1 Introduction

Graph theory is, in a fundamental sense, a visual aid to analyzing relations between objects. In broader terms, however, it is a mathematical tool utilized in numerous fields of study, including computer science, linguistics, biology, physics, and sociology. The fundamental notions of the theory can be explained to a student at an elementary level, but the intricacies of graphs are still being explored by mathematicians at the cutting edge of the field. The gentle learning curve (due to the intuitive nature of the subject) in conjunction with the countless present-day applications make graph theory a worthy candidate for any novice reader. While other fields in mathematics are restricted by formidable notation and idealistic assumptions (in regards to its practical application), graph theory contains visceral notation and is intrinsically conducive to physical examples. Because of this, we find the subject has seemingly unconventional uses– the last section in this paper providing an appropriate example. Without a basic understanding of graph theory, however, this application makes little sense. Hence comes the motivation of this paper. In the following sections, I will introduce necessary definitions and theorems of graphs for not only the application, but for a basic understanding of the potential of this field of mathematics.

2 Defining a Graph

Graphs are mathematical representations of pairwise relationships between objects. To refer to these graphs in a less abstract fashion, we must first standardize our notation and construction of graphs:

Definition 2.1. [4] Let V be a set and E a set of unordered pairs of elements of V. Then the pair $G = (V, E)$ is called a **graph** on the set V. The elements of the set V are called the **vertices of** G. If v_1 and v_2 are vertices of G, and $\{v_1, v_2\} \in E$, there is an **edge** in G joining v_1 and v_2. The elements of the set E are called the **edges of** G.

As we begin our discussion on the basic definitions of graph theory, it is important to note that I have made two simplifying assumptions. First, all graphs that are defined in this paper are finite. Second, for the graphs contained in the following two sections, every edge joins a distinct pair of vertices. Thus, a vertex cannot be connected to itself, and there can only be one edge connecting a pair of vertices. Graphs that satisfy these conditions are oftentimes referred to as *simple graphs*.

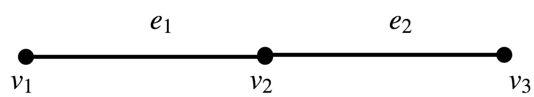


Figure 1: A Graph $G = (V, E)$ where $V = \{v_1, v_2, v_2\}$ and $E = \{\{v_1, v_2\}, \{v_2, v_3\}\}$.

Since the connections (or edges) in graphs from vertex to vertex are the representatives of object associations, is is necessary to define some terminology that quantifies the vertex and edge relations. The terms *incident* and *adjacent* will reappear throughout the paper, so we must first give a formal definition to each.

Definition 2.2. [4] Let u and v be vertices in G. If the pair $\{u, v\}$ is in E, then u and v are said to be **adjacent vertices**. The edge $\{u, v\}$ is said to be **incident** with the vertices u and v.

Now we are ready to quantify vertex connections:

Definition 2.3. [4] The **degree** of a vertex v is the number of edges incident with v. The degree of v is denoted by $\deg(v)$.

Figure 2 shows a graph G with two edges. The sum of the degrees of the vertices of G is four, in this case. In fact, there is an explicit relationship between the sum of the degrees and the number of edges:

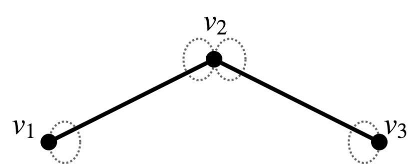
Theorem 2.4. Let $G = (V, E)$ be a graph. Let n be the number of edges of G. Then

$$\sum_{v \in V} \deg(v) = 2n.$$

Proof.

