Physics (PHYS2500J), Unit 4 Electromagnetic wave
1. Maxwell's equations and electromagnetic wave

Xiao-Fen Li
Associate Professor, SJTU

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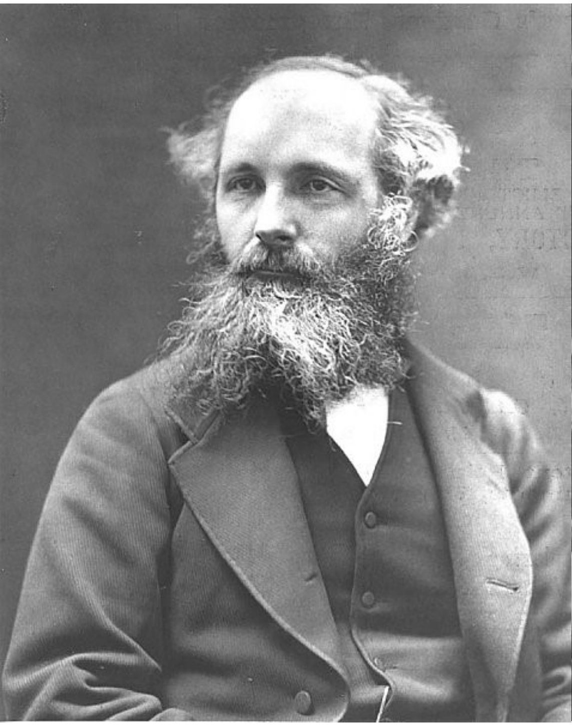
Maxwell



James Clerk Maxwell

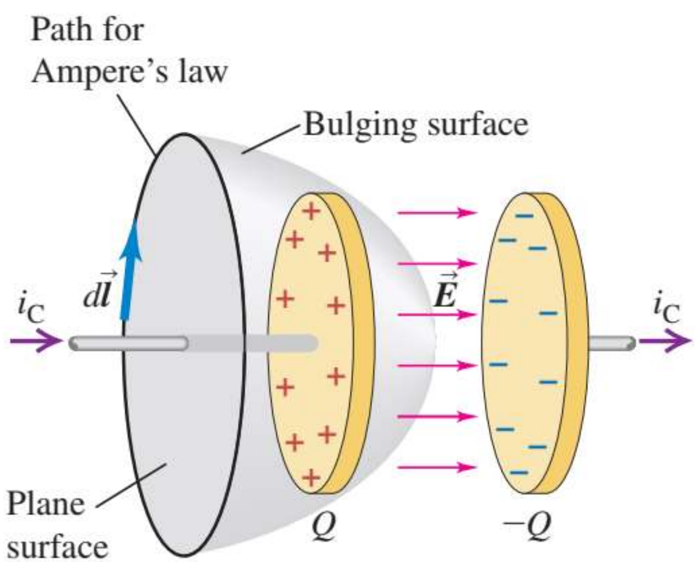
13 June 1831 – 5 November 1879

James Clerk Maxwell was a Scottish mathematician and scientist responsible for the classical theory of electromagnetic radiation, which was the first theory to describe electricity, magnetism and light as different manifestations of the same phenomenon. Maxwell's equations for electromagnetism have been called the "second great unification in physics" where the first one had been realized by Isaac Newton.

1855: starting to represent the electric and magnetic filed lines in mathematics
1862: suspecting the relationship between EM wave and light
1873: Maxwell’s equations systems

Defects in Ampere's circuital law

1. Current is not always closed, in case of a transient current through a capacitor, the plane surface and the bulging surface give different current.



$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I$$

2. Review the current continuity equations.

$$\iint \vec{J} \cdot \mathrm{d}\vec{S} = -\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint \epsilon_0 \vec{E} \cdot \mathrm{d}\vec{S}$$

A new vector (which have two components) has the property of always continuous.

$$\iint \vec{J} \cdot \mathrm{d}\vec{S} + \frac{\mathrm{d}}{\mathrm{d}t} \iint \epsilon_0 \vec{E} \cdot \mathrm{d}\vec{S} = 0$$

Maxwell suspected:

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I + \mu_0 \iint \frac{\mathrm{d}\epsilon \vec{E}}{\mathrm{d}t} \cdot \mathrm{d}\vec{S}$$

Displacement current

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I + \mu_0 \iint \frac{\mathrm{d}\epsilon \vec{E}}{\mathrm{d}t} \cdot \mathrm{d}\vec{S}$$

In terms of generation of curl magnetic field, the flux of $\varepsilon_0 \mathrm{d}E/\mathrm{d}t$ plays a similar role as current I, which is called a displacement current.

$$\epsilon_0 \iint \frac{\mathrm{d}\vec{E}}{\mathrm{d}t} \cdot \mathrm{d}\vec{S}$$

Maxwell's great contribution:

pointed out not only varying magnetic field can induce electric field, but varying electric field can also induce magnetic field.

