Chapter 8 Second-order Circuits

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8.1 Introduction

- In this chapter, we consider circuits containing two storage elements, known as second-order circuits.
- Examples of second-order circuits are shown in Fig. 8.1.

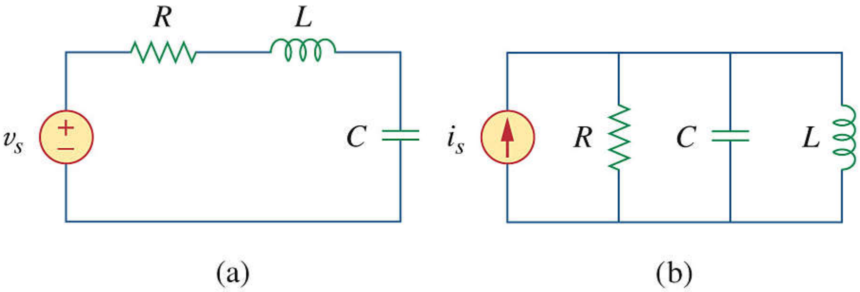


Figure 8.1 Typical examples of second-order circuits: (a) series RLC circuit, (b) parallel RLC circuit.

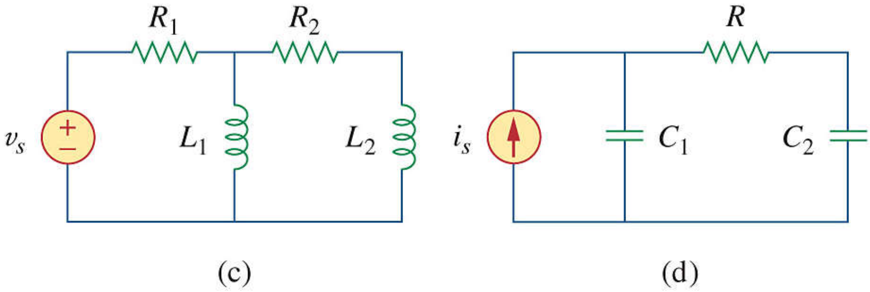


Figure 8.1 Typical examples of second-order circuits: (c) RLL circuit, (d) RCC circuit.

8.2 Finding Initial and Final Values

Example 8.1 The switch in Fig. 8.2 has been closed for a long time. It is open at $t = 0$. Find: (a) $i(0^+)$, $v(0^+)$, (b) $di(0^+)/dt$, $dv(0^+)/dt$, (c) $i(\infty)$, $v(\infty)$.

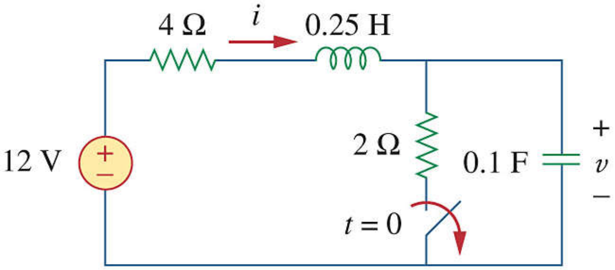


Figure 8.2

(a) $t = 0^-$

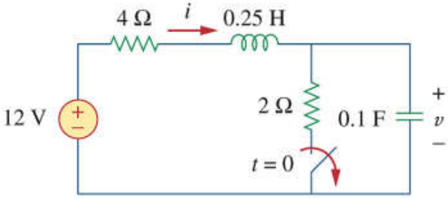


Figure 8.2

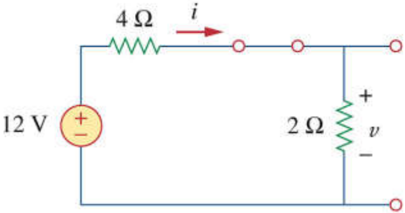


Figure 8.3 (a) Equivalent circuit of that in Fig. 8.2 for $t = 0^-$.

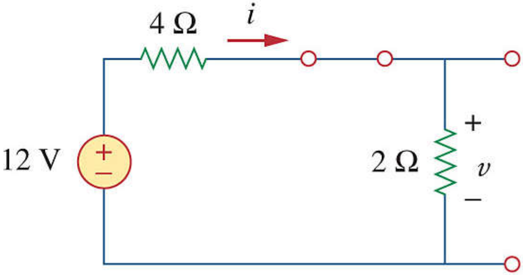


Figure 8.3 (a) Equivalent circuit of that in Fig. 8.2 for $t = 0^-$.

Solution :

(a)

$$i(0^+) = i(0^-) = \frac{12}{4+2} = 2 \text{ (A)}$$

$$v(0^+) = v(0^-) = 2i(0^-) = 4 \text{ (V)}$$

(b) $t = 0^+$

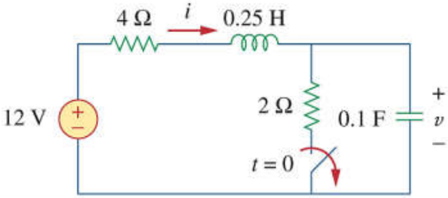


Figure 8.2

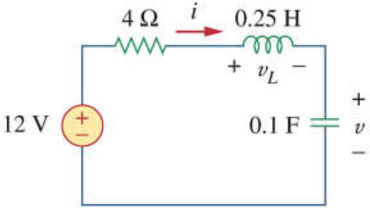


Figure 8.3 (a) Equivalent circuit of that in Fig. 8.2 for $t = 0^+$.

Represent dv/dt or di/dt in terms of v_C and/or i_L

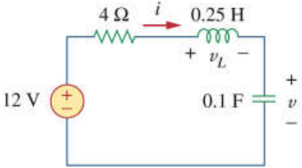


Figure 8.3 (a) Equivalent circuit of that in Fig. 8.2 for $t = 0^+$.

(b)

$$\begin{cases} i = 0.1\dfrac{dv}{dt} \\ 12 = 4i + 0.25\dfrac{di}{dt} + v \end{cases} \Rightarrow \begin{cases} \dfrac{dv}{dt} = \dfrac{i}{0.1} \\ \dfrac{di}{dt} = \dfrac{12 - 4i - v}{0.25} \end{cases}$$

$$dv(0^+)/dt = i(0^+)/0.1 = 2/0.1 = 20 \text{ (V/s)}$$

$$di(0^+)/dt = \left[12 - 4i(0^+) - v(0^+)\right]/0.25$$

$$= \left[12 - 4\times 2 - 4\right]/0.25 = 0 \text{ (A/s)}$$

(c) $t \rightarrow \infty$

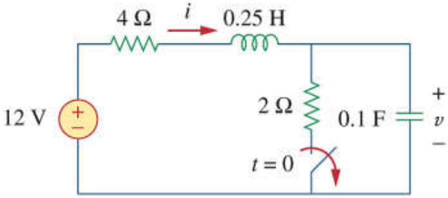


Figure 8.2

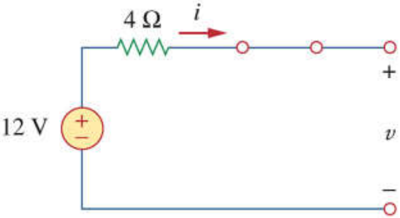


Figure 8.3 (c) Equivalent circuit of that in Fig. 8.2 for t = infinity.

