Σεπτέμβριος 2017 ΔΙΑΦΟΡΙΚΕΣ ΕΞΙΣΩΣΕΙΣ

Θέμα 1ο

α) $y' + 2y(1 - x\sqrt{y}) = 0 \iff y' + 2y - 2xy\sqrt{y} = 0 \iff$

$y' + 2y - 2xy^{3/2} = 0$ ~> Bernoulli με $\alpha = 3/2$

Θέτω $u = y^{1-\alpha} = y^{-1/2}$

$u' = (y^{-1/2})' = -\frac{1}{2}y^{-\frac{3}{2}} \cdot y' = -\frac{1}{2}y^{-\frac{3}{2}} \cdot (2xy^{\frac{3}{2}} - 2y) = -\frac{1}{2}\cdot 2x + \frac{1}{2}\cdot 2y^{-\frac{1}{2}} \iff$

$u' = x + u \iff u' - u + x = 0$: Γραμμική 1ης τάξης.

Θέτω $u = \varphi(x)\cdot U(x)$ όπου:

- $\varphi(x) = e^{-\int P dx} = e^{-\int -1 dx} = e^{x}$
- $U(x) = \int \frac{Q}{\varphi(x)} dx = \int \frac{-x}{e^{x}} dx = \int -xe^{-x} dx = [xe^{-x}] - \int e^{-x} = xe^{-x} + e^{-x} + c \iff$
  $= e^{-x}(x+1) + c$

Τελικά $u = e^{x}\cdot[e^{-x}(x+1) + c] = e^{0}(x+1) + e^{x}c = x + 1 + e^{x}c$

Άρα $y^{-1/2} = x + 1 + ce^{-x} \iff y = (x+1+ce^{-x})^{-2} = \frac{1}{(x+1+ce^{-x})^{2}}$

β) $y' = \frac{2y^4 + x^4}{xy^3}$ ~> Ομογενής

$= \frac{2y}{x} + \frac{x^3}{y^3} = 2\frac{y}{x} + \left(\frac{y}{x}\right)^{-3} = f\left(\frac{y}{x}\right)$

Θέτω $u = \frac{y}{x} \iff y = u\cdot x \iff y' = u'x + u \iff u'x + u = 2u + u^{-3} \iff$

$u' = \frac{u}{x} + \frac{u^{-3}}{x}$

Τελικά $u' - \frac{1}{x}u - \frac{1}{x}u^{-3} = 0$ ~> Bernoulli ή Χωριζομένων.

Θα το λύσω ως Bernoulli:

Θέτω $z = u^{1-\alpha} = u^{4}$

$z' = 4u^{3}\cdot u' = \frac{4(z+1)}{x} \iff z' - \frac{4}{x}z - \frac{4}{x} = 0$ ~> Γραμμική 1ης

Θέτω $z = \varphi\cdot Z$ όπου

- $\varphi(x) = e^{-\int P dx} = e^{\int \frac{4}{x} dx} = e^{4\ln x} = x^{4}$
- $Z(x) = \int \frac{Q}{\varphi} dx = \int \frac{\frac{4}{x}}{\frac{x^4}{1}} = \int \frac{4}{x^5} dx = 4\cdot\int x^{-5} dx = -x^{-4} + c$

Τελικά $z = x^{4}(-x^{-4} + c) = cx^{4} - 1$

Άρα $u^{4} = x^{4}(-x^{-4} + c) = cx^{4} - 1$

$u = \sqrt[4]{cx^{4} - 1}$

$y = u\cdot x \iff y = x\sqrt[4]{cx^{4} - 1}$

Θέμα 2ο

$(3x^2-y^2)\cdot y' = 2xy \Rightarrow (3x^2-y^2)\cdot y' - 2xy = 0 \Leftrightarrow 2xy\,dx + (-3x^2+y^2)dy = 0 \quad (I)$

↳ $P_y = 2x$, $Q_x = -6x$ Άρα μη ακριβής Δ.Ε

Ψάχνω μ ώστε $(\mu P)dx + (\mu Q)dy = 0$:

