Chapter 4 Circuit Theorems

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4.1 Introduction

- Even though the nodal analysis and mesh analysis are powerful techniques for solving circuits, we are still interested in methods that can be used to simplify circuits. Series-parallel reductions and wye-delta transformations are already on our list of simplifying techniques. In this chapter, we study more.

4.2 Linearity

- A linear circuit is one whose output (also called *response*) is linearly related to its input (also called *excitation*).
- The linearity property is a combination of both the homogeneity (scaling) property and the additivity property.

- Homogeneity requires that if the input is multiplied by a constant, then the output is multiplied by the same constant.
- Additivity property requires that the response to a sum of inputs is the sum of the responses to each input applied separately.
- A circuit is linear if it is both additive and homogeneous.

Homogeneous

- Assuming input x, output y
  If $x \rightarrow y$,
  then $kx \rightarrow ky$ (homogeneous)
  - Example: $y = ax$
    $a(kx) = k(ax) = ky$
  - Counterexample: $y = ax^2$
    $a(kx)^2 = ak^2x^2 = k^2(ax^2) = k^2y$
    The output is NOT multiplied by the same constant

Additive

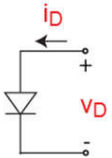
- Assuming input x_1 produces output y_1; input x_2 produces output y_2
  If $x_1 \rightarrow y_1$, $x_2 \rightarrow y_2$,
  then $x_1+x_2 \rightarrow y_1+y_2$ (additive)
  - Example: y=ax
    $y_1 = ax_1$, $y_2 = ax_2 \Rightarrow a(x_1+x_2) = ax_1+ax_2 = y_1+y_2$
  - Counterexample: y=ax^2
    $y_1 = ax_1^2$, $y_2 = ax_2^2 \Rightarrow a(x_1+x_2)^2 = ax_1^2+2ax_1x_2+ax_2^2 \neq y_1+y_2$

Circuits with nonlinear elements

- Example: a diode (like a nonlinear resistor).
  - very useful nonlinear components that are widely used in communication systems, high-speed electronics, power applications, etc.
  - Solar cells, photodetectors, and lasers are also examples of diodes.

Circuits with nonlinear elements

- **i-v characteristics of diode**
  - ❖Diode is a two terminal nonlinear resistor whose current is exponentially related to the voltage across its terminals.
  - ❖An analytical expression

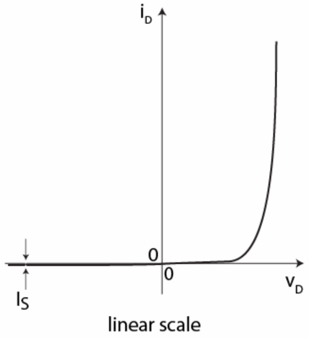


$$i_D = I_S(e^{qv_D/kT} - 1)$$

I_S: saturation current
q: electron charge ($1.6 * 10^{-19} C$)
k: Boltzmann constant ($1.38 * 10^{-23} J/K$)
T: temperature (K)
kT/q: thermal voltage ($26mV at 300K$)

Circuits with nonlinear elements

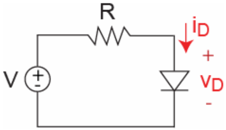
❖The function of the diode can be plotted as:



Obviously, the input-output is a nonlinear relation.

Circuits with nonlinear elements

- The circuit with nonlinear element



  - To determine v_D and i_D for different values of V and R, we can apply KCL

$$\frac{v_D - V}{R} + i_D = 0$$

  - And using the diode relation $\frac{v_D - V}{R} + I_S(e^{qv_D/kT} - 1) = 0$

Example 4.1 For the circuit in Fig. 4.2, find I_o when $v_s = 12$ V and $v_s = 24$ V. This example illustrate the homogeneity property.

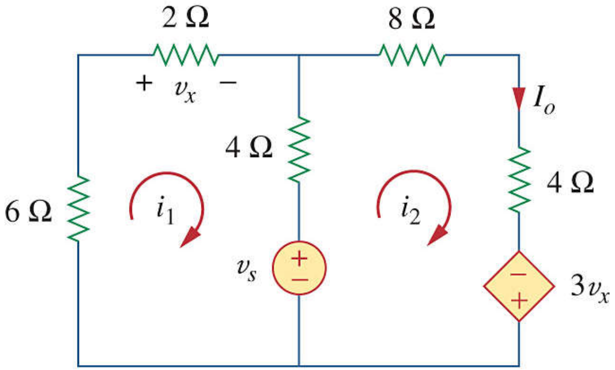


Figure 4.2

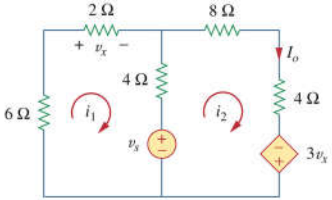


Figure 4.2

Solution :

$$\begin{bmatrix} 6+2+4 & -4 \\ -4 & 4+8+4 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -v_s \\ 3v_x + v_s \end{bmatrix}$$

$$v_x = 2i_1$$

$$\begin{bmatrix} 6+2+4 & -4 \\ -4 & 4+8+4 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -v_s \\ 6i_1 + v_s \end{bmatrix}$$

$$\begin{bmatrix} 6+2+4 & -4 \\ -4-6 & 4+8+4 \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -v_s \\ v_s \end{bmatrix}$$

$$\begin{bmatrix} 12 & -4 \\ -10 & 16 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -v_s \\ v_s \end{bmatrix}$$

$$\Delta = \begin{vmatrix} 12 & -4 \\ -10 & 16 \end{vmatrix} = 152$$

$$\Delta_2 = \begin{vmatrix} 12 & -v_s \\ -10 & v_s \end{vmatrix} = 2v_s$$

$$I_o = i_2 = \frac{\Delta_2}{\Delta} = \frac{v_s}{76}$$

When $v_s = 12$ V,

$$I_o = \frac{12}{76} = \frac{3}{19} \text{ (A)}$$

When $v_s = 24$ V,

$$I_o = \frac{24}{76} = \frac{6}{19} \text{ (A)}$$

The homogeneity property states that if $v_s = 12$ V gives $I_o = 3/19$ A, then $v_s = 2 \times 12$ V will give $I_o = 2 \times 3/19$ A $= 6/19$ A.

$v_s \rightarrow I_o$

$\Downarrow$ Homogeneous circuit

$2v_s \rightarrow 2I_o$

4.3 Superposition

- Superposition principle is based on additivity. It states that whenever a linear system is excited, or driven, by more than one independent source, the total response is the sum of the individual responses.

- The principle of superposition helps us to analyze a linear circuit with more than one independent source by calculating the contribution of each independent source separately and adding algebraically all the contributions to find the total contribution.

- To apply the superposition principle, we must keep two things in mind:
  - We consider one independent source at a time while all other independent sources are *turned off*. This implies that we replace every voltage source by 0 V (or a short circuit), and every current source by 0 A (or an open circuit).
  - Dependent sources are left intact because they are controlled by circuit variables.

Linear circuit: homogeneous and additive

- Linear circuit

  If $x_1 \rightarrow y_1$; $x_2 \rightarrow y_2$,

  then $ax_1+bx_2 \rightarrow ay_1+by_2$

  $\boxed{\text{Linear: homogeneous and additive}}$

- Superposition
  - (1) contribution from input x_1: $ax_1 \rightarrow ay_1$
  - (2) contribution from input x_2: $bx_2 \rightarrow by_2$

  The total contribution by adding algebraically all the contributions: ay_1+by_2

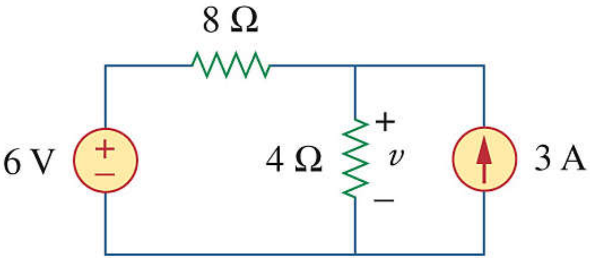


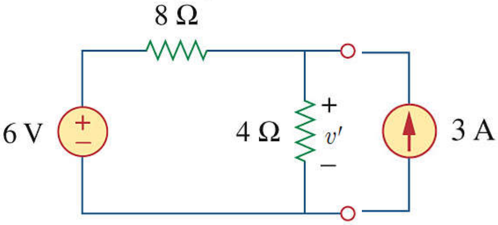
Figure 4.6

Solution :

Since there are two sources, let $v = v' + v''$, where v' and v'' are the contributions due to the voltage source and the current source, respectively.