(c)

$$i(\infty) = 0$$

$$v(\infty) = 12 \text{ (V)}$$

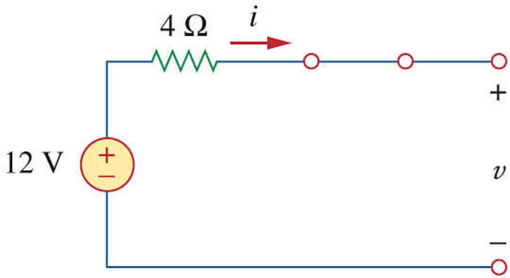


Figure 8.3 (c) Equivalent circuit of that in Fig. 8.2 for t = infinity.

8.3 The Source-Free Series *RLC* Circuit

Consider the circuit shown in Fig. 8.8. The circuit is being excited by the energy initially stored in the capacitor and inductor.

At $t = 0$,

$$v(0) = V_0, \quad i(0) = I_0$$

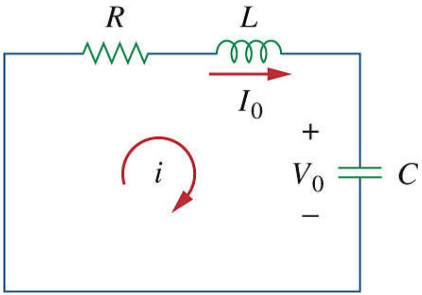


Figure 8.8 A source-free series RLC circuit.

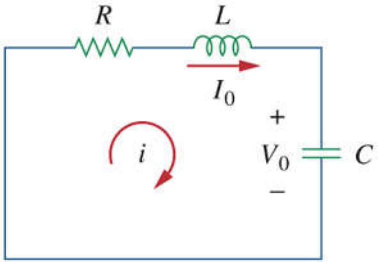


Figure 8.8 A source-free series RLC circuit.

$$iR + L\frac{di}{dt} + v = 0$$

$$iR + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i\,dt = 0$$

Represent the equation in terms of only one parameter *i*

$$\frac{di}{dt}R + L\frac{d^2 i}{dt^2} + \frac{1}{C}i = 0$$

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = 0$$

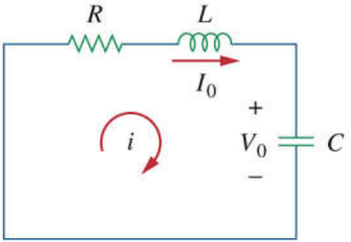


Figure 8.8 A source-free series RLC circuit.

The initial conditions are

$$i(0^+) = i(0^-) = I_0$$

$$i'(0^+) = -\frac{1}{L}\left(i(0^+)R + v(0^+)\right) \quad \longleftarrow \quad iR + L\frac{di}{dt} + v = 0$$

$$= -\frac{1}{L}\left(i(0^-)R + v(0^-)\right)$$

$$= -\frac{1}{L}\left(I_0 R + V_0\right)$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$s = \frac{-R/L \pm \sqrt{(R/L)^2 - 4\times 1\times\left(1/(LC)\right)}}{2\times 1}$$

$$= -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

where

$$\alpha = \frac{R}{2L}$$: neper frequency (damping factor),

Np/s (nepers per second)

$$\omega_0 = \frac{1}{\sqrt{LC}}$$: resonant frequency (undamped natural frequency), rad/s

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2},\ s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}:$$

natural frequencies, Np/s

Solution 1: Overdamped

There are three types of solutions:

1. If $\alpha > \omega_0$, $s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$, $s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$, we have the *overdamped* case,

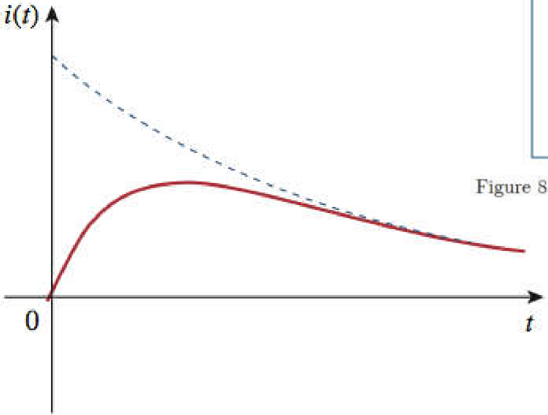
$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$S_1 < 0$, $S_2 < 0$
$S_1 \neq S_2$

where

$$A_1 = \frac{i'(0^+) - s_2 i(0^+)}{s_1 - s_2}$$

$$A_2 = \frac{s_1 i(0^+) - i'(0^+)}{s_1 - s_2}$$



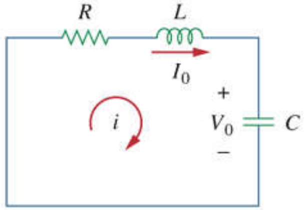


Figure 8.8 A source-free series RLC circuit.

1. no oscillation
2. region 1: $i(t)$ changes due to initially stored energy in L and C
3. region 2: steady state value should be 0 due to “zero input response”
4. $\alpha \uparrow$ (more damping) ➔ reaches steady state faster

Solution 2: Critically damped

2. If $\alpha = \omega_0$, $s_1 = s_2 = -\alpha$, we have the *critically damped* case,

$$i(t) = (B_1 t + B_2)e^{-\alpha t}$$

where

$$B_1 = i'(0^+) + \alpha i(0^+)$$

$$B_2 = i(0^+)$$

$S_1 < 0$, $S_2 < 0$
$S_1 = S_2$

$$\frac{d^2 i}{dt^2}+\frac{R}{L}\frac{di}{dt}+\frac{1}{LC}i=0$$

$$\Downarrow \quad \alpha=\frac{R}{2L} \qquad \omega_0=\frac{1}{\sqrt{LC}} \qquad \alpha=\omega_0$$

$$\frac{d^2i}{dt^2}+2\alpha\frac{di}{dt}+\alpha^2 i=0$$

or

$$\frac{d}{dt}\left(\frac{di}{dt}+\alpha i\right)+\alpha\left(\frac{di}{dt}+\alpha i\right)=0 \qquad (8.16)$$

