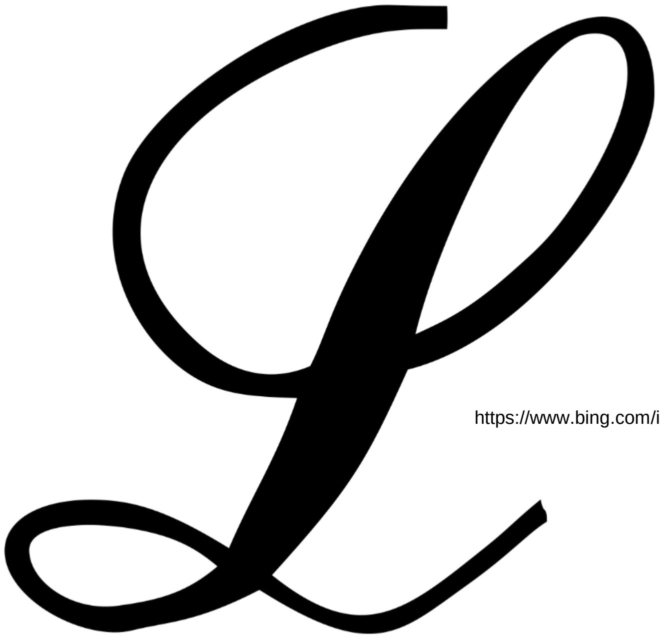
Department of Engineering

ENGR 311: System Dynamics

Fall 2020



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Chapter 2: Part 1

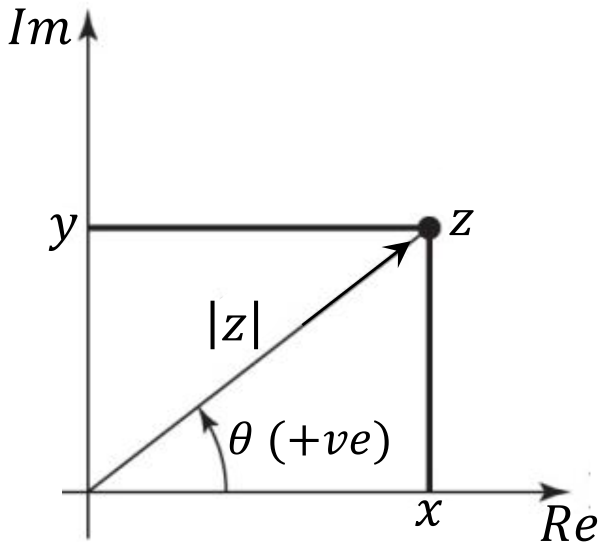
Dynamic Response and the Laplace Transform Method

Outline

- ❑ Complex Numbers, Variables and Functions
- ❑ Laplace Transformation
- ❑ Inverse Laplace Transformation
- ❑ Solving Linear, Time-invariant Differential Equations
- ❑ Additional Examples

Complex Numbers, Variables and Functions

- Complex numbers
  - Notation



$$z = x + jy \qquad j = \sqrt{-1} \quad or \quad j^2 = -1$$

$$Re(z) = x \quad \leftarrow \quad Real\ part\ (real\ constant)$$

$$Im(z) = y \quad \leftarrow \quad Imiginary\ part\ (real\ constant)$$

$$Magnitude\ (absolute\ value)\ of\ z, r = |z| = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \quad \rightarrow \quad Argument\ of\ z, \arg(z) \quad (-\pi, \pi] \ \ or \ \ (-180^o, 180^o]$$

  - $Euler's\ theorem$: $\cos\theta + j\sin\theta = e^{j\theta} \quad \rightarrow \quad \begin{cases} \cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta}) \\ \sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta}) \end{cases}$

  - Rectangular Forms

$$z = x + jy$$

$$z = |z|(\cos\theta + j\sin\theta)$$

$$\xleftarrow{\hspace{4em}}$$

$$x = |z|\cos\theta$$
$$y = |z|\sin\theta$$

  - Polar Forms

$$z = |z|e^{j\theta}$$

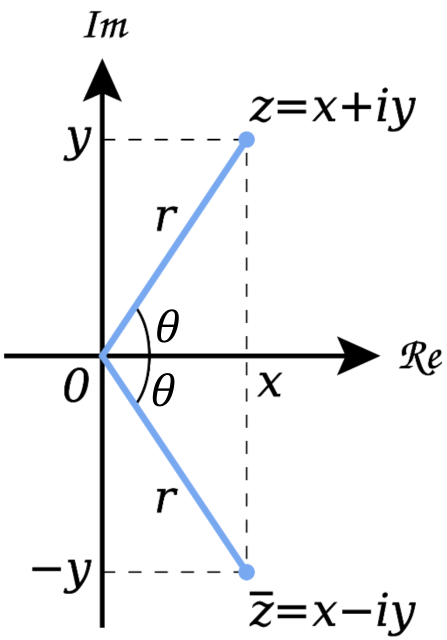
$$z = |z|\angle\theta$$

Complex Numbers, Variables and Functions

- Complex numbers
  - Complex Conjugate

$$z = x + jy = |z|(\cos\theta + j\sin\theta) = |z|e^{j\theta} = |z|\angle\theta$$

$$\bar{z} = x - jy = |z|(\cos\theta - j\sin\theta) = |z|e^{-j\theta} = |z|\angle -\theta$$



  - Complex Algebra

$$z_1 = x_1 + jy_1 \qquad\qquad z_2 = x_2 + jy_2$$

- Equality

$$z_1 = z_2 \quad \text{if and only if} \quad x_1 = x_2 \quad \text{and} \quad y_1 = y_2$$