Set the current source to zero, as shown in Fig. 4.7(a), we obtain

$$v' = \frac{4}{8+4} \times 6 = 2 \text{ (V)}$$



(a)

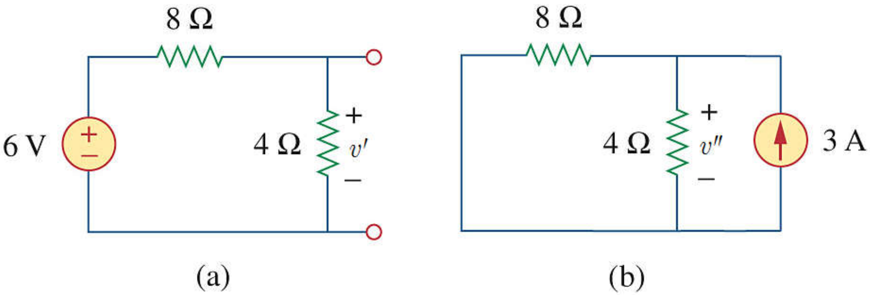
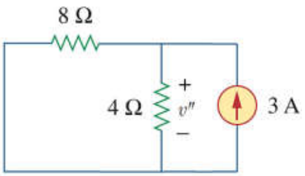


Figure 4.7

Set the voltage source to zero, as shown in Fig. 4.7(b), we obtain

$$v'' = 3 \times (4 \parallel 8) = 3 \times \frac{4 \times 8}{4+8} = 8 \text{ (V)}$$



Thus,

$$v = v' + v'' = 2 + 8 = 10 \text{ (V)}$$

This example shows that suposition helps reduce a complex circuit to simpler circuits through turning off independent sources.

Practice Problem 4.4 Use superposition to find v_x in the circuit of Fig. 4.11.

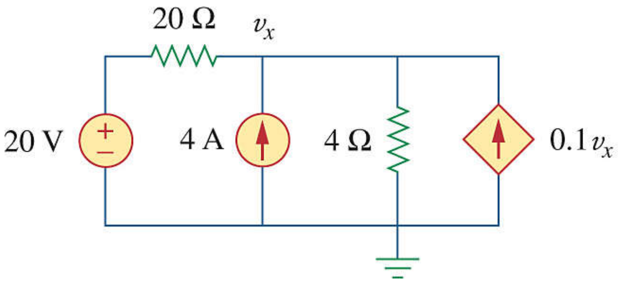


Figure 4.11

Solution :

Let $v_x = v'_x + v''_x$

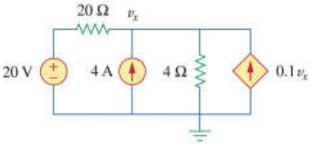


Figure 4.11

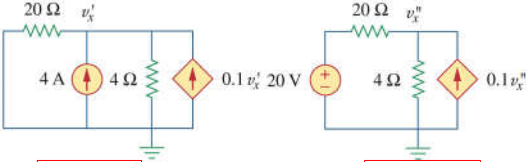
Set the voltage source to zero,

$$\frac{v'_x}{20} - 4 + \frac{v'_x}{4} - 0.1v'_x = 0 \Rightarrow v'_x = 20 \text{ (V)}$$

Set the current source to zero,

$$\frac{v''_x - 20}{20} + \frac{v''_x}{4} = 0.1v''_x \Rightarrow v''_x = 5 \text{ (V)}$$

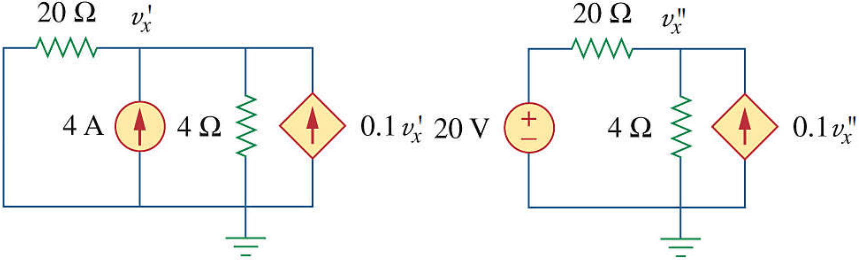
$$v_x = v'_x + v''_x = 25 \text{ (V)}$$



Input: 4A

For Practice Problem 4.4

Input: 20V



For Practice Problem 4.4

4.4 Source Transformation

- A source transformation is the process of replacing a voltage source in series with a resistor by a current source in parallel with a resistor, or vice versa.
- Source transformation is a tool for simplifying circuits.

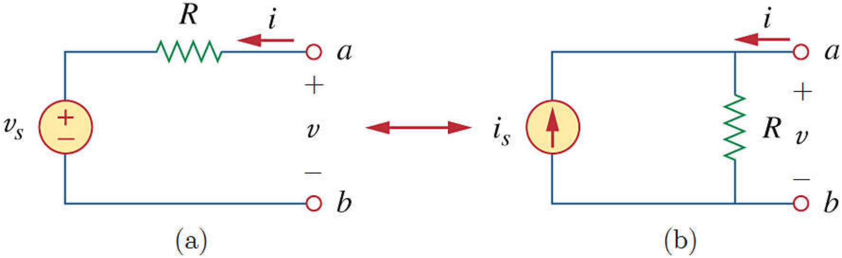


Figure 4.15 Transformation of independent sources .

$$v_s = i_s R \quad \text{or} \quad i_s = \frac{v_s}{R}$$

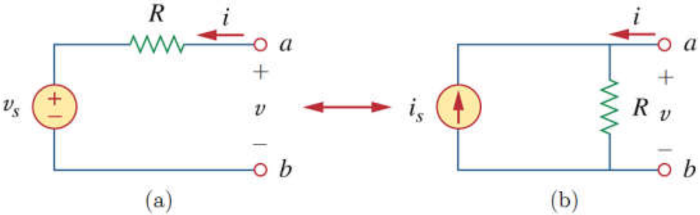


Figure 4.15 Transformation of independent sources.

Proof

Two circuits are said to be equivalent if they have the same i - v relation. — Terminal voltage and current

For circuit in Fig. 4.15(a), $v = iR + v_s$

For circuit in Fig. 4.15(b), $v = iR + i_s R$

Obviously, the two i - v relations are identical provided that $v_s = i_s R$ or $i_s = v_s / R$.

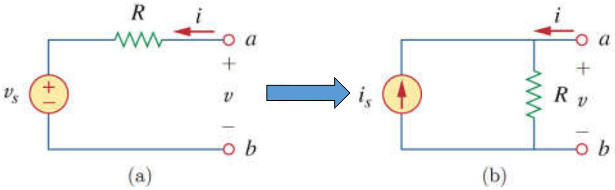


Figure 4.15 Transformation of independent sources.

$$i_s = \frac{v_s}{R}$$

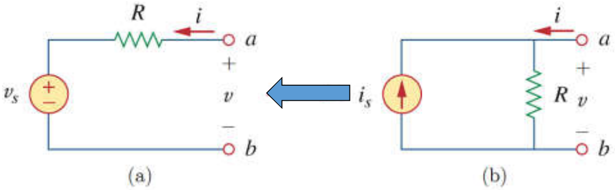
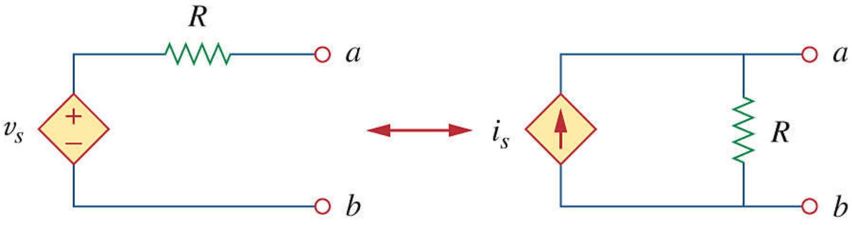


Figure 4.15 Transformation of independent sources .

$$v_s = i_s R$$

- Source transformation also applies to dependent sources, provided we carefully handle the dependent variable.



