Physics (PHYS2500J), Unit 1 Electrostatics:
3. Electric potential

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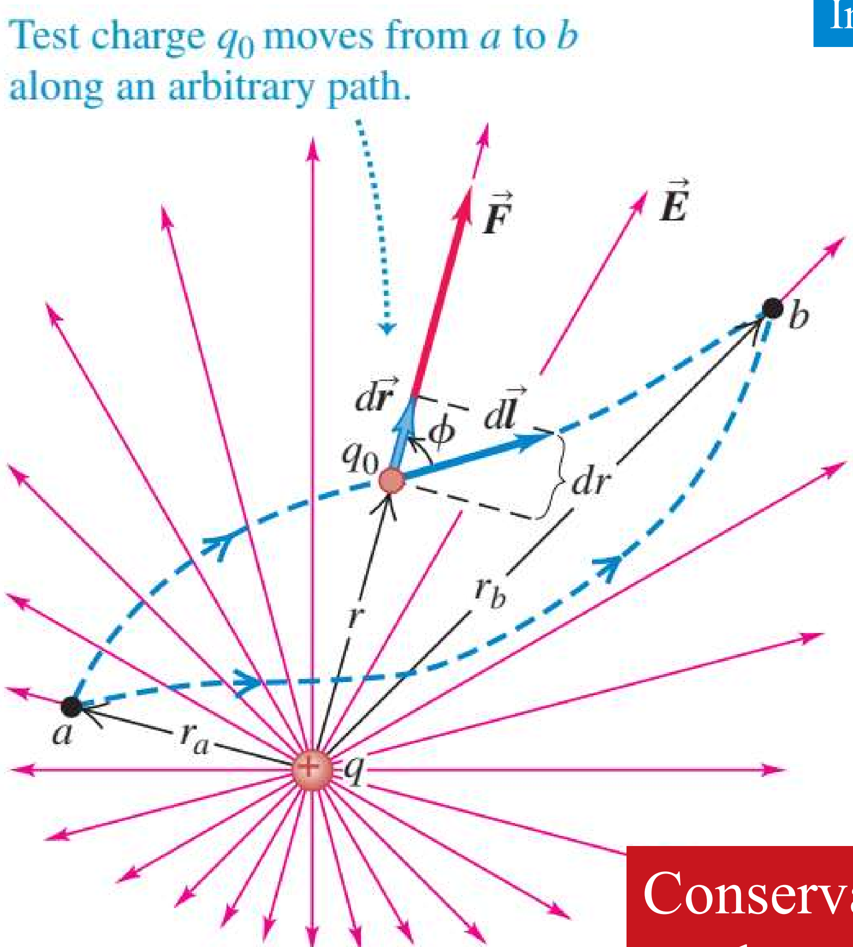
Electrical work done by the field of a charge



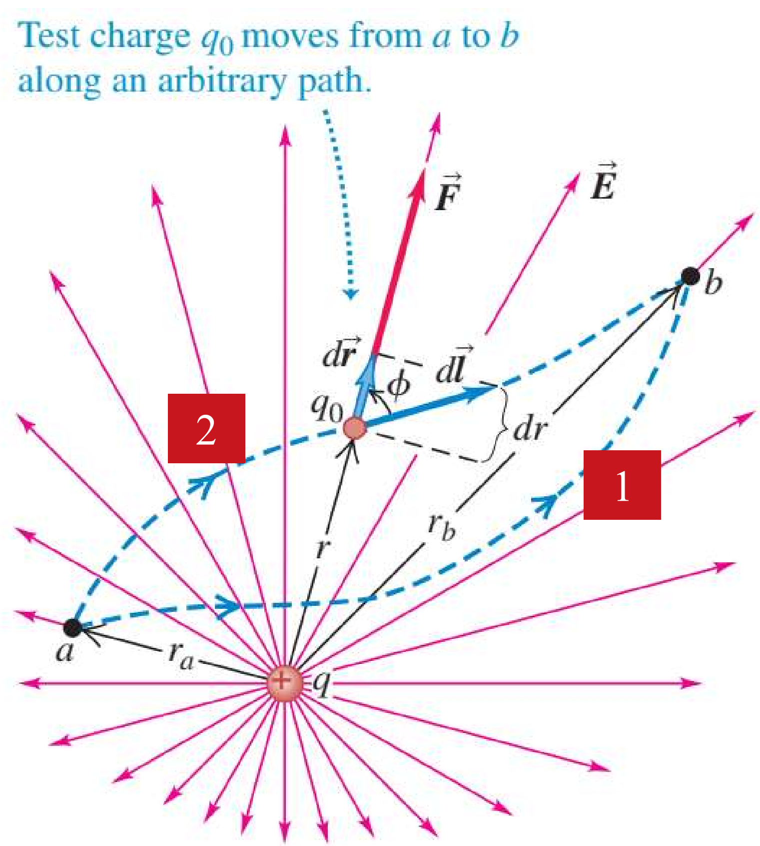
In spherical coordinate system

$$\begin{aligned} W &= \int_a^b \vec{F}\cdot \mathrm{d}\vec{l} \\ &= \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}\hat{r}\cdot(\mathrm{d}r\hat{r} + r\mathrm{d}\theta\hat{\theta} + r\sin\theta \mathrm{d}\phi\hat{\phi}) \\ &= \frac{1}{4\pi\epsilon_0}\int_a^b \frac{q}{r^2}\mathrm{d}r \\ &= \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_a} - \frac{1}{r_b}\right) \end{aligned}$$

Conservative field (force): the work done just depends on the starting and ending point, not on the path.

