HW-07 . 2024.12.7

1. 求下列优化问题的KKT点，并判断是否为最优解.

$\min \quad x_1^2 - x_2 - 3x_3$

$$\text{s.t.} \begin{cases} -x_1 - x_2 - x_3 \geq 0 \\ x_1^2 + 2x_2 - x_3 = 0 \end{cases}$$

$(u \geq 0)$

解: 由题可知: $L(x) = x_1^2 - x_2 - 3x_3 - u(-x_1 - x_2 - x_3) + \lambda(x_1^2 + 2x_2 - x_3)$

故有: $\nabla L = \begin{pmatrix} 2x_1 + u + 2\lambda x_1 \\ -1 + u + 2\lambda \\ -3 + u - \lambda \end{pmatrix}$, $\nabla^2 L = \begin{pmatrix} 2(\lambda+1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

相应的KKT条件为: $\begin{cases} \nabla L = 0 \\ x_1^2 + 2x_2 - x_3 = 0 \\ (-x_1 - x_2 - x_3) \cdot u = 0 \\ \cancel{(x_1^2 + 2x_2 - x_3)\lambda = 0} \\ -x_1 - x_2 - x_3 \geq 0 \\ u \geq 0 \end{cases}$

解得KKT点为: $\begin{cases} x_1 = -\frac{7}{2} \\ x_2 = -\frac{35}{12} \\ x_3 = \frac{77}{12} \\ u = \frac{7}{3} \\ \lambda = -\frac{2}{3} \end{cases}$ ①

~~将①代入$\nabla^2 L$中，$\nabla^2 L = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$~~

对于$f(x) = x_1^2 - x_2 - 3x_3$，其为凸函数($x_1^2$二阶导数大于0，$-x_2$与$-3x_3$为线性函数)，约束条件也均为凸函数(线性函数、凸函数+线性函数)

因此，该KKT点为全局最优解.

2. 以$(1,1)^T$为初始点，用共轭梯度法求解下列问题.

$\min \quad x_1^2 - 2x_1x_2 + 4x_2^2 + x_1 - 3x_2$

对$f(x) = x_1^2 - 2x_1x_2 + 4x_2^2 + x_1 - 3x_2$ 求梯度 $g_k = \nabla f(x) = [2x_1 - 2x_2 + 1 \quad -2x_1 + 8x_2 - 3]^T$

代入$(1,1)^T$，得: $g_0 = [1 \quad 3]^T$ ~~令$\varphi(\alpha) = f(x_0 + \alpha(-g_0)) = 1 - 10\alpha + 31\alpha^2$~~

~~令$\varphi'(\alpha) = 0$，得: $\alpha_0 =$~~ 则 $d_0 = -g_0 = [-1 \quad -3]^T$，且对于$f(x)$，有: $G = \begin{pmatrix} 2 & -2 \\ -2 & 8 \end{pmatrix}$，$b = (1 \quad -3)^T$

则: $\alpha_0 = -\frac{d_0^T g_0}{(d_0)^T G d_0} = -\frac{-10}{62} = \frac{5}{31}$ 则: $x_1 = x_0 + \alpha_0 d_0 = (\frac{26}{31} \quad \frac{16}{31})^T$

代入梯度中，得: $g_1 = (\frac{51}{31} \quad -\frac{17}{31})^T$，则: $\beta_0 = \frac{g_1^T g_1}{g_0^T g_0} = \frac{289}{961}$

则: $d_1 = -g_1 + \beta_0 d_0 = (-\frac{2870}{961} \quad -\frac{340}{961})^T$，$\alpha_1 = -\frac{d_1^T g_1}{d_1^T G d_1} = \frac{13495400}{961^2}$

则: $x_2 = x_1 + \alpha_1 \cdot d_1 = \begin{pmatrix} \frac{26}{31} - \frac{3.8318 \times 10^{10}}{961^3} \\ \frac{16}{31} - \frac{4.588436 \times 10^9}{961^3} \end{pmatrix}$，$g_2 = 0$ "二次终止性"

算法终止.