Physics (PHYS2500J), Unit 3 Electromagnetic induction: 1. Faraday's law

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Fall 2023

Learning EM induction

Electromagnetic induction is one of the most important discoveries:
power generators which transfer energy from mechanical form into electrical form are forming energy foundation of modern society.

In this chapter, we need to

1. Learn how to determined the direction and value of induced voltage (important), including induced emf and motional emf.

2. Understand example devices from the EM induction point of view;

3. Know that Faraday's law is one of the most reliable rules to refer to when confused with circuit / EM field related problems.

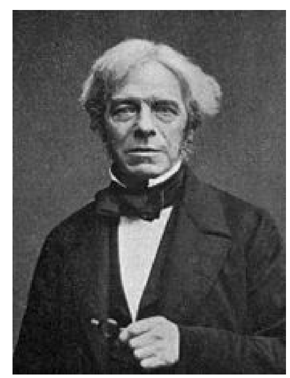
Contents

Hint from the history

1818	Ørsted	Looking for a connection between electricity and magnetism, but confused
1820	Ørsted	Published his discovery that a compass needle was deflected from magnetic north by a nearby electric current
1821	Davy & Wollaston	Tried, but failed, to design an electric motor
1821	Faraday	Built a homopolar motor
1823	Ampère	Discovered Ampère's circuital law

Hans Christian Ørsted


André-Marie Ampère


Michael Faraday

Hint from the history

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1821	Faraday	Built a homopolar motor
1823	Ampère	Discovered Ampère's circuital law
1824	Faraday	Study whether a magnetic field could regulate the flow of a current in an adjacent wire, but he found no such relationship
1831	Faraday	began his great series of experiments in which he discovered electromagnetic induction

https://en.wikipedia.org/wiki/Amp%C3%A8re%27s_circuital_law
https://en.wikipedia.org/wiki/Homopolar_motor
https://en.wikipedia.org/wiki/Homopolar_generator
https://en.wikipedia.org/wiki/Hans_Christian_%C3%98rsted
https://en.wikipedia.org/wiki/Michael_Faraday

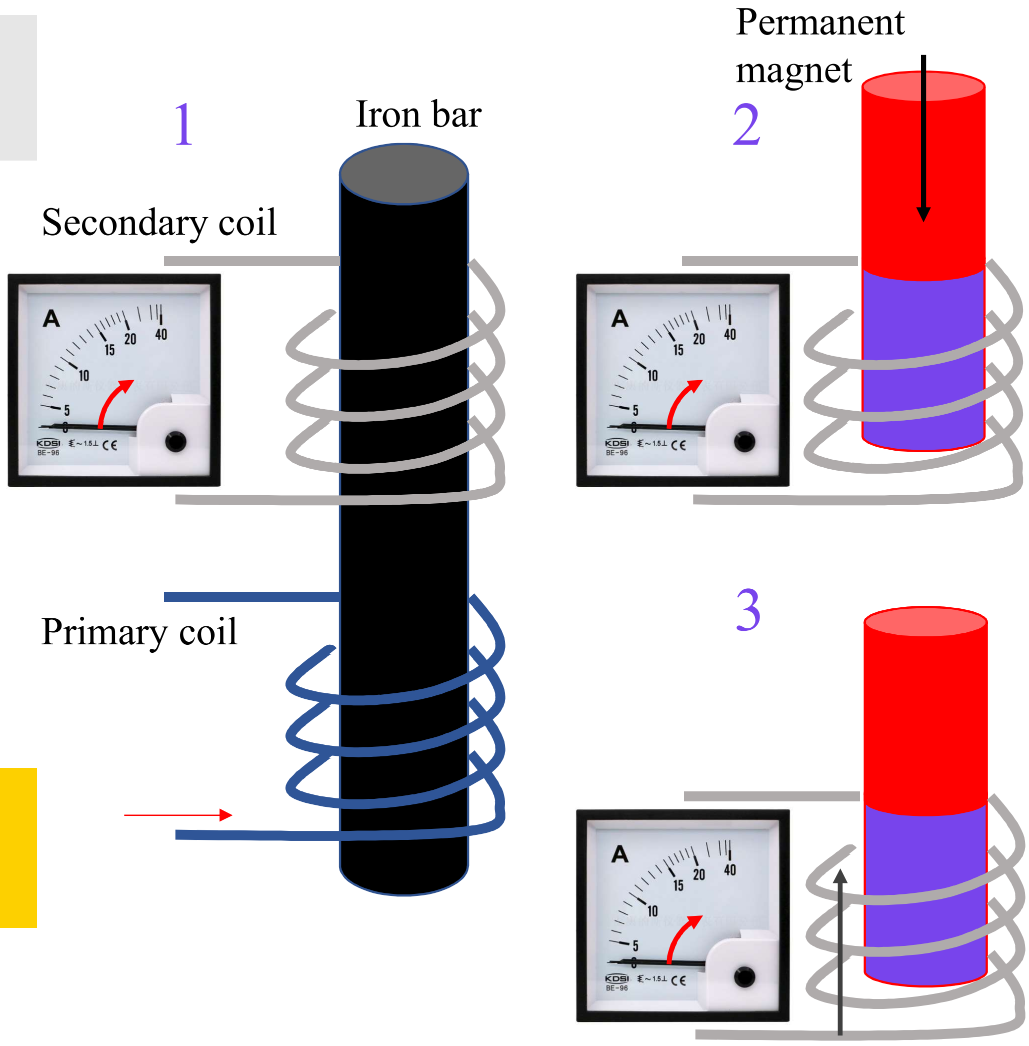
Discovery of electromagnetic induction

1. Faraday's breakthrough:
when current is supplied to a primary coil, induced current is found in the secondary coil.

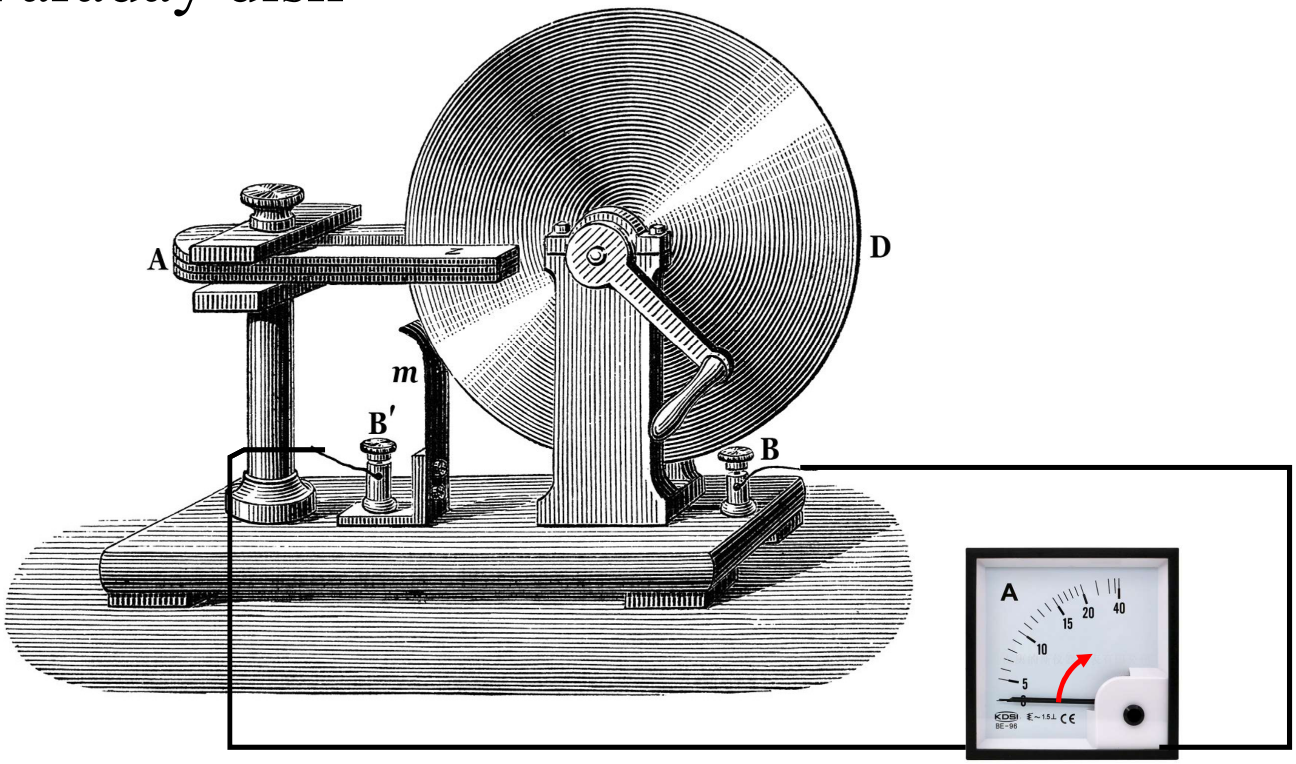
2. Following experiments:
the motion of a magnet can cause the same effect: induced current

3. the relative motion is the reason. No matter if the coil or the magnet is moved.

4. Faraday's conclusion:
varying magnetic field is the reason that current is induced in the coil



Faraday disk



Nobody tried to rotate the homopolar motor and do this experiment during the years!!!

