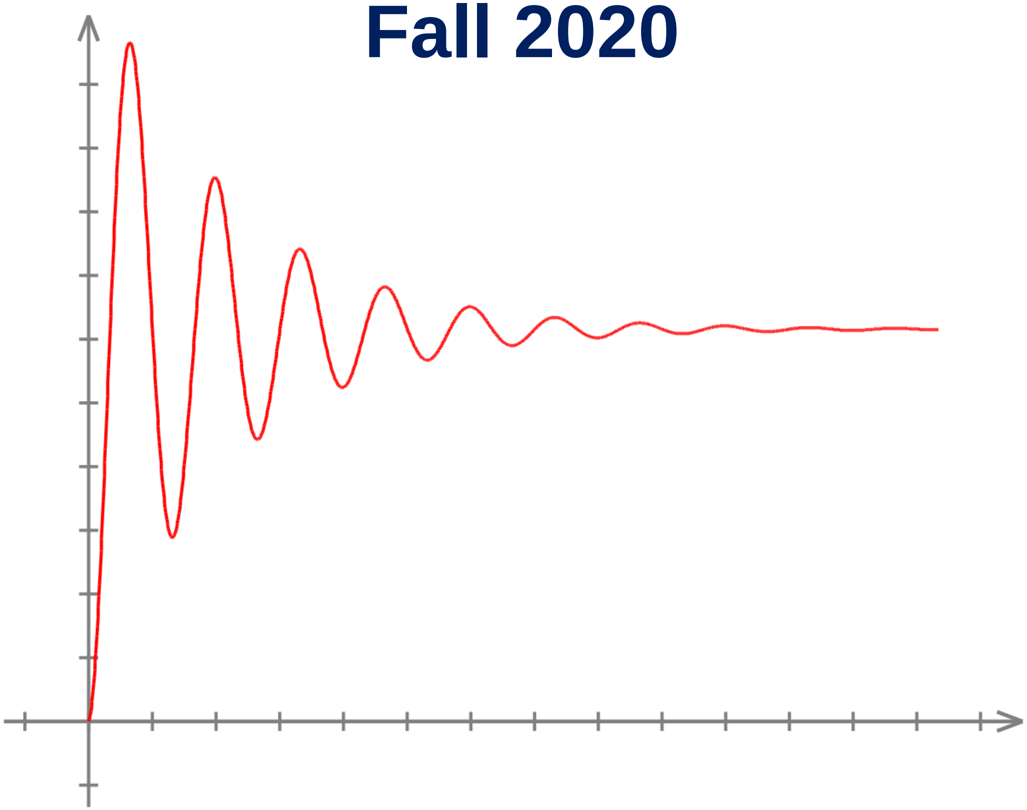
Department of Engineering

ENGR 311: System Dynamics

Fall 2020



Chapter 6

Time Domain Analysis of Dynamic Systems

23/11/2020

Outline

- Introduction

- Natural and forced response

- Transient-response analysis of first-order systems: Analytical and MATLAB

- Transient-response analysis of second-order systems: Analytical and MATLAB

- Transient-response analysis of higher-order systems: Analytical and MATLAB

- Solution of the state equation

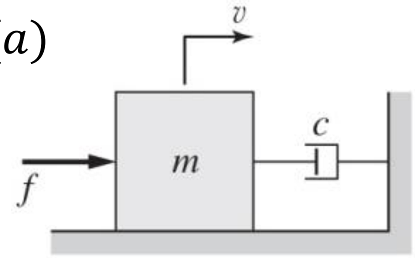
Time Domain Analysis of Dynamic Systems

First-order Systems

$$a_o \frac{dy}{dt} + a_1 y = bf(t)$$

Mechanical System: Translational

(a)

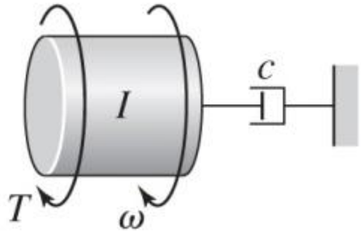


$$m\frac{dv}{dt} + cv = f$$

$$\tau = \frac{m}{c}$$

Mechanical System: Rotational

(b)

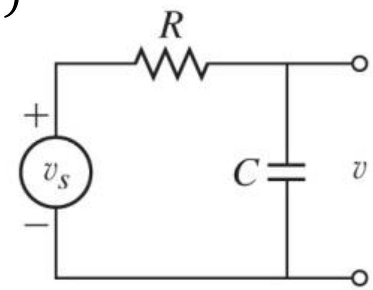


$$I\frac{d\omega}{dt} + c\omega = T$$

$$\tau = \frac{I}{c}$$

Electrical System

(c)

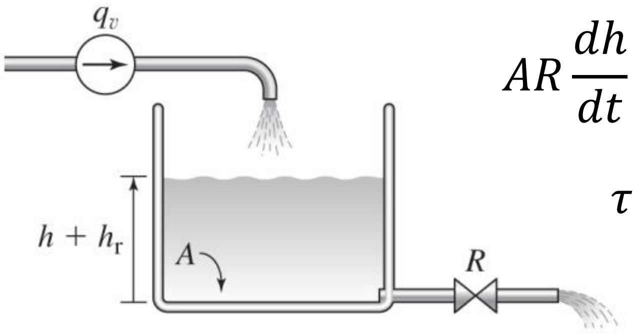


$$RC\frac{dv}{dt} + v = v_s$$

$$\tau = RC$$

Fluid System

(d)

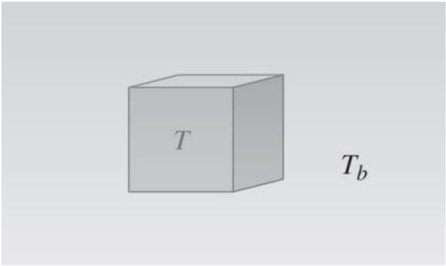


$$AR\frac{dh}{dt} + gh = Rq_v$$

$$\tau = \frac{AR}{g}$$

Thermal System

(e)



$$mc_pR\frac{dT}{dt} + T = T_b$$

$$\tau = mc_pR$$

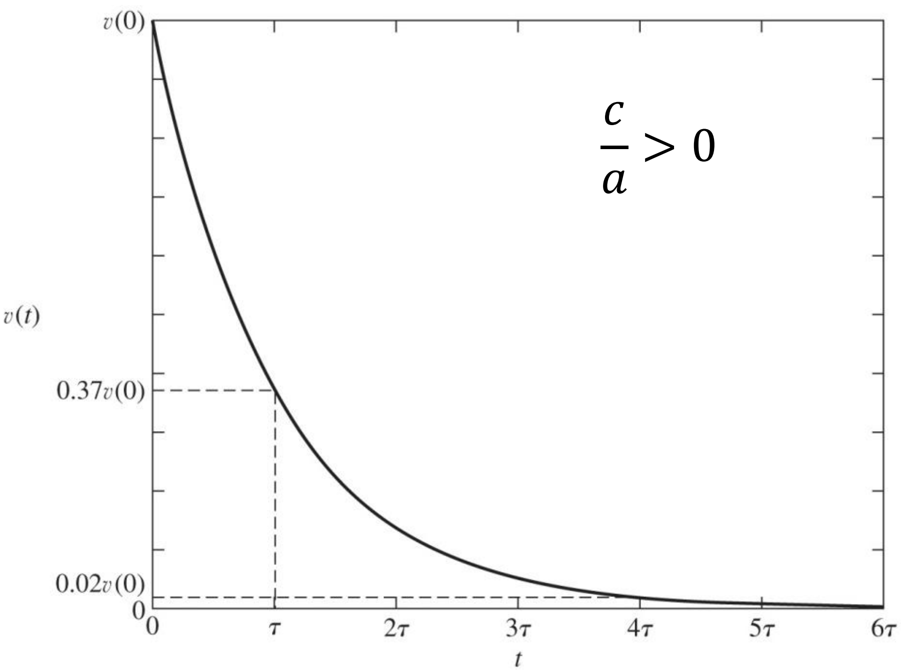
Time Domain Analysis of Dynamic Systems

First-order Systems

$$a\frac{dv}{dt} + cv = f(t)$$

$$f(t) = 0$$

$$v(t) = v_{free}(t) = v(0)e^{-ct/a}$$



$$\frac{c}{a} > 0$$

$\tau = a/c$

$c/a > 0$: stable

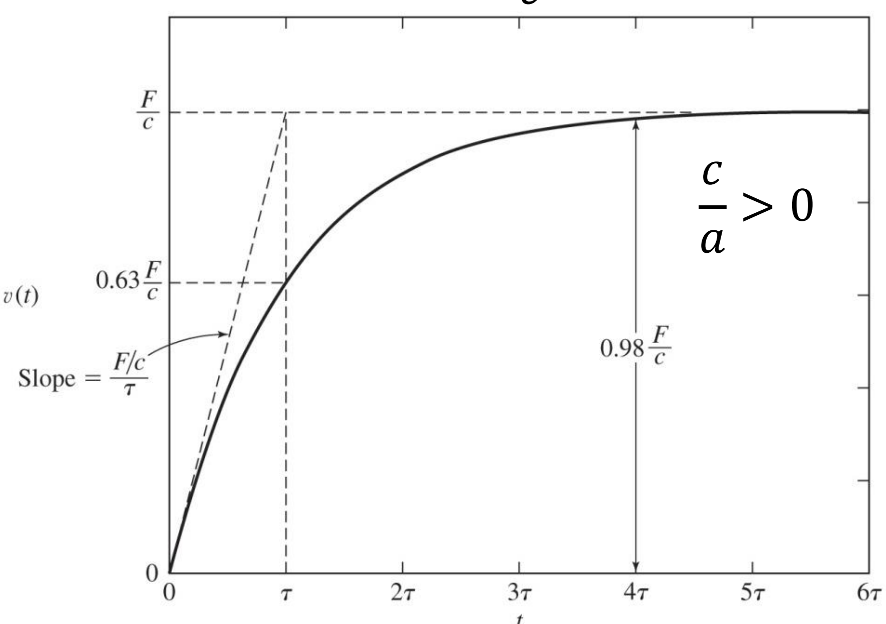
$c/a < 0$: unstable

$c/a = 0$: neutrally stable

$f(t) \neq 0$ (e.g., $f(t)$ is step with magnitude F)

