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❓ Problem 1

求解 $\frac{dy}{dx} = y \ln x$

✅ Solution Solution >

将方程中的变量 y 和 x 分离开来：

$$\frac{1}{y} dy = \ln x \, dx$$

两边积分，有：

$$\ln |y| = x \ln x - x + C$$

此即为通解，其中 C 为任意常数。
另有解

$$y = 0.$$

❓ Problem 2

求解 $\tan y dx - \cot x dy = 0$

✅ Solution Solution >

若 $\sin y, \cos x$ 均不为零，则 $\mathrm{d}(\cos x \sin y) = -\sin x \sin y dx + \cos x \cos y dy = 0$，故其通解为 $\cos x \sin y = C$，其中 C 为任意常数。
若 $\sin y = 0$ 或 $\cos x = 0$，则解为 $y = k\pi$ 或 $x = \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$.

❓ Problem 3

求解初值问题 $\begin{cases} (x^2 - 1)y' + 2xy^2 = 0, \\ y(0) = 1 \end{cases}$

✅ Solution Solution >

当 $y \neq 0, x^2-1 \neq 0$ 时，方程变为

$$-\frac{\mathrm{d}y}{y^2}=\frac{2x\,\mathrm{d}x}{(x^2-1)},$$

积分得

$$\frac{1}{y}=\ln\left|x^2-1\right|+C.$$

在通解中代入初值 $y(0)=1$ ，有 $C=1$ ．所求特解为 $y=\dfrac{1}{\ln\left|x^2-1\right|+1}$.

❓ Problem 4

求解 $(2x-4y+6)dx+(x+y-3)dy=0$

✅ Solution Solution >

作变换

$$x=\xi+1, \quad y=\eta+2$$

有

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi}=\frac{4\eta-2\xi}{\xi+\eta}$$

再作变换 $\mathrm{u}=\dfrac{\eta}{\xi}$ ，有

$$u+\xi\frac{du}{d\xi}=\frac{4u-2}{1+u}$$

变量分离，有

$$\frac{u+1}{(u-1)(u-2)}du=-\frac{d\xi}{\xi} \quad (u \neq 1,2)$$

两边积分，有

$$(u-2)\left(\frac{u-2}{u-1}\right)^2\xi=C$$

回代，得原方程通解

$$(y-2x)^3=C(y-x-1)^2$$

若 $u=1, u=2$ ， 即 $y=x+1, y=2x$, 则还有解 $y=x+1$.

❓ Problem 5

求解 $y\ln y dx+(x-\ln y)dy=0$

✅ Solution Solution ›

将方程变形为

$$\frac{\mathrm{d}x}{\mathrm{d}y}+\frac{x}{y\ln y}=\frac{1}{y}$$

所求通解为

$$\begin{aligned}
x&=\mathrm{e}^{-\int\frac{1}{y\ln y}dy}\left[\int\frac{1}{y}\mathrm{e}^{\int\frac{1}{y\ln y}dy}\,\mathrm{d}y+C\right]\\
&=\mathrm{e}^{-\ln(\ln y)}\left[\int\frac{1}{y}\mathrm{e}^{\ln(\ln y)}\mathrm{d}y+C\right]\\
&=\frac{1}{\ln y}\left[\int\frac{\ln y}{y}\,\mathrm{d}y+C\right]=\frac{\ln y}{2}+\frac{C}{\ln y}.
\end{aligned}$$

另有解

$$y=1.$$

❓ Problem 6

求解 $xdy-\left(y+xy^3(1+\ln x)\right)dx=0$

✅ Solution Solution ›

$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{y+xy^3(1+\ln x)}{x}, \dfrac{\mathrm{d}y}{\mathrm{d}x}-\dfrac{y}{x}=(1+\ln x)y^3$,
令 $z=y^{1-3}=y^{-2}$ ， 得

$$\begin{aligned}
&\frac{1}{y^3}\frac{\mathrm{d}y}{\mathrm{d}x}-\frac{1}{x}\cdot\frac{1}{y^2}=1+\ln x\\
&-\frac{1}{2}\left(y^{-2}\right)'-\frac{1}{x}\cdot\frac{1}{y^2}=1+\ln x, \frac{\mathrm{d}\left(y^{-2}\right)}{\mathrm{d}x}+\frac{2}{x}y^{-2}=-2(1+\ln x)
\end{aligned}$$

故

$$
\begin{aligned}
y^{-2} &= \mathrm{e}^{-\int \frac{2}{x}\,\mathrm{d}x}\left[C - 2\int (1+\ln x)\mathrm{e}^{2\ln x}\,\mathrm{d}x\right] \\
&= \mathrm{e}^{-2\ln x}\left[C - 2\int (1+\ln x)x^2\,\mathrm{d}x\right] \\
&= \frac{1}{x^2}\left[C - \frac{2}{3}x^3 - \frac{2}{3}\int \ln x\,\mathrm{d}\left(x^3\right)\right] \\
&= \frac{1}{x^2}\left(C - \frac{2}{3}x^3 - \frac{2}{3}x^3\ln x + \frac{2}{9}x^3\right)
\end{aligned}
$$

故所求通解为

$$
\frac{1}{y^2} = \frac{C}{x^2} - \frac{2}{3}x\ln x - \frac{4}{9}x.
$$

❓ Problem 7

求解 $(\ln y + 2x - 1)dy - 2ydx = 0$

✅ Solution Solution >

化为一阶线性微分方程

$$
\frac{dy}{dx} - \frac{1}{y}x = \frac{(\ln y - 1)}{2y}
$$

故可得：

$$
\begin{aligned}
x &= e^{-\int -\frac{1}{y}dy}\left(\int \frac{\ln y - 1}{2y}\cdot e^{\int -\frac{1}{y}dy}dy + C\right) = y\left(\int \frac{\ln y - 1}{2y^2}dy + C\right) \\
&= y\left(\int -\frac{\ln y - 1}{2}d\left(\frac{1}{y}\right) + C\right) = y\left(-\frac{\ln y - 1}{2y} + \frac{1}{2}\int \frac{1}{y^2}dy + C\right) \\
&= y\left(-\frac{\ln y - 1}{2y} - \frac{1}{2y} + C\right) = -\frac{\ln y}{2} + Cy
\end{aligned}
$$

即，$x = -\dfrac{\ln y}{2} + Cy.$