We will proceed by induction on the number of edges n. Consider the case where $n = 1$. Since there is only one edge $e_1 = \{v_1, v_2\}$, the two vertices v_1, v_2 must have a degree of one. thus $\sum_{v \in V} \deg(v) = \deg(v_1) + \deg(v_2) = 1 + 1 = 2 = 2n$. Now assume that for any graph $G = (V, E)$ having k edges, $\sum_{v \in V} \deg(v) = 2k$. Consider a graph G^* with $k + 1$ edges $e_1, e_2, ..., e_k, e_{k+1}$. The edge e_{k+1} is incident to two vertices $\{v_n, v_m\}$. Removing the edge e_{k+1}

from G produces a subgraph $G' = (V, E \setminus e_{k+1})$ having k edges. By the induction hypothesis, the sum of the degrees of the vertices in G' is $2k$. Since including the edge e_{k+1} increases the degree of v_n and v_m by 1, we know that for a graph with $k+1$ edges, $\sum_{v \in V} \deg(v) = 2k + 2 = 2(k+1)$. ∎



$$n = 2, \sum_{v \in V} \deg(v) = 4 = 2n$$

Figure 2: A Trivial Example of Theorem 2.4

We have the necessary definitions to construct a graph, but in order to analyze these graphs we must be able to navigate through their edges and vertices. Think of the vertices as "islands" and the edges as "bridges". If we want to determine the connectivity of two islands, the easiest way to go about this is to attempt to walk from one to the other via bridges and islands (edges and vertices).

Definition 2.5. [4] Let $G = (V, E)$ be a graph. Let u and v be vertices in G. A **walk** in G from u to v is an alternating list of vertices and edges in which each edge is incident with the vertices that come before and after it:

$$u, \{u, v_1\}, v_1, \{v_1, v_2\}, v_2, \{v_2, v_3\}, v_3, ..., v_{n-1}, \{v_{n-1}, v\}, v.$$

We will denote such a walk by

$$u \to v_1 \to v_2 \to ... \to v_{n-1} \to v$$

or simply $W = (V_W, E_W)$, where the set V_W consists of the vertices contained in W and the set E_W consists of the edges contained in W. The **length** of the walk is the number of edges in the walk.

Walks are oftentimes circuitous (hence the phrase "walking in circles") but a path is typically the shortest connection between two destinations. Similarly, a path in graph theory is a direct walk:

Definition 2.6. A **path** $P = (V_P, E_P)$ from a vertex v is a walk in which no vertex appears more than once. The set V_P consists of the vertices contained in P and the set E_P consists of the edges contained in P.

By definition, every path is a walk, but not all walks are paths. However, the next theorem proves the existence of a walk between two vertices ensures the existence of a path between them.

Theorem 2.7. Let $G = (V, E)$ be a graph. Let u and v be vertices of G Then there is a walk in G from u to v if and only if there is a path in G from u to v.

Proof.
($\Longrightarrow$) We will proceed by induction on the length n of the walk. Let there be a walk $u \to v$ of length $n = 1$. No vertex appears more than once in the walk, so $u \to v$ is a path. Assume for any walk $u \to v$ of length $n = k$ for some $k \in \mathbb{N}$, there is a path from $u \to v$. Consider a walk $u \to w$ of length $k+1$: $u \to w_1 \to w_2 \to ... \to w_k \to w$. By our base case, we know that a walk of length $n = 1$ is a path, so $w_k \to w$ is a path. We know by induction hypothesis that there exists a path $u \to w_k$. Thus, $u \to w_k \to w = u \to w$ is a path.

($\Longleftarrow$) Let there be a path from $u \to v$. The path is a walk by definition.

∎

Many physical paths start and end at the same place. Hiking trails, for example, typically contain a loop. In graph theory, we call such a path a *cycle*:

Definition 2.8. A **cycle** or **closed path** $C = (V_C, E_C)$ in a graph $G = (V, E)$ is a walk that satisfies the following properties:

1. it starts and ends at the same vertex v.
2. no vertex other than v appears more than once.
3. no edge is used more than once.

The set V_C consists of the vertices contained in C and the set E_C consists of the edges contained in C.

We call a graph without any cycles *acyclic*. It is very important in analysis to know whether a graph is *acyclic* or not, since cycles allow for alternate paths from between two points. Walks, paths, and cycles all play key roles in defining graphs with specified properties.

