Electrodynamics

Field :- A field is a function that describes the behaviour of a physical quantity at all points in a given region of space.

Scalar field :- A scalar field is specified by the magnitude only. Examples – Temperature, volume, pressure, electric potential, gravitational potential etc.

Vector field :- A vector field is specified by its magnitude and direction. Examples – Velocity, acceleration, electric field, magnetic field etc.

Vector Differential Operator (∇) :- [∇ is read as 'del' or 'nabla']
It is a differential operator that on operation with scalar and vector fields gives us variation of fields at different points in given region of space. It is not a vector but just an operator. In cartesian coordinate system it is given by,

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

∇ operation on Scalar & vector field results —

Field	Operation Name	Result
Scalar	① $\text{grad}\,\phi = \nabla\phi$	vector quantity
vector	① $\text{div}\,\vec{A} = \nabla\cdot\vec{A}$ ② $\text{Curl}\,\vec{A} = \nabla\times\vec{A}$	Scalar Quantity Vector Quantity

where ϕ is a scalar field & $\vec{A}$ is a vector field.

1) Gradient :- (Gradient of a scalar function)

Let a scalar field denoted by scalar function $\phi_{(x,y,z)}$. At points $(x+dx, y+dy, z+dz)$ there is a change $d\phi$ in the function $\phi_{(x,y,z)}$. This change is written as,

$$d\phi = \left(\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial \phi}{\partial z}\right)dz \quad \text{—— (1)}$$

Eqn (1) can also be written as a product,

$$d\phi = \left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right) \quad \text{—— (2)}$$

In eqn (2) the vector quantity $\left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}\right)$ is a gradient of scalar function $\phi_{(x,y,z)}$.

It is denoted by grad ϕ and also $\nabla\phi$.

$\therefore$

$$\boxed{\text{grad}\,\phi = \nabla\phi = \frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j} + \frac{\partial \phi}{\partial z}\hat{k}} \quad \text{—— (3)}$$

Eqn (2) can also be written as,

$$d\phi = \nabla\phi\cdot\vec{dr} \quad \text{—— (4)}$$

where

$\vec{dr} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ is a displacement Vector.

• Physical Significance of Gradient -

Gradient of scalar field results in vector Quantity. for a scalar field ϕ its gradient $\nabla\phi$ is a vector quantity. we have change in ϕ as,

$$d\phi = (\nabla\phi)\cdot\vec{dr} \quad \text{—— (1)}$$ which can be further written as

$$= |\nabla\phi||\vec{dr}|\cos\theta \quad \text{—— (2)}$$

In eq (2) θ is the angle between $\nabla\phi$ and $\vec{dr}$. If we fix the magnitude (making constant), the maximum change in ϕ occurs at $\theta = 0$ (ie; $\cos\theta = 1$). that means $d\phi$ is greatest when $\vec{r}$ move in the direction of $\nabla\phi$.

Therefore we can say, the gradient $\nabla\phi$ points in the direction from eq ② $|\nabla\phi| = d\phi/dr$ (when $\cos\theta = 1$)

i.e; the magnitude $|\nabla\phi|$ gives the slope (rate of increase) along this maximal direction.

2) **Divergence :-** (Divergence of a vector function)

Let 'V' is a vector function given as $V = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$.

The divergence of this vector function is written as,

$$\text{div}\,\vec{V} = \nabla\cdot\vec{V}$$

$$\nabla\cdot\vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left[V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right]$$

$$\therefore \quad \boxed{\nabla\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}} \quad \text{—— ①}$$

Looking at eqn ① it is clear that $\nabla\cdot\vec{V}$ gives a scalar Quantity.

- **Physical Significance of Divergence :—**

the divergence is a measure of how much the vector $\vec{V}$ spreads out (diverges) from a point or spreads in.

Example ① Consider the motion of fluid having velocity $\vec{V}$ then its divergence ($\text{div}\,\vec{V}$) gives the rate of flow (outflow) per unit volume at a point of the fluid.

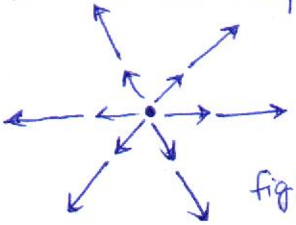


fig ①

Arrows pointed out (length increasing)
Its positive Divergence [$\nabla\cdot V > 0$]

② Electric lines of force ending at negative charge is a negative divergence of electric field vector. [$\nabla\cdot V < 0$]

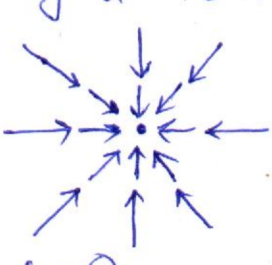


fig ②

③ when field do not spreads in or out. then its zero divergence situation.

