{L} = 0$$

$$r = -\frac{R}{L}$$

$$i = Be^{rt} = Be^{-\frac{R}{L}t} = Be^{-\frac{t}{(L/R)}}$$

$$i(0) = B = I_0$$

$$i = I_0 e^{-\frac{t}{(L/R)}}$$

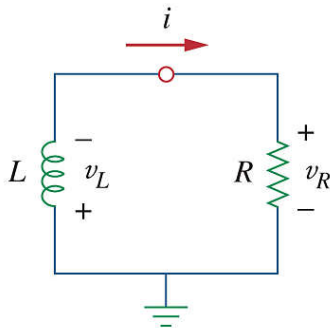


Figure 7.11 A source-free RL circuit.

Let $\tau = L / R$, we have

$$i = I_0 e^{-t/\tau}$$

The current response is shown in Fig. 7.12.

Since the response is due only to I_0 (i.e., the initial state of the circuit), it is called the zero-input response of the circuit.

= source free

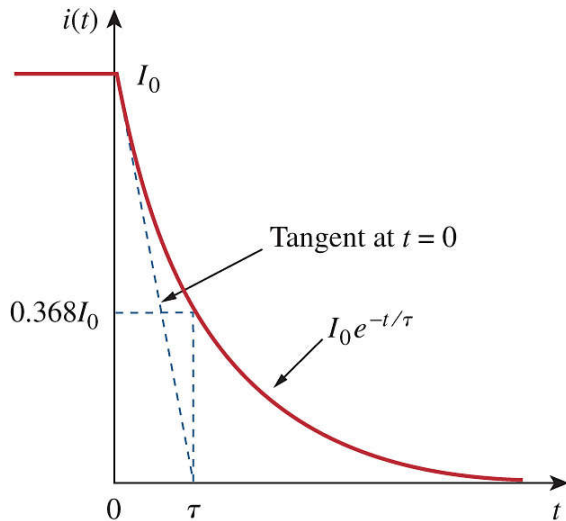


Figure 7.12 The current response of the source-free RL circuit.

With the current $i = I_0 e^{-t/\tau}$, we can find the resistor voltage

$$v_R = iR = I_0 R e^{-t/\tau}$$

The power dissipated in the resistor is

$$p = v_R i = I_0^2 R e^{-2t/\tau}$$

The energy absorbed by the resistor up to time t is

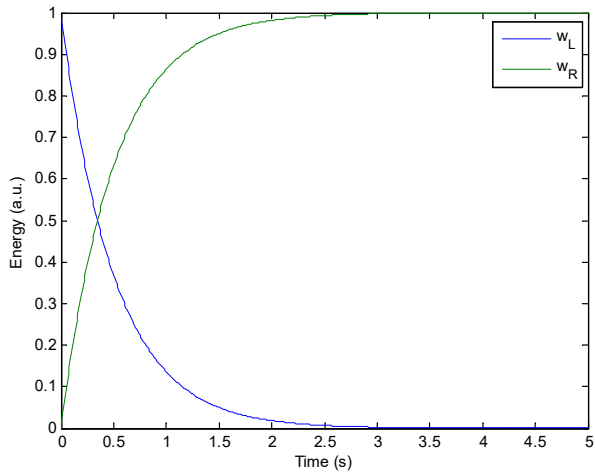
$$w_R = \int_0^t \left(I_0^2 R e^{-2t/\tau} \right) dt = I_0^2 R \frac{e^{-2t/\tau}}{-2/\tau} \bigg|_0^t$$
$$= \frac{1}{2} L I_0^2 \left(1 - e^{-2t/\tau} \right)$$

Notice that as $t \rightarrow \infty$, $w_R \rightarrow \frac{1}{2} L I_0^2 = w_L(0)$, the energy initially stored in the inductor.

The key to working with a source-free RL circuit is to find

1. The initial current $i(0) = I_0$ through the inductor.
2. The time constant $\tau = L / R$, where R is often the equivalent resistance at the terminals of L .

With the two items, we obtain the response as the inductor current $i_L = i = i(0)e^{-t/\tau}$. Once i_L is first obtained, other variables (v_L, v_R, i_R) can be obtained.



$\tau = 1\text{s}$

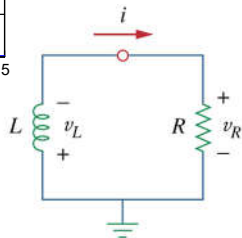


Figure 7.11 A source-free RL circuit.

Practice Problem 7.4 For the circuit in Fig. 7.18, find $i(t)$ for $t > 0$.

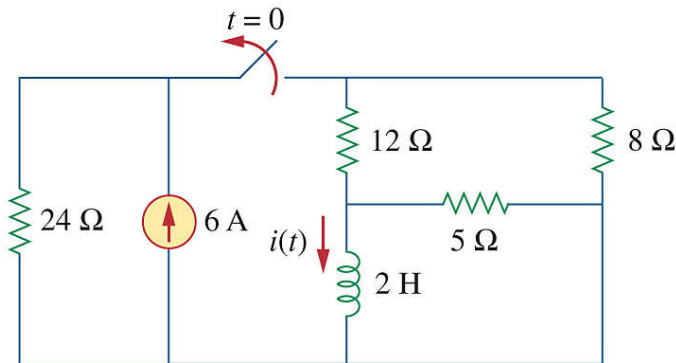


Figure 7.18

Solution :

When $t < 0$, the current through the $5\text{-}\Omega$ resistor is zero.

$$i(t) = 6 \times \frac{24 \parallel 8}{24 \parallel 8 + 12} = 2 \text{ (A)}$$

When $t > 0$,

$$R_{eq} = 5 \parallel (12 + 8) = 4 \text{ (}\Omega\text{)}$$

$$\tau = L / R_{eq} = 2 / 4 = 0.5 \text{ (s)}$$

$$i(t) = i(0)e^{-t/\tau} = 2e^{-t/0.5} = 2e^{-2t} \text{ (A)}$$

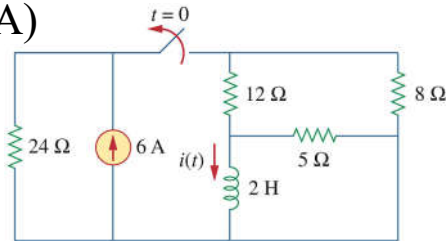


Figure 7.18

7.4 Singularity Functions

- Singularity functions (also called *switching functions*) are functions that either are discontinuous or have discontinuous derivatives. They serve as good approximations to switching operations.
- The three most widely used singularity functions in circuit analysis are the *unit step*, the *unit impulse*, and the *unit ramp* functions.

I. Unit Step

The unit step function $u(t)$ is 0 for negative values of t and 1 for positive values of t . In mathematical terms,

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

The unit step function is not defined at $t = 0$, where it changes abruptly from 0 to 1.

Figure 7.23 depicts the unit step function.

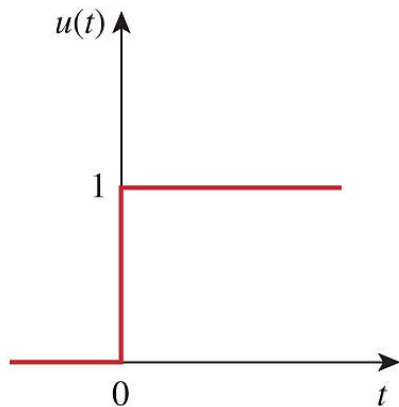


Figure 7.23 The unit step function.