3 Connectedness in Graphs

Referring back to our island-bridge analogy, it would be important to note which islands are connected by at least one bridge, and which islands are isolated (assumedly, it would not be favorable to end up on an unconnected one!). One way to check if all islands are connected would be to attempt to create a path between any two given islands and see if it exists. If a path exists for all possible combinations of island pairs, then the islands are all connected. In graph theory, we are typically interested in whether all the vertices are connected in a given graph.

Definition 3.1. [4] Let $G = (V, E)$ be a graph. A graph $G = (V, E)$ is said to be **connected** if for every pair of vertices u and v there is a path in G from u to v.

Trying to find all paths between any two given islands might start to become tedious, especially in an archipelago of islands similar to that of the Seychelles. It is therefore advantageous to use Theorem 3.2 as an alternate method in proving connectedness.

Theorem 3.2. Let $G = (V, E)$ be a graph. Then G is connected if and only if there is a walk that goes through all the vertices.

Proof.
($\Longrightarrow$) Let $G = (V, E)$ be connected. We will proceed by induction on the number of vertices n in G. Let there be $n = 2$ vertices in G. Define the two vertices to be v_1, v_2. The walk $v_1 \to v_2$ goes through all the vertices in G. Now assume for a graph G with $n = k$ vertices, there exists a walk that goes through the vertices $v_1, v_2, ..., v_k$. Let G contain $k + 1$ vertices. Since G is connected, we know there exists a path v_k to v_{k+1}. So by Theorem 2.7, there is a walk from $v_k \to v_{k+1}$. We know by induction hypothesis that

there exists a walk that goes through the vertices $v_1, v_2, ..., v_k$. Thus, there exists a walk in G that goes through vertices $v_1, v_2, ..., v_k, v_{k+1}$. Hence there exists a walk through all vertices for any connected graph.

($\Longleftarrow$) Let there be a walk that goes through all the vertices. Thus, for every pair of vertices u and v, there is a walk from u to v. Hence there is a path from u to v by Theorem 2.7. Therefore G is connected.

∎

A *tree* is a special type of connected graph. Figure 3 should give some intuition for why we call it thus. In fact, mathematicians draw further references to the analogy by referring to the vertex v_1 in Figure 3 as the *root* and the vertices v_4, v_5, v_6, v_7 as *leaves* (it should be obvious as to why they call them such!). Descendant relationships are oftentimes diagrammed as tree graphs. Formally, a tree is a connected graph with one condition:

Definition 3.3. A **tree** is a connected graph with no cycles.

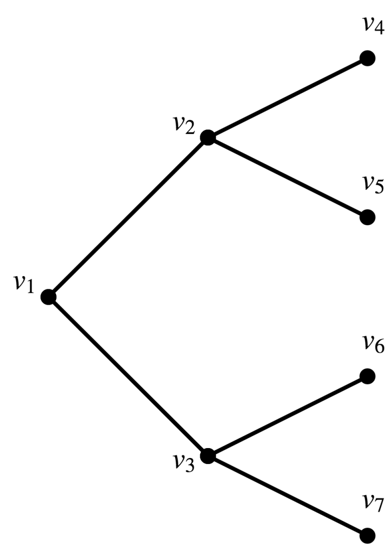


Figure 3: A Classic Example of a Tree.

The acyclic and connected nature of trees grant them very useful mathematical properties. One we will explore is the relationship between the number of vertices and edges. Notice how Figure 3 contains seven vertices and six edges. In fact, this sort of relationship between the number of vertices and the number of edges in a tree will also hold– the number of vertices is always one greater than the number of edges. To prove this, we must first detour to Lemma 3.4.

Lemma 3.4. Let $G = (V, E)$ be a tree. Any two vertices in G are connected by a unique path.

