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Two-parameter Odoma distribution with applications

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Abstract- ***This paper proposes a new lifetime distribution called two-parameter Odoma (TPO) distribution. The distribution is an extension of the one-parameter Odoma distribution. The TPO distribution enjoys a close form distributional expression which is more flexible than that of the one parameter Odoma distribution. Some of its interesting statistical properties are presented. The model parameters are estimated using the maximum likelihood estimation technique. The application of the distribution was illustrated with two real data-sets and its goodness-of-fit was compared with that of the Odoma (O), Two-parameter Lindley (TPL), Two-parameter Akash (TPA), Two-parameter Sujatha (TPS), Two-parameter Rama (TPR) and Two-parameter Pranav (TPP) distributions by using some model selection criteria and the results shows that the two-parameter Odoma distribution is the best candidate for the data-sets.***

Keywords- **Odoma distribution, Maximum likelihood estimator, Moments, Reliability measures, Order statistics, Stochastic ordering**

I. INTRODUCTION

Parametric statistical methods used for modelling lifetime data require that such data follows some specific statistical distributions. In practice, however, many classical statistical distributions have been observed to be inadequate in modelling certain kinds of data sets. For example, data sets with non-monotone hazard rate functions cannot be appropriately modelled by the exponential and gamma distributions due to the fact that they possess monotonic hazard rate functions.

Several studies have shown that many classical statistical distributions can be made more flexible to capture several real-life problems by the introduction of additional parameter(s). The role of additional shape parameter(s) is to vary the tail weight of the existing distribution, thereby inducing it with skewness [1]. Among the recent statistical distributions developed by addition of parameter(s) is the two-parameter Lindley distribution due to [2], two-parameter Akash distribution proposed by [3], two-parameter Rama distribution developed by [4], two-parameter Sujatha distribution proposed by [5], two-parameter Pranav distribution introduced by [6] and many others.

Recently, [7] introduced a new lifetime distribution called Odoma distribution. The probability density function (pdf) and the cumulative distribution function (cdf) of the Odoma distributed random variable X with parameter α are respectively given by

$$f(x;\alpha)=\frac{\alpha^5}{2(\alpha^5+\alpha^3+24)}\left(2x^4+\alpha x^2+2\alpha\right)e^{-\alpha x}\ ;x>0,\alpha>0 \tag{1}$$

and

$$F(x;\alpha)=1-\left[1+\frac{\alpha^2x^2\left(\alpha^2x^2+4\alpha x+12\right)}{\left(\alpha^5+\alpha^3+24\right)}+\frac{\alpha x^2\left(\alpha^4x+2\alpha^3+48\right)}{2\left(\alpha^5+\alpha^3+24\right)}\right]e^{-\alpha x};x>0,\alpha>0 \tag{2}$$

Notably, the Odoma distribution is a three-component mixture of the exponential(α), gamma $(3,\alpha)$and gamma $(5,\alpha)$ distributions with mixing proportions $\alpha^5\big/\left(\alpha^5+\alpha^3+24\right)$, $\alpha^3\big/\left(\alpha^5+\alpha^3+24\right)$ and $24\big/\left(\alpha^5+\alpha^3+24\right)$respectively. Reference [7] showed that the Odoma distribution provides better fits to lifetime data than the Pranav, Sujatha, Aradhana, Akash, Lindley and exponential distributions respectively. A major limitation of the Odoma distribution is that it depends on one parameter and as such cannot be flexible in modelling data with varieties of tails.

The aim of this paper is to propose a two-parameter Odoma distribution, which provides more flexibility in modelling lifetime data than the Odoma distribution itself. The paper is unfolded as follows: Section II offers the derivation of the probability density function and the cumulative distribution function. In Sections III and IV, the quantile function and the stochastic ordering are respectively derived. Statistical properties of the proposed distribution are presented in Section V. The reliability measures of the proposed model are introduced in Section VI. The Rényi entropy measure is derived in Section VII. The maximum likelihood estimators (MLEs) of parameters of the proposed model are obtained in Section VIII. The asymptotic confidence intervals of parameters of the proposed model are given in Section IX. Distributions of order statistics are presented in Section X. An application of the proposed model is discussed in Section XI. Finally, Section XII concludes the paper.

II. TWO-PARAMETER ODOMA DISTRIBUTION

A. Probability density function of the two-parameter Odoma distribution

Proposition 1. *The probability density function (pdf) of a two-parameter Odoma distributed random variable X with parameters α and β is given by*

$$f(x;\alpha,\beta)=\frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}(2x^4+\alpha x^2+2\alpha\beta)e^{-\alpha x};x>0,\alpha>0,\beta>0 \tag{3}$$

Proof. Motivated by the work of [7], we define the two-parameter Odoma distribution as a three-component mixture of exponential(α), gamma $(3,\alpha)$and gamma $(5,\alpha)$ as

$$f(x;\alpha,\beta)=p_1g_1(x;\alpha)+p_2g_2(x;3,\alpha)+p_3g_3(x;5,\alpha) \tag{4}$$

for $p_1+p_1+p_3=1$and $x\in(0,\infty)$, where the exponential(α), gamma $(3,\alpha)$and gamma $(5,\alpha)$ distributions are respectively defined as

$$g_1(\alpha)=\alpha e^{-\alpha x};x>0,\alpha>0 \tag{5}$$

$$g_2(3,\alpha)=\frac{\alpha^3x^2e^{-\alpha x}}{2};x>0,\alpha>0 \tag{6}$$

$$g_3(5,\alpha)=\frac{\alpha^5x^4e^{-\alpha x}}{24};x>0,\alpha>0 \tag{7}$$

In order to derive the new distribution, we redefine the mixing proportions given by [7] to accommodate additional parameter β as follows

$$p_1=\frac{\alpha^5\beta}{\alpha^5\beta+\alpha^3+24} \tag{8}$$

$$p_2=\frac{\alpha^3}{\alpha^5\beta+\alpha^3+24} \tag{9}$$

$$p_3 = \frac{24}{\alpha^5\beta+\alpha^3+24} \tag{10}$$

Substituting (5)-(10) into (4), one obtains

$$f(x;\alpha,\beta) = \frac{\alpha^5\beta}{\alpha^5\beta+\alpha^3+24}.\alpha e^{-\alpha x} + \frac{\alpha^3}{\alpha^5\beta+\alpha^3+24}\frac{\alpha^3 x^2 e^{-\alpha x}}{2} \quad ; x>0, \alpha>0, \beta>0$$
$$+\frac{24}{\alpha^5\beta+\alpha^3+24}\frac{\alpha^5 x^4 e^{-\alpha x}}{24} \tag{11}$$

Simplification of (11) gives (3) and this completes the proof of Proposition 1.

Corollary 1. *The pdf of the two-parameter Odoma distribution is a valid density function.*

Proof. Corollary 1 suffices to show that (3) satisfies the conditions $f(x;\alpha,\beta)\geq 0$ and $\int_0^\infty f(x;\alpha,\beta)\,dx = 1$.

In view of this, it is observed from (3) that

$$\frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}\left(2x^4+\alpha x^2+2\alpha\beta\right)e^{-\alpha x} > 0 \text{ for all } x$$

and

$$\int_0^\infty f(x;\alpha,\beta)\,dx = \frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}\int_0^\infty\left(2x^4+\alpha x^2+2\alpha\beta\right)e^{-\alpha x}\,dx$$

$$=\frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}\left[2\int_0^\infty x^{5-1}e^{-\alpha x}dx + \alpha\int_0^\infty x^{3-1}e^{-\alpha x}dx + 2\alpha\beta\int_0^\infty x^{1-1}e^{-\alpha x}dx\right]$$

$$=\frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}\left[\frac{2\Gamma(5)}{\alpha^5}+\frac{\alpha\Gamma(3)}{\alpha^3}+\frac{2\alpha\beta\Gamma(1)}{\alpha}\right] \because \int_0^\infty x^{p-1}e^{-\alpha x}dx = \frac{\Gamma(\alpha)}{\alpha^p}$$

$$=\frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}\left[\frac{48}{\alpha^5}+\frac{2\alpha}{\alpha^3}+\frac{2\alpha\beta}{\alpha}\right]$$

$$=\frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}\left[\frac{2(\alpha^5\beta+\alpha^3+24)}{\alpha^5}\right]=1$$

Hence, the two-parameter Odoma distribution defined in (3) is a valid density function.

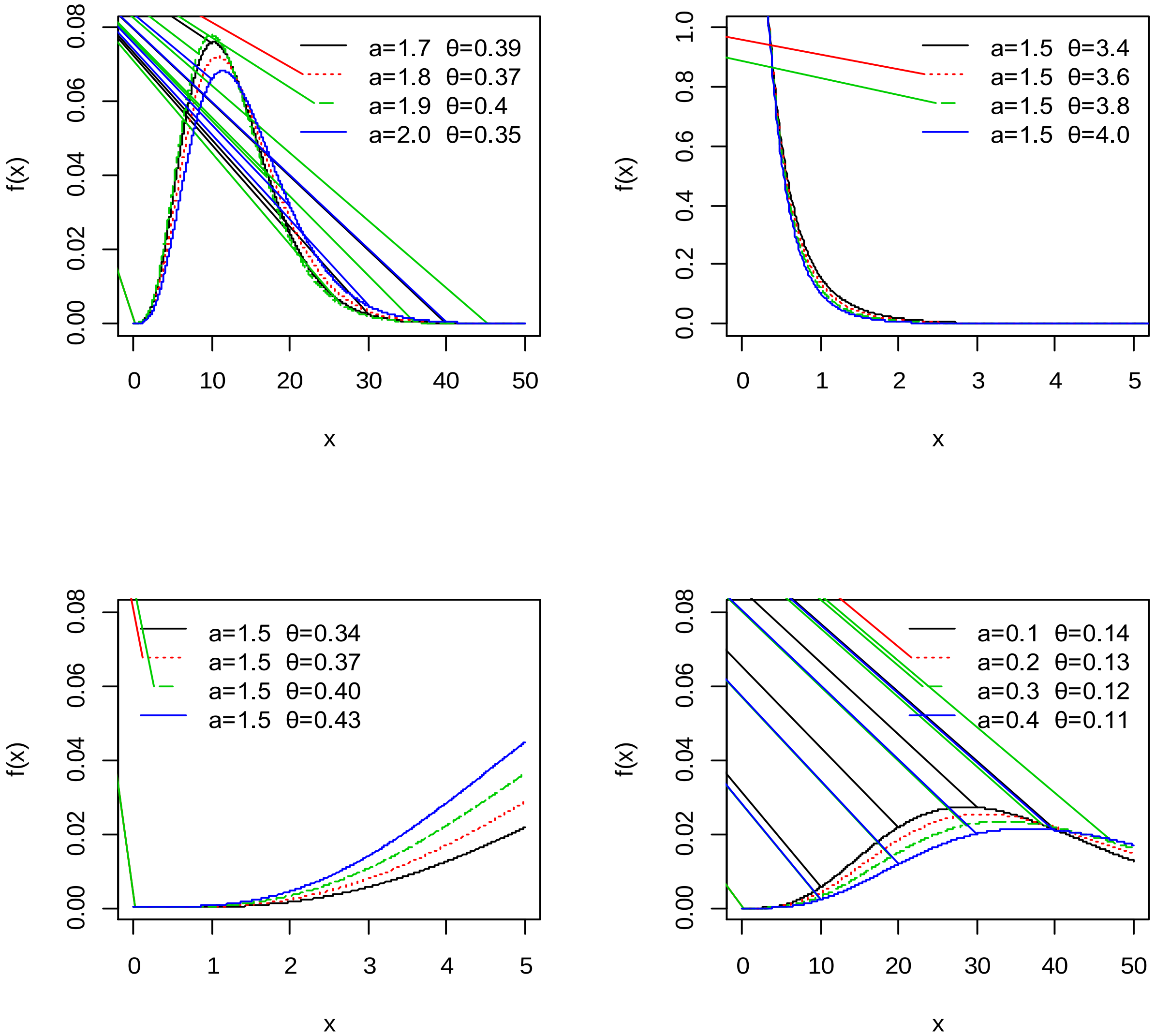

Fig. 1 Various shapes of the PDF of the two-parameter Odoma distribution

B. Cumulative distribution function of the two-parameter Odoma distribution

Proposition 2. *The cumulative distribution function (cdf) of a two-parameter Odoma distributed random variable X with parameters α and β is given by*

$$F(x;\alpha,\beta)=1-\left[1+\frac{2\alpha^2x^2\left(\alpha^2x^2+4\alpha x+12\right)+\alpha x^2\left(\alpha^4x+2\alpha^3+48\right)}{2\left(\alpha^5\beta+\alpha^3+24\right)}\right]e^{-\alpha x};x>0,\alpha>0,\beta>0 \qquad (12)$$