$$v(t) = v_{free}(t) + v_{forced}(t)$$

$$= v(0)e^{-ct/a} + \frac{F}{c}(1 - e^{-ct/a})$$



$$\frac{c}{a} > 0$$

$\tau = a/c$

$K = 1/c$

$$v_{transient}(t) = \left[v(0) - \frac{F}{c}\right]e^{-t/\tau}$$

$$v_{steady-state}(t) = \frac{F}{c}$$

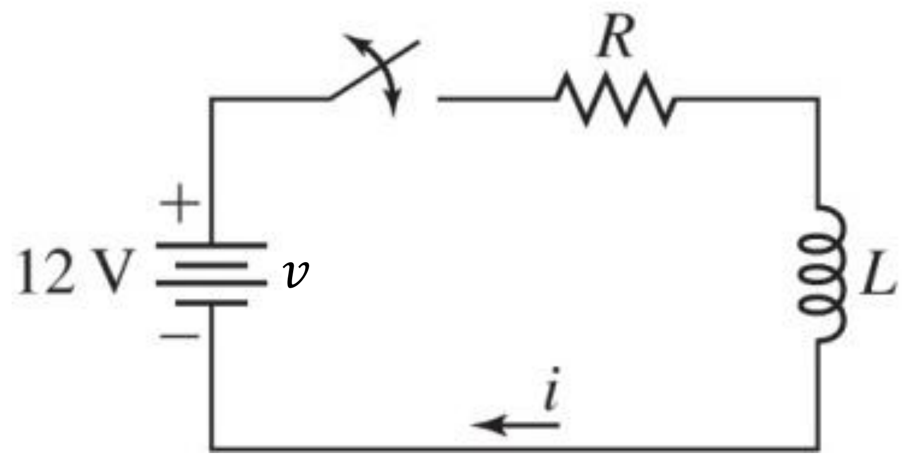
$$v(\infty) = v_{ss} = f_{ss}/c = F/c = FK$$

Time Domain Analysis of Dynamic Systems

Example 1

The Figure shows a circuit representation of a telegraph line. The resistance R is the line resistance and L is the inductance of the solenoid that activates the receiver's clicker. The switch represents the operator's key. $v(t)$ represents the input voltage and $i(t)$ is the current passing through the solenoid. Given the model of the system as

$$L\frac{di}{dt} + Ri = v(t)$$



Circuit representation of a telegraph line

(a) Determine:

i. the time constant

ii. Steady state current

iii. the DC gain

iv. the transfer function $I(s)/V(s)$

v. state-space model

(b) Assume that when sending a "dot," the key is closed for $0.1\ s$. Using the values $R = 20\ \Omega$ and $L = 4\ \mathrm{H}$, obtain the expression for the current $i(t)$ passing through the solenoid. What is the value for $i(0+)$ and $i(\infty)$?

Time Domain Analysis of Dynamic Systems

Example 2

Hardness and other properties of metal can be improved by the rapid cooling that occurs during *quenching*, a process in which a heated object is placed into a liquid bath (see the Figures). Consider a lead cube with a side length of $d = 20\,\mathrm{mm}$. The cube is immersed in an oil bath for which the convection coefficient $h = 200\,\mathrm{W/(m.\,{}^{o}C)}$. The oil temperature is T_b. The density of lead is $1.134 \times 10^4\ \mathrm{kg/m^3}$. Take the specific heat of lead to be $c_p = 129\,\mathrm{J/kg.\,{}^{o}C}$. The model of the cube's temperature as a function of the liquid temperature T_b, which is assumed to be known, is given as:

$$C\frac{dT}{dt} = -\frac{1}{R}(T - T_b), \qquad T > T_b$$

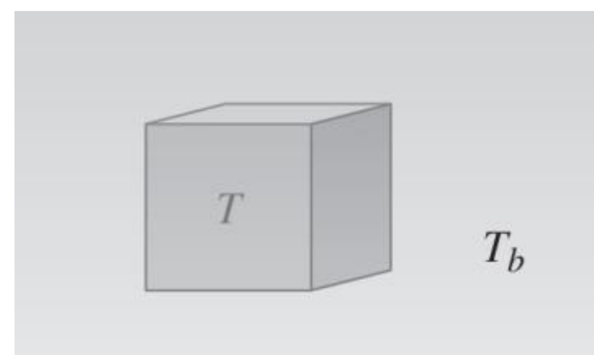
$$Thermal\ capacitance: C = mc_p = \rho V c_p$$

$$Thermal\ resistance: R = \frac{1}{hA}$$

(a) Show that the model can be written as: $24.4\dfrac{dT}{dt} + T = T_b$

(b) Hence, estimate the time the cube's temperature will reach T_b

Quenching Process



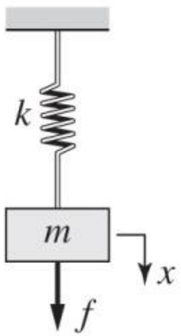
Quenching with a constant bath temperature

Time Domain Analysis of Dynamic Systems

Second-order Systems

Undamped Response: $\zeta = 0 \ (c = 0)$ $\qquad a\ddot{x} + bx = f(t) \quad$ or $\quad \ddot{x} + \omega_n^2 x = \frac{1}{a} f(t)$

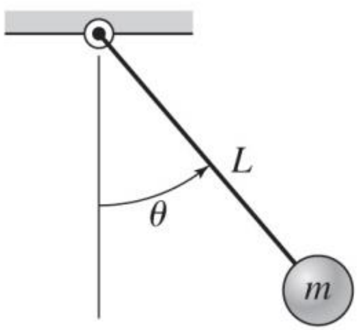
(a)



$$m\frac{d^2x}{dt^2} + kx = f(t)$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

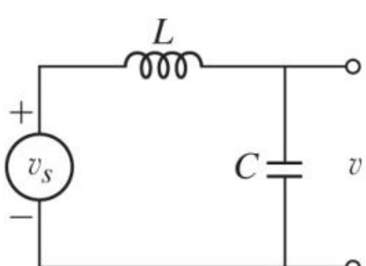
(b)



$$L\frac{d^2\theta}{dt^2} + g\theta = 0$$

$$\omega_n = \sqrt{\frac{g}{L}}$$

(c)



$$LC\frac{d^2v}{dt^2} + v = v_s$$

$$\omega_n = \frac{1}{\sqrt{LC}}$$

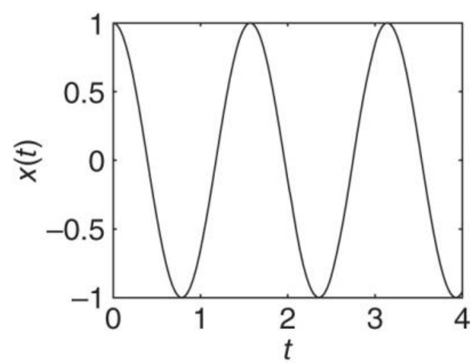
$$\omega_n = \sqrt{\frac{b}{a}} \qquad f = \frac{1}{2\pi}\sqrt{\frac{b}{a}} \qquad T = 2\pi\sqrt{\frac{a}{b}}$$

$$x(t) = \frac{\dot{x}(0)}{\omega_n}\sin\omega_n t + x(0)\cos\omega_n t$$

$$x(t) = A\sin(\omega_n t + \emptyset)$$

$$A = \sqrt{[x(0)]^2 + \left[\frac{\dot{x}(0)}{\omega_n}\right]^2} \qquad \sin\emptyset = \frac{x(0)}{A} \qquad \cos\emptyset = \frac{\dot{x}(0)}{A\omega_n}$$

The amplitude and phase angle depends on initial conditions

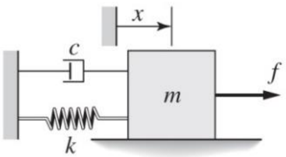


Time Domain Analysis of Dynamic Systems

Second-order Systems

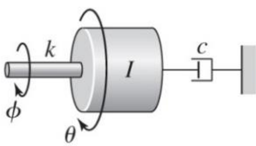
Response with damping: $\zeta \neq 0 \; (c \neq 0)$

(a)



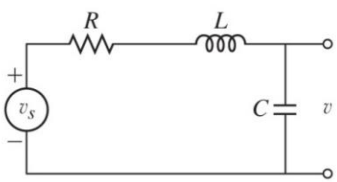
$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f$$