Proof.
The proof for this lemma is based on the one in [3]. We will proceed by contradiction. Let G be a tree. Pick two arbitrary vertices u and v in G. Assume there are two distinct paths P_1 and P_2 between two vertices u and v. Since the paths are distinct there exists an edge e in one that is not in the other. Without loss of generality, assume P_1 has edge $e = \{a, b\}$ that is not an edge in P_2. Thus there exists a path P that exhausts P_1 and P_2 without using the edge e. Therefore there exists a cycle C formed by P and e in G, so G is not a tree.

Theorem 3.5. Let $G = (V, E)$ be a graph with n vertices and e edges. If G is a tree, then $n = e + 1$.

Proof.
Let G be a tree with n vertices and e edges. Proceed by strong induction on n. Let $n = 1$. We cannot construct an edge in our graph with only one vertex. Thus, $e = 0$, and $n = 1 = e + 1$. Now assume that for all $k \in \mathbb{N}$ any tree having k vertices satisfies $k = e + 1$ provided that $k < n$ for some $n \in \mathbb{N}$. Let T be a tree with n vertices and e edges and let $e^* = \{u, v\}$ be an edge. By Lemma 3.4, the only path between u and v is e^*. Thus, $T - e^*$ consists of exactly two trees T_1 and T_2. Let n_1 and n_2 be the number of vertices in T_1 and T_2 and let e_1 and e_2 be the number of edges in T_1 and T_2, all respectively. Thus $n_1 + n_2 = n$ and $e_1 + e_2 = e - 1$. Since $n_1 < n$, $n_1 = e_1 + 1$ by induction hypothesis. Similarly, $n_2 = e_2 + 1$. Thus, $n = n_1 + n_2 = (e_1 + 1) + (e_2 + 1) = (e_1 + e_2) + 2 = e - 1 + 2 = e + 1$.

∎

4 Eulerian Graphs

While our discussion so far has been centered around simple graphs, it is important to note that there exist other types of graphs of higher complexity that describe relationships in ways that simple graphs cannot. These graphs are called *multigraphs.* Unlike a simple graph, a multigraph is allowed to contain multiple edges (or parallel edges) connecting one set of vertices. In other terms, for any two vertices $\{u, v\}$ there can exist edges $e_1, e_2..., e_k$ for some $k \in \mathbb{N}$ connecting $\{u, v\}$ in a multigraph G. There is a famous example (the quintessential example, perhaps) of graph theory that uses islands and bridges. The illustration will help us gain more insight into the application of multigraphs. See [1] and [3] for a more rigorous analysis of this example.

The Königsberg bridge problem was the first application of graph theory as it is known today. Leonhard Euler proved in a 1736 paper that it was impossible to cross the seven bridges of the city of Königsberg, Russia once and only once during a walk through town. He used a multigraph (as diagrammed in Figure 4) to represent the bridges, then used elementary graph theory to conclude that the route was in fact impossible. The specific walk that Euler was referring to is called a *closed Eulerian walk.* Recall that we introduced cycles previously in Definition 2.8, and that the terms cycle and closed walk are used interchangeably. A Eulerian walk is simply a walk that exhausts the edges in a graph:

Definition 4.1. Let $G = (V, E)$ be a multigraph. We call a walk W in G **Eulerian** if the walk uses all edges in G.

After a few moments of thought, it is clear that the Seven Bridges of Königsberg do not contain a closed Eulerian walk– it is impossible to completely trace the graph once over with no breaks, let alone while additionally starting and ending from the same place. Tracing is not a very "mathematical" procedure, however– we need a more formal way to approach this problem. Theorem 4.3 allows us to easily identify whether a multigraph contains a closed Eulerian walk.

Lemma 4.2. Let $G = (V, E)$ be a connected multigraph with a closed walk $C_1 \neq G$. Then there exists a vertex v such that C_1 does not contain all edges incident to v.