If we let

$\downarrow$ Reduced to 1st order DE

$$f=\frac{di}{dt}+\alpha i \qquad (8.17)$$

then Eq. (8.16) becomes

$$\frac{df}{dt}+\alpha f=0$$

which is a first-order differential equation with solution $f = A_1 e^{-\alpha t}$, where A_1 is a constant. Equation (8.17) then becomes

$$\frac{di}{dt} + \alpha i = A_1 e^{-\alpha t}$$

or

$$e^{\alpha t}\frac{di}{dt} + e^{\alpha t}\alpha i = A_1 \tag{8.18}$$

This can be written as

$$\frac{d}{dt}(e^{\alpha t} i) = A_1 \tag{8.19}$$

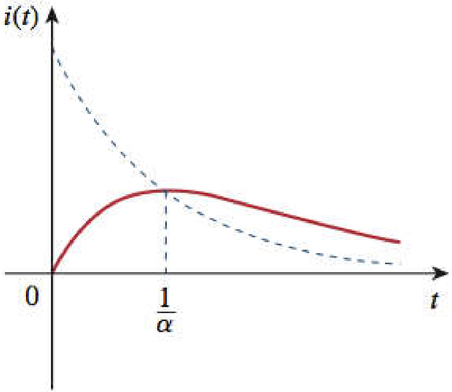
Integrating both sides yields

$$e^{\alpha t} i = A_1 t + \underline{A_2}$$

Integration constant

or

$$i = (A_1 t + A_2)e^{-\alpha t} \tag{8.20}$$



1. no oscillation
2. region 1: $i(t)$ changes due to initially stored energy in L and C
3. region 2: decays all the way to zero
4. $\alpha \uparrow$ (more damping) ➔ reaches steady state faster

Solution 3: Underdamped

3. If $\alpha < \omega_0$, $s_1 = -\alpha + j\omega_d$, $s_2 = -\alpha - j\omega_d$,

where $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$, which is called the *damping frequency*, we have the *underdamped* case,

$$i(t) = e^{-\alpha t}(C_1 \cos \omega_d t + C_2 \sin \omega_d t)$$

where

$$C_1 = i(0^+)$$

$$C_2 = \frac{i'(0^+) + \alpha i(0^+)}{\omega_d}$$

S_1, S_2 are complex conjugates

$$
\begin{aligned}
i(t) &= A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} \\
&= e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t})
\end{aligned} \tag{8.23}
$$

Using Euler's identities,

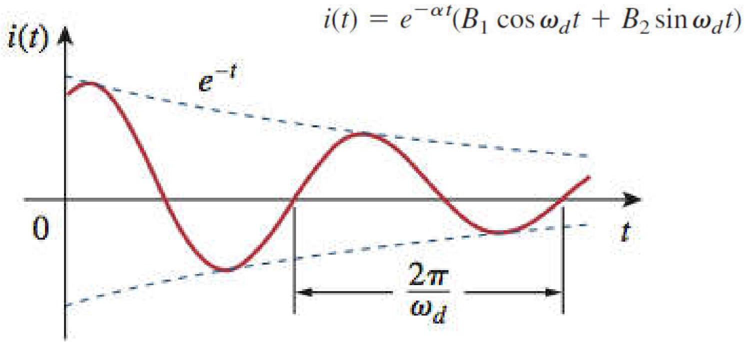
$$
e^{j\theta} = \cos\theta + j\sin\theta, \qquad e^{-j\theta} = \cos\theta - j\sin\theta \tag{8.24}
$$

we get

$$
\begin{aligned}
i(t) &= e^{-\alpha t}[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)] \\
&= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t]
\end{aligned} \tag{8.25}
$$

Replacing constants $(A_1 + A_2)$ and $j(A_1 - A_2)$ with constants B_1 and B_2, we write

$$
\boxed{i(t) = e^{-\alpha t}(B_1 \cos\omega_d t + B_2 \sin\omega_d t)} \tag{8.26}
$$



1. Oscillatory response
2. $\alpha \uparrow$ (more damping) ➔ reaches steady state faster
3. α: envelope
4. ω_d: oscillation frequency

Once the inductor current $i(t)$ is found,

other circuit quantities can be found,

$$v_R(t) = i(t)R$$

$$v_L(t) = L\frac{di(t)}{dt}$$

$$v_C(t) = \frac{1}{C}\int_0^t i(t)dt + v_C(0)$$

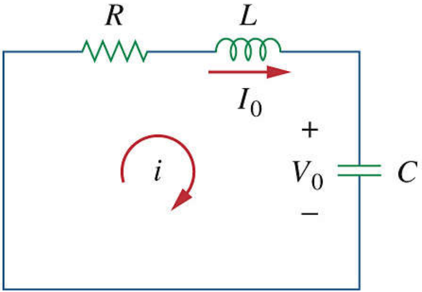


Figure 8.8 A source-free series RLC circuit.

Practice Problem 8.4 The circuit in Fig. 8.12 has reached steady state at $t = 0^-$. If the make-before-break switch moves to position b at $t = 0$, calculate $i(t)$ for $t > 0$.

Solution :

$$i(0^+) = i(0^-) = \frac{50}{10} = 5 \text{ (A)}$$

$$v(0^+) = v(0^-) = 0 \text{ (V)}$$

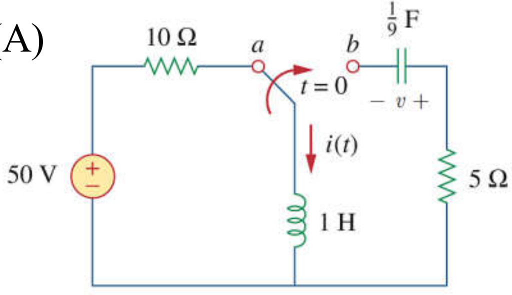


Figure 8.12

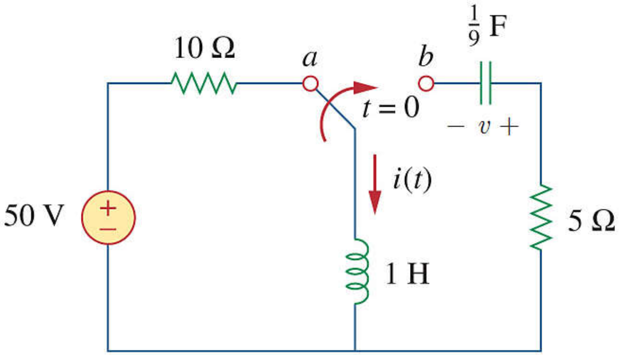


Figure 8.12

$$1\times\frac{di(t)}{dt}+i(t)\times 5+v(t)=0,\ i(t)=\frac{1}{9}\frac{dv(t)}{dt}$$

