Ve203 Discrete Mathematics (Spring 2023)

Assignment 2

Date Due: See canvas

This assignment has a total of **(28 points)**. Exercises with **0 pt** might be

- something that is very basic, but you should know anyway, or
- something that is too technical for this course, or
- ill-posed (i.e., wrong).

Meanwhile, this is not to say that other problems necessarily lack the above features.

Note: Unless specified otherwise, you must show the details of your work via logical reasoning for each exercise. Simply writing a final result (whether correct or not) will receive **0 point**. **Explain** (briefly) if you claim something is trivial or straightforward. Provide a counterexample if you are trying to disprove something. It is **NOT OK** to write something like "how do we know that blahblahblah is even true…"

Exercise 2.1 (2 pts) Show that concatenation of string is associative, i.e., $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in \Sigma^*$.

Exercise 2.2 (4 pts) Based on the recursive definition of string and string concatenation $\cdot$, use structural induction to show that for all strings $x, y, z \in \Sigma^*$,

(i) (2 pts) If $x \cdot y = x$, then $y = \varepsilon$.

(ii) (2 pts) If $x \cdot y = x \cdot z$, then $y = z$.

Exercise 2.3 (6 pts) Consider the binary/bivariate function $\star$ given by

$$(a, b) \star (c, d) := (a - b + \max\{b, c\}, d - c + \max\{b, c\})$$

(i) (2 pts) Show that $(\mathbb{N}^2, \star)$ is a magma.

(ii) (2 pts) If $(\mathbb{N}^2, \star, e)$ is a monoid, what is the identity element e?

(iii) (2 pts) Show that $(\mathbb{N}^2, \star, e)$ is indeed a monoid for the identity e found in (ii).

Exercise 2.4 (4 pts) Show that for any logical proposition φ using the connectives $\{\neg, \wedge, \vee, \to\}$, i.e., wffs, there exists a proposition that is logically equivalent to φ using only

(i) (2 pts) $\{\downarrow\}$, where $\downarrow$ is the Peirce arrow (NOR), with $p \downarrow q := \neg(p \vee q)$.

(ii) (2 pts) $\{|\}$, where $|$ is the Sheffer stroke (NAND), with $p \mid q := \neg(p \wedge q)$.

Exercise 2.5 (4 pts) Show by induction that the following two algorithms `msort` and `merge` are correct.

Input: $A[1 \dots n]$, unsorted array
Output: all the $A[i], 1 \le i \le n$ in increasing order

Function `msort`$(A[1 \dots n])$**:**
 if $n = 1$ **then**
  return A
 else
  $L \leftarrow$ `msort`$(1 \dots \lfloor \frac{n}{2} \rfloor)$
  $R \leftarrow$ `msort`$(\lfloor \frac{n}{2} \rfloor + 1 \dots n)$
  return `merge`(L, R)
 end
end

```
msort :: Ord a => [a] -> [a]
msort []  = []
msort [x] = [x]
msort xs  = merge (msort ys) (msort zs)
    where (ys, zs) = halve xs

halve :: [a] -> ([a], [a])
halve xs = (take n xs, drop n xs)
    where n = length xs `div` 2
    -- splitAt n xs
```

Input: $X[1 \dots n]$, $Y[1 \dots m]$, 2 sorted arrays
Output: $X \cup Y$ sorted with elements in increasing order

```
Function merge(X[1...n], Y[1...m]):
    if n = 0 then
        return Y
    else if m = 0 then
        return X
    else if X[1] < Y[1] then
        return X[1] followed by merge(X[2...n], Y)
    else
        return Y[1] followed by merge(X, Y[2...m])
    end
end
```

```haskell
merge :: Ord a => [a] -> [a] -> [a]
merge [] ys = ys
merge xs [] = xs
merge (x:xs) (y:ys)
    | x <= y    = x:merge xs (y:ys)
    | otherwise = y:merge (x:xs) ys
```

Exercise 2.6 (4 pts) Let

$$m \sim n \qquad \Leftrightarrow \qquad 2 \mid (n - m), \qquad\qquad m, n \in \mathbb{Z}.$$

(i) `(1 pt)` Show that $\sim$ is an equivalence relation.

(ii) `(1 pt)` What partition $\mathbb{Z}_2 \coloneqq \mathbb{Z}/\!\sim$ is induced by $\sim$?

(iii) `(2 pts)` Define addition and multiplication on $\mathbb{Z}_2$ by the addition and multiplication of class representatives, i.e.,

$$[m] + [n] \coloneqq [m + n], \qquad\qquad [m] \cdot [n] \coloneqq [m \cdot n].$$

Show that these operations are well-defined, i.e., independent of the representatives m and n of each class.

Exercise 2.7 (2 pts) Given a relation R on a nonempty set A, show that

(i) `(0 pts)` If R is transitive and symmetric, then R is reflexive.

(ii) `(2 pts)` If R is transitive and asymmetric, then R is irreflexive.

Exercise 2.8 (2 pts) Assume that Π is a partition of a set A. Define the relation R_Π as follows:

$$x R_\Pi y \Leftrightarrow (\exists B \in \Pi)(x \in B \wedge y \in B).$$

Show that R_Π is an equivalence relation on A.

The following exercises are not to be graded

Exercise 2.9* Show by induction that every nonempty finite set of real numbers has a smallest element.

Exercise 2.10* Let A be a set, and $f : A \times A \to A$ be a binary associative function, i.e., (A, f) is a semigroup. If for every $a \in A$, there exists an $e \in A$ such that $f(e, a) = a$, is (A, f, e) a monoid?

Exercise 2.11* Let $f : X \to Y$ be any function. Show that for all $A, B \subset X$, we have

(i) $f(A \cup B) = f(A) \cup f(B)$.

(ii) $f(A \cap B) \subset f(A) \cap f(B)$, where equality holds if f is injective.

(iii) $f(A) - f(B) \subset f(A - B)$, where equality holds if f is injective.

Show that for all $C, D \subset Y$, we have

(iv) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

(v) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

(vi) $f^{-1}(C - D) = f^{-1}(C) - f^{-1}(D)$.

Note that the function $f^{-1} : 2^Y \to 2^X$ has better behavior than $f : 2^X \to 2^Y$ with respect to union, intersection, and complementation. (Note that $f : 2^X \to 2^Y$ is induced by $f : X \to Y$, which is a not uncommmon overloading/abusing of notation.)