Ve203 Discrete Mathematics (Spring 2023)

Assignment 1

Date Due: See canvas

This assignment has a total of **(27 points)**. Exercises with **0 pt** might be

- something that is very basic, but you should know anyway, or
- something that is too technical for this course, or
- ill-posed (i.e., wrong).

Note: Unless specified otherwise, you must show the details of your work via logical reasoning for each exercise. Simply writing a final result (whether correct or not) will receive **0 point**. **Explain** (briefly) if you claim something is trivial or straightforward. Provide a counterexample if you are trying to disprove something. It is **NOT OK** to write something like "how do we know that blahblahblah is even true..."

Exercise 1.1 (6 pts) Given $\varphi = (A \to (B \to C)) \to (B \to (A \to C))$,

(i) (2 pts) Write the truth table for φ.

(ii) (2 pts) Write φ in disjunctive normal form.

(iii) (2 pts) Write φ in conjunctive normal form.

Exercise 1.2 (6 pts) The following shows the truth table for all $2^{2^2} = 16$ different binary logical operators φ_i, $i = 0, \dots, 15$.

p	q	φ_0	φ_1	φ_2	φ_3	φ_4	φ_5	φ_6	φ_7	φ_8	φ_9	φ_{10}	φ_{11}	φ_{12}	φ_{13}	φ_{14}	φ_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

Using infix notation, for example, φ_{13} can be represented as $\varphi_{13} = \to(p, q) = p \to q$.

A set S of logical operators is called *functionally complete* if every compound proposition is logically equivalent to a compound proposition involving only these logical operators in S. In this exercise, in order to show S is a functionally complete set, it suffices to verify that for all $i = 0, \dots, 15$, φ_i over logical variables p and q can be represented using only operators in S.

(i) (1 pt) Show that $\{\wedge, \vee, \neg\}$ is functionally complete.

(ii) (1 pt) Show that $\{\vee, \wedge\}$ is *not* functionally complete.

(iii) (4 pts) Suppose that the logical variables take on numerical values 0 and 1 as in **Exercise 1.2**, and consider $\varphi_i : \{0,1\}^2 \to \{0,1\}$ given by

$$\varphi_i(p, q) = \begin{cases} 1, & \text{if } a_i p + b_i q + c_i > 0 \\ 0, & \text{otherwise} \end{cases}, \qquad a_i, b_i, c_i \in \mathbb{R}$$

Find valid a_i, b_i, c_i for each $i = 0, \dots, 15$.

Exercise 1.3 (7 pts) Let M be a set and let $X, Y, Z, W \subset M$. We define the *symmetric difference*:

$$X \triangle Y := (X - Y) \cup (Y - X)$$

(i) (1 pt) Show that $X \triangle Y = (X \cup Y) - (X \cap Y)$.

(ii) (1 pt) Show that $(M - X) \triangle (M - Y) = X \triangle Y$.

(iii) (1 pt) Show that the symmetric difference is associative, i.e., $(X \triangle Y) \triangle Z = X \triangle (Y \triangle Z)$.

(iv) (1 pt) Show that $X \cap (Y \triangle Z) = (X \cap Y) \triangle (X \cap Z)$.

(v) (1 pt) Show that $X \triangle Y = Z \triangle W$ iff $X \triangle Z = Y \triangle W$.

(vi) (1 pt) Indicate the region of $X \triangle Y \triangle Z$ in a Venn diagram.

(vii) (1 pt) Sketch a Venn diagram for 4 distinct sets.

Exercise 1.4 (4 pts) Let X be a finite set, define the distance/metric $\varrho(A, B)$ of two sets $A, B \in 2^X$ by

$$\varrho(A, B) := |A \triangle B|.$$

Show that $(2^X, \varrho)$ is a *metric space* by verifying that for all $A, B, C \in 2^X$,

(i) `(1 pt)` $\varrho(A, B) = 0$ iff $A = B$;

(ii) `(1 pt)` $\varrho(A, B) = \varrho(B, A)$;

(iii) `(2 pts)` $\varrho(A, C) \leq \varrho(A, B) + \varrho(B, C)$.

Exercise 1.5 (2 pts) Let $x, y \in \mathbb{R}$, show that $\forall x \exists y (xy = 0) \Leftrightarrow \exists y \forall x (xy = 0)$

Exercise 1.6 (2 pts) Let (F_n) be the Fibonacci sequence with $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Let

$$\phi = \frac{1 + \sqrt{5}}{2}, \qquad \bar{\phi} = \frac{1 - \sqrt{5}}{2}$$

Show that $F_{m+n} = \phi^m F_n + \bar{\phi}^n F_m$ for all $m, n \in \mathbb{N}$, using

(i) `(0 pts)` the explicit formula (i.e., using powers of ϕ and $\bar{\phi}$) for F_n.

(ii) `(2 pts)` induction.

The following exercises are not to be graded

Exercise 1.7* Consider the table listing properties for $\vee$ and $\wedge$, do the same for max and min (choose a convenient domain of your own interest).

Exercise 1.8* Write four definitions for the **OR** function $\vee : \mathbb{B}^2 \to \mathbb{B}$ (not necessarily in Haskell).

Exercise 1.9* Let $\sum_{k=-\infty}^{\infty} := \lim_{n\to\infty} \sum_{k=-n}^{n}$ and $f(x) := \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x}$, where $c_k \in \mathbb{C}$ for all $k \in \mathbb{Z}$. Show that if $\sum_{k=-\infty}^{\infty} |k|^n |c_k| < \infty$, then the nth derivative of f is given by

$$f^{(n)}(x) = \sum_{k=-\infty}^{\infty} (2\pi i k)^n c_k e^{2\pi i k x}$$

(i) Show the base case for $n = 1$.

(ii) Show the inductive case for $n > 1$.

Exercise 1.10* What is wrong with the following **proof** of the "theorem"?

"Theorem". Given any $a > 0$, then $a^{n-1} = 1$ for all positive integer n.

"poof". If $n = 1$, $a^{n-1} = a^{1-1} = a^0 = 1$. By induction, assume that the theorem is true for $n = 1, 2, \ldots, k$, then for $n = k + 1$,

$$a^{(k+1)-1} = a^k = \frac{a^{k-1} \times a^{k-1}}{a^{(k-1)-1}} = \frac{1 \times 1}{1} = 1$$

therefore the theorem is true for all positive integers n. $\square$