Proof.
This proof is based on the one in [3]. We will proceed by contradiction. Assume that for all vertices v, C_1 does not contain all edges incident to v. Since C_1 contains fewer edges than G, and C_1 contains all edges incident to all its vertices, there must be a vertex u that is not in C_1. Since G is connected, for any vertex v, there exists a path from u to v in C_1. Thus there exists an edge incident to v that is not in C_1. Thus, there is always a vertex v such that C_1 does not contain all edges incident to v.

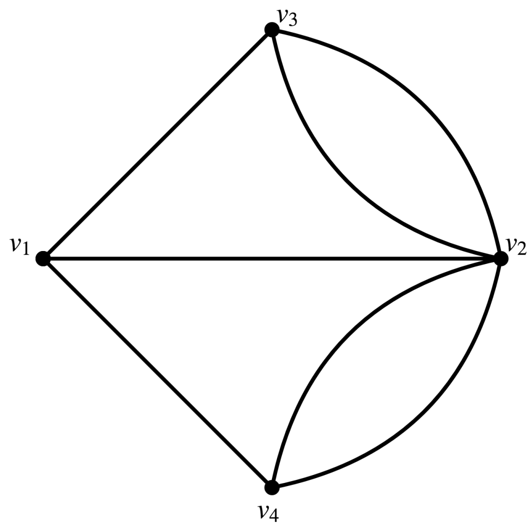


Figure 4: A Graph of the Seven Bridges of Königsberg.

∎

Theorem 4.3. A connected graph G has a closed Eulerian walk if and only if all vertices of G have even degree.

Proof.
This proof is based on the one in [3].

($\Longrightarrow$) Let G contain a closed Eulerian walk W. Since W is closed and Eulerian, we visit each vertex a certain number of times. Let v be a vertex that was not where W started, and let n be the number of times we visited v. For each visit, we had to enter and exit the vertex using different edges, so W visits v $2n$ times. Since W contains all edges of G, v cannot be incident to any additional edges, so $\deg(v) = 2n$. Thus, the degree of every vertex other than the start of W is even. Since W is closed, the starting vertex u of the walk must also be the ending vertex of the walk. If we visit u m additional times during the walk, the number of edges used for the vertex u would be $1 + 2m + 1 = 2(m + 1)$. Thus, the degree of u is even. Therefore, all vertices of G have an even degree.

($\Longleftarrow$) Assume all vertices of G have an even degree. Take any vertex v_1 in G. Begin a walk along an edge e_1 to v_2, then along another edge e_2... continue in this way (using unique edges) until a closed walk C_1 is formed. It is important to note that we cannot get "stuck" at some vertex before completing a closed walk– the degree of each vertex is even, so each time we enter a vertex, we can also leave it (with the exception of our starting vertex v_1). If $C_1 = G$, we are done. If not, then choose a vertex v_n in C_1 such that C_1 does not contain all edges incident to v_n. We know that such a vertex exists by Lemma 4.2. Now omit all edges of C_1 from G. Starting and ending at v_n, construct another closed walk C_2. We can then create a cumulative closed walk $C_1 \cup C_2$ by starting our walk at C_1, stopping at v_n, walking through C_2, then walking through the remaining portion of C_1. Refer to Figure 5 for a graphical explanation of the walk $C_1 \cup C_2$. If $C_1 \cup C_2 = G$, then we are done. If not then omit all edges of $C_1 \cup C_2$ from G, and construct another closed walk C_3. Continue this process until $C_1 \cup C_2 \cup ... \cup C_k = G$. ∎

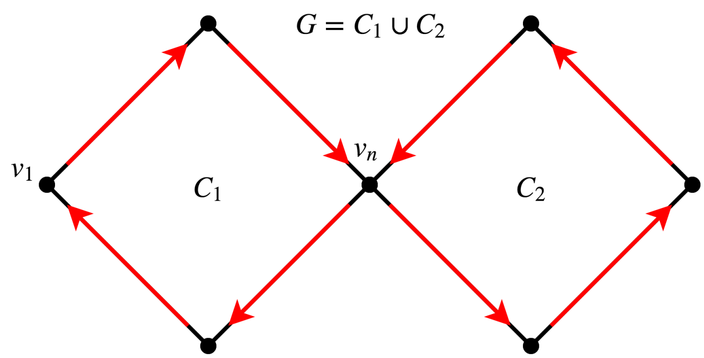


Figure 5: A Closed Eulerian Walk in G starting at v_1.

