# Ve203 Discrete Mathematics

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# Part II

## Counting and Algorithms

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# Linear Recurrence Relations

A sequence $(a_n) = (a_0, a_1, a_2, \dots)$ satisfies a (homogeneous) linear recurrence relation of order $d$ if there exists constants $c_1, c_{2,}, \dots, c_d$ with $c_d \neq 0$ such that

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}$$

for all $n \geq d$.

Example

- Fibonacci numbers. $F_0 = 0$, $F_1 = 1$.

$$F_n = F_{n-1} + F_{n-2}, \qquad n \geq 2$$

- Lucas numbers. $F_0 = 2$, $F_1 = 1$.

$$F_n = F_{n-1} + F_{n-2}, \qquad n \geq 2$$

- Geometric progression. $a_n = c_1 a_{n-1} = \cdots = c_1^n a_0$.

# Linear Recurrence Relations

Consider the second order case when $d = 2$: $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, $n \geq 2$, $c_2 \neq 0$. We call $\chi(t) = t^2 - c_1 t - c_2$ the ***characteristic polynomial*** of the linear recurrence relation. Let $r_1$ and $r_2$ be roots of $\chi$, i.e.,
$\chi(t) = (t - r_1)(t - r_2)$, or

$$r_{1,2} = \frac{c_1 \pm \sqrt{c_1^2 - 4c_2}}{2}$$

Note that $r_1 \neq 0$ and $r_2 \neq 0$.

### Theorem

*If $r_1 \neq r_2$, then there exist constants $\alpha_1$, $\alpha_2$ such that $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$*

### Remark

To solve for $\alpha_1$ and $\alpha_2$, plug in the values for $n = 0, 1$, and solve

$$a_0 = \alpha_1 + \alpha_2$$
$$a_1 = \alpha_1 r_2 + \alpha_2 r_2$$

# Linear Recurrence Relations

## Example

Consider the Fibonacci numbers, with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$. The characteristic polynomial is given by $\chi(t) = t^2 - t - 1$, with roots

$$r_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

hence

$$F_n = \alpha_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + \alpha_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

Plug in $n = 0$ and $n = 1$, we have

$$\begin{aligned} 0 &= \alpha_1 + \alpha_2 \\ 1 &= \alpha_1 \left( \frac{1 + \sqrt{5}}{2} \right) + \alpha_2 \left( \frac{1 - \sqrt{5}}{2} \right) \end{aligned}$$

# Linear Recurrence Relations

## Example (Cont.)

Thus we have $\alpha_1 = 1/\sqrt{5}$, and $\alpha_2 = -1/\sqrt{5}$. Hence

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

# Linear Recurrence Relations

## Example

Consider a 2-periodic sequence

$$(a_n)_{n=0}^{\infty} = (x, y, x, y, \dots)$$

which satisfies the lienar recurrence relation $a_n = a_{n-2}$ for $n \geq 2$.
The characteristic polynomial is given by $\chi(t) = t^2 - 1 = (t+1)(t-1)$. Thus

$$a_n = \alpha_1 1^n + \alpha_2 (-1)^n$$

for some constants $\alpha_1$, $\alpha_2$. We plug in the initial conditions and get

$$\alpha_1 = \frac{x+y}{2}, \qquad \alpha_2 = \frac{x-y}{2}$$

Thus

$$a_n = \frac{x+y}{2} + \frac{x-y}{2}(-1)^n$$

# Linear Recurrence Relations

## Proof (Formal Power Series).

Let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \sum_{n\geq 0} a_n x^n$$

then

$$\begin{aligned}
A(x) &= a_0 + a_1 x + \sum_{n\geq 2} a_n x^n = a_0 + a_1 x + \sum_{n\geq 2} (c_1 a_{n-1} + c_2 a_{n-2}) x^n \\
&= a_0 + a_1 x + c_1 \sum_{n\geq 2} a_{n-1} x^n + c_2 \sum_{n\geq 2} a_{n-2} x^n \\
&= a_0 + a_1 x + c_1 x \sum_{n\geq 2} a_{n-1} x^{n-1} + c_2 x^2 \sum_{n\geq 2} a_{n-2} x^{n-2} \\
&= a_0 + a_1 x + c_1 x \underbrace{\sum_{m\geq 1} a_m x^m}_{A(x)-a_0} + c_2 x^2 \underbrace{\sum_{m\geq 0} a_m x^m}_{A(x)}
\end{aligned}$$

# Linear Recurrence Relations

## Proof (Formal Power Series, Cont.)

Hence $A(x) = a_0 + a_1 x + c_1 x(A(x) - a_0) + c_2 x^2 A(x)$, hence

$$A(x) = \frac{a_0 + a_1 x - c_1 a_0 x}{1 - c_1 x - c_2 x^2} = \frac{a_0 + a_1 x - c_1 a_0 x}{(1 - r_1 x)(1 - r_2 x)}$$

We can use partial fraction to get (recall that $r_1 \neq r_2$)

$$A(x) = \frac{\alpha_1}{1 - r_1 x} + \frac{\alpha_2}{1 - r_2 x} = \alpha_1 \sum_{n \geq 0} (r_1 x)^n + \alpha_2 \sum_{n \geq 0} (r_2 x)^n$$

that is,

$$\sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} (\alpha_1 r_1^n + \alpha_2 r_2^n) x^n$$

Compare coefficients, we get $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n \geq 0$. □

## Remark

In general, $\displaystyle\sum_{n \geq 0} a_{n+k} x^n = \frac{1}{x^k}\left[A(x) - \sum_{n=0}^{k-1} a_n x^n\right]$ for $k \in \mathbb{N} \setminus \{0\}$.

# Linear Recurrence Relations

## Theorem

*For the second order linear recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2}$, if the characteristic polynomial $\chi$ has repeated roots $r$, i.e., $\chi(t) = (t-r)^2$, then there exist constants $\alpha_1$ and $\alpha_2$ such that $a_n = (\alpha_1 + \alpha_2 n) r^n$ for all $n \geq 0$.*

## Proof.

Same as before, we get

$$A(x) = \frac{a_0 + (a_1 - c_0 a_1 x)}{(1-rx)^2}$$

Then by partial fraction, there exist constants $\beta_1$, $\beta_2$ such that

$$\begin{aligned} A(x) &= \frac{\beta_1}{1-rx} + \frac{\beta_2}{(1-rx)^2} \\ &= \beta_1 \sum_{n\geq 0} (rx)^n + \beta_2 \sum_{n \geq 0} (n+1)(rx)^n \end{aligned}$$

Proof.