Proof. According to [8], the cumulative density function (cdf) of a continuous random variable X, denoted by $F(x)$, is given by

$$F(x)=P(X\le x)=\int_{-\infty}^{x} f(u)du \qquad (13)$$

Consequently, if $X \sim TPOD(\alpha,\beta)$, the cdf of X is obtained by putting (3) into (13) as follows

$$F(x)=\frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}\int_0^x\left(2u^4+\alpha u^2+2\alpha\beta\right)e^{-\alpha u}\,du$$

$$=\frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}\left[2\int_0^x u^4e^{-\alpha u}\,du+\alpha\int_0^x u^2e^{-\alpha u}\,du+2\alpha\beta\int_0^x e^{-\alpha u}\,du\right] \tag{14}$$

To evaluate (14), integration by parts is used to obtain the following

$$2\int_0^x u^4e^{-\alpha u}\,du=-\left(\frac{2x^4e^{-\alpha x}}{\alpha}+\frac{8x^3e^{-\alpha x}}{\alpha^2}+\frac{24x^2e^{-\alpha x}}{\alpha^3}+\frac{48xe^{-\alpha x}}{\alpha^4}+\frac{48e^{-\alpha x}}{\alpha^5}-\frac{48}{\alpha^5}\right) \tag{15}$$

$$\alpha\int_0^x u^2e^{-\alpha u}\,du=-\left(x^2e^{-\alpha x}+\frac{2xe^{-\alpha x}}{\alpha}+\frac{2e^{-\alpha x}}{\alpha^2}-\frac{2}{\alpha^2}\right) \tag{16}$$

$$2\alpha\beta\int_0^x e^{-\alpha u}\,du=2\beta\left(1-e^{-\alpha x}\right) \tag{17}$$

Plugging (15)-(17) into (14), the following is obtained

$$F(x)=\frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}\left[\begin{array}{l}-\left(\frac{2x^4e^{-\alpha x}}{\alpha}+\frac{8x^3e^{-\alpha x}}{\alpha^2}+\frac{24x^2e^{-\alpha x}}{\alpha^3}+\frac{48xe^{-\alpha x}}{\alpha^4}+\frac{48e^{-\alpha x}}{\alpha^5}-\frac{48}{\alpha^5}\right)\\-\left(x^2e^{-\alpha x}+\frac{2xe^{-\alpha x}}{\alpha}+\frac{2e^{-\alpha x}}{\alpha^2}-\frac{2}{\alpha^2}\right)+2\beta\left(1-e^{-\alpha x}\right)\end{array}\right] \tag{18}$$

Equation (18) can be arranged as follows

$$F(x)=\frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}\left[\left(\frac{48}{\alpha^5}+\frac{2}{\alpha^2}+2\beta\right)-\left(\begin{array}{l}\frac{48e^{-\alpha x}}{\alpha^5}+\frac{2e^{-\alpha x}}{\alpha^2}+2\beta e^{-\alpha x}+\frac{2x^4e^{-\alpha x}}{\alpha}+\frac{8x^3e^{-\alpha x}}{\alpha^2}\\+\frac{24x^2e^{-\alpha x}}{\alpha^3}+\frac{48xe^{-\alpha x}}{\alpha^4}+x^2e^{-\alpha x}+\frac{2xe^{-\alpha x}}{\alpha}\end{array}\right)\right]$$

$$=\frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}\left[\begin{array}{l}\frac{2(\alpha^5\beta+\alpha^3+24)}{\alpha^5}-\frac{2(\alpha^5\beta+\alpha^3+24)}{\alpha^5}e^{-\alpha x}\\-\left(\frac{(2\alpha^3x^4+8\alpha^2x^3+24\alpha x^2)}{\alpha^4}+\frac{(\alpha^4x^2+2\alpha^3x+48x)}{\alpha^4}\right)e^{-\alpha x}\end{array}\right]$$

$$F(x)=1-e^{-\alpha x}-\left(\frac{2\alpha^2x^4\left(2\alpha^2x^2+4\alpha x+12\right)+x\left(\alpha^4x+2\alpha^3+48\right)}{2(\alpha^5\beta+\alpha^3+24)}\right)e^{-\alpha x} \tag{19}$$

Further simplification of (19) yields (12), which completes the proof of Proposition 2.

Corollary 2. *The cdf in (12) satisfies the conditions* $F(-\infty)=0$ *and* $F(+\infty)=0$ *respectively.*

Proof. The proof of Corollary 2 follows directly from (12) by substituting $-\infty$ for x and $+\infty$ for x respectively.

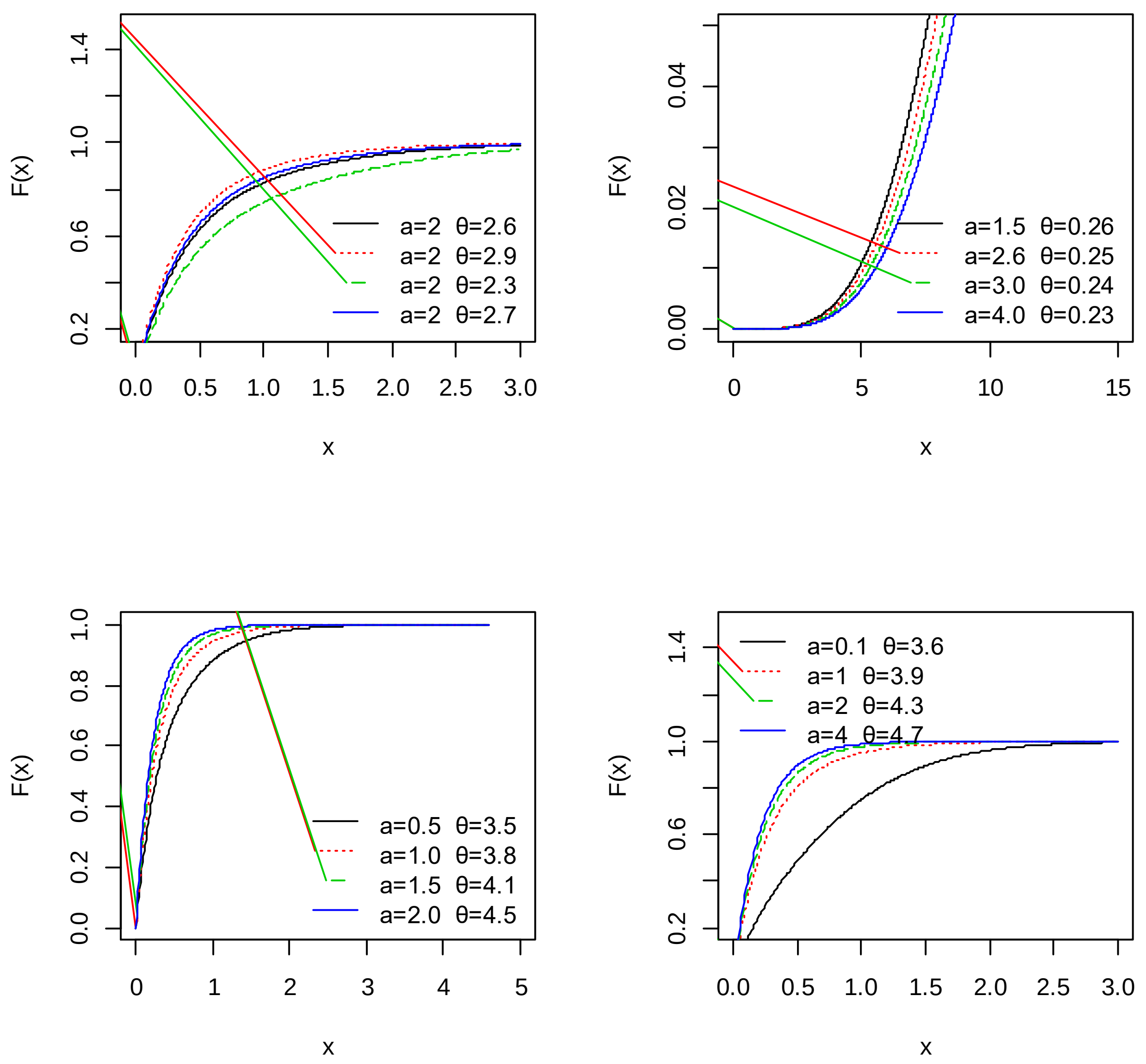

Fig. 2 Various shapes of the CDF of the two-parameter Odoma distribution

III. QUANTILE FUNCTION OF THE TWO-PARAMETER ODOMA DISTRIBUTION

Let $0<p<1$, then the pth quantile of the random variable $X \sim \text{TPOD}(\alpha,\beta)$ is the value of x that satisfies the relation:

$$p=F(x;\alpha,\beta) \tag{20}$$

Substituting (12) into (20) and solving for x, we get

$$x_p=\ln\left\{\frac{2\left(\alpha^5\beta+\alpha^3+24\right)-p\left[2\left(\alpha^5\beta+\alpha^3+24\right)\right]}{2\alpha^2x_p^2\left(\alpha^2x_p^2+4\alpha x+12\right)+\alpha x^2\left(\alpha^4x_p+2\alpha^3+48\right)+2\left(\alpha^5\beta+\alpha^3+24\right)}\right\}^{-\frac{1}{\alpha}} \tag{21}$$

The quantile function can be useful for random number generation, estimation based on percentiles and quantile regression methods. For random number generation, one can solve (21) for x_p for when p is a uniform random number between 0 and 1. The solving must be performed numerically, for example, using uniroot in the R software [9].