Figurev4.16 Transformation of dependent sources.

Example 4.6 Use source transformation to find v_o in the circuit of Fig. 4.17.

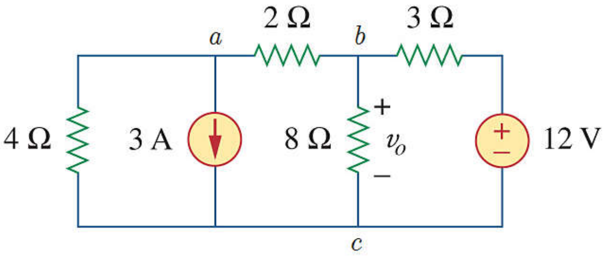


Figure 4.17

Solution : See Fig. 4.18. From the simpilified circuit, we have

$$v_o = 2 \times (8 \,||\, 2) = 2 \times \frac{8 \times 2}{8+2} = 3.2 \text{ (V)}$$

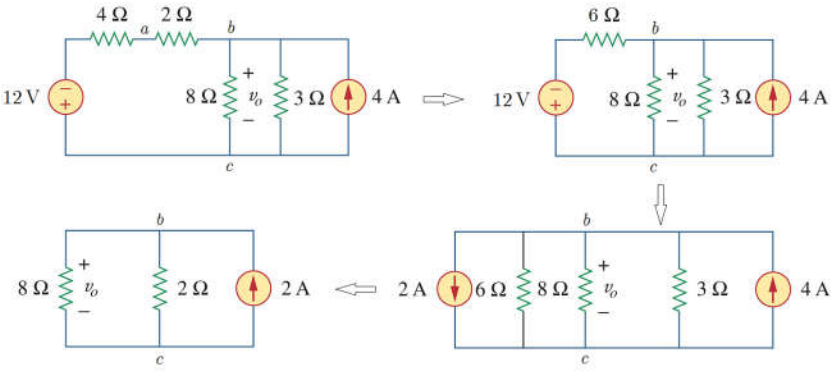


Figure 4.18.

Practice Problem Use source transformation to find i_x in the circuit shown in Fig. 4.22.

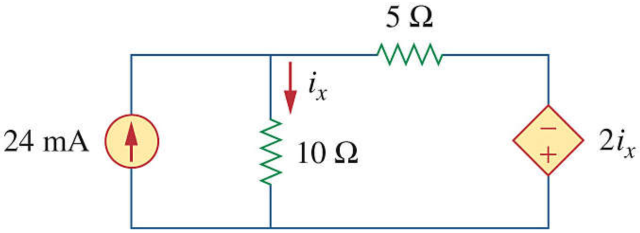


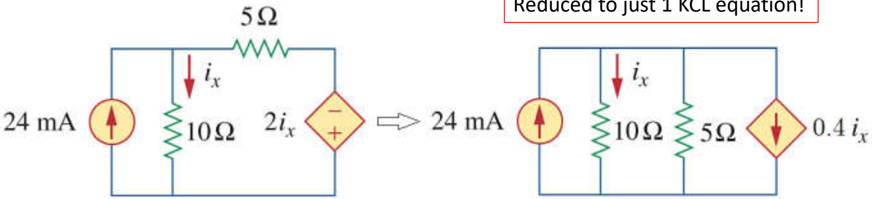
Figure 4.22

Solution : See the figure below. Apply KCL,

$$-24 + i_x + \frac{10i_x}{5} + 0.4i_x = 0$$

$$i_x = \frac{24}{3.4} \approx 7.06 \text{ (mA)}$$

Reduced to just 1 KCL equation!



For Practice Problem 4.7

4.5 Thevenin's Theorem

- At times in circuit analysis, we want to concentrate on what happens at a specific pair of terminals. For example, when we connect a load to a linear two-terminal circuit, we are interested primarily in the **voltage and current at the terminals** of the load. We have little or no interest in the effect that connecting the load has on voltages or currents elsewhere in the linear two-terminal circuit.

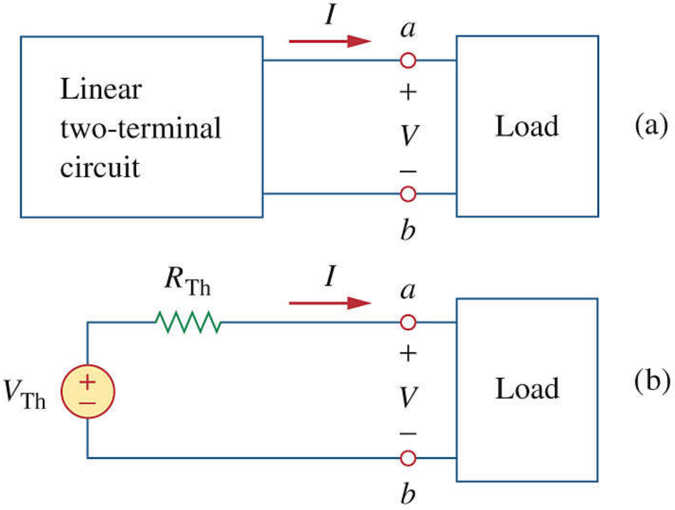


Figure 4.23 Replacing a linear two-terminal circuit by its Thevenin equivalent.

- Thevenin’s theorem provides a tool of simplifying circuit analysis.

Léon Charles Thévenin (30 March 1857, Meaux, Seine-et-Marne - 21 September 1926, Paris) was a French telegraph engineer who extended Ohm’s law to the analysis of complex electrical circuits.

Thevenin's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a voltage source V_{Th} in series with a resistor R_{Th}, where V_{Th} is the open-circuit voltage at the terminals and R_{Th} is the equivalent resistance at the terminals when the independent sources are turned off.

Proof (Section 4.7) :

Consider the circuit in Fig. 4.46(a). It is assumed that the circuit is resistive. Without loss of generality, we suppose the linear circuit conatins two independent voltage sources v_{s1} and v_{s2} and two independent current sources i_{s1} and i_{s2}.

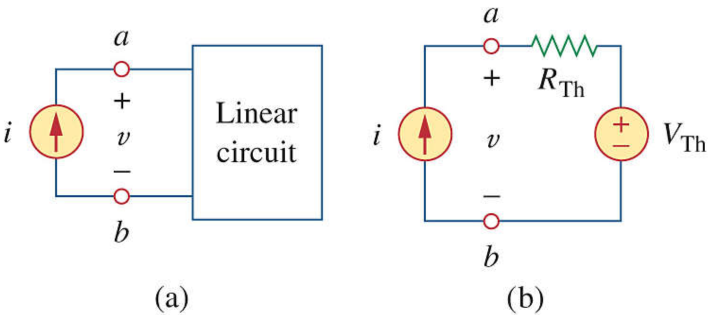


Figure 4.46 Derivation of Thevenin equivalent:
(a) a current-driven circuit, (b) its equivalent.

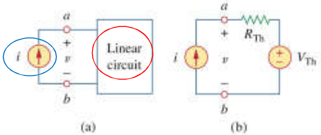


Figure 4.46 Derivation of Thevenin equivalent: (a) a current-driven circuit, (b) its equivalent.