Then by comparing coefficients, we have

$$a_n = \beta_1 r^n + \beta_2(n+1)r^n = (\beta_1 + \beta_2)r^n + \beta_2 n r^n$$

Finally, let $\alpha_1 = \beta_1 + \beta_2$ and $\alpha_2 = \beta_2$. $\square$

# Linear Recurrence Relations

## Linear Recurrence Relations of Higher order

Consider $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_d a_{n-d}$, $n \geq d$, with characteristic polynomial

$$\begin{aligned}\chi(t) &= t^d - c_1 t^{d-1} - c_2 t^{d-2} - \cdots - c_{d-1} t - c_d \\ &= (t - r_1)(t - r_2) \cdots (t - r_d).\end{aligned}$$

- If all $r_1, \ldots, r_d$ are distinct, then there exist $\alpha_1, \ldots, \alpha_d$ such that

$$a_n = \alpha_1 r_1^n + \cdots + \alpha_d r_d^n$$

- If the root $r_i$ appears with multiplicity $m_i$, then we have solutions

$$r_i^n, n r_i^n, n^2 r_i^n, \ldots, n^{m_i - 1} r_i^n$$

The final solution is obtained by taking their linear combinations.

# General Homogeneous Recurrence Relations

## Example

Consider a linear recurrence relations

$$(T-1)^5(T+1)^3(T-3)^2(T+8)(T-9)^4 a_n = 0$$

where $T$ is the ***translation operator*** such that $(Ta)_n = a_{n+1}$. The general solution is given by

$$\begin{aligned}
a_n = {} & \alpha_1 + \alpha_2 n + \alpha_3 n^2 + \alpha_4 n^3 + \alpha_5 n^4 \\
& + (\alpha_6 + \alpha_7 n + \alpha_8 n^2)(-1)^n \\
& + (\alpha_9 + \alpha_{10} n)3^n \\
& + \alpha_{11}(-8)^n \\
& + (\alpha_{12} + \alpha_{13} n + \alpha_{14} n^2 + \alpha_{15} n^3)9^n
\end{aligned}$$

with constants $\alpha_1, \alpha_2, \ldots, \alpha_{15}$.

# Inhomogeneous/Nonhomogeneous Equations

## General Strategy

Homogeneous solution + (any) particular solution

## Example

Find the general solution to

$$(T+2)(T-6)a_n = 3^n$$

- Homogeneous solution: $a_n^{(h)} = \alpha_1(-2)^n + \alpha_2 6^n$.
- Particular solution: Try $a_n^{(p)} = d3^n$. ($\Rightarrow d = -1/15$)

General solution

$$a_n = \alpha_1(-2)^n + \alpha_2 6^n - \frac{1}{15}3^n$$

# Inhomogeneous/Nonhomogeneous Equations

## Example (Cont.)

We can also try to use the generating function. Ignore the initial conditions (Set $a_0 = a_1 = 0$), and let $A(x) = \sum_{n\geq 0} a_n x^n$, we have

$$(x^{-1}+2)(x^{-1}-6)A^{(p)}(x) = \sum_{n\geq 0} 3^n x^n = \frac{1}{1-3x}$$

thus

$$A^{(p)}(x) = \frac{x^2}{(1-3x)(1+2x)(1-6x)} = \frac{-1/15}{1-3x} + \frac{\cdots}{1+2x} + \frac{\cdots}{1-6x}$$

hence a particular solutions is given by

$$a_n^{(p)} = -\frac{1}{15}3^n$$

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# Formal Power Series

## Definition

A ***formal power series*** is an expression

$$A(x) = \sum_{n \geq 0} a_n x^n$$

which is called the ***generating function*** of the sequence $(a_n)$, where $x$ is usually called the ***variable*** or ***indeterminate***. Specifically, we identify $x$ with the sequence $(0, 1, 0, 0, \dots)$. We also write the scalar coefficients as $[x^n]A(x) = a_n$. In general, the scalar coefficients could be taken as elements of a ring.

# Formal Power Series

## Properties of Formal Power Series

Let $A(x) = \sum_{n\geq 0} a_n x^n$, $B(x) = \sum_{n\geq 0} b_n x^n$, $C(x) = \sum_{n\geq 0} c_n x^n$,

- Equality: $A(x) = B(x) \Leftrightarrow a_n = b_n$ for all $n \geq 0$.
- Addition: $A(x) + B(x) = \sum_{n\geq 0}(a_n + b_n)x^n$
  - commutative: $A(x) + B(x) = B(x) + A(x)$
  - associative: $(A(x) + B(x)) + C(x) = A(x) + (B(x) + C(x))$
  - additive identity: $0 + A(x) = A(x)$ for all $A(x)$, where $0 = \sum_{n\geq 0} 0x^n$.
  - additive inverse: $A(x) + (-A(x)) = 0$, where $(-A)(x) \coloneqq \sum_{n\geq 0}(-a_n)x^n$

# Formal Power Series

## Properties of Formal Power Series (Cont.)

- Multiplication: $A(x)B(x) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)x^n$.
  - commutative: $A(x)B(x) = B(x)A(x)$
  - associative: $(A(x)B(x))C(x) = A(x)(B(x)C(x))$
  - multiplicative identity: $1 \cdot A(x) = A(x)$ for all $A(x)$, where $1 = 1 + 0x + 0x^2 \cdots$
- Distributivity: $A(x)(B(x) + C(x)) = A(x)B(x) + A(x)C(x)$

To summarize, formal power series forms a ***commutative ring***.

# Formal Power Series

## Example

Let

$$A(x) = B(x) = \sum_{n \geq 0} x^n$$

then

$$A(x) + B(x) = \sum_{n \geq 0} 2x^n$$

$$A(x)B(x) = \sum_{n \geq 0} \left( \sum_{i=0}^{n} 1 \cdot 1 \right) x^n = \sum_{n \geq 0} (n+1)x^n$$

# Formal Power Series

## Definition

A formal power series $A(x)$ is ***invertible*** if there exists $B(x)$ such that $A(x)B(x) = 1$.