Contents

The direction of induced voltage

1. Lentz's law:
   The induced emf resulting from a changing magnetic flux has the polarity that leads to an induced current whose direction is such that the induced magnetic field opposes the original flux change.

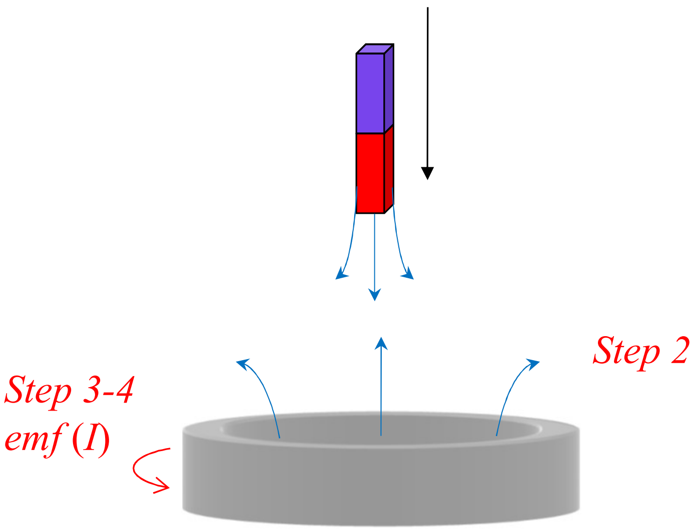
If the magnet is moving down, what is the direction of induce emf in the conductor ring?

Step 1: the direction of flux change?

Step 2: the direction of the flux generated by the current in the ring, if any (will be emf even if in open circuit).

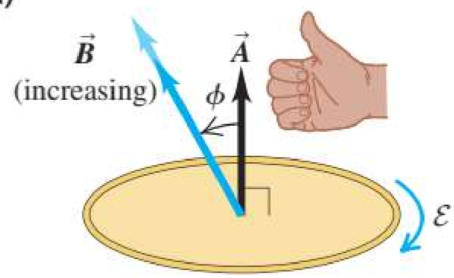
Step 3: the direction of the current, if any

Step 4: the direction of emf



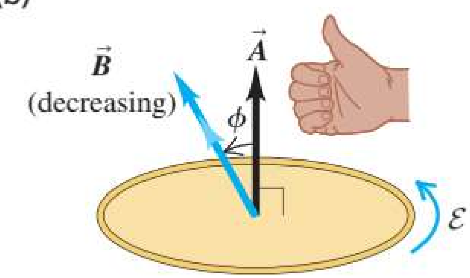
Different cases

(a)



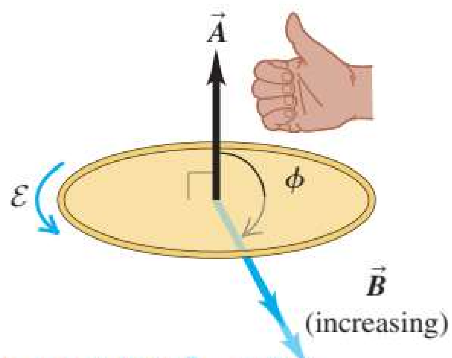
- Flux is positive ($\Phi_B > 0$) ...
- ... and becoming more positive ($d\Phi_B/dt > 0$).
- Induced emf is negative ($\mathcal{E} < 0$).

(b)



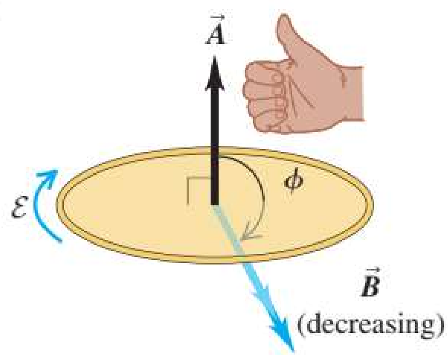
- Flux is positive ($\Phi_B > 0$) ...
- ... and becoming less positive ($d\Phi_B/dt < 0$).
- Induced emf is positive ($\mathcal{E} > 0$).

(c)



- Flux is negative ($\Phi_B < 0$) ...
- ... and becoming more negative ($d\Phi_B/dt < 0$).
- Induced emf is positive ($\mathcal{E} > 0$).

(d)



- Flux is negative ($\Phi_B < 0$) ...
- ... and becoming less negative ($d\Phi_B/dt > 0$).
- Induced emf is negative ($\mathcal{E} < 0$).

The value of induced emf

1. Experiments show that the induced emf in a loop is

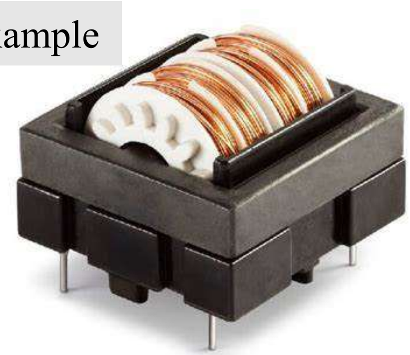
$$\mathcal{V} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

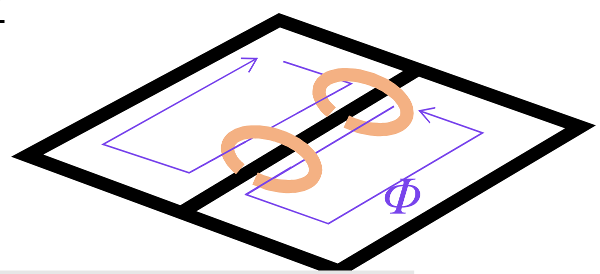
$\mathcal{V}$ is the emf and Φ is the magnetic flux through the loop. The positive direction of emf and flux satisfies right-hand rule.

2. For multi-turn coils, if the flux through each turn are the same,

$$\mathcal{V} = -n\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

example





The iron core makes sure that flux through each turn is the same

Equation for emf in transformer

$$\frac{\mathcal{V}_1}{\mathcal{V}_2} = \frac{-n_1\mathrm{d}\Phi/\mathrm{d}t}{-n_2\mathrm{d}\Phi/\mathrm{d}t} = \frac{n_1}{n_2}$$

Dimensions of the equation

$$\mathcal{V} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

$$\dim\ \Phi = \mathrm{T} \cdot \mathrm{m}^2 = \frac{\mathrm{N}}{\mathrm{A{\cdot}m}} \cdot \mathrm{m}^2 = \frac{\mathrm{J}}{\mathrm{A}}$$

Joule per Ampere

$$\dim\ \mathcal{V} = \dim\ \frac{\text{Power}}{\text{current}} = \frac{\mathrm{J}}{\mathrm{A{\cdot}s}}$$

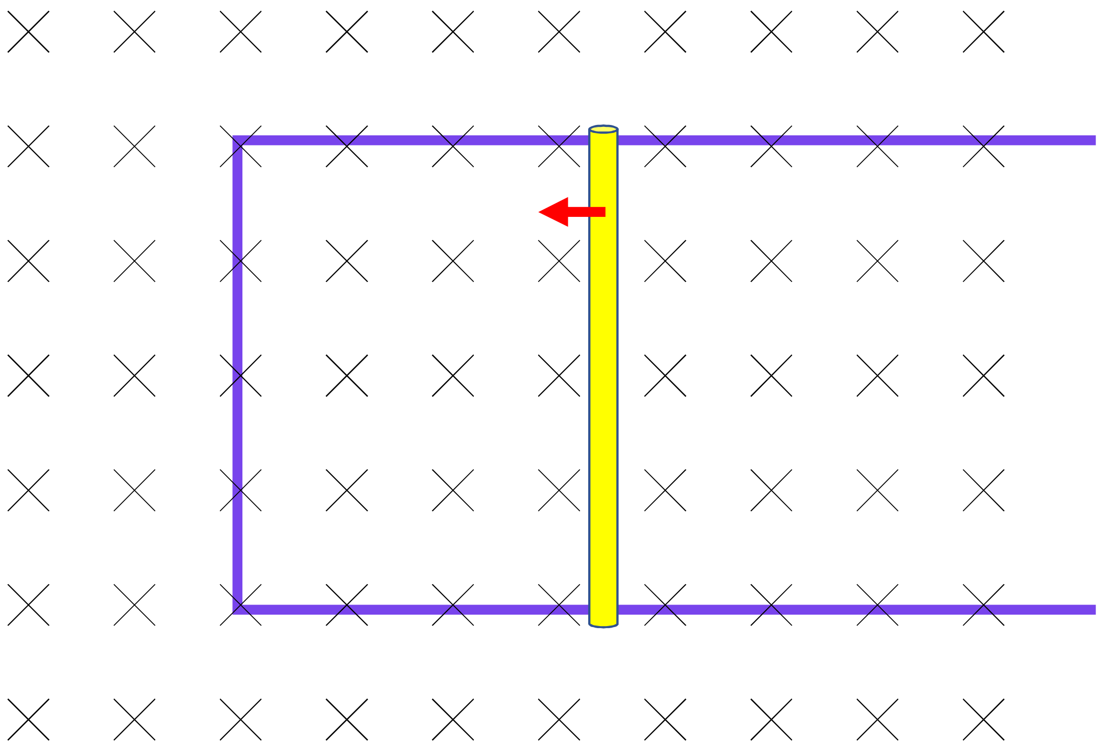
Why is there no constant?

Because of the definitions of Volt and Tesla.