IV. STOCHASTIC ORDERING OF THE TWO-PARAMETER ODOMA DISTRIBUTION

As pointed out by [10], stochastic ordering of positive continuous random variables plays significant role in assessing the comparative behaviour of random variables. The different types of stochastic orderings which are useful in ordering random variables include the usual stochastic order, the hazard rate order, the mean residual life order, and the likelihood ratio order for the random variables under a restricted parameter space. Suppose X and Y are independent random variables with cumulative distribution functions $F_X(x)$ and $F_Y(x)$ respectively, then X is said to be stochastically smaller $(X \leq_{st} Y)$ than if $F_X(x) \geq F_Y(x)$ for all x. Similarly, X is said to be stochastically smaller $(X \leq_{st} Y)$ than Y in the

(i) hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$, for all x

(i) mean residual life order $(X \leq_{mrl} Y)$ if $m_X(x) \leq m_Y(x)$ for all x

(iii) likelihood ratio order $(X \leq_{lr} Y)$ if $\dfrac{f_X(x)}{f_Y(x)}$ decreases in x.

According to [10] the following stochastic ordering relationship $X \leq_{lr} Y \Rightarrow \underset{\substack{\Downarrow \\ X \leq_{st} Y}}{X \leq_{hr} Y} \Rightarrow X \leq_{mlr} Y$ holds true. Also, the two-parameter Odoma distribution is ordered with respect to the strongest ‘‘likelihood ratio’’ ordering as shown in Proposition 3 below.

Proposition 3. *Let* $X \sim \mathrm{TPOD}(\alpha_1, \beta_1)$ *and* $Y \sim \mathrm{TPOD}(\alpha_2, \beta_2)$. *If* $\alpha_1 > \alpha_2$ *and* $\beta_2 = \beta_1$ *(or* $\alpha_1 = \alpha_2$ *and* $\beta_2 \geq \beta_1$*), then* $X \leq_{lr} Y$. *Hence,* $X \leq_{hr} Y$, $X \leq_{mrl} Y$ *and* $X \leq_{st} Y$.

Proof. To show the flexibility of the TPO distribution, the following likelihood ratio is defined:

$$\frac{f_X(x)}{f_Y(x)} = \frac{\dfrac{\alpha_1^5}{2\left(\alpha_1^5\beta_1 + \alpha_1^3 + 24\right)}\left(2x^4 + \alpha_1 x^2 + 2\alpha_1\beta_1\right)e^{-\alpha_1 x}}{\dfrac{\alpha_2^5}{2\left(\alpha_2^5\beta_2 + \alpha_2^3 + 24\right)}\left(2x^4 + \alpha_2 x^2 + 2\alpha_2\beta_2\right)e^{-\alpha_2 x}} \tag{22}$$

$$= \frac{\alpha_1^5\left(\alpha_2^5\beta_2 + \alpha_2^3 + 24\right)}{\alpha_2^5\left(\alpha_1^5\beta_1 + \alpha_1^3 + 24\right)}\left(\frac{2x^4 + \alpha_1 x^2 + 2\alpha_1\beta_1}{2x^4 + \alpha_2 x^2 + 2\alpha_2\beta_2}\right)e^{-(\alpha_1 - \alpha_2)x}$$

Taking the log of both sides of (22) gives the likelihood ratio as

$$\ln\left(\frac{f_X(x)}{f_Y(x)}\right) = \ln\left[\frac{\alpha_1^5\left(\alpha_2^5\beta_2 + \alpha_2^3 + 24\right)}{\alpha_2^5\left(\alpha_1^5\beta_1 + \alpha_1^3 + 24\right)}\right] + \ln\left(\frac{2x^4 + \alpha_1 x^2 + 2\alpha_1\beta_1}{2x^4 + \alpha_2 x^2 + 2\alpha_2\beta_2}\right) - (\alpha_1 - \alpha_2)x \tag{23}$$

Differentiating (23) with respect to x, we obtain

$$\frac{d}{dx}\ln\left(\frac{f_X(x)}{f_Y(x)}\right)=\frac{8x^3+2\alpha_1 x}{2x^4+\alpha_1 x^2+2\alpha_1\beta_1}-\frac{8x^3+2\alpha_2 x}{2x^4+\alpha_2 x^2+2\alpha_2\beta_2}-(\alpha_1-\alpha_2)$$

$$=\frac{8(\alpha_2-\alpha_1)x^5+4(\alpha_1-\alpha_2)x^4+16(\alpha_2\beta_1-\alpha_1\beta_2)x^3}{(2x^4+\alpha_1 x^2+2\alpha_1\beta_1)(2x^4+\alpha_2 x^2+2\alpha_2\beta_2)}-(\alpha_1-\alpha_2) \tag{24}$$

If $\alpha_1>\alpha_2$ *and* $\beta_2=\beta_1$ (*or* $\alpha_1=\alpha_2$ *and* $\beta_2\geq\beta_1$), then $\frac{d}{dx}\ln\left(\frac{f_X(x)}{f_Y(x)}\right)<0$. This means that $X\leq_{lr}Y$ and hence, $X\leq_{hr}Y$, $X\leq_{mrl}Y$ and $X\leq_{st}Y$.

V. STATISTICAL PROPERTIES OF THE TWO-PARAMETER ODOMA DISTRIBUTION

Most of the useful characteristics of a statistical distribution can be obtained through its moments. In this section, we present the raw and central moments of the two-parameter Odoma distribution. The mean, variance, coefficients of variation, index of dispersion, kurtosis, and skewness are equally presented. Apart from producing the moments, the moment generating function, characteristic function, incomplete moment, and mean deviations are presented.

A. Raw and central moments of the two-parameter Odoma distribution

Proposition 4. *Given that* $X\sim TPOD(\alpha,\beta)$ *the* kth *raw moment of* X *is given by*

$$\mu_k^1=\frac{2(k+4)!+\alpha^3(k+2)!+2\alpha^5\beta k!}{2\alpha^k(\alpha^5\beta+\alpha^3+24)} \tag{25}$$

Proof. The kth raw moment of a TPO distributed random variable X is given by

$$\mu_k^1=\int_0^\infty x^k f(x)\,dx$$

$$\mu_k^1=\frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}\int_0^\infty x^k(2x^4+\alpha x^2+2\alpha\beta)e^{-\alpha x}\,dx$$

$$=\frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}\left[2\int_0^\infty x^{(k+5)-1}e^{-\alpha x}\,dx+\alpha\int_0^\infty x^{(k+3)-1}e^{-\alpha x}\,dx+2\alpha\beta\int_0^\infty x^{(k+1)-1}e^{-\alpha x}\,dx\right]$$

$$=\frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}\left[\frac{2\Gamma(k+5)}{\alpha^{k+5}}+\frac{\alpha\Gamma(k+3)}{\alpha^{k+3}}+\frac{2\alpha\beta\Gamma(k+1)}{\alpha^{k+1}}\right]$$

$$=\frac{\alpha^5}{2(\alpha^5\beta+\alpha^3+24)}\left[\frac{2(k+4)!}{\alpha^{k+5}}+\frac{\alpha(k+2)!}{\alpha^{k+3}}+\frac{2\alpha\beta k!}{\alpha^{k+1}}\right] \tag{26}$$

Further simplification of (26) yields the result given in (25), which completes the proof of Proposition 3.

Corollary 3. The first four raw moments of the two-parameter Odoma distribution are respectively given by

$$\mu_1^1 = \frac{\alpha^5\beta + 3\alpha^3 + 120}{\alpha\left(\alpha^5\beta + \alpha^3 + 24\right)} = \mu \tag{27}$$

$$\mu_2^1 = \frac{2\left(\alpha^5\beta + 6\alpha^3 + 360\right)}{\alpha^2\left(\alpha^5\beta + \alpha^3 + 24\right)} \tag{28}$$

$$\mu_3^1 = \frac{6\left(\alpha^5\beta + 10\alpha^3 + 840\right)}{\alpha^3\left(\alpha^5\beta + \alpha^3 + 24\right)} \tag{29}$$

$$\mu_4^1 = \frac{24\left(\alpha^5\beta + 15\alpha^3 + 1680\right)}{\alpha^4\left(\alpha^5\beta + \alpha^3 + 24\right)} \tag{30}$$

Proof . The proof of (27)-(30) follows directly on substituting $k = 1, 2, 3$ and 4 respectively into (25).

Proposition 5. *Given that* $X \sim TPOD(\alpha, \beta)$ *the* kth *central moment of* X *is*

$$\mu_k = \sum_{j=0}^{k}\binom{k}{j}(-1)^k \frac{\left(\alpha^5\beta + 6\alpha^3 + 120\right)^{k-j}\left[2(j+4)! + \alpha^3(j+2)! + 2\alpha^5\beta\, j!\right]}{2\alpha^k\left(\alpha^5\beta + \alpha^3 + 24\right)^{k-j+1}} \tag{31}$$

Proof The kth central moment of a TPO distributed random variable X is given by

$$\mu_k = E\left[\left(X - \mu_1^1\right)^k\right] = \int_0^\infty \left(X - \mu_1^1\right)^k f(x;\alpha,\beta)\,dx$$

$$= \sum_{j=0}^{k}\binom{k}{j}\left(-\mu_1^1\right)^{k-1} E\left(X^j\right)$$

$$= \sum_{j=0}^{k}\binom{k}{j}(-1)^k\left(\frac{\alpha^5\beta + 3\alpha^3 + 120}{\alpha\left(\alpha^5\beta + \alpha^3 + 24\right)}\right)^{k-j} \frac{2(j+4)! + \alpha^3(j+2)! + 2\alpha^5\beta\, j!}{2\alpha^j\left(\alpha^5\beta + \alpha^3 + 24\right)} \tag{32}$$

Simplifying (32) yields the result defined in (31).

Corollary 4. *The first four central moments of the two-parameter Odoma distribution are respectively given by*

$$\mu_2 = \frac{\alpha^{10}\beta^2 + 8\alpha^8\beta + 528\alpha^5\beta + 3\alpha^6 + 288\alpha^3 + 2880}{\alpha^2\left(\alpha^5\beta + \alpha^3 + 24\right)^2} = \sigma^2 \tag{33}$$

$$\mu_3 = \frac{\begin{bmatrix} 2\alpha^{15}\beta^3 + 30\alpha^{13}\beta^2 + 3024\alpha^{10}\beta^2 - 18\alpha^{11}\beta + 2592\alpha^8\beta \\ +3456\alpha^5\beta + 6\alpha^9 + 1008\alpha^6 + 17280\alpha^3 + 138240 \end{bmatrix}}{\alpha^3\left(\alpha^5\beta + \alpha^3 + 24\right)^3} \tag{34}$$

$$\mu_4 = \frac{\begin{bmatrix} 9\alpha^{20}\beta^4 + 132\alpha^{18}\beta^3 + 23616\alpha^{15}\beta^3 - 126\alpha^{16}\beta^2 + 33840\alpha^{13}\beta^2 - 551232\alpha^{10}\beta^2 \\ -744\alpha^{14}\beta - 1104\alpha^{11}\beta - 45171312\alpha^5\beta - 2115072\alpha^8\beta + 12\alpha^{18}\beta^2 + 21600\alpha^9\beta \\ -34560\alpha^6\beta - 79272\alpha^9 - 180817920\alpha^3 - 4371840\alpha^6 - 531\alpha^{12} - 9720\alpha^{11} - 1308856320 \end{bmatrix}}{\alpha^4\left(\alpha^5\beta + \alpha^3 + 24\right)^4} \tag{35}$$

Proof. The proof of (33), (33) and (35) respectively follows directly on substituting $k = 1, 2, 3$ and 4 respectively into (31).