By superposition, the terminal voltage is

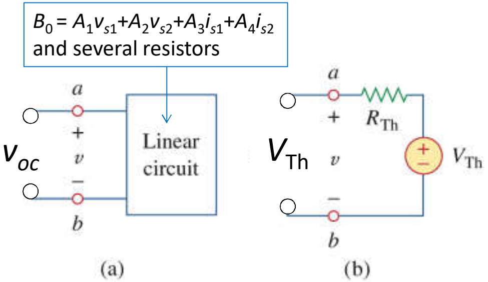
$$v = \underline{A_0 i} + \underline{A_1 v_{s1} + A_2 v_{s2} + A_3 i_{s1} + A_4 i_{s2}}$$

where A_0, A_1, A_2, A_3, and A_4 are constants.

Let $\underline{B_0 = A_1 v_{s1} + A_2 v_{s2} + A_3 i_{s1} + A_4 i_{s2}}$

$$v = \underline{A_0 i} + \underline{B_0} \qquad (1)$$

When i is turned off, $v = B_0$. Thus, B_0 is the open-circuit voltage v_{oc} of the linear circuit, which is the same as V_{Th}, so $B_0 = V_{Th}$.



$v = A_0 i + B_0$

Figure 4.46 Derivation of Thevenin equivalent: (a) a current-driven circuit, (b) its equivalent.

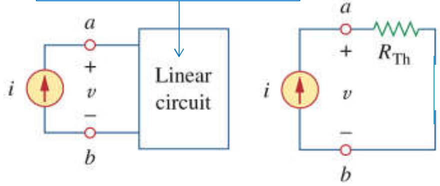
-<u>By figure</u>:
When i is turned off
→ $v = v_{oc} = V_{Th}$

-<u>By equation</u>:
When i is turned off
→ $v = B_0$

V_{Th} = open-circuit voltage of the linear circuit

Also, $B_0 = V_{Th}$

$B_0 = A_1 v_{s1} + A_2 v_{s2} + A_3 i_{s1} + A_4 i_{s2}$
and several resistors



$v = A_0 i + B_0$

Figure 4.46 Derivation of Thevenin equivalent: (a) a current-driven circuit, (b) its equivalent.

-<u>By figure</u>:

When B_0 is turned off

(a) Several resistors with all independent sources off → $\boldsymbol{v = R_{eq} i}$

(b) Previously, we just obtained $B_0 = V_{Th}$, so turning off V_{Th} → $\boldsymbol{v = R_{Th} i}$

-<u>By equation</u>:

When B_0 is turned off

→ $v = A_0 i$

$R_{Th} = R_{eq}$ with all independent sources off

Also, $A_0 = R_{Th}$

$B_0 = A_1 v_{s1} + A_2 v_{s2} + A_3 i_{s1} + A_4 i_{s2}$
and several resistors

$v = A_0 i + B_0$

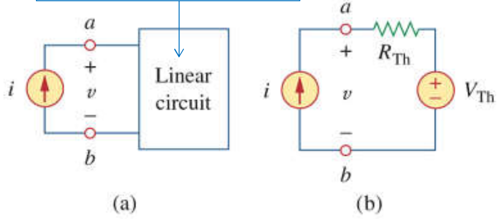


Figure 4.46 Derivation of Thevenin equivalent: (a) a current-driven circuit, (b) its equivalent.

We just obtained:

$B_0 = V_{Th}$

$A_0 = R_{Th}$

Case (a): $v = A_0 i + B_0 = R_{Th} i + V_{Th}$

Case (b): $v = R_{Th} i + V_{Th}$

Thus, they are equivalent!

When all the internal independent sources are turned off, $B_0 = 0,\ v = A_0 i,$ the linear circuit is equivalent to a resistor, whose resistance A_0 is the same as R_{Th}. Now Eq. (1) becomes $\boxed{v = R_{Th} i + V_{Th}}$, which expresses the voltage-current relation at terminals a and b of the circuit in Fig. 4.46(b). Thus, <u>the two circuits in Fig. 4.46 are equivalent.</u>

Fig. 4.24 shows the idea of finding the Thevenin voltage V_{Th} and the Thevenin resistance R_{Th}.

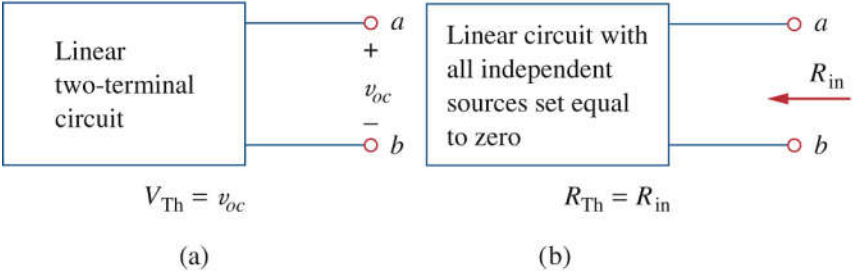


Figure 4.24 Finding V_{Th} and R_{Th}.

V_{Th} = open-circuit voltage of the linear circuit

R_{Th} = R_{eq} with all independent sources off

To apply the idea in finding the Thevenin resistance R_{Th}, we need to consider two cases.

1. If the network has no dependent sources, we turn off all independent sources. R_{Th} is the equivalent resistance looking between terminals a and b, as shown in Fig. 4.24 (b).

2. If the network has dependent sources, we turn off all independent sources. And we apply a voltage source v_o (or current source i_o) at terminals a and b and determine the resulting current i_o (or voltage v_o). Then $R_{Th} = v_o / i_o$, as shown in Fig. 4.25.

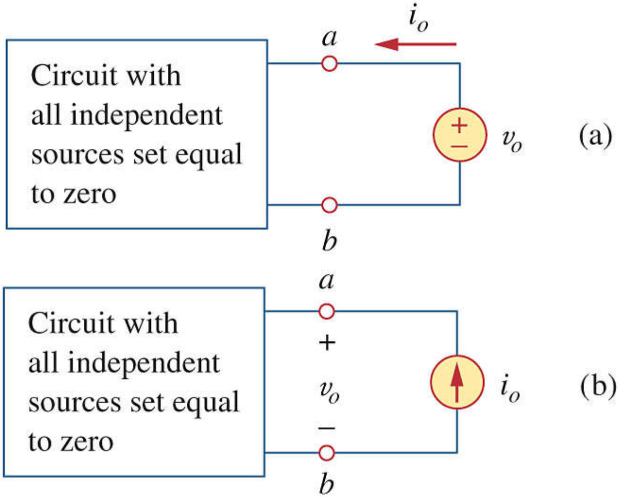


Figure 2.45 Finding R_{Th} when circuit has dependent sources. $R_{Th} = v_o/i_o$.

Example 4.8 Find the Thevenin equivalent circuit of the circuit shown in Fig. 4.27, to the left of the terminals *a* - *b*. then find the current through $R_L = 6, 16,$ and $36\ \Omega$.

