Chapter 9 Confidence intervals

Now that we have investigated the theoretical properties of a distribution of a sample means, we are ready to take the next step and apply this knowledge to the process of statistical inference.

Two methods of estimation are commonly used. The first is called point estimation; it involves using the sample data to calculate a single number to estimate the parameter of interest. ($\overline{X} \xrightarrow{estimate} \mu$) A point estimate does not provide any information about the inherent variability of the estimator; we do not know how close $\overline{X}$ is to μ in any given situation. While $\overline{X}$ is more likely to be near the true population mean if the sample on which it is based is large—recall the property of ***consistency***—a point estimate provides no information about the size of this sample. Consequently, a second method of estimation, known as interval estimation, is often preferred. This technique provides a range of reasonable values that are intended to contain the parameter of interest—the population mean μ, in this case—with a certain degree of confidence. This range of values is called the confidence interval.

9.1 Two-sided confidence intervals

To construct a confidence interval for μ, we draw on our knowledge of the *sampling distribution of the mean* from the preceding chapter.

Given r.v. $X \sim N\left(\mu, \sigma^2\right)$, sample size is *n*. By CLT, we can get that

$$\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

Standardizing X, then

$$Z = \frac{\overline{X} - \mu}{\sqrt{\sigma^2/n}} \overset{approximate}{\sim} N(0,1).$$

The 95% of the observations lie between -1.96 and 1.96.

i.e. $P(-1.96 \le Z \le 1.96) = 0.95$

$$\Rightarrow P\left(-1.96 \le \frac{\overline{X} - \mu}{\sqrt{\sigma^2/n}} \le 1.96\right) = 0.95$$

$$\Rightarrow P\left(-1.96 \times \sqrt{\sigma^2/n} \le \overline{X} - \mu \le 1.96 \times \sqrt{\sigma^2/n}\right) = 0.95$$

$$\Rightarrow P\left(-1.96 \times \sqrt{\sigma^2/n} - \overline{X} \le -\mu \le 1.96 \times \sqrt{\sigma^2/n} - \overline{X}\right) = 0.95$$

$$\Rightarrow P\left(1.96 \times \sqrt{\sigma^2/n} + \overline{X} \ge \mu \ge -1.96 \times \sqrt{\sigma^2/n} + \overline{X}\right) = 0.95$$

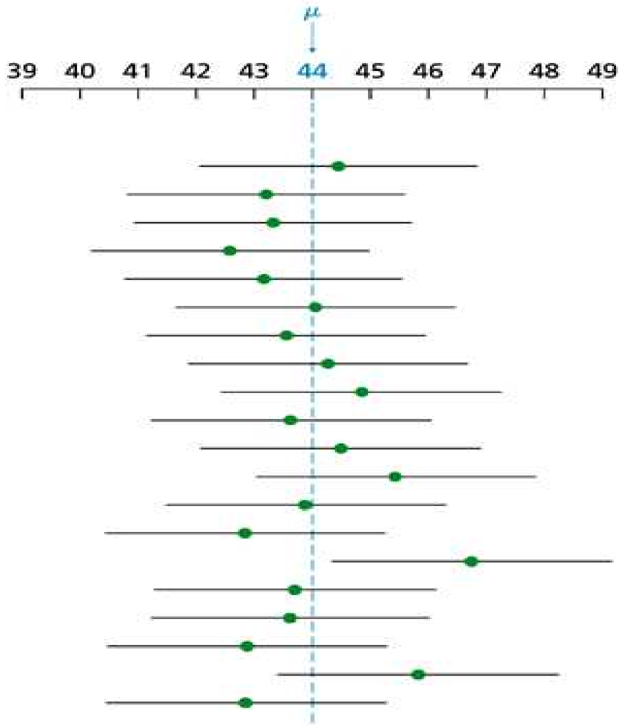
$$\Rightarrow P\left(\overline{X} - 1.96 \times \sqrt{\sigma^2/n} \le \mu \le \overline{X} + 1.96 \times \sqrt{\sigma^2/n}\right) = 0.95$$

The probability statement says something about μ.