(b)



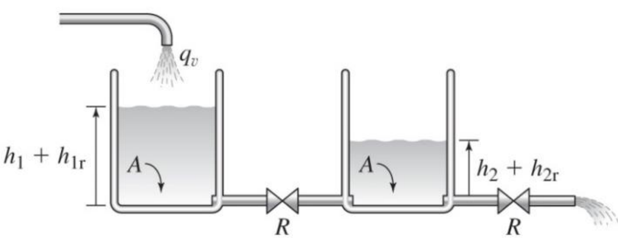
$$I\frac{d^2\theta}{dt^2} + c\frac{d\theta}{dt} + k\theta = k\phi$$

(c)



$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = v_s$$

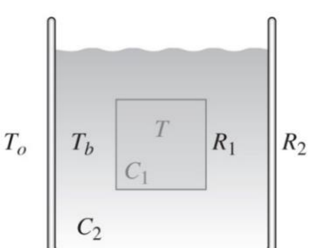
(d)



$$RA\frac{dh_1}{dt} + g(h_1 - h_2) = Rq_v$$

$$RA\frac{dh_2}{dt} + g(h_2 - h_1) + gh_2 = 0$$

(e)



$$R_1C_1\frac{dT}{dt} + T = T_b$$

$$R_1R_2C_2\frac{dT_b}{dt} + (R_1 + R_2)T_b = R_2T + R_1T_o$$

$$a\ddot{x} + c\dot{x} + bx = f(t)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = \frac{1}{a}f(t)$$

$$as^2 + cs + b = 0$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s = \frac{-c \pm \sqrt{c^2 - 4ab}}{2a}$$

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{ab}}$$

ζ: *Damping ratio/factor*

c: *actual damping value*

c_c/c_o: *critical damping value*

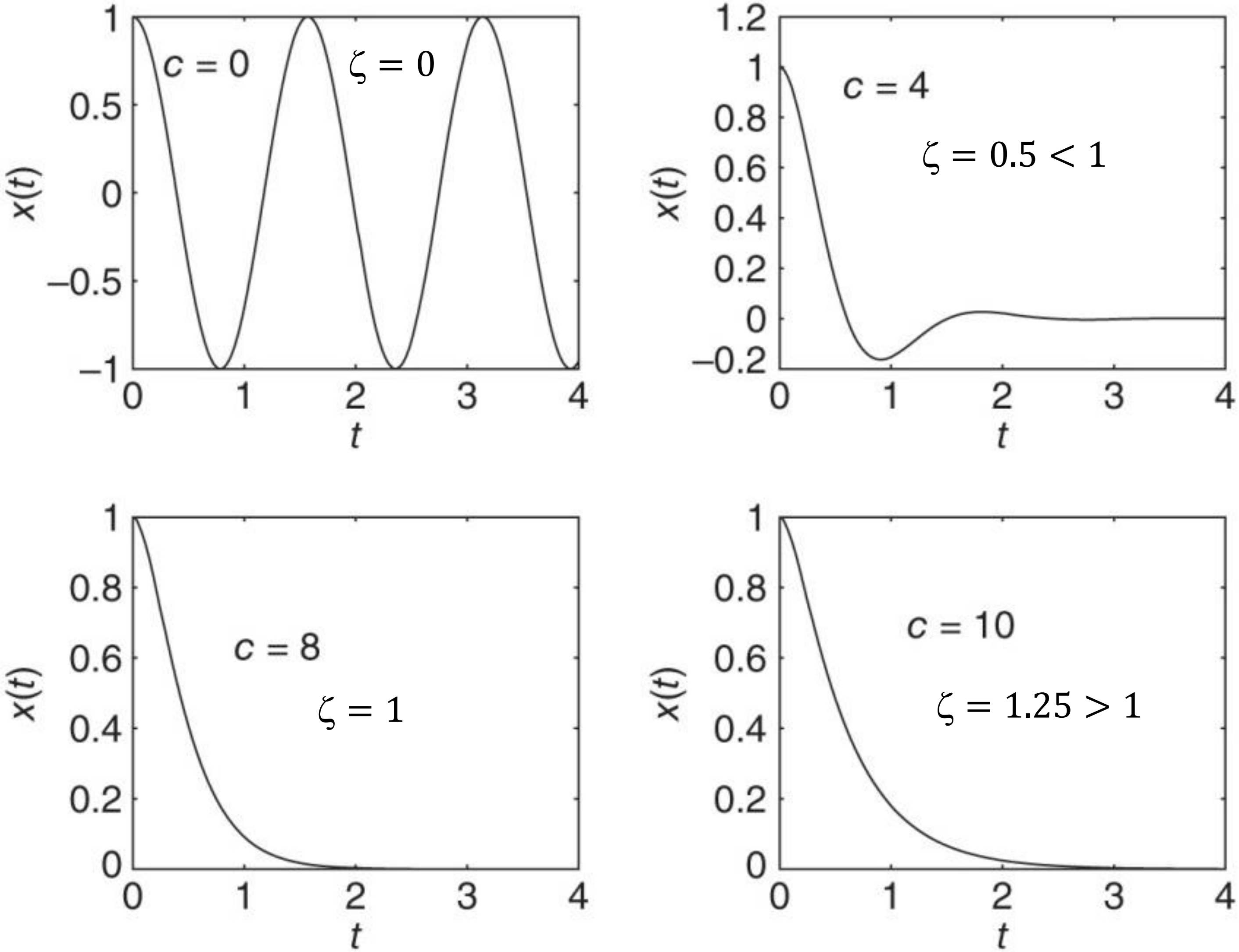
Time Domain Analysis of Dynamic Systems

Second-order Systems

$$s^2 + cs + 16 = 0$$

Response with damping: $\zeta \neq 0$ $(c \neq 0)$

$x(0) = 1$ and $\dot{x}(0) = 0$



Time Domain Analysis of Dynamic Systems

Second-order Systems

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Response with damping: $\zeta \neq 0\ (c \neq 0)$

$$c < c_c$$

$$a\ddot{x} + c\dot{x} + bx = 0$$

Underdamped Response

$$0 < \zeta < 1$$

Initial conditions

$x(0)$ and $\dot{x}(0)$

$$X(s) = \frac{(s + 2\zeta\omega_n)x(0) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

$$s = -\zeta\omega_n \pm \omega_d j$$

$$x(t) = e^{-\zeta\omega_n t}\left\{\left[\frac{\zeta}{\sqrt{1 - \zeta^2}}x(0) + \frac{1}{\omega_d}\dot{x}(0)\right]\sin\omega_d t + x(0)\cos\omega_d t\right\}$$

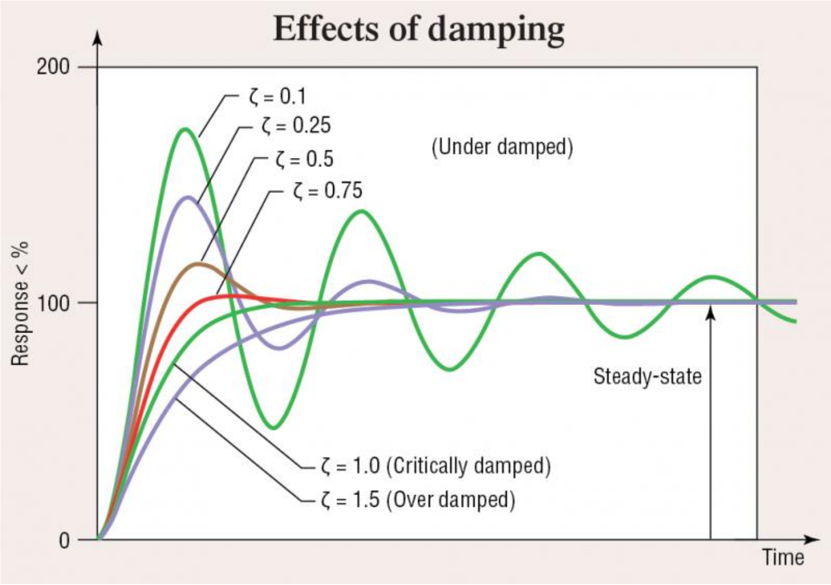
$$s = -\alpha \pm \omega_d j$$

$$x(t) = e^{-\zeta\omega_n t}A\sin(\omega_d t + \emptyset) \qquad 0 \leq \zeta < 1$$

$$A = \frac{1}{\omega_d}\sqrt{[\omega_d x(0)]^2 + [\dot{x}(0) + \zeta\omega_n x(0)]^2}$$

$$\sin\emptyset = \frac{x(0)}{A}$$

$$\text{or} \quad \cos\emptyset = \frac{\dot{x}(0) + \zeta\omega_n x(0)}{A\zeta\omega_n}$$



