Ve203 Discrete Mathematics (Spring 2023)

Assignment 5

Date Due: See canvas

This assignment has a total of **(44 points)**. Exercises with **0 pt** might be

- something that is very basic, but you should know anyway, or
- something that is too technical for this course, or
- ill-posed (i.e., wrong).

Meanwhile, this is not to say that other problems necessarily lack the above features.

Note: Unless specified otherwise, you must show the details of your work via logical reasoning for each exercise. Simply writing a final result (whether correct or not) will receive **0 point**. **Explain** (briefly) if you claim something is trivial or straightforward. Provide a counterexample if you are trying to disprove something. It is **NOT OK** to write something like "how do we know that blahblahblah is even true..."

Exercise 5.1 (6 pts) Show that the following are posets.

(i) (2 pts) Let J be the set of closed intervals of the real line, with the partial order defined on J by

$$[a, b] \leq_{\text{int}} [c, d] \Leftrightarrow b \leq c \text{ or } [a, b] = [c, d].$$

(ii) (2 pts) The set $\mathbb{N}^n$, $n \in \mathbb{N}$, with the lexicographic order defined on $\mathbb{N}^n$ by

$$\begin{aligned}(x_1, \dots, x_n) \preceq (y_1, \dots, y_n) \Leftrightarrow &(x_1, \dots, x_n) = (y_1, \dots, y_n)\\ &\text{or } \exists k \in \{1, \dots, n\} \text{ with } x_i = y_i \text{ for } i < k \text{ and } x_k < y_k\end{aligned}$$

(iii) (2 pts) Given a poset $(P, \leq_P)$, the dual of of P, denoted by P^{d}, with the dual order defined on P by

$$\leq_{P^{\mathrm{d}}} := \{(a, b) \mid b \leq_P a\}.$$

Exercise 5.2 (2 pts) Let $0 < a_1 < a_2 < \cdots < a_{sr+1}$ be $sr+1$ integers, $s, r \in \mathbb{N}$. Show that we can select either $s+1$ of them, no one of which divides any other, or $r+1$ of them, each dividing the following one.

Exercise 5.3 (4 pts) Define a relation $\preceq$ on $\mathbb{N}$ by

$$m \preceq n \Leftrightarrow (m = n) \vee (m \text{ even and } n \text{ odd}) \vee (m, n \text{ both even or both odd, and } m < n)$$

(i) (2 pts) Show that $(\mathbb{N}, \preceq)$ is a total order.

(ii) (2 pts) Sketch a Hasse diagram for $(\mathbb{N}, \preceq)$, and provide the explicit coordinates for each element in $\mathbb{N}$.

Exercise 5.4 (4 pts)

(i) (2 pts) Sketch all non-isomorphic simple graphs with 3 vertices.

(ii) (2 pts) Sketch all 11 non-isomorphic simple graphs with 4 vertices and identify all pairs of complement graphs (including self-complementary ones, i.e., one that is isomorphic to its own complement).

Exercise 5.5 (2 pts) Show that two simple graphs are isomorphic iff their complement graphs are isomorphic.

Exercise 5.6 (2 pts) How many cycles of length n are there in the complete graph K_n, $n \geq 1$?

Exercise 5.7 (2 pts) Consider the complete graph K_n, $n > 0$. Show that if $\sum_i n_i = n$, $n_i \in \mathbb{N}$, then $\binom{n}{2} \geq \sum_i \binom{n_i}{2}$.

Exercise 5.8 (2 pts) For $m, n \in \mathbb{N}$, let C_{2m+1} and C_{2n+1} be two cycles, show that there exists a graph homomorphism $f : C_{2m+1} \to C_{2n+1}$ iff $m \geq n$.

Exercise 5.9 (2 pts) Given a finite graph G with the degrees of every vertex at least 2, show that G contains a cycle (as a subgraph).

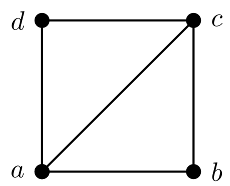
Exercise 5.10 (4 pts) A graph G is called **k-regular** if all vertices of G have the same degree k.