- Addition

$$z_1 + z_2 = (x_1 + jy_1) + (x_2 + jy_2) = (x_1 + x_2) + j(y_1 + y_2)$$

- Subtraction

$$z_1 - z_2 = (x_1 + jy_1) - (x_2 + jy_2) = (x_1 - x_2) + j(y_1 - y_2)$$

Complex Numbers, Variables and Functions

- Complex numbers
  - Complex Algebra

$$z = x + jy = |z|(\cos\theta + j\sin\theta) = |z|e^{j\theta} = |z|\angle\theta$$

$$z_1 = x_1 + jy_1 = |z_1|(\cos\theta_1 + j\sin\theta_1) = |z_1|e^{j\theta_1} = |z_1|\angle\theta_1$$

$$z_2 = x_2 + jy_2 = |z_2|(\cos\theta_2 + j\sin\theta_2) = |z_2|e^{j\theta_2} = |z_2|\angle\theta_2$$

- Multiplication
  - $kz = k(x + jy) = kx + jky \qquad k\text{: } real\ number$
  - $jz = j(x + jy) = jx + j^2y = -y + jx = |z|\underline{\angle\theta + 90^o}$
  - $z_1z_2 = (x_1 + jy_1)(x_2 + jy_2) = x_1x_2 + jx_1y_2 + jy_1x_2 + j^2y_1y_2$
    $= (x_1x_2 - y_1y_2) + j(x_1y_2 + y_1x_2)$
  - $z^2 = (x + jy)^2 = (x^2 - y^2) + j2xy$
  - $z\bar{z} = (x + jy)(x - jy) = x^2 + y^2 = |z|^2$
  - $z_1z_2 = |z_1||z_2|(\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)) = |z_1||z_2|e^{j(\theta_1 + \theta_2)}$
    $= |z_1||z_2|\underline{\angle\theta_1 + \theta_2}$

Complex Numbers, Variables and Functions

- Complex numbers

$$z = x + jy = |z|(\cos\theta + j\sin\theta) = |z|e^{j\theta} = |z|\angle\theta$$

  - <u>Complex Algebra</u>

$$z_1 = x_1 + jy_1 = |z_1|(\cos\theta_1 + j\sin\theta_1) = |z_1|e^{j\theta_1} = |z_1|\angle\theta_1$$

$$z_2 = x_2 + jy_2 = |z_2|(\cos\theta_2 + j\sin\theta_2) = |z_2|e^{j\theta_2} = |z_2|\angle\theta_2$$

    - <u>Division</u>

➢ $$\frac{z_1}{z_2} = \frac{x_1 + jy_1}{x_2 + jy_2} = \frac{(x_1 + jy_1)(x_2 - jy_2)}{(x_2 + jy_2)(x_2 - jy_2)} = \ \dots = \frac{(x_1x_2 + y_1y_2) + j(y_1x_2 - x_1y_2)}{x_2^2 \ + \ y_2^2}$$

➢ $$\frac{z}{j} = \frac{x + jy}{j} = ? = ? = y - jx = |z|\underline{\angle\theta - 90^o}$$

➢ $$\frac{z_1}{z_2} = \frac{|z_1|\angle\theta_1}{|z_2|\angle\theta_2} = \frac{|z_1|}{|z_2|}\underline{\angle\theta_1 - \theta_2} = \frac{|z_1|}{|z_2|}(\cos(\theta_1 - \theta_2) + j\sin(\theta_1 - \theta_2))$$

$$= \frac{|z_1|}{|z_2|}e^{j(\theta_1 - \theta_2)}$$

Complex Numbers, Variables and Functions

- Complex numbers

$$z = x + jy = |z|(\cos\theta + j\sin\theta) = |z|e^{j\theta} = |z|\angle\theta$$

  - Powers and Roots

    - $$z^n = (|z|(\cos\theta + j\sin\theta))^n = (|z|e^{j\theta})^n = (|z|\angle\theta)^n$$
      $$= |z|^n(\cos n\theta + j\sin n\theta) = |z|^n e^{jn\theta} = |z|^n\angle n\theta$$

De Moivre's Theorem

    - $$z^{1/n} = (|z|(\cos\theta + j\sin\theta))^{1/n} = (|z|e^{j\theta})^{1/n} = (|z|\angle\theta)^{1/n}$$
      $$= |z|^{1/n}(\cos\theta/n + j\sin\theta/n) = |z|^{1/n}e^{j\theta/n} = |z|^{1/n}\angle\theta/n$$

  - Some Properties

    - $|z_1 z_2| = |z_1|\,|z_2|$
    - $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$
    - $|z_1 + z_2| \neq |z_1| + |z_2|$

Complex Numbers, Variables and Functions

- Complex variable

$$s = \sigma + j\omega$$

$$\left.\begin{aligned} Re(s) &= \sigma \\ Im(s) &= \omega \end{aligned}\right\} \mathit{Both\ real\ quantities. At\ leat\ one\ is\ a\ variable}$$

- Complex function

$$F(s) = F_x + jF_y \qquad \bar{F}(s) = F_x - jF_y \qquad |F(s)| = \sqrt{F_x^2 + F_y^2}$$

$$\left.\begin{aligned} Re(F(s)) &= F_x \\ Im(F(s)) &= F_y \end{aligned}\right\} \mathit{Both\ real\ quantities} \qquad \theta = \tan^{-1}\left(\frac{F_y}{F_x}\right)$$