Maxwell's equations: integral form

Gauss's law

$$\iint \vec{E} \cdot \mathrm{d}\vec{S} = \frac{Q}{\epsilon_0}$$

Faraday's law

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}}{\mathrm{d}t} \iint \vec{B} \cdot \mathrm{d}\vec{S}$$

Magnetic Gauss's law

$$\iint \vec{B} \cdot \mathrm{d}\vec{S} = 0$$

Ampere's circuital law
+ Maxwell's displacement current

$$\oint \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 \iint \vec{J} \cdot \mathrm{d}\vec{S} + \mu_0 \epsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \iint \vec{E} \cdot \mathrm{d}\vec{S}$$

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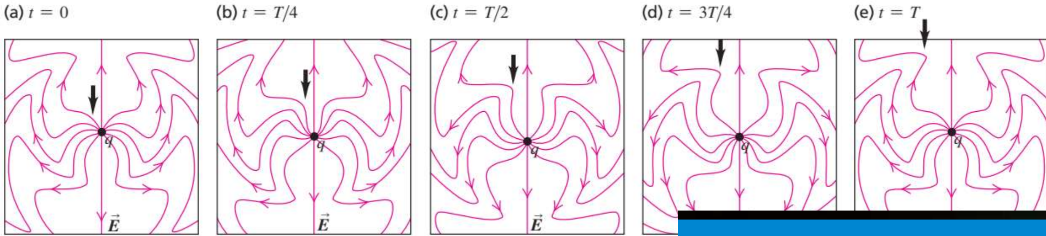
Sources of EM wave: EM wave is related to charged particles