$$i'(0^+)=-5i(0^+)-v(0^+)=-5\times 5-0$$

$$=-25\ \text{(A/s)}$$

$$1\times\frac{d^2i(t)}{dt^2}+\frac{di(t)}{dt}\times 5+\frac{dv(t)}{dt}=0$$

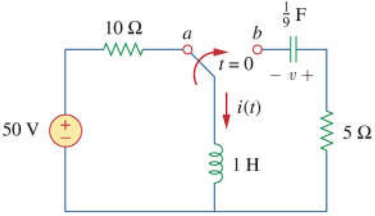


Figure 8.12

$$1\times\frac{d^2i(t)}{dt^2}+\frac{di(t)}{dt}\times 5+\frac{1}{1/9}i(t)=0$$

$$\frac{d^2i(t)}{dt^2}+5\frac{di(t)}{dt}+9i(t)=0$$

$$s = \frac{-5 \pm \sqrt{5^2 - 4\times 1\times 9}}{2\times 1} = \frac{-5 \pm j\sqrt{11}}{2}$$

$$i(t) = e^{-2.5t}(A_1 \cos\frac{\sqrt{11}}{2}t + A_2 \sin\frac{\sqrt{11}}{2}t)$$

$$i(0^+) = A_1 \Rightarrow A_1 = i(0^+) = 5$$

$$i'(0^+) = -2.5A_1 + \frac{\sqrt{11}}{2}A_2 \Rightarrow A_2 = \frac{i'(0^+) + 2.5A_1}{\sqrt{11}/2}$$

$$= \frac{-25 + 2.5\times 5}{\sqrt{11}/2} = -\frac{25}{\sqrt{11}}$$

$$i(t) \approx e^{-2.5t}(5\cos 1.6583t - 7.5378\sin 1.6583t)\ \text{(A)}$$

Steps for source-free 2nd order circuit

1. Plot the circuit at $t<0$, find initial conditions, $i(0^+)$ **and** $v(0^+)$
2. Plot the circuit at $t>0$, express di/dt or dv/dt in terms of i_L and v_c, find initial conditions $di(0^+)/dt$ **or** $dv(0^+)/dt$
3. Express the circuit in 2nd order D.E. with only one parameter (either i or v) and solve it.
4. Solve the coefficients using initial conditions.

8.4 The Source-Free Parallel *RLC* Circuit

Consider the circuit shown in Fig. 8.13. The circuit is being excited by the energy initially stored in the capacitor and inductor.

At $t = 0$,

$$v(0) = V_0, \quad i(0) = I_0$$

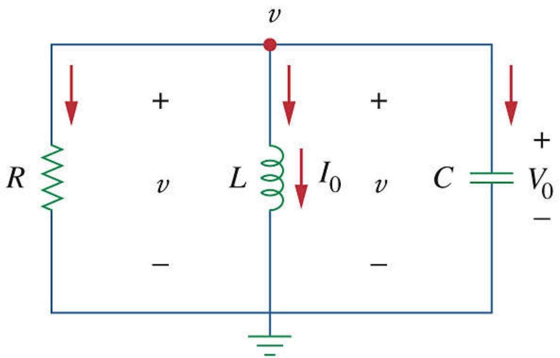


Figure 8.13 A source-free parallel RLC circuit.

$$\frac{v}{R} + i + C\frac{dv}{dt} = 0$$

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v dt + C\frac{dv}{dt} = 0$$

$$\frac{1}{R}\frac{dv}{dt} + \frac{1}{L}v + C\frac{d^2v}{dt^2} = 0$$

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

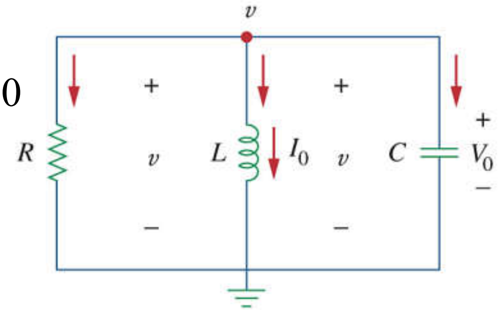


Figure 8.13 A source-free parallel RLC circuit.

The initial conditions are

$$v(0^+) = v(0^-) = V_0$$

$$v'(0^+) = -\frac{1}{C}\left(v(0^+)/R + i(0^+)\right) \longleftarrow \frac{v}{R} + i + C\frac{dv}{dt} = 0$$

$$= -\frac{1}{C}\left(v(0^-)/R + i(0^-)\right)$$

$$= -\frac{1}{C}\left(V_0/R + I_0\right)$$

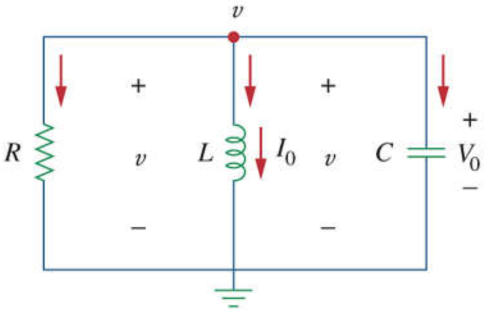


Figure 8.13 A source-free parallel *RLC* circuit.

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

$$s = \frac{-1/(RC) \pm \sqrt{1/(RC)^2 - 4\times 1\times \left(1/(LC)\right)}}{2\times 1}$$

$$= -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

where

$$\alpha = \frac{1}{2RC}: \text{ neper frequency (damping factor),}$$

Np/s (nepers per second)

$$\omega_0 = \frac{1}{\sqrt{LC}}: \text{resonant frequency (undamped}$$

natural frequency), rad/s

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}:$$

natural frequencies, Np/s

There are three types of solutions:

1. If $\alpha > \omega_0$, $s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$, we have the *overdamped* case,

$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

2. If $\alpha = \omega_0$, $s_1 = s_2 = -\alpha$, we have the *critically damped* case,

$v(t) = (B_1 t + B_2) e^{-\alpha t}$

3. If $\alpha < \omega_0$, $s_1 = -\alpha + j\omega_d$, $s_2 = -\alpha - j\omega_d$, we have the *underdamped* case,

$v(t) = e^{-\alpha t}(C_1 \cos \omega_d t + C_2 \sin \omega_d t)$

Once the capacitor voltage $v(t)$ is found, other circuit quantities can be found,

$$i_R(t) = \frac{v(t)}{R}$$

$$i_L(t) = \frac{1}{L}\int_0^t v(t)dt + i_L(0)$$

$$i_C(t) = C\frac{dv(t)}{dt}$$

Example 8.6 Find $v(t)$ for $t > 0$ in the *RLC* circuit of Fig. 8.15.