If the abrupt change occurs at $t = t_0$ (where $t_0 > 0$) instead of $t = 0$, the mathematical representation becomes

$$u(t - t_0) = \begin{cases} 0, & t < t_0 \\ 1, & t > t_0 \end{cases}$$

meaning that $u(t)$ is delayed by t_0 seconds.

If the abrupt change occurs at $t = -t_0$ (where $t_0 > 0$) instead of $t = 0$, the mathematical representation becomes

$$u(t + t_0) = \begin{cases} 0, & t < -t_0 \\ 1, & t > -t_0 \end{cases}$$

meaning that $u(t)$ is advanced by t_0 seconds.

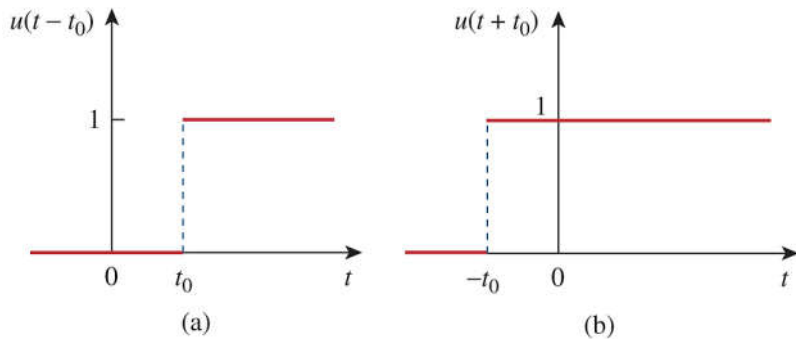


Figure 7.24 (a) The delayed version of the unit step, (b) the advanced version of the unit step.

In general, the step function is $Ku(t)$, where K is a constant.

Switching operations create abrupt changes in voltages and currents. An abrupt change can be represented by the step function. Two examples are given in Figures 7.25 and 7.26.

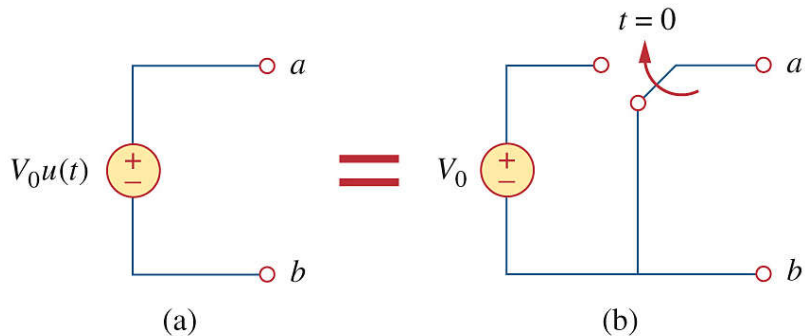


Figure 7.25 (a) Voltage source of $V_0 u(t)$, (b) its equivalent circuit.

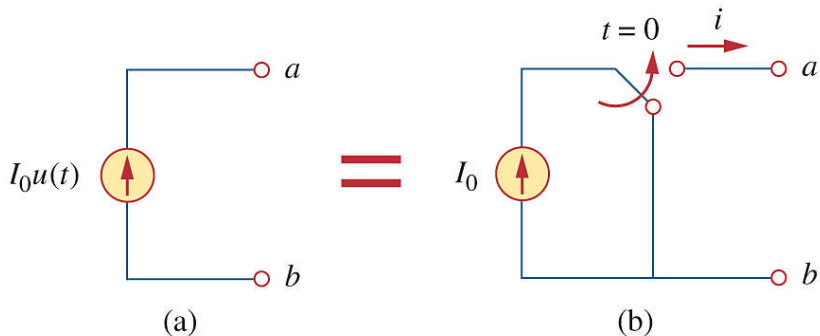


Figure 7.26 (a) Current source of $I_0 u(t)$, (b) its equivalent circuit.

II. Unit Impulse

Singularity functions can be used to write mathematical expressions for signals. For example, the *rectangular pulse* in Fig. 7(i) can be expressed as

$$x_{\Delta}(t) = \frac{1}{\Delta} (u(t + \Delta / 2) - u(t - \Delta / 2))$$

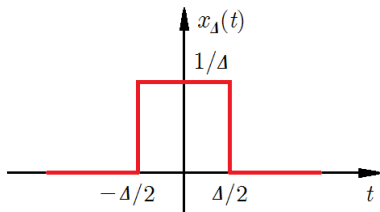


Figure 7(i) A rectangular pulse.

We view the *unit impulse* function $\delta(t)$ as the limiting form of any pulse, say the rectangular pulse $x_{\Delta}(t)$ in Fig. 7(i), that is an even function of time t with unit area:

$$\delta(t) = \lim_{\Delta \rightarrow 0} x_{\Delta}(t)$$

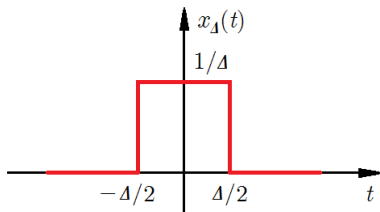


Figure 7(i) A rectangular pulse.

The derivative of the unit step function $u(t)$ is the unit impulse function $\delta(t)$, which we write as

$$\delta(t) = \frac{d}{dt}u(t)$$

That implies

$$u(t) = \int_{-\infty}^t \delta(\tau) d\tau$$

Proof :

Without loss of generality, we take the rectangular pulse as an example.

$$\begin{aligned}\delta(t) &= \lim_{\Delta \rightarrow 0} x_{\Delta}(t) \\ &= \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} (u(t + \Delta / 2) - u(t - \Delta / 2)) \\ &= \lim_{\Delta \rightarrow 0} \frac{u(t + \Delta / 2) - u(t - \Delta / 2)}{(t + \Delta / 2) - (t - \Delta / 2)} = \frac{d}{dt} u(t)\end{aligned}$$

An impulse function is a signal of infinite amplitude and zero duration. Such signals don't exist in nature, but some circuit signals come very close to approximating this definition, so we find a mathematical model of an impulse useful.

The unit impulse function is also called the *delta* function. We use an arrow to symbolize the unit impulse function, as in Fig. 7.27. The area under $\delta(t)$ is unity:

$$\int_{-\infty}^{\infty} \delta(t) dt = \int_{0^-}^{0^+} \delta(t) dt = 1$$

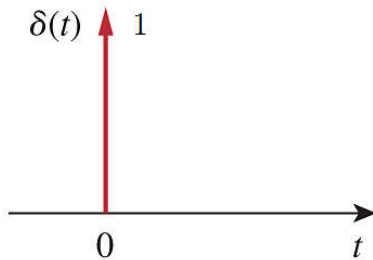


Figure 7.27 The unit impulse function.

where $t = 0^-$ denotes the time just before $t = 0$ and $t = 0^+$ denotes the time just after $t = 0$. For this reason, it is customary to write "1" beside the arrow. The area is known as the strength of the function. Figure 7.28 shows the impulse functions $5\delta(t + 2)$, $10\delta(t)$, and $-4\delta(t - 3)$.

