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上海交通大学

Physics (PHYS2500J), Unit 3 Electromagnetic induction: 2. Inductance

Xiao-Fen Li
Associate Professor, SJTU

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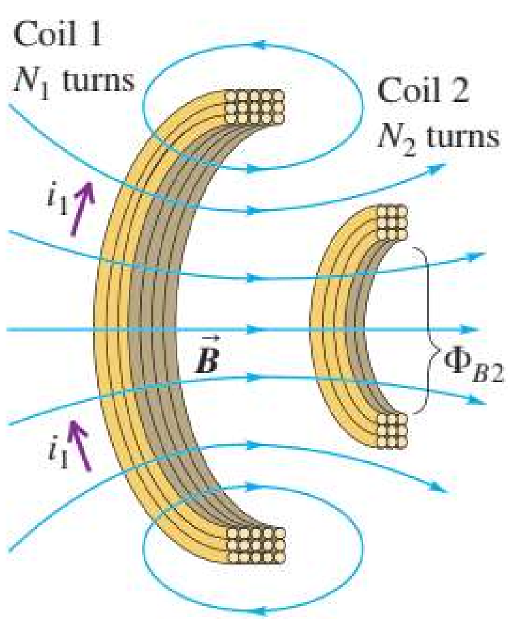
Mutual inductance

1. The scenario is like this: two coils are placed near each other. An AC current is applied to coil 1, what is the emf generated in coil 2?

30.1 A current i_1 in coil 1 gives rise to a magnetic flux through coil 2.

Mutual inductance: If the current in coil 1 is changing, the changing flux through coil 2 induces an emf in coil 2.



2. The magnetic field generated by coil 1 and picked up by coil 2 is defined as Φ_2

3. The magnetic field generated by coil 1 is proportional to the current in coil 1 i_1 and Φ_2. One can define a constant M_{12} relating the two as

$$N_2\Phi_2 \equiv \Psi_2 = M_{12} i_1$$

Flux linkage

4. Faraday's law gives the emf in coil 2

$$\mathcal{V}_2 = -N_2 \frac{\mathrm{d}\Phi_2}{\mathrm{d}t}$$

$$\mathcal{V}_2 = -M_{12} \frac{\mathrm{d}i_1}{\mathrm{d}t}$$

The unit of inductance

The unit of inductance is Henry, H

$$M_{12} = -\frac{n_2 \Phi_2}{i_1}$$

1 H = 1 Wb / 1 A

1 H of mutual inductance means 1 A in the primary coil will generate 1 Wb of flux link in the secondary coil.

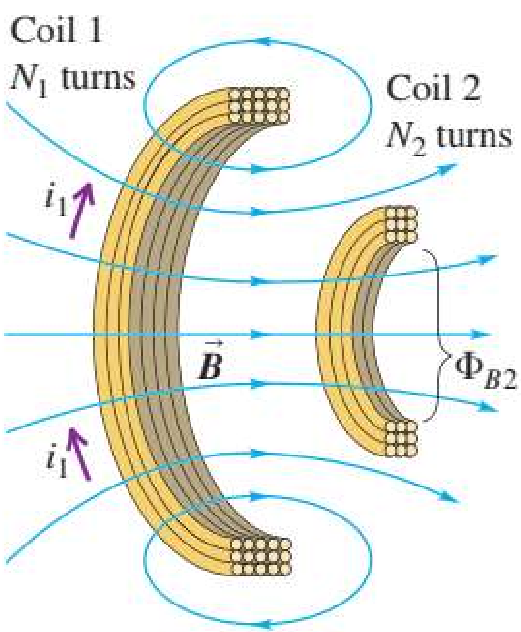
$$M_{12} = -\frac{\mathcal{V}_2}{\mathrm{d}i_1/\mathrm{d}t}$$

1 H = 1 V / (1 A / 1 s) = 1 Ωs

The reason to define inductance:
Inductance makes circuit problems with time varying current simpler.

How to have a larger mutual inductance?

1. What parameters affect the mutual inductance?



$$M_{21} \propto ?$$

1. Is it related to n_2?

It is proportional to n_2.