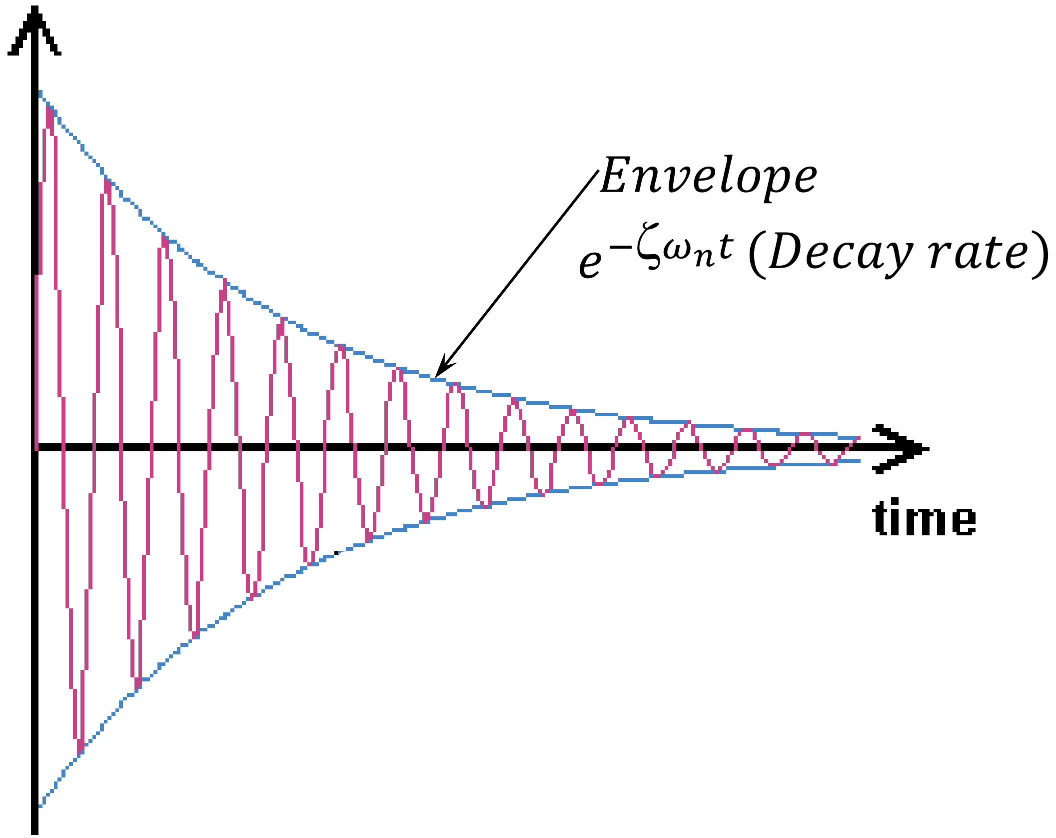
Time Domain Analysis of Dynamic Systems

Second-order Systems

Underdamped Response

$$c < c_c \quad \text{or} \quad \zeta < 1 \qquad\qquad s = -\zeta\omega_n \pm \omega_d j$$

$$x(t) = e^{-\zeta\omega_n t} A\sin(\omega_d t + \emptyset)$$



Time Domain Analysis of Dynamic Systems

Second-order Systems

Effect of Root Location: Graphical Interpretation

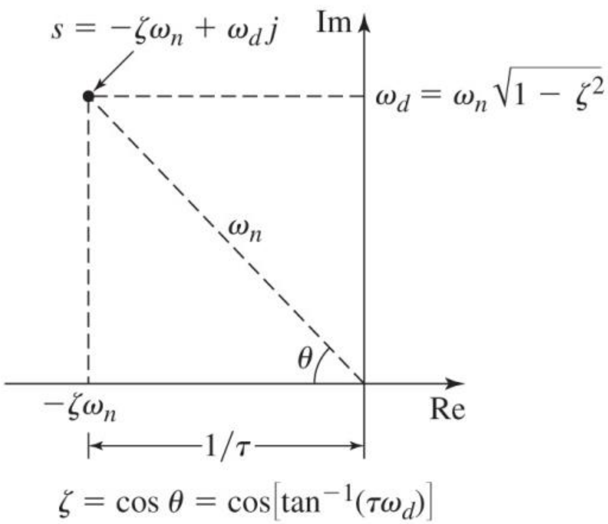
$$as^2 + cs + b = 0$$

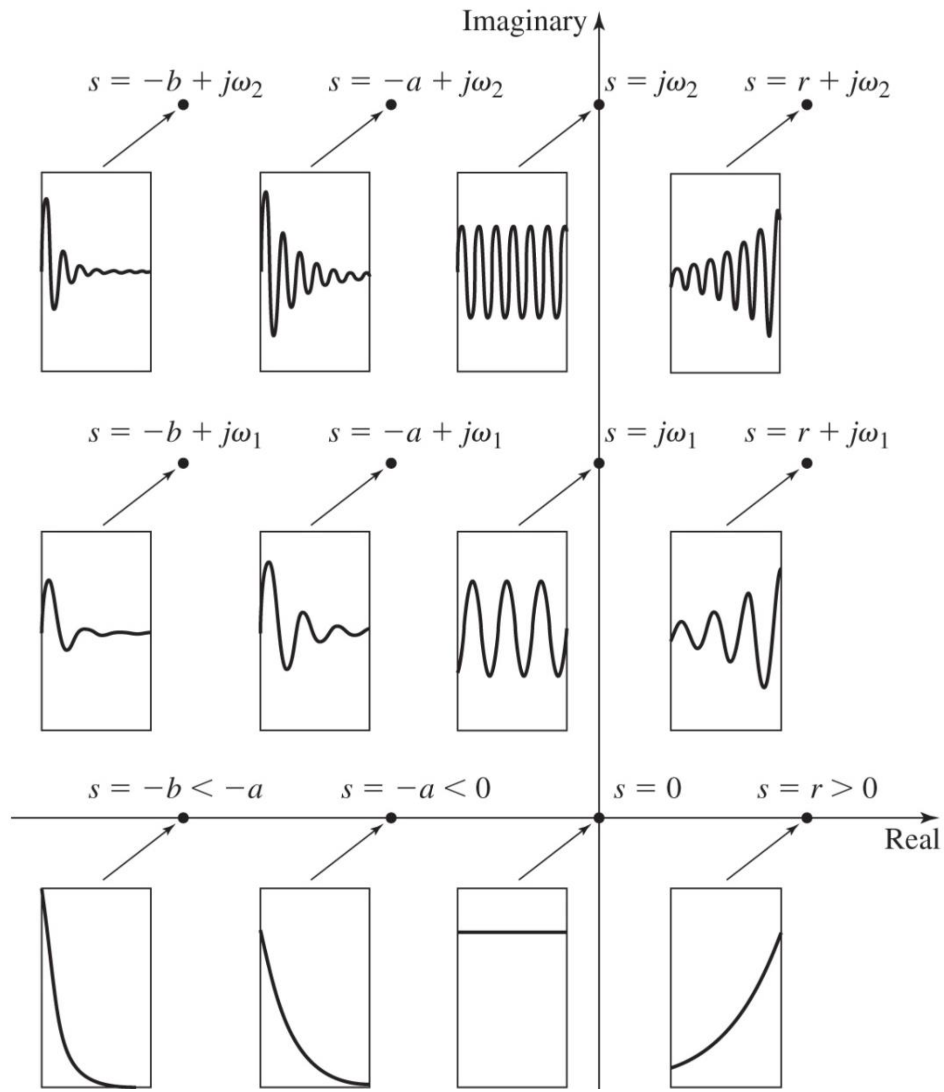
$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s = \frac{-c \pm \sqrt{c^2 - 4ab}}{2a}$$

