Ve203 Discrete Mathematics (Spring 2023)

Assignment 4

Date Due: None

Exercise 4.1 Consider the functions $f : B \to U$, count the number of functions and fill in the blanks below.

Elements of Domain	Elements of Codomain	Any f	Injective f	Surjective f
distinguishable	distinguishable			
indistinguishable	distinguishable			

where

(i) $B = [3]$ and $U = [5]$. (ii) $B = [5]$ and $U = [3]$.

Exercise 4.2 The *Euler's Totient Function*, or the *Euler phi function*, denoted $\varphi(n)$ or $\phi(n)$, counts the number of positive integers less than n and relatively prime to n, i.e.

$$\varphi(n) := |\{k \in \mathbb{N} \mid \gcd(k, n) = 1, 1 \leq k \leq n\}|$$

where "gcd" stands for "greatest common divisor".

Derive the following formula for the Euler's totient function φ

$$\varphi(n) = n \prod_{\substack{p \in \mathbb{P} \\ p \mid n}} \left(1 - \frac{1}{p}\right)$$

by applying the inclusion-exclusion principle to the set $[n]$. $\mathbb{P}$ is the set of prime numbers.

Exercise 4.3 Consider $n \in \mathbb{N}$, $n \geq 2000$.

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 \leq n$$

What are the number of integer solutions if

(i) $x_i > 0$ and $=$ holds; (ii) $x_i \geq 0$ and $=$ holds; (iii) $x_i > 0$ and $<$ holds;
(iv) $x_i \geq -1$ and $<$ holds; (v) $1 \leq x_i \leq 5$ and $<$ holds; (vi) $1 \leq x_i \leq 5$ and $=$ holds;

(Express the solution in closed forms. For example, results containing $\sum_{k=1}^{n}(\cdots)$ is not closed; results containing $\sum_{k=1}^{4}(\cdots)$ is closed.)

Exercise 4.4 Given a formal power series $A(x) = \sum_{n \geq 0} a_n x^n$, show that

(i) $A(x) = a_0$ if $DA = 0$. (ii) $A(x) = c \exp(x)$ if $DA = A$, where c is a constant and $\exp(x) := \sum_{n \geq 0} x^n/n!$.

Exercise 4.5 Let $\lambda(x)$ be a formal power series. Define the operator $\lambda D : \mathbb{C}[[x]] \to \mathbb{C}[[x]]$ by

$$(\lambda D)f := \lambda(Df) = \lambda f'$$

(i) Show that $(D\lambda)f = \lambda' f + \lambda f'$.
(ii) Show that $(xD)^n = \sum_{k=1}^{n} \left\{ {n \atop k} \right\} x^k D^k$ for $n \in \mathbb{N} \setminus \{0\}$.
(iii) Show that $(Dx)^n = \sum_{k=0}^{n} \left\{ {n+1 \atop k+1} \right\} x^k D^k$ for $n \in \mathbb{N}$.

Exercise 4.6 Given a formal power series $A(x) = \sum_{n \geq 0} a_n x^n$, show that

(i) If $k \in \mathbb{N} \setminus \{0\}$, then

$$\sum_{n \geq 0} a_{n+k} x^n = \frac{1}{x^k} \left[A(x) - \sum_{n=0}^{k-1} a_n x^n \right]$$

(ii) If p is a polynomial, then

$$(p(xD)A)(x) = \sum_{n \geq 0} p(n) a_n x^n$$

Exercise 4.7 For integers $n, k \geq 0$, prove Pascal's identity

$$\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$$

by verifying the following equalities of generating functions.

(i) $\sum_{k\geq 0} \binom{n+1}{k+1} x^k = \sum_{k\geq 0} \left[\binom{n}{k} + \binom{n}{k+1}\right] x^k$

(ii) $\sum_{n\geq 0} \binom{n+1}{k+1} x^n = \sum_{n\geq 0} \left[\binom{n}{k} + \binom{n}{k+1}\right] x^n$

Exercise 4.8 Find closed formulas (in the sense that there is no infinite sums) for the generating function $A(x)$ of the following sequences $(a_n)_{n\geq 0}$. You may want to consult tables for z-transform from signal and systems. α and ω are fixed scalars.

(i) $a_n = n$

(ii) $a_n = n^2$

(iii) $a_n = \alpha^n$

(iv) $a_n = n\alpha^n$

(v) $a_n = n^2\alpha^n$

(vi) $a_n = \cos \omega n$

(vii) $a_n = \alpha^n \sin \omega n$

(viii) $a_n = (n)_2 = n(n-1)$

(ix) $a_n = (n)_3 = n(n-1)(n-2)$

Exercise 4.9 (Series Multisection) Show that for $s, t \in \mathbb{N}$ with $0 \leq t < s$,

$$\sum_{m\geq 0} \binom{n}{t+sm} = \frac{1}{s} \sum_{j=0}^{s-1} 2^n \cos^n\left(\frac{\pi j}{s}\right) \cos \frac{\pi(n-2t)j}{s}$$

Exercise 4.10 Verify the following identities (if you like)

(i) $\displaystyle\sum_{n\geq 0} \binom{2n}{n} x^n = \frac{1}{\sqrt{1-4x}}$

(ii) $\displaystyle\sum_{n\geq 0} \binom{3n}{n} x^n = \frac{2\cos\left(\frac{1}{3}\arcsin\left(\frac{3}{2}\sqrt{3x}\right)\right)}{\sqrt{4-27x}}$