Contents

Motional emf

1. Experiments also show that current will be induced in the loop in the following experiment.



Moving conductor

Sliding contact

Background B field

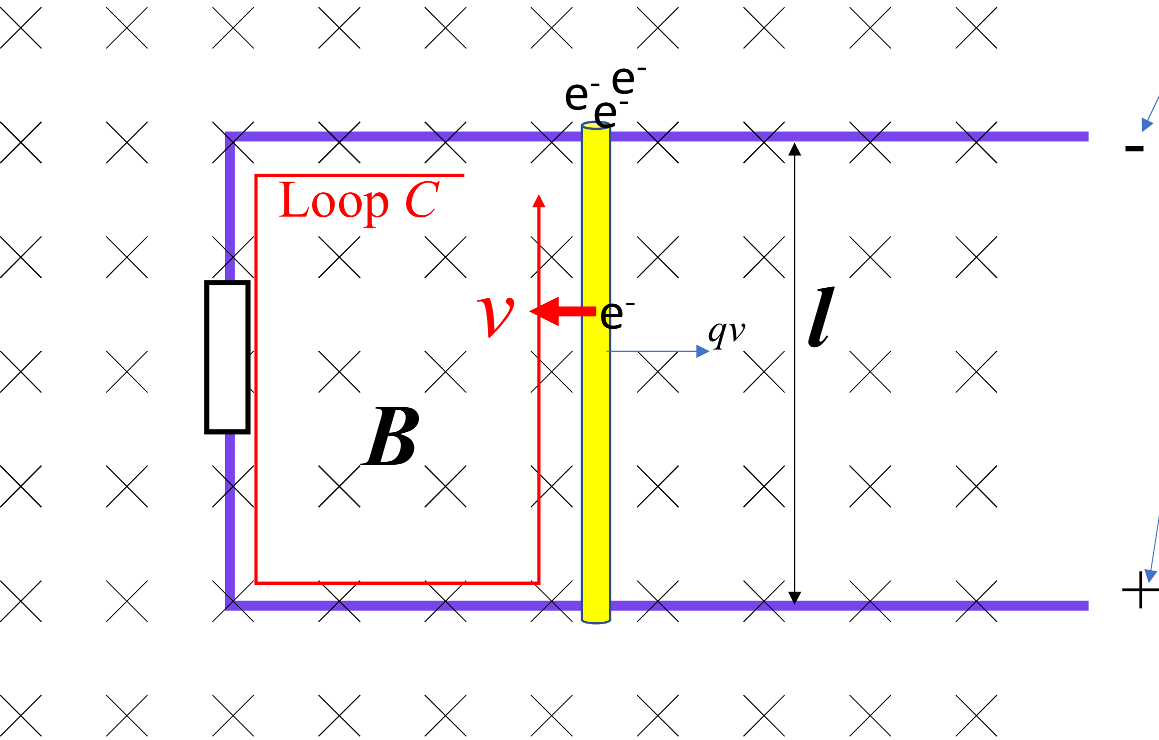
Current induced

2. What is the reason for the current?

The Lorentz force in magnetic field.

$$\vec{F}_L = q(\vec{E} + \vec{v} \times \vec{B})$$

The direction and value of motional emf



Direction of the emf is determined by right hand rule.

Magnitude of the emf of the loop

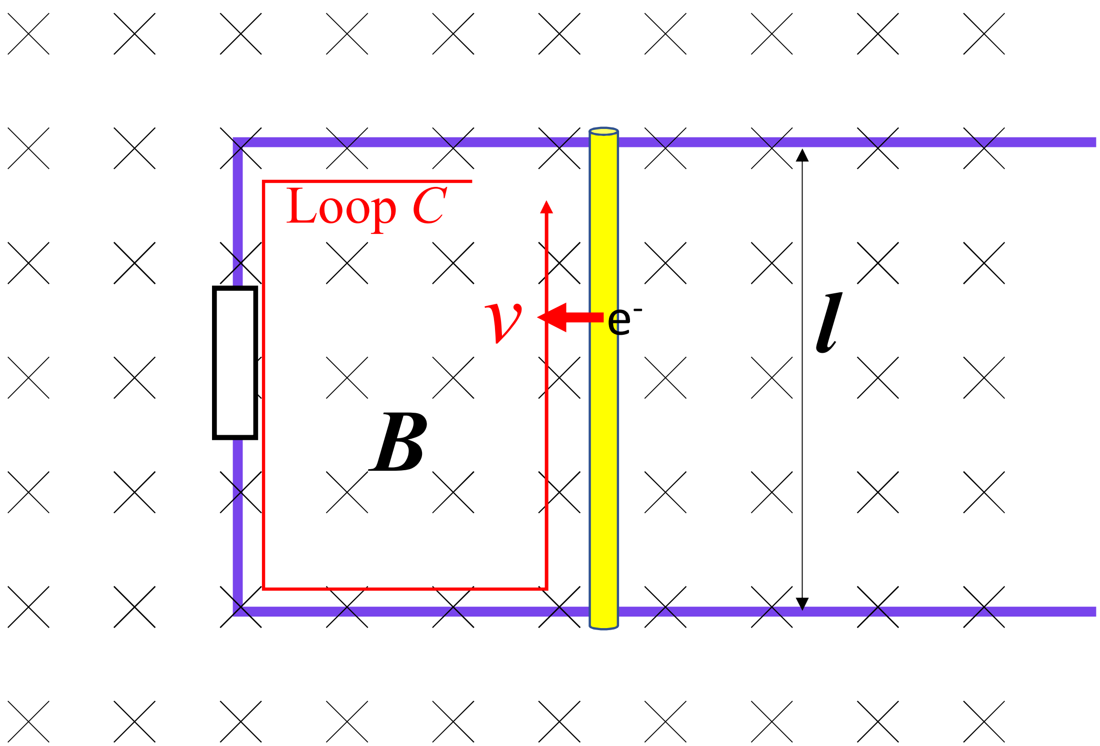
$$\mathcal{V} = \oint_C \vec{K} \cdot \mathrm{d}\vec{l} = \oint_C (\vec{v} \times \vec{B}) \cdot \mathrm{d}\vec{l}$$

The positive direction is defined with loop *C*, then the emf is

$$\mathcal{V} = -vBl$$

The motional emf follows the same induction emf law

Induction emf law: $\mathcal{V} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$



1. Use loop C to define the positive direction of emf and flux. Emf is positive if it is anticlockwise. B is positive if it points out of the paper.

2. Determine the direction of flux change: the loop shrinks with motion. Magnitude of Φ is reducing, but B is negative, which means $\mathrm{d}\Phi / \mathrm{d}t > 0$.

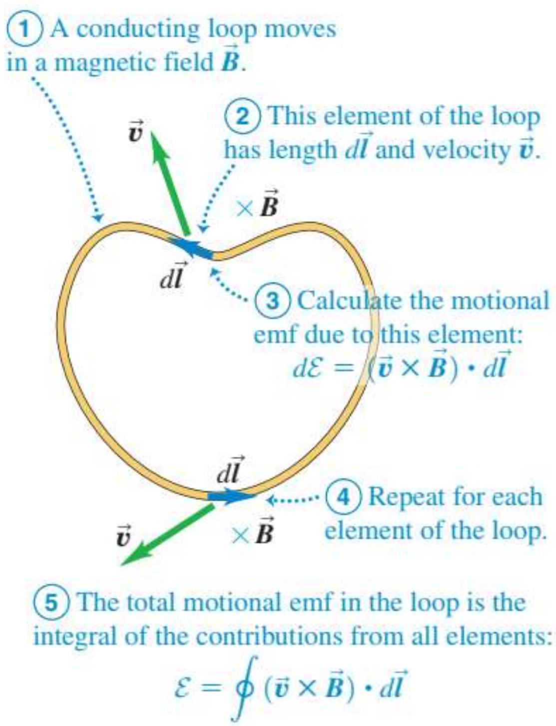
3. Determine the direction of emf: $\mathrm{d}\Phi / \mathrm{d}t > 0$, so emf < 0.

4. Determine the magnitude of emf:

$$-\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -B\frac{\mathrm{d}A}{\mathrm{d}t} = -Bvl$$

A more general case of motional emf

29.16 Calculating the motional emf for a moving current loop. The velocity $\vec{v}$ can be different for different elements if the loop is rotating or changing shape. The magnetic field $\vec{B}$ can also have different values at different points around the loop.