Circular linear integral

The work done to the charge when it moves in a circle.



$$\oint \vec{F} \cdot \mathrm{d}\vec{l} = q \oint \vec{E} \cdot \mathrm{d}\vec{l} = 0$$

Reverse path 2 and combine with path 1, then you get a circle.

Any circle can be broken into two reversed paths from point a to b.

The relative energy is a function of position and position only: potential energy (势能、位能)

Definition of potential

Conservative force: no energy consumption / accumulation during a circle

Gravitational force → Gravitational field → Gravitational potential

$$\vec{F} = -\frac{GMm}{r^2}\hat{r}$$

$$\vec{g} = -\frac{GM}{r^2}\hat{r}$$

$$\phi = -\frac{GM}{r}$$

Static electrical force → electric field → electric potential

$$\vec{F} = \frac{1}{4\pi\epsilon}\frac{Qq}{r^2}\hat{r}$$

$$\vec{E} = \frac{1}{4\pi\epsilon}\frac{Q}{r^2}\hat{r}$$

$$\phi = \frac{1}{4\pi\epsilon}\frac{Q}{r}$$

有心力 (centered force?):

The force field is pointing from / to an origin (center) and the magnitude of the force is a function of only the distance (in spherical coordinates, $F(r,\theta,\varphi)=F(r)$)

Inverse square law is a enhanced condition

Work done in a centered force field is only dependent on the starting and ending point of the loop. Or, a centered force is conservative.

What happens if the force field is not conservative?

The concept of electric potential is not valid for general cases of electric field, just for static electric field.

Potential difference and potential: reference point

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$$W = \int_a^b q\vec{E} \cdot \mathrm{d}\vec{l} = U_a - U_b$$

The loss of potential energy.

Work done by electric force (not overcoming the electric force).

$$\phi_a - \phi_b = \int_a^b \vec{E} \cdot \mathrm{d}\vec{l}$$, with $$\phi = \frac{U}{q}$$

U: electrical (potential) energy;
φ: electrical potential

$$V_a - V_b = \int_a^b \vec{E} \cdot \mathrm{d}\vec{l}$$

With infinity chosen as the reference point ($\Phi_{infinity}$=0)

Ground and infinity long are usually taken as reference points.

$$\phi_a = \int_a^\infty \vec{E} \cdot \mathrm{d}\vec{l} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r_a}$$

Work and potential in static electric field (several charges)

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$$\int_s \vec{F}\cdot \mathrm{d}\vec{r} = \int_s \sum_i \vec{F}_i \cdot \mathrm{d}\vec{r} = \sum_i \int_s \vec{F}_i \cdot \mathrm{d}\vec{r}$$

$$\int_s \vec{E}\cdot \mathrm{d}\vec{r} = \int_s \sum_i \vec{E}_i \cdot \mathrm{d}\vec{r} = \sum_i \int_s \vec{E}_i \cdot \mathrm{d}\vec{r}$$

$$U(P) = \sum_i U_i(P)$$

This answers why infinity (rather than, say, 5 cm away from a charge) is a good reference

