Physics (PHYS2500J), Unit 1 Electrostatics:
2. Electric field

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Coulomb’s law

$$F = k\frac{q_1 q_2}{r^2}$$

$$k = 8.987551787 \times 10^9\ \mathrm{N \cdot m^2/C^2} \cong 8.988 \times 10^9\ \mathrm{N \cdot m^2/C^2}$$

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

$$\epsilon_0 = 8.854 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2}$$

ε_0 known as the dielectric constant of vacuum.

How do two non-contacting object exert force on each other???

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It worked through some invisible medium (ether).

It just did, no need through any medium (no need of time either, compared with elastic force, propagating with velocity of sound), this is called an action at a distance
But how does the charge know the existence of the other charge?

$$F = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$$

1. Symmetrical for both charges
2. Proportional to the "test charge": hint for some "field"?
   (Cavendish used the term 'electric gas')

Charge ←→ Field ←→ Charge

$$\vec{E} = \frac{\vec{F}}{q}$$

Test charge

Field itself has energy and momentum.

Field can sustain them selves without charge.

Field itself is a form of matter

The electric field of a point charge

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$$

$$\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}}$$

Unit vector pointing from the source point to the field point

Unit of electric field: N/C

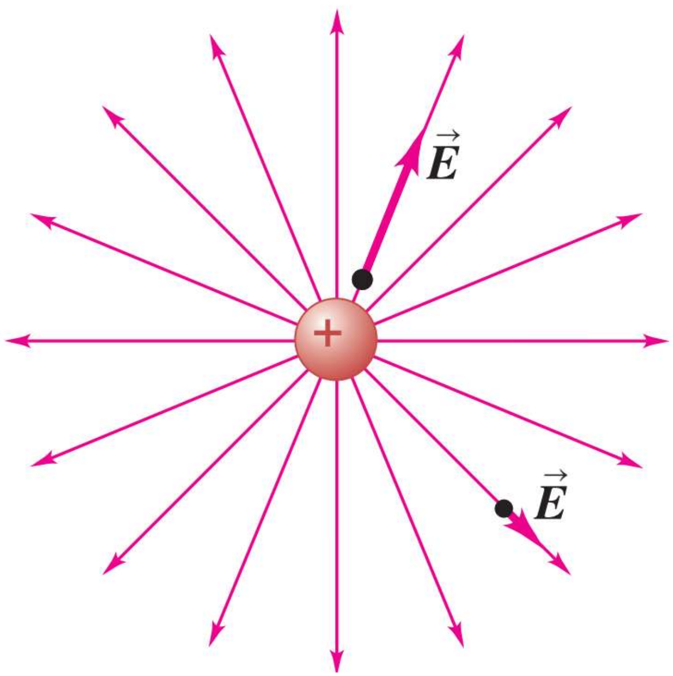
Point charge is useful for:

1. when the two charges in action is far away
2. complicated shape can be considered as the sum of many point charges.

Field lines of a point charge

Introduced by English scientist Michael Faraday

The field line of one charge



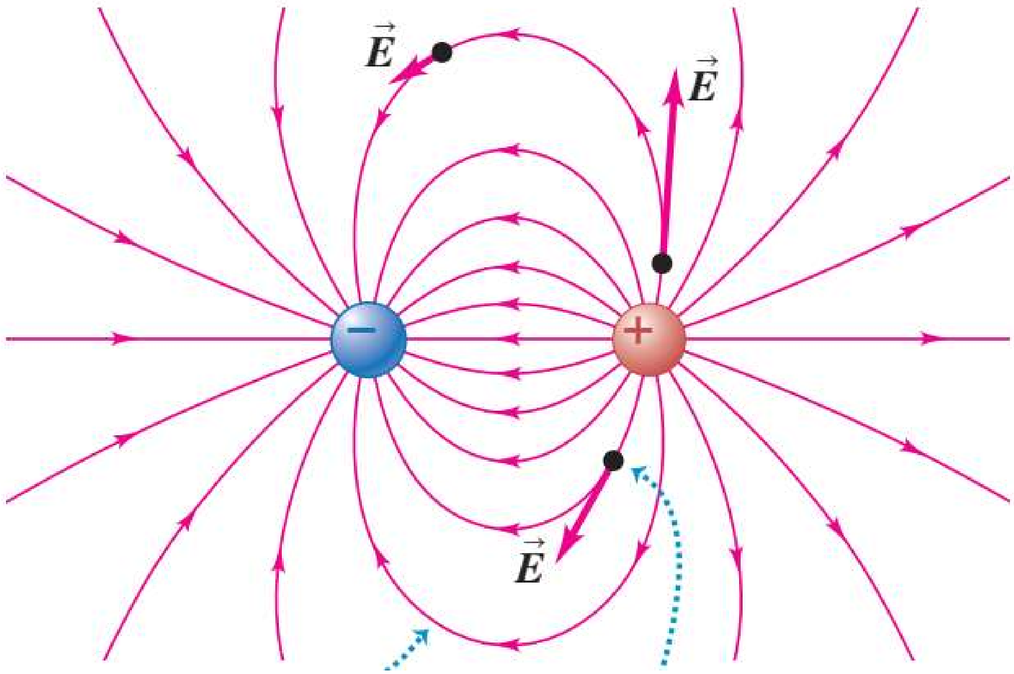
Direction: along the line of force

Magnitude: proportional to the density of the lines

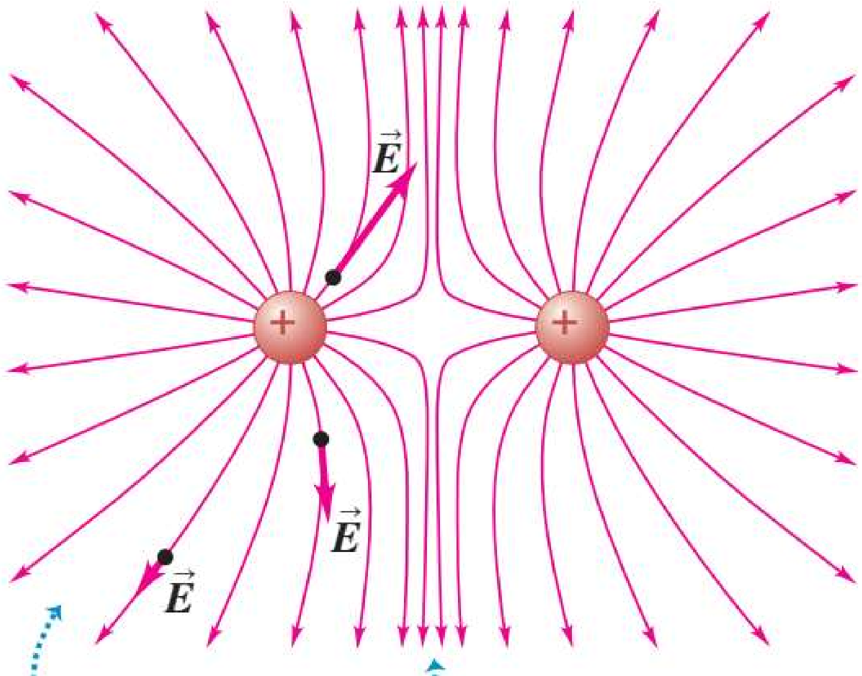
Never cross one another, unless at the singularity point

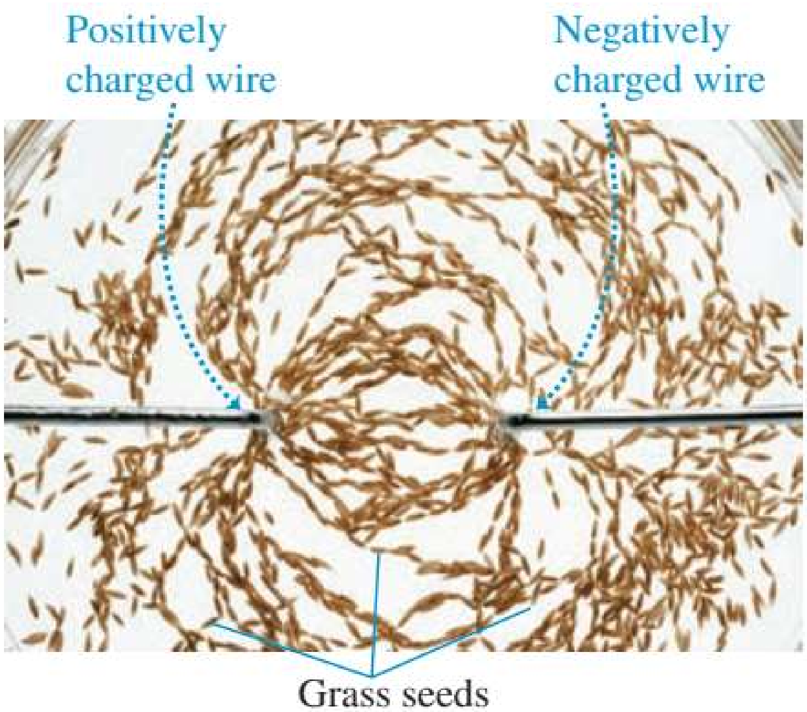
Electric field line of two charges with the same absolute value

(b) Two equal and opposite charges (a dipole)



(c) Two equal positive charges





The field lines can be seen. Mainly the direction.