Παρατηρώ ότι για $\mu=\mu(y)$: $\frac{Q_x-P_y}{P} = \frac{-8x}{2xy} = -4\cdot\frac{1}{y}$

Άρα $\frac{\mu'}{\mu} = -4\cdot\frac{1}{y} \Leftrightarrow \ln\mu = \ln y^{-4} \Leftrightarrow \mu = y^{-4}$

Η (I) γίνεται: $2xy^{-3}dx + (-3x^2y^{-4}+y^{-2})dy = 0$ με $P_y = Q_x = -6xy^{-4}$

Άρα:

- $f_x = P \Leftrightarrow f(x,y) = \int 2xy^{-3}dx = x^2y^{-3} + c(y)$
- $f_y = Q \Leftrightarrow (x^2y^{-3}+c(y))'_y = -3x^2y^{-4}+y^{-2} \Leftrightarrow$

$-3x^2y^{-4}+c'(y) = -3x^2y^{-4}+y^{-2} \Leftrightarrow c'(y) = y^{-2} \Leftrightarrow$

$c(y) = \int y^{-2}dy = -y^{-1} = -\frac{1}{y}$

Τελικά: $f(x,y) = x^2y^{-3} - \frac{1}{y}$

Θέμα 3ο

i) $x^2y'' - xy' + y = x^2$: Euler-Cauchy

Θέτω $x = e^t \Leftrightarrow t = \ln x$

↳ $y''_t - 2y'_t + y = e^{2t}$

↳ $p^2 - 2p + 1 = 0$, $\Delta = 4-4 = 0$ με $p = \frac{-(-2)}{2} = 1$

Άρα $y_0 = c_1e^{-t} + c_2te^{-t}$

Για y_p: Θέτω $y_p = \lambda e^{2t}$, $y_p' = 2\lambda e^{2t}$, $y_p'' = 4\lambda e^{2t}$

↳ $4\lambda e^{2t} - 2\cdot 2\lambda e^{2t} + \lambda e^{2t} = e^{2t} \Leftrightarrow \lambda = 1$

Άρα $y_p = e^{2t}$

Τελικά: $y = e^{2t} + c_1e^{-t} + c_2te^{-t}$

$y = x^2 + c_1x^{-1} + c_2t\cdot x^{-1}$

$y = x^2 + c_1\frac{1}{x} + c_2\cdot\frac{\ln x}{x}$

ii) $y'' + 3y' = 3^x$: Θέτω $y' = u$

↳ $u' + 3u = 3^x \Leftrightarrow u' + 3u - 3^x = 0$ ⇝ Γραμμ. 1ης τάξης

Θέτω $u = g(x)\cdot U(x)$ όπου

- $g(x) = e^{-\int 3dx} = e^{-3x}$
- $U(x) = \int\frac{-3^x}{e^{-3x}} = \int e^{3x}(-3^x)dx = \int e^{3x}\cdot e^{x\ln 3}dx = \int e^{3x+x\ln 3}dx =$

$= \int e^{x(\ln 3+3)}dx = \frac{1}{\ln 3+3}e^{x(\ln 3+3)} + c$

Άρα $u = e^{-3x}\cdot(e^{x(\ln 3+3)} + c) = e^{x\ln 3} + ce^{-3x} = 3^x + ce^{-3x}$

Άρα $y = \frac{c}{3}e^{-3x} + \frac{e^{x\ln 3}}{\ln 3} + c$

Θέμα 4ο

$y'' + 4y = 0$ με $y(0) = y'(0) = 2$

$L(y''+4y) = L(0) \iff L(y'') + L(4y) = L(0) \iff$

$s^2 Y - s\,y(0) - y'(0) + 4Y = 0 \iff Y = \frac{2s+2}{s^2+4} \iff$

$Y = \frac{2s}{s^2+4} + \frac{2}{s^2+4}$.

Άρα $L^{-1}(Y) = L^{-1}\left(\frac{2s}{s^2+4}\right) + L^{-1}\left(\frac{2}{s^2+4}\right) \iff$

$y = 2\cos(2x) + \sin(2x)$