Thus we have a methodology for many problems similar to the Königsberg bridges: we can simply count the degrees of each vertex. This proves invaluable for larger graphs of higher complexity. We will return to analysis of simple graphs in the next section to discuss the concept of a graph with specific weights attached to its edges.

5 Weighted Graphs

Consider a group of cities spread out across an expansive area. These cities all need to be connected by telephone lines. It would be a waste of resources to create a cycle of lines connecting these cities, since all we need is a path from any given city to another. However, we want all cities to be "on the grid", so to speak. Thus, we need to find a tree that connects all of the cities. The spanning tree is used to find the shortest path that connects all vertices in a graph.

Definition 5.1. [1] If G is a connected graph, we say that T is a *spanning tree* of G if G and T both have the same vertices and each edge of T is also an edge of G.

If G is a tree, then it is obviously its own spanning tree. If G is a connected graph with cycles, we proceed to omit an edge e^* such that G is still connected. We continue this process until our new graph G^* is a tree. This graph is a spanning tree of G. Figure 7 displays two spanning trees of the graph G in Figure 6.

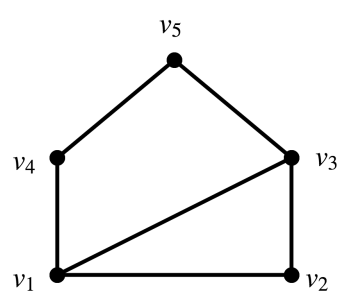


Figure 6: A Connected Graph $G = (V, E)$

Interpreting these graphs in terms of the telephone line problem, think of the vertices $\{v_1, v_2, v_3, v_4, v_6\}$ as cities, with the edges of the graphs representing the lines. It is safe to assume that these cities are not all equidistant from each other, however. It is possible to represent these varied distances using *weighted graphs*:

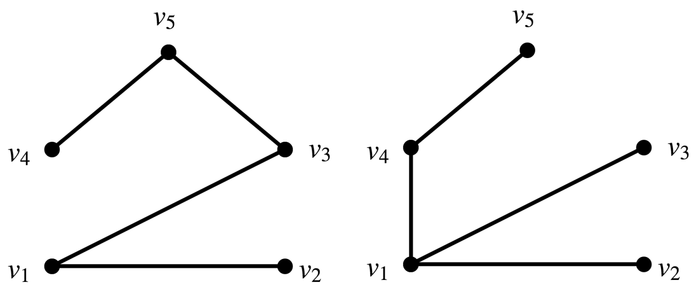


Figure 7: Two Spanning Trees of the Graph in Figure 6.

Definition 5.2. [3]. Let $G = (V, E)$ be a graph. With each edge e of G let there be associated a real number $w(e)$, called its weight, Then G, together with these weights on its edges, is called a **weighted graph**.

Think of the weights $w(e)$ in Figure 8 as the distance (in some unit) from city v_n to city v_m. Obviously, Figure 8 is not to scale (it is important to note that position of vertices in $\mathbb{R}^2$ adds no significance to *any* graph). Our goal is to minimize the units of telephone line used while still connecting all cities. To do this, we need to find an *optimal* (or *minimum*) *spanning tree.*

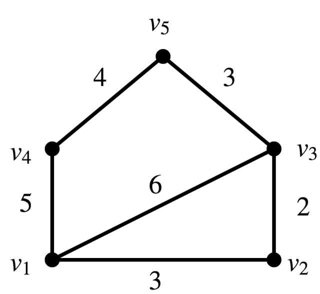


Figure 8: A Weighted Graph $G = (V, E, w(e))$.