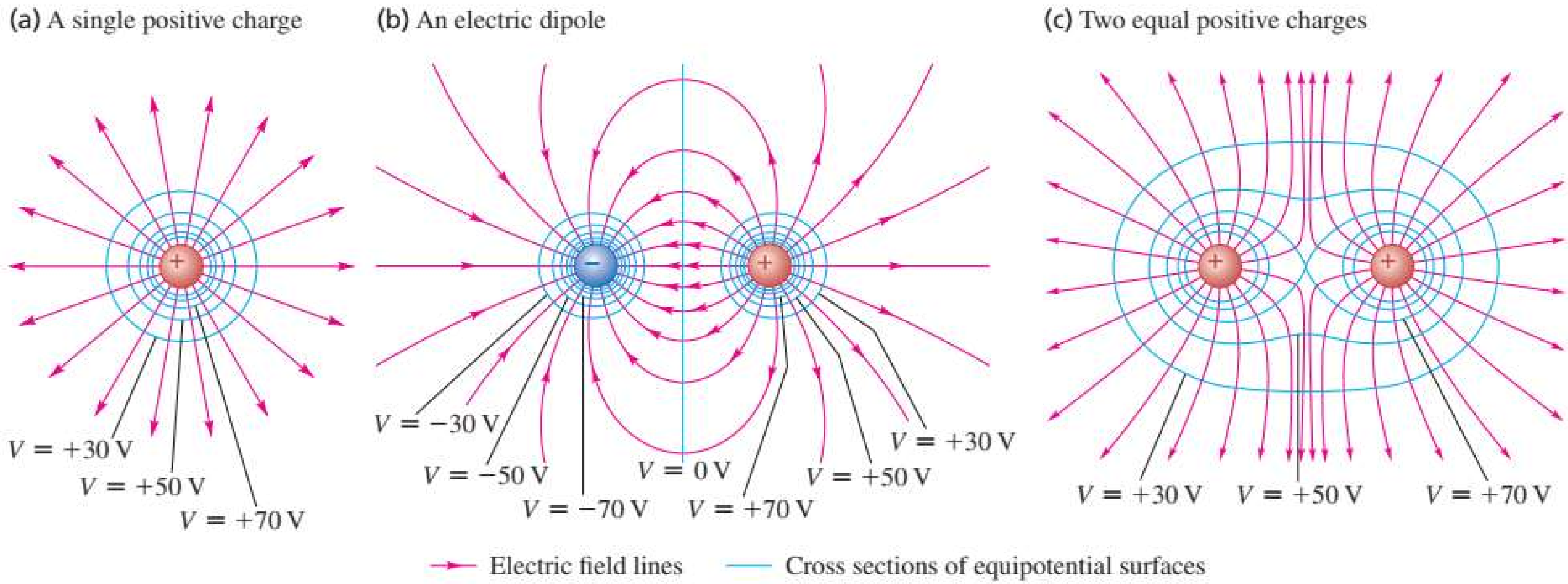
Equipotential surfaces

One charge

Two charges with opposite sign

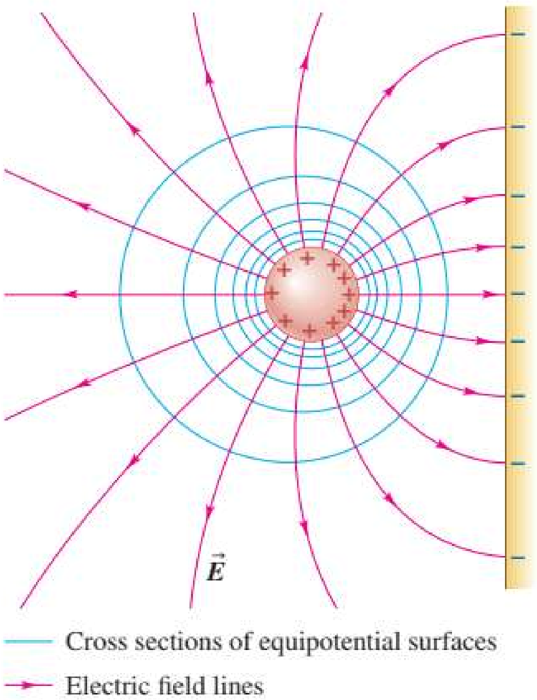
Two charges with the same sign

23.23 Cross sections of equipotential surfaces (blue lines) and electric field lines (red lines) for assemblies of point charges. There are equal potential differences between adjacent surfaces. Compare these diagrams to those in Fig. 21.28, which showed only the electric field lines.



Equipotential surfaces with ground

23.24 When charges are at rest, a conducting surface is always an equipotential surface. Field lines are perpendicular to a conducting surface.



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Field and potential

If field (and a reference point) is given, how do you calculate potential

$$\phi(\vec{r}_1) = \int_{\vec{r}_1}^{\vec{r}_0} \vec{E}(\vec{r})\mathrm{d}\vec{r} + \phi_0$$

If potential is known, how do you calculate the field?

1-D problem such as the gravitational potential: take derivative

E field is a 3-D problem

$$\vec{E} = -\vec{\nabla}\phi$$

Gradient

$$\vec{\nabla}\phi = \hat{x}\frac{\partial}{\partial x}\phi + \hat{y}\frac{\partial}{\partial y}\phi + \hat{z}\frac{\partial}{\partial z}\phi$$

Gradient

Gradient: the gradient of a ***Scalar*** field φ at point $\boldsymbol{r}$ is a vector

Direction is the fastest increasing direction of φ.

Magnitude is the spatial derivative of φ in this direction.

The gradient operation changes a scalar field into a vector field, e.g.

$$\vec{E} = -\vec{\nabla}\phi$$

Is gradient REALLY a vector?

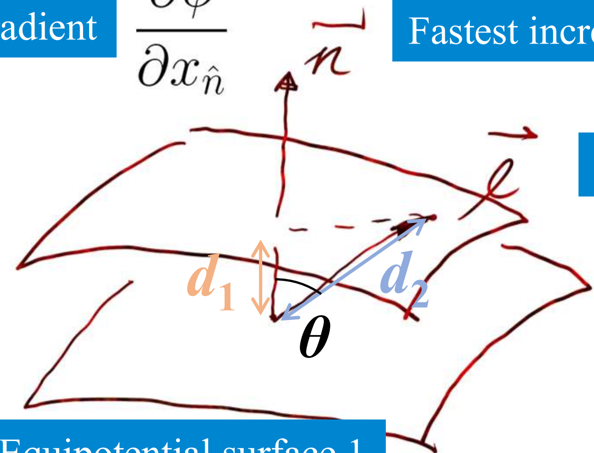
Not any 3 components make a vector. A vector has to be able to sum and decomposed like a vector.