(i) (2 pts) Show that a k-regular bipartite graph has no cut-edge for $k \geq 2$.

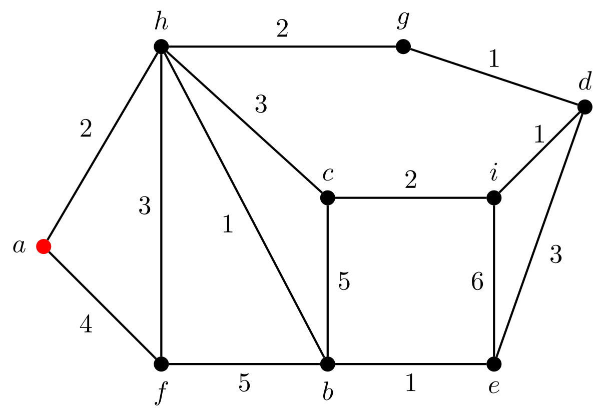
(ii) (2 pts) Show that a k-regular bipartite graph has a perfect matching for $k \geq 1$.

Exercise 5.11 (2 pts) Given graph G, show that G is a tree iff G is connected and e is a cut-edge for all $e \in E(G)$.

Exercise 5.12 (2 pts) Sketch all 8 spanning trees of the following graph.[1]



Exercise 5.13 (10 pts) Given the following simple connected graph, with weighted edges specified as follows,



(i) `(5 pts)` Find a minimum-weight spanning tree via Kruskal's algorithm. List the edges chosen in order and sketch the tree.

(ii) `(5 pts)` Given the root vertex a, find a shortest-path spanning tree via Dijkstra's algorithm. List the edges chosen in order, list the shortest path distance (from root vertex) to each vertex. Sketch the tree.

[1] For counting the number of spanning trees, check *Kirchhoff's matrix tree theorem.*

The following exercises are not to be graded

Exercise 5.14* Given Dilworth's Theorem as follows, find out what goes wrong with the proof.

Theorem (Dilworth's Theorem). *A finite poset $(P, \leq)$ of width k can be partitioned into k chains.*

"Poof". Induction on $n := |P|$, $n = 1$ is obvious. For the induction step n to step $n+1$, assume that Dilworth's theorem holds for posets with n elements and let P be a poset with $(n+1)$ elements. Let $m \in P$ be a maximal element. Then $|P - \{m\}| = n$. By induction hypothesis, there are chains $C_1, \ldots, C_{w(P-\{m\})}$ forming a partition of $P - \{m\}$. If $w(P - \{m\}) = k-1$, set $C_k := \{m\}$ and we are done. Otherwise, m has strict lower bounds and thus for some $i_0 \in \{1, \ldots, w(P - \{m\}\}$, we have that m is an upper bound of C_{i_0}. Then $C_{i_0} \cup \{m\}$ is a chain, and $C_1, \ldots, C_{i_0-1}, C_{i_0} \cup \{m\}, C_{i_0+1}, \ldots, C_{w(P-\{m\})}$ form a chain partition of P. □

Exercise 5.15* [SS10, ex. 4.4, Chap. 1] Decide whether it is possible to define a *total order* $\preceq$ on the complex numbers $\mathbb{C}$ such that,

(a) For all $z_1, z_2, z_3 \in \mathbb{C}$, $z_1 \preceq z_2$ implies $z_1 + z_3 \preceq z_2 + z_3$.

(b) For all $z_1, z_2, z_3 \in \mathbb{C}$ with $0 \prec z_3$, $z_1 \prec z_2$ implies $z_1 z_3 \preceq z_2 z_3$.

Exercise 5.16* [Gal11, p. 440] Given two matrices $A = (a_{ij}), B = (b_{ij}) \in M_{m\times m}(\mathbb{F}_2)$, i.e., A, B are $m \times m$ matrices with entries either 0 or 1, define

$$(A+B)_{ij} := a_{ij} \vee b_{ij}$$

$$(AB)_{ij} := \bigvee_{k=1}^{m} (a_{ik} \wedge b_{kj}) = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{im} \wedge b_{mj})$$

that is, interpret 0 as **FALSE**, 1 as **TRUE**, + as **OR**, and $\cdot$ as **AND**. Let $B^k := A + A^2 + \cdots + A^k$. Show that there is some $k_0 \in \mathbb{N}$ such that

$$B^{n+k_0} = B^{k_0}$$

for all $n \geq 1$. Describe the graph associated with the adjacency matrix B^{k_0}.