Field is affected by the seeds, so it is not exactly the field line.

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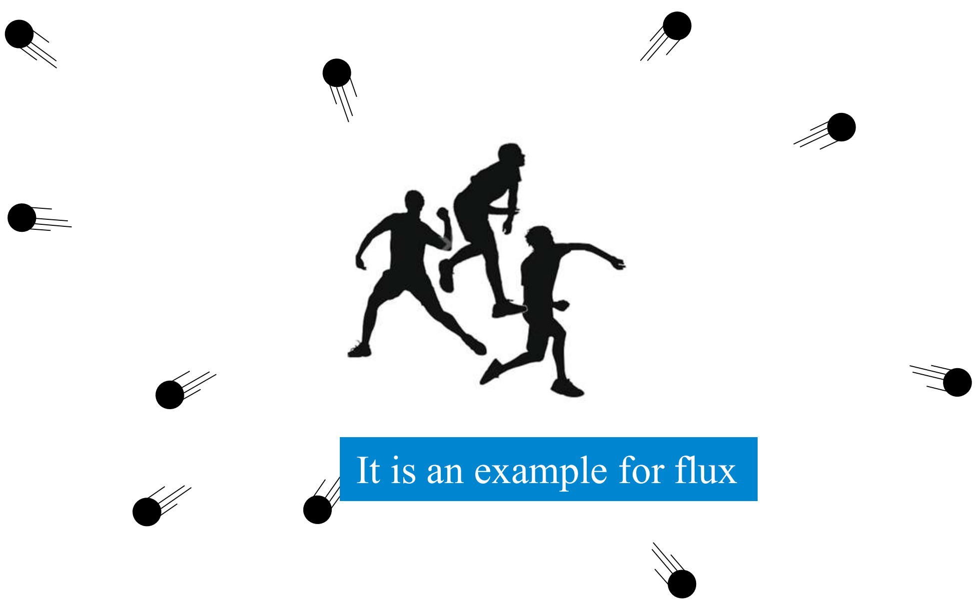
Rewriting the Coulomb's law

$$\vec{E} = \frac{1}{4\pi r^2}\frac{1}{\epsilon_0}Q\hat{r}$$

Inverse square law: r^{-2} hints about the surface of a sphere



It is an example for flux

What is the number of hit one get when the balls are not affected by aerodynamic drag and other forces.

$$\text{Number of hits} = \frac{tA_{\text{you}}p_{\text{throw}}}{r^2}$$

The number of hit you take is proportional to your size, the rate he throws the ball, the time you stand there, and inverse proportional to distance square.

Flux (通量)

Dictionary: the action or process of flowing or flowing out

Most common: flux of fluid (gas or liquid).

What is the unit for water (fluid) flux?

E.g. SCCM: standard cubic centimeter per minute

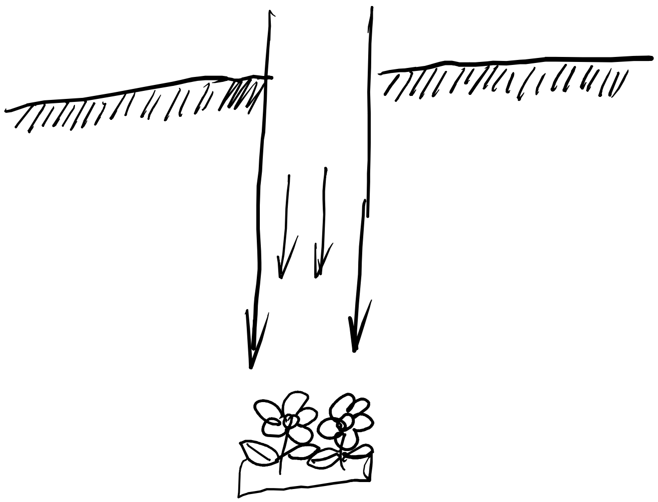
From kg/s to $(kg/m^2s)\cdot m^2$

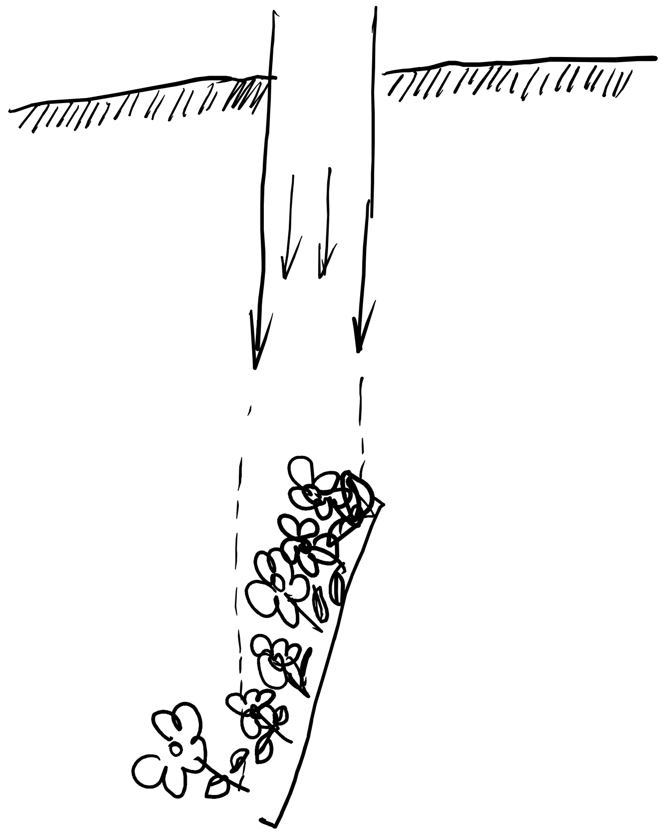
Can you name some other examples of flux?

Flux of energy (light or heat).

Flux of charge (current).

Vectorized flux



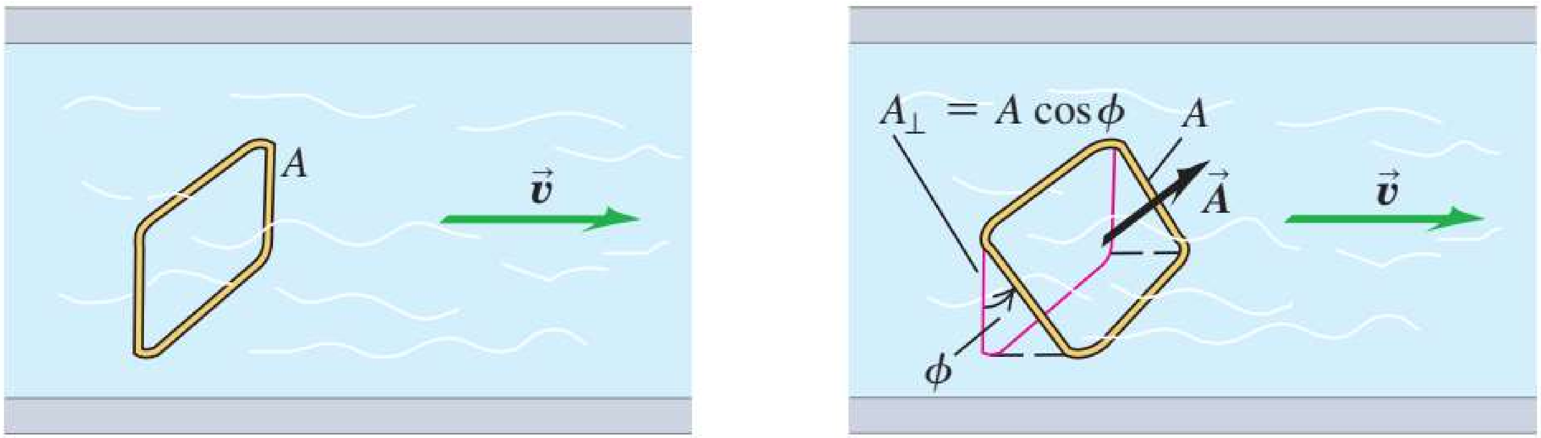


It is hotter in Hainan than Taiyuan. ☺

What is the fair way of doing it? a cosθ factor

(a) A wire rectangle in a fluid

(b) The wire rectangle tilted by an angle ϕ



$$\vec{v} \cdot \vec{A}$$

$$\iint_S \vec{v} \cdot \mathrm{d}\vec{A}$$

The (**total**) flux through **surface** S.

