Bernoulli Number and Power Sum

Wearry

Problem Description

Calculate:

$$S_k(n) = \sum_{i=0}^{n-1} i^k$$

Transform

$$\begin{aligned} S_{m+1}(n) + n^{m+1} &= S_{m+1}(n+1) \\ &= \sum_{k=0}^{n-1} (k+1)^{m+1} \\ &= \sum_{k=0}^{m+1} \binom{m+1}{k} S_k(n) \end{aligned} \tag{1.1}$$

$$\text{Let } \hat{S}_k(n) = \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j} B_j n^{k+1-j} \tag{1.2}$$

$$\text{Where } B_i \text{ follow} : \sum_{j=0}^{m} \binom{m+1}{j} B_j = [m=0] \tag{1.3}$$

Proof

$$\hat{S}_k(n) = S_k(n)$$

Considering proof below by induction, assume that $\forall\ 0 \leq j < m$: $\hat(S)_j(n) = S_j(n)$, now to proof $\hat{S}_m(n) = S_k(n)$.

Minus $S_{m+1}(n)$ from both left and right part of equation(1.1):

$$\begin{aligned}
n^{m+1} &= \sum_{k=0}^{m} \binom{m+1}{k} S_k(n) \\
&= \sum_{k=0}^{m} \binom{m+1}{k} \hat{S}_k(n) + (m+1)\Delta \\
&= \sum_{0 \leq j \leq k \leq m} \binom{m+1}{k} \binom{k+1}{k-j} \frac{B_{k-j} n^{j+1}}{k+1} + (m+1)\Delta \\
&= \sum_{k=0}^{m} \binom{m+1}{k} \frac{1}{k+1} \sum_{j=0}^{k} \binom{k+1}{j+1} B_{k-j} n^{j+1} + (m+1)\Delta \\
&= \sum_{k=0}^{m} \binom{m+1}{k} \sum_{j=0}^{k} \binom{k}{j} \frac{B_{k-j} n^{j+1}}{j+1} + (m+1)\Delta \\
&= \sum_{j=0}^{m} \binom{m+1}{j} \frac{1}{j+1} n^{j+1} \sum_{k=j}^{m} B_{k-j} \binom{m+1-j}{k-j} + (m+1)\Delta \\
&= n^{m+1} + (m+1)\Delta
\end{aligned}$$

$$\Delta = 0, \text{Q.E.D.}$$

Generating Function

Define :

$$G(x) = \sum_{i=1}^{\infty} \frac{1}{i!} x^i, F(x) = \sum_{i=0}^{\infty} \frac{B_i}{i!} x^i$$

Then consider the production of function F and function G:

$$
\begin{aligned}
(F * G)(x) &= \sum_{i=1}^{\infty} \left(\sum_{j=0}^{i-1} B_j \binom{i}{j} \right) x^i \\
&= \sum_{i=1}^{\infty} [i - 1 = 0] x^i \\
&= x \\
F(x) &= \frac{x}{e^x - 1}
\end{aligned}
$$