Homework 1 (20 Points)

Due: 2023/09/24

Submission Guidlines

1. You need to write your Question 2 and 3 in Rmd. (To help you get started, please go through the "Note for Rmarkdown" document)
2. For Question 1, if you write it in Rmd, you will get 10% bonus.
3. Please submit two files: one *Rmd and one generated *pdf file. If you hand write question 1, please merge the pdf file before submission.

Question 1: Linear Algebra Review

Let $f : \mathbf{R}^n \to \mathbf{R}$ be a multivariate function. Recall its gradient

is the n-vector of partial derivatives

$$\nabla_x f(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} f(x) \\ \vdots \\ \frac{\partial}{\partial x_n} f(x) \end{bmatrix},$$

and its Hessian is the symmetric $n \times n$ matrix

$$\nabla^2 f(x) = f''(x) = \begin{bmatrix} \frac{\partial}{\partial x_1^2} f(x) & \dots & \frac{\partial}{\partial x_1 \partial x_n} f(x) \\ \vdots & & \vdots \\ \frac{\partial}{\partial x_n \partial x_1} f(x) & \dots & \frac{\partial}{\partial x_n^2} f(x) \end{bmatrix}.$$

a. Let $f(x) = \frac{1}{2} x^T A x + b^T x$, where $A \in \mathbf{R}^{n \times n}$ is symmetric and $b \in \mathbf{R}^n$. What are $\nabla_x f(x)$ and $\nabla^2 f(x)$?
b. Let $f(x) = g(h(x))$, where $g : \mathbf{R} \to \mathbf{R}$ and $h : \mathbf{R}^n \to \mathbf{R}$ are both differentiable. What is $\nabla_x f(x)$?
c. Let $f(x) = g(a^T x)$, where $a \in \mathbf{R}^n$. What are $\nabla_x f(x)$ and $\nabla^2 f(x)$.

Qestion 2: 90th Percentile

Assessing estimates of the 90th percentile of 100 iid uniform(0,1) random variables.

The R function quantile() implements a somewhat complex interpolation method in order to estimate a particular quantile, such as the 90th percentile. We will compare the estimate in quantile() with simpler estimates.

a. Write a function that takes as input a vector of length 100 and outputs the 90th of the 100 values sorted from smallest to largest. Note that the input vector might not be sorted.
b. Write a function to find the 91st of the sorted vector of 100 values.
c. Write a function that outputs the average of the 90th and 91st of the sorted vector of 100 values.
d. For each of your functions in parts a-c, as well as the function **quantile(x,0.9)**, do the following:
    i. Generate 100 iid uniform (0,1) random variables, and calculate your estimate of the 90th percentile.
    ii. Repeat step (i) 10,000 times.

iii. Plot the sample mean of the first m of your estimates, as a function of m.

e. Report the ultimate sample mean of your 10,000 estimates, for each of the four estimates. In 1-2 sentences, indicate which of the 4 estimates appears to be the best, and why. (Note: this is open-ended)

Question 3: Estimating π

a. Write a function called **pi2(n)** that approximates π as a function of n, using the approximation

$$\pi = lim_{n\to\infty}\sqrt{[6\sum_{k=1}^{n}k^{-2}]}$$

. Evaluate $pi2(10^j)$ for $j = 0, 1, 2, \cdots, 6$

b. Write a function **pi3(n)** that approximates π as a function of n, by simulating random points in the square with vertices (-1,-1), (-1,1), (1,1), and (1,-1), seeing what fraction of them are in the unit circle (the circle with radius 1 centered at the origin), and then converting this fraction into an estimate of π. Evaluate $pi3(10^j)$ for $j = 0, 1, 2, \cdots, 6$

c. For j=6, plot your simulated points, using different plotting symbols for simulated points inside and outside the unit circle.