Flux through an enclosed surface: water flux and other sources/sinks

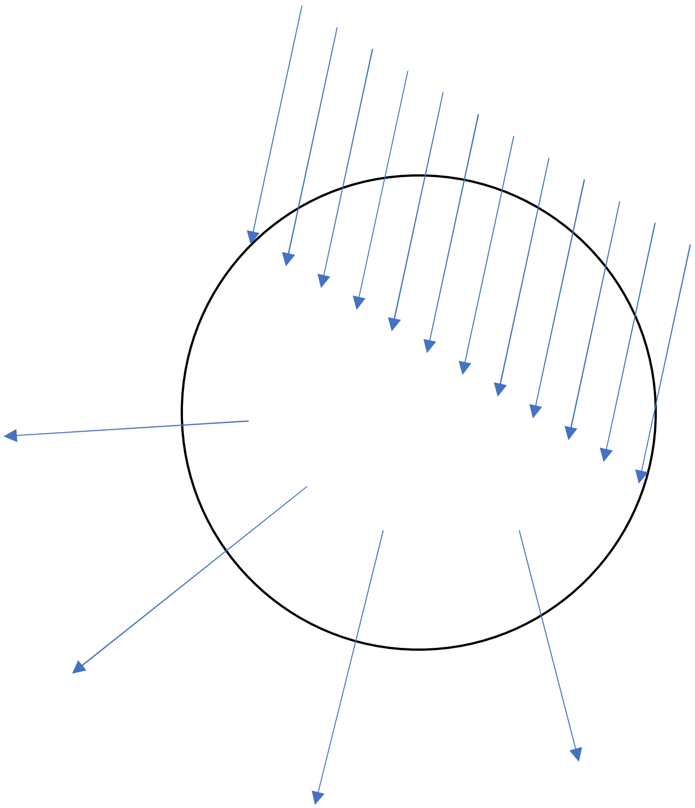
A heater

A light bulb

An AC

Chinese term for source and sink: 源与汇

Flux through an enclosed surface



Mathematical expression:

$$\oint_{\partial\Omega} \vec{A} \cdot \mathrm{d}\vec{S}$$

Sometimes $\oiint$

Definition of direction

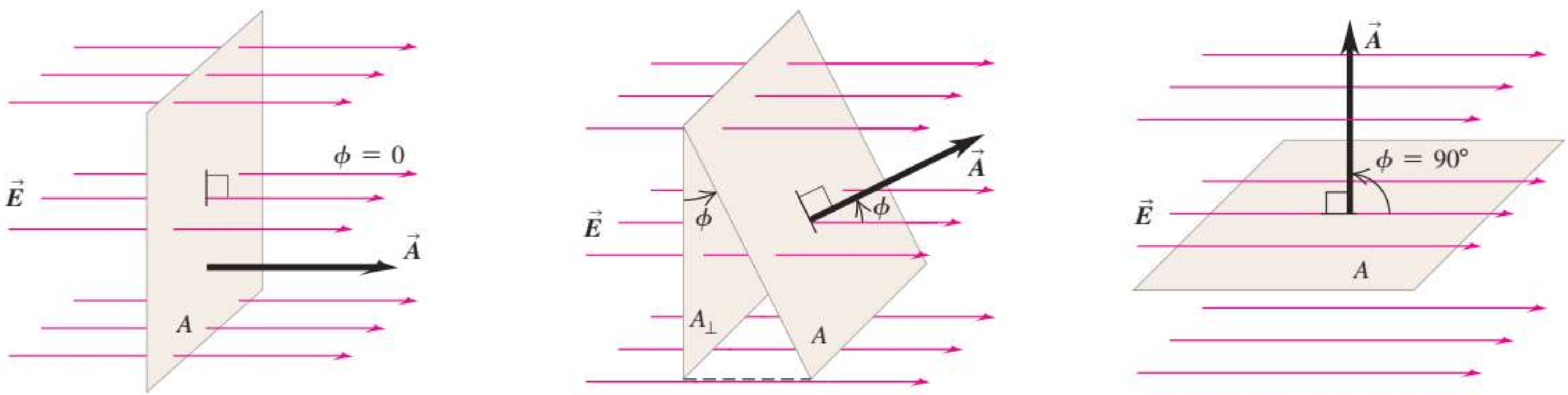
Pointing out as positive.

Some effects, example heat

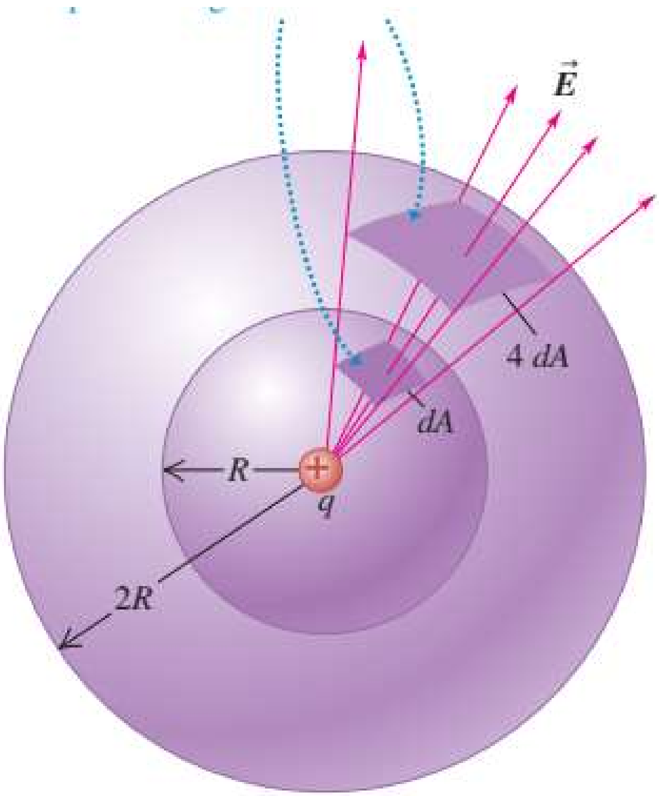
$$\oint_{\partial\Omega} k\vec{\nabla}T \cdot \mathrm{d}\vec{A} = C_\Omega \frac{\mathrm{d}T}{\mathrm{d}t}$$

Assuming the heat conduction is perfect inside the volume, to avoid integral on the right.

22.6 A flat surface in a uniform electric field. The electric flux Φ_E through the surface equals the scalar product of the electric field $\vec{E}$ and the area vector $\vec{A}$.



Flux through a sphere centered at the source charge

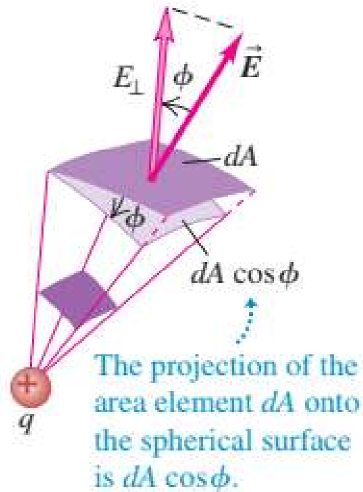


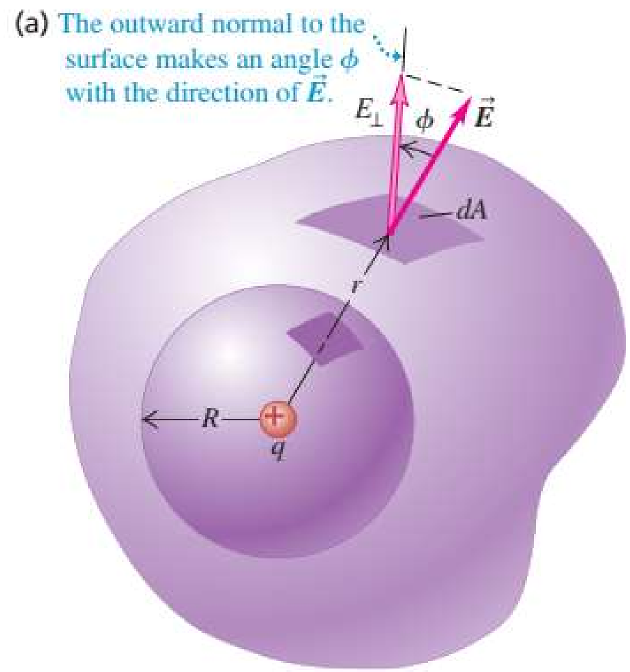
$$\Phi_e = \oiint \vec{E} \cdot \mathrm{d}\vec{A}$$

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

Independent on the radius!

