Physics (PHYS2500J), Unit 1 Electrostatics: 5. Electric field energy

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The interaction energy between point charges

Work done by the person who moved the charge (to the system).

$$W = \phi_1 Q_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 r_{12}}$$

It is also the increase of the system energy.

The interaction energy between point charges

Total work done by the person who moved the charge (to the system)

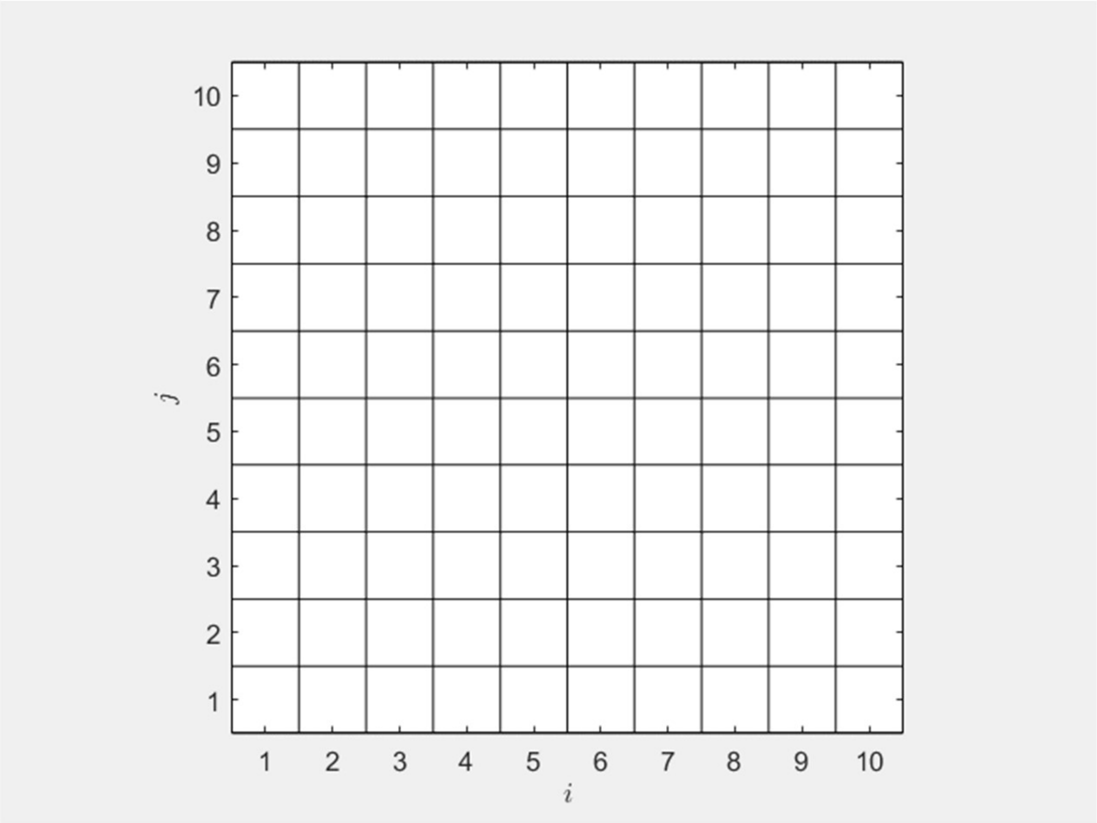
$$
\begin{aligned}
W &\\
&= \phi_1 Q_2 + \phi_{12} Q_3 \\
&= \frac{Q_1 Q_2}{4\pi\epsilon_0 r_{12}} + \frac{Q_1 Q_3}{4\pi\epsilon_0 r_{13}} + \frac{Q_2 Q_3}{4\pi\epsilon_0 r_{23}}
\end{aligned}
$$

The interaction energy between point charges

How do you write the energy of many charges system?

Eq. 1 $$W = \frac{1}{4\pi\epsilon_0}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\frac{Q_iQ_j}{r_{ij}}$$

i, the charge been moved. *j*, the charges arrive before *i*.



$$2W = \sum_i \sum_{j\neq i} \frac{q_j}{4\pi\epsilon_0 r_{ji}} q_i = \sum_i \phi_i q_i$$

The potential energy of 3D charge converges when the volume is reduced. The equal sign is valid for real integrations.

Self energy of a point charge

Energy of a point charge diverges

$$\mathcal{E} \propto \frac{1}{r}$$

Energy of a linear charge also diverges

$$\mathcal{E} \propto \ln r$$

Point or linear charges are not real situations.

General expression of static field energy

$$\mathcal{E} = \frac{1}{2}\sum \phi_i q_i$$

In continuous system:

$$\mathcal{E} = \frac{1}{2}\int \phi(r)\rho(r)\mathrm{d}V$$

Some mathematical trick

$$\begin{aligned}
&\mathcal{E} \\
&= \frac{1}{2}\int \phi\nabla\cdot\vec{D}\mathrm{d}V \\
&= \frac{1}{2}\int [\nabla\cdot(\phi\vec{D}) - \nabla\phi\cdot\vec{D}]\mathrm{d}V \\
&= \frac{1}{2}\oint \phi\vec{D}\cdot\mathrm{d}\vec{S} + \frac{1}{2}\int \vec{E}\cdot\vec{D}\mathrm{d}V
\end{aligned}$$

Field has energy!