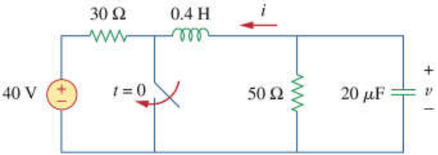


Figure 8.15

Solution :

$$v(0^+) = v(0^-) = 40 \times \frac{50}{30+50} = 25 \text{ (V)}$$

$$i(0^+) = i(0^-) = -\frac{40}{30+50} = -0.5 \text{ (A)}$$

Step 1

$$v'(0^+) = -\frac{1}{C}\left(v(0^+)/R + i(0^+)\right)$$

$$= -\frac{1}{20\times 10^{-6}}\left(25/50 + (-0.5)\right) = 0 \text{ (V/s)}$$

Step 2

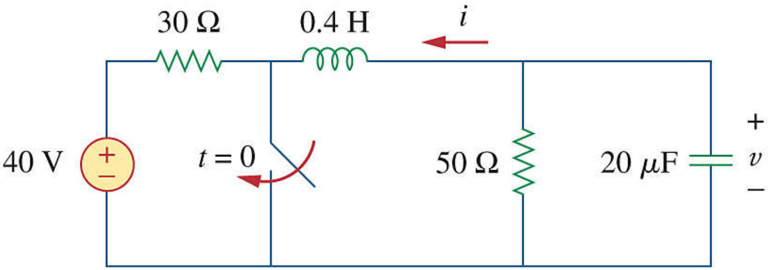


Figure 8.15

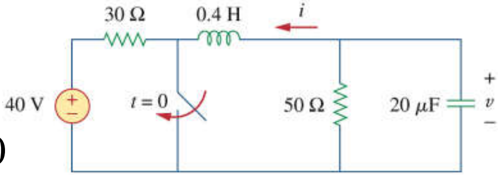


Figure 8.15

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0$$

$$\frac{1}{LC}=\frac{1}{0.4\times 20\times 10^{-6}}=125000$$

$$\frac{1}{RC}=\frac{1}{50\times 20\times 10^{-6}}=1000$$

$$\frac{d^2v}{dt^2}+1000\frac{dv}{dt}+125000v=0$$

$$s_{1,2}=\frac{-1000\pm\sqrt{1000^2-4\times 1\times 125000}}{2\times 1}$$

Step 3

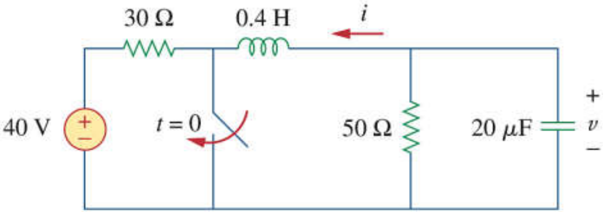


Figure 8.15

$$s_1 \approx -146.4466,\ s_2 \approx -853.5534$$

$$v(t) = A_1 e^{-146.4466t} + A_2 e^{-853.5534t}$$

Step 3

$$v(0^+) = A_1 + A_2 = 25$$

$$v'(0^+) = -146.4466 A_1 - 853.5534 A_2 = 0$$

$$A_1 \approx 30.1777,\ A_2 \approx -5.1777$$

$$v(t) \approx 30.18e^{-146.45t} - 5.18e^{-853.55t} \text{ (V)}$$

Step 4

8.5 Series *RLC* Circuit with Step Input

Consider the circuit in Fig. 8.18. For $t > 0$,

$$\begin{cases} V_s = iR + L\dfrac{di}{dt} + v \\ i = C\dfrac{dv}{dt} \end{cases}$$

$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = V_s$$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}V_s$$

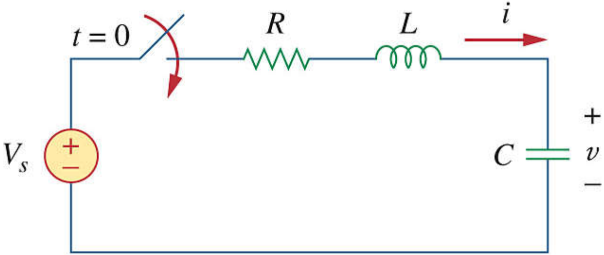


Figure 8.18 Step voltage applied to a series RLC circuit.

It can be shown that the solution has three possible forms:

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + V_s$$

(Overdamped)

$$v(t) = (A_1 + A_2 t)e^{-\alpha t} + V_s$$

(Critically damped)

$$v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) + V_s$$

(Underdamped)

Example 8.7 For the circuit in Fig. 8.19, find $v(t)$ for $t > 0$. Consider these cases: $R = 5\ \Omega$, $R = 4\ \Omega$, $R = 1\ \Omega$.

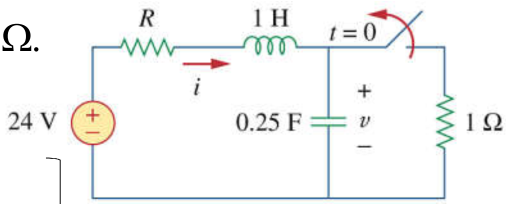


Figure 8.19

Solution :

$$i(0^+) = i(0^-) = \frac{24}{R+1}$$

$$v(0^+) = v(0^-) = 24 \times \frac{1}{R+1}$$

Step 1

$$i(t) = 0.25\frac{dv(t)}{dt} \Rightarrow v'(0^+) = \frac{1}{0.25}i(0^+)$$

Step 2

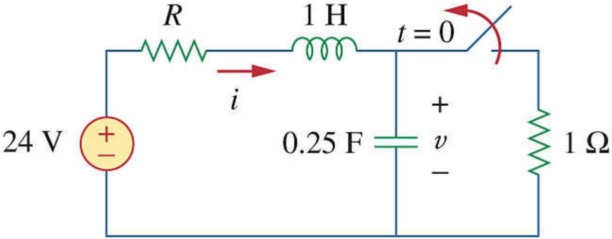


Figure 8.19

$$\left.\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{LC}v=\frac{1}{LC}V_s\right\}\text{ Step 3, 4, \& 5}$$