Non-spherical surface





Independent on the radius!

$$\mathrm{d}\Phi_e = \vec{E} \cdot \mathrm{d}\vec{A} = E \mathrm{d}A \cos\phi = \frac{q}{\epsilon_0} \frac{\mathrm{d}\omega}{4\pi}$$

, where dω is the solid angle of the small patch

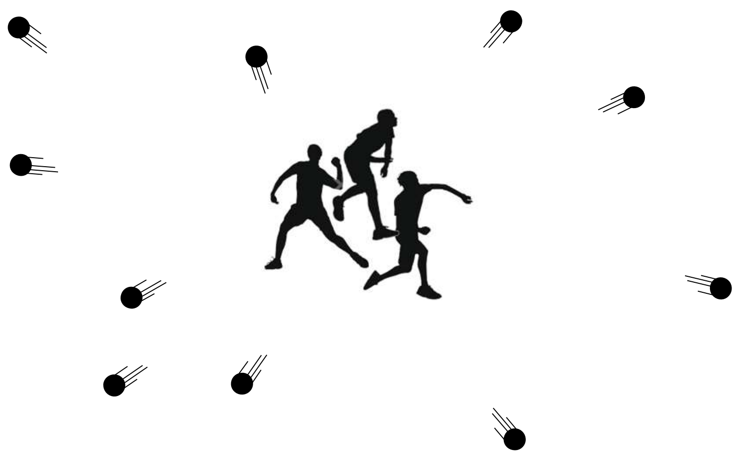
The flux on any enclosed surface

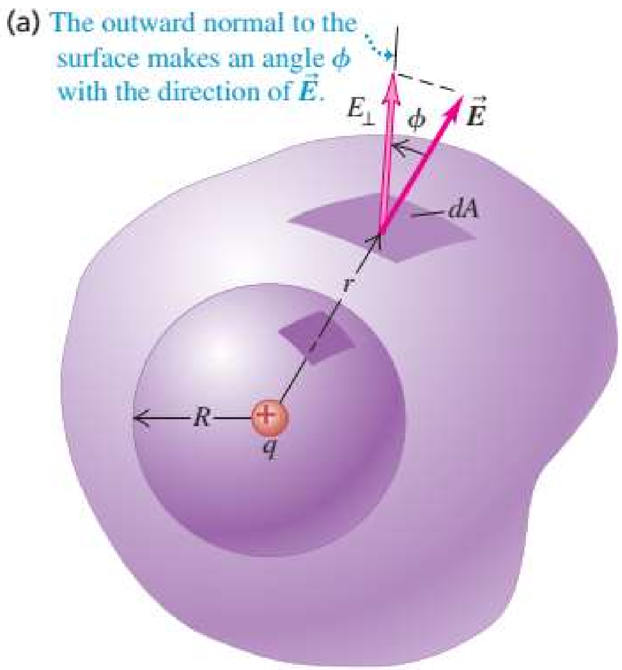
$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

Similarity and difference

Similarity:
The velocity vector and the E vector does not turn, so for the same solid angle, what goes through the spherical surface goes through the small patch.

Difference: nothing really goes through the surface in the electric field case.





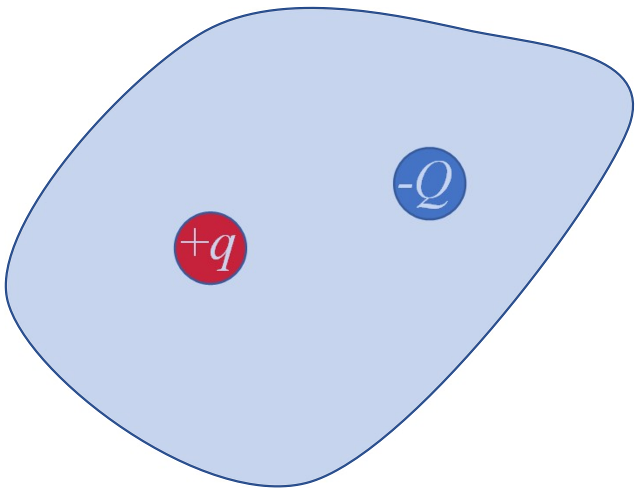
Gauss's law

Carl Friedrich Gauss

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q}{\epsilon_0}$$

For any enclosed surface, and q is the total charge inside.

Is this right?



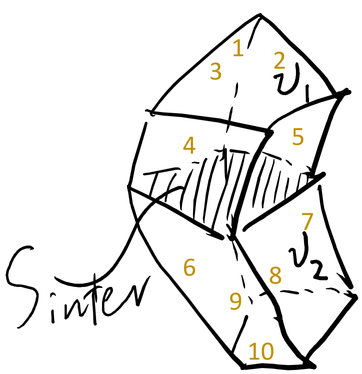
The flux through the surface is $$\frac{q-Q}{\epsilon_0}$$

Is this right?

Logical proof of Gauss's law

Method 1. Use the superposition of electric field

Method 2. Sum of the Gauss surface



$$\oint_{\partial V_1} = \int_{s1} + \int_{s2} + \int_{s3} + \int_{s4} + \int_{s5} - \int_{s\ \ inter}$$

$$\oint_{\partial V_2} = \int_{s6} + \int_{s7} + \int_{s8} + \int_{s9} + \int_{s10} + \int_{s\ \ inter}$$

$$\oint_{\partial V_{12}} = \sum_{i=1}^{10} \int_{s_i} = \oint_{\partial V_1} + \oint_{\partial V_2}$$

The notation is not strict, just to indicate the enclosed integral is the sum of all the surfaces. The key is that the **flux on the interface is canceled** for both direct integration on the total surface, and the sum of two sub surfaces, due to the definition of positive direction. **The enclosed surface integral is summable, in the same manner as volume.**

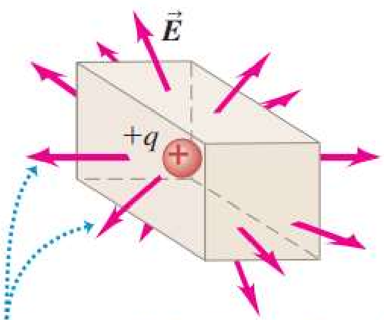
$$V_{12} = V_1 + V_2$$

Examples for Gauss's law

Flux vs. charge

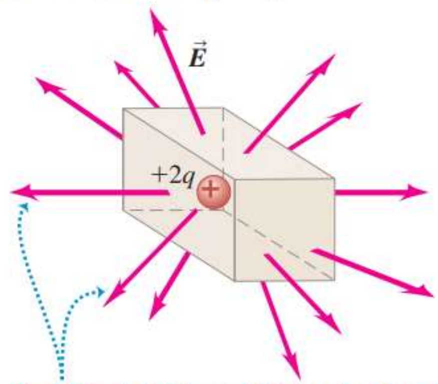
22.4 Three boxes, each of which encloses a positive point charge.

(a) A box containing a positive point charge $+q$



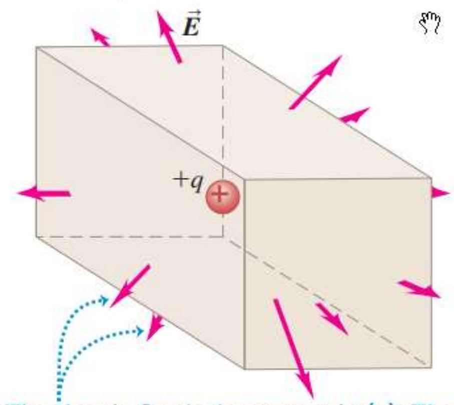
There is outward electric flux through the surface.

(b) The same box as in (a), but containing a positive point charge $+2q$



Doubling the enclosed charge also doubles the magnitude of the electric field on the surface, so the electric flux through the surface is twice as great as in (a).

(c) The same positive point charge $+q$, but enclosed by a box with twice the dimensions

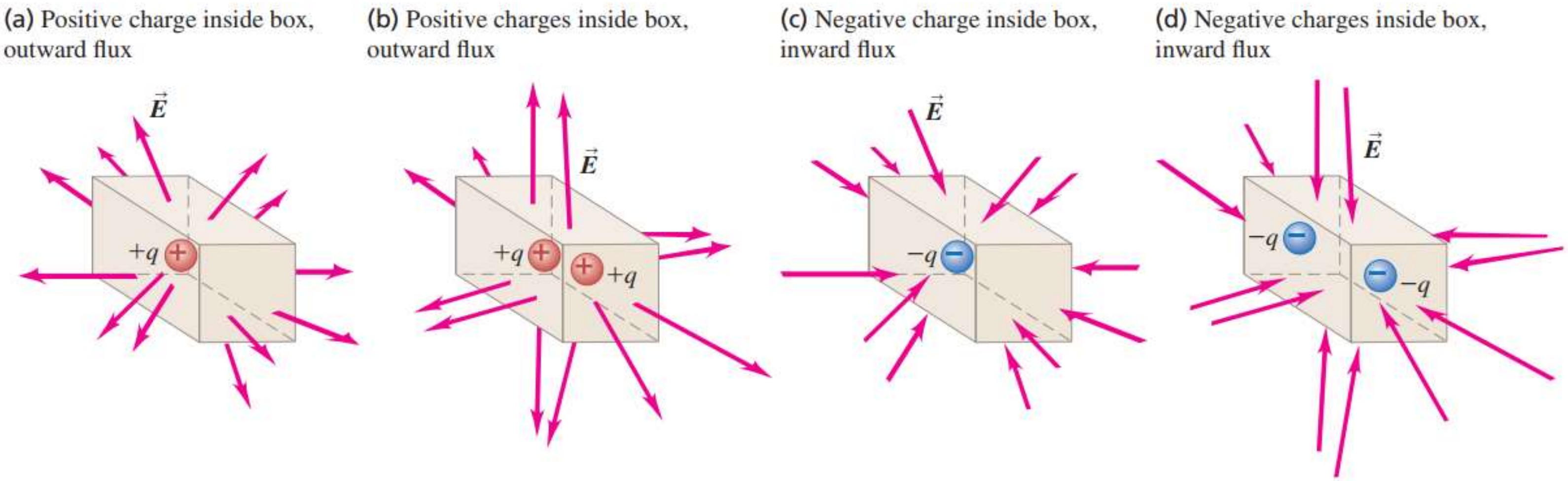


The electric flux is the same as in (a): The magnitude of the electric field on the surface is $\frac{1}{4}$ as great as in (a), but the area through which the field "flows" is 4 times greater.