B. Skweness, Kurtosis, Coefficient of Variation and index of dispersion of the two-parameter Odoma distribution

The skweness (γ_1), kurtosis (γ_2), coefficient of variation (γ) and index of dispersion (φ) of the two-parameter Odoma distribution are respectively defined as:

$$\gamma_1 = \sqrt{\beta_1} = \frac{\mu_3}{\sigma^3} = \frac{\left[\alpha^2\left(\alpha^5\beta + \alpha^3 + 24\right)\right]^{\frac{3}{2}} \begin{bmatrix} 2\alpha^{15}\beta^3 + 30\alpha^{13}\beta^2 + 3024\alpha^{10}\beta^2 - 18\alpha^{11}\beta + 2592\alpha^8\beta \\ +3456\alpha^5\beta + 6\alpha^9 + 1008\alpha^6 + 17280\alpha^3 + 138240 \end{bmatrix}}{\alpha^3\left(\alpha^5\beta + \alpha^3 + 24\right)^3\left(\alpha^{10}\beta^2 + 8\alpha^8\beta + 528\alpha^5\beta + 3\alpha^6 + 288\alpha^3 + 2880\right)^{\frac{3}{2}}} \tag{36}$$

$$\gamma_2 = \beta_2 = \frac{\mu_3}{\sigma^4} = \frac{\begin{bmatrix} 9\alpha^{20}\beta^4 + 132\alpha^{18}\beta^3 + 23616\alpha^{15}\beta^3 - 126\alpha^{16}\beta^2 + 33840\alpha^{13}\beta^2 - 551232\alpha^{10}\beta^2 \\ -744\alpha^{14}\beta - 1104\alpha^{11}\beta - 45171312\alpha^5\beta - 2115072\alpha^8\beta + 12\alpha^{18}\beta^2 + 21600\alpha^9\beta \\ -34560\alpha^6\beta - 79272\alpha^9 - 180817920\alpha^3 - 4371840\alpha^6 - 531\alpha^{12} - 9720\alpha^{11} - 1308856320 \end{bmatrix}}{\left(\alpha^{10}\beta^2 + 8\alpha^8\beta + 528\alpha^5\beta + 3\alpha^6 + 288\alpha^3 + 2880\right)^2} \tag{37}$$

$$\gamma = \frac{\sigma}{\mu} = \frac{\left(\alpha^{10}\beta^2 + 8\alpha^8\beta + 528\alpha^5\beta + 3\alpha^6 + 288\alpha^3 + 2880\right)^{\frac{1}{2}}}{\alpha^5\beta + 3\alpha^3 + 120} \tag{38}$$

$$\varphi = \frac{\sigma^2}{\mu} = \frac{\alpha^{10}\beta^2 + 8\alpha^8\beta + 528\alpha^5\beta + 3\alpha^6 + 288\alpha^3 + 2880}{\alpha\left(\alpha^5\beta + \alpha^3 + 24\right)\left(\alpha^5\beta + 3\alpha^3 + 120\right)} \tag{39}$$

C. Moment generating function and characteristic function of the two-parameter Odoma distribution

Proposition 6. *The moment generating function (mgf) of a two-parameter Odoma distributed random variable X with parameters α and β is given by*

$$M_X(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{2(k+4)! + \alpha^3(k+2)! + 2\alpha^5\beta\, k!}{2\alpha^k\left(\alpha^5\beta + \alpha^3 + 24\right)} \tag{40}$$

Proof . The moment generating function of a TPO distributed random variable X is given by

$$M_X(t) = E\left(e^{tX}\right) = \int_0^\infty e^{tx} f(x;\alpha,\beta)\,dx$$

$$M_X(t) = \frac{\alpha^5}{2\left(\alpha^5\beta+\alpha^3+24\right)}\int_0^\infty e^{tx}\left(2x^4+\alpha x^2+2\alpha\beta\right)e^{-\alpha x}dx$$

$$= \frac{\alpha^5}{2\left(\alpha^5\beta+\alpha^3+24\right)} 2\int_0^\infty x^4 e^{-(\alpha-t)x}dx + \alpha\int_0^\infty x^2 e^{-(\alpha-t)x}dx + 2\alpha\beta\int_0^\infty e^{-(\alpha-t)x}dx$$

$$= \frac{\alpha^5}{2\left(\alpha^5\beta+\alpha^3+24\right)}\left[\frac{2\Gamma(5)}{(\alpha-t)^5}+\frac{\alpha\Gamma(3)}{(\alpha-t)^3}+\frac{2\alpha\beta\Gamma(1)}{(\alpha-t)}\right]$$

$$= \frac{\alpha^5}{2\left(\alpha^5\beta+\alpha^3+24\right)}\left[\frac{48}{\alpha^5}\sum_{k=0}^\infty\binom{k+4}{k}\left(\frac{t}{\alpha}\right)^k+\frac{2}{\alpha^2}\sum_{k=0}^\infty\binom{k+4}{k}\left(\frac{t}{\alpha}\right)^k+2\beta\sum_{k=0}^\infty\left(\frac{t}{\alpha}\right)^k\right]$$

$$= \frac{\alpha^5}{2\left(\alpha^5\beta+\alpha^3+24\right)}\left[\frac{2}{\alpha^5}\sum_{k=0}^\infty (k+4)!\frac{t^k}{k!}+\frac{1}{\alpha^2}\sum_{k=0}^\infty (k+2)!\frac{t^k}{k!}+2\alpha\beta k!\sum_{k=0}^\infty\frac{t^k}{k!}\right] \tag{41}$$

Applying little algebra on (41) leads to the required result as defined in (40). Hence, the proof of Proposition 6 has been attained.

Corollary 5. *The characteristic function (cf) of a two-parameter Odoma distributed random variable X with parameters α and β is given by*

$$\phi_X(t) = \sum_{k=0}^\infty \frac{(it)^k}{k!}\,\frac{2(k+4)!+\alpha^3(k+2)!+2\alpha^5\beta\,k!}{2\alpha^k\left(\alpha^5\beta+\alpha^3+24\right)} \tag{42}$$

Proof. The proof of Corollary 5 follows directly from (40) by replacing the index (t) with an index (it).

D. Incomplete moments the two-parameter Odoma distribution

The kth incomplete moment of the random variable $X \sim \text{TPOD}(\alpha,\beta)$, denoted by $I(z,k)$, is given by

$$I(z,k) = \int_{-\infty}^{z} x^k f(x;\alpha,\beta)\,dx$$

$$= \frac{\alpha^5}{2\left(\alpha^5\beta+\alpha^3+24\right)}\int_0^z x^k\left(2x^4+\alpha x^2+2\alpha\beta\right)e^{-\alpha x}dx \tag{43}$$

Putting $z=\alpha x$, $x = z/\alpha$ and $dx = dz/\alpha$ into (43) and simplifying the integral yields

$$I(z,k)=\frac{2\alpha^5\beta\gamma(k+1,\alpha z)+\alpha^3\gamma(k+3,\alpha z)+2\gamma(k+5,\alpha z)}{2\alpha^k\left(\alpha^5\beta+\alpha^3+24\right)} \tag{44}$$

where $\gamma(s,x)=\int_0^x t^{s-1}e^{-t}dt$ is the lower incomplete gamma function.

E. Mean deviations of the two-parameter Odoma distribution

The mean deviation is a special kind of the incomplete moments commonly used to measure the amount of scatter in a population. In view of this, the mean deviation of $X \sim \text{TPOD}(\alpha,\beta)$ about the mean $\mu=E(X)$, denoted by $\text{MD}(\mu)$, is given by

$$\text{MD}(\mu)=E|X-\mu| \tag{45}$$

which according to [11] may be written as

$$\text{MD}(\mu)=2\left\{\mu F(\mu)-\int_0^\mu x\,f(x;\alpha,\beta)\,dx\right\} \tag{46}$$

where μ is given in (27) while $F(\mu)$ and $\int_0^\mu x\,f(x;\alpha,\beta)\,dx$ are respectively given by

$$F(\mu)=1-\left[1+\frac{2\alpha^2\mu^2\left(\alpha^2\mu^2+4\alpha\mu+12\right)+\alpha\mu^2\left(\alpha^4\mu+2\alpha^3+48\right)}{2\left(\alpha^5\beta+\alpha^3+24\right)}\right]e^{-\alpha\mu} \tag{47}$$

and

$$\begin{aligned}\int_0^\mu x\,f(x;\alpha,\beta)\,dx=\frac{\alpha^5\beta+3\alpha^3+60}{\alpha\left(\alpha^5\beta+3\alpha^3+24\right)}-\left[\frac{\alpha^5\mu^5+\alpha^3\left(\alpha^3+20\right)\mu^3+3\alpha^2\left(\alpha^3+20\right)\mu^2}{2\alpha\left(\alpha^5\beta+3\alpha^3+120\right)}\right]e^{-\alpha\mu}\\ -\left[\frac{5\alpha^4\mu^4+2\alpha\left(\alpha^5\beta+3\alpha^3+60\right)\mu+2\left(\alpha^5\beta+3\alpha^3+60\right)}{2\alpha\left(\alpha^5\beta+3\alpha^3+120\right)}\right]e^{-\alpha\mu}\end{aligned} \tag{48}$$

On putting (47) and (48) into (46), the mean deviation of the random variable $X \sim \text{TPOD}(\alpha,\beta)$ about the mean is obtained to be:

$$\text{MD}(\mu)=\frac{\left\{\begin{array}{l}2\alpha\left(\alpha^5\beta+\alpha^3+24\right)\mu-2\left(\alpha^5\beta+3\alpha^3+60\right)\\ -\left[\begin{array}{l}\alpha^5\mu^5-5\alpha^4\mu^4+4\alpha^3\mu^3+\alpha^2\left(8\alpha^2-\alpha-12\right)\mu^2\\ -4\alpha\left(\alpha^3+180\right)\mu-2\left(\alpha^5\beta+3\alpha^3+60\right)\end{array}\right]e^{-\alpha\mu}\end{array}\right\}}{\alpha\left(\alpha^5\beta+\alpha^3+24\right)} \tag{49}$$

Similarly, the mean deviation of the random variable $X \sim \text{TPOD}(\alpha,\beta)$ about the median m, denoted by $\text{MD}(m)$, is given by

$$\text{MD}(m)=\frac{\left\{\begin{matrix}2\alpha\left(\alpha^5\beta+\alpha^3+24\right)m-2\left(\alpha^5\beta+3\alpha^3+60\right)\\ -\left[\begin{matrix}\alpha^5m^5-5\alpha^4m^4+4\alpha^3m^3+\alpha^2\left(8\alpha^2-\alpha-12\right)m^2\\ -4\alpha\left(\alpha^3+180\right)m-2\left(\alpha^5\beta+3\alpha^3+60\right)\end{matrix}\right]e^{-\alpha m}\end{matrix}\right\}}{\alpha\left(\alpha^5\beta+\alpha^3+24\right)} \tag{50}$$

F. Lorenz and Bonferroni Curves for the two-parameter Odoma distribution

The Lorenz curve was introduced by [12] whereas Bonferroni curve was developed by [13]. The two curves are used to measure the inequality of the distribution of a variable. They are applicable in the field of economics, reliability, demography, insurance and medicine.