- Typically, such functions have the form:

$$F(s) = K\frac{a_m s^m + a_{m-1}s^{m-1} + \cdots + a_o}{b_n s^n + b_{n-1}s^{n-1} + \cdots + b_o} = K\frac{(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

$$F(s) = 0 \;\rightarrow\; s = -z_1, -z_2, \ldots, -z_m \;\; (zeros).\ Some\ may\ be\ at\ infinity\ (\infty)$$

$$F(s) = \infty \;\rightarrow\; s = -p_1, -p_2, \ldots, -p_n \;\; (poles, simple)$$

$$F(s) = \infty; (s+p)^k \;\rightarrow\; s = -p\ (multiple/repeated\ pole)$$

Complex Numbers, Variables and Functions

- Example 1

Find the zeros and poles of the complex functions:

$$(a) \qquad F(s) = K\frac{(s+2)(s+10)}{s(s+1)(s+5)(s+15)^2}$$

$$(b) \qquad F(s) = \frac{7(s-4)}{s^2\,(s^2-6s-27)}$$

$$(c) \qquad F(s) = \frac{1}{(s^2+1)(s^2+4s+20)}$$

Laplace Transformation

- The ***Laplace transform*** is an operational method that can be used advantageously in solving LTI ODEs.
- Another advantage is that it converts linear differential equations in t domain into algebraic equations in s domain.
- In solving the differential equation, the initial conditions are automatically taken care of, and both the particular solution and the complementary solution can be obtained simultaneously.
- The Laplace transform $\mathcal{L}[f(t)]$ of a function $f(t)$, real function of time, is defined as follows:

$$f(t) = \begin{cases} ?, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\mathcal{L}[f(t)] = \lim_{T\to\infty}\left[\int_0^T f(t)e^{-st}dt\right]$$

$s: complex\ variable$

$$\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt = F(s)$$

$\mathcal{L}:\ Laplace\ transform\ operator$

$\mathcal{L}^{-1}:\ inverse\ Laplace\ transform\ operator$

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

Laplace Transformation

- For some relatively simple functions either the Laplace transform does not exist (such as for e^{t^2} and $1/t$), or it cannot be represented as an algebraic expression.

- The Linearity Property

If $\mathcal{L}[f_1(t)]$ and $\mathcal{L}[f_2(t)]$ exist, then

$$\mathcal{L}[f_1(t) + f_2(t)] = \mathcal{L}[f_1(t)] + \mathcal{L}[f_2(t)]$$

- Also,

$$\mathcal{L}^{-1}\,[F_1(s) + F_2(s)] = \mathcal{L}^{-1}[f_1(t)] + \mathcal{L}^{-1}[f_2(t)]$$

- Transforms of Common Functions

<u>Constant</u>

$$f(t) = \begin{cases} A, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

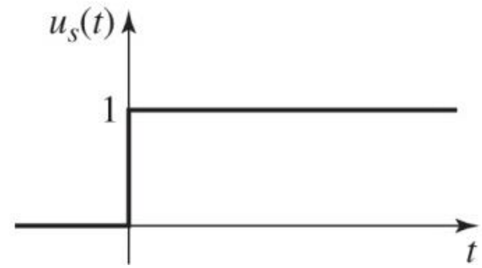
$$\mathcal{L}[A] = \int_0^{\infty} Ae^{-st}dt = A\int_0^{\infty} e^{-st}dt = A\left[-\frac{e^{-st}}{s}\right]_0^{\infty} = A\left[0 + \frac{1}{s}\right]$$

$$\boxed{\mathcal{L}[A] = \frac{A}{s}}$$

$$\mathcal{L}[A] = \lim_{T\to\infty}\left[A\int_0^{T} e^{-st}dt\right] = A\lim_{T\to\infty}\left\{\left[-\frac{e^{-st}}{s}\right]_0^{T}\right\} = A\lim_{T\to\infty}\left\{-\frac{e^{-sT}}{s} + \frac{1}{s}\right\}$$

Laplace Transformation

- Transforms of Common Functions



Step Function

$$f(t) = \begin{cases} A, & t > 0 \\ 0, & t < 0 \end{cases}$$

$$\boxed{\mathcal{L}[A] = \frac{A}{s}}$$

Unit-step Function

$$f(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \quad \text{Or} \quad u_s(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

The **step function**, a discontinuous function, models an input that rapidly reaches a constant value.

$$\boxed{\mathcal{L}[u_s(t)] = \frac{1}{s}}$$

$$f(t) = Au_s(t)$$

*The **unit step function**, or the **Heaviside step function**, usually denoted by H or θ (but sometimes u, 1 or 𝟙).*

Exponential Function

$$f(t) = \begin{cases} Ae^{-\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\mathcal{L}[Ae^{-\alpha t}] = \int_0^\infty Ae^{-\alpha t}e^{-st}dt = \int_0^\infty Ae^{-(s+\alpha)t}dt$$

$$\boxed{\mathcal{L}[Ae^{-\alpha t}] = \frac{A}{s+\alpha}}$$

$\alpha > 0$

$$= A\left[-\frac{e^{-(s+\alpha)t}}{s+\alpha}\right]_0^\infty$$

$$f(t) = \begin{cases} Ae^{\alpha t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\boxed{\mathcal{L}[Ae^{\alpha t}] = \frac{A}{s-\alpha}}$$

$$= A\left[0 + \frac{1}{s+\alpha}\right]$$

Laplace Transformation

- Transforms of Common Functions

Ramp Function

$$f(t) = \begin{cases} At, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\boxed{\mathcal{L}[At] = \frac{A}{s^2}}$$

$$\mathcal{L}[At] = A\int_0^\infty te^{-st}dt = A\left\{\left[-t\frac{e^{-st}}{s}\right]_0^\infty - \int_0^\infty -\frac{e^{-st}}{s}dt\right\}$$

An input that changes at a constant rate is modeled by the ***ramp function***.