Examples for Gauss's law

Charge distribution

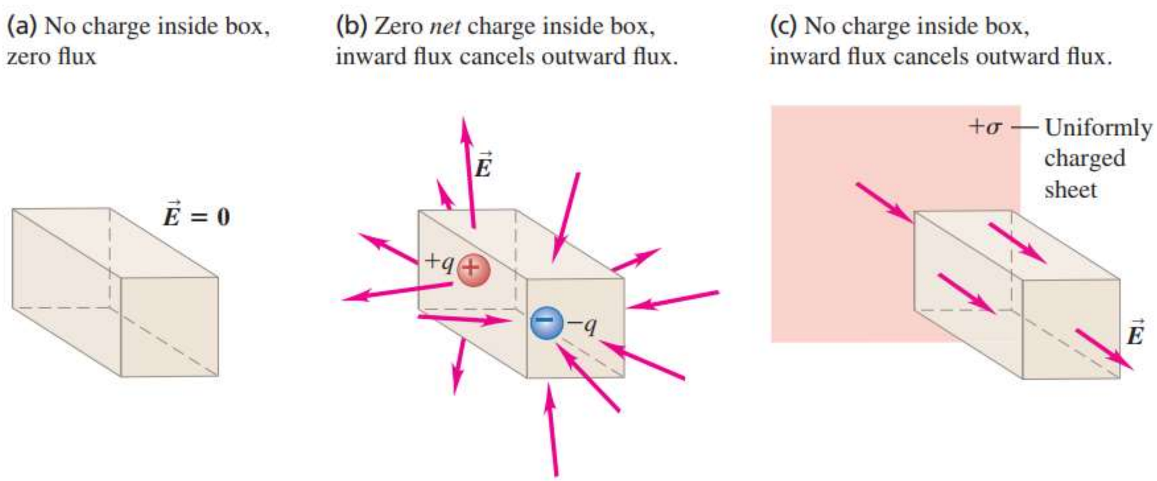
22.2 The electric field on the surface of boxes containing **(a)** a single positive point charge, **(b)** two positive point charges, **(c)** a single negative point charge, or **(d)** two negative point charges.



Examples for Gauss's law

Cases of zero flux

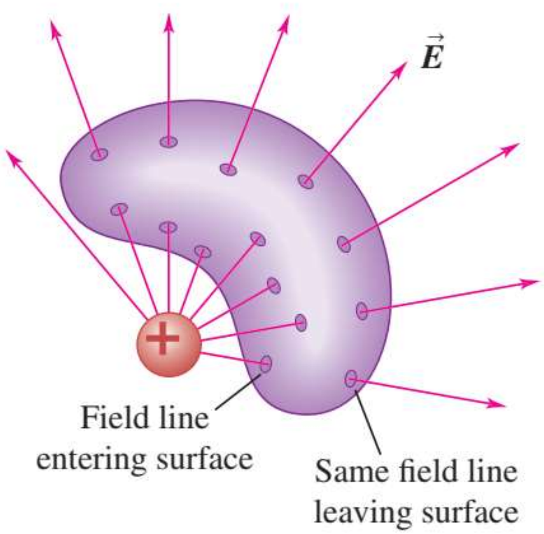
22.3 Three cases in which there is zero *net* charge inside a box and no net electric flux through the surface of the box. (a) An empty box with $\vec{E} = \mathbf{0}$. (b) A box containing one positive and one equal-magnitude negative point charge. (c) An empty box immersed in a uniform electric field.

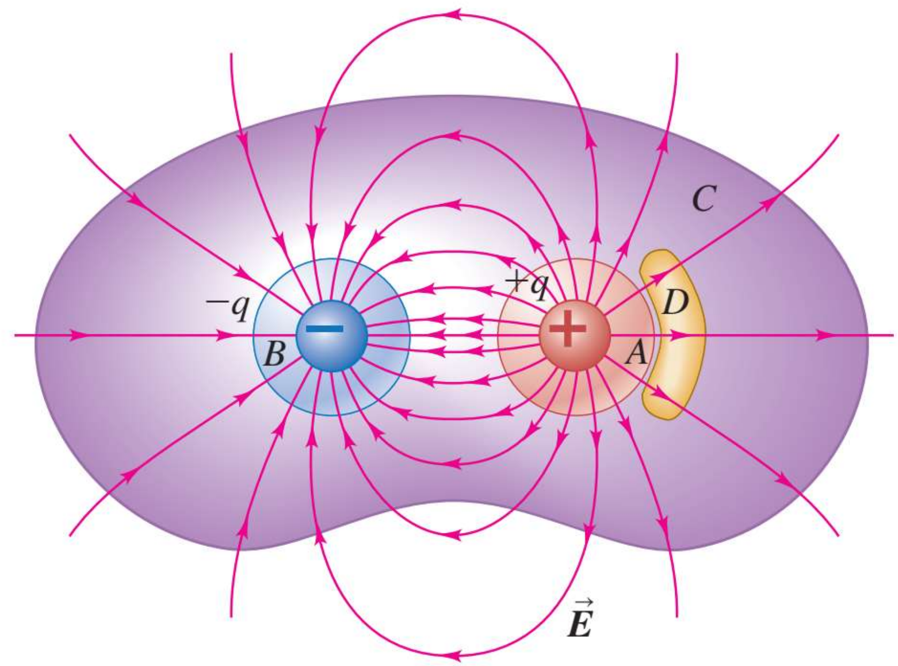


Examples for Gauss's law

With more precise field lines

22.13 A point charge *outside* a closed surface that encloses no charge. If an electric field line from the external charge enters the surface at one point, it must leave at another.