$$s = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Underdamped

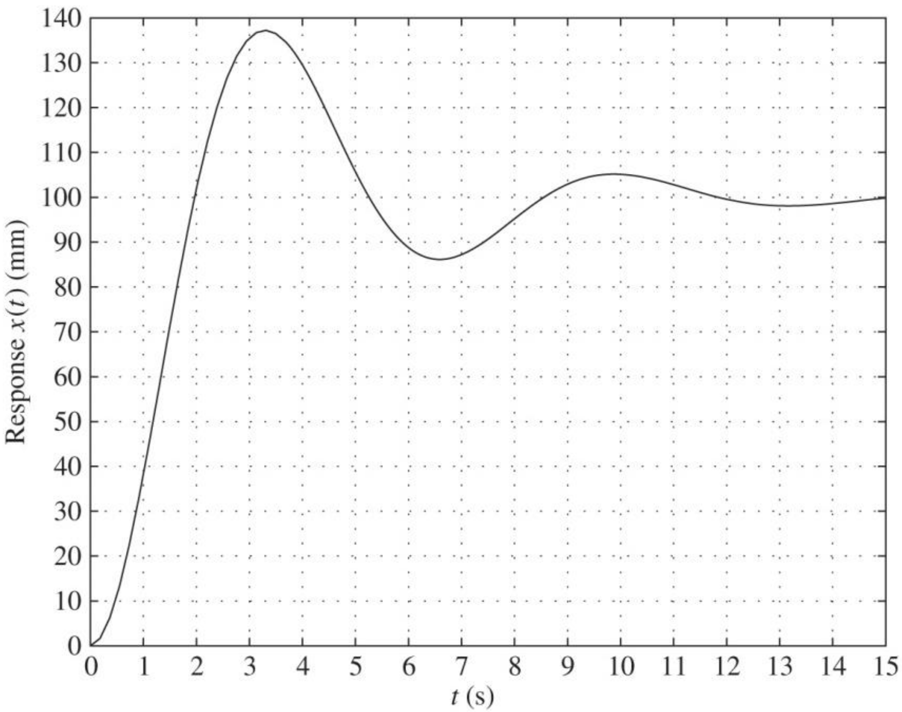


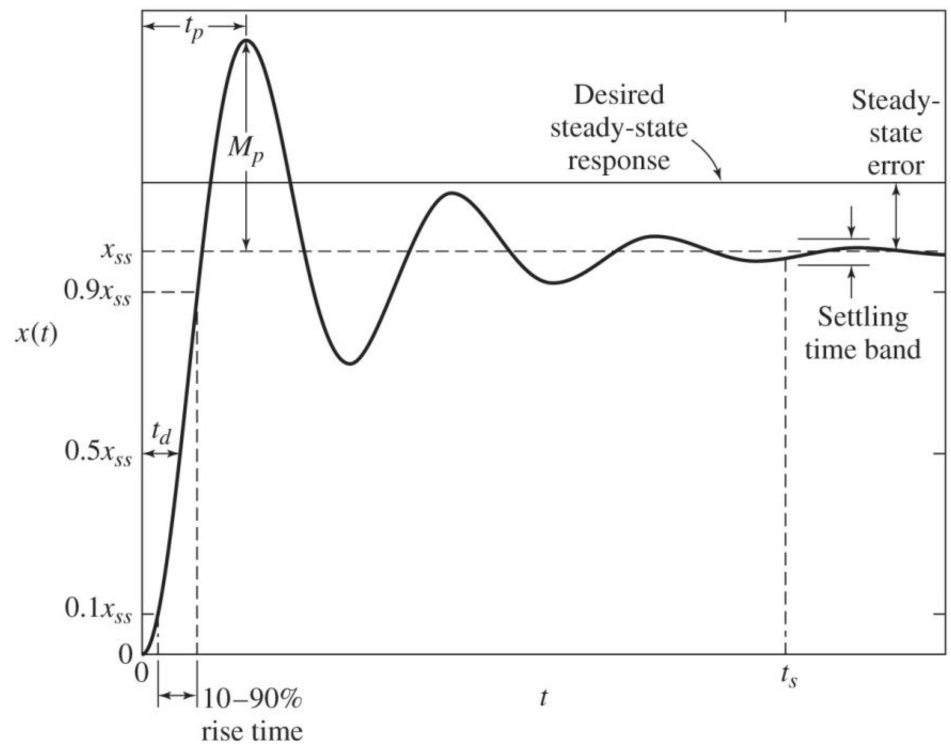


Time Domain Analysis of Dynamic Systems

Second-order Systems

- Consider the underdamped step response





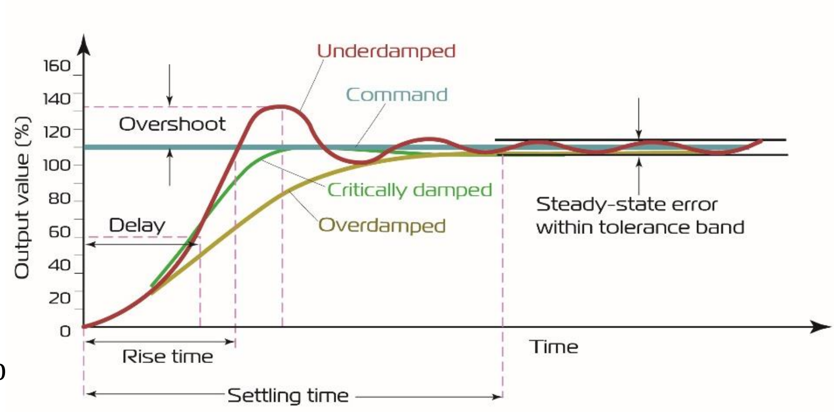
- Maximum or peak overshoot, M_p
- Percent maximum overshoot, $P.O$ or $M_{\%}$
- Peak value, x_p or x_{max}
- Peak time, t_p
- Rise time, t_r
- Settling time, t_s
- Delay time, t_d
- Steady state value, x_{ss}
- Steady state error, e_{ss}

Time Domain Analysis of Dynamic Systems

Second-order Systems

- Maximum or peak overshoot, M_p
  - When a signal or function exceeds its target. It is often associated with ringing.

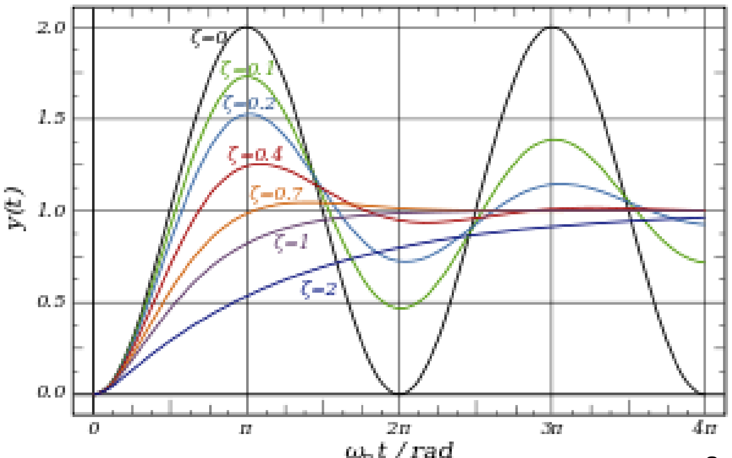
$$\boxed{M_p = x_{max} - x_{ss}} \qquad \boxed{M_p = x_p - x_{ss}}$$

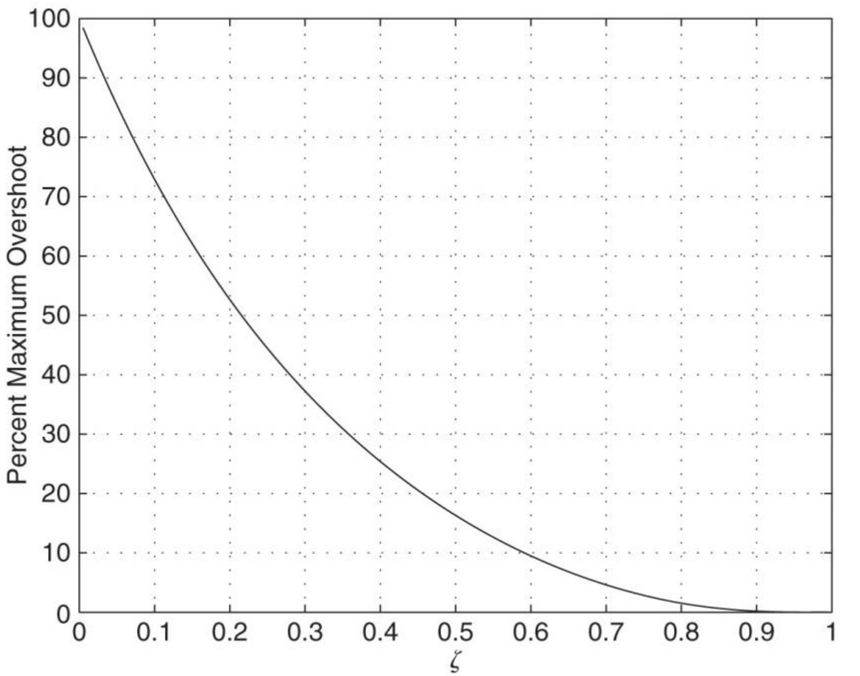


- Percent maximum overshoot, $P.O$ or $M_{\%}$

$$\boxed{M_{\%} = \left(\frac{x_{max} - x_{ss}}{x_{ss}}\right)100 = 100e^{-\pi\zeta/\sqrt{1-\zeta^2}}}$$

$$\boxed{\zeta = \frac{R}{\sqrt{\pi^2 + R^2}} \qquad R = \ln\left(\frac{100}{M_{\%}}\right)}$$



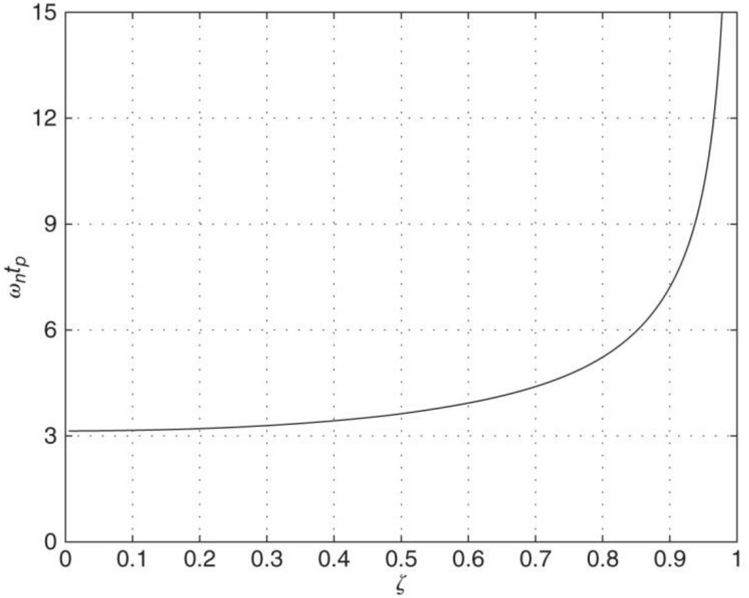


Time Domain Analysis of Dynamic Systems

Second-order Systems

- **Peak time, t_p**
  - Time required for the response to reach the first peak of the overshoot.