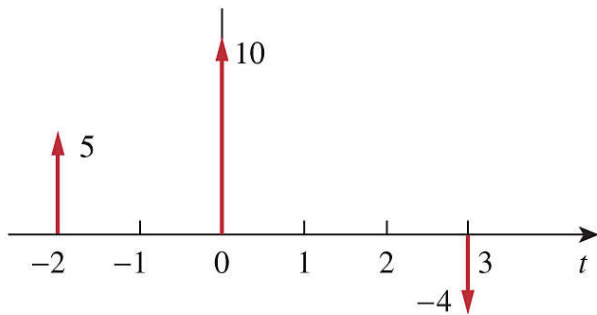


Figure 7.28

The unit impulse function has a property:

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

where $f(t)$ is a function that is continuous at $t = t_0$.

Proof :

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt &= \int_{-\infty}^{\infty} f(t_0)\delta(t-t_0)dt \\ &= f(t_0)\int_{-\infty}^{\infty} \delta(t-t_0)dt = f(t_0)\end{aligned}$$

This shows that when a function is integrated with the impulse function, we obtain the value of the function at the point t_0 where the impulse occurs. This property is known as the *sifting* property.

When $t_0 = 0$, the equation becomes

$$\int_{-\infty}^{\infty} f(t) \delta(t) dt = f(0)$$

III. Unit Ramp

The *unit ramp function* is zero for negative values of t and has a unit slope for positive values of t .

$$r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}$$

Figure 7.29 shows the unit ramp function.

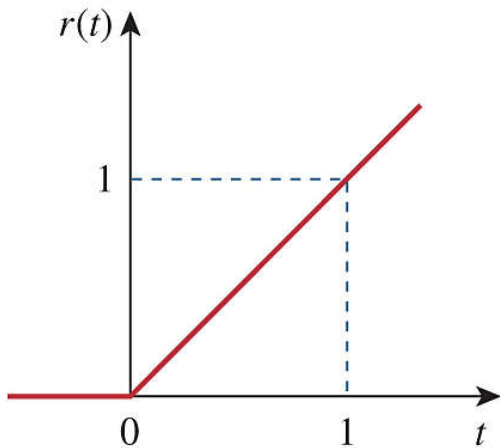


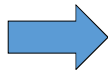
Figure 7.29 The unit ramp function.

Unit step

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

Unit ramp

$$r(t) = \begin{cases} 0, & t \leq 0 \\ t, & t \geq 0 \end{cases}$$


$$r(t) = tu(t)$$

Unit step

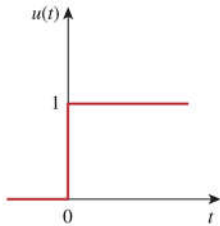


Figure 7.23 The unit step function.

Unit ramp

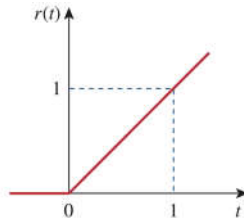
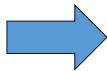


Figure 7.29 The unit ramp function.



$$r(t) = \int_{-\infty}^t u(\tau) d\tau$$

Integrating the unit step function $u(t)$ results in the unit ramp function $r(t)$; we write

$$r(t) = \int_{-\infty}^t u(\tau) d\tau = tu(t)$$

The unit ramp function may be delayed or advanced as shown in Fig. 7.30.

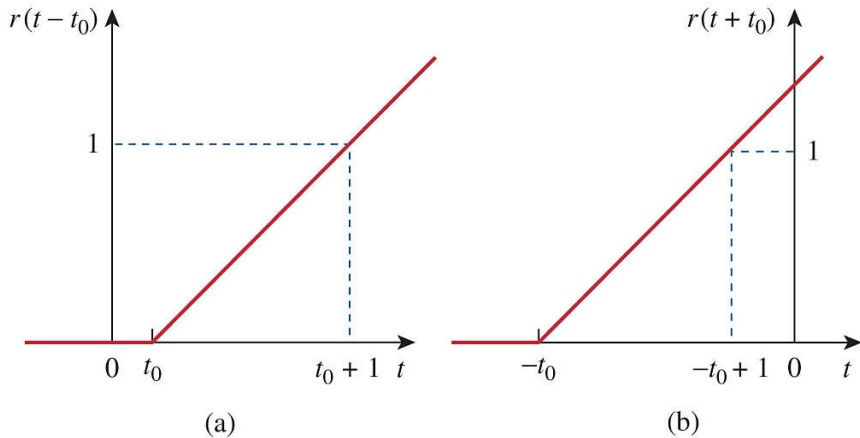


Figure 7.30 The unit ramp function (a) delayed by t_0 , (b) advanced by t_0 .

Example 7.6 Express the voltage pulse in Fig. 7.31 in terms of the unit step. Calculate its derivative and sketch it.

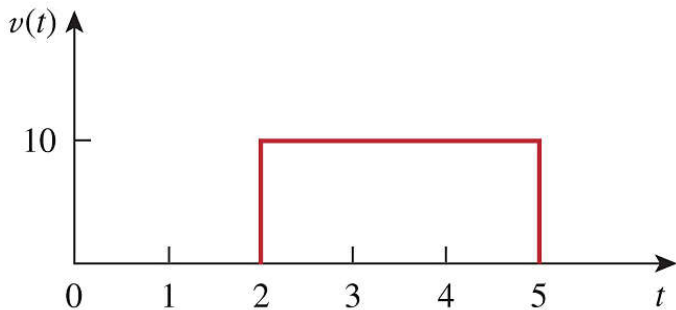


Figure 7.31

Solution :

$$v(t) = 10[u(t-2) - u(t-5)]$$

$$\frac{dv}{dt} = 10[\delta(t-2) - \delta(t-5)], \text{ see Fig. 7.32(b).}$$

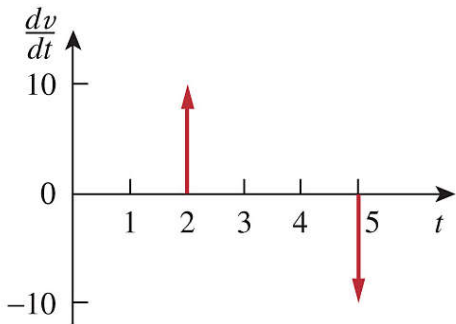


Figure 7.32(b)

Example 7.7 Express the *sawtooth* function shown in Fig. 7.35 in terms of singularity functions.