$$= A\left[0 + \frac{1}{s}\int_0^\infty e^{-st}dt\right]$$

$$= \frac{A}{s}\left[\left[0 + \frac{1}{s}\right]\right]$$

Sinusoidal Function

$$f(t) = \begin{cases} A\sin\omega t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\boxed{\mathcal{L}[A\sin\omega t] = \frac{A\omega}{s^2 + \omega^2}}$$

$$\mathcal{L}[A\sin\omega t] = A\int_0^\infty \sin\omega t e^{-st}dt$$

$$= \frac{A}{2j}\int_0^\infty (e^{j\omega t} - e^{-j\omega t})e^{-st}dt$$

$$= \frac{A}{2j}\left\{\left[-\frac{e^{-(s-j\omega)t}}{s-j\omega}\right]_0^\infty - \left[-\frac{e^{-(s+j\omega)t}}{s+j\omega}\right]_0^\infty\right\}$$

$$= \frac{A}{2j}\left\{\left[\frac{1}{s-j\omega}\right] - \left[\frac{1}{s+j\omega}\right]\right\}$$

$$f(t) = \begin{cases} A\cos\omega t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$\boxed{\mathcal{L}[A\cos\omega t] = \frac{As}{s^2 + \omega^2}}$$

Laplace Transformation

- Table of Laplace Transform Pairs

	$X(s)$	$x(t), t \geq 0$
1.	1	$\delta(t)$, unit impulse
2.	$\frac{1}{s}$	$u_s(t)$, unit step
3.	$\frac{c}{s}$	constant, c
4.	$\frac{e^{-sD}}{s}$	$u_s(t-D)$, shifted unit step
5.	$\frac{n!}{s^{n+1}}$	t^n
6.	$\frac{1}{s+a}$	e^{-at}
7.	$\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!}t^{n-1}e^{-at}$
8.	$\frac{b}{s^2+b^2}$	$\sin bt$
9.	$\frac{s}{s^2+b^2}$	$\cos bt$
10.	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
11.	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
12.	$\frac{a}{s(s+a)}$	$1-e^{-at}$
13.	$\frac{1}{(s+a)(s+b)}$	$\frac{1}{b-a}(e^{-at}-e^{-bt})$
14.	$\frac{s+p}{(s+a)(s+b)}$	$\frac{1}{b-a}[(p-a)e^{-at}-(p-b)e^{-bt}]$
15.	$\frac{1}{(s+a)(s+b)(s+c)}$	$\frac{e^{-at}}{(b-a)(c-a)}+\frac{e^{-bt}}{(c-b)(a-b)}+\frac{e^{-ct}}{(a-c)(b-c)}$
16.	$\frac{s+p}{(s+a)(s+b)(s+c)}$	$\frac{(p-a)e^{-at}}{(b-a)(c-a)}+\frac{(p-b)e^{-bt}}{(c-b)(a-b)}+\frac{(p-c)e^{-ct}}{(a-c)(b-c)}$
17.	$\frac{b}{s^2-b^2}$	$\sinh bt$
18.	$\frac{s}{s^2+b^2}$	$\cosh bt$ $\rightarrow$ $\frac{s}{s^2-b^2}$
19.	$\frac{a^2}{s^2(s+a)}$	$at-1+e^{-at}$
20.	$\frac{a^2}{s(s+a)^2}$	$1-(at+1)e^{-at}$
21.	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\omega_n\sqrt{1-\zeta^2}t$
22.	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$	$-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t-\phi\right), \phi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$
23.	$\frac{\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right)$
24.	$\frac{1}{s[(s+a)^2+b^2]}$	$\frac{1}{a^2+b^2}\left[1-\left(\frac{a}{b}\sin bt+\cos bt\right)e^{-at}\right], \phi=\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$
25.	$\frac{b^2}{s(s^2+b^2)}$	$1-\cos bt$
26.	$\frac{b^3}{s^2(s^2+b^2)}$	$bt-\sin bt$
27.	$\frac{2b^3}{(s^2+b^2)^2}$	$\sin bt-bt\cos bt$
28.	$\frac{2bs}{(s^2+b^2)^2}$	$t\sin bt$
29.	$\frac{s^2-b^2}{(s^2+b^2)^2}$	$t\cos bt$
30.	$\frac{s}{\left(s^2+b_1^2\right)\left(s^2+b_2^2\right)}$	$\frac{1}{b_2^2-b_1^2}(\cos b_1t-\cos b_2t), \quad \left(b_1^2\neq b_2^2\right)$
31.	$\frac{s^2}{(s^2+b^2)^2}$	$\frac{1}{2b}(\sin bt+bt\cos bt)$

Laplace Transformation

Example 2

Obtain the Laplace transform of the following functions:

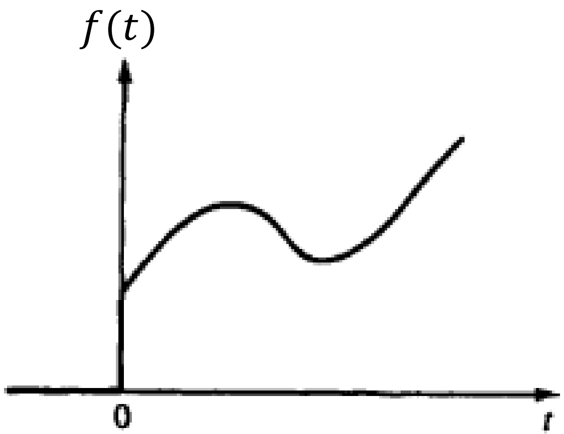
$$(a) \quad f(t) = \begin{cases} 2t^4, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

