Σεπτέμβριος 2016 ΔΙΑΦΟΡΙΚΕΣ ΕΞΙΣΟΣΕΙΣ

Θέμα 1ο

$$x(3-xy)y' = y(xy-1) \iff x(3-xy)\frac{dy}{dx} = y(xy-1) \iff$$

$$[y(xy-1)]dx + [-x(3-xy)]dy = 0 \iff$$

$$(xy^2-y)dx + (-3x+x^2y)dy = 0 \quad (I)$$

$$\left.\begin{aligned} P_y &= 2xy-1 \\ Q_x &= -3+2xy \end{aligned}\right\} \boxed{P_y \neq Q_x}$$

Πρέπει να βρω μ ώστε $(\mu P)dx + (\mu Q)dy = 0$

Παρατηρώ ότι $\mu=\mu(xy)=\mu(w)$: $\frac{\mu'}{\mu} = \frac{-3+2xy-2xy+1}{x^2y^2-xy+3xy-x^2y^2} = -\frac{2}{2xy} = -\frac{1}{xy} = \frac{1}{w}$

$$\int\frac{\mu'}{\mu} = \int -\frac{1}{w} \iff \ln\mu = \ln w^{-1} \iff \mu = w^{-1} \iff \boxed{\mu = \frac{1}{xy}}$$

Έτσι η (I) γίνεται: $\left(y - \frac{1}{x}\right)dx + \left(-\frac{3}{y} + x\right)dy = 0$

με $P_y = 1 = Q_x$ → Ακριβής Δ.Ε

Άρα:

- $f_x = P \iff f(x,y) = \int y - \frac{1}{x}dx = yx - \ln x + c_1(y)$
- $f_y = Q \iff (yx - \ln x + c_1(y))'_y = -\frac{3}{y} + x \iff x + c_1'(y) = -\frac{3}{y}+x \iff$

$$c_1'(y) = -\frac{3}{y} \iff c_1(y) = -3\ln y + c_2$$

Τελικά $\boxed{f(x,y) = yx - \ln x - 3\ln y + c_2}$

Θέμα 2ο

α) $x^2-2y^2+xyy'=0 \iff x^2-2y^2+xy\frac{dy}{dx}=0 \iff (x^2-2y^2)dx + (xy)dy = 0 \ (I)$

↳ $P_y = -4y \neq y = Q_x$

Πρέπει να βρω μ. ώστε: $(\mu P)dx + (\mu Q)dy = 0$

Παρατηρώ ότι για $\mu=\mu(x)$: $\frac{P_y-Q_x}{Q} = \frac{-4y-y}{xy} = -\frac{5}{x}$

Άρα $\frac{\mu'}{\mu} = -\frac{5}{x} \iff \ln\mu = -5\ln x = \ln x^{-5} \iff \boxed{\mu = x^{-5} = \frac{1}{x^5}}$

Έτσι το (I) γίνεται $[x^{-3}-2y^2x^{-5}]dx + [yx^{-4}]dy = 0$, με $P_y = Q_x$

Άρα:

- $f_x = P \iff f(x,y) = \int x^{-3} - 2y^2x^{-5}dx = \frac{x^{-2}}{-2} - \frac{2y^2x^{-4}}{-4} + c_1(y)$
- $f_y = Q \iff \left(-\frac{1}{2}\cdot x^{-2} - \frac{1}{2}y^2x^{-4} + c_1(y)\right)'_y = xy \iff$

$$-yx^{-4} + c_1'(y) = xy \iff c_1'(y) = xy + yx^{-4} \iff$$

$$c_1(y) = \frac{xy^2}{2} + \frac{y^2x^{-4}}{2}$$

Τελικά: $\boxed{f(x,y) = -\frac{1}{2}x^{-2} - \frac{1}{2}y^2x^{-4} + \frac{xy^2}{2} + \frac{y^2x^{-4}}{2} = \frac{1}{2}(xy^2 - x^{-2})}$

β) $x^2y'+xy+\sqrt{y}=0 \iff y'+\frac{1}{x}y+\frac{1}{x^2}y^{1/2}=0$

Θέτω $y^{1-1/2}=u \iff u=y^{1/2} \iff u'=(y^{1/2})'=\frac{1}{2}y'\cdot y^{-\frac{1}{2}} \iff$

$u'=\frac{1}{2}\left[-\frac{1}{x}y-\frac{1}{x^2}y^{\frac{1}{2}}\right]\cdot y^{-\frac{1}{2}}=\frac{1}{2}\left[-\frac{1}{x}y^{\frac{1}{2}}-\frac{1}{x^2}\right] \iff$

