

Bernoulli Number and Power Sum

Wearry

Problem Description

Calculate:

$$S_k(n) = \sum_{i=0}^{n-1} i^k$$

Transform

$$\begin{aligned} S_{m+1}(n) + n^{m+1} &= S_{m+1}(n+1) \\ &= \sum_{k=0}^{n-1} (k+1)^{m+1} \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} S_k(n) \end{aligned} \tag{1.1}$$

$$\text{Let } \hat{S}_k(n) = \frac{1}{k+1} \sum_{j=0}^k \binom{k+1}{j} B_j n^{k+1-j} \tag{1.2}$$

$$\text{Where } B_i \text{ follow : } \sum_{j=0}^m \binom{m+1}{j} B_j = [m=0] \tag{1.3}$$

Proof

$$\hat{S}_k(n) = S_k(n)$$

Considering proof below by induction, assume that $\forall 0 \leq j < m :$
 $\hat{S}_j(n) = S_j(n)$, now to proof $\hat{S}_m(n) = S_m(n)$.

Minus $S_{m+1}(n)$ from both left and right part of equation(1.1):

$$\begin{aligned}
n^{m+1} &= \sum_{k=0}^m \binom{m+1}{k} S_k(n) \\
&= \sum_{k=0}^m \binom{m+1}{k} \hat{S}_k(n) + (m+1)\Delta \\
&= \sum_{0 \leq j \leq k \leq m} \binom{m+1}{k} \binom{k+1}{k-j} \frac{B_{k-j} n^{j+1}}{k+1} + (m+1)\Delta \\
&= \sum_{k=0}^m \binom{m+1}{k} \frac{1}{k+1} \sum_{j=0}^k \binom{k+1}{j+1} B_{k-j} n^{j+1} + (m+1)\Delta \\
&= \sum_{k=0}^m \binom{m+1}{k} \sum_{j=0}^k \binom{k}{j} \frac{B_{k-j} n^{j+1}}{j+1} + (m+1)\Delta \\
&= \sum_{j=0}^m \binom{m+1}{j} \frac{1}{j+1} n^{j+1} \sum_{k=j}^m B_{k-j} \binom{m+1-j}{k-j} + (m+1)\Delta \\
&= n^{m+1} + (m+1)\Delta
\end{aligned}$$

$\Delta = 0$, Q.E.D.