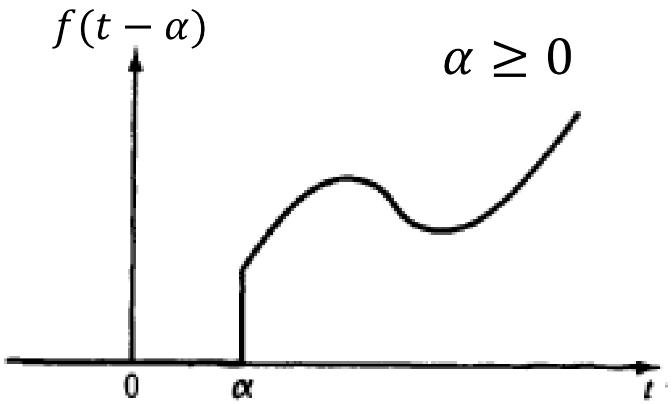
$$(b) \quad f(t) = \begin{cases} \frac{5}{3}e^{2t} - 7\sin 5t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$(c) \quad f(t) = \begin{cases} 8t - e^{-t} + 3\cosh 2t, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Laplace Transformation

- Properties of the Laplace Transform
  - Time Shifted/Translated Function





$$f(t) = \begin{cases} g(t)u_s(t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

$$f(t-\alpha) = \begin{cases} g(t-\alpha)u_s(t-\alpha), & t \geq \alpha \\ 0, & t < \alpha \end{cases}$$

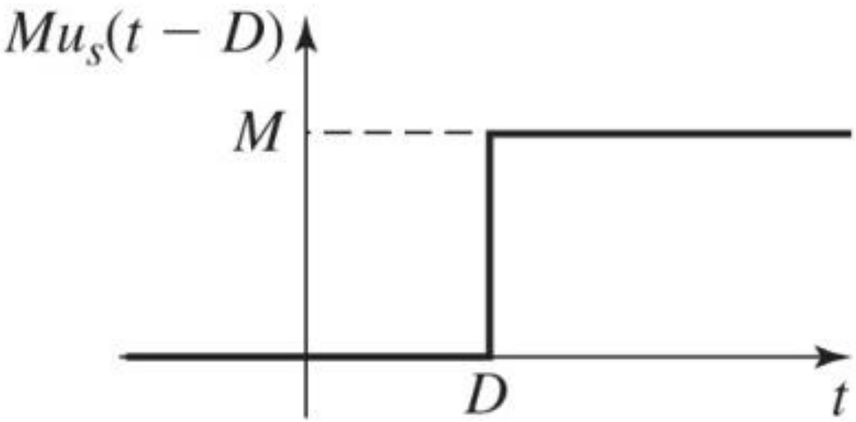
$$\mathcal{L}[f(t-\alpha)] = \mathcal{L}[g(t-\alpha)u_s(t-\alpha)] = \int_0^{\infty} g(t-\alpha)u_s(t-\alpha)e^{-st}dt$$

$$\boxed{\mathcal{L}[f(t-\alpha)] = \mathcal{L}[g(t-\alpha)] = \mathcal{L}[g(t-\alpha)u_s(t-\alpha)] = e^{-\alpha s}G(s)}$$

- Translation of $g(t)$ by $\alpha \rightarrow$ Multiplication of $G(s)$ by $e^{-\alpha s}$

Laplace Transformation

- Properties of the Laplace Transform
  - Time Shifted/Translated Function
    - Shifting Step Function



$$f(t) = \begin{cases} Mu_s\,(t-D), & t > D \\ 0, & t < D \end{cases}$$

$$\boxed{\mathcal{L}[Mu_s\,(t-D)] = \frac{Ae^{-Ds}}{s}}$$

- Shifting Unit-step Function

$$\boxed{\mathcal{L}[u_s(t-D)] = \frac{e^{-Ds}}{s}}$$

Laplace Transformation

- Properties of Laplace Transforms
  - <u>Multiplication by an Exponential</u>

$$\mathcal{L}[e^{-\alpha t}f(t)] = \int_0^{\infty} e^{-\alpha t}f(t)e^{-st}dt = \int_0^{\infty} f(t)e^{-(s+\alpha)t}dt$$

$$\boxed{\mathcal{L}[e^{-\alpha t}f(t)] = F(s + \alpha)}$$

Shifting along the s-axis

<u>Illustration</u>

Derive the Laplace transform of the function $te^{-\alpha t}$

$$\mathcal{L}[te^{-\alpha t}] = \left.\frac{1}{s^2}\right|_{s \to s+\alpha} = \frac{1}{(s+\alpha)^2}$$

  - <u>Multiplication by t</u>

$$\mathcal{L}[tf(t)] = \int_0^{\infty} tf(t)e^{-st}dt$$

$$\boxed{\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)}$$

<u>Illustration</u>

Derive the Laplace transform of the function $t\cos\omega t$

$$\mathcal{L}[t\cos\omega t] = -\frac{d}{ds}\left(\frac{s}{s^2+\omega^2}\right) = \frac{s^2-\omega^2}{(s^2+\omega^2)^2}$$

$$\boxed{\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s)}$$

Laplace Transformation

- **Properties of Laplace Transforms**
  - Comments on the lower limit of the Laplace integral
    - In some cases, $f(t)$ possesses a discontinuity in the function at $t = 0$.
    - This phenomenon occurs in models having impulse inputs and in models containing derivatives of a discontinuous input, such as a step function.
    - Then the lower limit of the Laplace integral must be clearly specified as to whether it is $0-$ or $0+$; The Laplace transforms of $f(t)$ will differ for these two lower limits.

$$\mathcal{L}_+[f(t)] \neq \mathcal{L}_-[f(t)], \qquad \text{since} \quad \int_{0-}^{0+} f(t)e^{-st}dt \neq 0$$