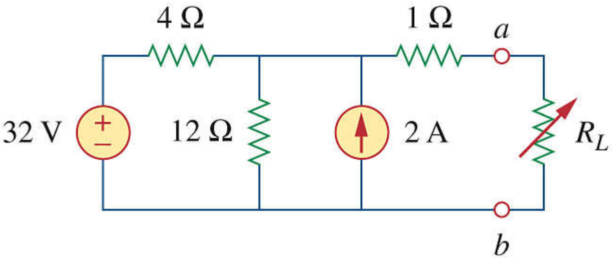


Figure 4.27

Solution :

Turn off all independent sources, the circuit becomes what is shown in Fig. 4.28(a). Thus,

$$R_{Th} = 4 \parallel 12 + 1 = \frac{4 \times 12}{4 + 12} + 1 = 4\ (\Omega)$$

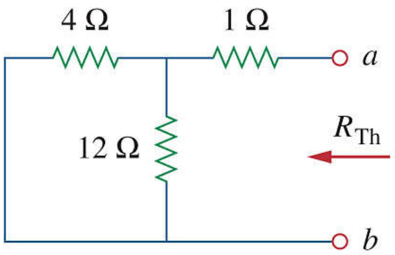


Figure 4.28(a)

Open-circuiting R_L, the circuit becomes the one in Fig. 4.28(b).

$$\frac{V_{Th}-32}{4}+\frac{V_{Th}}{12}-2=0 \Rightarrow V_{Th}=30 \text{ (V)}$$

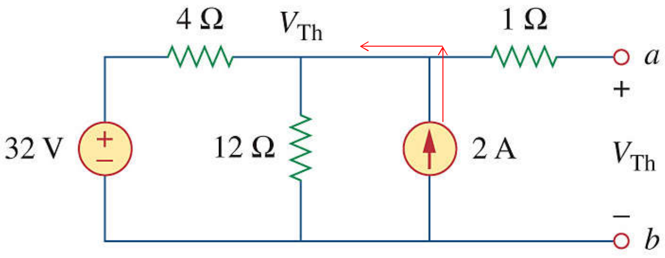


Figure 4.28(b)

The circuit in Fig. 4.27 can be replaced by the circuit shown in Fig. 4.29.

$$I_L = \frac{V_{Th}}{R_{Th} + R_L} = \frac{30}{4 + R_L} = 3, 1.5, 0.75 \text{ (A)}$$

when $R_L = 6, 16, 26\ \Omega.$

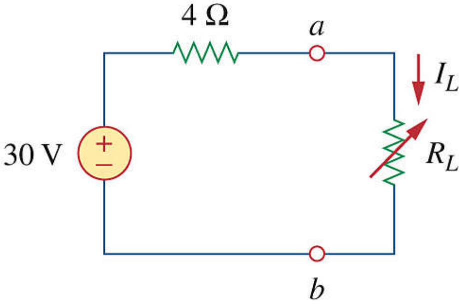


Figure 4.29

Practice Problem 4.9 Find the Thevenin equivalent circuit of the circuit in Fig. 4.34 to the left of the terminals.

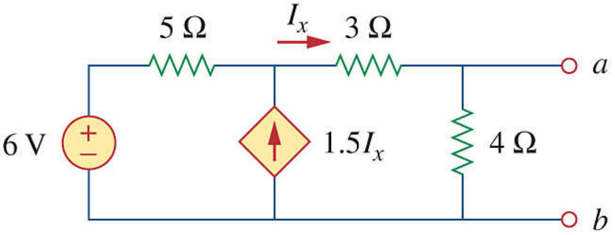
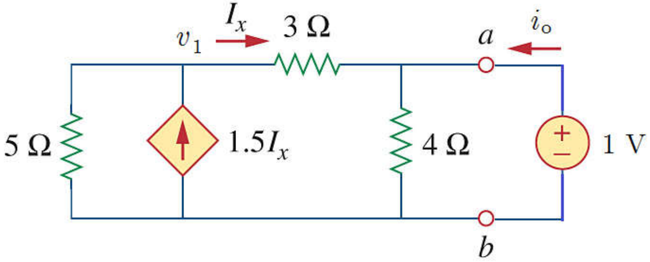


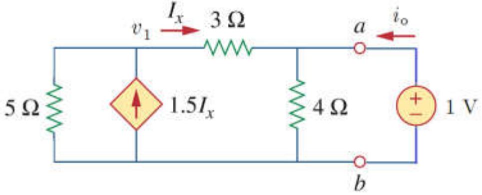
Figure 4.34

Solution :

To find R_{Th}, we turn off the independent voltage source and apply a test voltage source at the terminals a - b.



For Practice Problem 4.9: Find the Thevenin resistance



For Practice Problem 4.9: Find the Thevenin resistance

$$\begin{cases} I_x = \dfrac{v_1 - 1}{3} \\ \dfrac{v_1}{5} + I_x = 1.5 I_x \end{cases} \Rightarrow I_x = -2 \text{ (A)}$$

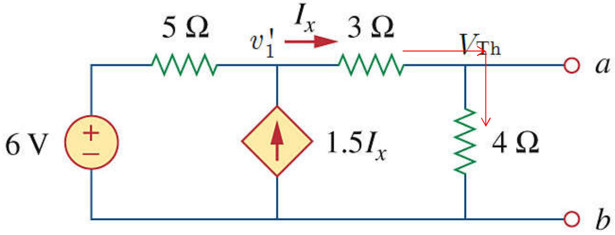
$$i_o = -I_x + \frac{1}{4} = \frac{9}{4} \text{ (A)}$$

$$R_{Th} = \frac{1}{i_o} = \frac{4}{9} \approx 0.44 \ (\Omega)$$

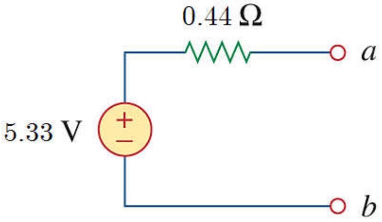
V_{Th} is the terminal voltage $V_{ab} = 4I_x$.

$$\begin{cases} I_x = \dfrac{v_1'}{3+4} \\ \dfrac{v_1' - 6}{5} + I_x = 1.5 I_x \end{cases} \Rightarrow I_x = \frac{4}{3} \text{ (A)}$$

$$V_{Th} = 4I_x = \frac{16}{3} \approx 5.33 \text{ (V)}$$



For Practice Problem 4.9: Find the Thevenin voltage.



For Practice Problem 4.9:
The Thevenin Equivalent.

It often occurs that R_{Th} takes a negative value. In this case, the negative resistance implies that the circuit is supplying power. This is possible in a circuit with dependent sources. Example 4.10 will illustrates this.

Example 4.10 Determine the Thevenin equivalent of the circuit in Fig. 4.35(a) at terminals a - b.

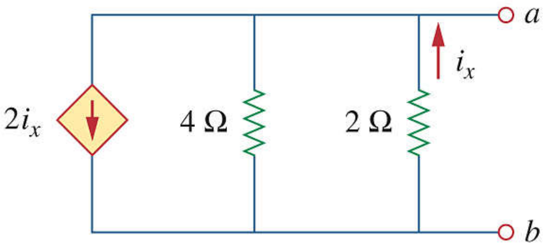


Figure 4.35(a)

Solution :

$$\begin{cases} v_o = -2i_x \\ 2i_x + \dfrac{v_o}{4} = i_x + i_o \end{cases} \Rightarrow v_o = -4i_o$$

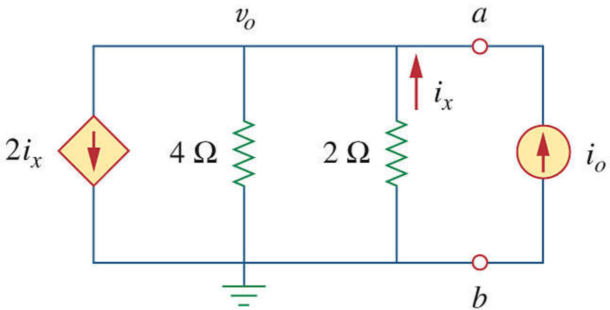


Figure 4.35(b)

$$R_{Th} = \frac{v_o}{i_o} = -4\ (\Omega)$$

$$V_{Th} = 0$$

The negative value of R_{Th} tells us that the circuit is supplying power. Of course, the resistors in the circuit cannot supply power; it is the dependent source that supplies the power.

$V_{Th} = 0$

Method 1: $V_{Th} = B_0 = A_1 v_{s1} + A_2 v_{s2} + A_3 i_{s1} + A_4 i_{s2}$,
where v_{s1}, v_{s2}, i_{s1}, and i_{s2} are independent sources
Thus, no independent sources → $B_0 = 0$ → $V_{Th} = 0$

Method2: $V_{Th} = V_{oc}$

$2i_x = i_x + i_4$

$i_4 = i_x$

$V_{oc} = -4i_x = -2i_x$

$i_x = 0$

$V_{oc} = 0$

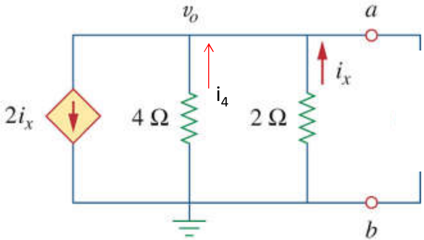


Figure 4.35(b)

4.6 Norton’s Theorem

- Norton’s theorem is similar to Thevenin’s theorem.