## Remark

If $B(x)$ exists, then it is unique, and $B(x)A(x) = 1$. We usuall write

$$B(x) = A(x)^{-1} = \frac{1}{A(x)}$$

# Formal Power Series

## Example (Geometric Series)

Let

$$A(x) = \sum_{n \geq 0} x^n, \qquad B(x) = 1 - x$$

Recall that

$$A(x)B(x) = \sum_{n \geq 0} \left( \sum_{i=0}^{n} a_i b_{n-i} \right) x^n$$

- If $n = 0$: $\sum_{i=0}^{0} a_i b_{0-i} = a_0 b_0 = 1$.
- If $n \geq 1$: $\sum_{i=0}^{n} a_i b_{n-i} = 1 - 1 = 0$.

So overall, $A(x)B(x) = 1$ and $B(x)A(x) = 1$. We write

$$\frac{1}{1-x} = \sum_{n \geq 0} x^n$$

# Formal Power Series

### Theorem

*A formal power series $A(x)$ is invertible iff $a_0 \neq 0$.*

### Remark

Note that $A(x) = x$ is NOT invertible.

# Formal Power Series

## Definition

Let $A(x)$ and $B(x)$ be formal power series, $a_0 = 0$ or $B$ is a polynomial, then the ***composition*** is given by

$$(B \circ A)(x) = B(A(x)) = \sum_{n \geq 0} b_n A(x)^n$$

# Formal Power Series

## Definition

Let $A(x) = \sum_{n \geq 0} a_n x^n$ be a formal power series, then the ***formal derivative*** of $A(x)$ is given by

$$DA(x) = \sum_{n \geq 0} n a_n x^{n-1} = \sum_{n \geq 0} (n+1) a_{n+1} x^n$$

## A Few Properties

Given $A(x)$ and $B(x)$ formal power series, then

- $D(\alpha A + \beta B) = \alpha DA + \beta DB$, where $\alpha, \beta$ are scalars
- $D(AB) = (DA)B + A(DB)$
- $D(B \circ A) = (DB \circ A)(DA)$ given $a_0 = 0$
- $D(1/A) = -DA/A^2$ given $a_0 \neq 0$
- $D(A^n) = nA^{n-1}(DA)$

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# Binomial Theorem

## Lemma

*Let $A(x)$ be a formal power series such that $A(0) = 1$, $d \in \mathbb{N} \setminus \{0\}$. Then there exists a* ***unique*** *formal power series $B(x)$ such that $B(0) = 1$ and $B(x)^d = A(x)$. We write $B(x) = A(x)^{1/d}$.*

## Remark

A consequence of the ***uniqueness*** of the $d$th root of a power series is that for $c \in \mathbb{Z} \setminus \{0\}$, $d \in \mathbb{N} \setminus \{0\}$, the expression $A(x)^{c/d}$ is well-defined, by either $(A(x)^{1/d})^c$ or $(A(x)^c)^{1/d}$. Also cf., Baby Rudin, Chapter 1, exercise 6.

## Definition

Let $m \in \mathbb{Q}$, define $\binom{m}{0} := 1$, and

$$\binom{m}{k} := \frac{m(m-1)\cdots(m-k+1)}{k!}$$

where $k \in \mathbb{N} \setminus \{0\}$. Note that if $m \in \mathbb{N} \setminus \{0\}$, then $\binom{m}{k} = \frac{m!}{k!(m-k)!}$.

# Binomial Theorem

## Theorem (Binomial Theorem)

*Let* $m \in \mathbb{Q}$*, then*

$$(1+x)^m = \sum_{n \geq 0} \binom{m}{n} x^n$$

## Lemma

*Let* $m \in \mathbb{Q}$*, and* $A(0) = 1$*, then* $D(A(x)^m) = m \cdot (DA)(x) \cdot A(x)^{m-1}$

## Proof.

Let $m = p/q$, with $p \in \mathbb{Z}$, $q \in \mathbb{Z}_+$, then

$$\begin{aligned} p \cdot (DA)(x) \cdot A(x)^{p-1} &= D(A(x)^p) = D((A(x)^m)^q) \\ &= q \cdot (A(x)^m)^{q-1} \cdot D(A(x)^m) \end{aligned}$$

thus

$$D(A(x)^m) = \frac{p \cdot D(A)(x) \cdot A(x)^{p-1}}{q \cdot (A(x)^m)^{q-1}} = m \cdot (DA)(x) \cdot A(x)^{m-1} \qquad \square$$

# Binomial Theorem

## Proof of Binomial Theorem.

By previous lemma,

$$D((1+x)^m) = m(1+x)^{m-1}$$

thus (by induction, say)

$$D^n((1+x)^m) = m(m-1)\cdots(m-n+1)(1+x)^{m-n}$$

Then the theorem follows by noting that

$$[x^n](1+x)^m = \frac{D^n(1+x)^m(0)}{n!} = \frac{m(m-1)\cdots(m-n+1)}{n!} = \binom{m}{n} \qquad \square$$

# Binomial Theorem

### Example

If $A(0) = 1$, $m \in \mathbb{Q}$, then

$$A(x)^m = (1 + (A(x) - 1))^m = \sum_{n \geq 0} \binom{m}{n} (A(x) - 1)^n$$

### Example

Let $m \in \mathbb{N} \setminus \{0\}$, then $\binom{m}{n} = 0$ for $n > m$, hence

$$(1 + x)^m = \sum_{n \geq 0} \binom{m}{n} x^n = \sum_{n=0}^{m} \binom{m}{n} x^n$$

# Binomial Theorem

## Example

If $m = -1$, then

$$(1+x)^{-1} = \sum_{n \geq 0} \binom{-1}{n} x^n = \sum_{n \geq 0} (-1)^n x^n$$

where we have calculated

$$\binom{-1}{n} = \frac{(-1)(-2)\cdots(-1-n+1)}{n!} = \frac{(-1)^n(1)(2)\cdots(n)}{n!} = (-1)^n$$

Or we can compose the geometric series

$$(1-x)^{-1} = \sum_{n \geq 0} x^n$$

with $-x$ (as usually done in analysis).

# Binomial Theorem

## Example

Let $m = -d$, $d \in \mathbb{N} \setminus \{0\}$,

$$\binom{-d}{n} = \frac{(-d)(-d-1)\cdot(-d-n+1)}{n!} = \frac{(-1)^n d(d+1)\cdots(d+n-1)}{n!}$$
$$= (-1)^n \frac{(d+n-1)!}{(d-1)!n!} = (-1)^n \binom{d+n-1}{n}$$

thus

$$(1+x)^{-d} = \sum_{n\geq 0} \binom{-d}{n} x^n = \sum_{n\geq 0} (-1)^n \binom{d+n-1}{n} x^n$$
$$(1-x)^{-d} = \sum_{n\geq 0} \binom{-d}{n} (-x)^n = \sum_{n\geq 0} \binom{d+n-1}{n} x^n$$

# Binomial Theorem

The following two identities are worth mentioning,

$$
\begin{aligned}
(1+x)^n &= \sum_{k\geq 0} \binom{n}{k} x^k \\
&= \sum_{k=0}^{n} \binom{n}{k} x^k \qquad \text{if } n \in \mathbb{N} \\
\frac{x^k}{(1-x)^{k+1}} &= \sum_{n\geq 0} \binom{n}{k} x^n = \sum_{n\geq k} \binom{n}{k} x^n
\end{aligned}
$$

where $k \in \mathbb{N}$.