Solution :

$$v(t) = 5r(t) - 5r(t - 2) - 10u(t - 2).$$

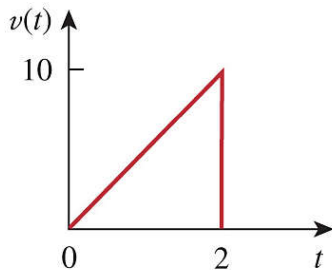
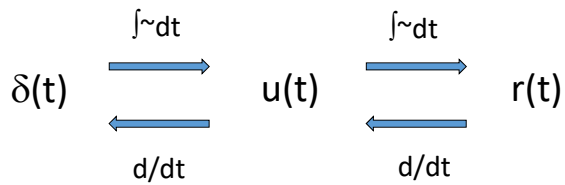
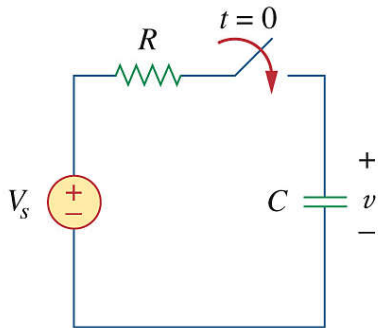


Figure 7.35

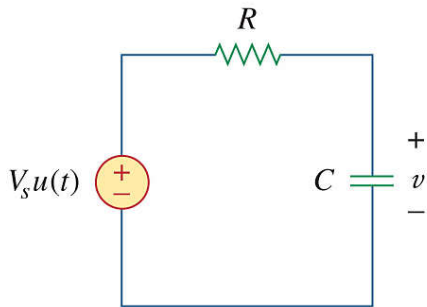


7.5 An RC Circuit with Step Input

Consider the RC circuit in Fig. 7.40(a) which can be replaced by the circuit in Fig. 7.40(b), where V_s is a constant dc voltage source. Again, we select the capacitor voltage v as the circuit response to be determined. We assume an initial voltage V_0 on the capacitor.



(a)



(b)

Figure 7.40 An RC circuit with voltage step input.

Since the voltage across the capacitor cannot change instantaneously,

$$v(0^+) = \underline{v(0^-)} = V_0$$

For $t > 0$,

Initial condition

$$\left(C \frac{dv}{dt} \right) R + v = V_s$$

or

$$\frac{dv}{dt} + \frac{1}{\tau} v = \frac{V_s}{\tau}, \text{ where } \tau = RC$$

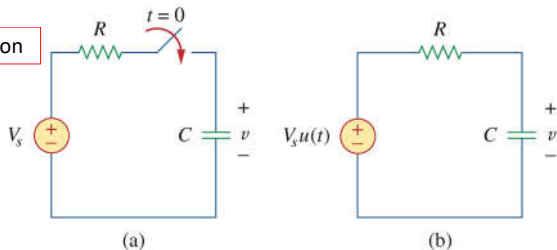


Figure 7.40 An RC circuit with voltage step input.

Method1

$$\rightarrow dv/dt = -1/\tau \times (v - V_s) \rightarrow dv/(v - V_s) = -dt/\tau$$

Integrating both sides and introducing the initial conditions,

$$\ln(v - V_s) \Big|_{V_0}^{v(t)} = -\frac{t}{RC} \Big|_0^t$$

$$\ln(v(t) - V_s) - \ln(V_0 - V_s) = -\frac{t}{RC} + 0$$

or

$$\ln \frac{v - V_s}{V_0 - V_s} = -\frac{t}{RC}$$

Taking the exponential of both sides

$$\frac{v - V_s}{V_0 - V_s} = e^{-t/\tau}, \quad \tau = RC$$

$$v - V_s = (V_0 - V_s)e^{-t/\tau}$$

or

$$v(t) = V_s + (V_0 - V_s)e^{-t/\tau}, \quad t > 0$$

Method2

$$r + \frac{1}{\tau} = 0 \Rightarrow r = -\frac{1}{\tau}$$

The homogeneous solution or *natural response*

$$v_n(t) = Ae^{-t/\tau}$$

Suppose the particular solution or *forced response*

$$v_f(t) = B$$

$$\frac{dB}{dt} + \frac{1}{\tau} B = \frac{V_s}{\tau} \Rightarrow B = V_s$$

The complete solution or *complete response*
or *total response*

$$v(t) = v_n(t) + v_f(t) = Ae^{-t/\tau} + V_s$$

When $t = 0^+$,

$$v(0^+) = A + V_s = V_0 \Rightarrow A = V_0 - V_s$$

Hence,

$$v(t) = (V_0 - V_s)e^{-t/\tau} + V_s$$

This is the response of the RC circuit to a sudden application of a constant dc source, assuming the capacitor is initially charged. Assuming that $V_s > V_0 > 0$, a plot of $v(t)$ is shown in Fig. 7.41.

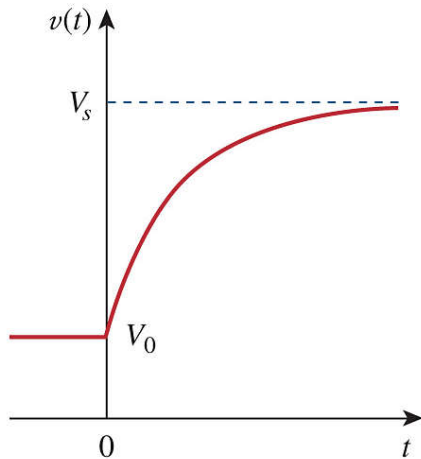


Figure 7.41 Response of an RC circuit.

If the capacitor is uncharged initially, i.e.,

$V_0 = 0$, then

$$v(t) = V_s(1 - e^{t/\tau}), \quad t > 0$$

or

$$v(t) = V_s(1 - e^{t/\tau})u(t)$$

Initial $V_0 = 0$

This is the zero - state response. The zero-state response corresponding to a unit-step input is called the *unit - step response*.

	RC circuit	RL circuit
Zero-input response	$V_s = 0$	
Zero-state response	$v_c(0) = 0$	
Unit-step response	$V_s = ku(t)$ $v_c(0) = 0$	

Interpretation 1: natural and forced responses

The complete response consists of two components:

$$(a) \ v(t) = v_n(t) + v_f(t)$$

where $v_n(t) = \underline{(V_0 - V_s)}e^{-t/\tau}$ is the natural response (homogeneous solution), and $v_f(t) = \underline{V_s}$ is the forced response (particular solution). Basically, $v_f(t)$ has the same form as the input (external "force").

Interpretation 2: temporary and steady state

$$(c) \ v(t) = v_t(t) + v_{ss}(t)$$

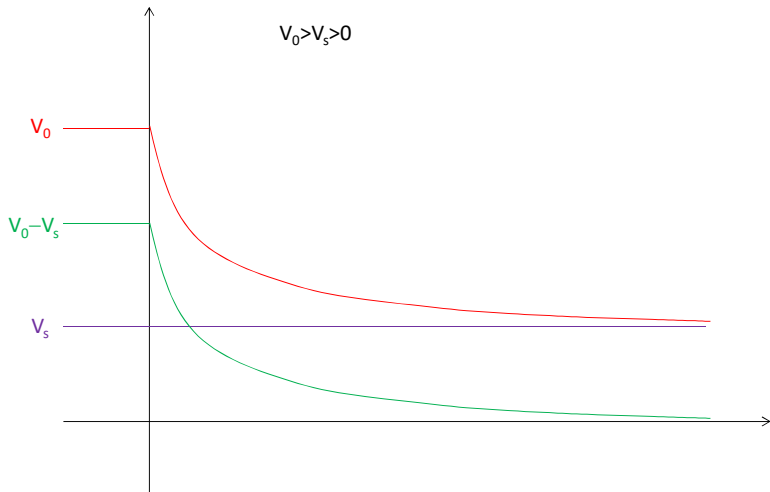
where $v_t(t) = \underline{(V_0 - V_s)}e^{-t/\tau}$ is the temporary response (will die out with time) and $v_{ss}(t) = \underline{V_s}$ is the response as $t \rightarrow \infty$.