    - If such a distinction of the lower limit of the Laplace integral is necessary, we use the notations

$$\mathcal{L}_+[f(t)] = \int_{0+}^{\infty} f(t)e^{-st}dt \qquad \text{and} \qquad \mathcal{L}_-[f(t)] = \int_{0-}^{\infty} f(t)e^{-st}dt$$

$$= \int_{0-}^{0+} f(t)e^{-st}dt + \mathcal{L}_+\,[f(t)]$$

Laplace Transformation

- Properties of Laplace Transforms

  - <u>Differentiation theorem</u>

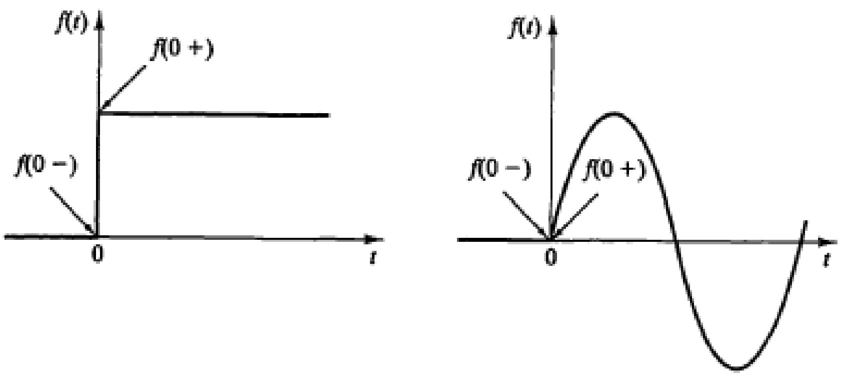
    - To solve differentiation equations, we will need to obtain the transforms of derivatives.

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = \int_0^\infty \frac{df(t)}{dt}e^{-st}dt$$

$$\boxed{\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0)}$$

$$\boxed{\mathcal{L}_{\pm}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0\pm)}$$

    - For a given function $f(t)$, the values of $f(0+)$ and $f(0-)$ may be the same or different. $f(t)$ may have a discontinuity at $t = 0$, and in such a case, $df(t)/dt$ will involve an impulse function at $t = 0$; $f(0+) \neq f(0-)$.



<u>Step function and sine function indicating initial values at $t = 0-$ and $t = 0+$</u>

$$\boxed{\mathcal{L}_{\pm}\left[\frac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)}$$

$$\boxed{\mathcal{L}_{\pm}\left[\frac{d^n}{dt^n}f(t)\right] = s^nF(s) - s^{n-1}f(0\pm) - s^{n-2}\dot{f}(0\pm) - \cdots - f^{n-1}(0\pm)}$$

Laplace Transformation

Example 3

Derive the $X(s)$ from the differential equations:

$$(a) \quad 5\dot{x} - 7x = e^{2t}, \qquad x(0) = 2$$

$$(b) \quad \ddot{x} - 4\dot{x} + x = t, \qquad x(0) = 1, \qquad \dot{x}(0) = 3$$

Laplace Transformation

- Properties of Laplace Transforms
  - Final Value Theorem
    - To find the limit of the function $f(t)$ as $t \to \infty$, we can use the ***final value theorem***. The theorem states that

$$\boxed{f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0}[sF(s)]}$$

    - The theorem applies if and only if the limit exists.
    - The final-value theorem relates the *steady-state* behavior of $f(t)$ to the behavior of $sF(s)$ in the neighborhood of $s = 0$.
    - The function $f(t)$ will approach a constant value if all the roots of the denominator of $sF(s)$ have negative real parts.
    - If all poles of $sF(s)$ lie in the left half s-plane, then $\lim_{t\to\infty} f(t)$ exists, but if $sF(s)$ has poles on the imaginary axis or in the right half s plane, $f(t)$ will contain oscillating or exponentially increasing time functions, respectively, and $\lim_{t\to\infty} f(t)$) will not exist. The final-value theorem does not apply to such cases.

Laplace Transformation

- Properties of Laplace Transforms

  - Initial Value Theorem

    - Sometimes we will need to find the value of the function $f(t)$ at $t = 0+$ (a time infinitesimally greater than 0), given the transform $F(s)$.

    - The answer can be obtained with the ***initial value theorem***, which states that

$$\boxed{f(0+) = \lim_{t\to 0+} f(t) = \lim_{s\to\infty}[sF(s)]}$$

$s \to \infty$ is taken along the real axis

    - The conditions for which the theorem is valid are that the latter limit exists and that the transforms of $f(t)$ and df/dt exist.

    - If $F(s)$ is a rational function and if the degree of the numerator of $F(s)$ is less than the degree of the denominator, then the theorem will give a finite value for $f(0+)$. If the degrees are equal, then the initial value is undefined, and the initial value theorem is invalid.

    - In applying the initial-value theorem, we are not limited as to the locations of the poles of $sF(s)$. Thus, the theorem is valid for the sinusoidal function.

$$\boxed{\dot{f}(0+) = \lim_{t\to 0+} \dot{f}(t) = \lim_{s\to\infty}\{s[sF(s) - f(0+)]\}}$$

$$\boxed{\dot{f}(0+) = \lim_{s\to\infty}\{s[sF(s)]\}}$$

$$f(0+) = 0$$

Laplace Transformation

- Properties of Laplace Transforms
  - Final and Initial Value Theorems
    - Note that the initial-value and the final-value theorems provide a convenient check on the solution, since they enable us to predict the system behavior in the time domain without actually transforming functions in s back to time functions.