## Example

We show that

$$\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}$$

where $m, n \in \mathbb{N}$.

# Binomial Theorem

## Proof 1.

We show

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{k}{m}x^n = \sum_{n=0}^{\infty}\binom{n+1}{m+1}x^n$$

Indeed, note that by interchanging the order of summation,

$$\begin{aligned}
\text{LHS} &= \sum_{k=0}^{\infty}\binom{k}{m}\sum_{n\geq k}x^n = \sum_{k=0}^{\infty}\binom{k}{m}\frac{x^k}{1-x} = \frac{1}{1-x}\sum_{k=0}^{\infty}\binom{k}{m}x^k \\
&= \frac{1}{1-x}\cdot\frac{x^m}{(1-x)^{m+1}} = \frac{x^m}{(1-x)^{m+2}} \\
\text{RHS} &= \frac{1}{x}\sum_{n=0}^{\infty}\binom{n+1}{m+1}x^{n+1} = \frac{1}{x}\cdot\frac{x^{m+1}}{(1-x)^{m+2}} = \frac{x^m}{(1-x)^{m+2}}
\end{aligned}$$

and we are done. $\square$

# Binomial Theorem

## Proof 2.

We show

$$\sum_{m=0}^{\infty}\sum_{k=0}^{n}\binom{k}{m}x^m = \sum_{m=0}^{\infty}\binom{n+1}{m+1}x^m$$

Indeed, again by interchanging the order of summation,

$$\begin{aligned}
\text{LHS} &= \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{k}{m}x^m = \sum_{k=0}^{n}(1+x)^k \\
&= \frac{1-(1+x)^{n+1}}{1-(1+x)} = \frac{(1+x)^{n+1}-1}{x} \\
\text{RHS} &= \frac{1}{x}\sum_{m+1\geq 1}^{\infty}\binom{n+1}{m+1}x^{m+1} = \frac{(1+x)^{n+1}-1}{x}
\end{aligned}$$

and we are done. □

# Counting Subsets

## Proposition

Given $[n] = \{1, 2, \ldots, n\}$, the number of subsets of $[n]$ of size $k$ is $\binom{n}{k}$.

## Proof.

Expand $(1 + x)^n$ as $(1 + x)(1 + x) \cdots (1 + x)$. For each $1 + x$, choosing 1 represents not in the subset, and choosing $x$ represents in the subset. Thus the number of subsets of $[n]$ of size $k$ is given by

$$[x^k](1 + x)^n = \binom{n}{k} \qquad \square$$

# Pascal's Identity

## Proposition

For positive natural numbers $n$ and $k$,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

## Proof.

Note that $(1+x)^n = (1+x)^{n-1}(1+x)$, thus by comparing coefficients

$$[x^k](1+x)^n = [x^k](1+x)^{n-1}(1+x)$$

which is the identity. $\square$

# Multinomial Theorem

## Definition

Given $k_1 + k_2 + \cdots + k_d = n$, $k_i \in \mathbb{N}$, $i = 1, \ldots, d$, the ***multinomial coefficient*** is given by

$$\binom{n}{k_1, k_2, \ldots, k_d} = \frac{n!}{k_1! k_2! \cdots k_d!}$$

## Theorem (Multinomial Theorem)

*Let $x_1, \ldots, x_d$ be variables, then*

$$(x_1 + \cdots + x_d)^n = \sum_{k_1 + \cdots + k_d = n} \binom{n}{k_1, k_2, \ldots, k_d} x_1^{k_1} \cdots x_d^{k_d}$$

# Multinomial Theorem

## Proof.

Induction on $d$. Base case: $d = 1$. Both sides are $x_1^n$. Inductive case: $d > 1$, assume the IH that the theorem holds for $d - 1$ variables. Let $x = x_1 + \cdots + x_{d-1}$ and $y = x_d$, then by binomial theorem and IH,

$$
\begin{aligned}
(x_1 + \cdots + x_d)^n &= \sum_{m=0}^{n} \binom{n}{m} (x_1 + \cdots + x_{d-1})^m x_d^{n-m} \\
&= \sum_{m=0}^{n} \binom{n}{m} \sum_{k_1+\cdots+k_{d-1}=m} \binom{m}{k_1, k_2, \ldots, k_{d-1}} x_1^{k_1} \cdots x_{d-1}^{k_{d-1}} x_d^{n-m}
\end{aligned}
$$

The rest follows by setting $k_d = n - m$, and note that

$$
\begin{aligned}
\binom{n}{n-k_d}\binom{m}{k_1, k_2, \ldots, k_{d-1}} &= \frac{n!}{(n-k_d)!k_d!} \cdot \frac{(n-k_d)!}{k_1! \cdots k_{d-1}!} \\
&= \binom{n}{k_1, k_2, \ldots, k_d}
\end{aligned}
$$

$\square$

# Counting Multisets

## Proposition

The number of multisets of $[n]$ of size $k$ is $\binom{n+k-1}{k}$.

## Proof.

Expand $(1 + x + x^2 + \cdots)^n$ and calculate the coefficient of $x^k$. Thus

$$
\begin{aligned}
[x^k](1 + x + x^2 + \cdots)^n &= [x^k]\left(\sum_{\ell \geq 0} x^\ell\right)^n = [x^k](1-x)^{-n} \\
&= (-1)^k \binom{-n}{k} = \binom{n+k-1}{k} \qquad \square
\end{aligned}
$$

# Counting Integer Solutions

## Example

Consider

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \leq 538$$

What are the number of integer solutions if

1. $x_i > 0$ and $=$ holds;
2. $x_i \geq 0$ and $=$ holds;
3. $x_i > 0$ and $<$ holds;
4. $x_i \geq 0$ and $<$ holds;
5. $x_i \geq 0$.

## Remark

- $x_i > 0 \Rightarrow x_i \geq 1$.