Goes 0 at infinity, φ decreases in manner of $1/r$, D decreases in manner of $1/r^2$, S increases in manner of r^2.

Energy is measured as field (not any more potential and charge)

The space of integral extends to the whole space!

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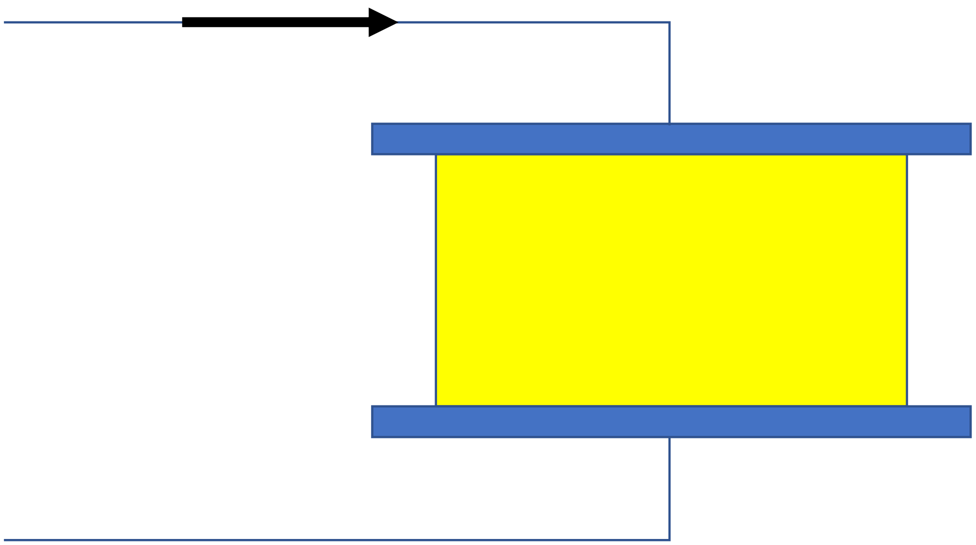
Work done by the circuit



$$\phi = V$$

$$\phi = 0$$

If energy dissipation is neglected, dQ is moved from 0 to V.

$$\mathrm{d}W = V\mathrm{d}Q$$

Write it with circuit variables V and I

$$\mathrm{d}W = VI\mathrm{d}t$$
$$P = VI$$

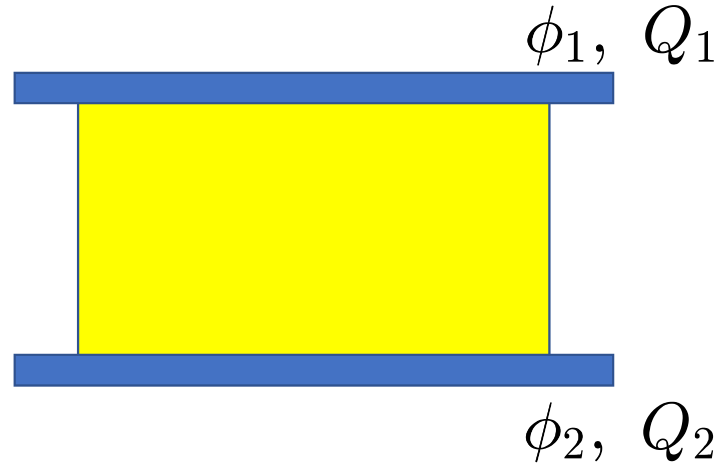
Energy storage in a capacitor (a circuit point of view)

Starting with no charge

$$
\begin{aligned}
& \int_0^{t_0} V I \mathrm{d}t \\
= & \int_0^{Q_0} V \mathrm{d}Q \\
= & \int_0^{Q_0} \frac{Q}{C} \mathrm{d}Q \\
= & \frac{Q^2}{2C} = \frac{CV^2}{2}
\end{aligned}
$$

What determines the upper limit of energy storage in a capacitor?

Energy storage in a capacitor (a potential energy point of view)



$$
\begin{aligned}
& \frac{1}{2}(\phi_1 Q_1 + \phi_2 Q_2) \\
&= \frac{1}{2}(\phi_1 Q - \phi_2 Q) \\
&= \frac{1}{2} Q V = \frac{1}{2} C V^2
\end{aligned}
$$

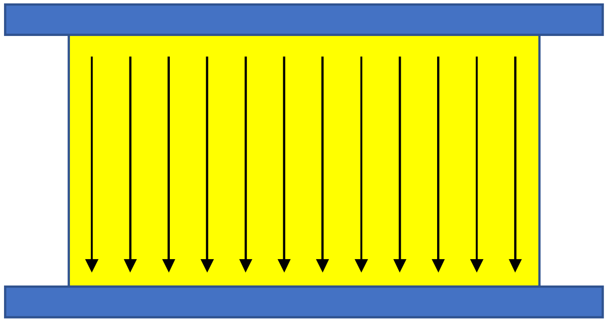
Potential difference is voltage

Energy storage in a capacitor (a field energy point of view)

$$\underbrace{\frac{1}{2}\epsilon_0 E^2}_{\text{Energy density}}\ \underbrace{Sd}_{\text{volume}}$$

$$= \frac{1}{2}\frac{\epsilon_0 S}{d}E^2 d^2$$

$$= \frac{1}{2}CV^2$$



Thanks