[$\nabla\cdot V = 0$]

3> Curl :- (Curl of a vector field - rate of rotation)

The curl of vector $\vec{V}(x,y,z)$ is written as $\nabla \times \vec{V}$. It is a measure of how much the vector $\vec{V}$ curls around the point.

$$\nabla \times \vec{V} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left[V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right] \quad \text{—} ①$$

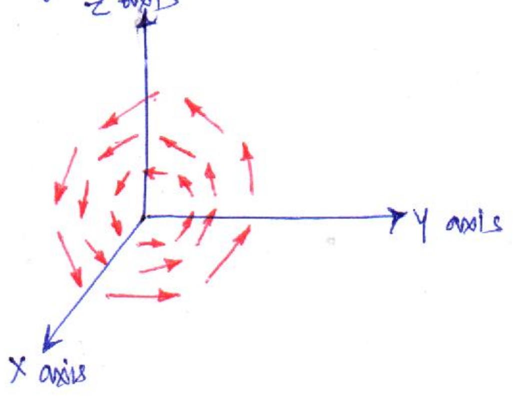
Eq^n ① can be written in determinant form as

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_x & V_y & V_z \end{vmatrix}$$

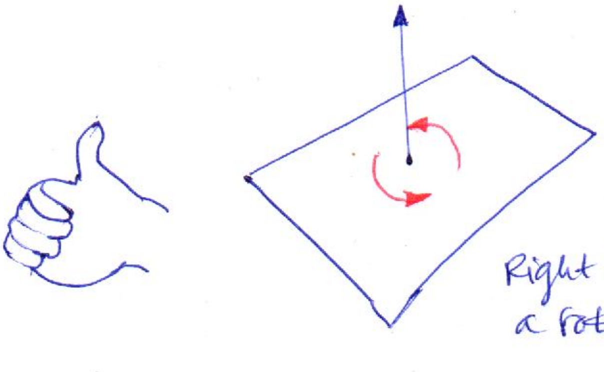
$$\nabla \times \vec{V} = \hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) - \hat{j}\left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) \quad \text{—} ②$$

- Physical Significance of Curl of a vector:-

The curl measures the degree to which the fluid (vector field is velocity-vector of fluid) is rotating about a given point. The curl of vector field is a vector having magnitude equal to the maximum circulation and the direction is perpendicular to the plane of circulation/rotation.



Example - A paddlewheel put on river not only move ahead but also spins around. The rotation of water velocity vector is same as of rotation of a pt on paddlewheel. Below figures shows rotation of velocity vector and the direction by Right hand.



Right hand rule to determine the direction of a rotation vector.

Imp

* Divergence of a Curl of a vector is Zero :—

for any vector field $\vec{B}$ the divergence of curl of $\vec{B}$ is written as.

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \left(\hat{i}_x\frac{\partial}{\partial x}+\hat{i}_y\frac{\partial}{\partial y}+\hat{i}_z\frac{\partial}{\partial z}\right)(\vec{\nabla}\times\vec{B})$$

$$= \left(\hat{i}_x\frac{\partial}{\partial x}+\hat{i}_y\frac{\partial}{\partial y}+\hat{i}_z\frac{\partial}{\partial z}\right)\left[\hat{i}_x\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)+\hat{i}_y\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)+\hat{i}_z\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)\right]$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \frac{\partial}{\partial x}\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right)$$

$$\therefore \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$$

* Divergence Theorem and Stokes theorem :-

① Divergence Theorem :- The volume integral of the divergence of a vector field $\vec{A}$ taken over any volume V is equal to the surface integral of $\vec{A}$ taken over the surface enclosing the volume V.

$$\int_V \vec{\nabla}\cdot\vec{A}\,dv = \oint_S \vec{A}\cdot\vec{ds}$$

Significance :- The total outward flux of a vector field $\vec{A}$ through the closed surface 's' is the same as the volume integral of divergence of $\vec{A}$.

② Stokes theorem :- The surface integral of curl of a vector field $\vec{A}$ over an open surface is equal to the line integral of the vector field over the closed curve bounding the surface area.

$$\int_S (\vec{\nabla}\times\vec{A})\cdot\vec{ds} = \oint_C \vec{A}\cdot\vec{dl}$$

Maxwell's Equations :—

There are two forms of each Maxwell's equation namely integral form and differential form or point form. These four equations are the laws of electromagnetism which helps in study of electromagnetic waves, their propagation, transmission lines and antenna.

following are the Maxwell's Equations.