2. Is it related to n_1?

It is proportional to n_1, too. Since inductance is defined by flux per unit current (not per unit current turn).

3. It is also related to the distance between the coils.

4. And the permeability of the material inserted (air for air cored coils, but can insert iron core, too).

Mutual inductance is symmetric

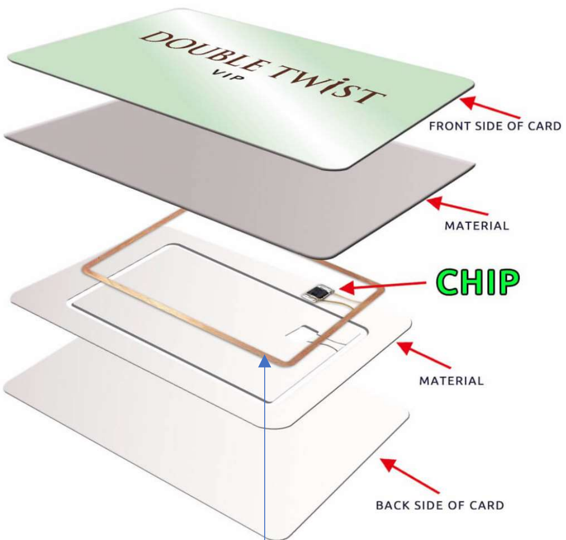
1. Mutual inductance from coil 1 to coil 2 is the same as that from coil 2 to coil 1.

$$M_{21} = M_{12} = M$$

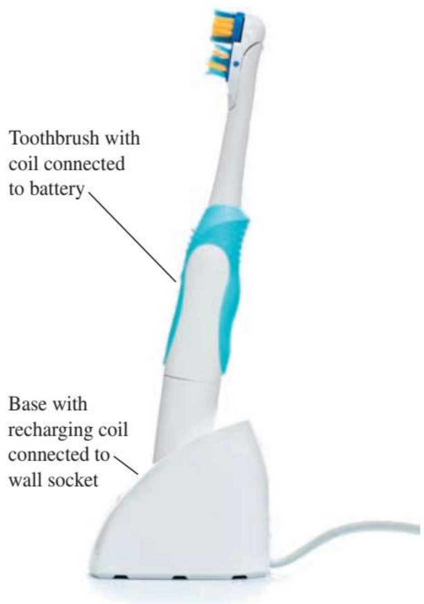
2. The proof involves the vector potential $\boldsymbol{A}$. Interested student can refer to electrodynamics or electromagnetic field books.

Applications

1. Emf generated by mutual inductance can be used as low power source for charging or electronic chips.



This coil is the energy receiving coil.



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“Mutual inductance” between a coil and itself

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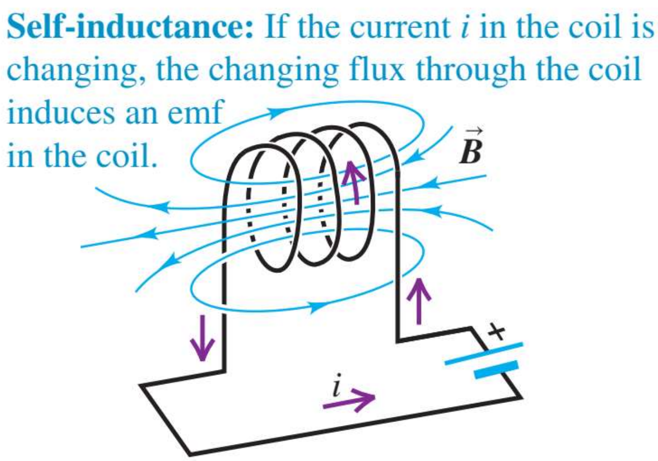
1. A coil can generate flux through itself, which can be used to define the self-inductance L, as

$$n\Phi = Li$$

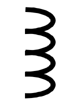


and of course, emf due to the current variation

$$\mathcal{V} = -L\frac{\mathrm{d}i}{\mathrm{d}t}$$

Self inductance is one of the 3 major passive circuit components (the other two being capacitor and resistor)

Symbol in circuit diagram for inductors.