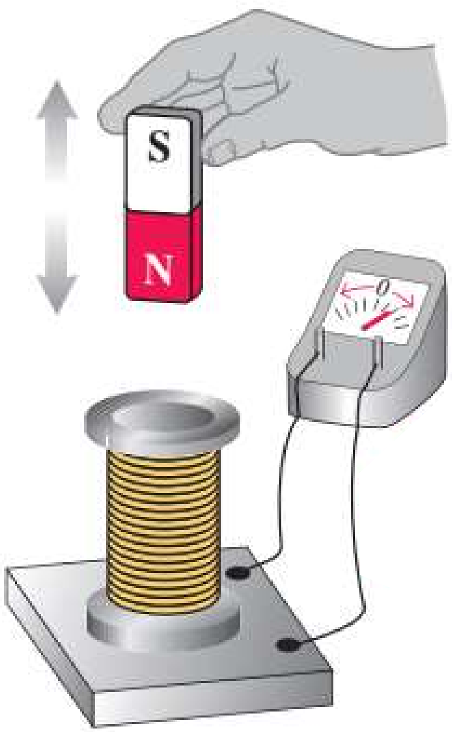
$$\mathcal{V} = \oint_C \vec{K} \cdot \mathrm{d}\vec{l} = \oint_C (\vec{v} \times \vec{B}) \cdot \mathrm{d}\vec{l}$$

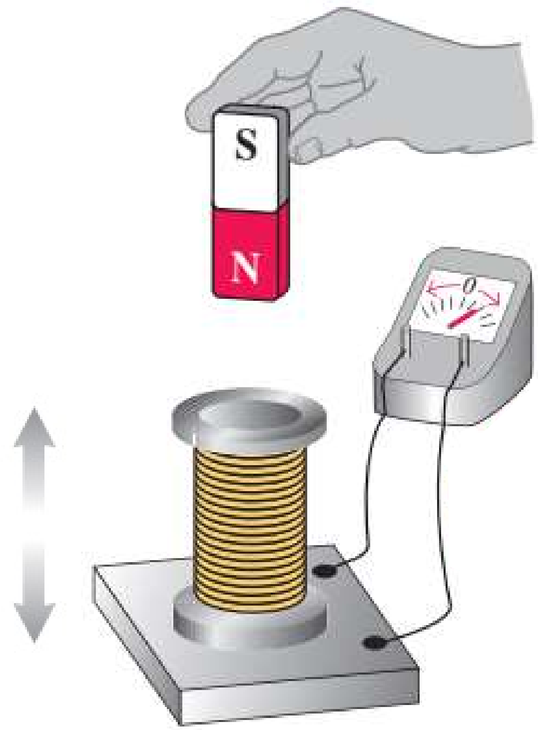
Emf is the closed loop linear integral of non-electrostatic field, and here the non-electrostatic force is the Lorentz force.

Motional or induced emf?

1. In this experiment, do we see induced or motional emf?

2. What about in the coil's reference frame?

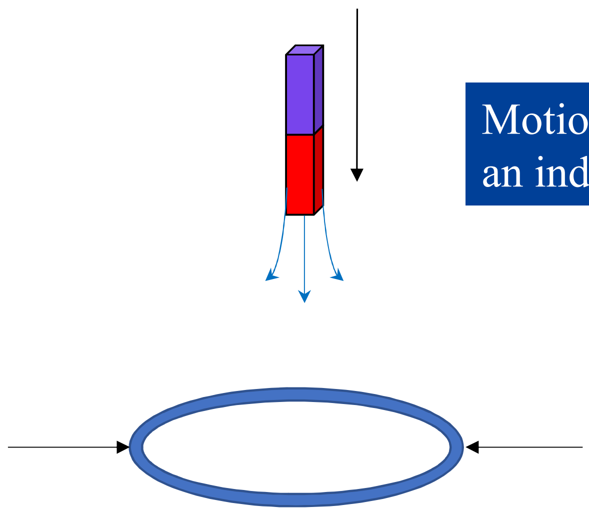




It is not always true that one can distinguish between motional and induced emf.

The induction emf law is suitable for more generalized cases

And in some situations, a combination of induced and motional emf is found.
e.g. A squeezed coil in a varying magnetic field



Motion of the magnet is causing an induced emf in the loop.

The deformation of the loop is considered a motional emf.

$$\mathcal{V} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

This equation is good for cases where both co-exists.

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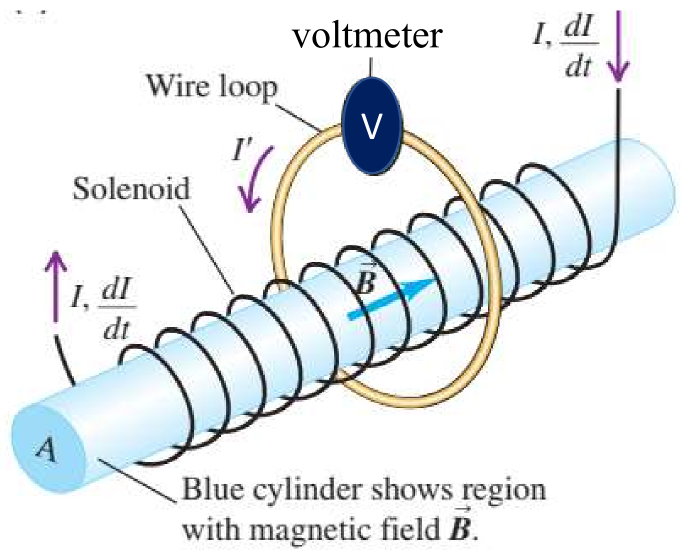
emf in the following case

1. A long, tight wound solenoid (N turns per m, radius R) is fed with an increasing current $I = kt$, from $t = 0$ to $t = 5$ s. Please calculate the magnitude of the induced emf (which is the same as the voltage measured in the voltmeter, why?) during this period.



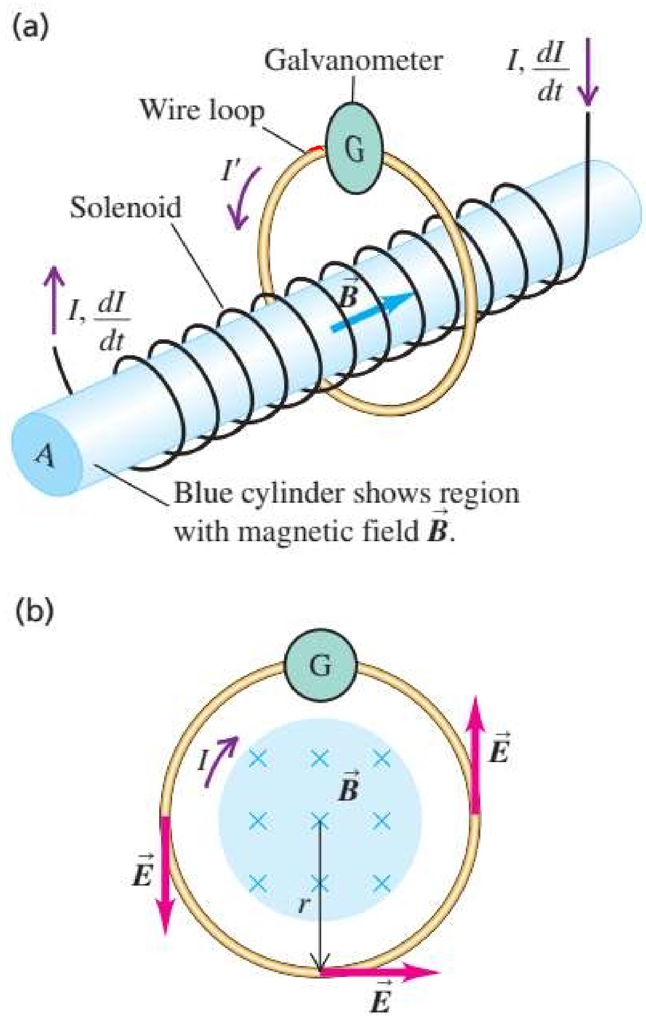
$$\Phi = BA = \pi R^2 B = \pi R^2 \mu_0 N I$$

$$\mathcal{V} = -\dot{\Phi} = -\pi R^2 \mu_0 N \dot{I} = -\pi R^2 \mu_0 N k$$

2. The direction of the emf can be determined by Lentz's law.

Where is the magnetic field by long solenoid distributed?

What is the nature of K for induced emf



In the whole loop, magnetic field is not found. Also, no velocity of charged particle is involved.

So, the "non-electrostatic field" for induced emf is not Lorentz force field

It is an electric field (induced electric field, 感生电场)

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = \oint \vec{E}_i \cdot \mathrm{d}\vec{l} + \oint \vec{E}_s \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} + 0$$

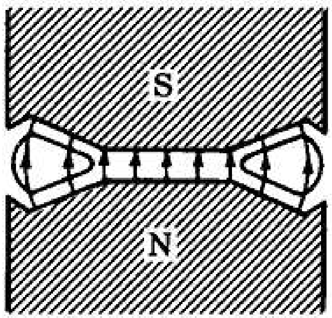
Induced electric field: induction accelerator

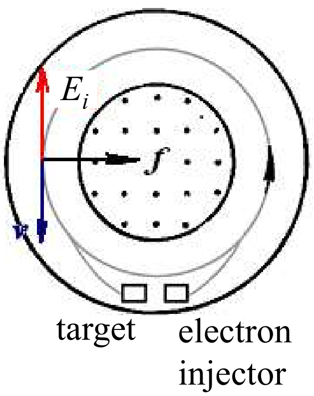
1. Electrons are injected into the accelerator shown as below, where the magnetic field is $B=kt$ from $t=0$ to $t=t_0/4$. If the electrons are on the circular trajectory with radius r, please give the velocity of the charge as a function of time.

$$2\pi r E_i = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mathrm{d}\overline{B}}{\mathrm{d}t}\pi r^2$$

$$E_i = -\frac{\mathrm{d}\overline{B}}{\mathrm{d}t}\frac{r}{2}$$

$$v_e = -\frac{E_i e t}{m_e} = \frac{\mathrm{d}\overline{B}}{\mathrm{d}t}\frac{ret}{2m_e} = \frac{kret}{2m_e}$$





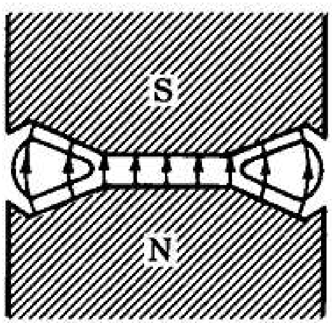
2. If the electrons can move in a circle under the Lorentz force, what is the field B_1 that they feel?