1) Gauss law (for static electric field) :-

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$$

; Where ρ = volume charge density

ϵ_0 = Permittivity of free space.

$\epsilon_0 = 8.85\times10^{-12}\ C^2/Nm^2$

Significance :- The total electric flux through a closed surface enclosing a charge is equal to $\frac{1}{\epsilon_0}$ times the magnitude of the charge enclosed.

2) Gauss law (for static magnetic field) :-

$$\vec{\nabla}\cdot\vec{B} = 0$$

Significance :— The net magnetic flux through a closed surface is zero. That means magnetic mono poles donot exist.

3) Faraday's law (for electromagnetic field) :-

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

Significance :- Changing magnetic field creates an electric field.

4) Ampere's Law or Modified Ampers law (magneto electric field) :-

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \epsilon_0\frac{\partial\vec{E}}{\partial t}$$

; $\vec{J}$ = Current density

μ_0 = Permeability of free space

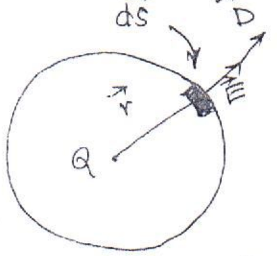
$\mu_0 = 4\pi\times10^{-7}\ (N/A^2)$

The magnetic effect of changing electric field or of a current.

* Derivation of Maxwell's Equations :—

1) First Equation (Gauss Law for static electric field) :—

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.



Mathematically, the surface integral of the normal component of electric flux density ($\vec{D}$) over any closed surface equals the charge enclosed.

Therefore,
$$\oint_S \vec{D}\cdot\vec{ds} = Q_{enclosed} = \int_V \rho \, dv \quad \text{—} (1)$$

Where ρ is charge density

Using Gauss Divergence thm,
$$\oint_S \vec{D}\cdot\vec{ds} = \int_V (\vec{\nabla}\cdot\vec{D})\,dv \quad \text{—} (2)$$

from eq^n (1) and (2)
$$\oint_S \vec{D}\cdot\vec{ds} = \int_V (\vec{\nabla}\cdot\vec{D})\,dv = \int_V \rho \, dv$$

$$\therefore \int_V (\vec{\nabla}\cdot\vec{D})\,dv = \int_V \rho\,. dv$$

$$\therefore \boxed{\vec{\nabla}\cdot\vec{D} = \rho} \quad \text{—} (3)$$

Eq^n (3) can also be written as,
$$\vec{\nabla}\cdot(\epsilon_0 \vec{E}) = \rho$$
$$\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0} \quad \text{—} (4)$$

Eq^n (3) or (4) is Maxwell's first eq^n i.e; Gauss law of static electric field in differential form.
Eq^n (1) is its integral form.

2> Gauss law for static magnetic field (2nd Equations):-

Same as of electrostatics, the magnetic flux through an element of area 'ds' is given by the dot product of '$\vec{B}$' with $\vec{ds}$. For an arbitrary surface ds bounded by a closed contour S, the total magnetic flux passing through the surface is given by,

$$\phi = \oint_S \vec{B} \cdot \vec{ds}$$

The lines of $\vec{B}$ have neither begining nor ending. The no. of lines emerging from any volume bounded by a closed surface S is always equal to the no. of lines entering the volume. Hence, the flux of $\vec{B}$ through any closed surface is equal to zero. Thus,

$$\oint_S \vec{B} \cdot \vec{ds} = 0 \quad \text{——— (1)}$$

using Stokes theorem,

$$\oint_S \vec{B} \cdot \vec{ds} = \int_V (\vec{\nabla} \cdot \vec{B})\, dv \quad \text{——— (2)}$$

using (1)

$$\therefore \int_V (\vec{\nabla} \cdot \vec{B})\, dv = 0 \quad \text{——— (3)}$$

eqn (3) can be written as,

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0} \quad \text{——— (4)}$$

The net magnetic flux through a closed surface is zero. It implies that magnetic poles do not exist separately in the way as electric charges do.

Thus, in other words, magnetic monopoles do not exist.

3) Faraday's law (3rd Equation):-

According to faraday's law, electromagnetic force (emf) induced in a closed loop is negative rate of change of the magnetic flux.