Accelerated motion of charge

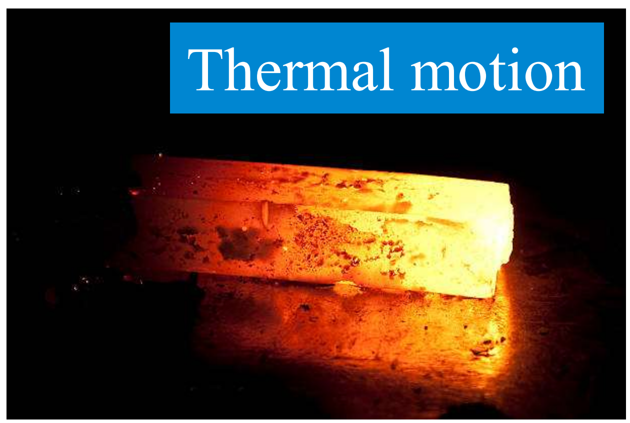


Energy level transition

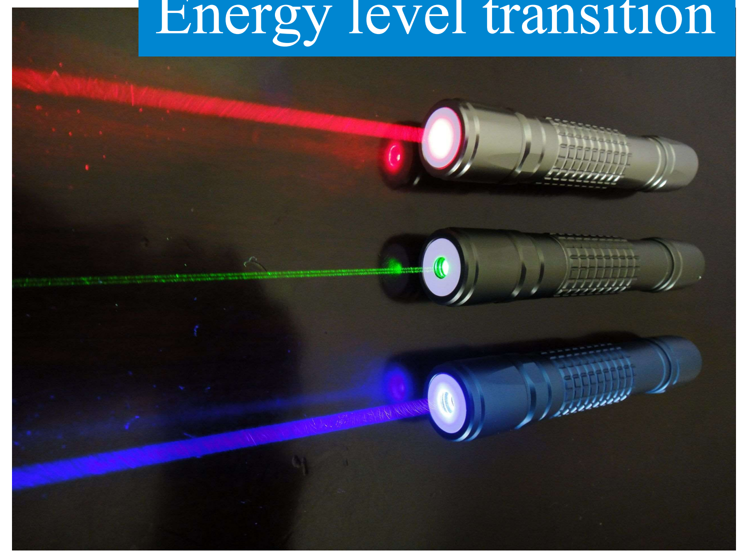
oscillations

Energy level transition

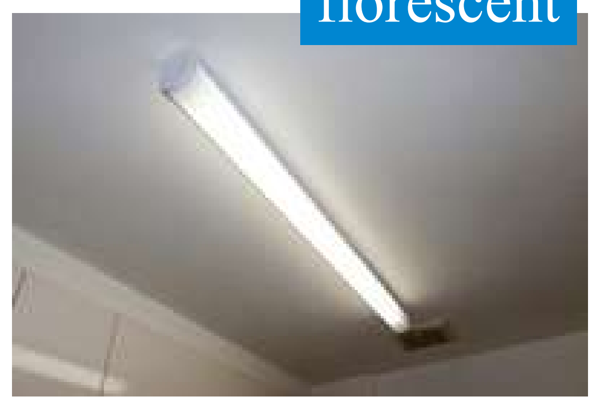
Thermal motion



Energy level transition



florescent



Spectrum

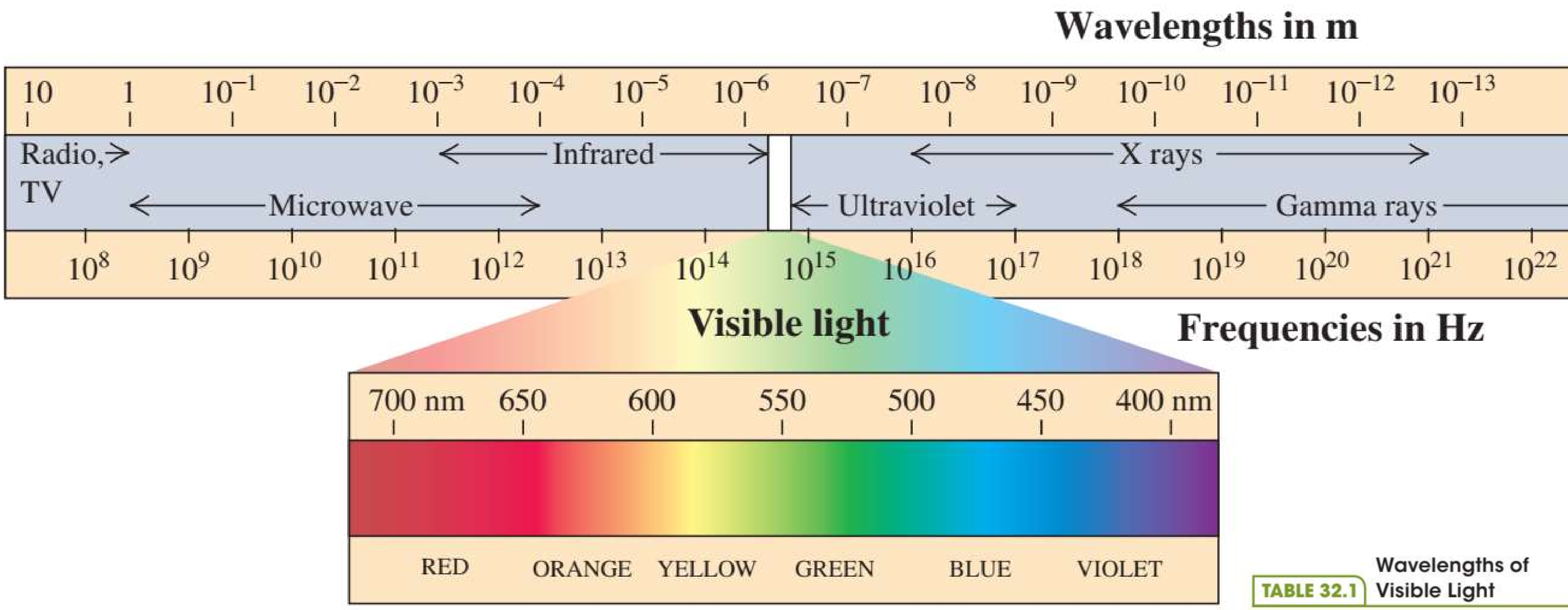


TABLE 32.1 Wavelengths of Visible Light

380–450 nm	Violet
450–495 nm	Blue
495–570 nm	Green
570–590 nm	Yellow
590–620 nm	Orange
620–750 nm	Red