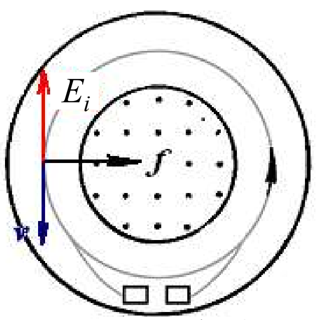
$$m_e\frac{v_e^2}{r} = e v_e B_1$$

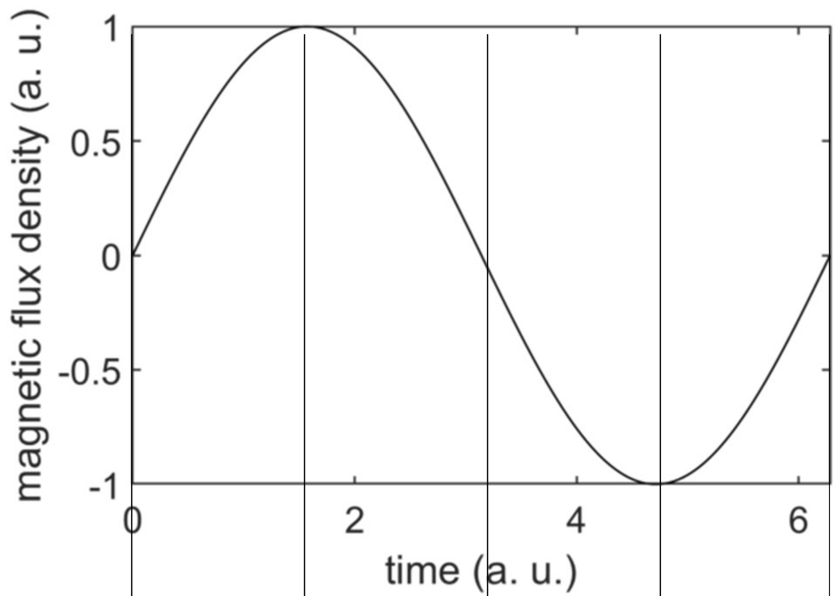
$$B_1 = \frac{m_e v_e}{re} = \frac{kt}{2}$$

If the magnetic field on the circle is well controlled to be half of the average field in the circle. The electrons can be accelerated while kept in constant trajectory.

1. If the magnetic field is a sinusoidal function, which is easier to achieve in reality, what happens to the electrons, what if they are positrons?







Direction of E_i	↻	↺	↺	↻
Electron motion	acc. ↺	dec. ↺	acc. ↻	dec. ↻
Positron motion	acc. ↻	dec. ↻	acc. ↺	dec. ↺

Faraday's law

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

Generally speaking, electric field *E* is not divided into the induced *E* part and static *E* part, since the static part is contributed by the distributed surface charge and very difficult to get directly.

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = -\int \frac{\mathrm{d}\vec{B}}{dt} \cdot \mathrm{d}\vec{S}$$

Faraday's law is a very basic rule for electric field and magnetic field

Maxwell's equations

On the left are the integration forms of Maxwell's equations, on the right are the differential forms of Maxwell's equations.

$$\oint_{\partial\Omega} \vec{E} \cdot \mathrm{d}\vec{S} = \frac{\int_\Omega \rho \mathrm{d}V}{\epsilon_0} \qquad \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

$$\oint_{\partial A} \vec{E} \cdot \mathrm{d}\vec{l} = -\int_A \frac{\mathrm{d}\vec{B}}{\mathrm{d}t} \cdot \mathrm{d}\vec{S} \qquad \nabla \times \vec{E} = -\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}$$

$$\oint_{\partial\Omega} \vec{B} \cdot \mathrm{d}\vec{S} = 0 \qquad \nabla \cdot \vec{B} = 0$$

$$\oint_{\partial A} \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 \int_A \vec{J} \cdot \mathrm{d}\vec{S} \qquad \nabla \times \vec{B} = \mu_0 \vec{J}$$

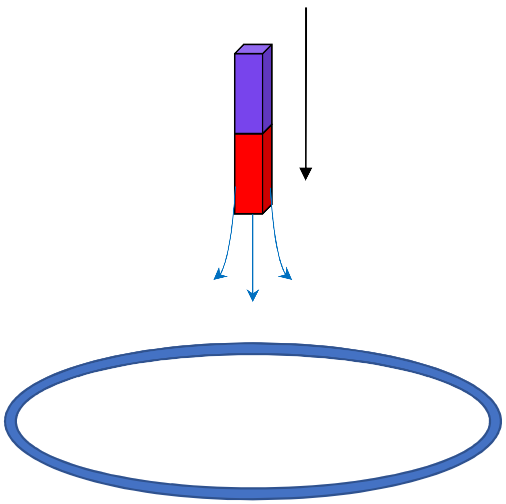
Each line is a physics law and the relation between left and right is pure mathematical.

The last line is not complete yet.

Contents

Flux in a perfect conducting loop never changes

1. For a perfectly conducting ring, if a magnet is approaching, electromotive force and thus current will be induced. From Faraday's law



$$\oint \vec{E}_i \cdot \mathrm{d}\vec{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

2. From Ohm's law, we have

$$\vec{E} = \rho \vec{J} = 0$$

Resistivity for a perfect conductor (ideal conductor) is 0, so electric field is 0 everywhere inside the conductor.

3. It is derived that

$$\frac{\partial \Phi}{\partial t} = -\oint \vec{E} \cdot \mathrm{d}\vec{l} = 0$$

Which means that the flux inside a perfect conductor ring never changes.

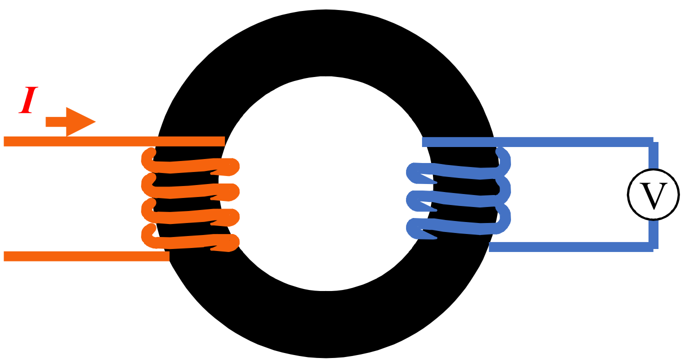
Acceleration of electrons is missing here.
Strictly speaking, the flux in the loop may vary a little.

To measure the flux density inside a closed magnetic circuit

1. Formerly we introduced the hysteresis curve of magnetic material (*H*-*B* curve)

To measure the magnetic flux inside a closed system, one can use Faraday's law.

$$H = \frac{nI}{2\pi R}$$



$$\mathcal{V} = -n\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

$$\Phi = \Phi(t=0) - \int_0^t \frac{\mathcal{V}}{n}\mathrm{d}t$$

$$B = B(0) - \int_0^t \frac{\mathcal{V}}{nA}\mathrm{d}t$$

2. Integral has to be done to obtain *B* as a function of time.

Vibrational sample magnetometer (VSM)

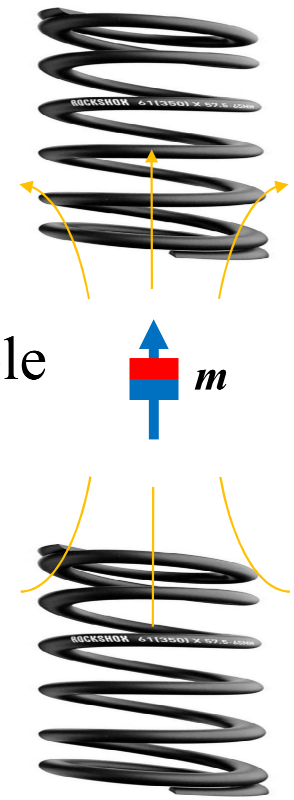
1. To study the magnetic properties of materials (which is important to physicist and material scientist), one need to measure the magnetic moment of small samples. A VSM is a device to measure dipole moments, where the sample geometry is neglected and just taken as a small magnetic dipole.