Proposition 7. *For the random variable X with the two-parameter Odoma distribution having parameters α* and β, *the Lorenz curve is given by*

$$L(x)=\frac{\alpha^5\beta+3\alpha^3+60}{2\left(\alpha^5\beta+3\alpha^3+120\right)}-\left\{\frac{\begin{matrix}\alpha^5x^5+5\alpha^4x^4+\alpha^3\left(\alpha^3+20\right)x^3+3\alpha^2\left(\alpha^3+20\right)x^2\\ +2\alpha\left(\alpha^5\beta+3\alpha^3+60\right)x+2\alpha\left(\alpha^5\beta+3\alpha^3+60\right)\end{matrix}}{2\left(\alpha^5\beta+3\alpha^3+120\right)}\right\}e^{-\alpha x} \tag{51}$$

Proof. Given that X is a nonnegative random variable with pdf $f(x)$, then the Lorenz curve may be defined as

$$L(x)=\frac{1}{\mu}\int_{-\infty}^{x} t\, f(t)\,dt \tag{52}$$

where $\mu=E(X)$ and $x=F^{-1}(t)$. So, for the TPO distribution, one obtains the following Lorenz curve

$$\begin{aligned}L(x)&=\frac{\alpha^5}{2\mu\left(\alpha^5\beta+\alpha^3+24\right)}\int_0^x t\left(2t^4+\alpha t^2+2\alpha\beta\right)e^{-\alpha t}\,dt\\ &=\frac{\alpha^5}{2\mu\left(\alpha^5\beta+\alpha^3+24\right)}\left[2\int_0^x t^5e^{-\alpha t}\,dt+\alpha\int_0^x t^3e^{-\alpha t}\,dt+2\alpha\beta\int_0^x te^{-\alpha t}\,dt\right]\\ &=\frac{\alpha^5}{2\mu\left(\alpha^5\beta+\alpha^3+24\right)}\left[2\int_0^x t^5e^{-\alpha t}\,dt+\alpha\int_0^x t^3e^{-\alpha t}\,dt+2\alpha\beta\int_0^x te^{-\alpha t}\,dt\right]\end{aligned} \tag{53}$$

To evaluate (53), we using integration by parts to evaluate the following integrals

$$2\int_0^x t^5e^{-\alpha t}\,dt=\frac{-x^5e^{-\alpha x}}{\alpha}-\frac{5x^4e^{-\alpha x}}{\alpha^2}-\frac{20x^3e^{-\alpha x}}{\alpha^3}-\frac{60x^2e^{-\alpha x}}{\alpha^4}-\frac{120xe^{-\alpha x}}{\alpha^5}-\frac{120e^{-\alpha x}}{\alpha^6}+\frac{120}{\alpha^6} \tag{54}$$

$$\alpha\int_0^x t^3e^{-\alpha t}\,dt = -x^3e^{-\alpha x} - \frac{3x^2e^{-\alpha x}}{\alpha} - \frac{6xe^{-\alpha x}}{\alpha^2} - \frac{6e^{-\alpha x}}{\alpha^3} + \frac{6}{\alpha^3} \tag{55}$$

$$2\alpha\beta\int_0^x te^{-\alpha t}\,dt = -2\beta xe^{-\alpha x} - \frac{2\beta e^{-\alpha x}}{\alpha} + \frac{2\beta}{\alpha} \tag{56}$$

Putting (54), (55) and (56) into (53) yields

$$L(x) = \frac{\alpha^5}{2\mu\left(\alpha^5\beta+\alpha^3+24\right)}\left\{\frac{2\left(\alpha^5\beta+3\alpha^3+60\right)}{\alpha^6} - \left[\begin{array}{c}\dfrac{\alpha^5x^5+5\alpha^4x^4+\alpha^3\left(\alpha^3+20\right)x^3+3\alpha^2\left(\alpha^3+20\right)x^2}{\alpha^6} \\ \dfrac{+2\alpha\left(\alpha^5\beta+3\alpha^3+60\right)x+2\left(\alpha^5\beta+3\alpha^3+60\right)}{\alpha^6}\end{array}\right]e^{-\alpha x}\right\} \tag{57}$$

Substituting (27) into (57), one obtains the result defined in (51) and that completes the proof of Proposition 7.

Similarly, for the random variable X that follows the two-parameter Odoma distribution with parameters α and β, its Bonferroni curve is given by

$$B(x) = \frac{L(x)}{x} = \frac{\alpha^5\beta+3\alpha^3+60}{2\left(\alpha^5\beta+3\alpha^3+120\right)x} - \left\{\frac{\begin{array}{c}\alpha^5x^5+5\alpha^4x^4+\alpha^3\left(\alpha^3+20\right)x^3+3\alpha^2\left(\alpha^3+20\right)x^2 \\ +2\alpha\left(\alpha^5\beta+3\alpha^3+60\right)x+2\alpha\left(\alpha^5\beta+3\alpha^3+60\right)\end{array}}{2\left(\alpha^5\beta+3\alpha^3+120\right)x}\right\}e^{-\alpha x} \tag{58}$$

VI. RELIABILITY MEASURES OF THE TWO-PARAMETER ODOMA DISTRIBUTION

In this section, we present some useful reliability measures.

A. Survival function of the two-parameter Odoma distribution

Proposition 8. *The survival function of a two-parameter Odoma distributed random variable X with parameters α and β is given by*

$$S(x;\alpha,\beta) = \left[1+\frac{2\alpha^2x^2\left(\alpha^2x^2+4\alpha x+12\right)+\alpha x\left(\alpha^4x+2\alpha^3+48\right)}{2\left(\alpha^5\beta+\alpha^3+24\right)}\right]e^{-\alpha x};\, x>0, \alpha>0, \beta>0 \tag{59}$$

Proof By definition, the survival function of a continuous probability distribution is given by

$$S(x;\alpha,\beta) = 1 - F(x;\alpha,\beta) \tag{60}$$

Thus, if $X \sim TPOD(\alpha,\beta)$, the survival function of X defined in (59) is obtained by substituting (12) into (60) and simplify to get the complete proof of Proposition 8.

Figure 3 shows the plot of the survival function for various values of parameters of the proposed model.

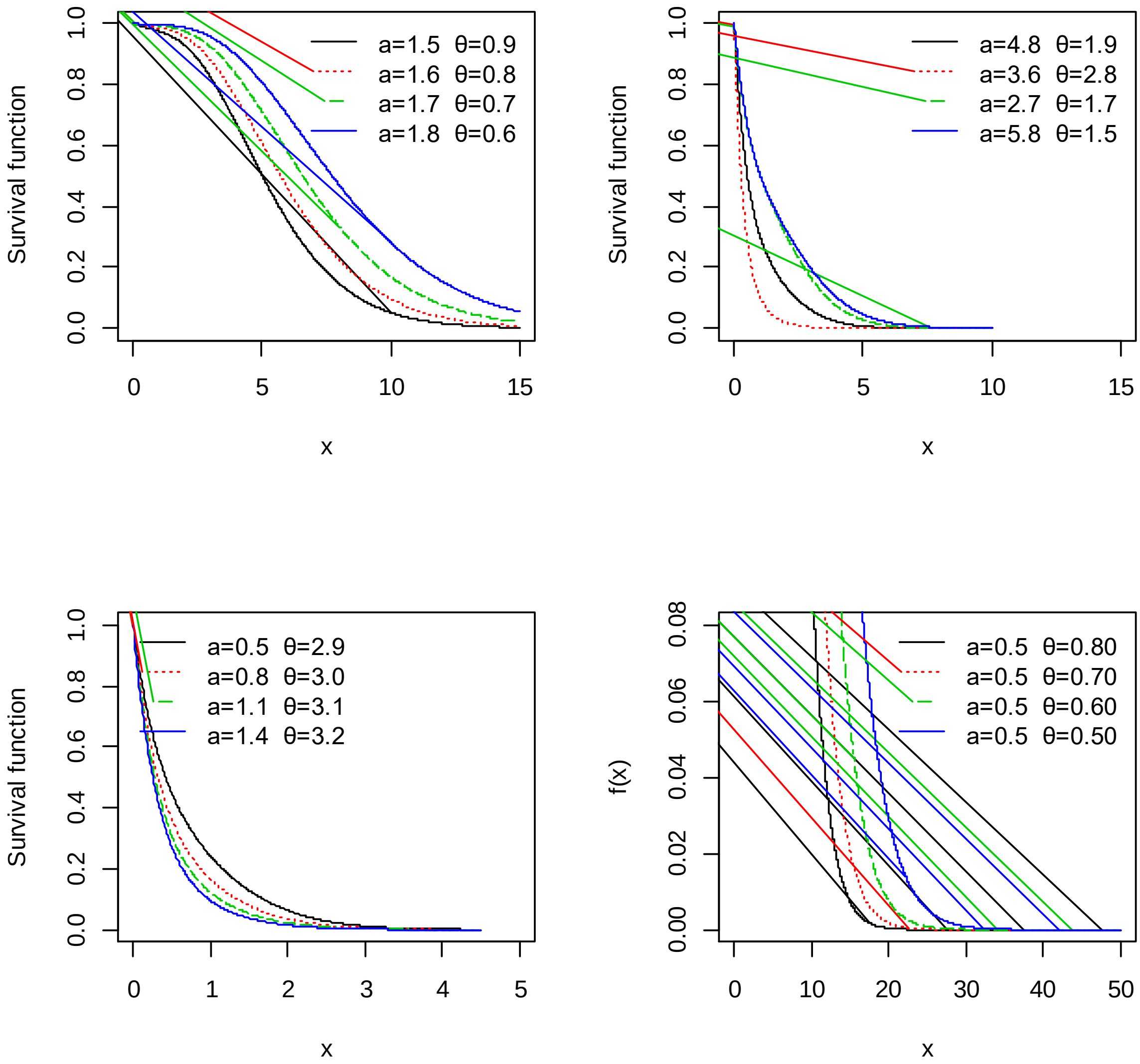

Fig. 3 Various shapes of the survival function of the two-parameter Odoma distribution

B. Hazard function of the two-parameter Odoma distribution

Proposition 9 *The hazard function of a two-parameter Odoma distributed random variable* X *with parameters* α *and* β *is given by*

$$h(x;\alpha,\beta)=\frac{\alpha^5\left(2x^4+\alpha x^2+2\alpha\beta\right)}{2\alpha^2x^2\left(\alpha^2x^2+4\alpha x+12\right)+\alpha x\left(\alpha^4x+2\alpha^3+48\right)+2\left(\alpha^5\beta+\alpha^3+24\right)};x>0,\alpha>0,\beta>0 \quad (61)$$

Proof By definition, the hazard function of a continuous probability distribution is given by

$$h(x;\alpha,\beta)=\frac{f(x;\alpha,\beta)}{S(x;\alpha,\beta)} \quad (62)$$

If $X \sim Odoma(\alpha,\beta)$, the hazard function of X defined in (61) is obtained by substituting (3) and (59) into (62) and the proof of Proposition 9 is complete.

Figure 4 shows the plot of the hazard function for various values of parameters of the proposed model.