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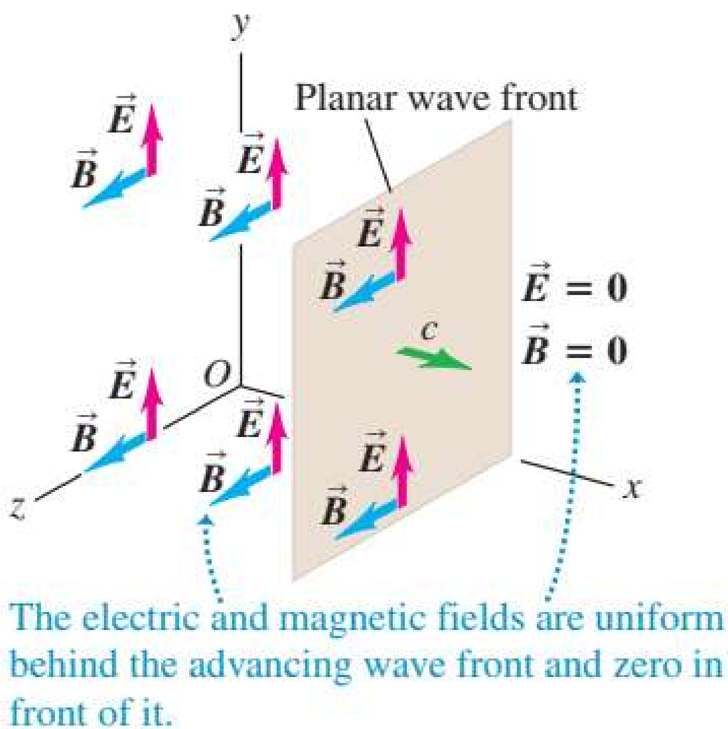
A simple example of plane wave

1. space is divided into two regions by a plane parallel to *yoz* plane. At every point to the left of this plane there are a uniform electric field *E* in the +*y* direction and a uniform magnetic field *B* in the +*z* direction.

The boundary plane, which we call the wave front, moves to the right in the +*x* direction with a constant speed *c*.



2. With the setup, check the compatibility with Maxwell's equations.

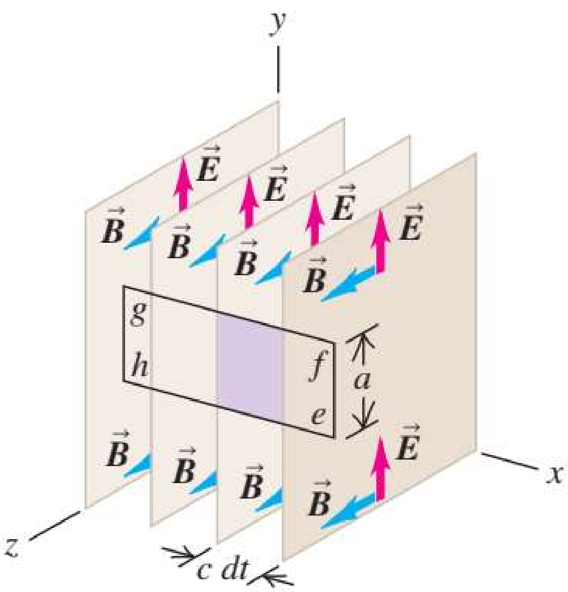
vacuum

$$\oint \vec{E} \cdot d\vec{S} = 0$$

$$\oint \vec{B} \cdot d\vec{S} = 0$$

Conclusion, as long as neither *E* or *B* has *x* component, the two equations are satisfied.

Faraday's law



$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$$

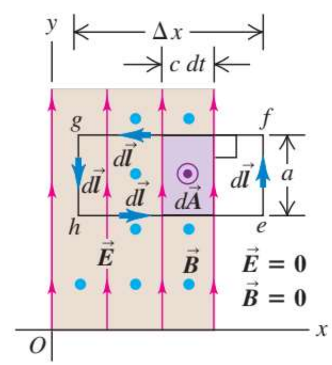
Loop totally in the left part, flux does not change, homogeneous electric field does not have loop.

Loop totally in the right part, flux does not change (always 0). No electric field, no loop integral.