Upper coil:

see increasing flux

$V_1 < 0$

Moving dipole



Lower coil:

see decreasing flux

$V_2 > 0$

2. Two pick-up coils are oppositely connected in series, so that

$$V = V_1 - V_2$$

3. Sample vibrates, and signal is measured by a lock-in amplifier (a precision AC volt-meter with given frequency).

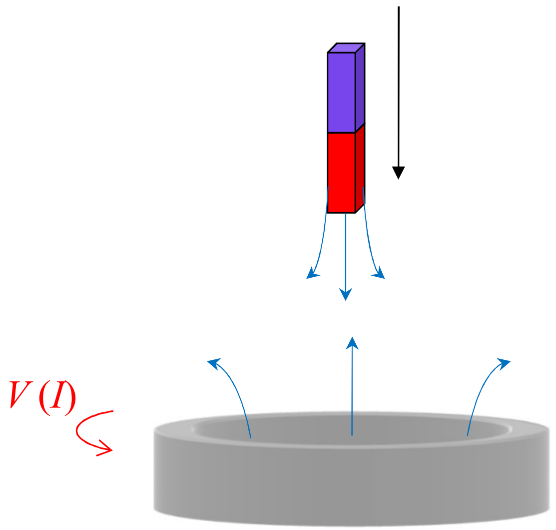
Beautiful strategy given here:

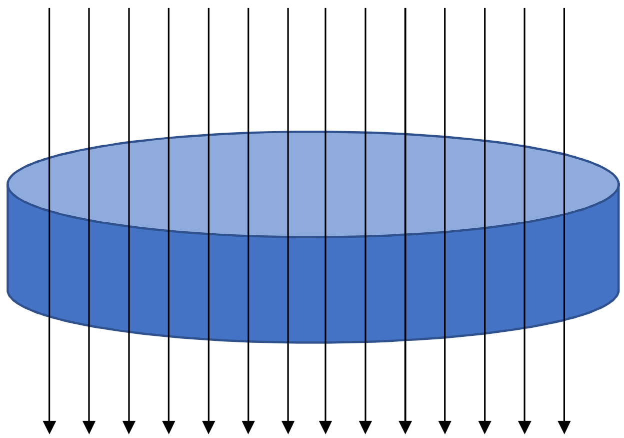
1. to move the sample to create signal
2. two coils canceling the background noise
3. lock-in amplifier to reduce noise.

Not only the signal from the sample is enhanced, but also the signal from background field is canceled.

Eddy current

1. Eddy current (涡流) is a common phenomenon, which could be trouble making in a lot of cases. But can also be used deliberately for heating, deceleration, etc. Eddy current is the induced loop current in (usually continuous) metal.





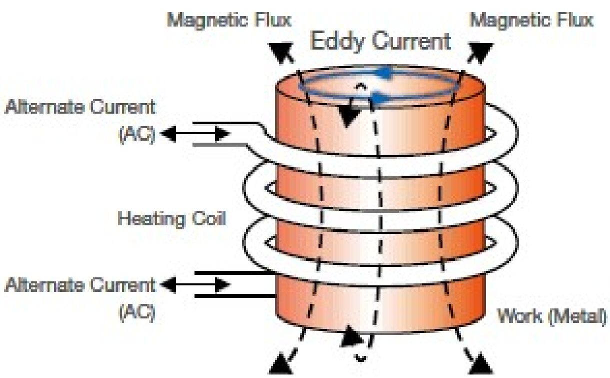
2. Case 1, Induced current in a loop (usually not called eddy current)

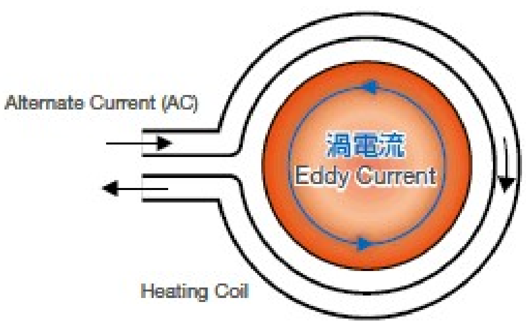
3. Case 2, Induced current in a disk with varying homogeneous B (eddy current). A typical feature of eddy current is the flow of current is complicated. Even in this simple case here, the direction of the flow is known but the amplitude is difficult to obtain.

Application of eddy current 1: induction heating

1. Induction heating can be used for not only home cooking but also industrial metal heating (for compressing 挤压成型, melting). It is high power (fast) and non-contacting.

2. Can be applied to materials with proper resistivity (not too high or too low), or with hysteresis losses.

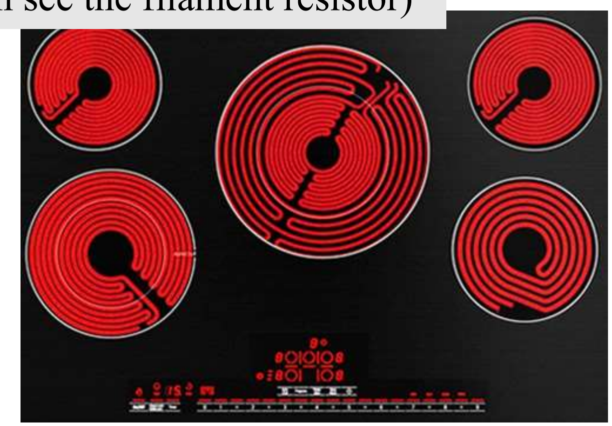




Industrial heater

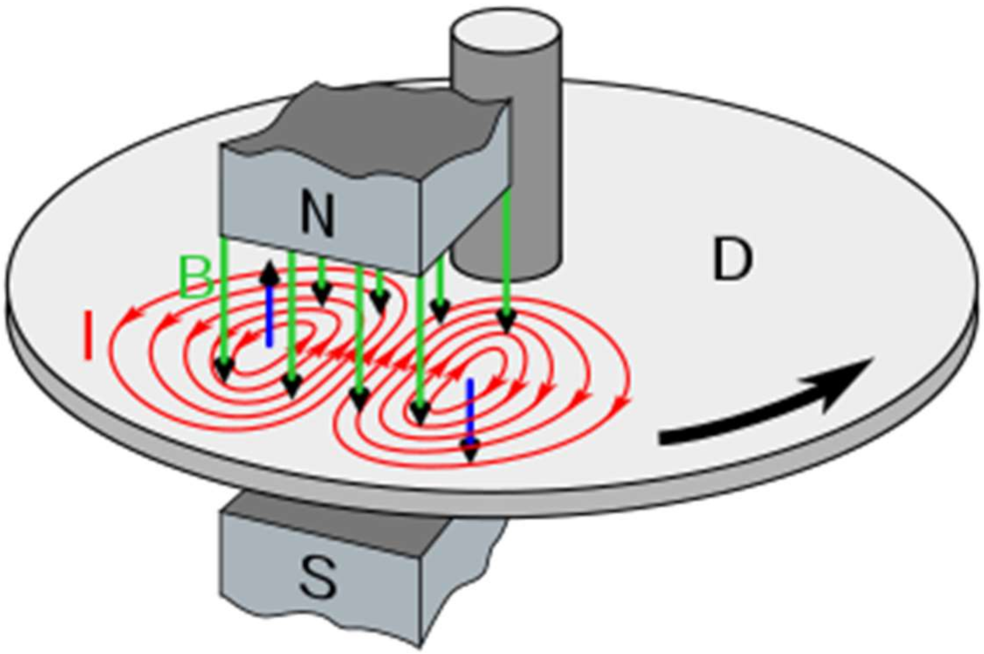
Induction heating cooker

Resistive heating cooker
(you can see the filament resistor)



Eddy current break

1. Eddy current can also be used as a break. For the rotation disk, if a magnet is place near. Eddy current will appear



2. For the front edge (on the left), the disc sees an increasing field, the induced eddy current generates reversed flux; but for the other edge, the disc sees a decreasing field, the induced eddy current generates flux in the same direction as the applied field.

3. In the center region, eddy current is in negative radial direction. The Lorentz force is towards the reversed direction as rotation.

4. One can also consider the two eddy current loop as to magnets and see the direction of the force.

5. Eddy current generates Joule power, kinetic energy is converted into heat.

Lentz's law

The induced emf resulting from a changing magnetic flux has the polarity that leads to an induced current whose direction is such that the induced magnetic field opposes the original flux change.

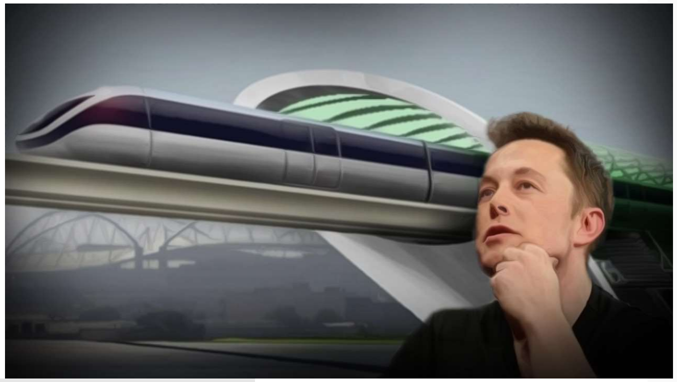
Extended version (intuitive)

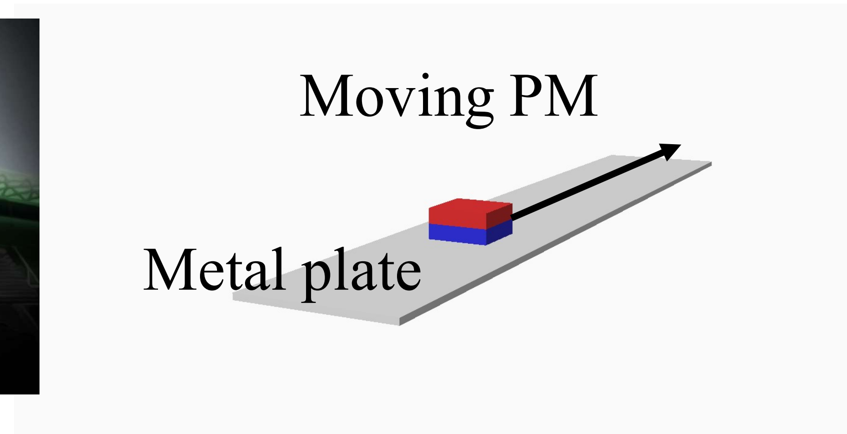
Metal / closed loop circuit tries to prevent flux change to happen.