Edward Lawry Norton (28 July 1898, Rockland, Maine–28 January 1983, Chatham, New Jersey) was an accomplished Bell Labs engineer and scientist famous for developing the concept of the Norton equivalent circuit.

Norton's theorem states that a linear two-terminal circuit can be replaced by an equivalent circuit consisting of a currrent source I_N in parallel with a resistor R_N, where I_N is the short-circuit current through the terminals and R_N is the equivalent resistance at the terminals when the independent sources are turned off.

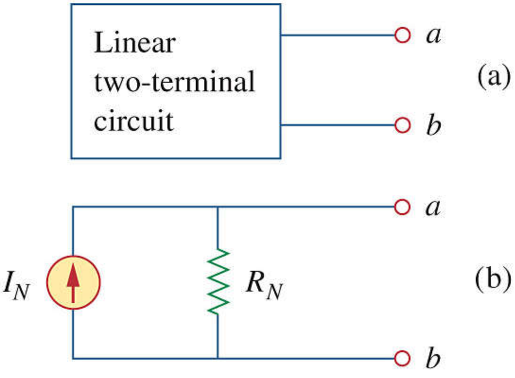
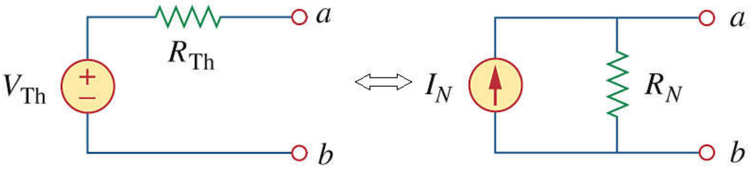


Figure 4.37 (a) Original circuit,
(b) Norton equivalent circuit.

A Norton equivalent circuit consists of an independent current source in parallel with the Norton equivalent resistance. We can derive it from a Thevenin equivalent circuit simply by making a source transformation.



From what we know about the source transformation, the Thevenin and Norton resistances are equal and the Norton current equals the Thevenin voltage divided by the Thevenin resistance; that is,

$$R_N = R_{Th}$$

$$I_N = \frac{V_{Th}}{R_{Th}}$$

It is evident that the short-circuit current in Fig. 4.37(b) is I_N. This must be the same short-circuit current from terminal a to b in Fig. 4.37(a), since the two circuits are equivalent. Thus,

$$I_N = i_{sc}$$

shown in Fig. 4.38.

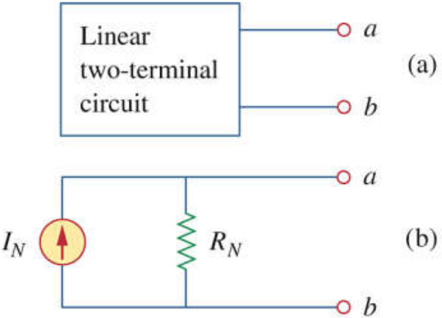


Figure 4.37 (a) Original circuit, (b) Norton equivalent circuit.

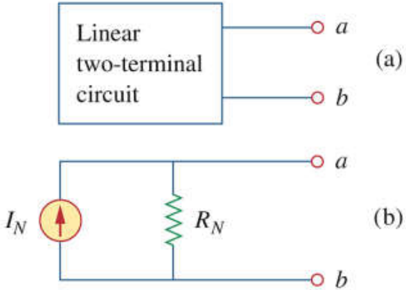


Figure 4.37 (a) Original circuit, (b) Norton equivalent circuit.

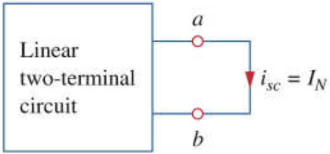
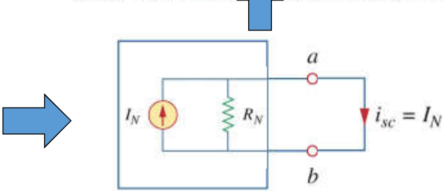


Figure 4.38 Finding the Norton current I_N.



From $I_N = V_{Th} / R_{Th}$, we have

$$R_{Th} = \frac{V_{Th}}{I_N}$$

But $V_{Th} = v_{oc}$ and $I_N = i_{sc}$, so

$$R_{Th} = \frac{v_{oc}}{i_{sc}}$$

Thus the Thevenin or Norton resistance is the ratio of the open-circuit voltage to the short-circuit current.

Practice Problem 4.11 Find the Norton equivalent circuit for the circuit in Fig. 4.42, at terminals *a* - *b*.

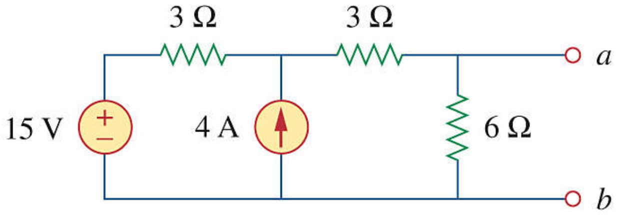


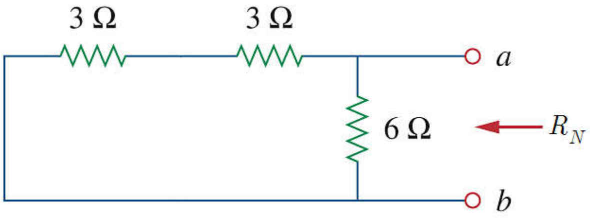
Figure 4.42

Solution :

Turn off the voltage and current sources,

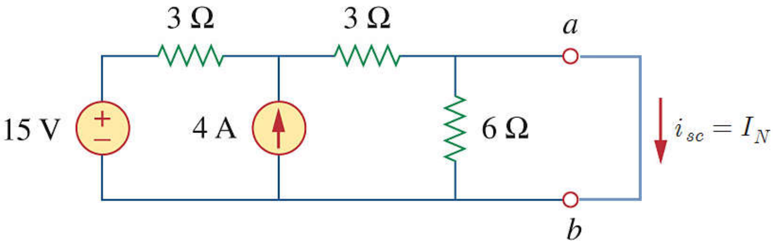
the Norton resistance is

$$R_N = (3+3) \parallel 6 = 6 \parallel 6 = 3\ (\Omega)$$

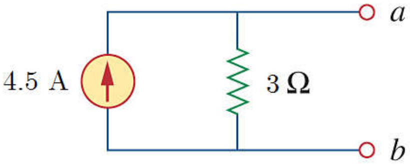


Short-circuit terminals a and b, ignore the 6-Ω resistor, apply superposition principle,

$$I_N = I'_N + I''_N = \frac{15}{3+3} + 4 \times \frac{3}{3+3} = \frac{9}{2} = 4.5 \text{ (A)}$$



The Norton equivalent is shown below.



4.8 Maximum Power Transfer

- In many practical situations, a circuit is designed to provide power to a load.
- **I. Maximum power efficiency**
  Power utility systems are concerned with the generation, transmission, and distribution of large quantities of electric power. This type of systems emphasizes the efficiency of the power transfer.

- **II. Maximum power transfer**
  In communication and instrumentation systems, the amount of power being transferred is **small**, so the efficiency is not a primary concern. It is often desirable to transmit as much of power as possible to the load. (E.g., a cell phone)
- We now consider maximum power transfer in systems with the aid of the circuit shown in Fig. 4.48.