Loop including the wave front,

$$d\Phi = Bacdt$$

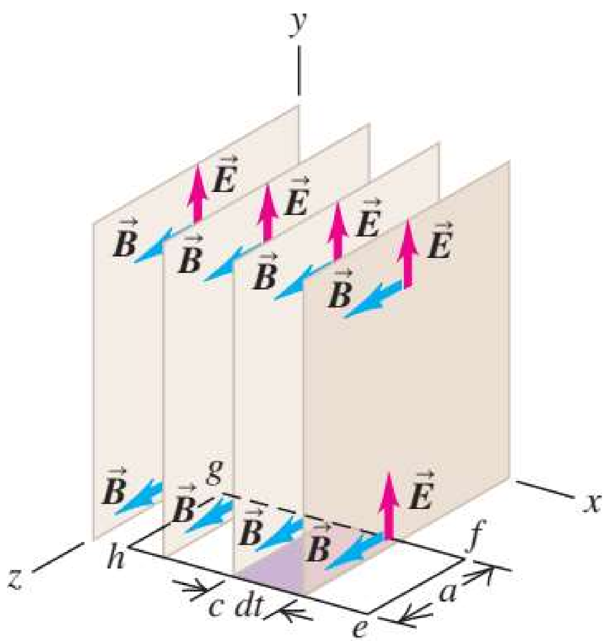
$$\oint \vec{E} \cdot d\vec{l} = -Ea$$



Faraday's law means

$$Bc = E$$

Displacement current



$$\oint \vec{B} \cdot d\vec{l} = \mu_0\epsilon_0 \int \frac{d\vec{E}}{dt} \cdot d\vec{S}$$

Loop totally in the left part, flux of *E* does not change, homogeneous magnetic field does not have loop.

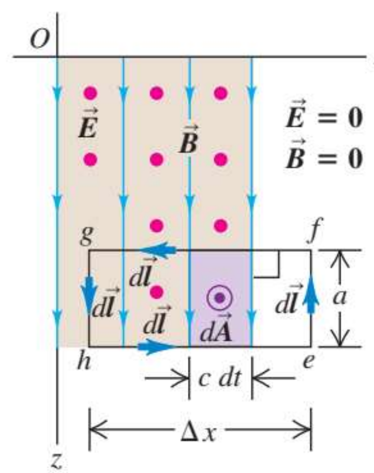
Loop totally in the right part, flux of *E* does not change (always 0). No magnetic field, no loop integral.

Loop including the wave front,

$$\int \frac{d\vec{E}}{dt} \cdot d\vec{S} = Eacdt$$

$$\oint \vec{B} \cdot d\vec{l} = Ba$$

$$B = \mu_0\epsilon_0 Ec$$



$$\left.\begin{aligned} Bc &= E \\ B &= \mu_0\epsilon_0 Ec \end{aligned}\right\}$$

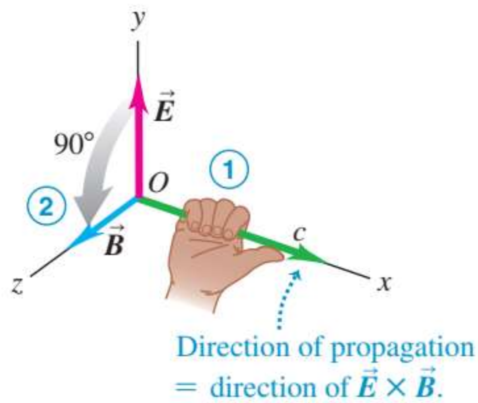
Speed of light in vacuum *c* satisfies

$$\frac{1}{c^2} = \mu_0\epsilon_0$$

If a negative sign is applied to either B, E or velocity, the two equations (Faraday's law and displacement current) can not be satisfied.

The key features of an EM wave* (important)

1. The wave is *transverse*; both ***E*** and ***B*** are perpendicular to the direction of propagation of the wave. The electric and magnetic fields are also perpendicular to each other. The direction of propagation is the direction of the vector product of ***E*** and ***B***



2. There is a definite ratio between the magnitudes of ***E*** and ***B***: $E = cB$. (also notice the unit)

$$q(\vec{E} + \vec{v} \times \vec{B})$$

Can also be seen from the equation for Lorentz force.

3. The wave travels in vacuum with a definite and unchanging speed.

4. Unlike mechanical waves, which need the particles of a medium such as air to transmit a wave, electromagnetic waves require no medium.

Energy of EM wave

EM wave carries energy, is it stored more in the electric field or more in the magnetic field?

$$u = \tfrac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2$$

$$B = \frac{E}{c} = \sqrt{\epsilon_0\mu_0}\,E$$

$$u = \tfrac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}\left(\sqrt{\epsilon_0\mu_0}\,E\right)^2 = \epsilon_0 E^2$$

Total energy is double the energy in electric field, equally stored in electric and magnetic fields.