Mathematically, $$e = -\frac{d\phi}{dt} \quad \text{——} (1)$$

Also, total magnetic flux on any arbitrary surface S can be written as,

$$\phi = \oint_S \vec{B} \cdot \vec{ds} \quad \text{——} (2)$$

eqn (1) is now,

$$\therefore \quad e = -\frac{d}{dt}\left[\oint_S \vec{B} \cdot \vec{ds}\right] = -\oint_S \left[\frac{d\vec{B}}{dt}\right] \vec{ds} \quad \text{——} (3)$$

The electromotive force is the work done in carrying a unit charge around the closed loop

$$\therefore \quad e = \oint_l \vec{E} \cdot dl \quad \text{——} (4)$$

Eqn (3) with (4)

$$\therefore \quad \oint_l \vec{E} \cdot dl = -\oint_S \left[\frac{d\vec{B}}{dt}\right] ds \quad \text{——} (5)$$

Now, by using Stoke's theorem;

$$\oint_l \vec{E} \cdot dl = \oint_S (\vec{\nabla} \times \vec{E}) \vec{ds} \quad \text{——} (6)$$

$\therefore$ eqn (5) becomes

$$\oint_S (\vec{\nabla} \times \vec{E}) \vec{ds} = -\oint_S \left[\frac{d\vec{B}}{dt}\right] \vec{ds}$$

$$\therefore \quad \boxed{\vec{\nabla} \times \vec{E} = -\frac{d\vec{B}}{dt}} \quad \text{——} (7)$$

(08)

4) Ampere's Law and Modified Ampere's law (4^{th} equation):

The line integral of magnetic field intensity ($\vec{H}$) around a closed path is exactly equal to the direct current (I) enclosed by that path.

Mathematically;

$$\oint_c \vec{H}\cdot\vec{dl} = I \quad \text{—— (1)}$$

Since $\vec{B} = \mu_0 \vec{H}$, eq^n (1) becomes.

$$\oint \vec{B}\cdot\vec{dl} = \mu_0 I$$

Here 'I' can be replaced by the surface integral of conduction current density ($\vec{J}$) over the surface area bounded by the path of integration of $\vec{H}$ or $\vec{B}$

$$\therefore \oint \vec{B}\cdot\vec{dl} = \mu_0 \int_s \vec{J}\cdot\vec{ds} \quad \text{—— (2)}$$

Using Stokes theorem,

$$\oint_s (\vec{\nabla}\times\vec{H})\cdot\vec{ds} = \int_s \vec{J}\cdot\vec{ds} \quad \text{—— (3)}$$

$\therefore$ Here $\vec{\nabla}\times\vec{H} = \vec{J}$ or $\vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$

$\therefore$ we can write

$$\therefore \boxed{\vec{\nabla}\times\vec{B} = \mu_0\vec{J}} \quad \text{this is Ampere's law. —— (4)}$$

This eq^n holds validity for steady state. ~~But~~ for varying field we need to take its divergence. But Div of curl is zero.

$$\nabla\cdot(\vec{\nabla}\times\vec{B}) = 0$$

$$\Rightarrow \mu_0\vec{J} = 0$$

But this is in contrast with continuity eq^n

$$\vec{\nabla}\cdot\vec{J} + \frac{\partial\rho}{\partial t} = 0$$

$$\text{or } \vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t} \quad \text{—— (5)}$$

To correct this, Maxwell suggested the total current density needs an additional component say $\vec{J}'$

$$\therefore \quad \vec{\nabla}\times\vec{B} = \mu_0(\vec{J}+\vec{J}') \qquad \text{—— (6)}$$

Now, if divergence is taken then

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = \mu_0\vec{\nabla}\cdot(\vec{J}+\vec{J}')$$

$$0 = \mu_0\vec{\nabla}\cdot(\vec{J}+\vec{J}')$$

$$\Rightarrow \quad \therefore \mu_0\vec{\nabla}\cdot\vec{J} = -\mu_0\vec{\nabla}\cdot\vec{J}' \qquad \text{—— (7)}$$

But from eq (5) $\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$ and also from Maxwell's 1st eq^n $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 \Rightarrow \vec{\nabla}\cdot(\varepsilon_0\vec{E}) = \rho$ —— (8)

above eq^n can be written as

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial}{\partial t}\left[\vec{\nabla}\cdot(\varepsilon_0\vec{E})\right] = -\nabla\cdot\frac{\partial\vec{D}}{\partial t} = -\nabla\cdot\frac{\partial(\varepsilon_0\vec{E})}{\partial t}$$