2. What are the parameters that affects L?

Self inductance L is proportional to the square of the number of turns

$$L \propto n^2$$

One order comes from the fact that for more turns, 1 A generates stronger field, the other one comes from the fact that for more turns, the same field generates more flux link.

Self inductance L is affected by the shape and insertion materials.

Connecting two coils in series

1. If the coils are connected in the same direction (mutual inductance is positive).

The total flux is:

$$\Psi = n_1\Phi_1 + n_2\Phi_2 = (L_1 i_1 + M i_2) + (L_2 i_2 + M i_1)$$

Since current is the same (serial connection)

$$L = L_1 + L_2 + 2M$$

2. If the coils are connected in the opposite direction (mutual inductance defined as a positive value but the flux are in opposite ways).

$$L = L_1 + L_2 - 2M$$

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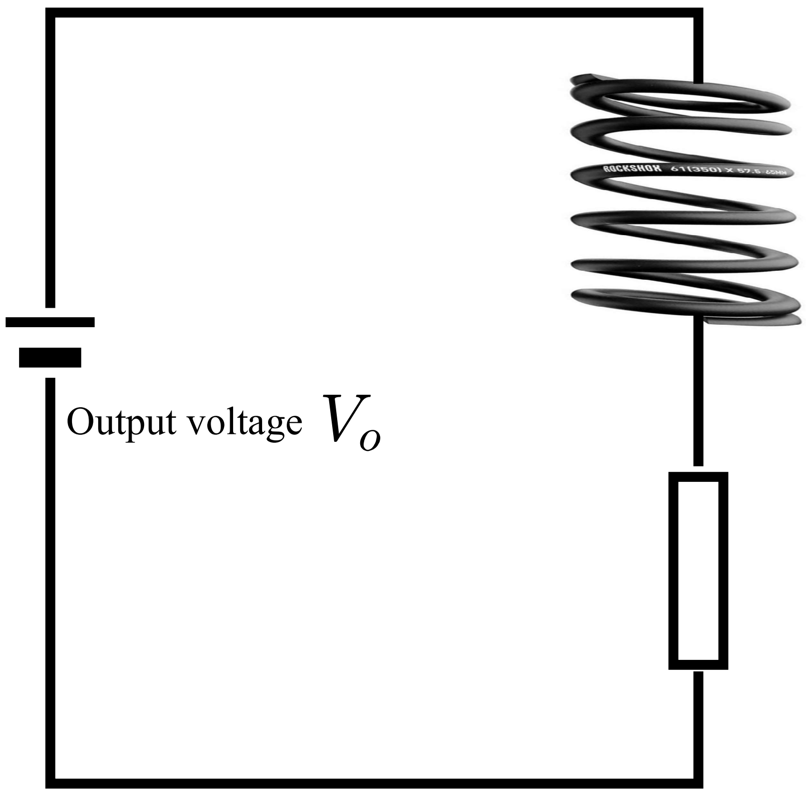
Question: why do we need inductance?

Inductive voltage



1. A simple circuit with an inductor, establish the circuit equation for it.

2. Starting from concepts. An inductor has self inductance L and no resistance. Any real coil can be considered a resistor and an inductor connected in series.

3. When the current in the circuit is steady, no voltage drop on the inductor.

4. When the current is varying with time.

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = iR - V_o$$

Electric field in a pure inductor is 0, since resistance is 0. Notice the direction of the resistive voltage drop and that inside the battery.

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Psi}{\mathrm{d}t} = -L\frac{\mathrm{d}i}{\mathrm{d}t}$$

Faraday's law.

$$V_o = L\frac{\mathrm{d}i}{\mathrm{d}t} + IR$$

Inductive voltage drop

Resistive voltage drop

Several points about inductive voltage

1. About ideal inductor: in reality, an ideal inductor does not exist. The circuit model just moves the resistive voltage drop into other parts of the circuit. From our knowledge on resistance, this operation will not alter the conclusion that an inductor has an equivalent voltage drop.