Contents

Maxwell's equations

The differential forms of Maxwell's equations.

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\nabla \times \vec{E} = -\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\mathrm{d}\vec{E}}{\mathrm{d}t}$$

$\xrightarrow{\text{vacuum}}$

$$\vec{\nabla} \cdot \vec{E} = 0$$

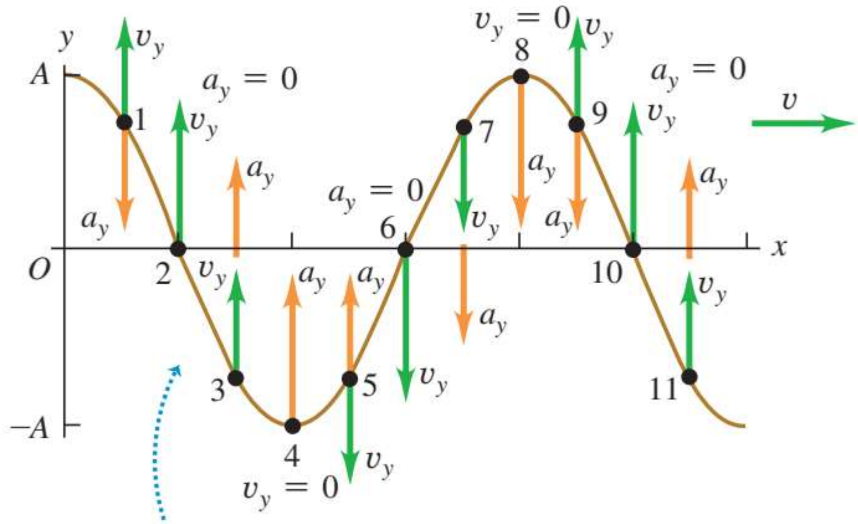
$$\nabla \times \vec{E} = -\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\nabla \times \vec{B} = \mu_0 \epsilon_0 \frac{\mathrm{d}\vec{E}}{\mathrm{d}t}$$

What is a wave equation?

Acceleration is upward where string curves upward, downward where string curves downward.



$$\frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y(x,t)}{\partial t^2}$$

Wave equation of a sting transverse wave. The dynamic equation determines the speed of wave.

1. Wave is a propagation of status (not real material, in this case, the status of motion) ;
2. Wave equation is a dynamic equation (left side of the equation is force and right is the acceleration);
3. The solution to wave equation is in the form of $f(x, t)=f(x-vt)$.

Wave equation is one of the most important equations in physics

Wave equation for electric field

Vector differential equation

$$\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$$

Put it in the Maxwell's equations

$$\begin{aligned}
&\nabla \times (\nabla \times \vec{E}) \\
&= -\nabla \times \frac{\partial \vec{B}}{\partial t} \\
&= -\frac{\partial}{\partial t}\left[\mu_0 \vec{J}_f + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \frac{\partial \vec{P}}{\partial t} + \mu_0 (\nabla \times \vec{M})\right]
\end{aligned}$$

Including every possible source of E field

If in simple medium

$$\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

Similar operation can be done to magnetic field

Electromagnetic wave in homogeneous medium

No free charge, insulating medium (or vacuum)

$$\boxed{\nabla^2 \vec{E} - \mu\epsilon \frac{\partial^2 \vec{E}}{\partial t^2} = 0}$$

$$\boxed{\nabla^2 \vec{H} - \mu\epsilon \frac{\partial^2 \vec{H}}{\partial t^2} = 0}$$

The equation is describes the time and spatial derivative of electric field (or magnetic field). They are the wave equation for EM wave.

Speed of light is determined as

$$\frac{1}{c^2} = \mu_0 \epsilon_0$$

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/s}$$

Also suggest to read book p1057-1059

By comparing the two derivation processes, you can get a better understanding of Stokes' theorem and the differential form of Maxwell's equations.

Speed of light in medium

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{1}{\sqrt{\mu_r \epsilon_r}} c$$

1. Refraction index

$$n = \frac{c}{v} = \sqrt{\mu_r \epsilon_r} \approx \sqrt{\epsilon_r}$$

(most transparent materials are not magnetic)

It is why the frequency dependence of dielectric constant is usually called dispersion curve (色散曲线)

2. Phase velocity and group velocity