Magnitude of gradient $\dfrac{\partial \phi}{\partial x_{\hat{n}}}$

Fastest increasing direction

Any other direction

Equipotential surface 2



Equipotential surface 1

What is the projection of $\dfrac{\partial \phi}{\partial x_{\hat{n}}}\hat{n}$ in $\hat{l}$ direction?

$$\frac{\partial \phi}{\partial x_{\hat{n}}}\cos\theta = \frac{\Delta\phi}{d_1/\cos\theta} = \frac{\Delta\phi}{d_2} = \frac{\partial \phi}{\partial x_{\hat{l}}}$$

It decomposes in a vector way.

The sum of two gradient vector involves two field, you can prove it yourself.

The gradient of potential

$$-\vec{\nabla}\phi = \vec{E}$$

$$\Delta\phi = -\vec{E}\cdot\Delta\vec{r} = -E_x\Delta x - E_y\Delta y - E_z\Delta z$$

$$\frac{\partial\phi}{\partial x} = -\frac{E_x\Delta x + E_y\Delta y + E_z\Delta z}{\Delta x}\Big|_{\Delta y=0,\Delta z=0}$$

What does \partial mean?

All depends on the fact that gradient **IS** a vector

Some tips about “nabla”

It behaves like a vector

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

But it is not, it is an operator.

It is easy to note this way.

Applied it on a scalar (field), you get a vector (field).

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z}$$

$$\vec{E} = -\vec{\nabla}\phi$$

Use it as dot product, change a vector (field) into a scalar (field).

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

It also works in the cross product way (curl).

Equivalence of the following statement

$$\int \vec{E} \cdot \mathrm{d}\vec{l}$$

is independent of the path chosen.

$$\oint \vec{E} \cdot \mathrm{d}\vec{l} = 0$$

$$\vec{E} = \nabla \phi$$

Vector field ***E*** is conservative.

$$\nabla \times \vec{E} = 0$$

Example of using potential to calculate field

$$\nabla \frac{1}{4\pi\epsilon_0}\frac{Q}{r} = \frac{Q}{4\pi\epsilon_0}\left(\frac{\partial}{\partial x}\frac{1}{r}\hat{i} + \frac{\partial}{\partial y}\frac{1}{r}\hat{j} + \frac{\partial}{\partial z}\frac{1}{r}\hat{k}\right)$$

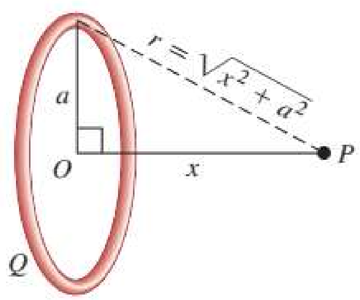
$$\frac{\partial}{\partial x}\frac{1}{r} = -\frac{\partial r/\partial x}{r^2} = -\frac{x}{r^3}$$

$$-\nabla\phi(r) = \frac{Q}{4\pi\epsilon_0}\frac{x\hat{i}+y\hat{j}+z\hat{k}}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\frac{\vec{r}}{r}$$

Using spherical coordinate system is easier.

Example of using potential to calculate field

The field of a charged ring:



23.20 All the charge in a ring of charge Q is the same distance r from a point P on the ring axis.

$$V = \frac{1}{4\pi\epsilon_0}\int\frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\frac{1}{\sqrt{x^2 + a^2}}\int dq = \frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{x^2 + a^2}}$$

$$E_x = -\frac{\partial V}{\partial x} = \frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2 + a^2)^{3/2}}$$

Charge, *E* field and potential

Charge density $\rho(r)$

$$\nabla \cdot \vec{E} = \rho/\epsilon_0$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$

$$\phi = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

$$\nabla^2\phi = -\frac{\rho}{\epsilon_0}$$

$$\phi = \int \vec{E} \cdot \mathrm{d}\vec{l}$$

E field $E(r)$

$$\vec{E} = -\vec{\nabla}\phi$$

Potential $\Phi(r)$

Poisson's equation for electrostatic potential

The derivation of Poisson's equation

$$\nabla \cdot (\nabla \phi) = \nabla \cdot (-\vec{E}) = -\frac{\rho}{\epsilon_0}$$

Some vector calculation will show that

$$\nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Two simpler ways of noting the Laplace operator

$$\nabla \cdot (\nabla \phi) \equiv \nabla^2$$

$$\nabla \cdot (\nabla \phi) \equiv \Delta$$

The Poisson's equation, called Laplace's equation when charge is absent

$$\nabla^2 \phi = -\frac{\rho}{\epsilon_0}$$

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Behavior of metal in electric field

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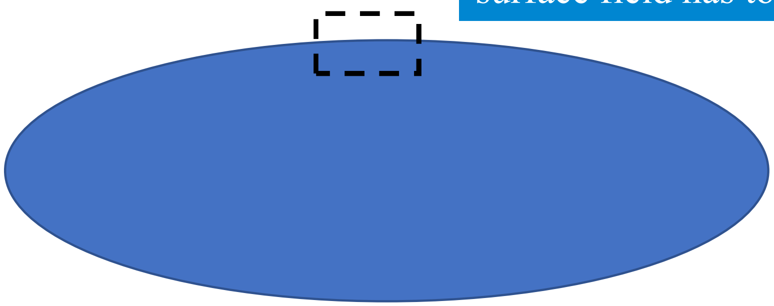
1. No electric field allowed inside bulk (homogeneous, static) metal. ⟷ Bulk metal is a equipotential body.