2. From the power point of view. The power source is doing work, the power of which is the same as $V_o i$ (Due to the definition of emf for the power supply, besides the Joule heating in the power source, the rest of work has the form of $V_o i$.), which mathematically includes an inductive part $L\mathrm{d}i/\mathrm{d}t$ and a resistive part iR. The two parts corresponds to the inductive energy storage and Joule heating on the resistor.

3. However, from microscopic point of view, EJ contains only resistive heating part of the power. The reason is that in AC circuit,
$$V \neq \int \vec{E} \cdot \mathrm{d}\vec{l}$$
actually, there is no clear definition of potential for an AC circuit, since E is not a conservative field any more.

4. In other words, microscopic Ohm's law is always correct that $E = \rho J$, since the relation is reestablished in the order of 10s of femto-seconds.

What is the inductive power used for?

1. From the battery's point of view, the inductive power is used for magnetic energy storage.

$$P = Vi = Li\frac{\mathrm{d}i}{\mathrm{d}t}$$

2. For an inductor, starting from 0 current status, the total stored energy is (important)

$$\mathcal{E}_m = \int_0^t P\mathrm{d}t = L\int i\frac{\mathrm{d}i}{\mathrm{d}t}\mathrm{d}t = L\int i\mathrm{d}i = \frac{LI^2}{2}$$

I is the final current, i is the current as a function of time.

3. For a long solenoid, length L, total number of turns NL, area A. The energy can be calculated as

$$L = \frac{Nl\Phi}{i} = lN^2\mu_0 A$$

$$\mathcal{E}_m = \frac{LI^2}{2} = lA\frac{N^2\mu_0^2 I^2}{2\mu_0} = lA\frac{B^2}{2\mu_0}$$

Volume of the magnetic field region.

Field energy intensity. Unit J/m^3

4. Magnetic field energy density. (important)

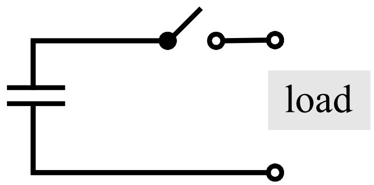
$$\frac{B^2}{2\mu_0} = \frac{\mu_0 H^2}{2}$$

magnetic and electric field energy storage

$$E_e = \frac{1}{2}CV^2 = \iiint \mathrm{d}V \frac{\epsilon_0 E^2}{2}$$

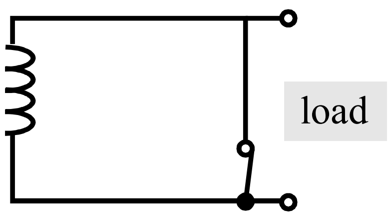
$$E_m = \frac{1}{2}LI^2 = \iiint \mathrm{d}V \frac{B^2}{2\mu_0}$$

1. Question: how to release the stored energy from a capacitor bank?



Close the switch, energy will be released to the load.

