Ve203 Discrete Mathematics (Spring 2023)

Assignment 3

Date Due: See canvas

This assignment has a total of **(32 points)**. Exercises with **0 pt** might be

- something that is very basic, but you should know anyway, or
- something that is too technical for this course, or
- ill-posed (i.e., wrong).

Meanwhile, this is not to say that other problems necessarily lack the above features.

Note: Unless specified otherwise, you must show the details of your work via logical reasoning for each exercise. Simply writing a final result (whether correct or not) will receive **0 point**. **Explain** (briefly) if you claim something is trivial or straightforward. Provide a counterexample if you are trying to disprove something. It is **NOT OK** to write something like "how do we know that blahblahblah is even true…"

Exercise 3.1 (8 pts)

(i) `(2 pts)` Prove that the function $f : \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ defined by

$$f(a_0a_1\cdots a_n\cdots, b_0b_1\cdots b_n\cdots) = a_0b_0a_1b_1\cdots a_nb_n\cdots$$

is a bijection, where $a_i, b_i \in \{0,1\}$, and $\{0,1\}^{\mathbb{N}}$ is the set of countably infinite sequences of 0 and 1.

(ii) `(2 pts)` Represent the reals in $(0,1)$ by their decimal expansions **WITHOUT** the infinite suffix $99999\cdots$. Define the function $h : (0,1) \times (0,1) \to (0,1)$ by

$$h(0.r_0r_1\cdots r_n\cdots, 0.s_0s_1\cdots s_n\cdots) = 0.r_0s_0r_1s_1\cdots r_ns_n\cdots$$

with $r_i, s_i \in \{0,1,2,\ldots,9\}$. Prove that h is injective but not surjective.

(iii) `(2 pts)` If we pick in (ii) the decimal representations ending **WITH** the infinite suffix $99999\cdots$ rather that an infinite string of 0's, prove that h is also injective but still not surjective.

(iv) `(2 pts)` Show that there exists a bijection between $(0,1)\times(0,1)$ and $(0,1)$.

Exercise 3.2 (8 pts) Let R and S be two relations on a set X.

(i) `(2 pts)` Show that if R and S are both reflexive, then $R \circ S$ is reflexive.

(ii) `(2 pts)` Show that if R and S are both symmetric and if $R\circ S = S\circ R$, then $R\circ S$ is symmetric.

(iii) `(2 pts)` Show that R is transitive iff $R\circ R \subset R$. Show that if R and S are both transitive and if $R\circ S = S\circ R$, then $R\circ S$ is transitive. Can the hypothesis $R\circ S = S\circ R$ be omitted?

(iv) `(2 pts)` Show that if R and S are both equivalence relations and if $R\circ S = S\circ R$, then $R\circ S$ is the smallest equivalence relation containing R and S.

Exercise 3.3 (8 pts) Let X, Y, Z be any three nonempty sets and let $g : Y \to Z$ be any function. Define the function $L_g : Y^X \to Z^X$ (L_g, as a reminder that we compose with g on the left), by $L_g(f) = g\circ f$ for every function $f : X \to Y$.

(i) `(2 pts)` Show that if $Y = Z$ and $g = \mathrm{id}_Y$, then $L_{\mathrm{id}_Y}(f) = f$ for all $f : X \to Y$.

(ii) `(2 pts)` Let T be another nonempty set and let $h : Z \to T$ be any function. Show that $L_{h\circ g} = L_h \circ L_g$.

(iii) `(2 pts)` Show that if $g : Y \to Z$ is injective, then $L_g : Y^X \to Z^X$ is also injective.

(iv) `(2 pts)` Show that if $g : Y \to Z$ is surjective, then $L_g : Y^X \to Z^X$ is also surjective.

Exercise 3.4 (6 pts) Let X be a finite set.

(i) `(2 pts)` Show that every injection $f : X \to X$ is a bijection.

(ii) `(2 pts)` Show that every surjection $f : X \to X$ is a bijection.

(iii) `(2 pts)` Given counterexamples to (i) and (ii) when X is infinite.

Exercise 3.5 (2 pts) Given sets A, B such that $A \subset B$ and $|B| < \infty$, use pigeonhole principle to show that $A = B$.

The following exercises are not to be graded

Exercise 3.6*

(i) Let $\pi : \mathbb{N}^2 \to \mathbb{N}$ be Cantor's pairing function. Find $m, n \in \mathbb{N}$ such that $\pi(m, n) = 2023$.

(ii) Give an explicit formula for a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$.

Exercise 3.7* Briefly explain why the collection $\{x \mid \operatorname{card} x = 1\}$ is not a set.

Exercise 3.8* Find an explicit bijection between $(0, 1]$ and $[0, 1]$. Sketch the graph of the function.[1]

Exercise 3.9* Explain why the diagonal argument does not work for rational numbers.

Exercise 3.10* Consider a periodic function $f : \mathbb{R} \to \mathbb{C}$ with period $T > 0$. Note that for any $k \in \mathbb{N} \setminus \{0\}$, kT is also a period of f. Assume f has a smallest period, consider the Fourier series of f given by

$$f(x) \sim \sum_{n \in \mathbb{Z}} c_n e^{2\pi i n x / T} \tag{1}$$

where

$$c_n = \frac{1}{T} \int_0^T f(x) e^{-2\pi i n x / T} \mathrm{d}x$$

Show that Eq. (1) is independent of the choice of the period T of f.

[1] Good luck if you are after a continuous function, i.e., continuous in the usual sense.