- Interpretation 1 is similar to interpretation 2

$$v(t) = \underbrace{V_s}_{\text{Forced response; Steady state}} + \underbrace{(V_0 - V_s)e^{-t/\tau}}_{\text{Homogeneous response; Temporary response}}, \quad t > 0$$

Forced response;
Steady state

Homogeneous response;
Temporary response



$$v(t) = \underbrace{V_s}_{\text{Forced response; Steady state}} + \underbrace{(V_0 - V_s)e^{-t/\tau}}_{\text{Homogeneous response; Temporary response}}, \quad t > 0$$

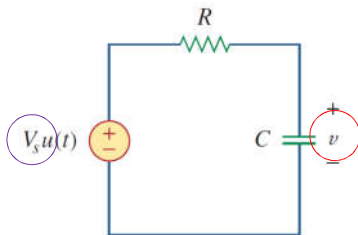
Forced response;
Steady state

Homogeneous response;
Temporary response

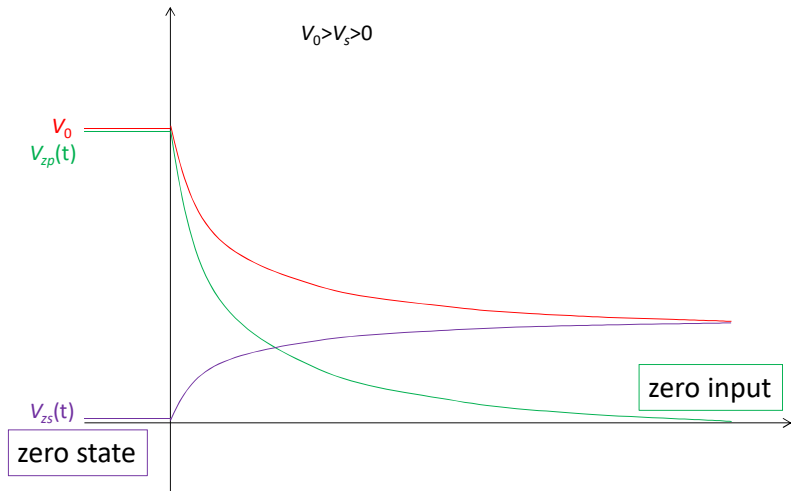
Interpretation 3: $v_c(0)$ and V_s

$$(b) \ v(t) = v_{zp}(t) + v_{zs}(t)$$

where $v_{zp}(t) = V_0 e^{-t/\tau}$ is the *zero-input response* (due to the initial state), and $v_{zs}(t) = V_s(1 - e^{-t/\tau})$ is the zero-state response (due to the input).



(b)

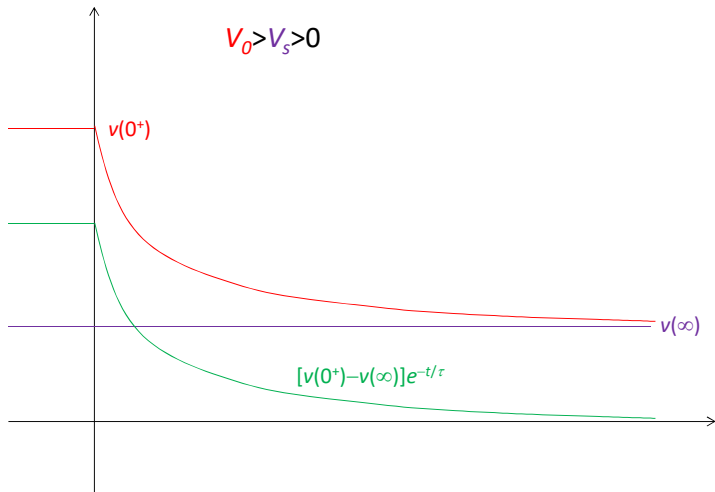


Whichever way we look at it, the complete response may be written as

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

where $v(0^+)$ is the *initial value* (i.e., the value at $t = 0^+$) and $v(\infty)$ is the *final value* (or steady-state value).

$$\begin{aligned}\text{At } t = 0, v(0^+) &= v(\infty) + [v(0^+) - v(\infty)] \times 1 = v(0^+) \\ \text{At } t \rightarrow \infty, v(\infty) &= v(\infty) + [v(0^+) - v(\infty)] \times 0 = v(\infty)\end{aligned}$$



Thus, to find the response of a first-order RC circuit requires three things:

1. The initial capacitor voltage $v(0^+)$.
2. The final capacitor voltage $v(\infty)$.
3. The time constant $\tau = RC$.

This is known as the *three - factor method*.

Source-free RC circuit is a special case when $v(\infty) = 0$

Example 7.10 The switch in Fig.7.43 has been in position *A* for a long time. At $t = 0$, the switch moves to *B*. Determine $v(t)$ for $t > 0$ and calculate its value at $t = 1$ s and 4s.

Solution :

For $t < 0$,

$$v(t) = 24 \times \frac{5}{3+5} = 15 \text{ (V)}$$

$$v(0^-) = 15 \text{ V}$$

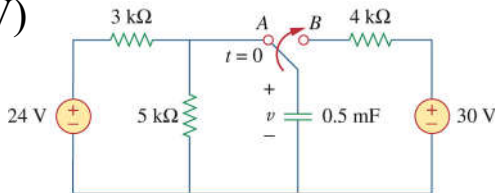


Figure 7.43

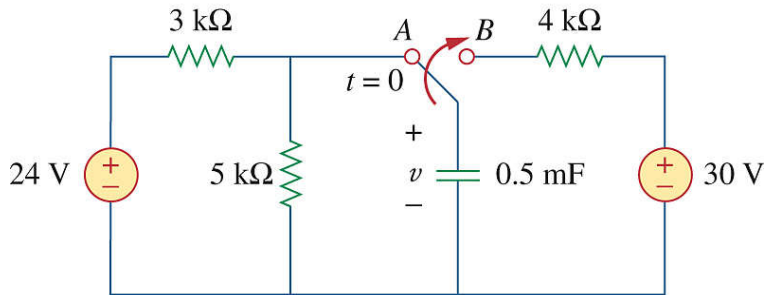


Figure 7.43

For $t > 0$,

$$v(0^+) = v(0^-) = 15 \text{ V}$$

$$v(\infty) = 30 \text{ V}$$

$$\tau = 4 \times 10^3 \times 0.5 \times 10^{-3} = 2 \text{ (s)}$$

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

$$= 30 + (15 - 30)e^{-t/2} = 30 - 15e^{-0.5t} \text{ (V)}$$

$$v(1) = 30 - 15e^{-0.5 \times 1} \approx 20.90 \text{ (V)}$$

$$v(4) = 30 - 15e^{-0.5 \times 4} \approx 27.97 \text{ (V)}$$

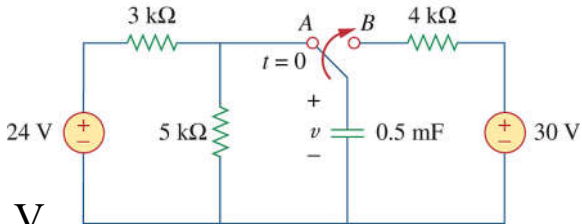


Figure 7.43

Practice Problem 7.11 The switch in Fig. 7.43 is closed at $t = 0$. Find $i(t)$ and $v(t)$ for all time.