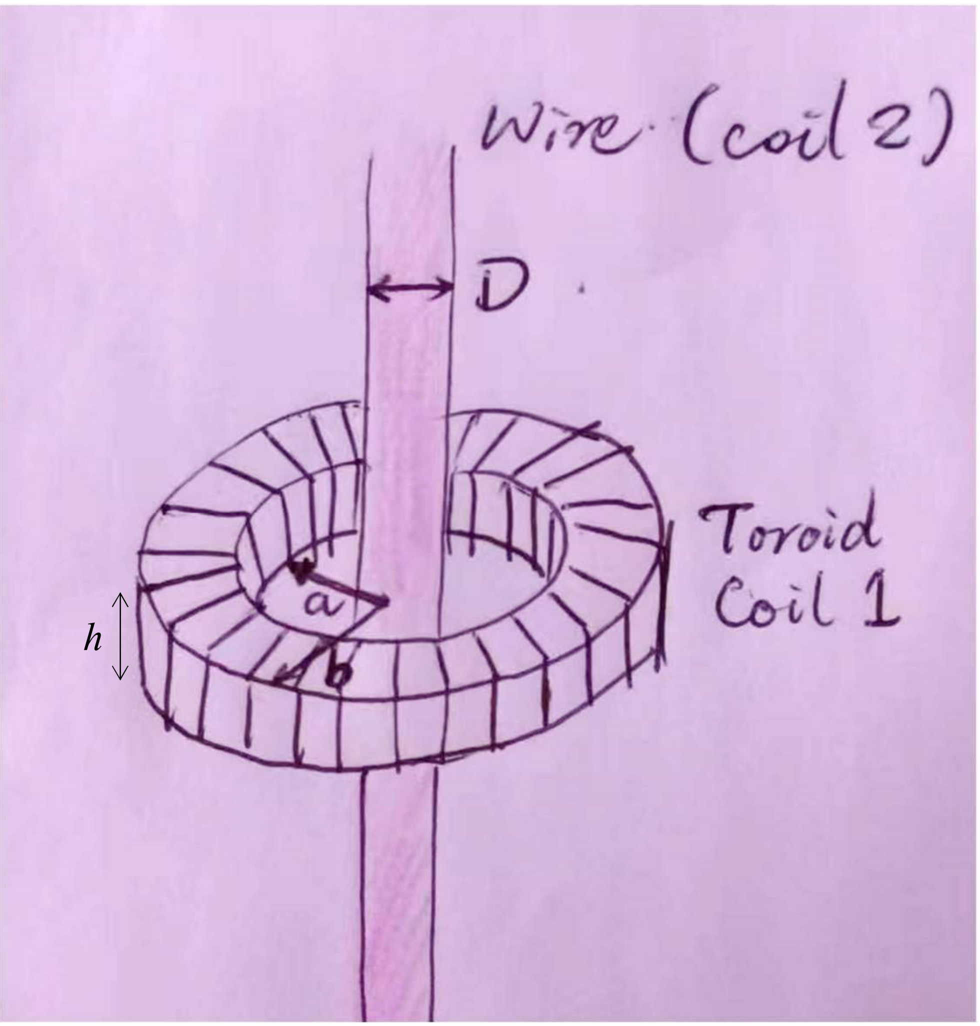
2. Question: how to release the stored energy from an inductive energy storage?



Open the switch, energy will be released to the load.

Example problems

If the toroidal coil has n turns, please calculate the mutual inductance between the coil and the long straight wire.



Method 1: calculated the flux linkage from the wire to the coil

Ampere's circuital law:

$$B(r) = \frac{\mu_0 I}{2\pi r}$$

$$\Phi = h\int_a^b B(r)\mathrm{d}r = \frac{h\mu_0 I}{2\pi}\ln\frac{b}{a}$$

$$M = \frac{n\Phi}{I} = \frac{nh\mu_0}{2\pi}\ln\frac{b}{a}$$

Method 2: calculated the flux linkage from the coil to the wire

The key is to find which parts of the flux is linked to the wire? Refer to the physical meaning of long wire, current returns in the shield of the co-axial cable, so all flux in the toroid is linked (only once) to the wire.

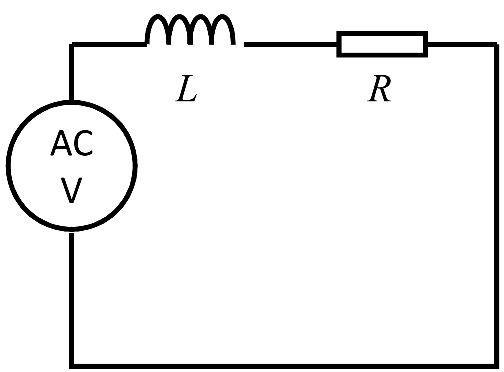
Ampere's circuital law: $$B(r) = \frac{\mu_0 n I}{2\pi r}$$

$$M = \frac{\Phi}{I} = \frac{nh\mu_0}{2\pi}\ln\frac{b}{a}$$

Mutual inductance from two calculations are the same

Example problems

1. For the following circuit, if the voltage of the source is $V=V_0\cos\omega t$, what is the current and the voltage drop on each component V_R and V_L?