$$\boxed{\text{(a) } R = 5\ \Omega}$$

$$i(0^+)=4\text{ A},\ v(0^+)=4\text{ V},\ v'(0^+)=16\text{ V/s}$$

$$\frac{d^2v}{dt^2}+5\frac{dv}{dt}+4v=96$$

$$s^2+5s+4=0\Rightarrow s_1=-1,\ s_2=-4$$

$$\underline{v_n(t)=A_1e^{-t}+A_2e^{-4t}}$$

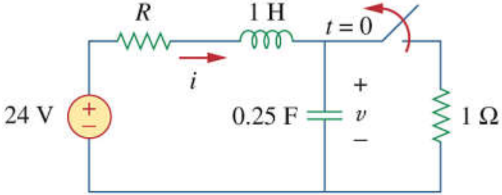


Figure 8.19

$$v_f(t) = B \Rightarrow B = 96/4 = 24$$

$$v(t) = v_n(t) + v_f(t) = A_1 e^{-t} + A_2 e^{-4t} + 24$$

$$v(0^+) = A_1 + A_2 + 24 = 4$$

$$v'(0^+) = -A_1 - 4A_2 = 16$$

$$A_1 = -\frac{64}{3},\ A_2 = \frac{4}{3}$$

$$v(t) = -\frac{64}{3}e^{-t} + \frac{4}{3}e^{-4t} + 24 \text{ (V)}$$

$$i(t) = \frac{16}{3}e^{-t} - \frac{4}{3}e^{-4t} \text{ (A)}$$

$\boxed{\text{(b) } R = 4\ \Omega}$

$$i(0^+) = 4.8 \text{ A}, v(0^+) = 4.8 \text{ V}, v'(0^+) = 19.2 \text{ V/s}$$

$$\frac{d^2v}{dt^2} + 4\frac{dv}{dt} + 4v = 96$$

$$s^2 + 4s + 4 = 0 \Rightarrow s_1 = s_2 = -2$$

$$v_n(t) = (A_1 + A_2 t)e^{-2t}$$

$$v_f(t) = B \Rightarrow B = 96/4 = 24$$

$$v(t) = v_n(t) + v_f(t) = (A_1 + A_2 t)e^{-2t} + 24$$

$$v(0^+) = A_1 + 24 = 4.8$$

$$v'(0^+) = A_2 - 2A_1 = 19.2$$

$$A_1 = A_2 = -19.2$$

$$v(t) = (-19.2 - 19.2t)e^{-2t} + 24 \text{ (V)}$$

$$i(t) = 4.8(1 + 2t)e^{-2t} \text{ (A)}$$

$$\boxed{\text{(c) } R = 1\ \Omega}$$

$$i(0^+) = 12 \text{ A}, v(0^+) = 12 \text{ V}, v'(0^+) = 48 \text{ V/s}$$

$$\frac{d^2v}{dt^2} + \frac{dv}{dt} + 4v = 96$$

$$s^2 + s + 4 = 0 \Rightarrow s_{1,2} = -\frac{1}{2} \pm j\frac{\sqrt{15}}{2}$$

$$v_n(t) = e^{-t/2}(A_1 \cos\frac{\sqrt{15}}{2}t + A_2 \sin\frac{\sqrt{15}}{2}t)$$

$$v_f(t) = B \Rightarrow B = 96/4 = 24$$

$$v(t) = v_n(t) + v_f(t)$$

$$= e^{-t/2}(A_1 \cos\frac{\sqrt{15}}{2}t + A_2 \sin\frac{\sqrt{15}}{2}t) + 24$$

$$v(0^+) = A_1 + 24 = 12$$

$$v'(0^+) = -\frac{1}{2}A_1 + \frac{\sqrt{15}}{2}A_2 = 48$$

$$A_1 = -12,\ A_2 = \frac{84}{\sqrt{15}} \approx 21.689$$

$$v(t) = e^{-t/2}(-12\cos\frac{\sqrt{15}}{2}t + \frac{84}{\sqrt{15}}\sin\frac{\sqrt{15}}{2}t) + 24\ (\mathrm{V})$$

$$i(t) = e^{-t/2}(12\cos\frac{\sqrt{15}}{2}t + \frac{12}{\sqrt{15}}\sin\frac{\sqrt{15}}{2}t)\ (\mathrm{A})$$

	(a)	(b)	(c)
R	5Ω	4Ω	1Ω
α	2.5	2	0.5
v_f	24V	24V	24V
	Overdamped	Critically damped	Underdamped

Figure 8.20 plots the responses for the three cases. From this figure, we observe that the critically damped response approaches the step input of 24 V the fastest.

Overall, the speed to reach steady state value:
Critically damped > Overdamped > Underdamped

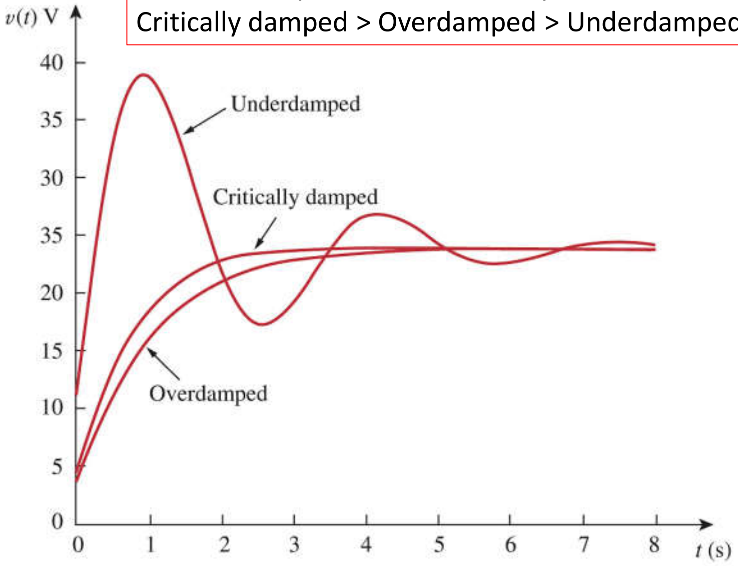


Figure 8.20

Steps for 2^{nd} order circuit with *step input*

1. Plot the circuit at $t<0$, find initial conditions, $i(0^+)$ **and** $v(0^+)$
2. Plot the circuit at $t>0$, express di/dt or dv/dt in terms of i_L and v_c, find initial conditions $di(0^+)/dt$ **or** $dv(0^+)/dt$
3. Express the circuit in 2^{nd} order D.E. with only one parameter (either i or v) and solve it.
4. Find steady state values $i(\infty)$, $v(\infty)$ (i.e., forced response i_f, v_f)
5. Solve the coefficients using initial conditions.

8.6 Parallel *RLC* Circuit with Step Input

Consider the circuit in Fig. 8.22. We want to find i due to a sudden application of a dc current. For $t > 0$,

$$I_s = \frac{v}{R} + i + C\frac{dv}{dt}, \quad v = L\frac{di}{dt}$$

$$LC\frac{d^2 i}{dt^2} + \frac{L}{R}\frac{di}{dt} + i = I_s$$

$$\frac{d^2 i}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = \frac{1}{LC}I_s$$

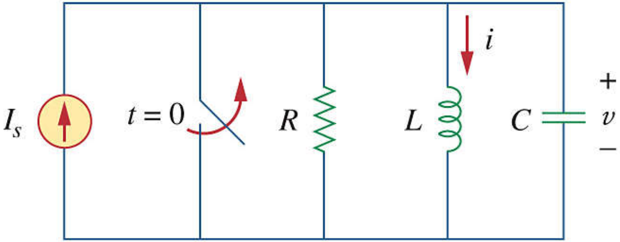


Figure 8.22 Parallel RLC circuit with an applied current.

It can be shown that the solution has three possible forms:

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + I_s$$

(Overdamped)

$$i(t) = (A_1 + A_2 t)e^{-\alpha t} + I_s$$

(Critically damped)

$$i(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t) + I_s$$

(Underdamped)

Practice Problem 8.8 Find $i(t)$ and $v(t)$ for $t > 0$ in the circuit of Fig. 8.24.