- ✓ *The quantities* $\left(\overline{X} - 1.96 \times \sqrt{\sigma^2/n}, \overline{X} + 1.96 \times \sqrt{\sigma^2/n}\right)$ *is 95% confidence interval for the population mean. We are 95% confident that the interval will cover* μ.

- ✓ The statement does ***not*** imply that μ is a random variable that assumes a value within the interval 95% of the time, nor that 95% of the population values lie between these limits; rather, <u>it means that if we were to select 100 random samples from the population and use these samples to calculate 100 different confidence intervals for μ, approximately 95 of the intervals would cover the true population mean and 5 would not.</u>

Sample	$\bar{x}$	95.44% CI	μ in CI?
1	44.45	42.06 to 46.85	yes
2	43.21	40.81 to 45.61	yes
3	43.33	40.93 to 45.73	yes
4	42.59	40.19 to 44.99	yes
5	43.17	40.77 to 45.57	yes
6	44.07	41.67 to 46.47	yes
7	43.56	41.16 to 45.96	yes
8	44.28	41.88 to 46.68	yes
9	44.87	42.48 to 47.27	yes
10	43.61	41.22 to 46.01	yes
11	44.51	42.11 to 46.91	yes
12	45.45	43.05 to 47.85	yes
13	43.88	41.48 to 46.28	yes
14	42.85	40.45 to 45.25	yes
15	46.73	44.33 to 49.13	no
16	43.70	41.30 to 46.10	yes
17	43.60	41.20 to 46.00	yes
18	42.88	40.48 to 45.28	yes
19	45.82	43.42 to 48.22	yes
20	42.84	40.45 to 45.24	yes



- ✓ The estimator $\overline{X}$ is a random variable, whereas the parameter μ is a constant. The interval $\left(\overline{X}-1.96\times\sqrt{\sigma^2/n},\overline{X}+1.96\times\sqrt{\sigma^2/n}\right)$ is random and has a 95% chance of covering μ ***before*** a sample is selected. Once a sample has been drawn and the confidence interval $\left(\bar{x}-1.96\times\sqrt{\sigma^2/n},\bar{x}+1.96\times\sqrt{\sigma^2/n}\right)$ have been calculated, either μ is within the interval or it is not.
- ✓ Can you construct the 99% confidence interval?
- ✓ $(1-\alpha)$: confidence level
- ✓ The general form of $(1-\alpha)\times 100\%$ confidence interval for μ.

$$P\left(-Z_{\alpha/2}\le Z\le Z_{\alpha/2}\right)=0.95$$

$$\Rightarrow P\left(-Z_{\alpha/2} \le \frac{\overline{X}-\mu}{\sqrt{\sigma^2/n}} \le Z_{\alpha/2}\right) = (1-\alpha)\times 100\%$$

$$\Rightarrow P\left(-Z_{\alpha/2}\times\sqrt{\sigma^2/n} \le \overline{X}-\mu \le Z_{\alpha/2}\times\sqrt{\sigma^2/n}\right) = (1-\alpha)\times 100\%$$

$$\Rightarrow P\left(-Z_{\alpha/2}\times\sqrt{\sigma^2/n}-\overline{X} \le -\mu \le Z_{\alpha/2}\times\sqrt{\sigma^2/n}-\overline{X}\right) = (1-\alpha)\times 100\%$$

$$\Rightarrow P\left(\overline{X}-Z_{\alpha/2}\times\sqrt{\sigma^2/n} \le \mu \le \overline{X}+Z_{\alpha/2}\times\sqrt{\sigma^2/n}\right) = (1-\alpha)\times 100\%$$

$\left(\overline{x}-Z_{\alpha/2}\times\sqrt{\sigma^2/n}, \overline{x}+Z_{\alpha/2}\times\sqrt{\sigma^2/n}\right)$ is the $(1-\alpha)\times 100\%$ confidence interval for population mean.

✓ Standard error (標準誤): $\sqrt{\sigma^2/n}$.

✓ The correlation of standard error and sample size. Page 217: table.

✓ The correlation of confidence interval and confidence level.