$$V_R + V_L = V_0 \cos \omega t$$

$$V_R = iR$$

$$V_L = L\frac{\mathrm{d}i}{\mathrm{d}t}$$

Assume the solution is

$$i = i_0 \cos(\omega t + \alpha)$$

$$V_R + V_L$$
$$= i_0[(R\cos\alpha - \omega L \sin\alpha)\cos\omega t - (R\sin\alpha + \omega L\cos\alpha)\sin\omega t]$$

If the former expression is the same as $V_0\cos\omega t$

$$R\sin\alpha + \omega L\cos\alpha = 0$$

and

$$i_0(R\cos\alpha - \omega L\sin\alpha) = V_0$$

From:

$$R\sin\alpha + \omega L\cos\alpha = 0$$

We have

$$\alpha = -\mathrm{atan}\frac{\omega L}{R}, \quad \cos\alpha = \frac{R}{\sqrt{R^2+\omega^2 L^2}}, \quad \sin\alpha = \frac{-\omega L}{\sqrt{R^2+\omega^2 L^2}}$$

Put values in

$$i_0(R\cos\alpha - \omega L\sin\alpha) = V_0$$

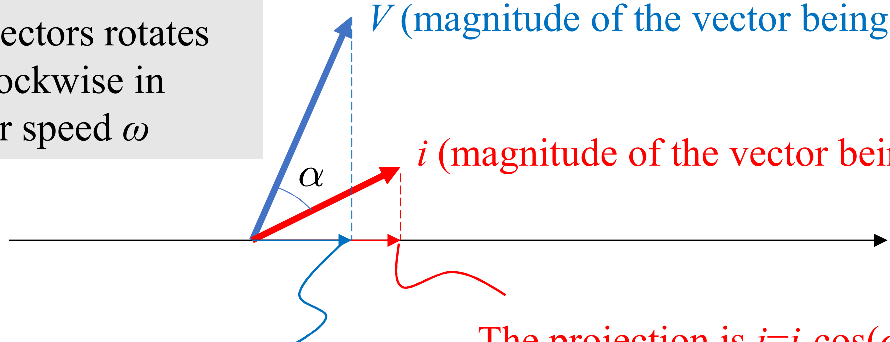
$$i_0 = \frac{V_0}{\sqrt{R^2+\omega^2 L^2}}$$

With both i_0 and α, the problem of current is solved.

Geometrical meaning of the solution:

Both vectors rotates anti-clockwise in angular speed ω

V (magnitude of the vector being V_0)



i (magnitude of the vector being i_0)

Real axis

The projection is $i=i_0\cos(\omega t-|\alpha|)$

The projection is $V=V_0\cos\omega t$

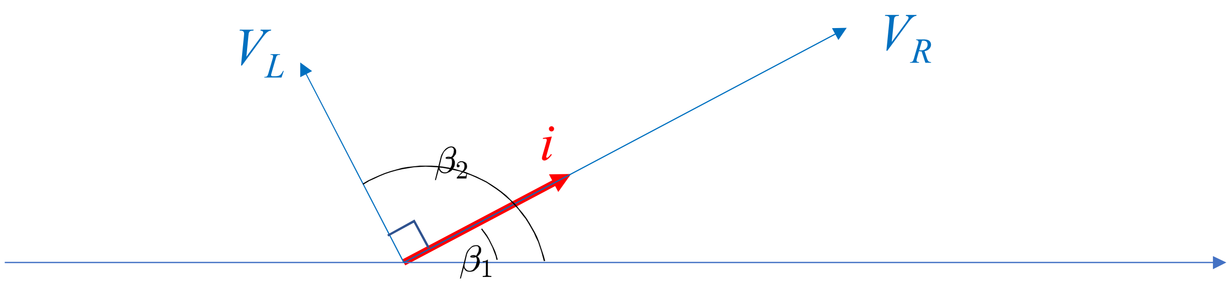
V and i vectors rotates, real value of voltage/current is the projection of the vectors on the real axis