Kruskal's Algorithm gives us a means to construct such a tree. While the proof that the algorithm does indeed create an optimal spanning tree is beyond the scope of this paper, we can still use the result to solve the telephone line problem. For a rigorous proof of the algorithm's effectiveness, refer to [3].

Kruskal's Algorithm

1. Choose an edge e_1 such that $w(e_1)$ is as small as possible.

2. If edges $e_1, e_2, ..., e_i$ have been chosen, then choose an edge e_{i+1} from $E \setminus \{e_1, e_2, ..., e_i\}$ in such a way that

   (i) The graph with edges$\{e_1, e_2, ...e_{i+1}\}$ does not contain a cycle;

   (ii) $w(e_{i+1})$ is as small as possible subject to (i).

3. Stop when step 2 cannot be implemented further.

Applying this process to Figure 8, we choose the smallest edge $e_1 = \{v_2, v_3\}$ with $w(e_1) = 2$. In this case, it doesn't matter whether we choose $\{v_1, v_2\}$ or $\{v_3, v_5\}$ first, as they will be the next two edges as small as possible subject to e_1. Choose $e_2 = \{v_1, v_2\}$ and $e_3 = \{v_3, v_5\}$. Next we choose $e_4 = \{v_4, v_5\}$. The edges formed by vertices $\{v_1, v_4\}$ and $\{v_1, v_3\}$, if chosen, will create a cycle. Thus our spanning tree consists of our defined edges $\{e_1, e_2, e_3, e_4\}$, with a total sum weight of $2 + 3 + 3 + 4 = 12$ (See Figure 9). Notice that the total sum weight of the spanning trees in Figure 7 are 16 and 18 (in order from left to right), provided that we assigned the same weights to the edges as we did in Figure 8. These are just two examples of sub-optimal spanning trees.

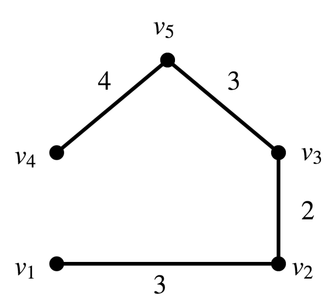


Figure 9: The Optimal Spanning Tree of the Graph in Figure 6.

6 Application: Musical Graphs

The natural appearance of mathematics in music is not always truly recognized-rather, it is the emotional interpretation of the organized sound that takes the center stage. Goffredo Haus and Alberto Pinto utilized graph theory to model any musical idea in [2]. Translating pitch, duration, and accentuation to a single graph allows us to analyze the musical similarities of different melodies. Before we introduce the definition of this *musical graph*, we must first briefly discuss *directed graphs*.

When modeling relations, oftentimes a connection is not bidirectional. For example, on the social media platform Twitter, you can follow an account, but that does not necessarily mean that they follow you back. The converse is also true. Figure 10 shows a basic directed graph with the ordered pairs $\{(v_1, v_2), (v_1, v_3), (v_3, v_2)\}$. A graph in which each edge is assigned a direction is called a *directed graph.* The pairs of vertices in each edge are therefore *ordered* pairs, the first term being the tail of the edge, and the second being the head (see *directed graphs* in [1]). These are called *directed edges.*

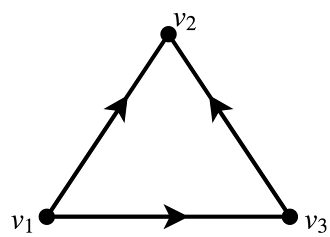


Figure 10: A directed graph.