Example 4

(a) Use the final and initial value theorems to determine $g(\infty)$ and $g(0+)$, respectively, for the transforms:

$$G(s) = \frac{7s + 2}{s(s + 6)}$$

(b) Given the following model, for a unit-step input $f(t) = u_s(t)$, what are the values of $y(0+)$ and $\dot{y}(0+)$?

$$\frac{Y(s)}{F(s)} = \frac{s}{2s^2 + 14s + 20}$$

Laplace Transformation

- Properties of Laplace Transforms

  - <u>Integration Theorem</u>

    - The integration theorem is given as

$$\boxed{\mathcal{L}\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{\left[\int f(t)dt\right]_{t=0}}{s}}$$

$$\boxed{\mathcal{L}\left[\int_0^t f(t)dt\right] = \frac{F(s)}{s}}$$

where $F(s) = \mathcal{L}[f(t)]$

    - Note that, if $f(t)$ involves an impulse function at $t = 0$, then the following must be observed:

$$\boxed{\mathcal{L}_+\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{\left[\int f(t)dt\right]_{t=0+}}{s}}$$

$$\boxed{\mathcal{L}_+\left[\int_{0+}^t f(t)dt\right] = \frac{\mathcal{L}_+[f(t)]}{s}}$$

$$\boxed{\mathcal{L}_-\left[\int f(t)dt\right] = \frac{F(s)}{s} + \frac{\left[\int f(t)dt\right]_{t=0-}}{s}}$$

$$\boxed{\mathcal{L}_-\left[\int_{0-}^t f(t)dt\right] = \frac{\mathcal{L}_-[f(t)]}{s}}$$

Laplace Transformation

- Properties of Laplace Transforms Table

$x(t)$	$X(s) = \int_0^\infty f(t)e^{-st}\,dt$
1. $af(t) + bg(t)$	$aF(s) + bG(s)$
2. $\dfrac{dx}{dt}$	$sX(s) - x(0)$
3. $\dfrac{d^2x}{dt^2}$	$s^2X(s) - sx(0) - \dot{x}(0)$
4. $\dfrac{d^nx}{dt^n}$	$s^nX(s) - \sum_{k=1}^{n} s^{n-k}g_{k-1}$ $g_{k-1} = \left.\dfrac{d^{k-1}x}{dt^{k-1}}\right\vert_{t=0}$
5. $\int_0^t x(t)\,dt$	$\dfrac{X(s)}{s} + \dfrac{g(0)}{s}$ $g(0) = \left.\int x(t)\,dt\right\vert_{t=0}$
6. $x(t) = \begin{cases} 0 & t < D \\ g(t-D) & t \geq D \end{cases}$ $= u_s(t-D)g(t-D)$	$X(s) = e^{-sD}G(s)$
7. $e^{-at}x(t)$	$X(s+a)$
8. $tx(t)$	$-\dfrac{dX(s)}{ds}$
9. $x(\infty) = \lim_{s\to 0} sX(s)$	
10. $x(0+) = \lim_{s\to\infty} sX(s)$	

For Entries 2, 3, 4, and 5, if $x \neq 0$ for $t < 0$, then replace the initial conditions at $t = 0$ with the pre-initial condtions at $0-$.

1	$\mathscr{L}[Af(t)] = AF(s)$
2	$\mathscr{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$
3	$\mathscr{L}_\pm\left[\dfrac{d}{dt}f(t)\right] = sF(s) - f(0\pm)$
4	$\mathscr{L}_\pm\left[\dfrac{d^2}{dt^2}f(t)\right] = s^2F(s) - sf(0\pm) - \dot{f}(0\pm)$
5	$\mathscr{L}_\pm\left[\dfrac{d^n}{dt^n}f(t)\right] = s^nF(s) - \sum_{k=1}^{n} s^{n-k}\overset{(k-1)}{f}(0\pm)$ where $\overset{(k-1)}{f}(t) = \dfrac{d^{k-1}}{dt^{k-1}}f(t)$
6	$\mathscr{L}_\pm\left[\int f(t)\,dt\right] = \dfrac{F(s)}{s} + \dfrac{\left[\int f(t)\,dt\right]_{t=0\pm}}{s}$
7	$\mathscr{L}_\pm\left[\iint f(t)\,dt\,dt\right] = \dfrac{F(s)}{s^2} + \dfrac{\left[\int f(t)\,dt\right]_{t=0\pm}}{s^2} + \dfrac{\left[\iint f(t)\,dt\,dt\right]_{t=0\pm}}{s}$
8	$\mathscr{L}_\pm\left[\int \cdots \int f(t)(dt)^n\right] = \dfrac{F(s)}{s^n} + \sum_{k=1}^{n} \dfrac{1}{s^{n-k+1}}\left[\int \cdots \int f(t)(dt)^k\right]_{t=0\pm}$
9	$\mathscr{L}\left[\int_0^t f(t)\,dt\right] = \dfrac{F(s)}{s}$
10	$\int_0^\infty f(t)\,dt = \lim_{s\to 0} F(s)$ if $\int_0^\infty f(t)\,dt$ exists
11	$\mathscr{L}[e^{-at}f(t)] = F(s+a)$
12	$\mathscr{L}[f(t-\alpha)1(t-\alpha)] = e^{-\alpha s}F(s) \quad \alpha \geq 0$
13	$\mathscr{L}[tf(t)] = -\dfrac{dF(s)}{ds}$
14	$\mathscr{L}[t^2f(t)] = \dfrac{d^2}{ds^2}F(s)$
15	$\mathscr{L}[t^nf(t)] = (-1)^n\dfrac{d^n}{ds^n}F(s) \quad n = 1, 2, 3, \ldots$
16	$\mathscr{L}\left[\dfrac{1}{t}f(t)\right] = \int_s^\infty F(s)\,ds$ if $\lim_{t\to 0}\dfrac{1}{t}f(t)$ exists
17	$\mathscr{L}\left[f\left(\dfrac{t}{a}\right)\right] = aF(as)$