Solution :

For $t < 0$,

$$i(t) = 0 \text{ A}$$

$$v(t) = 20 \text{ V}$$

$$v(0^-) = 20 \text{ V}$$

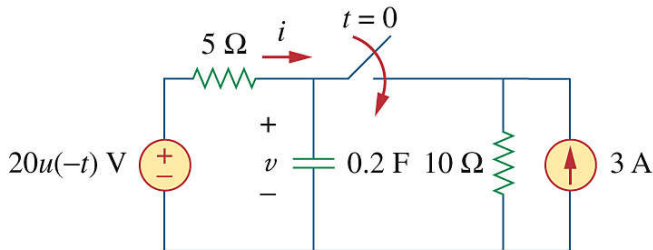


Figure 7.47

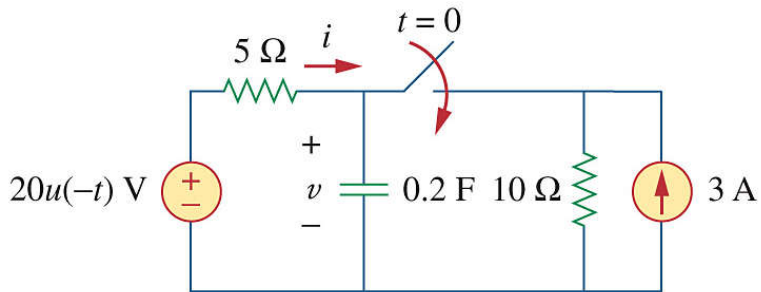


Figure 7.47

For $t > 0$,

$$v(0^+) = v(0^-) = 20 \text{ V}$$

$$v(\infty) = 3 \times (5 \parallel 10) = 10 \text{ (V)}$$

$$\tau = (5 \parallel 10) \times 0.2 = \frac{2}{3} \text{ (s)}$$

$$v(t) = v(\infty) + [v(0^+) - v(\infty)]e^{-t/\tau}$$

$$= 10 + (20 - 10)e^{-t/(2/3)} = 10(1 + e^{-1.5t}) \text{ (V)}$$

$$i(t) = -\frac{v(t)}{5} = -2(1 + e^{-1.5t}) \text{ (A)}$$

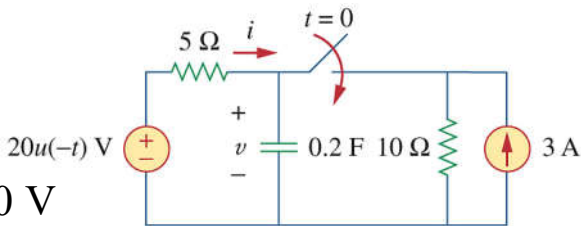


Figure 7.47

$v(0^+)$: by the value of $v(0^-)$
 $v(\infty)$: by the circuit at $t > 0$

7.6 An RL Circuit with Step Input

Consider the RL circuit in Fig. 7.48(a) which can be replaced by the circuit in Fig. 7.48(b), where V_s is a constant dc voltage source. We select the inductor current i as the circuit response to be determined. We assume an initial current I_0 in the inductor.

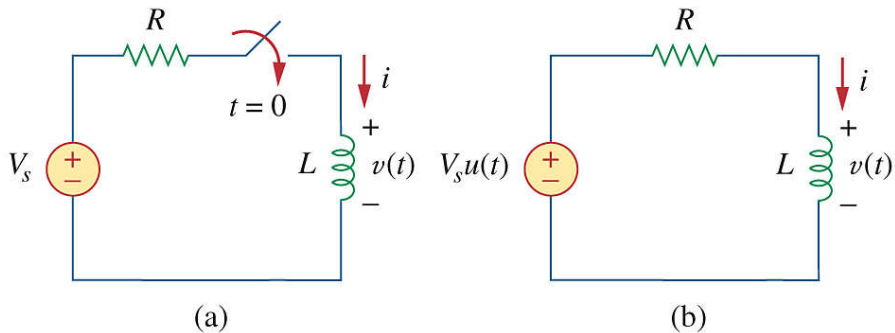


Figure 7.48 An RL circuit with a step input voltage.

Since the current through the inductor cannot change instantaneously,

$$i(0^+) = i(0^-) = I_0$$

For $t > 0$,

$$iR + L \frac{di}{dt} = V_s$$

or

$$\frac{di}{dt} + \frac{1}{\tau} i = \frac{V_s / R}{\tau}, \text{ where } \tau = \frac{L}{R}$$

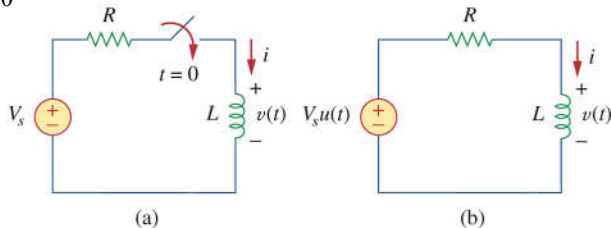


Figure 7.48 An RL circuit with a step input voltage.

$$\frac{di}{dt} + \frac{1}{\tau} i = \frac{V_s / R}{\tau}, \text{ where } \tau = \frac{L}{R}$$

$$\frac{dv}{dt} + \frac{1}{\tau} v = \frac{V_s}{\tau}, \text{ where } \tau = RC$$

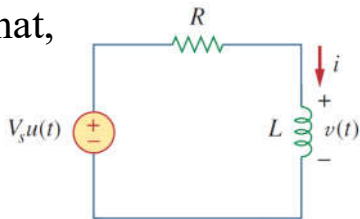
Since this differential equation has the same form as that describing the RC circuit,

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R} \right) e^{-t/\tau}$$

$$v(t) = (V_0 - V_s) e^{-t/\tau} + V_s$$

It is evident from Fig. 7.48 that,

$$i(\infty) = \frac{V_s}{R}$$



(b)

The complete response can be written as

$$i(t) = i(\infty) + \left[i(0^+) - i(\infty) \right] e^{-t/\tau}$$

Thus, to find the response of a first-order RL circuit requires three things:

1. The initial inductor current $i(0^+)$.
2. The final inductor current $i(\infty)$.
3. The time constant $\tau = L / R$.