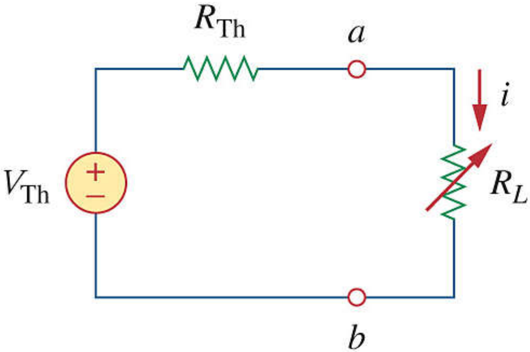
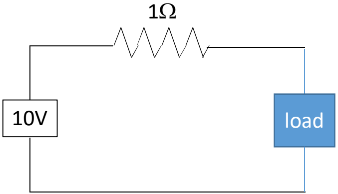


Figure 4.48 The circuit used for maximum power transfer.

Circuit types	I. Power utility systems	II. Communication & instrumentation systems
Power scale	Large quantities of electric power (generation, etc.)	Small amount of power is being transferred
Primary concern of power	Efficiency of the power transfer	Transmit as much of power as possible to the load

Assuming the same source for easy comparison

I. Power utility systems



Care electricity bills

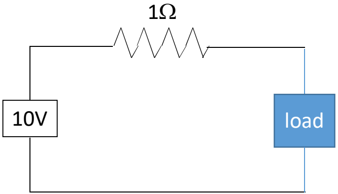
$R_L = 9\Omega$

Power generated from the source = 10W

Power delivered to the load = 9W

Efficiency of power transfer **90%** (loss=10%)

II. Communication & instrumentation systems



Care supplying sufficient power

$R_L = 1\Omega$

Power generated from the source = 50W

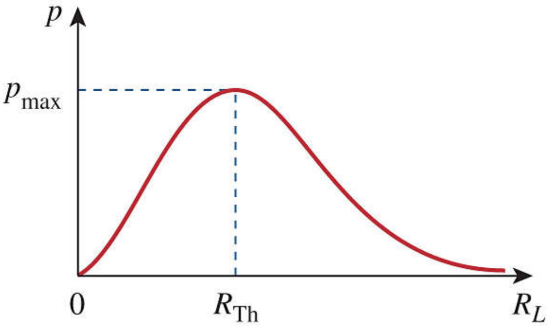
Power delivered to the load = **25W**

Efficiency of power transfer 50% (loss=50%)

The power delivered to the load is

$$p = i^2 R_L = \left(\frac{V_{Th}}{R_{Th} + R_L} \right)^2 R_L$$

as sketched in Fig. 4.49.



The maximum power theorem states that the maximum power is transferred to the load when the load resistance equals the Thevenin resistance as seen from the load ($R_L = R_{Th}$).

Proof : Let

$$\frac{dp}{dR_L} = V_{Th}^2 \frac{R_{Th} - R_L}{(R_{Th} + R_L)^3} = 0$$

We have $R_L = R_{Th}$.

$$\left.\frac{d^2 p}{dR_L^2}\right|_{R_L = R_{Th}} = V_{Th}^2 \left.\frac{2R_L - 4R_{Th}}{(R_{Th} + R_L)^4}\right|_{R_L = R_{Th}} = -\frac{V_{Th}^2}{8R_{Th}^3}$$

$\left.\frac{d^2 p}{dR_L^2}\right|_{R_L = R_{Th}} < 0$ implies that at $R_L = R_{Th}, p$ takes

the maximum value.

The maximum power transferred is

$$p_{\max} = \frac{V_{Th}^2}{4R_{Th}}$$

It should be noted that delivering the maximum power to the load results in significant internal losses.

Example 4.13 Find the value of R_L for maximum power transfer in the circuit of Fig. 4.50. Find the maximum power.

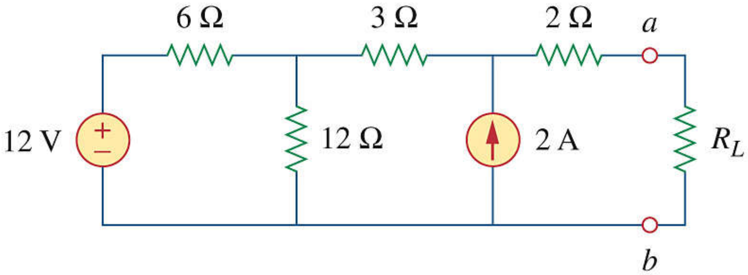


Figure 4.50

Solution :

$$R_{Th} = 2 + 3 + 6 \parallel 12 = 5 + \frac{6 \times 12}{6 + 12} = 9\ (\Omega)$$

For maximum power transfer,

$$R_L = R_{Th} = 9\ \Omega$$

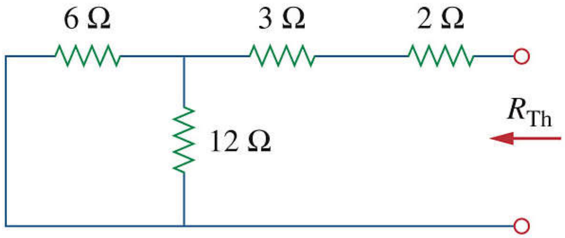


Figure 4.51(a)

$$V_{Th} = 12 \times \frac{12}{6+12} + 2 \times (3 + 6 \,||\, 12) = 22 \text{ (V)}$$

The maximum power transferrd

$$p_{\max} = \frac{V_{Th}^2}{4R_{Th}} = \frac{22^2}{4 \times 9} = \frac{121}{9} \approx 13.44 \text{ (W)}$$

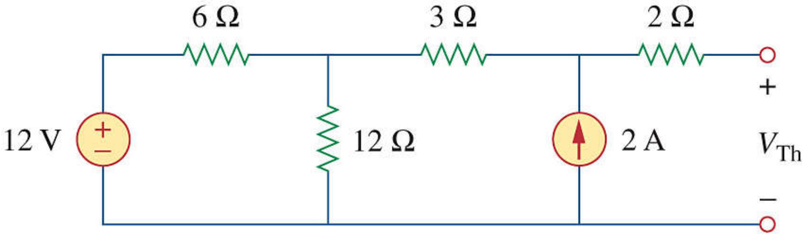


Figure 4.51(b)

Source Modeling (Section 4.10)

A practical voltage source is modeled by an ideal voltage source v_s in series with an internal (or a source) resistance R_s, a practical current source is modeled by an ideal current source i_s in parallel with an internal resistance R_p, as shown in Fig. 4.58.

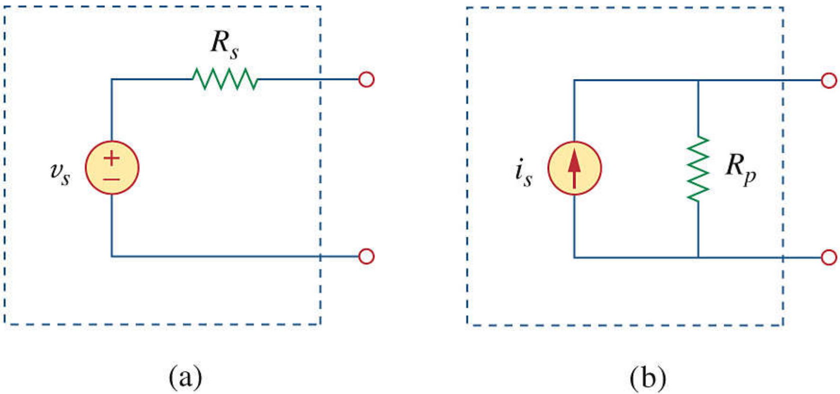
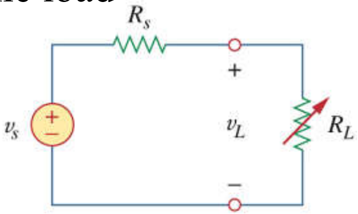


Figure 4.58 (a) Practical voltage source, (b) practical current source.

