Δημητριάδης
Βασίλειος

1. $f(x,\lambda)=\begin{cases}\lambda e^{-\lambda x}, & x>0\\ 0, & x<0\end{cases}, \lambda\in\mathbb{R}$

α) Αρκεί να δείξω ότι $f_X(x)\geq 0\ \forall x\in\mathbb{R}$:

i) Για $x<0$: $f(x)=0\geq 0$

ii) Για $x>0$: $f(x)=\lambda e^{-\lambda x}$ όπου

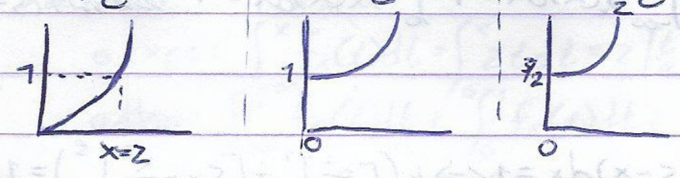
Αν $\lambda<0$, $f(x)<0$, άρα για $\lambda<0$ <u>δεν είναι</u>

Αν $\lambda\geq 0$, $f(x)\geq 0$, άρα για $\lambda\geq 0$ <u>είναι</u>

Με την χρήση ολοκλ. βλέπουμε ότι $\int_0^{+\infty}\lambda e^{-\lambda x}dx=1$. Άρα ισχύει $\forall \lambda\in\mathbb{R}$?

Δηλαδή η $f(x,\lambda)$ είναι συνάρτηση πυκνότητας πιθανότητας για $\lambda\geq 0$.

β) $\lambda=0.5 \rightsquigarrow e^{-\frac{x}{2}}$ | $\lambda=1 \rightsquigarrow e^{-x}$ | $\lambda=1.5 \rightsquigarrow \frac{3}{2}e^{-\frac{3}{2}x}$



γ) Για $\lambda=1$ έχω:

$f_X(x)=\begin{cases} e^{-x}, & x>0\\ 0, & x<0\end{cases}$ | Ισχύει ότι: $\int_w^z e^{-x}dx=[-e^{-x}+c]_w^z$

$\boxed{P(X\leq 4)}$: $\int_{-\infty}^{4} f(x) = \int_0^4 f(x)dx = e^{-4}+e^{0} = 1-\frac{1}{e^4}$

$\boxed{P(2\leq X\leq 5)}$: $\int_2^5 f(x)dx = [-e^{-x}]_2^5 = -e^{-5}+e^{-2}$

$\boxed{P(X<3)}$: $\int_{-\infty}^{3} f(x)dx = \int_0^3 f(x)dx = 1-\frac{1}{e^3}$

$\boxed{P(X>2)}$: $1-P(X\leq 2) = 1-\int_0^2 f(x)dx = 1-1+\frac{1}{e^2}=\frac{1}{e^2}=e^{-2}$

$\boxed{P(4<X\leq 6)}$: $\int_4^6 f(x)dx = \frac{e^2-1}{e^6}$

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2. α. $\int_{-\infty}^{+\infty} f_X(x)dx=1 \iff \int_{-\infty}^{0}\cancel{f_X(x)}^{0}dx+\int_0^1 f_X(x)dx+\int_1^{+\infty}\cancel{f_X(x)}^{0}dx=\int_0^1 \kappa[x-x^2]dx \iff$

$\kappa\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1=1 \iff \boxed{\kappa=6}$ Άρα έχω: $f_X(x)=\begin{cases}6(1-x), & 0\leq x\leq 1\\ 0, & \text{αλλού}\end{cases}$

β. $\boxed{P(0.25<X)}$: $\int_{0.25}^{+\infty} f_X(x)dx = \int_{0.25}^{1} f_X(x)dx = 0.83$

$\boxed{P(X\le 0{,}41)}$: $\int_{-\infty}^{5} f_X(x)\,dx = \cancel{\int_{-\infty}^{0} f_X(x)\,dx} + \int_0^1 f_X(x)\,dx + \cancel{\int_1^5 f_X(x)\,dx} = 3$ $\qquad \int_{-\infty}^{0,41} f_X(x)\,dx = 0.38$