(Not only from the EM field point of view, but also mechanical.)

Eddy current levitation

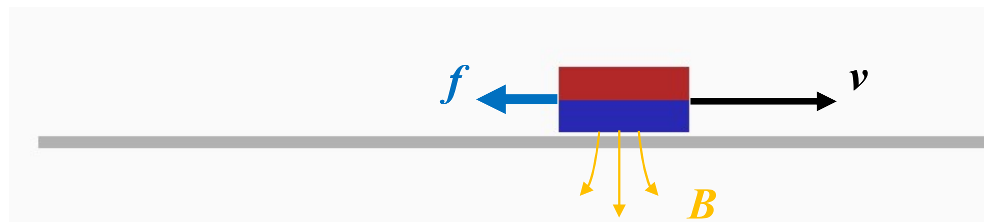
1. Elon Musk's idea of maglev has a very simple structure (original idea belongs to MIT, mag-plane): to move a train loaded with magnets on top of a metal plate.





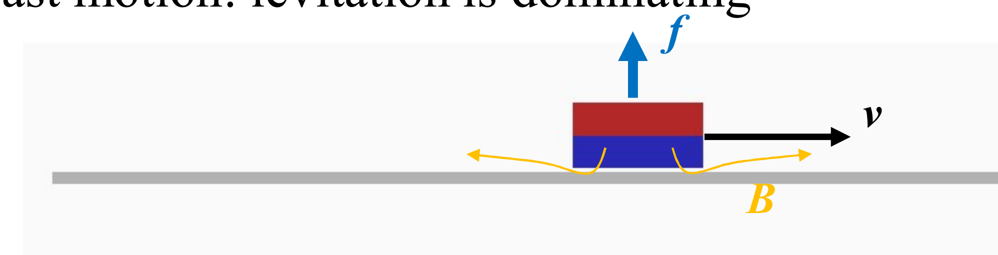
2. From generalized Lentz's law point of view, the concept is easy to catch.

Slow motion: drag force is dominating



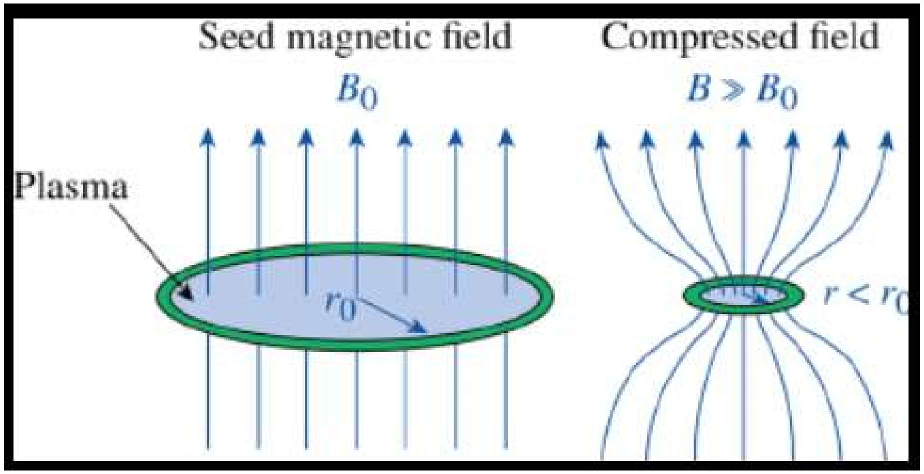
3. If the motion is slow, as the former example, drag force will dominate since flux has already penetrated into the metal plate. So the BEST way to keep the flux unchanged is to keep the magnet where it was.

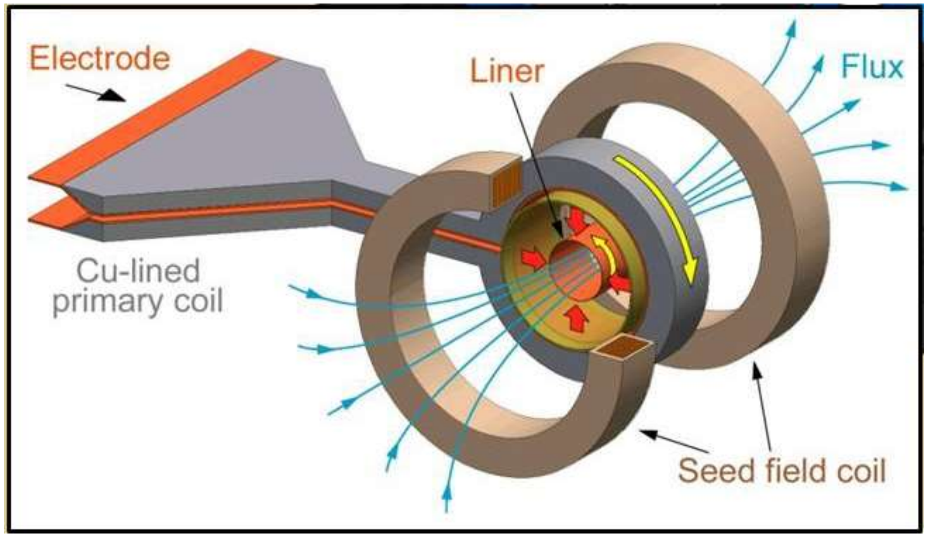
Fast motion: levitation is dominating



4. If the motion is fast, on the other hand, levitation force will dominate since flux hasn't got enough time to penetrate into the metal plate. So the BEST way to keep the flux unchanged (flux free) is to keep the magnet away.

Electromagnetic flux compression (explosive pulsed high field)





1. The highest magnetic field human can make is generated by such Electromagnetic Flux Compression (EMFC) devices (up to 1000 T).

2. The principle is first put some flux (seed field) into the a metallic cylinder (liner), followed by sudden compress of the liner, which will compress the flux inside into high density. The compression happens in a few microseconds.

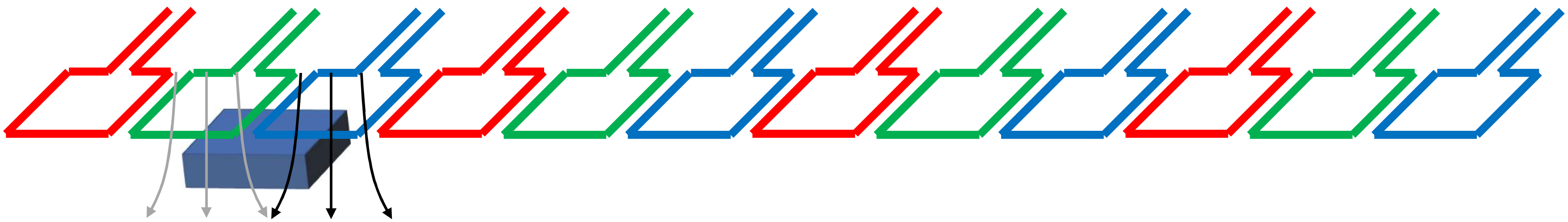
3. 4 T flux density was first generated as "seed field", by the seed field coil.

4. The compression force is provided by a pulsed current in the primary coil. After the compression the sample and the liner are both destructed.

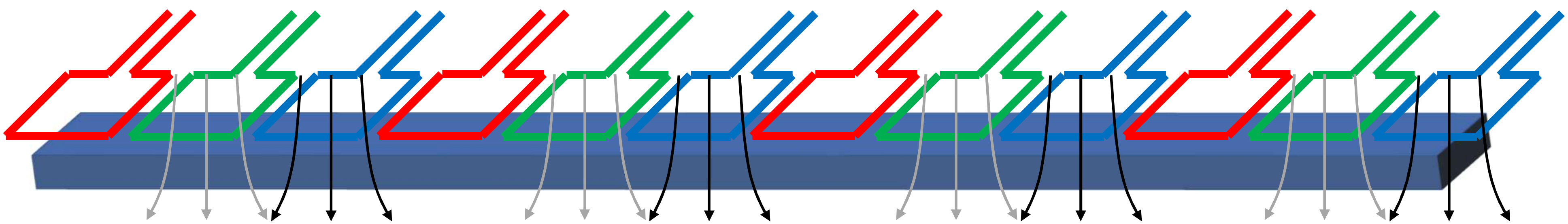
5. Two EM induction process is involved here. a. the compression of the seed field is a inductive process; b. the compression force is provided by a separate inductive process.