Now we can begin to introduce Haus and Pinto's musical graphs. Their musical graph is created from a thematic fragment. A thematic fragment is musically defined as a small musical phrase. For example, the musical phrase that famously opens Beethoven's Fifth Symphony would consist of the four notes {G,G,G,E♭}. This fragment M would have a length $m = 4$. Haus and Pinto gave analytic meaning to the thematic fragment's pitch, length, and accent. The metric space (V, d) for pitches can vary, but the most likely candidate is the twelve-tone system. This set contains the integers $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, corresponding to the pitches {C,C♯,D,D♯,E,F,F♯,G,G♯,A,A♯,B}. The distance function in the twelve-tone metric space uses arithmetic modulo twelve, similar to a twelve-hour clock. For example, the distance between the pitches C (mapped to zero) and A♯ (mapped to ten) is two, not ten. These mapped numeric values are not conjecture– rather, this technique is referred to as twelve-tone serialism, a compositional process made famous by Arnold Schoenberg. Assigning numeric values to lengths and accents is an extension of twelve-tone serialism. This is referred to as *total serialism*. A popular mapping of note lengths to the natural numbers is Pierre Boulez's durational "scale". For example, a thirty-second note is mapped to the number one. Accents, such as sforzandos and staccatos, can be mapped similarly to the integers. We can create a graph $G_1 = (V, A_{G_1})$ with pitches $h_1, h_2, ..., h_m \in V$ as the vertices and $a_1 = (h_1, h_2), a_2 = (h_2, h_3), ..., a_m = (h_m, h_1) \in A_{G_1}$ as the edges. It is important to note that these edges in A_{G_1} are *directed*, since the musical fragment is analyzed in one direction– we are typically not interested in its reversal. Furthermore, we can create a weight function with the mapped accents and lengths, assigning a duration and velocity to each edge. Thus, this weight function p maps from the elements $a_1, a_2, ..., a_{m-1} \in A_{G_1}$ to the cartestian product of the two sets $\{d_1, d_2, ...d_{m-1}\} \in \mathbb{Q}^+$ (where d_n represents the duration or length of the n^{th} note) and $\{b_1, b_2, ..., b_{m-1}\} \in \mathbb{Q}^+$ (where b_n represents the accent associated with the n^{th} note). Figure 11 provides a generalized musical graph of length $m = 4$. Haus and Pinto rigorously define musical graphs in their definition as stated below:

Definition 6.1. [2] Let M be a thematic fragment of length $m = |M|$ and consider three characterizing sequences of observable: pitches $\{h_s\}_{s \in I}$, lengths $\{d_s\}_{s \in I}$, and accents $\{b_s\}_{s \in I}$, $\{I = 1, ..., m\}$. Then let (V, d) be a metric space on a finite set of elements V. V and d depends upon the musical system we are considering.

Now, let's consider the graph G obtained by associating a vertex labelled by h_s to every element $h_s \in V$ and an edge a_s to every couple (h_s, h_{s+1}).

We can then define a weight function $p : A_{G_1} \to \mathbb{Q} \times \mathbb{Q}$ by:

$$p : a_s \to (d_s, b_s) \in \mathbb{Q}^+ \times \mathbb{Q}^+ \; \forall \, s = 1, ..., m-1$$

where $a_m = (h_m, h_1)$ and $p(a_m) = (d_1, b_1)$.

Haus and Pinto introduce a similarity function on two musical graphs that is able to determine how much of one thematic fragment is contained

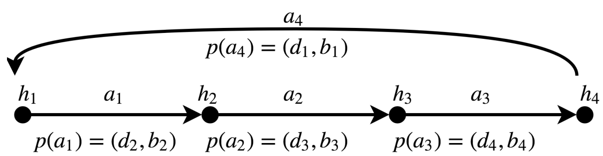


Figure 11: The Graph of a Thematic Fragment M of length $m = 4$.

in the other (see [2]). Applying this function to a score, we could create an algorithm that easily picks out similar melodies and countermelodies that reappear later in the piece. This would be invaluable to any music theorist. This is how graph theory is utilized in many disciplines- a scenario is theorized mathematically using graphs, then a constructed program runs a series of functions based on these graphs. It is graph theory that provides a bridge between the physical world and the computational one, a bridge that is rapidly narrowing at present.

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