$\boxed{P(0{,}25<X<0{,}41)}$: $\int_{0,25}^{0,41} f_X(x)\,dx = 0{,}21$

3. • $f(x)=2e^{-x}\ge 0 \ \forall x\in\mathbb{R}$, άρα είναι

• $f(x)=\frac{1}{\pi(1+x^2)}\ge 0 \ \forall x\in\mathbb{R}$ αφού $(1+x^2)>0$ και $\pi>0$. Άρα είναι.

4. α. $\int_0^4 c(4-x)\,dx=1 \iff c\int_0^4(4-x)\,dx=1 \iff c\left[4x-\frac{x^2}{2}\right]_0^4=1 \iff c=\frac{1}{8}$

Τελικά $f(x)=\frac{1}{8}(4-x)=\frac{1}{2}-\frac{x}{8}$, $0\le x\le 4$

β. $\boxed{P(X>3{,}3)}$: $1-P(X\le 3{,}3)=1-\int_{-\infty}^{3,3} f_X(x)\,dx = 1-\int_0^{3,3} f_X(x)\,dx = 1-0{,}969375=0{,}030625$

5. α. $\int_{-\infty}^{+\infty} f(x)\,dx=1 \iff \int_0^1 \kappa x\,dx+\int_1^5 \kappa(5-x)\,dx=1 \iff \kappa\left(\left[\frac{x^2}{2}\right]_0^1+\left[5x-\frac{x^2}{2}\right]_1^5\right)=1 \iff$

$\kappa\left(\frac{1}{2}+25-\frac{25}{2}-5+\frac{1}{2}\right)=1 \iff \kappa\left(21-\frac{25}{2}\right)=1 \iff \kappa=\frac{2}{17}$

β. $\boxed{P(X>2)}$: $\int_2^5 f_X(x)\,dx=0{,}495$

$\boxed{P(X<3)}$: $\int_0^1 f_X(x)\,dx+\int_1^3 f_X(x)\,dx=0{,}715$

$\boxed{P(0{,}5<X<2)}$: $\int_{0,5}^1 f_X(x)\,dx+\int_1^2 f_X(x)\,dx=0{,}42625$

γ. $\int_0^\mu f(x)\,dx=\int_0^1 f(x)\,dx+\int_1^\mu f(x)\,dx=\left[\kappa\frac{x^2}{2}\right]_0^1+\left[\kappa\left(5x-\frac{x^2}{2}\right)\right]_1^\mu=\dots=\kappa\left(5\mu-\frac{\mu^2}{2}-4\right)=0{,}5 \iff$

$10\mu-\mu^2-17=0 \rightarrow \mu_{1,2}$: $\mu_1=0.937$, $\mu_2=9.81$, απορρ. αφού πρέπει $0<x<5$

Τελικά $\mu=0.937$

δ. $\int_{-\infty}^{x_p} f_X(x)\,dx=\frac{75}{100} \iff \int_{-\infty}^0 f_X(x)\,dx+\int_0^1 f_X(x)\,dx+\int_1^{x_p} f_X(x)\,dx + \dots = 0{,}75 \iff$

$\frac{2}{17}\left(5x_p-\frac{x_p^2}{2}-4\right)=0.75 \iff 10x_p-x_p^2-20{,}75=0$: $p_1=2.938$, δεκτό; $p_2=7.0615$

6. $f(x)=cx^3,\ 0<x<2$

α. $\int_0^2 f_x(x)dx = \int_0^2 cx^3dx = \frac{c}{4}[x^4]_0^2 = \frac{c}{4}\cdot 2^4 = \frac{16}{4}c = 4c \Leftrightarrow 4c=1 \Leftrightarrow c=\frac{1}{4}$