https://mgsl.issp.u-tokyo.ac.jp/topics/magnet_en.html

An induction motor

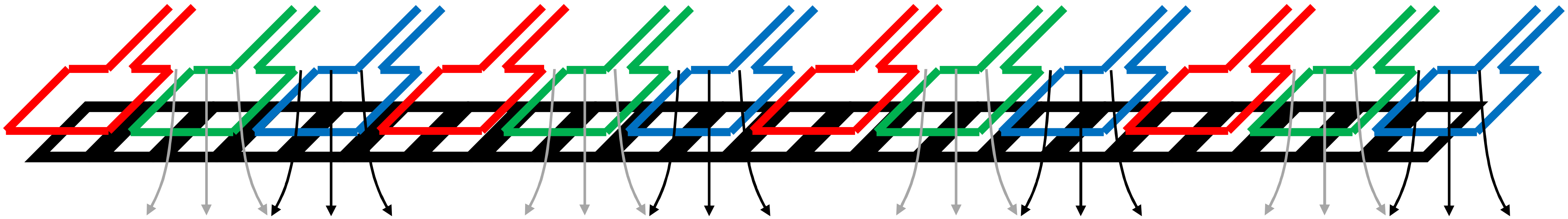


1. Apply current in the coils consecutively. The metal block is moving with the flux line.

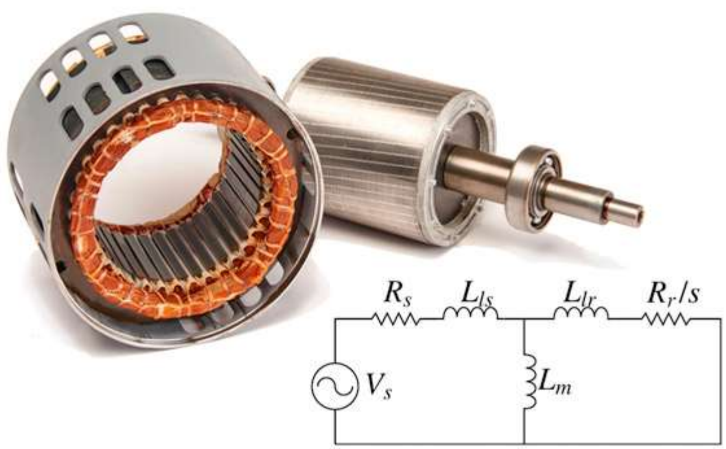


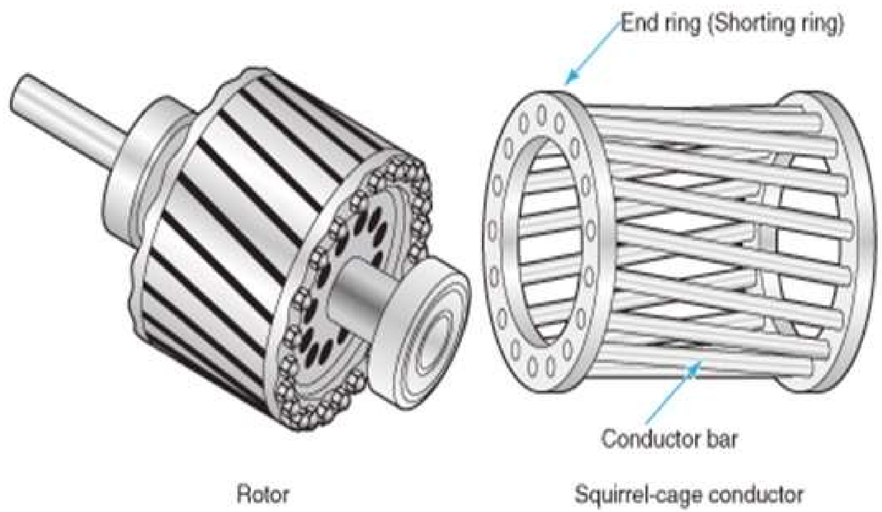
2. Extend the block. Apply AC current with phase lag.

An induction motor



3. Change the block into a grid of metal

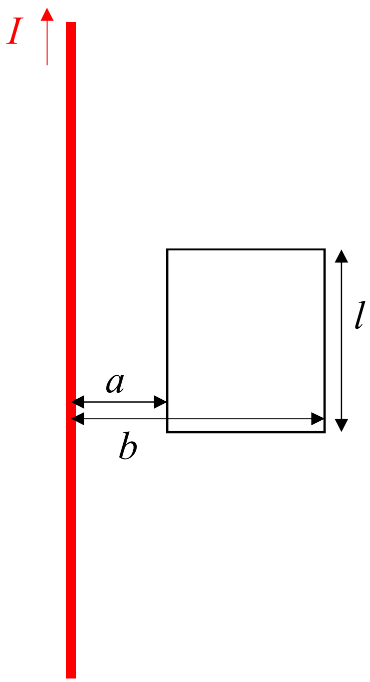




4. Wrap the linear structure up, into a circular structure: squirrel-cage motor.

Example problems

1. For a current carrying long straight wire ($I = I_0 \sin\omega t$), and a rectangular loop, with the dimensions noted in the figure, what is the induced emf in the loop (assuming that clockwise is defined the positive direction for the loop).



2. Firstly, calculate the flux in the loop as a function of the current in the wire.

$$B_\theta(r) = \frac{\mu_0 I}{2\pi r}$$

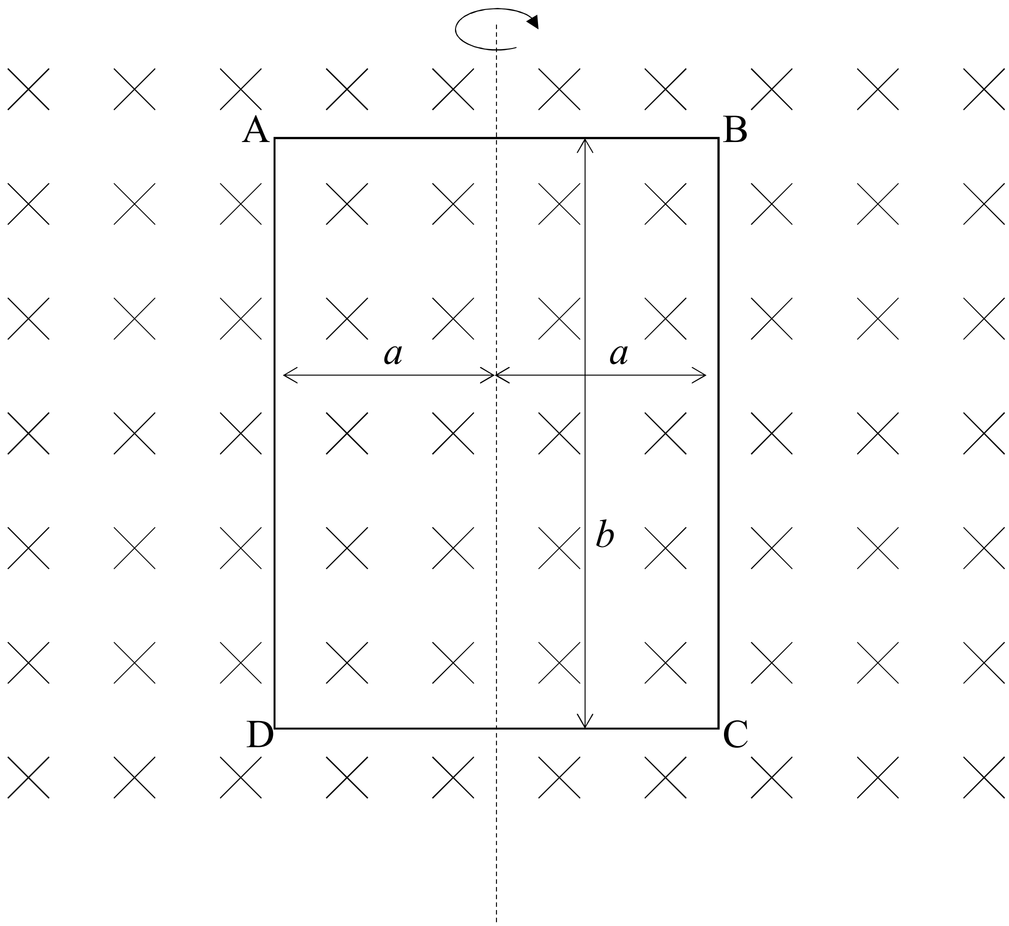
$$\Phi = l \int_a^b \mathrm{d}r B_\theta(r) = l \int_a^b \frac{\mu_0 I}{2\pi r}\mathrm{d}r = \frac{\mu_0 I l}{2\pi} \ln \frac{b}{a}$$

3. Then calculate the induced emf as

$$\mathcal{V} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -\frac{\mu_0 l}{2\pi} \ln \frac{b}{a} \frac{\mathrm{d}I}{\mathrm{d}t} = -\frac{\mu_0 l \omega I_0}{2\pi} \ln \frac{b}{a} \cos\omega t$$

Example problems

1. For a rotating wire loop inside a homogeneous magnetic field (B_z=B_0), with the rotation speed ω, initial angle as pictured, positive direction of emf defined as ABCDA, dimensions noted in the picture, what is the induced emf?



The cross section of the loop, described as a vector, is

$$\vec{S} = 2ab\cos\omega t\hat{z}$$

So the flux through the loop is

$$\Phi(t) = \vec{S}\cdot\vec{B} = 2abB_0\cos\omega t$$

And the induce emf

$$\mathcal{V} = -\frac{\mathrm{d}\Phi(t)}{\mathrm{d}t} = -2\omega abB_0\sin\omega t$$

One can also use the method of four edges cutting the magnetic flux lines, and get the same result.