Hence (7) becomes

$$\mu_0\vec{\nabla}\cdot\vec{J} = -\mu_0\vec{\nabla}\cdot\vec{J}' = -\nabla\cdot\frac{\partial(\varepsilon_0\vec{E})}{\partial t}$$

Hence $$\vec{J}' = \frac{\partial(\varepsilon_0\vec{E})}{\partial t} \qquad \text{—— (9)}$$

therefore Maxwell's 4th eq^n is now called as Ampere's modified law using eq^n (9)

$\therefore$ eq (6) becomes

$$\boxed{\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J}+\frac{\partial(\varepsilon_0\vec{E})}{\partial t}\right)} \qquad \text{—— (10)}$$

OR

$$\boxed{\vec{\nabla}\times\vec{H} = \vec{J}+\frac{\partial\vec{D}}{\partial t}} \qquad \text{—— (11)}$$

Where $\frac{\vec{B}}{\mu_0} = \vec{H}$ & $\varepsilon_0\vec{E} = \vec{D}$

Maxwell's Equations in Integral form :-

(1) Maxwell's first equation —

We have $\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0$ or $\vec{\nabla}\cdot\vec{D} = \rho$ — (1)

Taking volume Integration

$$\int_V \vec{\nabla}\cdot\vec{D}\, dv = \int_V \rho\, dv \quad \text{— (2)}$$

using Gauss divergence theorem

$$\int_V \vec{\nabla}\cdot\vec{D}\, ds = \oint_S \vec{D}\cdot ds \quad \text{— (3)}$$

(2) becomes :

$$\oint_S \vec{D}\cdot ds = \int_V \rho\, dv$$

$$\therefore \boxed{\oint_S \vec{D}\cdot ds = Q} \quad \text{— (4)} \quad \left(\because \int_V \rho\, dv = Q\right)$$

(2) Maxwell's second Equation —

We have $\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \int_V \vec{\nabla}\cdot\vec{B} = 0$ — (1)

Using Gauss Divergence theorem

$$\int_V \vec{\nabla}\cdot\vec{B}\, dv = \oint_S \vec{B}\cdot ds = 0$$

$$\therefore \boxed{\oint_S \vec{B}\cdot ds = 0} \quad \text{— (2)}$$

(3) Maxwell's Third Equation —

Differential form $\rightarrow$ $\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$ — (1), taking $\int_S$ on b.s.

$$\int_S (\vec{\nabla}\times\vec{E})\, ds = -\int_S \frac{\partial \vec{B}}{\partial t}\, ds$$

and now using Stokes thm.

$$\boxed{\oint_C \vec{E}\cdot dl = -\frac{\partial}{\partial t}\int \vec{B}\cdot ds} \quad \text{— (2)}$$

④ Maxwell's fourth Equation —

Differential form of 4^{th} eq^n.

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \frac{\partial(\epsilon_0 \vec{E})}{\partial t}$$ or

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \qquad \text{—}(1)$$

Taking Integration over surface

$$\int_S (\vec{\nabla} \times \vec{H}) \vec{ds} = \int_S \left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right] \vec{ds} \qquad \text{—}(2)$$

left side of eq ② can be replaced with contour integral according to Stokes theorem.

$$= \boxed{\oint_C \vec{H} \cdot dl = \int_S \left[\vec{J} + \frac{\partial \vec{D}}{\partial t}\right] \cdot \vec{ds}} \qquad \text{—}(3)$$

Maxwell's Equations

The general set of four Maxwell's equations for time varying electromagnetic fields are listed below.

Table 4.2

Differential (Point) form	Integral form	Significance
$\vec{\nabla} \cdot \vec{D} = \rho$	$\oint \vec{D} \cdot \vec{ds} = \int_V \rho \, dv$	Gauss's law for electrostatics
$\vec{\nabla} \cdot \vec{B} = 0$	$\oint \vec{B} \cdot \vec{ds} = 0$	Gauss's law for magnetostatics
$\vec{\nabla} \times \vec{E} = -\dot{\vec{B}}$	$\oint \vec{E} \cdot \vec{dl} = -\int_S \dot{\vec{B}} \cdot \vec{ds}$	Faraday's law
$\vec{\nabla} \times \vec{H} = \vec{J} + \dot{\vec{D}}$	$\oint \vec{H} \cdot \vec{dl} = \int_S (\vec{J} + \dot{\vec{D}}) \cdot \vec{ds}$	Ampere's law
Supplementary equation		
$\vec{\nabla} \cdot \vec{J} = -\dot{\rho}$	$\oint_S \vec{J} \cdot \vec{ds} = -\int_V \dot{\rho} \, dv$	Continuity equation