2. No tangential component of electric field is allowed on metal surface (either outer or inner surface). ⟷ Metal surface is a equipotential surface.

3. Charge has to be distributed only on the surfaces. If any charge is found in the inner surface, there must be the same quantity of opposite charge in the cavity.

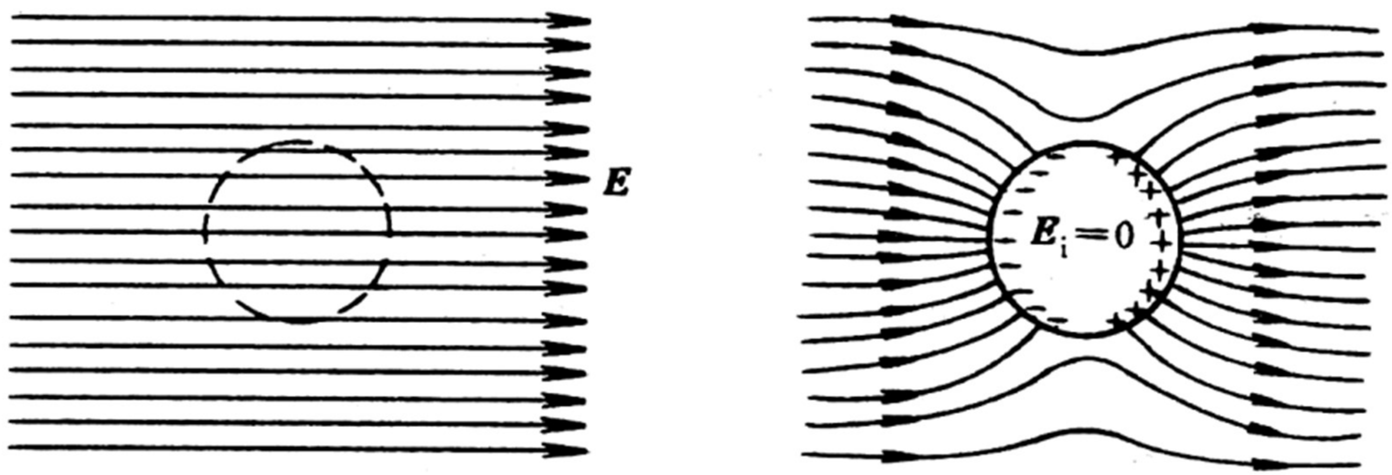
4. Surface field.



Using Gauss's law, the surface field has to be

$$\vec{E} = \hat{n}\frac{\sigma_e}{\epsilon_0}$$

Draw the electric field lines around a metal ball



Calculate the field in this case

$+q$ $(0, 0, h)$

The induced charge will:

1. cancel the lateral (tangential field generated by $+q$), by having lower and lower density with increasing x.
2. Compensate the field pointing down by $+q$.

From 2 and the gaussian surface, we have

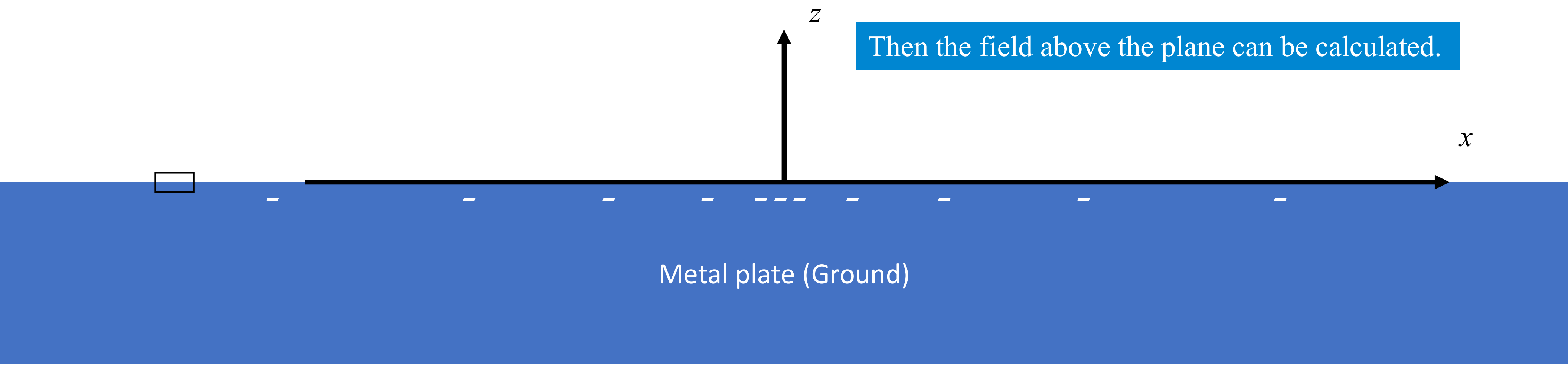
Since no normal component of E can be generated by charges elsewhere.