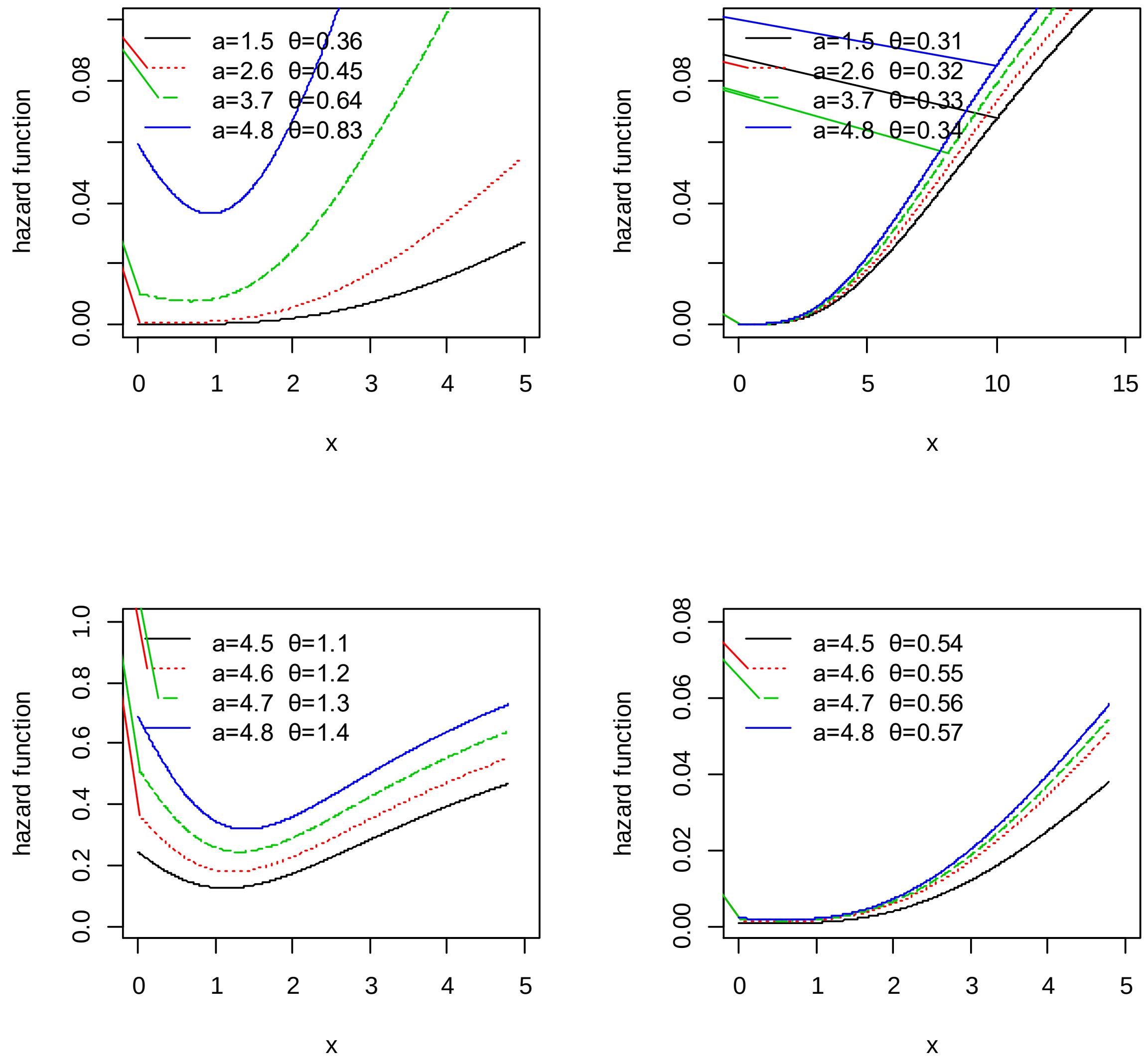

Fig. 4 Various shapes of the hazard function of the two-parameter Odoma distribution

C. Mean residual life function of the two-parameter Odoma distribution

Proposition 10. *The mean residual life function of a two-parameter Odoma distributed random variable* X *with parameters* α *and* β *is given by*

$$m(x;\alpha,\beta)=\frac{2\alpha^4x^4+16\alpha^3x^3+\left(72\alpha^2+\alpha^5\right)x^2+\left(192\alpha+4\alpha^4\right)x+2\left(\alpha^5\beta+3\alpha^3+120\right)}{\alpha\left[2\alpha^2x^2\left(\alpha^2x^2+4\alpha x+12\right)+\alpha x\left(\alpha^4x+2\alpha^3+48\right)+2\left(\alpha^5\beta+\alpha^3+24\right)\right]} \quad (63)$$

Proof. By definition, the mean residual life function of a continuous probability distribution is given by

$$m(x)=\frac{1}{1-F(x;\alpha,\beta)}\int_x^{\infty}\left[1-F(t;\alpha,\beta)\right]dt$$

$$m(x)=\frac{1}{S(x;\alpha,\beta)}\int_x^\infty\left[\frac{2\alpha^2t^2\left(\alpha^2t^2+4\alpha t+12\right)+\alpha t^2\left(\alpha^4t+2\alpha^3+48\right)+2\left(\alpha^5\beta+\alpha^3+24\right)}{2\left(\alpha^5\beta+\alpha^3+24\right)}\right]e^{-\alpha t}\,dt$$

$$=\frac{1}{S(x;\alpha,\beta)}\cdot\frac{\left[2\alpha^4x^4+16\alpha^3x^3+\left(72\alpha^2+4\alpha^5\right)x^2+\left(192\alpha+4\alpha^4\right)x+2\left(\alpha^5\beta+3\alpha^3+120\right)\right]e^{-\alpha x}}{2\left(\alpha^5\beta+\alpha^3+24\right)} \quad (64)$$

Substituting (59) into (64), we obtain (63) and the proof of Proposition 10 is complete.

VII. RENYI ENTROPY OF THE TWO PARAMETER ODOMA DISTRIBUTION

Proposition 11. *The Rényi entropy of a two-parameter Odoma distributed random variable* X *with parameters* α *and* β *is given by*

$$E_R(\rho)=(1-\rho)^{-1}\log\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\rho}{i}\binom{i}{j}\frac{\beta^{\rho-j}\alpha^{6\rho-2i-3j-1}2^{j-i}}{\left(\alpha^5\beta+\alpha^3+24\right)^{\rho}}\frac{(2i+2j)!}{\rho^{2i+2j+1}}\right] \quad (65)$$

Proof. Reference [14] defines entropy of a continuous random variable X as

$$E_R(\rho)=\frac{1}{1-\rho}\log\left(\int_{-\infty}^{\infty}f^{\rho}(x)\,dx\right)$$

$$\Rightarrow E_R(\rho)=(1-\rho)^{-1}\log\left[\int_0^\infty\left(\frac{\alpha^5}{2\left(\alpha^5\beta+\alpha^3+24\right)}\right)^{\rho}\left(2x^4+\alpha x^2+2\alpha\beta\right)^{\rho}e^{-\rho\alpha x}dx\right]$$

$$=(1-\rho)^{-1}\log\left[\frac{(2\alpha\beta)^{\rho}\alpha^{5\rho}}{2^{\rho}\left(\alpha^5\beta+\alpha^3+24\right)^{\rho}}\int_0^\infty\left(1+\frac{x^4}{\alpha\beta}+\frac{x^2}{2\beta}\right)^{\rho}e^{-\rho\alpha x}dx\right] \quad (66)$$

On using binomial expansion on (66), we get

$$\left(1+\frac{x^4}{\alpha\beta}+\frac{x^2}{2\beta}\right)^{\rho}=\sum_{i=0}^{\infty}\binom{\rho}{i}\left(\frac{x^4}{\alpha\beta}+\frac{x^2}{2\beta}\right)^{i}=\sum_{i=0}^{\infty}\binom{\rho}{i}\left(\frac{x^2}{2\beta}\right)^{i}\left(1+\frac{2x^2}{\alpha}\right)^{i}$$

$$=\sum_{i=0}^{\infty}\binom{\rho}{i}2^{-i}\beta^{-i}x^{2i}\sum_{j=0}^{\infty}\binom{i}{j}\left(\frac{2x^2}{\alpha}\right)^{j}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\rho}{i}\binom{i}{j}2^{j-i}\alpha^{-j}\beta^{-i}x^{2(i+j)} \quad (67)$$

Plugging (67) into (66) leads to

$$E_R(\rho)=(1-\rho)^{-1}\log\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\rho}{i}\binom{i}{j}\frac{\beta^{\rho-j}\alpha^{6\rho-j}2^{j-i}}{\left(\alpha^5\beta+\alpha^3+24\right)^{\rho}}\int_0^\infty x^{2(i+j)}e^{-\rho\alpha x}dx\right]$$

$$=(1-\rho)^{-1}\log\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\rho}{i}\binom{i}{j}\frac{\beta^{\rho-j}\alpha^{6\rho-j}2^{j-i}}{\left(\alpha^5\beta+\alpha^3+24\right)^{\rho}}\int_0^\infty x^{[2(i+j)+1]-1}e^{-\rho\alpha x}dx\right]$$

$$=(1-\rho)^{-1}\log\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\rho}{i}\binom{i}{j}\frac{\beta^{\rho-j}\alpha^{6\rho-j}2^{j-i}}{\left(\alpha^5\beta+\alpha^3+24\right)^{\rho}}\frac{\Gamma\left(2\left(i+j\right)+1\right)}{\left(\rho\alpha\right)^{2(i+j)+1}}\right]\because\int_0^{\infty}y^{\varphi-1}e^{-\eta y}=\frac{\Gamma(\varphi)}{\eta^{\varphi}}$$

$$=(1-\rho)^{-1}\log\left[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{\rho}{i}\binom{i}{j}\frac{\beta^{\rho-j}\alpha^{6\rho-2i-3j-1}2^{j-i}}{\left(\alpha^5\beta+\alpha^3+24\right)^{\rho}}\frac{2\left(i+j\right)\Gamma\left[2\left(i+j\right)\right]}{\rho^{2i+2j+1}}\right]\because\Gamma(\varphi+1)=\varphi\Gamma(\varphi)$$

Further simplification of (68) results to (65) and the proof of Proposition 11 is complete.