# Counting Integer Solutions

## Example

How many nonnegative integer solutions to the equation

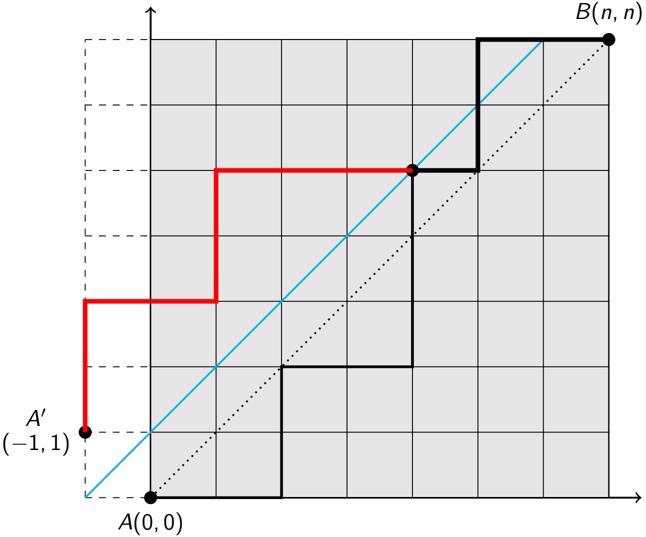
$$x_1 + x_2 + x_3 + x_4 = 63$$

such that $x_1, x_2 \geq 0$, $2 \leq x_3 \leq 5$, $x_4 > 0$.
The solution is given by

$$\begin{aligned}
&[x^{63}](1 + x + x^2 + \cdots)^2(x^2 + x^3 + x^4 + x^5)(x + x^2 + x^3 + \cdots) \\
&\quad = [x^{63}]\frac{x^3 + x^4 + x^5 + x^6}{(1-x)^3}
\end{aligned}$$

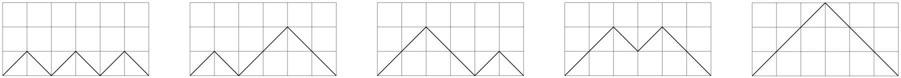
# Catalan Numbers

How many lattice paths from $(0, 0)$ to $(n, n)$ that never go above the diagonal?
$C_n = \binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$.

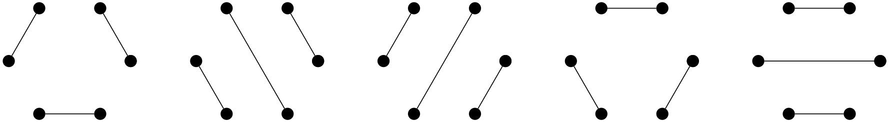


# Catalan Numbers

## Mountain Range/Dyck Paths



## Noncrossing Handshakes

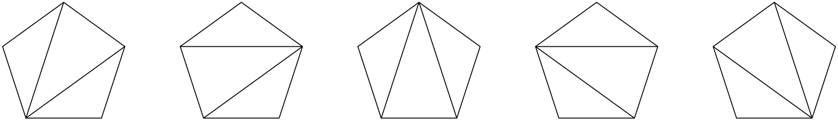


## Paired Parentheses

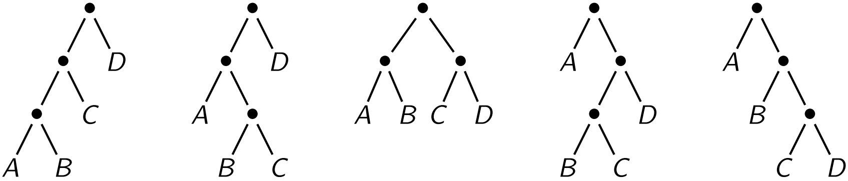
- ()()()
- (())()
- ()(())
- (()())
- ((()))

# Catalan Numbers

## Polygon Triangulation



## Full Binary Trees




## Matrix Chain Multiplication

- $((AB)C)D$
- $(A(BC))D$
- $(AB)(CD)$
- $A((BC)D)$
- $A(B(CD))$

# Catalan Numbers

## Segner's recurrence relation

We can establish the following recurrence relation starting with $C_0 = 1$, and

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \text{ for } n \geq 0,$$

We recognize the RHS is a convolution. Now consider the following generating function

$$c(x) := \sum_{n \geq 0} C_n x^n$$

then $c(x) = 1 + xc(x)^2$, and

$$c(x) = \frac{1 \pm \sqrt{1-4x}}{2x} = \frac{1 \pm (1 - 2x - 2x^2 + \cdots)}{2x}$$

Since $x$ is not invertible, the numerator must have vanishing constant term.

# Catalan Numbers

## Segner's recurrence relation (Cont.)

Thus we have

$$c(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

which we expand as

$$\begin{aligned} c(x) &= \frac{1}{2x}(1 - \sqrt{1-4x}) = \frac{1}{2x}\left[1 - \sum_{n\geq 0}\binom{1/2}{n}(4x)^n\right] \\ &= \frac{1}{2x}\cdot 2\sum_{n\geq 1}\frac{(-1)^n}{4^n}\binom{2n-2}{n-1}\frac{(-4x)^n}{n} = \sum_{n\geq 1}\binom{2n-2}{n-1}\frac{x^{n-1}}{n} \\ &= \sum_{n\geq 0}\binom{2n}{n}\frac{x^n}{n+1} = \sum_{n\geq 0} C_n x^n. \end{aligned}$$

# Twelvefold Way

Distribute $k$ balls into $n$ urns. ($f : B \to U$, $|B| = k$, $|U| = n$)

| Balls (domain) | Urns (codomain) | unrestricted (any function) | $\leq 1$ (injective) | $\geq 1$ (surjective) |
|---|---|---|---|---|
| labeled | labeled | $n^k$ | $n^{\underline{k}}$ | $n!\left\{ {k \atop n} \right\}$ |
| unlabeled | labeled | $\left(\!\binom{n}{k}\!\right)$ | $\binom{n}{k}$ | $\left(\!\binom{n}{k-n}\!\right)$ |
| labeled | unlabeled | $\sum_{i=1}^{n} \left\{ {k \atop i} \right\}$ | $[k \leq n]$ | $\left\{ {k \atop n} \right\}$ |
| unlabeled | unlabeled | $\sum_{i=1}^{n} p_i(k)$ | $[k \leq n]$ | $p_n(k)$ |