$$\vec{E}(x,0,0+) = 0\hat{x} + 0\hat{y} - \frac{1}{4\pi\epsilon_0}\frac{q}{x^2+h^2}\frac{h}{\sqrt{x^2+h^2}}\hat{z} + \frac{\sigma_e(x,y=0)}{2\epsilon_0}\hat{z}$$

$$\vec{E}(x,0,0-) = 0\hat{x} + 0\hat{y} - \frac{1}{4\pi\epsilon_0}\frac{q}{x^2+h^2}\frac{h}{\sqrt{x^2+h^2}}\hat{z} - \frac{\sigma_e(x,y=0)}{2\epsilon_0}\hat{z} = 0$$

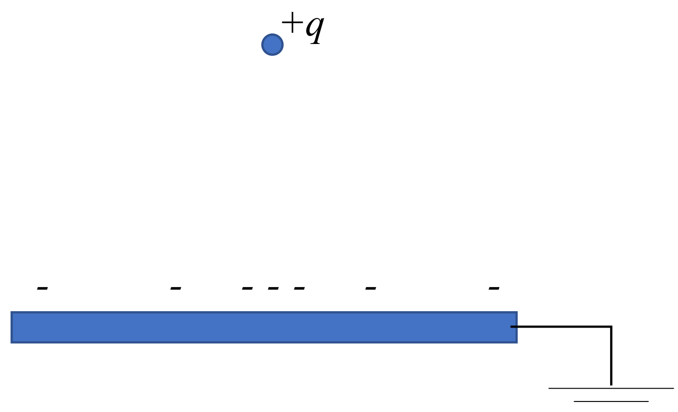
or $$\sigma_e(x,y=0) = -\frac{1}{2\pi}\frac{qh}{\sqrt{x^2+h^2}^{3/2}}$$

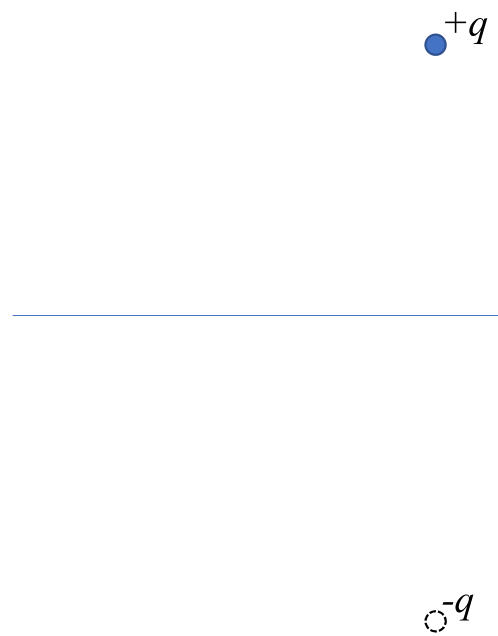
Then the field above the plane can be calculated.



A quicker solution?

comparing the two cases

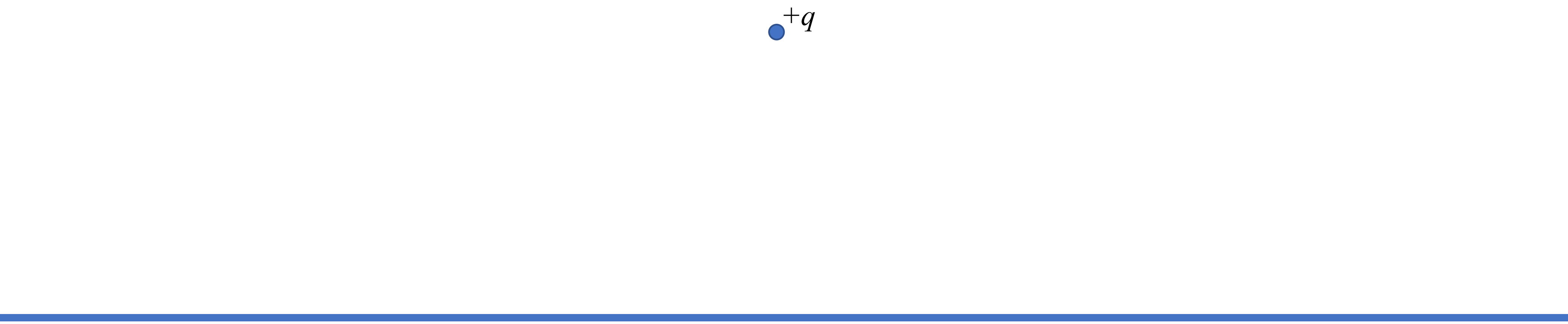




An imaginary charge

They are equivalent in term of creating a plane of 0 potential

The uniqueness theorem



If I know the **potential of this plane** (has to be 0 if it is infinitely large), and the **charge +q to be the only one** in the space above the plane, DO I know the electric field **everywhere in the space above the plane**?

The answer is YES! It is called the uniqueness theorem of electrostatic field.

Three key elements:

1. the boundary condition (potential is one type of boundary condition)
2. the charge distribution in the space
3. know which space you are talking about!