VIII. MAXIMUM LIKELIHOOD ESTIMATION OF PARAMETERS OF THE TWO-PARAMETER ODOMA DISTRIBUTION

Several methods of estimation exist in statistical literature. However, this work adopted the method of maximum likelihood due to the fact that estimators derived by the maximum likelihood method possess desirable properties such as consistency, asymptotic efficiency and invariance property. To use the method of maximum likelihood estimation, it is assumed that $X_1, X_2, ..., X_n$ constitutes a random sample of size n from the two-parameter Odoma distribution. On taking the natural logarithm of the pdf in (3), one obtains

$$\ln f\left(x;\alpha,\beta\right)=\ln\left(\frac{\alpha^5}{2\left(\alpha^5\beta+\alpha^3+24\right)}\left(2x^4+\alpha x^2+2\alpha\beta\right)e^{-\alpha x}\right)$$

$$=5\ln(\alpha)-\ln(2)-\ln\left(\alpha^5\beta+\alpha^3+24\right)+\ln\left(2x^4+\alpha x^2+2\alpha\beta\right)-\alpha x \qquad (69)$$

Summing both sides of (69) yields the log-likelihood function of the random sample as

$$\ln L(\alpha,\beta)=\sum_{i=1}^{n}\ln f\left(x;\alpha,\beta\right)$$

$$=n\left[5\ln\alpha-\ln(2)-\ln\left(\alpha^5\beta+\alpha^3+24\right)\right]+\sum_{i=1}^{n}\ln\left(2x_i^4+\alpha x_i^2+2\alpha\beta\right)-\alpha\sum_{i=1}^{n}x_i \qquad (70)$$

Differentiating (70) with respect to α gives

$$\frac{\partial\ln L(\alpha,\beta)}{\partial\alpha}=\frac{5n}{\alpha}-\frac{n\alpha^2\left(5\alpha^2\beta+3\right)}{\alpha^5\beta+\alpha^3+24}+\sum_{i=1}^{n}\frac{x_i^2+2\beta}{2x_i^4+\alpha x_i^2+2\alpha\beta}-\sum_{i=1}^{n}x_i \qquad (71)$$

Similarly, a differentiation of (70) with respect to β gives

$$\frac{\partial\ln L(\alpha,\beta)}{\partial\beta}=-\frac{n\alpha^5}{\alpha^5\beta+\alpha^3+24}+\sum_{i=1}^{n}\frac{2\alpha}{2x_i^4+\alpha x_i^2+2\alpha\beta} \qquad (72)$$

To obtain the maximum likelihood estimates of parameters α and β one is required to solve the nonlinear systems of equations $\partial\ln L(\alpha,\beta)/\partial\alpha=0$ and $\partial\ln L(\alpha,\beta)/\partial\beta=0$. A close look at (71) and (72) shows that the resulting systems of equations cannot provide closed form solutions. However, the solutions can be found numerically using some specialized numerical optimization method such as the Quasi-Newton method. For ease of computation, the BFGS iterative method which is implemented in the R software scheme based on the Quasi-Newton method is utilized.

IX. ASYMPTOTIC CONFIDENCE INTERVALS FOR THE MAXIMUM LIKELIHOOD ESTIMATES OF PARAMETERS OF THE TWO-PARAMETER ODOMA DISTRIBUTION

For confidence interval estimation, we first determine the second-order partial derivatives of the log-likelihood function in (70) with respect to each of parameters α and β. Thus,

$$\frac{\partial^2 \ln L(\alpha,\beta)}{\partial \alpha^2} = -\frac{5n}{\alpha^2} + \frac{n\left(5\alpha^8\beta^2 + 4\alpha^6\beta + 3\alpha^4 - 480\alpha^3\beta - 144\alpha\right)}{\left(\alpha^5\beta + \alpha^3 + 24\right)^2} - \sum_{i=1}^{n} \frac{\left(x_i^2 + 2\beta\right)^2}{\left(2x_i^4 + \alpha x_i^2 + 2\alpha\beta\right)^2} \tag{73}$$

$$\frac{\partial^2 \ln L(\alpha,\beta)}{\partial \beta^2} = \frac{n\alpha^{10}}{\left(\alpha^5\beta + \alpha^3 + 24\right)^2} - 4\sum_{i=1}^{n} \frac{\alpha^2}{\left(2x_i^4 + \alpha x_i^2 + 2\alpha\beta\right)^2} \tag{74}$$

$$\frac{\partial^2 \ln L(\alpha,\beta)}{\partial \alpha\, \partial \beta} = -\frac{2n\alpha^4\left(\alpha^3 + 60\right)}{\left(\alpha^5\beta + \alpha^3 + 24\right)^2} + 4\sum_{i=1}^{n} \frac{x_i^2}{\left(2x_i^4 + \alpha x_i^2 + 2\alpha\beta\right)^2} \tag{75}$$

Next, the Fisher Information matrix, denoted by $I(\alpha,\beta)$, is defined as

$$I(\alpha,\beta) = \begin{bmatrix} I_{\alpha\alpha} & I_{\alpha\beta} \\ I_{\beta\alpha} & I_{\beta\beta} \end{bmatrix} \tag{76}$$

where $I_{\alpha\alpha} = -E\left[\frac{\partial^2 L(\alpha,\beta)}{\partial \alpha^2}\right]$, $I_{\beta\beta} = -E\left[\frac{\partial^2 L(\alpha,\beta)}{\partial \beta^2}\right]$, $I_{\alpha\beta} = -E\left[\frac{\partial^2 L(\alpha,\beta)}{\partial \alpha \partial \beta}\right]$ and $I_{\beta\alpha} = -E\left[\frac{\partial^2 L(\alpha,\beta)}{\partial \beta \partial \alpha}\right]$ respectively.

Since the maximum likelihood estimators of the unknown parameters α and β cannot be derived in closed form, it is not easy to derive the exact distributions of the maximum likelihood estimators. Hence, the exact confidence intervals for the parameter cannot be obtained. It is customary to use the large sample approximation to derive the asymptotic distribution of the maximum likelihood estimators.

Consequently, the asymptotic distribution of the MLE's of α and β is

$$\left[\sqrt{n}\left(\hat{\alpha}_{MLE} - \alpha\right), \sqrt{n}\left(\hat{\beta}_{MLE} - \beta\right)\right] \to N_2\left(0, I^{-1}(\alpha,\beta)\right)$$

where $I^{-1}(\alpha,\beta)$ is the inverse of the Fisher Information Matrix $I(\alpha,\beta)$, which also correspond to the variance-covariance matrix of the unknown parameters α and β, defined as

$$I^{-1}(\alpha,\beta) = \begin{bmatrix} \operatorname{var}(\hat{\alpha}) & \operatorname{cov}\left(\hat{\alpha},\hat{\beta}\right) \\ \operatorname{cov}\left(\hat{\beta},\hat{\alpha}\right) & \operatorname{var}\left(\hat{\beta}\right) \end{bmatrix} \tag{77}$$

where $\operatorname{var}(\hat{\alpha})$ and $\operatorname{var}\left(\hat{\beta}\right)$ are the variances of the maximum likelihood estimates of parameters α and β respectively, whereas $\operatorname{cov}\left(\hat{\alpha},\hat{\beta}\right) = \operatorname{cov}\left(\hat{\beta},\hat{\alpha}\right)$ is the covariance between the maximum likelihood estimates of parameters α and β.

The approximate $100(1-\tau)\%$ two-sided confidence intervals for the parameters α and β are given by

$$\hat{\alpha} \pm Z_{\tau/2}\sqrt{\operatorname{var}(\hat{\alpha})} \tag{78}$$

and

$$\hat{\beta} \pm Z_{\tau/2}\sqrt{\operatorname{var}(\hat{\beta})} \tag{79}$$

where $Z_{\tau/2}$ is the upper $(\tau/2)$th percentile of a standard normal distribution.

X. DISTRIBUTIONS OF ORDER STATISTICS FOR THE TWO-PARAMETER ODOMA DISTRIBUTION

In many fields of statistics, the order statistics are found very useful in statistical modelling. Let $X_1, X_2, ..., X_n$ be a random sample of size n from the two-parameter Odoma distribution with cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$. Then, $X_{(1)}, X_{(2)}, ..., X_{(n)}$ denotes the corresponding order statistics, where $X_{(1)} \le X_{(2)}, ... \le X_{(n)}$, $X_{(1)} = min(X_1, X_2, ..., X_n)$ and $X_{(n)} = max(X_1, X_2, ..., X_n)$. The probability density function (pdf) of the kth order statistic is given by

$$\begin{aligned}
f_{X_{(r)}}(x;\alpha,\beta) &= \frac{n!}{(r-1)!(n-r)!}\left[F(x)\right]^{r-1}\left[1-F(x;\alpha,\beta)\right]^{n-r} f(x;\alpha,\beta) \\
&= \frac{n!}{(r-1)!(n-r)!}\sum_{l=0}^{n-r}\binom{n-r}{l}(-1)^l\left[F(x;\alpha,\beta)\right]^{r+l-1} f(x;\alpha,\beta) \\
&= \frac{n!\alpha^5\left(2x^4+\alpha x^2+2\alpha\beta\right)e^{-\alpha x}}{2\left(\alpha^5\beta+\alpha^3+24\right)(r-1)!(n-r)!}\sum_{l=0}^{n-r}\binom{n-r}{l}(-1)^l \\
&\quad \times\left\{1-\left[1+\frac{2\alpha^2x^2\left(\alpha^2x^2+4\alpha x+12\right)+\alpha x^2\left(\alpha^4x+2\alpha^3+48\right)}{2\left(\alpha^5\beta+\alpha^3+24\right)}\right]e^{-\alpha x}\right\}^{r+l-1}
\end{aligned} \tag{80}$$

Putting $r=1$ and $r=n$ into (80) gives the probability density function (pdf) of the first order and nth order statistics respectively for the two-parameter Odoma distribution as

$$\begin{aligned}
f_{X_{(1)}}(x;\alpha,\beta) &= n\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l\left[F(x;\alpha,\beta)\right]^{l} f(x;\alpha,\beta) \\
&= \frac{n\alpha^5\left(2x^4+\alpha x^2+2\alpha\beta\right)e^{-\alpha x}}{2\left(\alpha^5\beta+\alpha^3+24\right)}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l \\
&\quad \times\left\{1-\left[1+\frac{2\alpha^2x^2\left(\alpha^2x^2+4\alpha x+12\right)+\alpha x^2\left(\alpha^4x+2\alpha^3+48\right)}{2\left(\alpha^5\beta+\alpha^3+24\right)}\right]e^{-\alpha x}\right\}^{n+l-1}
\end{aligned} \tag{81}$$

and

$$f_{X_{(n)}}(x;\alpha,\beta) = n\left[F(x;\alpha,\beta)\right]^{n+l-1} f(x;\alpha,\beta)$$

$$= \frac{n\alpha^5\left(2x^4+\alpha x^2+2\alpha\beta\right)e^{-\alpha x}}{2\left(\alpha^5\beta+\alpha^3+24\right)}\left\{1-\left[1+\frac{2\alpha^2x^2\left(\alpha^2x^2+4\alpha x+12\right)+\alpha x^2\left(\alpha^4x+2\alpha^3+48\right)}{2\left(\alpha^5\beta+\alpha^3+24\right)}\right]e^{-\alpha x}\right\}^{n+l-1} \quad (82)$$

XI. APPLICATIONS OF THE TWO-PARAMETER ODOMA DISTRIBUTION

In this section, the application of the proposed two-parameter Odoma distribution is demonstrated by analyzing two real-life data sets.

Data Set 1 -The first data set which is reported in [6] represents the lifetime data relating to time (in months from 1st January, 2013 to 31st July, 2018) of 105 patients who were diagnosed with hypertension and received at least one treatment-related to hypertension in the hospital where death is the event of interest. The data are as given in Table I below.