In Fig. 4.59(a), a practical voltage source is connected to a variable load. The load voltage is

$$v_L = \frac{R_L}{R_s + R_L} v_s$$



This equation tells us that (1) v_L will be constant if $R_s = 0$ or at least $R_s << R_L$. In other words, the smaller R_s is compared with R_L, the closer the voltage source is to being ideal.

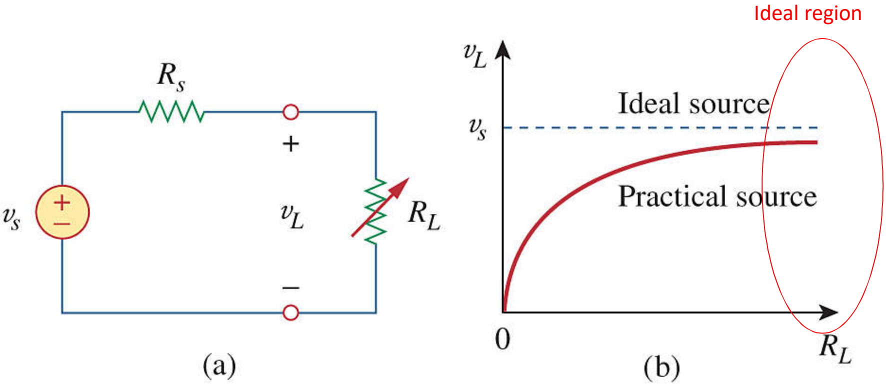
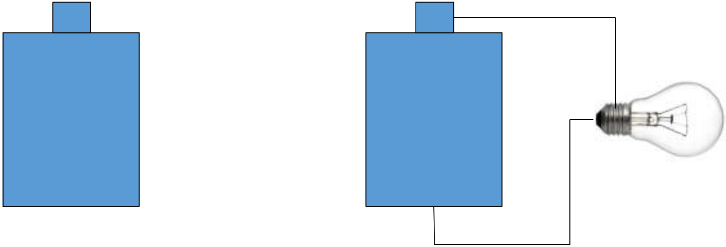


Figure 4.59 (a) Practical voltage connected to a load R_L, (b) load voltage decreases as R_L decreases.

(2) When the voltage source is open-circuited, $v_{oc} = v_s$. Thus, v_s may be regarded as the *unloaded source voltage*. The connection of R_L causes the terminal voltage to drop in magnitude; this is known as the *loading effect*.

9V

Slightly < 9V



Same or different voltages?

To find v_s and R_s, we follow the procedure illustrated in Fig. 4.61.

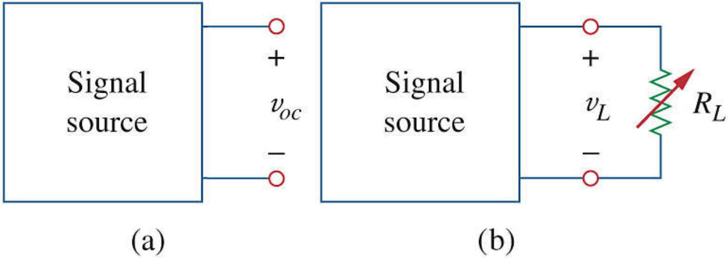


Figure 4.61 (a) Measuring the open-circuit voltage, (b) measuring the load voltage.

First, we measure the open-circuit voltage

v_{oc} and set

$v_s = v_{oc}$

Then, we connect a variable load R_L across the terminals. We adjust R_L until $v_L = v_{oc} / 2$. At this point, we disconnect R_L and measure it. We set

$R_s = R_L$

Example 4.16 The terminal voltage of a voltage source is 12 V when connected to a 2-W load. When the load is disconnected, the terminal voltage rises to 12.4 V. (a) Calculate the source voltage v_s and internal resistance R_s. (b) determine the voltage when an 8-Ω load is connected to the source.

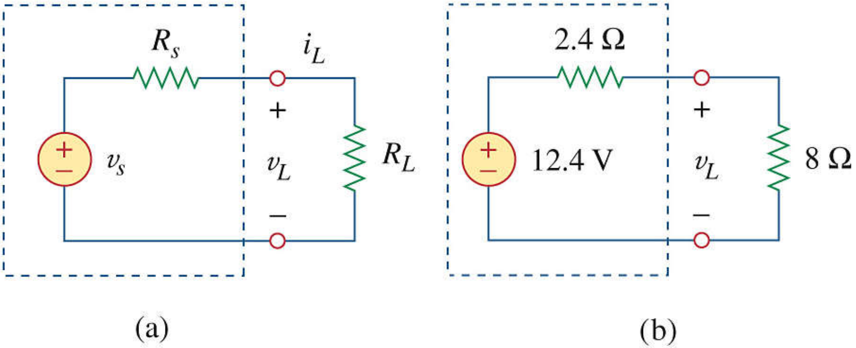


Figure 4.62

Solution :

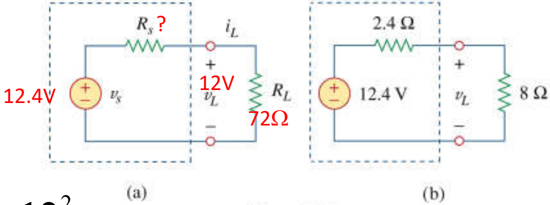


Figure 4.62

$$\text{(a) } v_s = v_{oc} = 12.4 \text{ V}$$

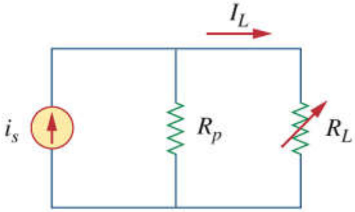
$$p_L = \frac{v_L^2}{R_L} \Rightarrow R_L = \frac{v_L^2}{p_L} = \frac{12^2}{2} = 72\ (\Omega)$$

$$v_L = v_s \frac{R_L}{R_s + R_L} \Rightarrow R_s = R_L\left(\frac{v_s}{v_L} - 1\right) = 72 \times \left(\frac{12.4}{12} - 1\right) = 2.4\ (\Omega)$$

$$\text{(b) } v_L = v_s \frac{R_L}{R_s + R_L} = 12.4 \times \frac{8}{2.4 + 8} \approx 9.54\ (\text{V})$$

In Fig. 4.60(a), a practical current source is connected to a variable load. The load current is

$$i_L = \frac{R_p}{R_p + R_L} i_s$$



This equation tells us that (1) i_L will be constant if $R_p = \infty$ or at least $R_p >> R_L$. In other words, the larger R_p is compared with R_L, the closer the current source is to being ideal.

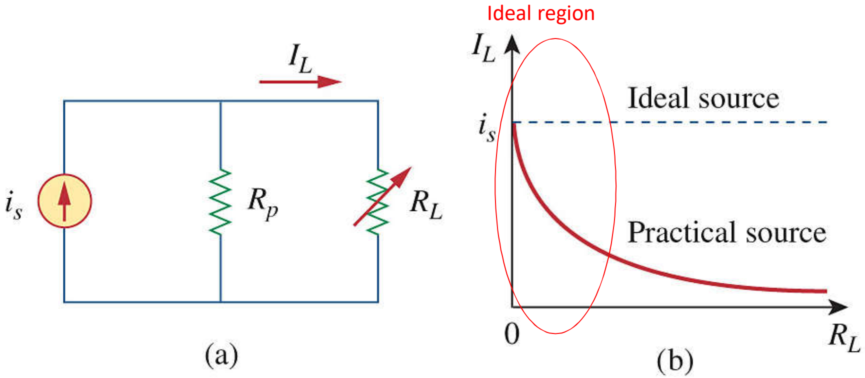


Figure 4.60 (a) practical current connected to a load R_L,
(b) load current decreases as R_L increases.

(2) When the current source is short-circuited, $i_{sc} = i_s$. Thus, i_s may be regarded as the *unloaded source current*. The connection of R_L causes the terminal current to drop in magnitude; this is known as the *loading effect*.