How to use the uniqueness theorem

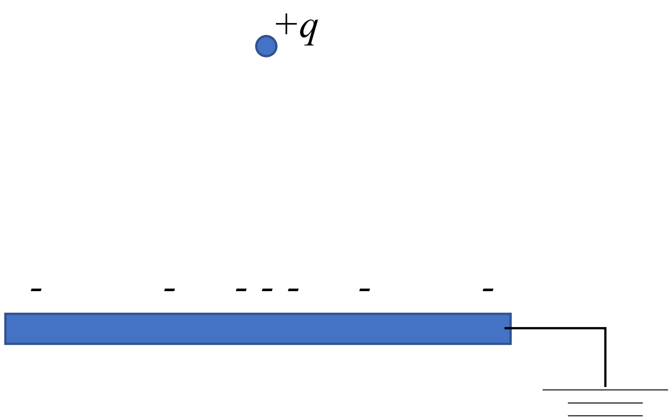
If the field can be determined by boundary potential and internal charge distribution.

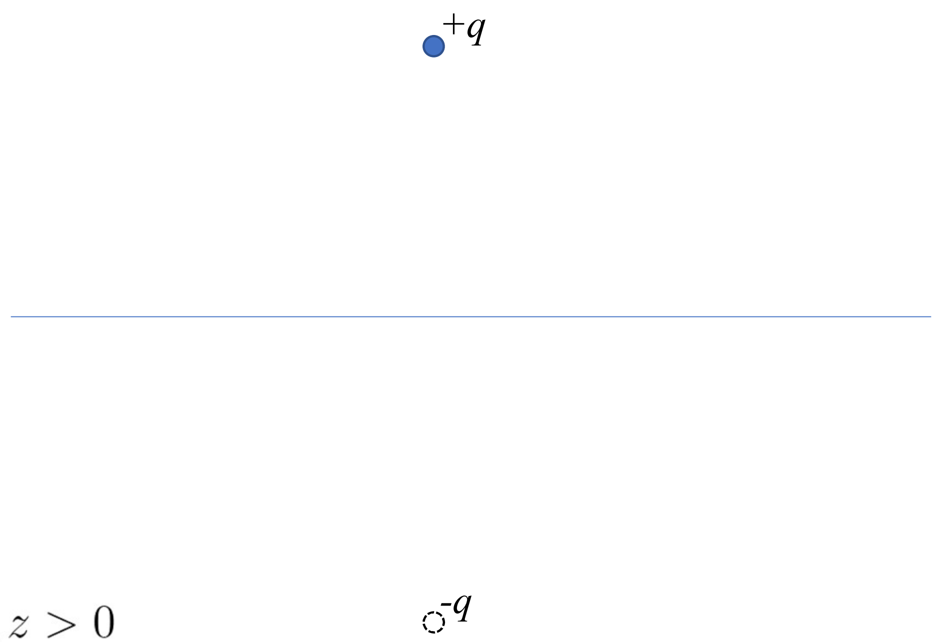
And potential on the boundary can be constructed with an "imaginary" charge OUT of the interesting space.

Can we use the imaginary charge <u>**outside**</u> the interesting space to determine the field <u>**in**</u> the interesting space?

Yes!

How?





$$\phi = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+(z-h)^2}} - \frac{1}{\sqrt{r^2+(z+h)^2}}\right], z > 0$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{r}{\sqrt{r^2+(z-h)^2}^3} - \frac{r}{\sqrt{r^2+(z+h)^2}^3}\right]\hat{r}$$
$$+\frac{q}{4\pi\epsilon_0}\left[\frac{z-h}{\sqrt{r^2+(z-h)^2}^3} - \frac{z+h}{\sqrt{r^2+(z+h)^2}^3}\right]\hat{z}, z > 0$$

It is called the image charge method!