TABLE I

TIME (IN MONTHS) OF DIAGNOSIS OF HYPERTENSION

45, 37, 14, 64, 67, 58, 67, 55, 64, 62, 9, 65, 65,43, 13, 8, 31, 30, 66, 9, 10, 31, 31, 31, 46, 37, 46, 44,45, 30, 26, 28, 45, 40, 47, 53, 47, 41, 39, 33, 38, 26, 22, 31, 46, 47, 66, 61, 54, 28, 9, 63, 56, 9, 49, 52, 58, 49, 53, 63, 16, 67, 61, 67, 28, 17, 31, 46, 52, 50, 30, 33, 13, 63, 54, 63, 56, 32, 33, 37, 7, 56, 1, 67, 38, 33, 22, 25, 30, 34, 53, 53, 41, 45, 59, 59, 60, 62, 14, 57, 56, 57, 40, 44, 63.

Data Set 2-The second data was first reported in [4] and it represents lifetime data relating to times (in months) of 200 patients who were diagnosed with Hepatitis B where second visit is the event of interest. Table II gives the data.

TABLE II

TIME (IN MONTHS) OF DIAGNOSIS OF HEPATITIS B

27, 32, 8, 30, 34, 23, 41, 36, 28,16, 30, 3, 24, 77, 30, 33, 17, 38, 36, 29, 17, 14, 7, 27, 12, 32, 32, 26, 15, 31, 34, 28, 27, 6, 7, 28, 44, 31, 27, 32, 7, 32, 35, 26, 16, 3, 8, 28, 35, 32, 29, 28, 27, 32, 33, 10, 14, 10, 1, 26, 23, 32, 29, 27, 31, 32, 36, 28, 39, 10, 2, 26, 23, 31, 29, 30, 31, 28, 34, 23, 39, 10, 9, 26 ,18, 31, 46, 30, 39, 33, 34, 31, 19, 3, 6, 27, 23, 30, 46, 38, 39, 33, 37, 30, 29, 3, 6, 32, 26, 30, 33, 48, 40, 32, 30, 25, 30, 3, 7

The two-parameter Odoma (TPO) distribution with five other competitive lifetime distributions would be fitted to the above the data-sets. In view of this, the probability density functions of the competing distribution provided below:

1) Odoma distribution introduced by [7] with pdf:

$$f(x;\alpha) = \frac{\alpha^5}{2\left(\alpha^5+\alpha^3+24\right)}\left(2x^4+\alpha x^2+2\alpha\right)e^{-\alpha x}$$

2) Two-parameter Lindley distribution developed by [2] with pdf:

$$f(x;\alpha,\theta)=\frac{\theta^2}{\alpha\theta+1}(\alpha+x)e^{-\theta x}$$

3) Two-parameter Akash distribution proposed by [3] with pdf:

$$f(x;\theta,\alpha)=\frac{\theta^3}{\alpha\theta^2+2}(\alpha+x^2)e^{-\theta x}$$

4) Two-parameter Rama distribution introduced by [4] with pdf:

$$f(x;\theta,\alpha)=\frac{\theta^4}{\alpha\theta^3+6}(\alpha+x^3)e^{-\theta x}$$

5) Two-parameter Sujatha distribution developed by [5] with pdf:

$$f(x;\theta,\alpha)=\frac{\theta^3}{\theta^2+\alpha\theta+2}(1+\alpha x+x^2)e^{-\theta x}$$

6) A two-parameter Pranav distribution proposed by [6] with pdf:

$$f(x;\theta,\alpha)=\frac{\theta^4}{\alpha\theta^4+6}(\alpha\theta+x^3)e^{-\theta x}$$

It is essential to determine whether the two data-sets being considered in this work actually comes from the proposed model or not. To achieve this purpose, we utilize goodness-of-fit test based on the Cramer von-Mises (CVM) test statistic with its corresponding p-value. The computational formula for this goodness-of-fit test is given by

$$\text{CVM}=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{2i-1}{2n}-\hat{F}(x_i)\right]^2 \tag{83}$$

where $\hat{F}(x_i)$ is the estimated distribution function under the ordered data.

Once it is verified that the data-sets come from the proposed distribution, we then move to estimation of some discrimination criteria, which enables one to select the best model for the data-sets. In this paper, we consider four discrimination criteria, based on the log-likelihood function evaluated at the maximum likelihood estimates, which includes the Akaike information criterion (AIC), Bayesian information criterion (BIC), Hannan-Quinn information criterion (HQIC), and corrected Akaike information criterion (CAIC) respectively. To compute the discrimination criteria the following formulae are used:

$$\text{AIC}=-2l+2k \tag{84}$$

$$\text{BIC}=-2l+k\log(n) \tag{85}$$

$$\text{HQIC}=-2l+2k\log(\log(n)) \tag{86}$$

$$\text{CAIC}=AIC+\frac{2k(k+1)}{n-k-1} \tag{87}$$

where l denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters in the statistical model and n is the sample size of the fitted data respectively. All the computations for (83)-(87) were performed using optim () R-function with argument method= “BFGS”. Generally, for the given data-sets, we consider a distribution to be best among all competing distributions if it has smaller values of the discrimination criteria.

The proposed distribution and other competing models are applied to the two datasets presented in Tables I and II. The estimates of the model parameters, their corresponding standard errors and 95% confidence intervals are presented in Tables III and V for datasets 1 and 2 respectively. The discrimination measures of the fitted models are provided in Tables IV and VI for datasets 1 and 2 respectively.

TABLE III

MAXIMUM LIKELIHOOD ESTIMATES WITH STANDARD ERRORS OF THE FITTED DISTRIBUTION FOR DATA 1

Fitted Distributions	**Parameters**	**Parameter Estimates**	**Standard Error of Estimates**	**Lower 95% Confidence Interval**	**Upper 95% Confidence Interval**
TPOD	β	107706.4	8428.402	91186.732	124226.1
	α	0.1117217	0.004343043	0.1032093	0.120234
OD	α	0.11862468	0.0051765	0.1084787	0.128771
TPRD	β	381.2325	318.2483	-242.5342	1004.999
	α	0.09156654	0.004943675	0.0818769	0.101256
TPAD	β	10.71554757	11.831339	-12.47388	33.90497
	α	0.069947421	0.0041071	0.0618975	0.077997
TPLD	β	0.001	1.1021262	-2.159167	2.161167
	α	0.047441153	0.003499	0.0405831	0.054299
TPSD	β	0.10020882	0.1219608	-0.138834	0.339252
	α	0.06930254	0.0040622	0.0613406	0.077264
TPPD	β	4163.472	3201.471	-2111.411	10438.36
	α	0.09156618	0.004799345	0.0821595	0.100973

TABLE IV

THE ANALYTICAL MEASURES OF THE FITTED MODELS FOR DATA 1

Distribution	**TPOD**	**OD**	**TPLD**	**TPAD**	**TPRD**	**TPSD**	**TPPD**
l	457.1554	467.8207	463.7301	45.4631	472.6460	464.0105	459.4631
AIC	918.3109	937.6415	931.4603	922.9262	949.2920	932.021	922.2341
BIC	923.6188	937.6803	931.5779	923.0438	949.4096	932.1386	928.2341
CAIC	918.4285	937.6803	931.5779	923.0438	949.4096	932.1386	923.0438
HQIC	920.4617	938.7169	933.6112	925.0770	951.4428	934.1718	925.0770
CVM	0.1233	0.2288	0.1366	0.1293	0.5973	0.4966	0.1293
p-value	0.5179	0.2293	0.4635	0.4926	0.0126	0.0300	0.4926

TABLE V

MAXIMUM LIKELIHOOD ESTIMATES WITH STANDARD ERRORS OF THE FITTED DISTRIBUTION FOR DATA 2

Fitted Distributions	Parameters	Parameter Estimates	Standard Error of Estimates	Lower 95% Confidence Interval	Upper 95% Confidence Interval
TPOD	β	27647.98	10326.75	7407.55	47888.41
	α	0.171548	0.008622	0.15465	0.188446
OD	α	0.194397	0.009717	0.175352	0.213442
TPRD	β	20.67996	13.35014	-5.48631	46.84624
	α	0.108177	0.008009	0.092479	0.123875
TPAD	β	334.6642	174.4921	-7.34032	676.6687
	α	0.139958	0.009071	0.122179	0.157737
TPLD	β	0.077992	0.977762	-1.83842	1.994406
	α	0.077478	0.006776	0.064198	0.090758
TPSD	β	0.050105	0.035254	-0.01899	0.119204
	α	0.10716	0.007932	0.091613	0.122707
TPPD	β	2391.21	1354.966	-264.523	5046.943
	α	0.1399576	0.009101	0.122121	0.157794

TABLE VI

THE ANALYTICAL MEASURES OF THE FITTED MODELS FOR DATA 1

Distribution	TPOD	OD	TPLD	TPAD	TPRD	TPSD	TPPD
l	311.6306	334.9436	319.734	315.1193	325.1761	319.9957	315.1193
AIC	627.2613	671.8873	643.4681	634.2385	654.3521	643.9914	634.2385
BIC	632.0253	674.2693	648.2321	639.0026	659.1162	648.7555	639.0026
CAIC	627.4171	671.9385	643.6239	634.3944	654.5080	644.1473	634.3944
HQIC	629.1713	672.8423	645.3781	636.1486	656.2622	645.9015	636.1486
CVM	0.1103	0.1767	0.1243	0.1163	0.5230	0.4949	0.1163
p-value	0.7138	0.2767	0.2224	0.7138	0.0404	0.0404	0.7138

Comparing the results from Tables IV and VI, one can see that the proposed model with the smallest discrimination criteria appears to be the best candidate for the data under consideration. In other words, the TPO distribution outclasses other competing models fitted in this paper. This is evident from Figures 1 through 4, which shows that the two-parameter Odoma distribution has the ability to model lifetime data with varieties of hazard rate shapes. In particular, the TPO distribution has the ability to model lifetime data sets with increasing and bathtub shaped hazard rates. As a result of the flexibility of the shape of the proposed model, it is obvious that the proposed distribution provides the superior fit to the two data-sets than the baseline distribution.

XII. CONCLUDING REMARKS

This paper proposed a two-parameter Odoma distribution, which is more flexible than that the one-parameter Odoma distribution and some existing lifetime distributions. The quantile function and stochastic ordering of the proposed distribution have been derived. Some of the statistical properties of this distribution have been derived, such as the moments, mean, variance, coefficient of variation, index of dispersion, skewness, kurtosis, moment generating function, characteristic function, incomplete moment, mean deviations, Lorenz and Bonferroni curves, Rényi entropy measure and order statistics. Further, some useful reliability properties like survival function, hazard function and mean residual function are derived. The model parameters are estimated using the maximum likelihood estimation method. Two lifetime data are used to illustrate application of the proposed two-parameter Odoma distribution. The comparison of the proposed distribution is conducted with some well-known lifetime distributions such as the Odoma distribution (OD), Two-parameter Lindley distribution (TPLD), Two-parameter Akash distribution (TPAD) and Two-parameter Sujatha distribution (TPSD), Two-parameter Rama distribution (TPRD) and Two-parameter Pranav distribution (TPPD) respectively. The results from data analysis show that TPO distribution provides a better fit to the data sets analyzed than other competing models. It is hoped that this new distribution can be used for effective modelling of varieties of lifetime data as well as in modelling other relevant datasets.

It is worth mentioning that the results in this paper can be extended in some ways. For example, Exponentiated, Kumaraswamy, beta, alpha-power and Transmuted versions of the Odoma distribution may be studied with several fundamental properties explored.

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