- $n^{\underline{k}} = (n)_k = P(n, k)$
- $\binom{n}{k} = C(n, k)$
- $\left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$
- $\left\{ {k \atop n} \right\} = \#$ partition of $[k]$ into $n$ parts.
- $p_n(k) = \#$ partition of $k$ into $n$ parts.
- $[k \leq n]$: Iverson bracket

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# Finite Calculus

## Finite Calculus

- $\Delta f(x) = f(x+1) - f(x)$
- If $g = \Delta G$, then

$$\sum\nolimits_a^b g(x)\,\delta x = G(x)\Big|_a^b = G(b) - G(a)$$

## Infinite Calculus

- $Df(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
- If $g = DG$, then

$$\int_a^b g(x)\,dx = G(x)\Big|_a^b = G(b) - G(a)$$

For integers $b \geq a$, we should put

$$\sum\nolimits_a^b g(x)\,\delta x = \sum_{k=a}^{b-1} g(k) = \sum_{a\leq k<b} g(k)$$

Then for integers $m, n \geq 0$, we have

- $\Delta(x^{\underline{m}}) = mx^{\underline{m-1}}$
- $\sum_{0\leq k<n} k^{\underline{m}} = \frac{k^{\underline{m+1}}}{m+1}\Big|_0^n = \frac{n^{\underline{m+1}}}{m+1}$

# Finite Calculus

## Example

- $\displaystyle\sum_{0\le k<n} k = \frac{n^{\underline{2}}}{2} = \frac{n(n-1)}{2}$.
- Note that $k^2 = k^{\underline{2}} + k^{\underline{1}}$, then

$$\sum_{0\le k<n} k^2 = \frac{n^{\underline{3}}}{3} + \frac{n^{\underline{2}}}{2} = \tfrac{1}{3}n(n-1)(n-2+\tfrac{3}{2}) = \tfrac{1}{3}n(n-\tfrac{1}{2})(n-1)$$

- Note that $k^3 = k^{\underline{3}} + 3k^{\underline{2}} + k^{\underline{1}}$, then

$$\sum_{a\le k<b} k^3 = \frac{n^{\underline{4}}}{4} + n^{\underline{3}} + \frac{n^{\underline{2}}}{2}\Big|_a^b$$

cf., Dijkstra, "Why numbering should start at zero".

# Stirling Number of the Second Kind

We can always express ordinary powers using factorial powers via Stirling numbers, i.e., for $n \geq 0$,

$$x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}}$$

## Definition

For $n, k \in \mathbb{N}$, $\left\{ {n \atop k} \right\}$ is the number of ways to partition a set with $n$ elements into $k$ disjoint, nonempty subsets. $\left\{ {n \atop k} \right\}$ is called a ***Stirling number of the second kind***. Reads "$n$ subset $k$".

## Remark

The power conversion above can be proved by induction based on the following recurrence relation

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$

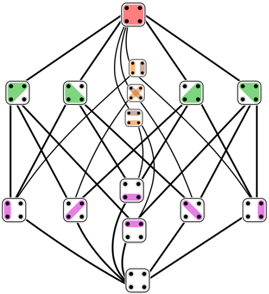
# Partition of A Set and Partition of A Number

## Partition of A Set

- $\left\{ {4 \atop 3} \right\} = 6$.

  12|3|4, 13|2|4, 14|2|3, 23|1|4, 24|1|3, 34|1|2.
- $\left\{ {4 \atop 2} \right\} = 7$.

  123|4, 124|3, 134|2, 234|1, 12|34, 13|24, 14|23.



## Partition of An Integer

A partition of the number $k$ is a tuple of ***positive*** integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $\sum_{i=1}^{n} \lambda_i = k$ and $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$. We use $p(k)$ for the number of partitions of $k$, and $p_n(k)$ for the number of partitions of $k$ into exactly $n$ parts. Note that $p(k) = \sum_{i=1}^{k} p_i(k)$.

- $p_2(7) = 3$.

  $7 = 6 + 1 = 5 + 2 = 4 + 3.$

- $p_3(7) = 4$.

  $$\begin{aligned} 7 &= 5 + 1 + 1 = 4 + 2 + 1 \\ &= 3 + 3 + 1 = 3 + 2 + 2. \end{aligned}$$

# Particular Values of Stirling Number of the Second Kind

- $\left\{ {n \atop k} \right\} = 0$ if $k > n$.
- $\left\{ {n \atop 0} \right\} = 0$.
- $\left\{ {n \atop 1} \right\} = 1$.
- $\left\{ {n \atop n} \right\} = 1$.
- $\left\{ {n \atop n-1} \right\} = \binom{n}{2}$.
- $\left\{ {n \atop 2} \right\} = 2^{n-1} - 1$.

| $n$ | $\left\{ {n \atop 0} \right\}$ | $\left\{ {n \atop 1} \right\}$ | $\left\{ {n \atop 2} \right\}$ | $\left\{ {n \atop 3} \right\}$ | $\left\{ {n \atop 4} \right\}$ | $\left\{ {n \atop 5} \right\}$ | $\left\{ {n \atop 6} \right\}$ | $\left\{ {n \atop 7} \right\}$ | $\left\{ {n \atop 8} \right\}$ | $\left\{ {n \atop 9} \right\}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | |
| 1 | 0 | 1 | | | | | | | | |
| 2 | 0 | 1 | 1 | | | | | | | |
| 3 | 0 | 1 | 3 | 1 | | | | | | |
| 4 | 0 | 1 | 7 | 6 | 1 | | | | | |
| 5 | 0 | 1 | 15 | 25 | 10 | 1 | | | | |
| 6 | 0 | 1 | 31 | 90 | 65 | 15 | 1 | | | |
| 7 | 0 | 1 | 63 | 301 | 350 | 140 | 21 | 1 | | |
| 8 | 0 | 1 | 127 | 966 | 1701 | 1050 | 266 | 28 | 1 | |
| 9 | 0 | 1 | 255 | 3025 | 7770 | 6951 | 2646 | 462 | 36 | 1 |

By inclusion-exclusion principle (on this later),

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$$

# Stirling Number of the Second Kind

**Theorem**

*For $n \geq k \geq 1$,*

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$

**Proof by double counting.**

- By definition, the LHS counts the ways of partitioning of an $n$-element set into $k$ subsets.
- Consider whether the element $n$ is alone in its own set.
  - If yes, then there are $\left\{ {n-1 \atop k-1} \right\}$ ways of partition the remaining $n-1$ elements into $k-1$ subsets.
  - If no, then there are $\left\{ {n-1 \atop k} \right\}$ ways of partition the remaining $n-1$ elements into $k$ subsets, and there are $k$ choices to insert the element $n$ into any of the $k$ subsets. □