Quick review

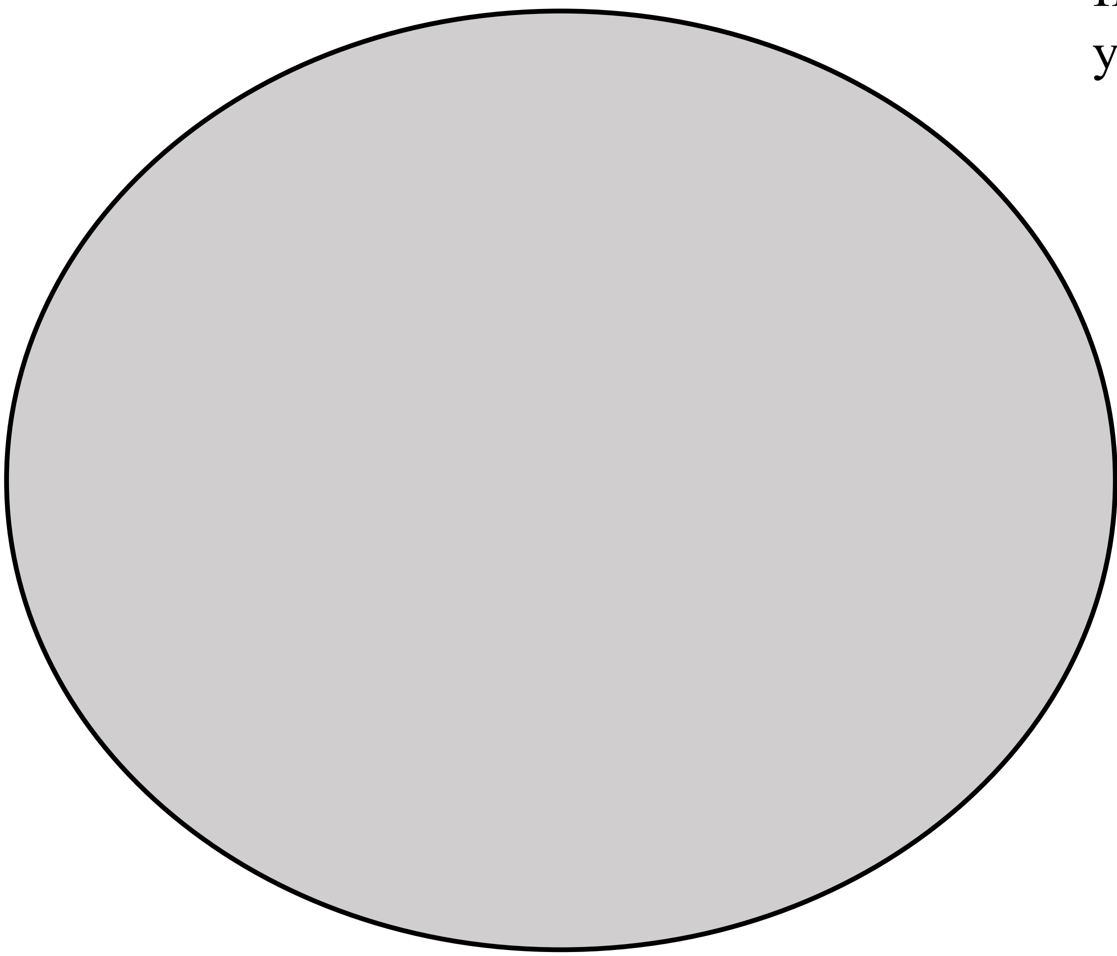
The image charge is based on the uniqueness theorem.

If you can use an image charge OUTSIDE the target space to create a right boundary potential.

Then the field created by the image charge and the real charge in the target space can create correct field IN the target space.

The image charge creates WRONG field outside the interesting space!

A way of looking at the uniqueness theorem



If you draw the electric field line in the space, your line has to start / end

1. from / at the charge inside
   internal charge distribution

2. from / at the boundary
   boundary potential
   (sometimes inner boundary potential)

3. form a loop
   No such thing for static electric field

Use proof by contradiction, one can prove the uniqueness theorem.

Assume two field distributions, E_1 and E_2, both satisfies the same internal charge distribution and the boundary potential.

Then, a new field $E = E_1 - E_2$ have some interesting properties:

$$\nabla \cdot \vec{E} = \nabla \cdot (\vec{E}_1 - \vec{E}_2) = \nabla \cdot \vec{E}_1 - \nabla \cdot \vec{E}_2 = \frac{\rho}{\epsilon_0} - \frac{\rho}{\epsilon_0} = 0$$

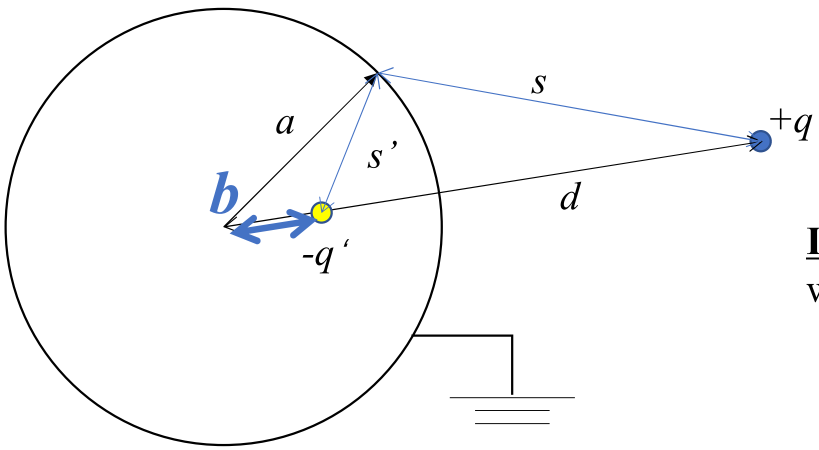
and that the boundary for E is equipotential

⇒ E starts at nowhere and ends at nowhere. E is 0 everywhere.

so $E_1 = E_2$, only one field distribution is allowed.

The case with a metal ball

A metal ball (radius *a*) is grounded, and a point charge *q* exists outside the sphere. Please solve the field outside the sphere.



IF there is a working image charge (what is a working image charge), where it is?

$$\frac{-q'}{4\pi\epsilon_0 s'} + \frac{q}{4\pi\epsilon_0 s} = 0$$

Surprisingly, there is a solution:

$$q' = aq/d$$
$$b = a^2/d$$

q and *q'* are image charges for each other

What if the sphere is not grounded?

Each cases for image charge method is a miracle

It probably take some very clever people a long time to solve one problem like this, so do not ask why you can not think of it, just use it.

Only very limited problems can be solved by the image charge method.