Laplace Transformation

Pulse and Impulse Functions

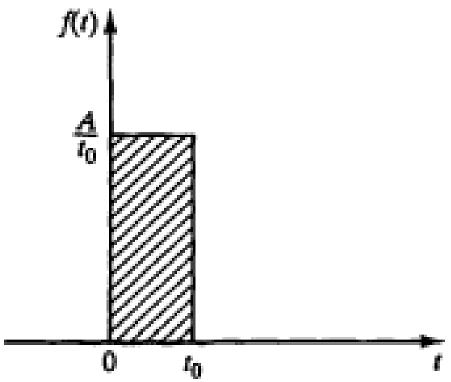
- Consider the ***(rectangular) pulse function*** shown

$$f(t) = \begin{cases} \dfrac{A}{t_o} & 0 < t < t_0 \\ 0 & t < 0, \quad t_0 < t \end{cases}$$

A and t_0 are constants

t_0 is the $\boldsymbol{pulse\ duration}$

*The **rectangular pulse function** models a constant input that is suddenly removed.*



Pulse function

$$The\ \boldsymbol{strength}\ of\ the\ pulse\ =\ \boldsymbol{area}\ under\ the\ pulse\ = A.$$

- The pulse function can be written as

$$f(t) = \frac{A}{t_0} u_s(t) - \frac{A}{t_0} u_s(t - t_0)$$

$$\mathcal{L}[f(t)] = \int_0^\infty f(t) e^{-st} dt$$

$$\mathcal{L}[f(t)] = \mathcal{L}\left[\frac{A}{t_0} u_s(t)\right] - \mathcal{L}\left[\frac{A}{t_0} u_s(t - t_0)\right]$$

$$= \frac{A}{t_0} \int_0^{t_0} e^{-st} dt = \frac{A}{t_0}\left[-\frac{e^{-s\,t}}{s}\right]_0^{t_0}$$

$$= \frac{A}{t_0 s} - \frac{A}{t_0 s} e^{-st_0}$$

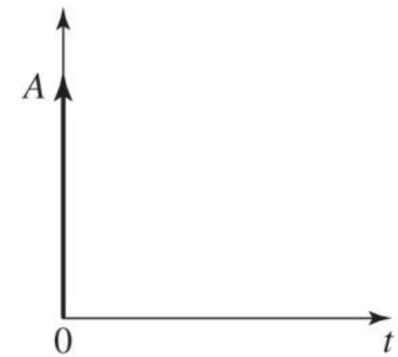
$$\boxed{\mathcal{L}[f(t)] = \frac{A}{t_0 s}\left(1 - e^{-st_0}\right)}$$

Laplace Transformation

- Pulse and Impulse Functions
  - Consider the ***impulse function*** shown

$$f(t) = \begin{cases} \lim\limits_{t_o \to 0} \dfrac{A}{t_o} & 0 < t < t_0 \\ 0 \qquad t < 0, & t_0 < t \end{cases}$$

*The **impulse** is similar to the pulse function, but it models an input that is suddenly applied and removed after a very short time.*



  - It is a special limiting case of the pulse function. It is a limiting case of the pulse function as t_o approaches zero; has an infinite magnitude for an infinitesimal time.
  - It is a mathematical function only and has no physical counterpart; fiction.

$$\mathcal{L}[f(t)] = \lim_{t_o \to 0}\left[\frac{A}{t_0 s}\left(1 - e^{-st_0}\right)\right]$$

L'Hopital's limit rule

$$\boxed{\mathcal{L}[f(t)] = A}$$

$$= \lim_{t_o \to 0}\left[\frac{\dfrac{d}{dt_0}\left[A\left(1 - e^{-st_0}\right)\right]}{\dfrac{d}{dt_0}(t_0 s)}\right] = \lim_{t_o \to 0}\left[\frac{Ase^{-st_0}}{s}\right] = \frac{As}{s} = A$$

Laplace Transformation

▪ Pulse and Impulse Functions

○ ***Unit impulse*** *or* ***Dirac delta function, $\delta(t)$***

- The strength of $\delta(t)$ is unity; $A = 1$.

$$\boxed{\delta(t) = \frac{d}{dt}u_s(t)}$$

- The unit-impulse function occurring at $t\ = t_o$ is usually denoted by $\delta(t\ - t_o)$, which satisfies the following conditions:

$$\delta(t\ - t_o) = 0 \quad for\ t \neq t_0$$

$$\delta(t\ - t_o) = \infty \quad for\ t = t_0$$

$$\int_{-\infty}^{\infty} \delta(t\ - t_o)dt = 1$$

$$\boxed{\delta(t\ - t_o) = \frac{d}{dt}u_s(t\ - t_o)}$$

- The unit-impulse function $\delta(t\ - t_o)$ can be considered the derivative of the unit-step function $1(t\ - t_o)$ at the point of discontinuity $t\ = t_o$.

$$\boxed{\mathcal{L}[\delta(t)] = 1}$$

$$\boxed{\mathcal{L}\left[\frac{d}{dt}\delta(t)\right] = s}$$

$$\boxed{\mathcal{L}\left[\frac{d^2}{dt^2}\delta(t)\right] = s^2}$$

- Impulse function often appears in the analysis of dynamic systems. It is an analytically convenient approximation of an input applied for only a very short time, such as when a high-speed object strikes a stationary object.

- The impulse is also useful for estimating the system's parameters experimentally and for analyzing the effect of differentiating a step or any other discontinuous input function.

Laplace Transformation

Example 5

(a) Given the following model, for a unit-impulse input, $f(t) = \delta(t)$, what is the value of $y(0+)$?

$$\frac{Y(s)}{F(s)} = \frac{1}{s+5}$$

(b) Given the following model, and initial conditions of $x(0) = 5$ and $\dot{x}(0) = 10$, obtain an expression for $X(s)$, hence determine the values of $x(0+)$ and $\dot{x}(0+)$.

$$\ddot{x} = \delta(t)$$