Solution :

Step1

$$i(0^+) = i(0^-) = 0$$

$$v(0^+) = v(0^-) = 0$$

Step2

$$v(0^+) = 5\frac{di(0^+)}{dt} \Rightarrow i'(0^+) = \frac{1}{5}v(0^+) = 0$$

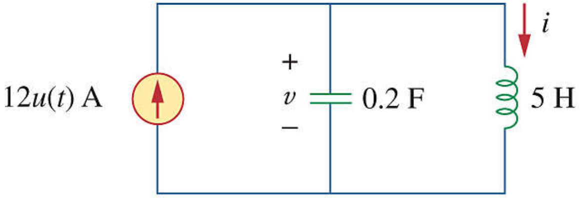


Figure 8.24 An LC circuit.

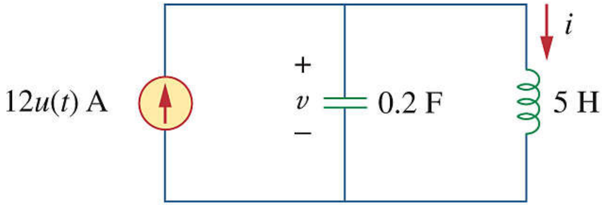


Figure 8.24 An LC circuit.

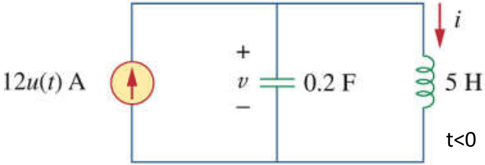


Figure 8.24 An *LC* circuit.

Step3

$$12 = 0.2\frac{dv}{dt} + i, \quad v = 5\frac{di}{dt}$$

$$\frac{d^2 i}{dt^2} + i = 12$$

$$s^2 + 1 \Rightarrow s_{1,2} = \pm j$$

$$i_n(t) = A_1 \cos t + A_2 \sin t$$

Step4

$$i_p(t) = 12$$

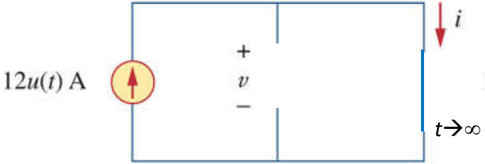


Figure 8.24 An *LC* circuit.

$$i(t) = i_n(t) + i_p(t) = A_1 \cos t + A_2 \sin t + 12$$

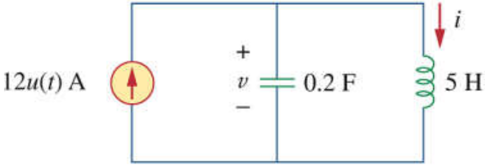


Figure 8.24 An LC circuit.

Step5

$$i(0^+) = A_1 + 12 = 0$$

$$i'(0^+) = -A_2 = 0$$

$$A_1 = -12,\ A_2 = 0$$

$$i(t) = -12\cos t + 12 = 12(1-\cos t)\ \text{(A)}$$

$$v(t) = 5\frac{di(t)}{dt} = 60\sin t\ \text{(V)}$$

8.7 General Second-Order Circuits

Practice Problem 8.10 For $t > 0$, obtain $v_o(t)$ in the circuit of Fig. 8.32. (*Hint* : First find v_1 and v_2.)

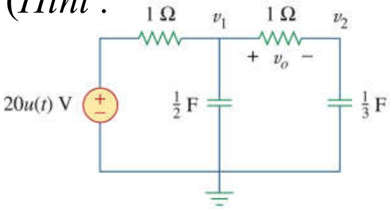


Figure 8.32 An *RCC* circuit.

Solution :

Step1

$$v_1(0^+) = v_2(0^+) = 0$$

Step2

$$\frac{20 - v_1(0^+)}{1} = \frac{1}{2}\frac{dv_1(0^+)}{dt} + \frac{v_1(0^+) - v_2(0^+)}{1}$$

$$v_1'(0^+) = 2[20 - 2v_1(0^+) + v_2(0^+)] = 40 \text{ (V/s)}$$

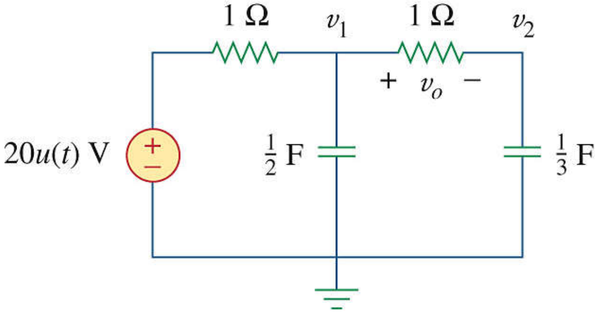


Figure 8.32 An RCC circuit.

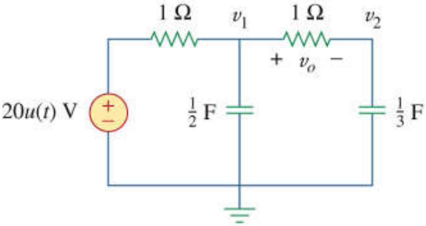


Figure 8.32 An RCC circuit.

Step4

$$v_1(\infty) = v_2(\infty) = 20 \text{ (V)}$$

Step3

$$\frac{20 - v_1}{1} = \frac{1}{2}\frac{dv_1}{dt} + \frac{v_1 - v_2}{1}, \quad \frac{v_1 - v_2}{1} = \frac{1}{3}\frac{dv_2}{dt}$$

$$\frac{d^2 v_1}{dt^2} + 7\frac{dv_1}{dt} + 6v_1 = 120$$

$$s^2 + 7s + 6 = 0 \Rightarrow s_1 = -1, s_2 = -6$$

$$v_{1h}(t) = A_1 e^{-t} + A_2 e^{-6t}$$

$$v_{1p}(t) = 20$$

Step5

$$v_1(t) = A_1 e^{-t} + A_2 e^{-6t} + 20$$

$$v_1(0^+) = A_1 + A_2 + 20 = 0$$

$$v_1'(0^+) = -A_1 - 6A_2 = 40$$

$$A_1 = -16,\ A_2 = -4$$

$$v_1(t) = -16e^{-t} - 4e^{-6t} + 20$$

Solve $v_2(t)$ using v_1 v_2 relation

$$v_2(t) = -24e^{-t} + 4e^{-6t} + 20$$

$$v_o(t) = v_1(t) - v_2(t) = 8e^{-t} - 8e^{-6t} \ \text{(V)}$$

8.10 Duality

- The concept of duality is a time-saving, effort-effective measure of solving circuit problems.
- Two circuits are said to be duals of one another if they are described **by the same characteristic equations with dual pairs interchanged**.
- Dual pairs are shown in Table 8.1.