Source-free RL circuit is a special case when $i(\infty) = 0$

The response of the RL circuit to a sudden application of a constant dc source is shown in Fig, 4.49, assuming that

$$I_0 > \frac{V_s}{R} > 0$$

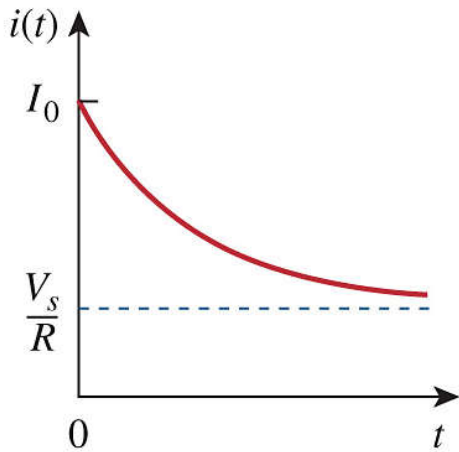
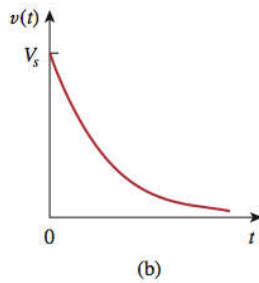
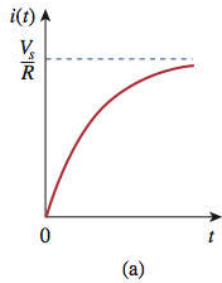
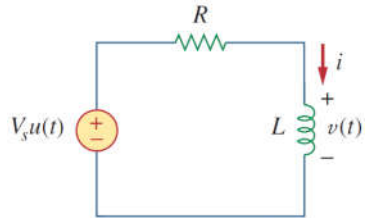


Figure 7.49 Total response of the RL circuit.



Example 7.13 At $t = 0$, switch 1 in Fig. 7.53 is closed, and switch 2 is closed 4 s later. Find $i(t)$ for $t > 0$. Calculate i for $t = 2$ s and $t = 5$ s.

Solution :

For $t < 0$,

$$i(t) = 0$$

$$i(0^-) = 0$$

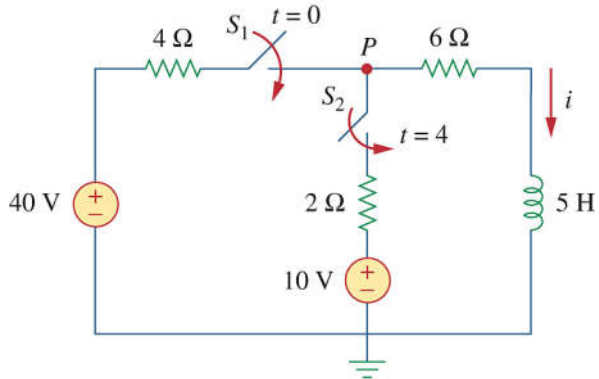


Figure 7.53

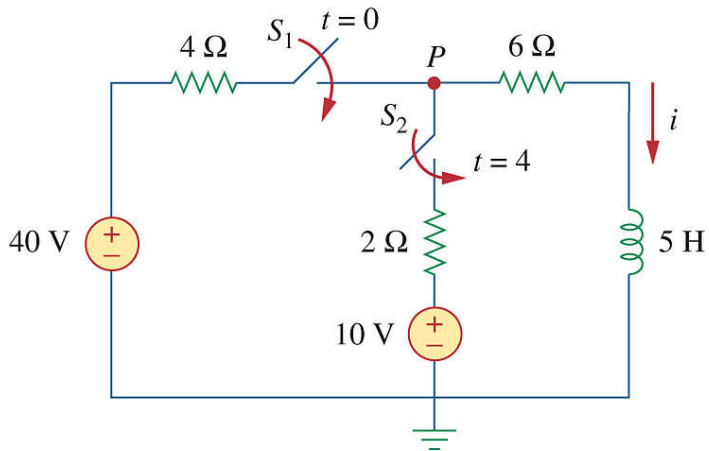


Figure 7.53

For $t > 0$,

$$i(0^+) = i(0^-) = 0$$

For $0 < t \leq 4$,

$$i(\infty) = \frac{40}{4+6} = 4 \text{ (A)}$$

$$\tau = \frac{5}{4+6} = 0.5 \text{ (s)}$$

$$i(t) = 4 + (0 - 4)e^{-t/0.5} = 4(1 - e^{-2t}) \text{ (A)}$$

$$i(2) = 4(1 - e^{-2 \times 2}) \approx 3.93 \text{ (A)}$$

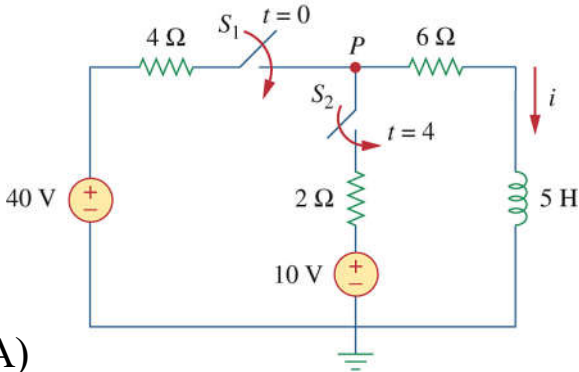


Figure 7.53

For $t > 4$,

$$i(4^+) = i(4^-) = 4(1 - e^{-2 \times 4}) = 4(1 - e^{-8}) \text{ (A)}$$

$$v_P(\infty) = 40 \times \frac{2 \parallel 6}{4 + 2 \parallel 6} + 10 \times \frac{4 \parallel 6}{2 + 4 \parallel 6}$$

$$= \frac{180}{11} \text{ (V)}$$

$$i(\infty) = \frac{v_P(\infty)}{6} = \frac{30}{11} \text{ (A)}$$

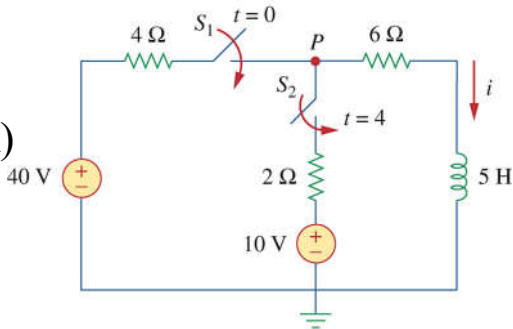


Figure 7.53

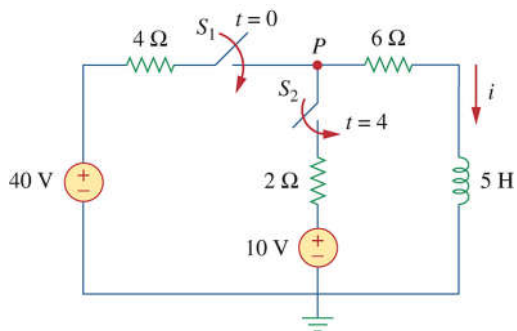


Figure 7.53

$$\tau = \frac{5}{6 + 4 \parallel 2} = \frac{15}{22} \text{ (s)}$$

$$i(t) = \frac{30}{11} + \left(4(1 - e^{-8}) - \frac{30}{11} \right) e^{-\frac{(t-4)}{(15/22)}}$$

Time shift property

$$\approx 2.7273 + 1.2714e^{-1.4667(t-4)} \text{ (A)}$$

$$i(5) = 2.7273 + 1.2714e^{-1.4667(5-4)} \approx 3.02 \text{ (A)}$$

7.7 First-order Op Amp Circuits

Example 7.14 For the op amp circuit in Fig. 7.55(a), find v_o for $t > 0$, given that $v(0) = 3$ V. Let $R_f = 80$ k Ω , $R_1 = 20$ k Ω , and $C = 5$ μ F.