Application of Gauss's law

1. Faraday's icepail and examination of Coulomb's law

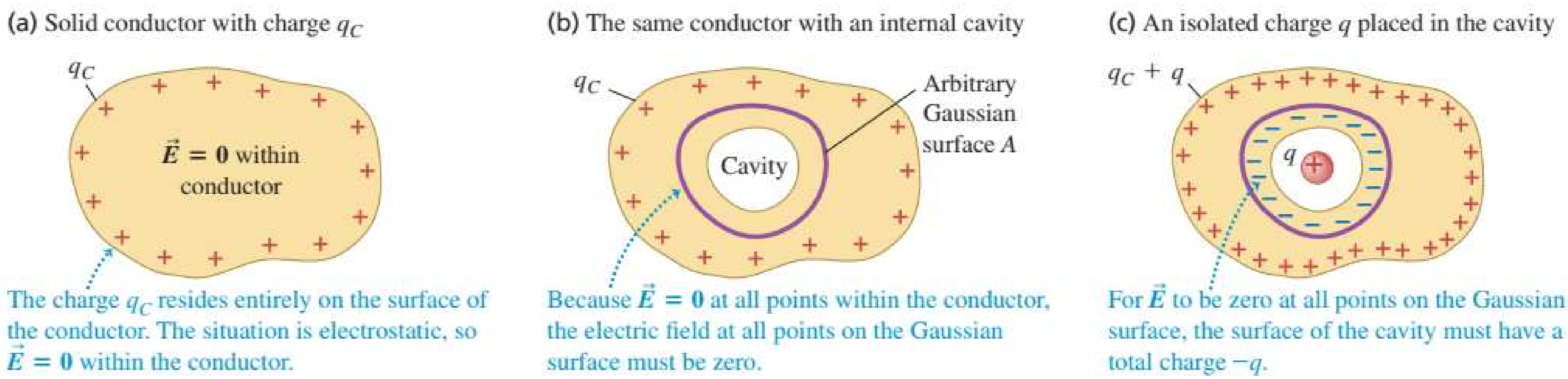
Gauss' law applied to a metal cavity

Metal in electric field

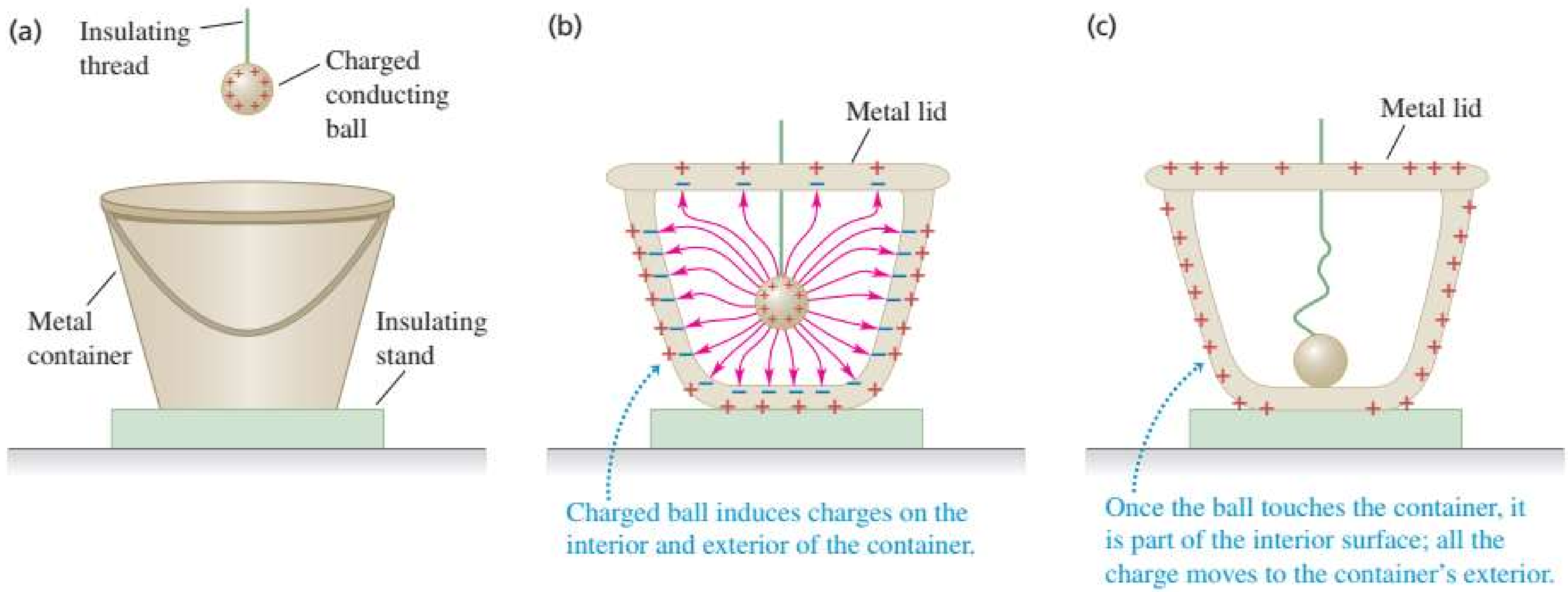
For metal which satisfies two conditions
1. Static (no flow of charge)
2. Homogeneous metal (the same composition, the same temperature)

In the bulk: No electric field; No net charge;
On the surface: field and charge are allowed but no **tangential** field.

Otherwise?	Ohm's law
Otherwise?	Thermal voltage
Otherwise?	Unbalanced force, driving electrons to move.



Faraday's icepail experiment



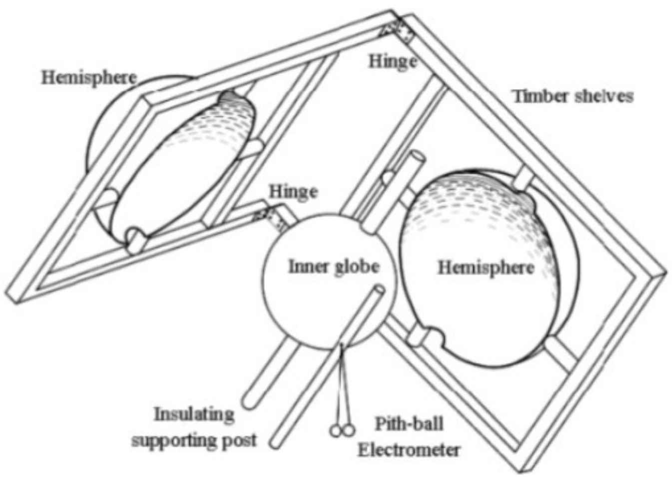
The more you can trust Gauss's law, the less residual charge you will find on the ball.

Cavendish's work

Faraday's icepail experiment is one of the "zero residual" type of experiment to prove Coulomb's law. **Cavendish firstly used a similar experiment for the verification of Coulomb's law.**

How early?

Earlier than Coulomb's law ☺



If the outer sphere produced interior electric fields, then charge would naturally migrate to the inner sphere. However, the electrometer failed to show any significant charge on the inner sphere, confirming the hypothesis that electric conductors cannot produce interior electric fields.

Henry Cavendish
1731-1810

The inverse square law is fundamental to physics

$$F \propto r^{-2\pm\delta}$$

If delta is not 0, what happens?

Coulomb's law → Gauss's law → Maxwell's equations → Structure of time and space

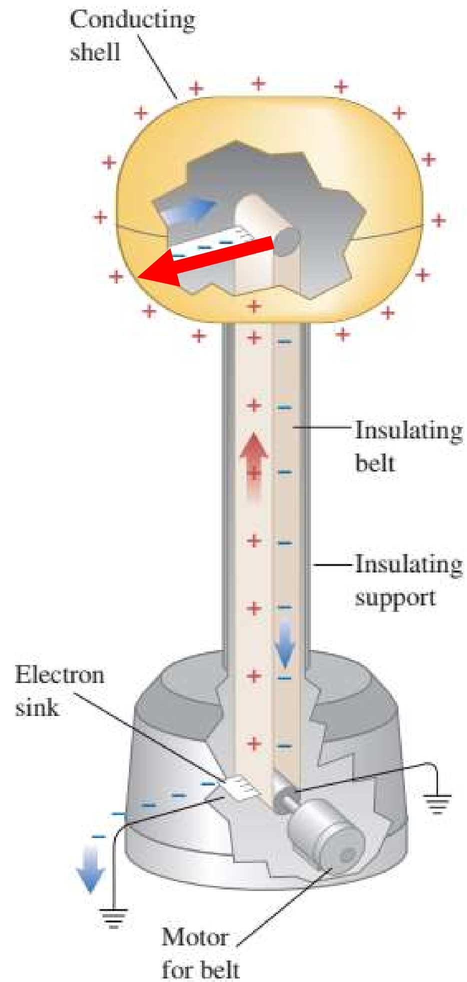
1772	Cavandish	2×10^{-2}
1879	Maxwell	5×10^{-5}
1936	Plimpton & Lawton	2×10^{-9}
1968	Cochran & Franken	9.2×10^{-12}
1970	Bartlett 等	1.3×10^{-13}
1971	Williams 等	$(2.7\pm3.1)\times10^{-16}$

Page51, 赵凯华、陈熙谋,《新概念物理教程——电磁学》(2nd edition), 高等教育出版社, 2006.12

2. Van de Graaff electrostatic generator

Although the voltage of the cavity is high, positive charge still moves from the belt to the surface (actually what moves is the electrons). A high voltage is built up quickly.

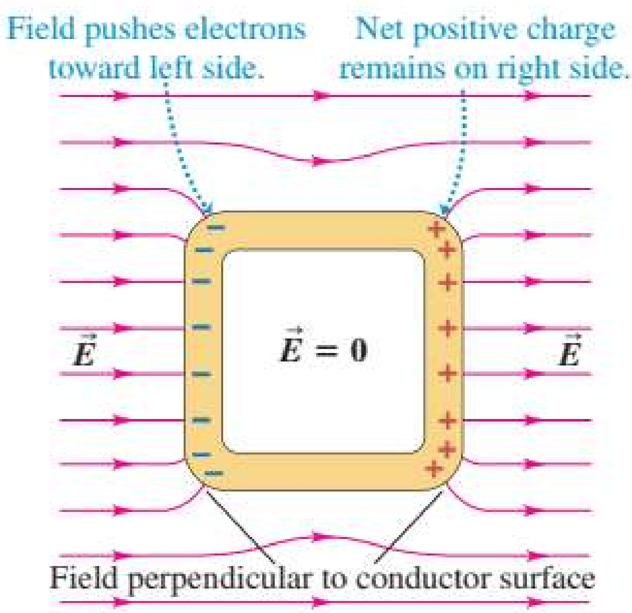
A low voltage source to charge the belt with positive charge.



A Faraday cage: to screen electric field

22.27 (a) A conducting box (a Faraday cage) immersed in a uniform electric field. The field of the induced charges on the box combines with the uniform field to give zero total field inside the box. (b) This person is inside a Faraday cage, and so is protected from the powerful electric discharge.

(a)



(b)

TEST YOUR UNDERSTANDING OF SECTION 22.5 A hollow conducting sphere has no net charge. There is a positive point charge q at the center of the spherical cavity within the sphere. You connect a conducting wire from the outside of the sphere to ground. Will you measure an electric field outside the sphere?

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Distribution of charge

How do you calculate E for any distribution of charge?

$$\vec{E} = \sum_i \vec{E}_i \qquad \vec{E} = \iiint d\vec{E}, \quad d\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r^2}\hat{r}$$

Discrete → Continuous, every dq is a small portion of charge.