Τελικά $f(x)=\frac{x^3}{4},\ 0<x<2$

β. $\int_{0,5}^1 f(x)dx = 0.0585$

γ. i) $\int_{-\infty}^{+\infty} x f(x)dx = \int_{-\infty}^{+\infty} x\cdot\frac{x^3}{4}dx = \int_0^2 \frac{x^4}{4}dx = \frac{1}{20}\cdot[x^5]_0^2 = 1.6$

ii) $Var(x) = E(x-\mu)^2 = E(x^2)-\mu^2 = E(x^2)-(E(x))^2 \overset{*}{=} 2-(1.6)^2 = -0.56$

$*\ \ E(x^2) = \int_0^2 x^2\cdot\frac{x^3}{4}dx = \frac{1}{4}\left[\frac{x^6}{6}\right]_0^2 = \frac{1}{32}[x^6]_0^2 = 2$

8. α) $\int_0^1 f_x(x)dx = \int_0^1 cx\,dx = c\left[\frac{x^2}{2}\right]_0^1 = \frac{c}{2} \Leftrightarrow \frac{c}{2}=1 \Leftrightarrow c=2$

β. Για $x<0$: $F(x)=0$

$0\le x\le 1$: $\int_{-\infty}^x f_x(t)dt = \int_0^x 2t\,dt = 2\left[\frac{t^2}{2}\right]_0^x = x^2$

αλλού: $\int_{-\infty}^0 f_x(t)dt + \int_0^1 f_x(t)dt + \int_1^x f_x(t) = 1$

Τελικά $F(x) = \begin{cases} 0, & x<0 \\ x^2, & 0\le x\le 1 \\ 1, & \text{αλλού} \end{cases}$

γ. $Y=\ln x \rightsquigarrow x=e^Y,\ \frac{dx}{dy}=e^Y$

Άρα $f_y(Y) = f(x(y))\cdot\left|\frac{dx}{dy}\right| = 2(e^Y)^2\cdot|e^Y| = 2e^{3Y}$

7. Διευκρίνιση: Το ολοκλήρωμα και η συνάρτηση είναι π.π μόνο για $x>0$. Έτσι έχω:

α. $c\int_0^{+\infty} x^2e^{-x^3}dx = \frac{c}{3}\cdot\int_0^{+\infty} 3x^2 e^{-x^3}dx = -\frac{1}{3}\cdot c\int_0^{+\infty}(-x^3)' e^{-x^3}dx = \frac{c}{3}\left[-e^{-x^3}\right]_0^{+\infty}$

Είναι: $\frac{c}{3}\left[-e^{-x^3}\right]_0^{+\infty} = 1 \Leftrightarrow \frac{c}{3}\lim_{r\to+\infty}\left[-e^{-x^3}\right]_0^r = 1 \Leftrightarrow -c\left[\lim_{r\to+\infty} e^{-r^3} - e^{-0}\right] = 3 \Leftrightarrow$ $\boxed{c=3}$

Τελικά $f(x)=3x^2e^{-x^3},\ x>0$

β. $3\int_{-\infty}^{x_p} x^2e^{-x^3} = 0.5 \Leftrightarrow \left[-e^{-x^3}\right]_0^{x_p} = 0.5 \Leftrightarrow \boxed{\lim_{r\to\infty}[e^{-r^3}-1]=0.5} \Leftrightarrow$

$-e^{-x_p^3}+1=0.5 \Leftrightarrow e^{-x_p^3}=0.5 \Leftrightarrow -x_p^3 = \ln\frac{1}{2} \Leftrightarrow x_p = \left(-\ln\frac{1}{2}\right)^3$

γ. $3\int_{-\infty}^{x_p} x^2e^{-x^3} = 0.2$ ή $F_x(x_{20})=0.2$ όπου $F_x(x) = -\frac{1}{3}e^{-x^3}$

$\hookrightarrow -\frac{1}{3}e^{-x_{20}^3} = 0.2 \Leftrightarrow e^{-x_{20}^3} = -0.6 \Leftrightarrow x_{20}^3 = -\ln(-0.6) \Leftrightarrow$

$x_{20} = \sqrt[3]{-\ln(-0,6)}$