

function

function, in mathematics, an expression, rule, or law that defines a relationship between one variable (the independent variable) and another variable (the dependent variable). Functions are ubiquitous in mathematics and are essential for formulating physical relationships in the sciences. The modern definition of function was first given in 1837 by the German mathematician Peter Dirichlet:

If a variable y is so related to a variable x that whenever a numerical value is assigned to x , there is a rule according to which a unique value of y is determined, then y is said to be a function of the independent variable x .

This relationship is commonly symbolized as $y = f(x)$ —which is said “ f of x ”—and y and x are related such that for every x , there is a unique value of y . That is, $f(x)$ can not have more than one value for the same x . To use the language of set theory, a function relates an element x to an element $f(x)$ in another set. The set of values of x is called the domain of the function, and the set of values of $f(x)$ generated by the values in the domain is called the range of the function. In addition to $f(x)$, other abbreviated symbols such as $g(x)$ and $P(x)$ are often used to represent functions of the independent variable x , especially when the nature of the function is unknown or unspecified.

Common functions

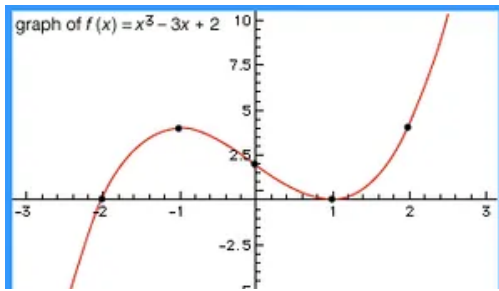
Many widely used mathematical formulas are expressions of known functions. For example, the formula for the area of a circle, $A = \pi r^2$, gives the dependent variable A (the area) as a function of the independent variable r (the radius). Functions involving more than two variables (called multivariable or multivariate functions) also are common in mathematics, as can be seen in the formula for the area of a triangle, $A = bh/2$, which defines A as a function of both b (base) and h (height). In these examples, physical constraints force the independent variables to be positive numbers. When the independent variables are also allowed to take on negative values—thus, any real number—the functions are known as real-valued functions.

The formula for the area of a circle is an example of a polynomial function. The general form for such functions is $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where the coefficients ($a_0, a_1,$

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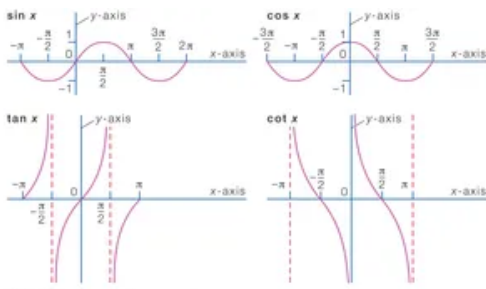
a_2, \dots, a_n) are given, x can be any real number, and all the powers of x are counting numbers (1, 2, 3,...). (When the powers of x can be any real number, the result is known as an algebraic function.) Polynomial functions have been studied since the earliest times because of their versatility—practically any relationship involving real numbers can be closely approximated by a polynomial function. Polynomial functions are characterized by the highest power of the independent variable. Special names are commonly used for such powers from one to five—linear, quadratic, cubic, quartic, and quintic for the highest powers being 1, 2, 3, 4, and 5, respectively.



cubic equation

Polynomial functions may be given geometric representation by means of analytic geometry. The independent variable x is plotted along the x -axis (a horizontal line), and the dependent variable y is plotted along the y -axis (a vertical line). When the graph of a relation between x and y is plotted in the x - y plane, the relation is a function if a vertical line

always passes through only one point of the graphed curve; that is, there would be only one point $f(x)$ corresponding to each x , which is the definition of a function. The graph of the function then consists of the points with coordinates (x, y) where $y = f(x)$. For example, the graph of the cubic equation $f(x) = x^3 - 3x + 2$ is shown in the figure.



graphs of some trigonometric functions

Another common type of function that has been studied since antiquity is the trigonometric functions, such as $\sin x$ and $\cos x$, where x is the measure of an angle (*see* figure). Because of their periodic nature, trigonometric functions are often used to model behaviour that repeats, or “cycles.”

The exponential function is a relation of the form $y = a^x$, with the independent variable x ranging over the entire real number line as the exponent of a positive number a . Probably the most important of the exponential functions is $y = e^x$, sometimes written $y = \exp(x)$, in which e (2.7182818...) is the base of the natural system of logarithms (\ln). By definition x is a logarithm, and there is thus a logarithmic function that is the inverse of the exponential function. Specifically, if $y = e^x$, then $x = \ln y$.

Nonalgebraic functions, such as exponential and trigonometric functions, are also known as transcendental functions.

Complex functions



point in the complex plane

Practical applications of functions whose variables are complex numbers are not so easy to illustrate, but they are nevertheless very extensive. They occur, for example, in electrical engineering and aerodynamics. If the complex variable is represented in the form $z = x + iy$, where i is the imaginary unit (the square root of -1) and x and y are real

variables (*see figure*), it is possible to split the complex function into real and imaginary parts: $f(z) = P(x, y) + iQ(x, y)$.

Inverse functions

By interchanging the roles of the independent and dependent variables in a given function, one can obtain an inverse function. Inverse functions do what their name implies: they undo the action of a function to return a variable to its original state. Thus, if for a given function $f(x)$ there exists a function $g(y)$ such that $g(f(x)) = x$ and $f(g(y)) = y$, then g is called the inverse function of f and given the notation f^{-1} , where by convention the variables are interchanged. For example, the function $f(x) = 2x$ has the inverse function $f^{-1}(x) = x/2$.

Other functional expressions

A function may be defined by means of a power series. For example, the infinite series

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots$$

could be used to define these functions for all complex

values of x . Other types of series and also infinite products may be used when convenient.

An important case is the Fourier series, expressing a function in terms of sines and cosines:

$$f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \cdots \\ + b_1 \sin x + b_2 \sin 2x + \cdots$$

Such representations are of great importance in physics, particularly in the study of wave motion and other oscillatory phenomena.

Sometimes functions are most conveniently defined by means of differential equations. For example, $y = \sin x$ is the solution of the differential equation $d^2y/dx^2 + y = 0$ having $y = 0$, $dy/dx = 1$ when $x = 0$; $y = \cos x$ is the solution of the same equation having $y = 1$, $dy/dx = 0$ when $x = 0$.

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