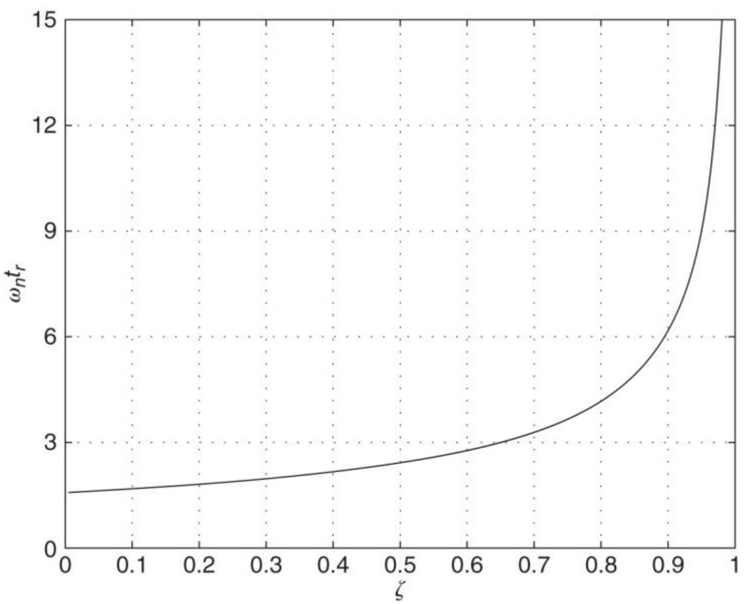
$$\boxed{t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}}$$



- **Rise time, t_r**
  - Time required for a signal to change from a specified low value to a specified high value.

$$\boxed{t_r = \frac{2\pi - \emptyset}{\omega_n\sqrt{1-\zeta^2}}} \qquad \emptyset = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) + \pi$$

  - This is for 100% rise time. No closed form expression exists for the 10%-90% rise time.

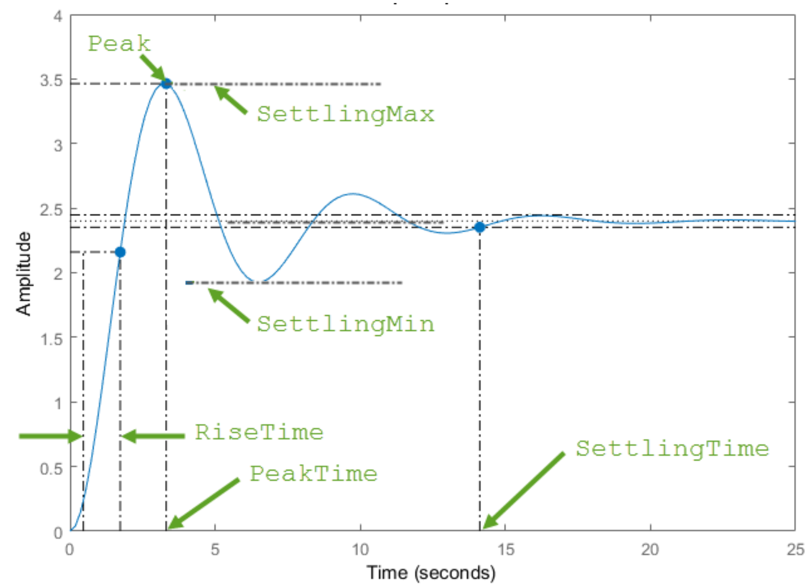


Time Domain Analysis of Dynamic Systems

Second-order Systems

- Settling time, t_s
  - Time elapsed from the application of an ideal instantaneous step input to the time at which the output has entered and remained within a specified error band.

$$\boxed{t_s = \frac{4}{\zeta\omega_n}} \qquad \boxed{\tau = \frac{1}{\zeta\omega_n}} \qquad \boxed{t_s = 4\tau}$$



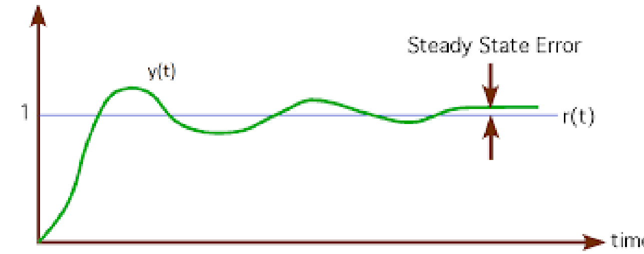
- Delay time, t_d
  - The delay time is the time required for the response to reach half the final value the very first time.

$$\boxed{t_d \approx \frac{1 + 0.7\zeta}{\omega_n}}$$

- Steady-state error, e_{ss}
  - The difference between the desired final output and the actual one when the system reaches a steady state, when its behavior may be expected to continue if the system is undisturbed.

$$\boxed{e_{ss} = x_{desired} - x_{ss}}$$

$$\boxed{e_{ss} = x_{reference} - x_{ss}}$$



Time Domain Analysis of Dynamic Systems

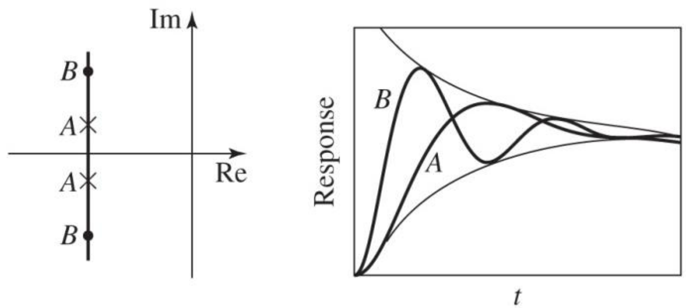
Second-order Systems

$$s = -\zeta\omega_n \pm \omega_d j$$

Effect of Root Location: Graphical Interpretation

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

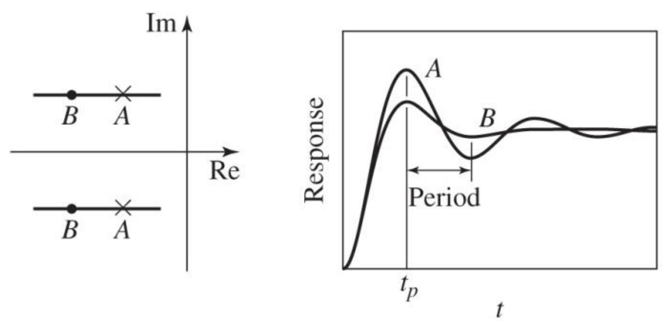
Models A and B have the same real part (on the same vertical line), the same time constant, and the same decay time.



$$\tau = \frac{1}{\zeta\omega_n} \qquad t_s = 4\tau = \frac{4}{\zeta\omega_n}$$

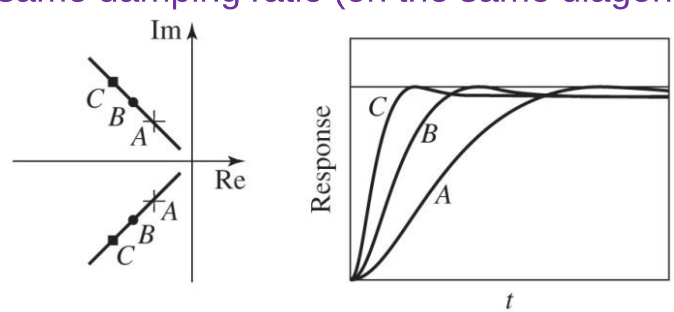
$$Decay\ rate\colon e^{-\zeta\omega_n t}$$

Models A and B have the same imaginary part (on the same horizontal line), the same period, and same peak time.



$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

Models A, B and C have same damping ratio (on the same diagonal/radial line) and same overshoot.



$$M_{\%} = 100e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Time Domain Analysis of Dynamic Systems

Example 3

The Figure shows the response of a forced spring-mass-damper system to a step input of magnitude $6 \text{ x } 10^3 \text{N}$.

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

Determine or estimate:

(a) the peak value

(b) the peak time

(c) steady state value

(d) k

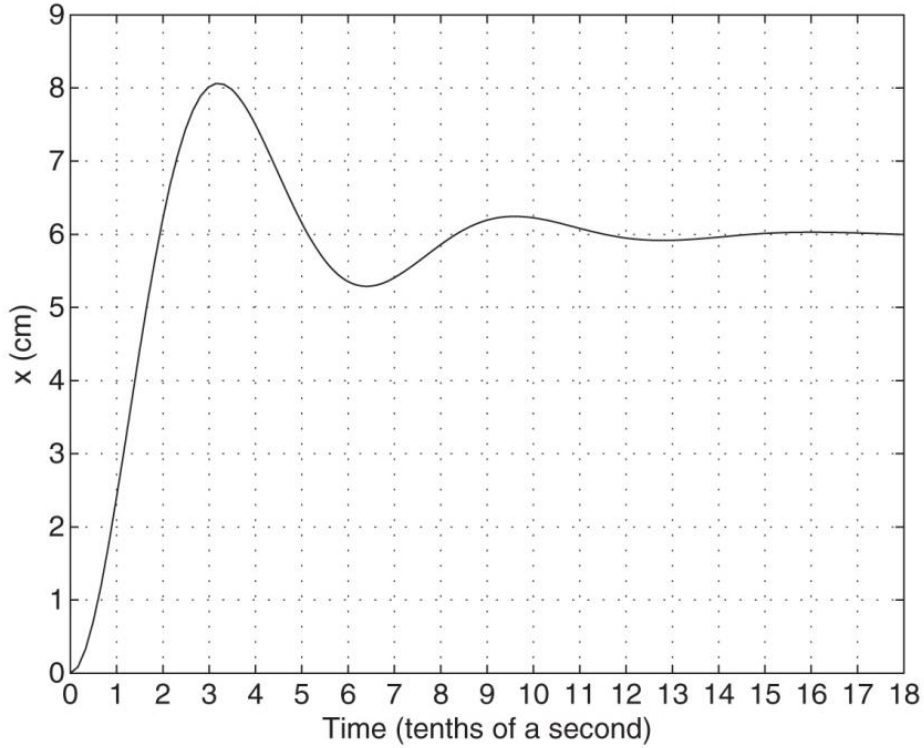
(e) the maximum percentage overshoot

(f) the damping ratio

(g) damped natural frequency

(h) undamped natural frequency

(i) m and c

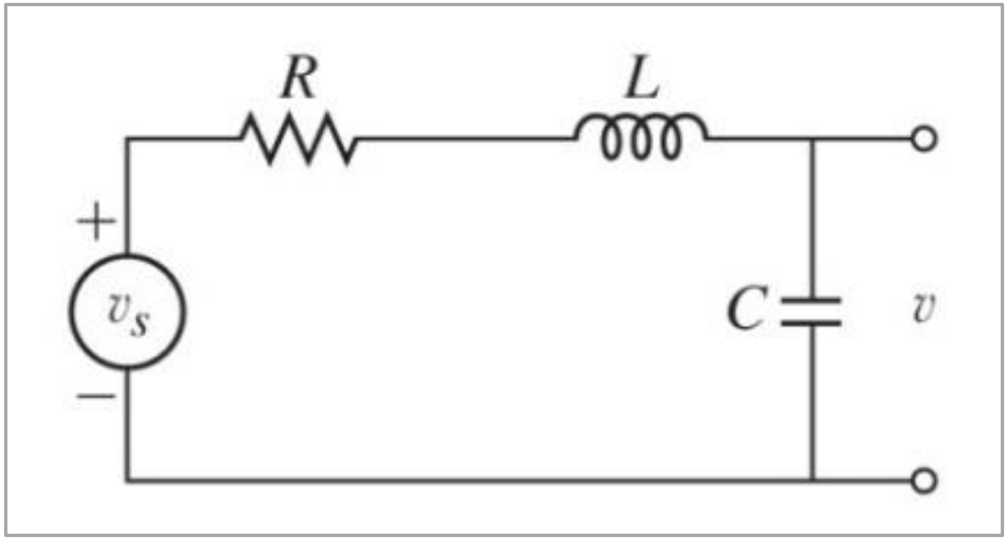


Time Domain Analysis of Dynamic Systems

Example 4

The Figure shows the response of an electrical system subjected to a step input voltage of 10 V.

$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = v_s \qquad R = 400\ \Omega, \qquad C = 20\ \mu\text{F}, \qquad L = 4\ \text{H}$$



Determine or estimate the:

- (a) peak value
- (b) peak time
- (c) steady state value
- (d) steady state error
- (e) overshoot
- (f) maximum percentage overshoot
- (g) rise time
- (h) settling time
- (i) delay time
- (j) damping ratio
- (k) damped natural frequency
- (l) undamped natural frequency
- *(m)* R and C, given $L = 4$ H

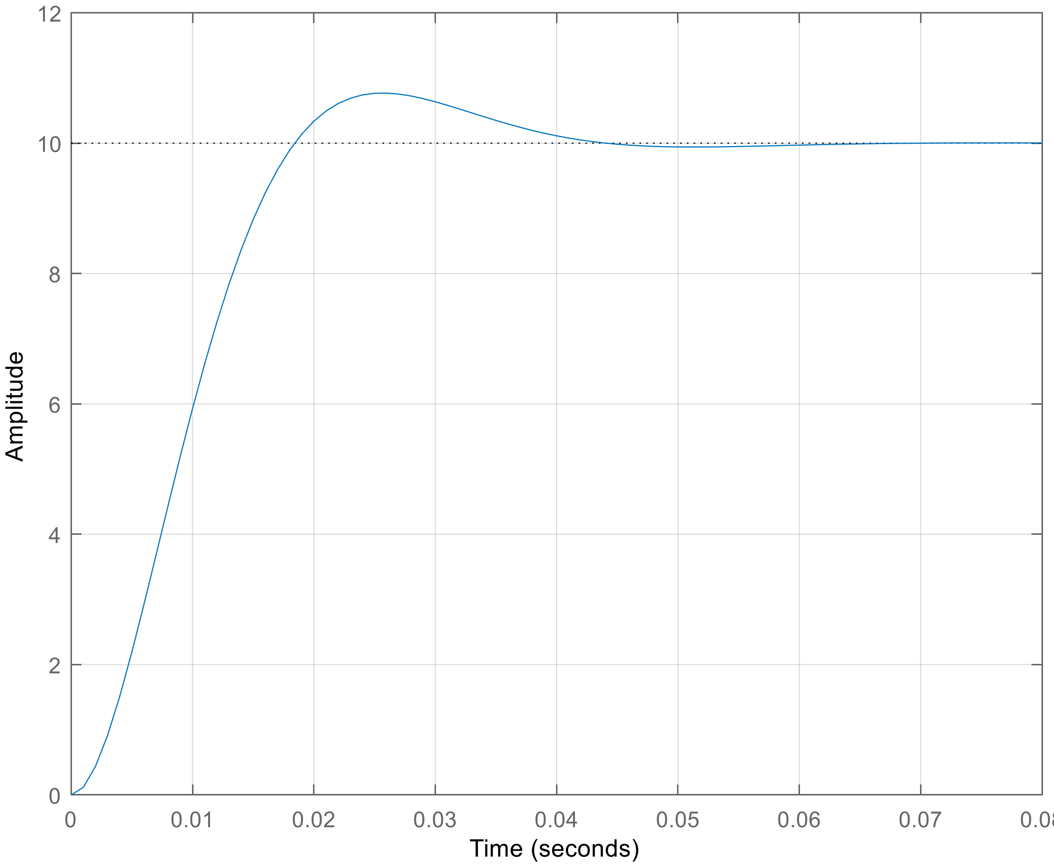
Time Domain Analysis of Dynamic Systems

Example 4

The Figure shows the response of an electrical system to a step input voltage of 10 V.

$$LC\frac{d^2v}{dt^2} + RC\frac{dv}{dt} + v = v_s$$

$$\frac{V(s)}{V_s(s)} = \frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$



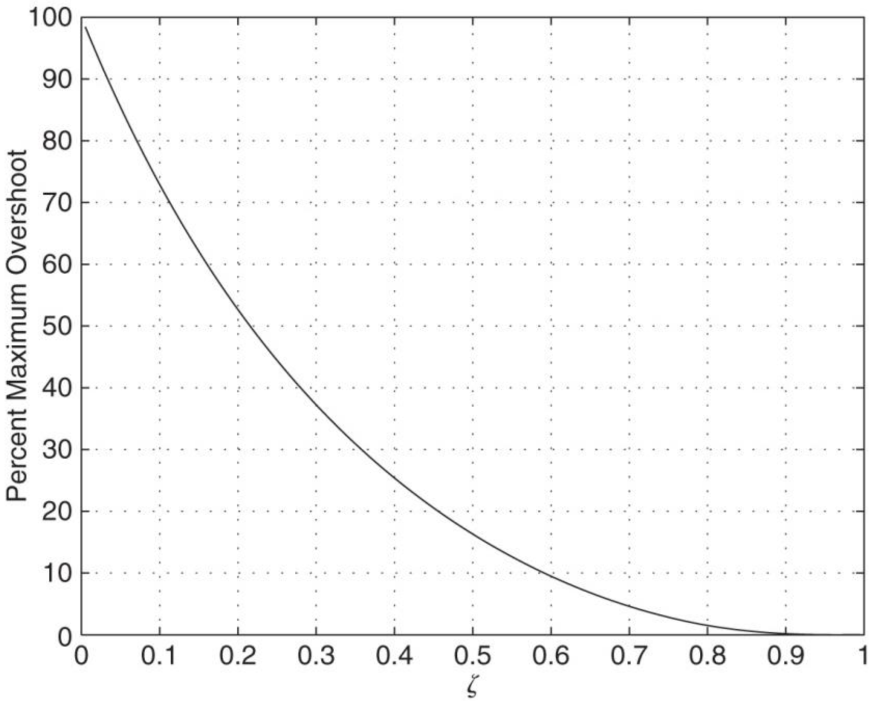
$R = 400\ \Omega,\qquad C = 20\ \mu\text{F},\qquad L = 4\ \text{H}$