$u'=-\frac{1}{2x}\cdot u-\frac{1}{x^2} \iff u'+\frac{1}{2x}\cdot u+\frac{1}{x^2}=0 \iff$

Θέτω $u=g\cdot U$ όπου

- $g(x)=e^{-\int\frac{1}{2x}dx}=e^{-\frac{1}{2}\ln x}=e^{\ln x^{-\frac{1}{2}}}=x^{-\frac{1}{2}}$
- $U(x)=\int\frac{\frac{1}{x^2}}{\frac{x^{-\frac{1}{2}}}{1}}dx=\int\frac{1}{x^2\cdot x^{-1/2}}dx=\int\frac{1}{x^{3/2}}=\int x^{-3/2}dx=-\frac{2}{\sqrt{x}}+c$

Τελικά $u=x^{-\frac{1}{2}}\cdot\left(-\frac{2}{\sqrt{x}}+c\right)$, όμως $u=y^{1/2}$:

$y^{1/2}=x^{-\frac{1}{2}}\cdot\left(-\frac{2}{\sqrt{x}}+c\right) \iff$

$\boxed{y=x^{-1}\cdot\left(-\frac{2}{\sqrt{x}}+c\right)^2}$

Θέμα 3ο

- $y''-3y'+2y=4e^{4x}+3e^{x}$

$\hookrightarrow p^2-3p+2=0$, $\Delta=9-8=1$ $\quad p_{1,2}=\frac{3\pm1}{2}=\begin{cases}2\\1\end{cases}$

Τελικά $\boxed{y_0=c_1e^{2x}+c_2e^{x}}$

Για $R_1=4e^{4x}$:

$\kappa=4\neq2,1$ Άρα $y_{p_1}=\lambda e^{4x}$, $y_{p_1}'=4\lambda e^{4x}$, $y_{p_1}''=16\lambda e^{4x}$

Άρα: $16\lambda e^{4x}-12\lambda e^{4x}+2\lambda e^{4x}=4e^{4x} \iff$

$6\lambda e^{4x}=4e^{4x} \iff \lambda=\frac{4}{6}=\frac{2}{3}$

Τελικά $\boxed{y_{p_1}=4\cdot\frac{2}{3}e^{4x}}$

Για $R_2=3e^{x}$:

$\kappa=1$, Άρα $y_{p_2}=\lambda xe^{\cancel{3x}}$, $y_{p_2}'=\lambda(e^x+xe^x)$, $y_{p_2}''=\lambda(2e^x+xe^x)$

Άρα: $2\lambda e^x+\lambda xe^x-3\lambda e^x-3\lambda xe^x+2\lambda xe^x=3e^x \iff$

$-\lambda e^x=3e^x \iff \lambda=-3$

Τελικά $\boxed{y_{p_2}=-3xe^{x}}$

Άρα $\boxed{y_0=\cancel{4}\cdot\frac{2}{3}e^{4x}+(-3xe^{x})}$

Τελικά $\boxed{y=c_1e^{2x}+c_2e^{x}+\frac{8}{3}e^{4x}-3xe^{x}}$

Θεμα 4

i) $L^{-1}\left(\frac{s/2}{s^2+16}\right) = \frac{1}{2} L^{-1}\left(\frac{s}{s^2+16}\right) = \frac{1}{2}\cos(4x)$

ii) $L(y'') + 16L(y) = 2L(\sin(4x))$, $y(0) = y'(0) = 1$

$$s^2Y - sy(0) - y'(0) + 16Y = 2\frac{4}{s^2+16} \iff$$

$$Y(s^2+16) - s - 1 = 8\frac{1}{s^2+16} \iff$$

$$Y = \left(8 \cdot \frac{1}{s^2+16} + s + 1\right) / (s^2+16) \iff$$

$$Y = 8 \cdot \frac{1}{(s^2+16)^2} + \frac{s}{s^2+16} + \frac{1}{s^2+16} \iff$$

$$L^{-1}(Y) = \frac{8}{64 \cdot 2} L^{-1}\left(\frac{2 \cdot 4^3}{(s^2+4^2)^2}\right) + L^{-1}\left(\frac{s}{s^2+16}\right) + \frac{1}{4} L^{-1}\left(\frac{4}{s^2+4^2}\right) \iff$$

$$\boxed{y = \frac{1}{16}[\sin(4x) - 4x\cos(4x)] + \frac{1}{2}\cos(4x) + \frac{1}{4}\sin(4x)}$$