Solution :

$$\left(C \frac{dv}{dt} \right) R_1 + v = 0, \quad t > 0$$

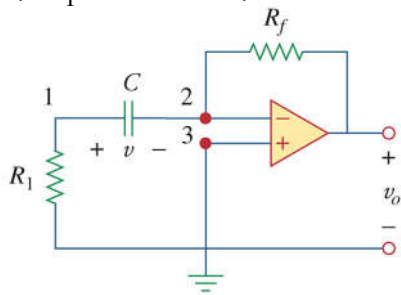


Figure 7.55(a)

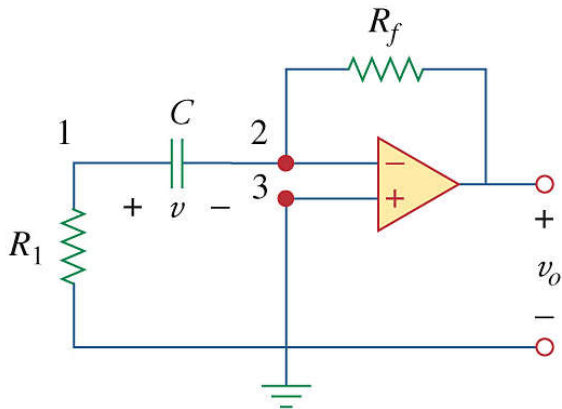


Figure 7.55(a)

or

$$\frac{dv}{dt} + \frac{1}{\tau}v = 0$$

where,

$$\tau = R_1 C = 20 \times 10^3 \times 5 \times 10^{-6} = 0.1 \text{ (s)}$$

$$v = v(0)e^{-t/\tau} = 3e^{-t/0.1} = 3e^{-10t} \text{ (V)}$$

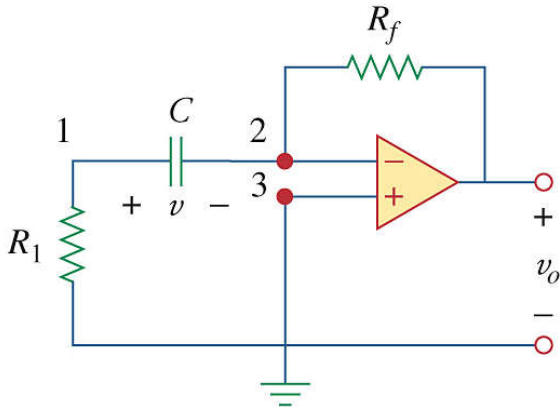


Figure 7.55(a)

$$v_o = -\left(C \frac{dv}{dt}\right) R_f$$

$$= -\left(5 \times 10^{-6} \frac{d}{dt}(3e^{-10t})\right) \times 80 \times 10^3$$

$$= -0.4 \times (3 \times (-10e^{-10t}))$$

$$= 12e^{-10t} \text{ (V)}$$

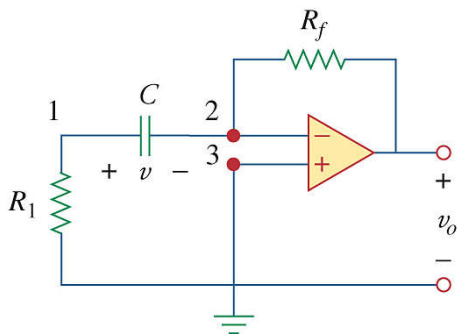


Figure 7.55(a)

Example 7.15 Determine $v(t)$ and $v_o(t)$ in the circuit of Fig. 7.57.

Solution :

For $t < 0$,

$$v_1(t) = 0, v_o(t) = 0, v(t) = 0$$

$$v(0^-) = 0$$

For $t > 0$,

$$v_1(t) = 3 \times \frac{20}{10 + 20} = 2 \text{ (V)}$$

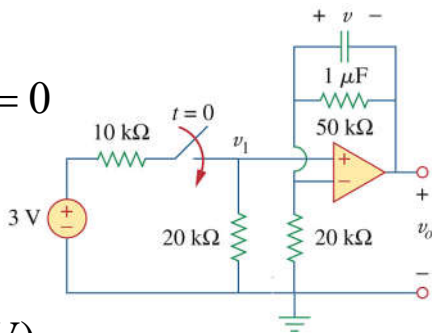


Figure 7.57

Circuit for $t < 0$

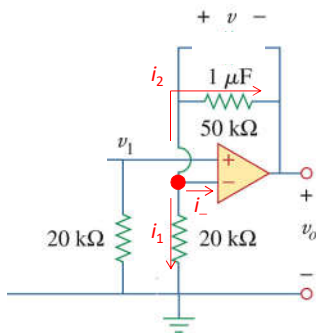


Figure 7.57

$$i_+ = 0 \rightarrow v_1 = 0 \rightarrow v_- = 0 \rightarrow i_1 = 0$$

$$\text{KCL: } i_1 + i_2 + i_- = 0 \rightarrow i_2 = 0$$

$$\rightarrow v(0^+) = v(0^-) = 0$$

$$\rightarrow v_o(0^+) = 0 - 0 = 0$$

OR



Non-inverting amplifier

$$v_o = 0$$

$$\rightarrow v(0^+) = v(0^-) = 0$$

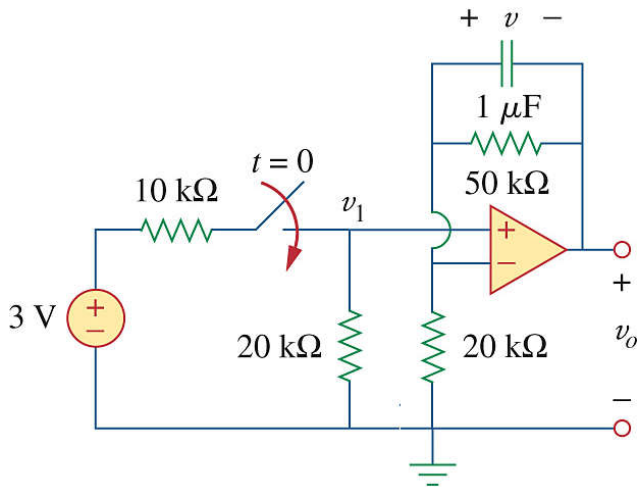


Figure 7.57

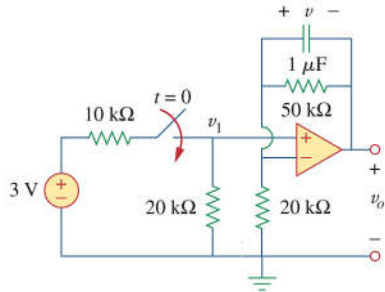


Figure 7.57

$$(1 \times 10^{-6}) \frac{dv}{dt} + \frac{v}{50 \times 10^3} = -\frac{2}{20 \times 10^3}$$

$$\frac{dv}{dt} + 20v = -100 \quad \boxed{dB/dt + 20B = -100}$$

$$v(t) = Ae^{-20t} + B = 5e^{-20t} - 5 \text{ (V)}$$

$$v_o(t) = -v(t) + 2 = 7 - 5e^{-20t} \text{ (V)}$$