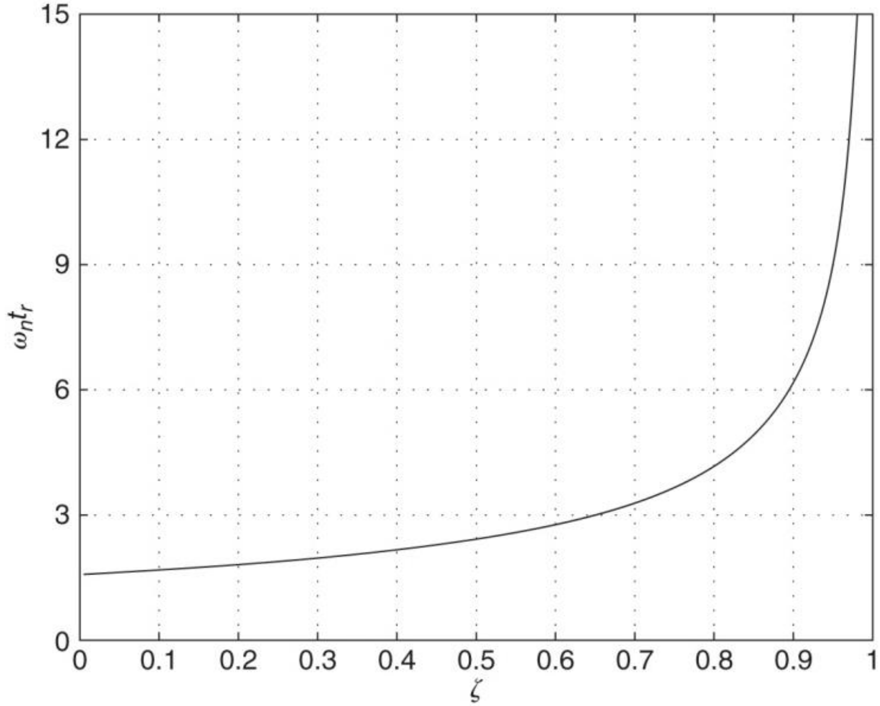
Time Domain Analysis of Dynamic Systems

Example 5

Compute the maximum percent overshoot, the maximum overshoot, the peak time, the 100% rise time, the delay time, and the 2% settling time for the following model. The initial conditions are zero. Time is measured in seconds.

$$\ddot{x} + 4\dot{x} + 8x = 2u_s(t)$$

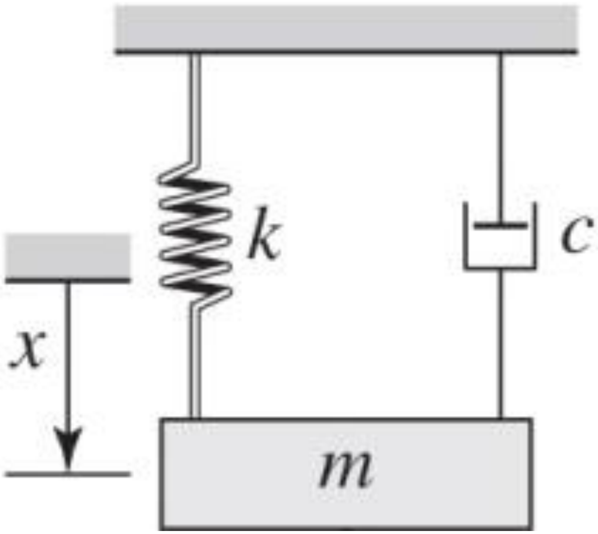




Time Domain Analysis of Dynamic Systems

Example 6

In the Figure shown, assume that $m = 1\ \text{kg}$, $c = 2\ \text{N-s/m}$, and $k = 100\ \text{N/m}$. The mass is displaced $0.05\ \text{m}$ and released without initial velocity. The displacement x is measured from the equilibrium position. Find the frequency observed in the vibration. Hence, find the analytical solution for $x(t)$.



$$X(s) = \frac{(s + 2\zeta\omega_n)x(0) + \dot{x}(0)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$x(t) = e^{-\zeta\omega_n t}\left\{\left[\frac{\zeta}{\sqrt{1-\zeta^2}}x(0) + \frac{1}{\omega_d}\dot{x}(0)\right]\sin\omega_d t + x(0)\cos\omega_d t\right\}$$

$$x(t) = e^{-\zeta\omega_n t}A\sin(\omega_d t + \emptyset)$$

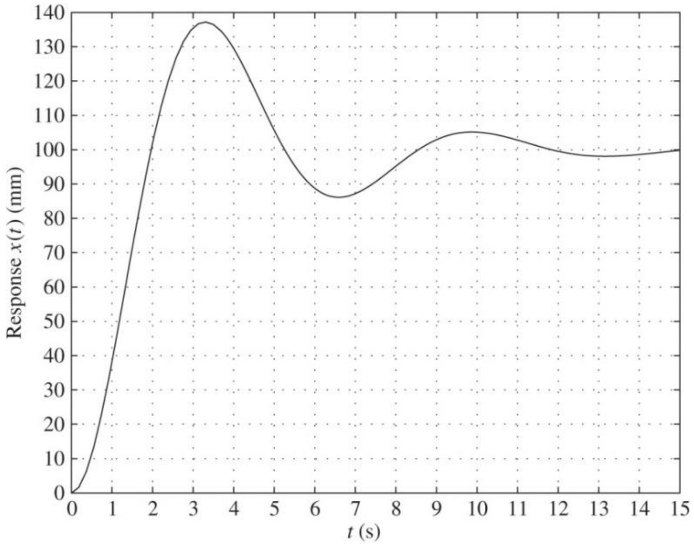
$$A = \frac{1}{\omega_d}\sqrt{[\omega_d x(0)]^2 + [\dot{x}(0) + \zeta\omega_n x(0)]^2}$$

$$\sin\emptyset = \frac{x(0)}{A} \qquad \text{or} \qquad \cos\emptyset = \frac{\dot{x}(0) + \zeta\omega_n x(0)}{A\zeta\omega_n}$$

Time Domain Analysis of Dynamic Systems

Second-order Systems

- Experimental determination of Damping ratio, ζ
  - Usually the damping coefficient c is the parameter most difficult to estimate.
  - ***Logarithmic decrement*** δ provides a good way to estimate the damping ratio ζ, from which we can compute c ($c = 2\zeta\sqrt{mk}$)..
  - For underdamped system



$$s = -\zeta\omega_n \pm \omega_d j, \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$$x(t) = Be^{-\zeta\omega_n t}\sin(\omega_d t + \emptyset) \qquad\qquad T = \frac{2\pi}{\omega_d}$$

$$\boxed{\delta = \ln\left(\frac{x(t)}{x(t+T)}\right)} = \ln\left(\frac{Be^{-\zeta\omega_n t}\sin(\omega_d t + \emptyset)}{Be^{-\zeta\omega_n (t+T)}\sin(\omega_d t + \omega_d T + \emptyset)}\right) = \ln e^{\zeta\omega_n T} = \zeta\omega_n T$$

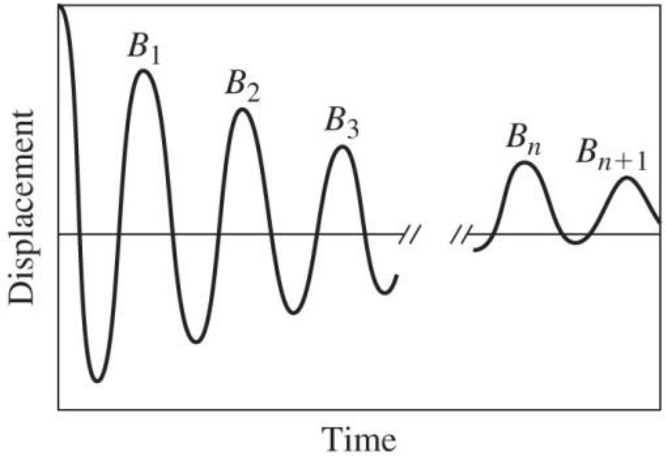
Ratio of two successive amplitudes

$$\boxed{\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}} \qquad \boxed{\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}}}$$

Time Domain Analysis of Dynamic Systems

Second-order Systems

- Experimental determination of Damping ratio, ζ
  - The plot of $x(t)$ will contain some measurement error, and for this reason, the preceding method is usually modified to use measurement of two peaks n cycles apart.



$$\ln\left(\frac{B_1}{B_2}\frac{B_2}{B_3}\frac{B_3}{B_4}\ldots\frac{B_n}{B_{n+1}}\right) = \ln\left(\frac{B_1}{B_{n+1}}\right)$$

$$\ln\left(\frac{B_1}{B_2}\right) + \ln\left(\frac{B_2}{B_3}\right) + \ln\left(\frac{B_3}{B_4}\right) + \cdots + \ln\left(\frac{B_n}{B_{n+1}}\right) = \ln\left(\frac{B_1}{B_{n+1}}\right)$$

$$\delta + \delta + \delta + \cdots + \delta = n\delta = \ln\left(\frac{B_1}{B_{n+1}}\right)$$

$$\boxed{\delta = \frac{1}{n}\ln\left(\frac{B_1}{B_{n+1}}\right)}$$

$$\boxed{\delta = \frac{1}{n}\ln\left(\frac{x_1}{x_{n+1}}\right)}$$

  - We normally take the first peak to be B_1, because this is the highest peak and least subject to measurement error, but this is not required. The formula applies to any two points n cycles apart.

Time Domain Analysis of Dynamic Systems

Example 7

Measurement of the free response of a certain spring-mass-damper system whose mass is $500\ \mathrm{kg}$ shows that after six cycles the amplitude of the displacement is 10% of the first amplitude. Also, the time for these six cycles to occur was measured to be $30\ s$. Estimate the system's damping c and stiffness k.

$$m\ddot{x} + c\dot{x} + kx = 0$$