Resistive voltage

$$V_R = Ri = Ri_0 \cos(\omega t - |\alpha|)$$

Inductive voltage

$$V_L = L\frac{\mathrm{d}i}{\mathrm{d}t} = -\omega L i_0 \sin(\omega t - |\alpha|)$$



Real axis

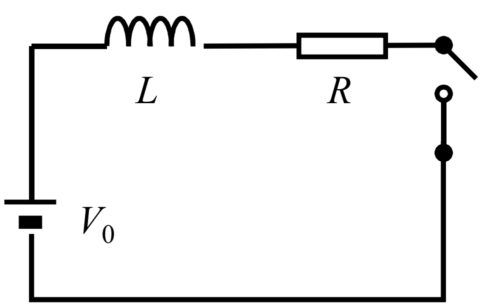
1. At any time, both *V* vectors rotates, real value of voltage is the projection of the vectors on the real axis

$$V_{L0}\cos\beta_2 = V_{L0}\cos(\beta_1 + \pi/2) = -V_{L0}\sin\beta_1$$

That is why the inductive voltage vector seems π/2 ahead of the current vector.

2. The inductive voltage is varying with frequency, the higher is the frequency, the higher is the inductive voltage.

Example problems



1. If the switch is closed at time t=0, what is the current as a function of time

$$L\frac{\mathrm{d}i}{\mathrm{d}t} = V_0 - iR$$

$$L\frac{\mathrm{d}(i - V/R)}{\mathrm{d}t} = -R(i - V_0/R)$$

$$\frac{\mathrm{d}(i - V/R)}{i - V_0/R} = -\frac{\mathrm{d}tR}{L}$$

$$\ln(i - V/R) = -\frac{R}{L}t + C$$

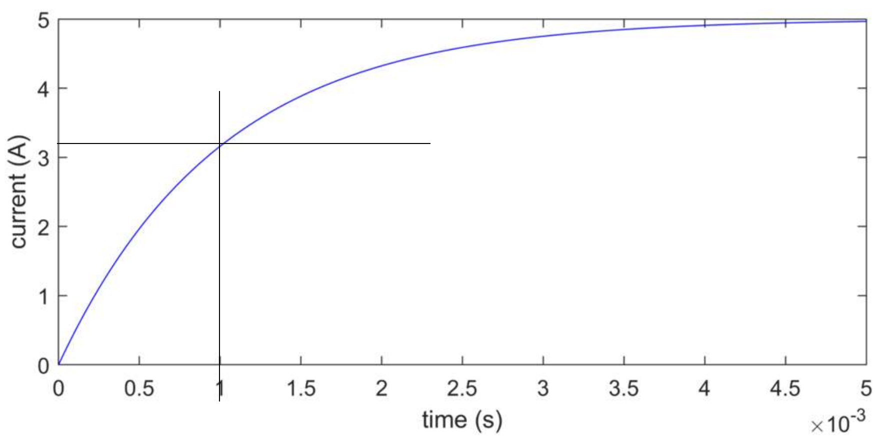
C being an integral constant

$$i = \frac{V}{R} + e^{C}e^{-Rt/L}$$

Include the initial condition, where i=0 at t=0, one can obtain the value of C, as

$$i = \frac{V}{R} - \frac{V}{R}e^{-Rt/L}$$

In case where V = 5 V, R = 1 Ohm and L = 1 mH



1. The current is 0 at t=0 and gradually reached 5 A in about 5 ms. The maximum current will be 5 A, as generally found in a DC circuit, $I = V/R$.

2. The time related term in the index can be re-written as

$$e^{-Rt/L} = e^{-t/\tau}$$

$$\tau \equiv \frac{L}{R}$$

τ is called a time constant (for LR circuit)

3. At $t = \tau = 1$ ms, the deviation of current from the stable value (5 A) is decreased from 5 A to (5-3.16) A,

$$5\ \mathrm{A} \times \frac{1}{e} = 1.839\ \mathrm{A} = (5\ \mathrm{A} - 3.16\ \mathrm{A})$$

in the next 1 ms, the deviation is going to decrease by $1/e$ again.