Exercise 5.17* [Gal11, p. 254] The purpose of this problem is to prove that the number of spanning trees of the complete graph K_n, $n \geq 2$, is n^{n-2}, a formula due to Cayley (1889).

(i) Let $T(n; d_1, \ldots, d_n)$ be the number of trees with $n \geq 2$ vertices $v_1, \ldots, v_n$, and degrees $d(v_1) = d_1$, $d(v_2) = d_2$, $\ldots$, $d(v_n) = d_n$, with $d_i \geq 1$. Show that

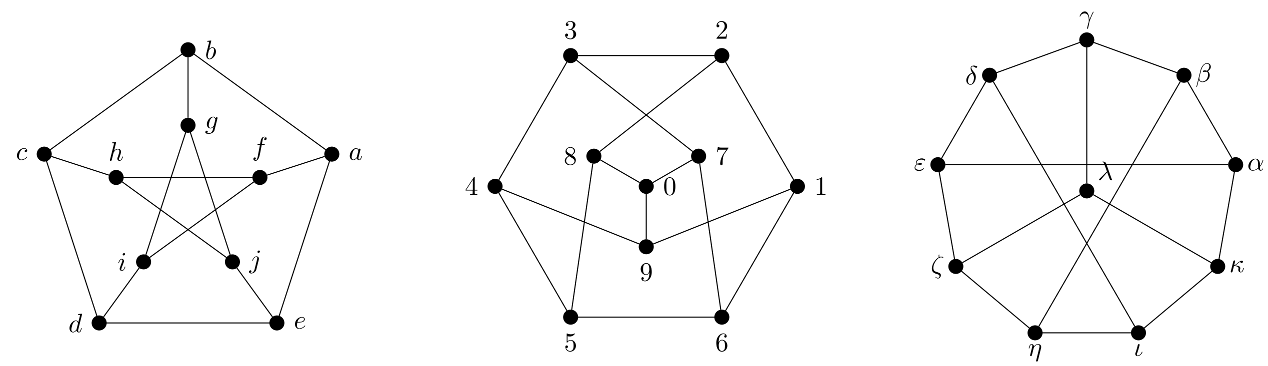
$$T(n; d_1, \ldots, d_n) = \binom{n-2}{d_1 - 1, d_2 - 1, \ldots, d_n - 1}$$

(ii) Show that $d_1, \ldots, d_n$, with $d_i \geq 1$, are degrees of a tree with n vertices iff

$$\sum_{i=1}^{n} d_i = 2(n-1)$$

(iii) Use (i) and (ii) prove that the number of spanning trees of K_n is n^{n-2}.

Exercise 5.18* Show that the following 3 drawings (of the Petersen graph) are isomorphic.



Exercise 5.19* (Birkhoff–von Neumann theorem) A *permutation matrix* is a square binary matrix that has exactly one entry of 1 in each row and each column and 0s elsewhere.

A *doubly stochastic matrix* (also called *bistochastic matrix*), is a square matrix $A = (a_{ij})$ of nonnegative real numbers, each of whose rows and columns sums to 1, i.e., $\sum_i a_{ij} = \sum_j a_{ij} = 1$.

Show that every doubly stochastic matrix is a convex combination of permutation matrices. In other words, if A is a doubly stochastic matrix, then there exists $\alpha_1, \ldots, \alpha_k \geq 0$ with $\sum_{i=1}^k \alpha_i = 1$, and permutation matrices $P_1, \ldots, P_k$ such that $A = \sum_{i=1}^k \alpha_i P_i$.

References

[Gal11] J. Gallier. *Discrete Mathematics.* Universitext. Springer, 2011 (Cited on page 3).

[SS10] E.M. Stein and R. Shakarchi. *Complex Analysis.* Princeton lectures in analysis. Princeton University Press, 2010 (Cited on page 3).