# Stirling Number of the Second Kind

**Theorem**

*For $m, n \geq 0$,*

$$m^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} m^{\underline{k}}$$

$$\begin{tikzcd}
{[n]} \arrow[r, "f"] \arrow[d, two heads, "q"'] & {[m]} \\
{[k] \cong [n]/\sim} \arrow[ur, dashed, "\tilde{f}"'] &
\end{tikzcd}$$

**Proof by double counting (Method I).**

- By definition, the LHS counts the number of functions from $[n] = \{1, \dots, n\}$ to $[m] = \{1, \dots, m\}$.
- Consider the partition of the domain induced by the function $f : [n] \to [m]$, the size of the partition ranges from 0 to $n$. Now for each fixed partition of size $k$, the induced function $\tilde{f} : [n]/\sim \to [m]$ is injective. There are $m^{\underline{k}}$ such injections, with $\left\{ {n \atop k} \right\}$ choices of different domains of size $k$ (Note that $q$ is surjective). $\square$

# Stirling Number of the Second Kind

**Theorem**

*For $m, n \geq 0$,*

$$m^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} m^{\underline{k}}$$

$$\begin{tikzcd} {[n]} \arrow[r, "f"] \arrow[d, two heads, "s"'] & {[m]} \\ {[k] \cong \operatorname{im} f} \arrow[ur, hook, dashed, "i"'] & \end{tikzcd}$$

**Proof by double counting (Method II).**

- By definition, the LHS counts the number of functions from $[n] = \{1, \ldots, n\}$ to $[m] = \{1, \ldots, m\}$.
- For each function $f : [n] \to [m]$, we can write $f = i \circ s$, where $s : [n] \twoheadrightarrow \operatorname{im} f$ is surjective, and $i : \operatorname{im} f \hookrightarrow [m]$ is injective (called an inclusion map). For each $\operatorname{im} f$ of fixed size $k$, the number of surjection $s$ is given by $k!\left\{ {n \atop k} \right\}$, and the choice of $\operatorname{im} f \subset [m]$ is given by $\binom{m}{k} = m^{\underline{k}}/k!$. $\square$

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# Inclusion-Exclusion Principle (PIE)

```
primes = filterPrime [2..] where
  filterPrime (p:xs) = p : filterPrime [x | x <- xs, x `mod` p /= 0]
```

- 2 sets: $|A \cup B| = |A| + |B| - |A \cap B|$
- 3 sets: $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$


## Applications

- Sieve of Eratosthenes
- Euler's totient function
- . . .

# Inclusion-Exclusion Principle

## Notation

Given $I \subset [n]$, we let

$$A_I := \bigcap_{i \in I} A_i,$$

where $A_i \subset X$ for all $i \in I$. For example, $A_{\{1,2,4\}} = A_1 \cap A_2 \cap A_4$. In particular, $A_\varnothing = X$. Or equivalently, take

$$A_I := \bigcap \left( \{X\} \cup \bigcup_{i \in I} \{A_i\} \right)$$

to justify the notation $A_\varnothing$. For example,

$$A_{\{1,2,4\}} = \bigcap \{X, A_1, A_2, A_4\} = A_1 \cap A_2 \cap A_4$$

and

$$A_\varnothing = \bigcap \{X\} = X$$

# Inclusion-Exclusion Principle

## Theorem (Inclusion-Exclusion Principle)

*Let $A_1, \ldots, A_n$ be subsets of $X$. Then the number of elements of $X$ which lie in none of the subsets $A_i$ is*

$$\sum_{I \subset [n]} (-1)^{|I|} |A_I| = \sum_{r \geq 0} (-1)^r \sum_{|I|=r} |A_I|$$

## Proof.

Note that there is a one-to-one correspondence between a set $A$ and its indicator function $1_A$, where

$$1_A(x) := \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

# Inclusion-Exclusion Principle

## Proof.

Thus

$$\begin{aligned} 1_{(A_1\cup\cdots\cup A_n)^c} &= \prod_{i=1}^{n} 1_{A_i^c} = \prod_{i=1}^{n}(1_X - 1_{A_i}) \\ &= 1_X - \sum_{1\le i\le n} 1_{A_i} + \sum_{1\le i<j\le n} 1_{A_i}1_{A_j} \\ &\quad - \sum_{1\le i<j<k\le n} 1_{A_i}1_{A_j}1_{A_k} + \cdots + (-1)^n 1_{A_1}\cdots 1_{A_n} \\ &= 1_X - \sum_{1\le i\le n} 1_{A_i} + \sum_{1\le i<j\le n} 1_{A_i\cap A_j} \\ &\quad - \sum_{1\le i<j<k\le n} 1_{A_i\cap A_j\cap A_k} + \cdots + (-1)^n 1_{A_1\cap\cdots\cap A_n} \qquad \square \end{aligned}$$

# Inclusion-Exclusion Principle

## Proof.

Note that $|A_I|$ represents the number of elements are in ***at least*** $A_i$'s where $i \in I$ (possibly also in other $A_i$ where $i \notin I$). Let $J_x := \{i \in [n] \mid x \in A_i\}$, then $x \in A_i$ iff $i \in J_x$, i.e., $x \in A_I$ iff $I \subset J_x$ (or $I \in \mathcal{P}(J_x)$). Also

$$N_r := \sum_{|I|=r} |A_I| = \sum_{|I|=r} \sum_{x \in X} 1_{A_I}(x) = \sum_{x \in X} \sum_{|I|=r} 1_{\mathcal{P}(J_x)}(I)$$
$$= \sum_{x \in X} \binom{|J_x|}{r} = \sum_{t \geq 0} \binom{t}{r} \sum_{\substack{x \in X \\ |J_x|=t}} 1 = \sum_{t \geq 0} \binom{t}{r} e_t$$

where $e_t := |\{x \in X \mid |J_x| = t\}|$ represents the number of elements that are in ***exactly*** $t$ subsets $A_i$'s where $i \in J_x$. Now let

$$N(x) := \sum_{r \geq 0} N_r x^r \qquad \text{and} \qquad E(x) := \sum_{t \geq 0} e_t x^t$$

# Inclusion-Exclusion Principle

## Proof (Cont.)

Thus

$$\begin{aligned} N(x) &= \sum_{r\geq 0}\sum_{t\geq 0}\binom{t}{r} e_t x^r = \sum_{t\geq 0} e_t \sum_{r\geq 0}\binom{t}{r} e^r \\ &= \sum_{t\geq 0} e_t (x+1)^t = E(x+1) \end{aligned}$$