TABLE 8.1 Dual Pairs

Resistance	Conductance
Inductance	Capacitance
Voltage	Current
Voltage source	Current source
Node	Mesh
Series path	Parallel path
Open circuit	Short circuit
KVL	KCL
Thevenin	Norton

Given a palnar circuit, we construct the dual circuit by taking the following steps:

1. Place a node at the center of each mesh of the given circuit. Place the reference node of the dual circuit outside the given circuit.
2. Draw lines between the nodes such that each line across an element. Replace the element by its dual.

3. To determine the polarity of voltage sources and direction of current sources, follow this rule: A voltage source that produces a positive (clockwise) mesh current has as its dual a current source whose reference direction is from the ground to the nonreference node. (vice versa) In case of doubt, one may verify the dual circuit by writing the nodal or mesh equations.

Example 8.14 Construct the dual of the circuit in Fig. 8.44.

Solution : See Fig. 8.45.

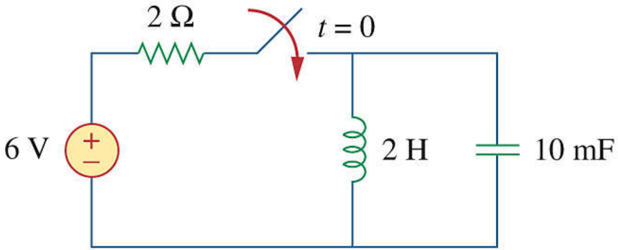


Figure 8.44

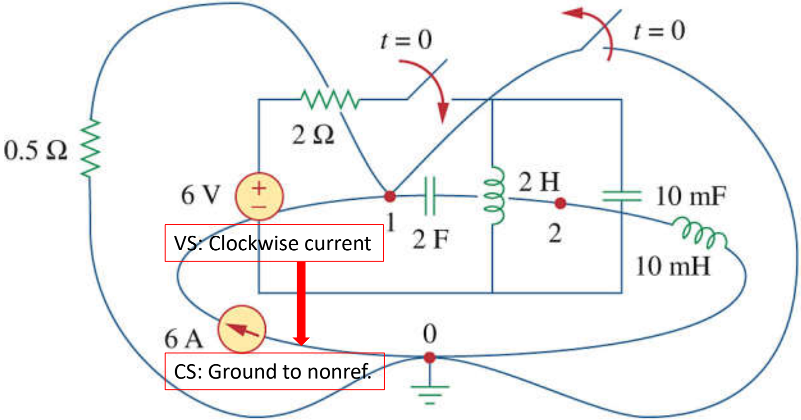


Figure 8.45(a) Construction of the dual circuit of Fig. 8.44.

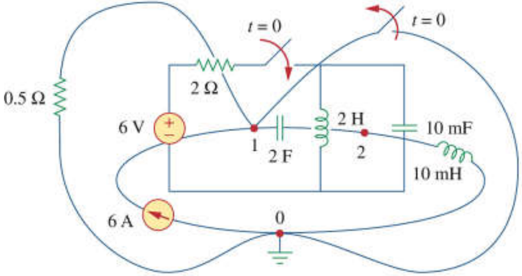


Figure 8.45(a) Construction of the dual circuit of Fig. 8.44.

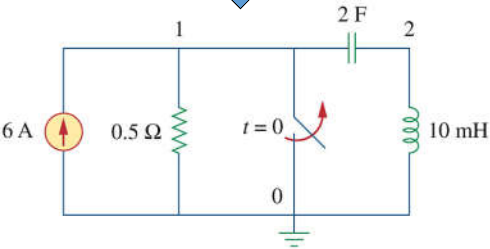


Figure 8.45(b) Dual circuit redrawn.

Example 8.15 Obtain the dual of the circuit in Fig. 8.48.

Solution : See Fig. 8.49.

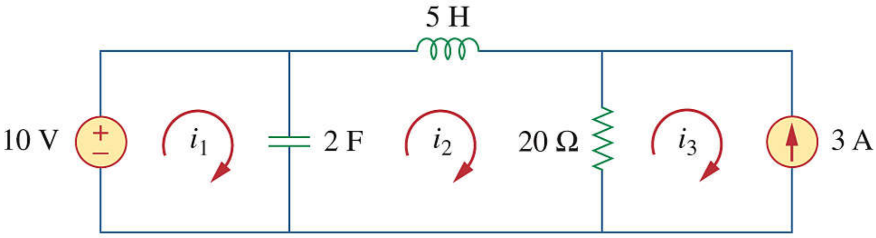


Figure 8.48

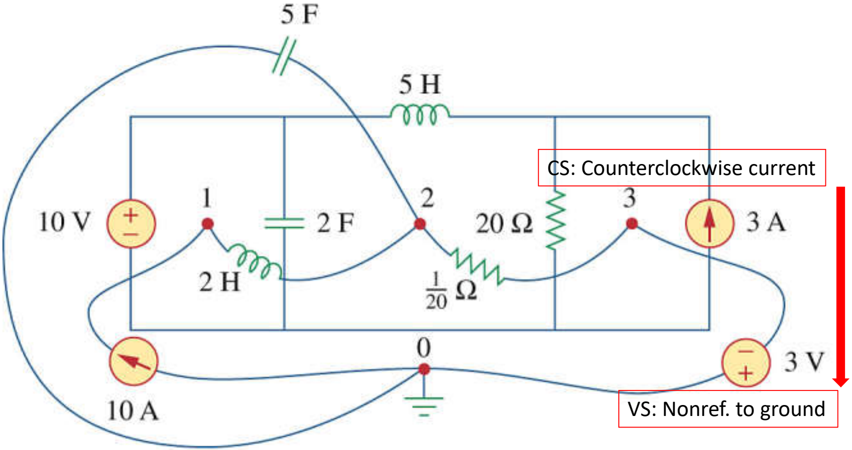


Figure 8.49(a) Construction of the dual circuit of Fig. 8.48.

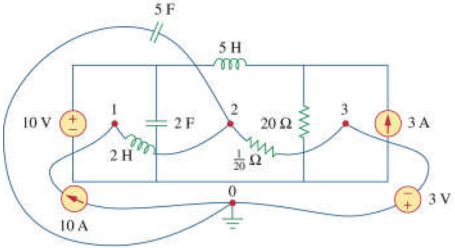


Figure 8.49(a) Construction of the dual circuit of Fig. 8.48.

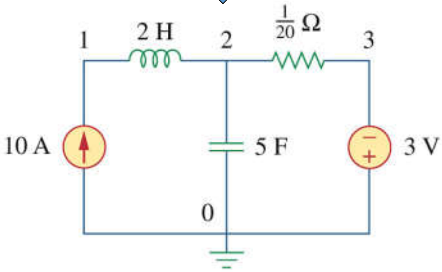


Figure 8.49(b) Dual circuit redrawn.