Depending on the actual distribution the integral could be 1D, 2D, or 3D.

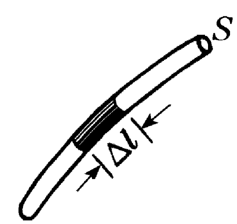
$$\mathrm{d}q = \rho_e \mathrm{d}V$$

ρ_e is the (volumetric) charge density

In some other cases, sheet charge density σ_e

or linear charge density λ_e

$$\mathrm{d}q = \lambda_e \mathrm{d}l$$

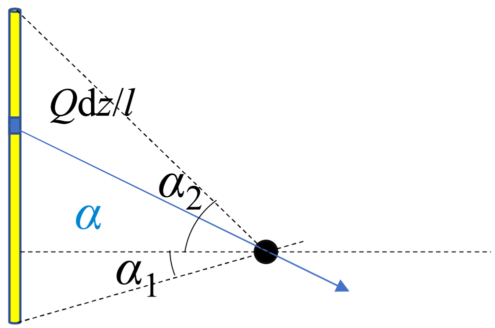


$$\mathrm{d}q = \sigma_e \mathrm{d}S$$

Apparently, ρ_e σ_e λ_e have different units.

Field of a charged rod

Homogeneously charge rod (along *z*, finite length). Calculate the field at point (*x*, 0, 0).



Trigonometry

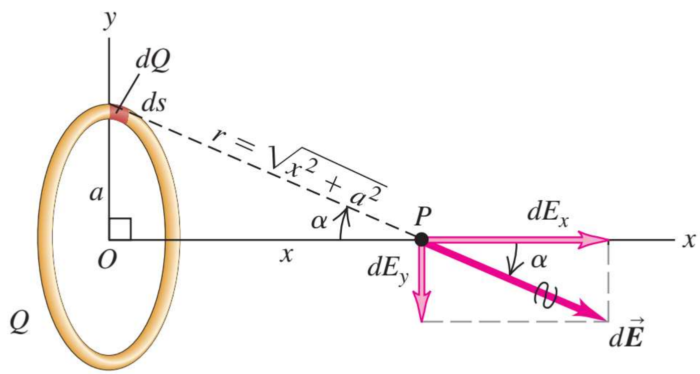
$$z = x\tan\alpha,\ \ \mathrm{d}z = x\frac{\mathrm{d}\alpha}{\cos^2\alpha}$$

$$E_x = \frac{1}{4\pi\epsilon_0}\frac{Q}{xL}(\sin\alpha_2 - \sin\alpha_1)$$

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{Q}{xL}(\cos\alpha_1 - \cos\alpha_2)$$

The field of a charged ring

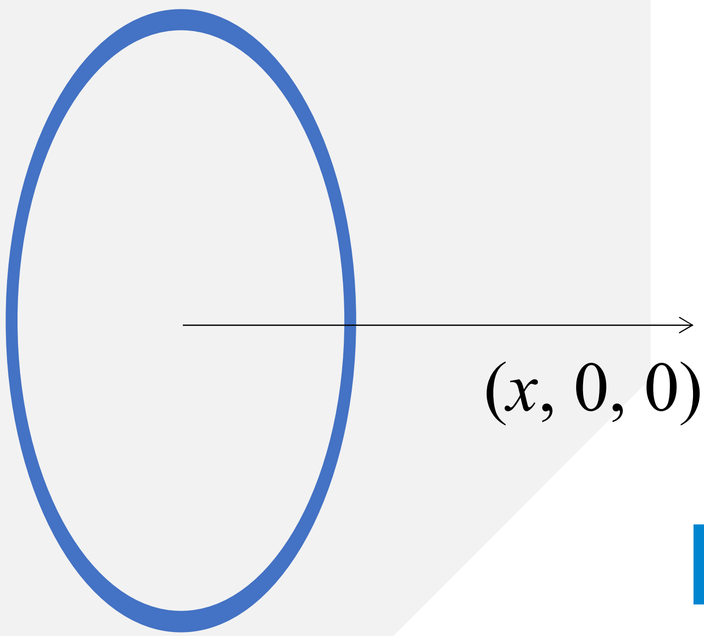
Ring on the *yz* plane, calculate the field at (*x*, 0, 0)



$$\vec{E} = \hat{x}\frac{1}{4\pi\epsilon_0}\frac{Qx}{(x^2+a^2)^{3/2}}$$

The field of a charged plane

Infinite plane, σ_e



How to write the (differential) charge?

$$\mathrm{d}q = \sigma_e 2\pi r \mathrm{d}r$$

$$\vec{E} = \int_0^\infty \frac{1}{4\pi\epsilon_0} \frac{2\pi r \mathrm{d}r \sigma_e x}{(x^2+r^2)^{3/2}}$$

Again, trigonometry

$$r = x\tan\alpha, \ \mathrm{d}r = x\frac{\mathrm{d}\alpha}{\cos^2\alpha}$$

$$\vec{E} = \hat{x}\frac{\sigma_e}{2\epsilon_0}$$

Application of Gauss' law to determine electric field

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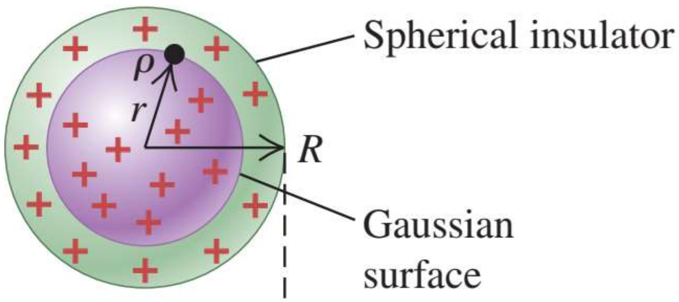
Application of Gauss' law: 1. field of homogeneously charged sphere



$$\vec{E} = \frac{Qr}{4\pi\epsilon_0 R^3}\hat{r}, r < R$$

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}, r \geq R$$

Using the Gaussian integration surface.

1. closed surface;
2. highly symmetric;

Application of Gauss' law: 2. field of homogeneously charged **LONG** cylinder

Cylindrical coordinate system (ρ, θ, z)

$$\vec{E} = \frac{Q}{4\pi\epsilon_0 \rho L}\hat{\rho}$$

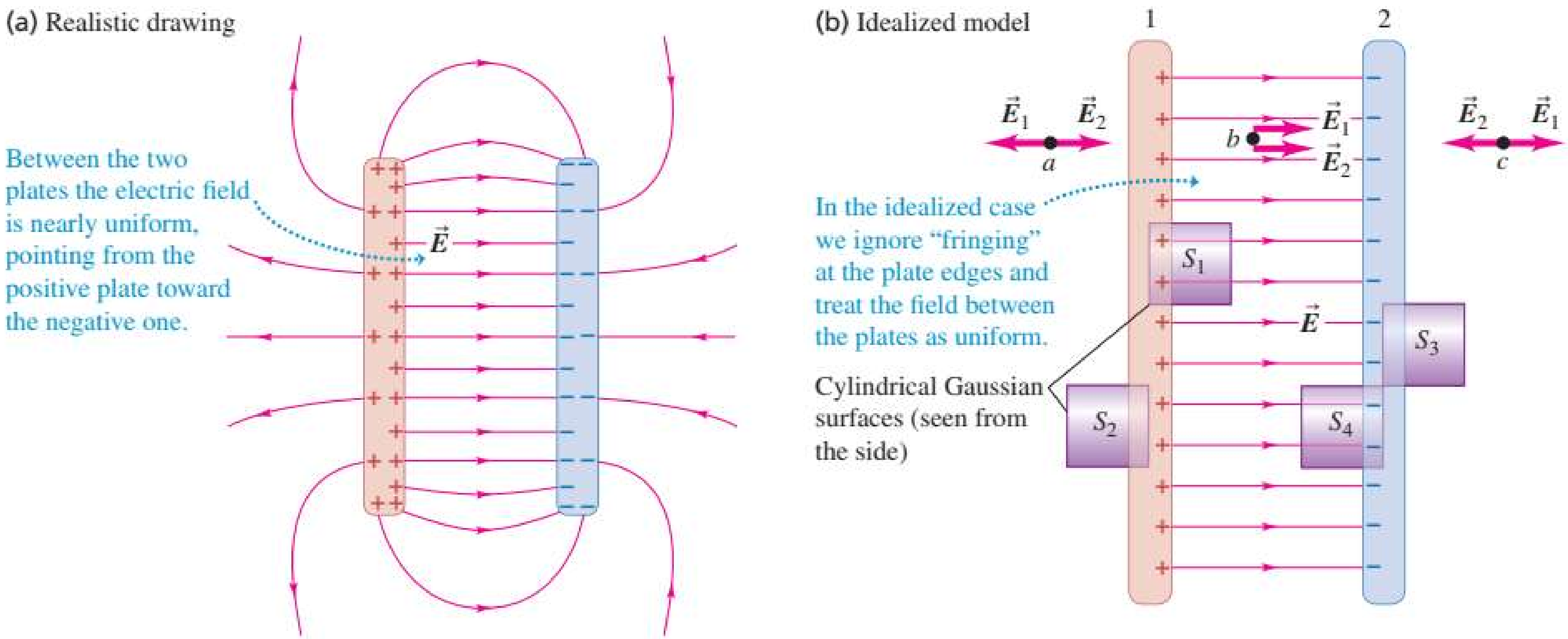
How long should it be to use the Gauss' law?