Hence $E(x) = N(x-1)$, so for $j \geq 0$, we get the **sieve formula** (cf., Gallier, Theorem 4.4),

$$e_j = \frac{E^{(j)}(0)}{j!} = \frac{N^{(j)}(-1)}{j!} = \sum_{t\geq 0}(-1)^{t-j}\frac{(n)_j}{j!}N_t = \sum_{t\geq 0}(-1)^{t-j}\binom{t}{j}N_t$$

and in particular,

$$e_0 = E(0) = N(-1) = \sum_{t\geq 0}(-1)^t N_t \qquad \square$$

# Inclusion-Exclusion Principle

## Proof (Not by induction).

Re-wirte the sum as

$$\sum_{I\subset[n]}(-1)^{|I|}|A_I| = \sum_{I\subset[n]}\sum_{x\in A_I}(-1)^{|I|} = \sum_{x\in X}\sum_{I:x\in A_I}(-1)^{|I|}$$

Let $J_x := \{i \in [n] \mid x \in A_i\}$, then $x \in A_i$ iff $i \in J_x$, i.e., $x \in A_I$ iff $I \subset J_x$.

- If $x \in X - \cup_{i=1}^{n} A_i$, then $J_x = \varnothing$. Hence $\sum_{I\subset J_x}(-1)^{|I|} = \sum_{I=\varnothing}(-1)^0 = 1$.
- Otherwise, $J \neq \varnothing$. Thus by binomial theorem,

$$\sum_{I\subset J_x}(-1)^{|I|} = \sum_{i=0}^{|J_x|}\binom{|J_x|}{i}(-1)^i = (1-1)^{|J_x|} = 0$$

Sum the terms and we are done. $\square$

# Inclusion-Exclusion Principle

## Corollary

*Let $A_1, \ldots, A_n$ be a sequence of (not necessarily distinct) sets, then*

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{\varnothing \neq I \subset [n]} (-1)^{|I|+1} |A_I|.$$

## Proof.

Take the complement of both sides of previous therem within the set $X = A_\varnothing$, that is,

$$\begin{aligned} |A_1 \cup \cdots \cup A_n| &= |A_\varnothing| - \sum_{I \subset [n]} (-1)^{|I|+1} |A_I| \\ &= \sum_{\varnothing \neq I \subset [n]} (-1)^{|I|+1} |A_I| \end{aligned}$$

□

# Inclusion-Exclusion Principle

## Special Case

The formula is a lot simpler when

$$|I| = |J| \Rightarrow |A_I| = |A_J|,$$

that is, $|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$ depends only on $k$, where $I = \{i_1, i_2, \ldots, i_k\}$. Now the formula becomes

$$|A_1 \cup \cdots \cup A_n| = \sum_{|I|=1}^{n} (-1)^{|I|+1} \binom{n}{|I|} |A_I|$$

# Derangement

## Definition

A permutation $\sigma \in S_n$ over the set $[n]$ is called a derangement if $\sigma(i) \neq i$ for all $i = 1, \ldots, n$.

## Theorem

*The number of derangements of the set $[n]$ is given by*

$$d_n = \sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$$

## Proof.

Take $A_i := \{\sigma \in S_n \mid \sigma(i) = i\}$, thus $|A_i| = (n-1)!$. Note that for general set $I \subset \{1, 2, \ldots, n\}$, $|A_I| = (n - |I|)!$. The rest follows by inclusion-exclusion principle and

$$\binom{n}{i}(n-i)! = \frac{n!}{i!(n-i)!}(n-i)! = \frac{n!}{i!} \qquad \square$$

# Derangement

## Asymptotics

Assume that each $\sigma \in S_n$ happens equally likely, what is the probability that $\sigma$ is a derangement?
Note that $|S_n| = n!$, thus

$$\lim_{n\to\infty} \frac{d_n}{n!} = \lim_{n\to\infty} \sum_{i=0}^{n} \frac{(-1)^i}{i!} = e^{-1} = \frac{1}{e} \approx \frac{1}{3}$$

# Counting Surjections

## Theorem

*Let $k \geq n$. The number of surjections $f : [k] \to [n]$ is given by*

$$S_{k,n} = \sum_{i=0}^{n-1} (-1)^i \binom{n}{i} (n-i)^k$$

## Proof.

Take $A_i = \{f \mid f(j) \neq i \text{ for all } j\} = \{f \mid i \notin \operatorname{im} f\}$. $\square$

# Counting Surjections

## Example

What is $S_{5,3} = |\{f : [5] \to [3] \mid f \text{ surjective}\}|$?

## Method I.

$$S_{5,3} = \binom{3}{0}(3-0)^5 - \binom{3}{1}(3-1)^5 + \binom{3}{2}(3-2)^5 - \underbrace{\binom{3}{3}(3-3)^5}_{=0} \qquad \square$$

## Method II.

We first calculate that

$$\left\{ {5 \atop 3} \right\} = \binom{5}{3} + 3\binom{5}{4} = 25$$

thus $S_{5,3} = 3!\left\{ {5 \atop 3} \right\} = 150$. $\square$

# Dimension of Vector Spaces

Given finite dimensional vector spaces $U$, $V$, and $W$, then?

- $\dim(U + V) = \dim U + \dim V - \dim(U \cap V)$.
- $$\begin{aligned}\dim(U + V + W) = {} & \dim U + \dim V + \dim W \\ & - \dim(U \cap V) - \dim(U \cap W) - \dim(V \cap W) \\ & + \dim(U \cap V \cap W).\end{aligned}$$

# Maximum-minimums Identity

Let $S = \{x_1, x_2, \ldots, x_n\}$, then

$$\max\{x_1, x_2, \ldots, x_n\} = \sum_{i=1}^{n} x_i - \sum_{i<j} \min\{x_i, x_j\} + \sum_{i<j<k} \min\{x_i, x_j, x_k\} - \cdots$$
$$\cdots + (-1)^{n+1} \min\{x_1, x_2, \ldots, x_n\},$$

or similarly,

$$\min\{x_1, x_2, \ldots, x_n\} = \sum_{i=1}^{n} x_i - \sum_{i<j} \max\{x_i, x_j\} + \sum_{i<j<k} \max\{x_i, x_j, x_k\} - \cdots$$
$$\cdots + (-1)^{n+1} \max\{x_1, x_2, \ldots, x_n\}.$$