1. much longer than the distance between the rod and the field point

2. so long that you can forget about the end effects.
(no axial component of field, distance to both ends much longer than distance between the rod and the field point).

Application of Gauss' law: 3. field of infinitely large plate

22.21 Electric field between oppositely charged parallel plates.



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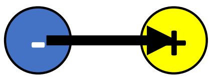
Electric dipole: book page 701, 707-710



A dipole consists of two opposite charges, of the same magnitude.

$$\vec{P} = q\vec{l}$$

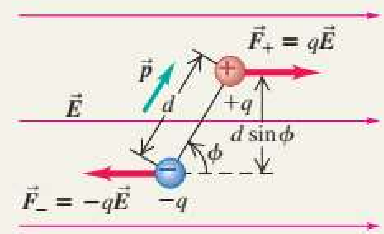
dipole moment: proportional to the charge and the distance between them (vector, from negative to positive).

Electric dipoles: An electric dipole is a pair of electric charges of equal magnitude q but opposite sign, separated by a distance d. The electric dipole moment $\vec{p}$ has magnitude $p = qd$. The direction of $\vec{p}$ is from negative toward positive charge. An electric dipole in an electric field $\vec{E}$ experiences a torque $\vec{\tau}$ equal to the vector product of $\vec{p}$ and $\vec{E}$. The magnitude of the torque depends on the angle ϕ between $\vec{p}$ and $\vec{E}$. The potential energy U for an electric dipole in an electric field also depends on the relative orientation of $\vec{p}$ and $\vec{E}$. (See Examples 21.13 and 21.14.)

$$\tau = pE \sin\phi \qquad (21.15)$$

$$\vec{\tau} = \vec{p} \times \vec{E} \qquad (21.16)$$

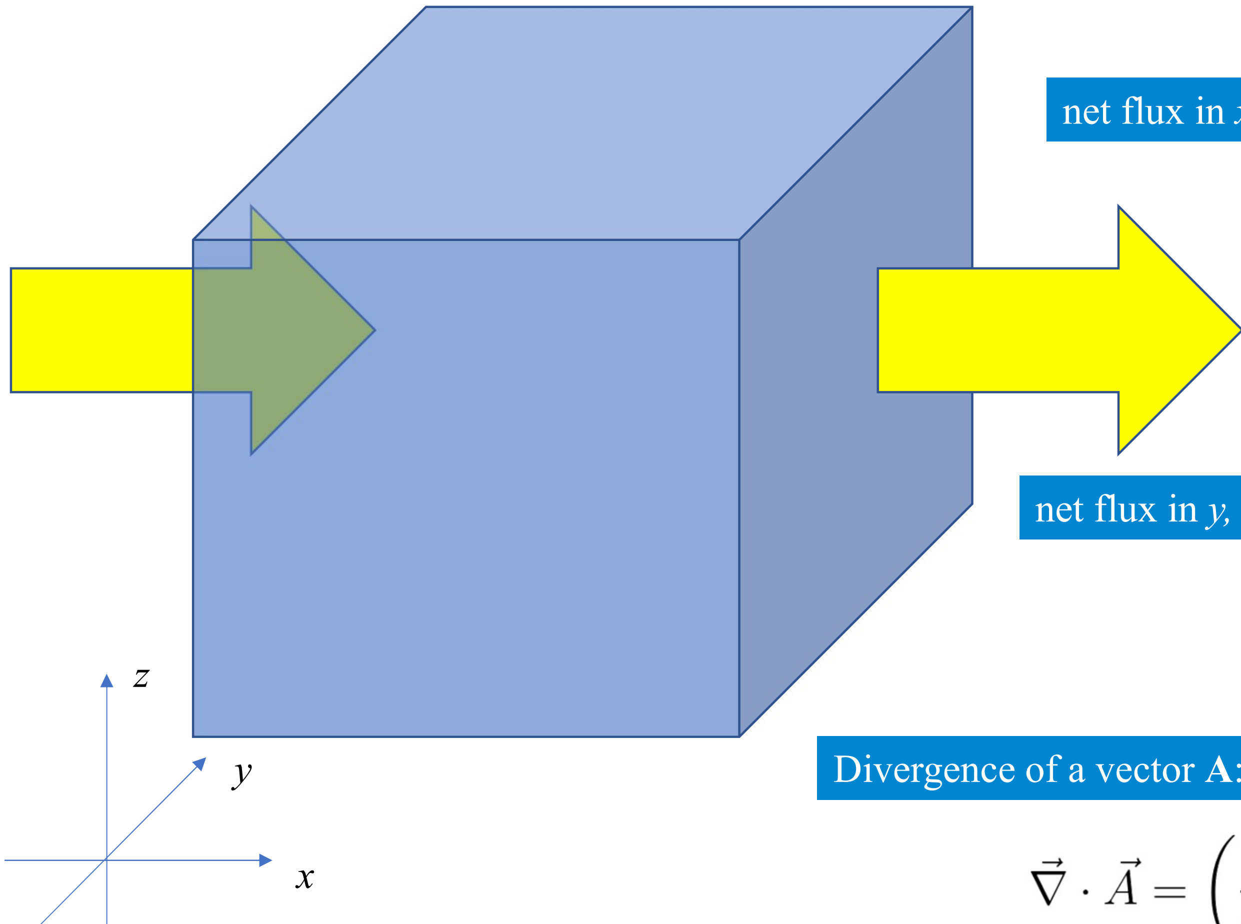
$$U = -\vec{p} \cdot \vec{E} \qquad (21.18)$$



Read about electric dipole:
The field of an electric dipole.
The force, torque and energy of dipole in field.

Contents

The Gaussian flux of vector **A** of a small volume



net flux in *x* direction (outbound):

$$[A_x(x+\mathrm{d}x) - A_x(x)]\mathrm{d}y\mathrm{d}z$$
$$= \frac{\partial A_x}{\partial x}\mathrm{d}x\mathrm{d}y\mathrm{d}z = \frac{\partial A_x}{\partial x}\mathrm{d}V$$

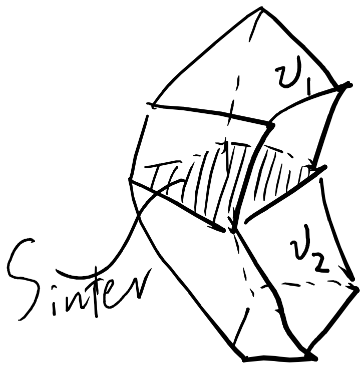
net flux in *y, z* directions (outbound):

$$\frac{\partial A_y}{\partial y}\mathrm{d}V, \quad \frac{\partial A_z}{\partial z}\mathrm{d}V$$

Divergence of a vector **A**:

$$\vec{\nabla}\cdot\vec{A} = \left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)\cdot(A_x\hat{x} + A_y\hat{y} + A_z\hat{z})$$
$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Divergence theorem



enclosed outbound flux itself is a quantity that can sum up like volume.

Gaussian flux: (outbound) flux on an enclosed surface

Mathematically, the divergence (something measured in "volume") of **A** is the density of Gaussian flux of **A**.

analogy

$$V_1 + V_2 = V_{12}$$

you can sum volume

$$\rho_m V_1 + \rho_m V_2 = \rho_m V_{12}$$

you can sum mass, which is density times volume

$$(\nabla \cdot \vec{A})V_1 + (\nabla \cdot \vec{A})V_2 = (\nabla \cdot \vec{A})V_{12}$$

If not homogeneous, summation changes into integral.

you can sum Gaussian flux, which is divergence times volume

∇ operator in different coordinate systems

Cartesian

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

Cylindrical

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{\partial}{\partial z}\hat{z}$$

Spherical

$$\nabla = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}$$

Conservation of charge

Test your understanding of flux and divergence

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

How to write Gauss's law in differential form

Test your understanding of flux and divergence

$$\Phi_E = \oint \vec{\boldsymbol{E}} \cdot d\vec{\boldsymbol{A}} = \frac{q}{\epsilon_0}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

summary

Field of a point charge

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

Flux

$$\Phi = \iint_S \vec{E} \cdot \mathrm{d}\vec{A}$$

Gaussian flux of a small volume =divergence * volume.

$$\nabla \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Gauss's law: integral and differential form

$$\oiint \vec{E} \cdot \mathrm{d}\vec{S} = \frac{Q}{\epsilon_0} \qquad \nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

Using Gauss's law to calculate field

to find a enclosed, highly symmetric surface