- *Example* : $\sigma = 46(mg/100ml)$, n=12, $\overline{x} = 217$, construct 95% confidence interval (C. I.) for mean.

- *Example* : $\sigma = 46(mg/100ml)$, n=12, $\bar{x} = 217$, construct 90% confidence interval (C. I.) for mean.

- How large a sample would we need to reduce the length of this interval to only 20 (mg/100ml)?

Sample Size for Estimating μ

The sample size required for a $(1-\alpha)$-level confidence interval for μ with a specified margin of error, E, is given by the formula

$$n = \left(\frac{z_{\alpha/2} \cdot \sigma}{E}\right)^2,$$

rounded up to the nearest whole number.

9.2 One-Sided Confidence Intervals

In some situations, we are concerned with either an <u>upper</u> limit for the population mean μ or a <u>lower</u> limit for μ, but not both.

Example: we consider the distribution of hemoglobin levels—hemoglobin is an oxygen-bearing protein found in red blood cells—for the population of children under the age of 6 who have been exposed to high levels of lead. We know that children who have lead poisoning tend to have much lower levels of hemoglobin than children who do not. We might be interested in finding an upper bound for μ.

Constructing a one-sided confidence interval, we consider the area in one tail of the standard normal distribution only. We find that 95% of the observation for a standard normal random variable.

$$
\begin{aligned}
&P(Z \ge -1.645) = 0.95 \\
&\Rightarrow P\left(\frac{\overline{X} - \mu}{\left(\frac{\sigma}{\sqrt{n}} \right)} \ge -1.645 \right) = 0.95 \\
&\Rightarrow P\left(\overline{X} - \mu \ge -1.645 \times \left(\frac{\sigma}{\sqrt{n}} \right) \right) = 0.95 \\
&\Rightarrow P\left(-\mu \ge -1.645 \times \left(\frac{\sigma}{\sqrt{n}} \right) - \overline{X} \right) = 0.95 \\
&\Rightarrow P\left(\mu \le \overline{X} + 1.645 \times \left(\frac{\sigma}{\sqrt{n}} \right) \right) = 0.95
\end{aligned}
$$

- ✓ $\overline{X} + 1.645 \times \left(\frac{\sigma}{\sqrt{n}} \right)$ is an upper 95% confidence bound for μ. Similarly, we could show that $\overline{X} - 1.645 \times \left(\frac{\sigma}{\sqrt{n}} \right)$ is the corresponding lower 95% confidence bound.

- ✓ n=74, $\bar{x} = 10.6$, one-sided 95% confidence interval for μ:

9.3 Student's t Distribution

When computing confidence intervals for an unknown population mean μ, we have to this point assumed that σ was known. If μ is unknown, σ is probably unknown as well. Instead of using the standard normal distribution, however, the analysis depends on a probability distribution known as Student's t distribution. The name Student is the pseudonym of the statistician who originally discovered this distribution.

$$t = \frac{\overline{X} - \mu}{\left(\frac{s}{\sqrt{n}}\right)}$$

Where s is the sample standard deviation.

- ✓ If X is normally distribution and a sample of size n is randomly chosen from this underlying population, the probability distribution of the random variable

  $$t = \frac{\overline{X} - \mu}{\left(\frac{s}{\sqrt{n}}\right)}$$

  is known as Student's t distribution with n-1 degrees of freedom(df). We represent this using the notation t_{n-1}.
- ✓ Like the standard normal distribution, the t distribution is unimodal and symmetric around its mean of 0. The total area under the curve is equal to 1.(同)
- ✓ It has thicker tails than the normal distribution; extreme values are more likely to occur with the t distribution than with the standard normal. (異)

- ✓ The shape of the t distribution reflects the extra variability introduced by the estimate s. (異)
- ✓ The t distribution has a property called the degrees of freedom. It measure the amount of information available in the data that can be used to estimate σ^2; hence, they measure the reliability of s^2 as an estimate of σ^2.
- ✓ The distributions with smaller degrees of freedom are more spread out; *as df increases, the t distribution approaches the standard normal*.(if n is very large, knowing the value of s is nearly equivalent to knowing σ)

Figure 9.2

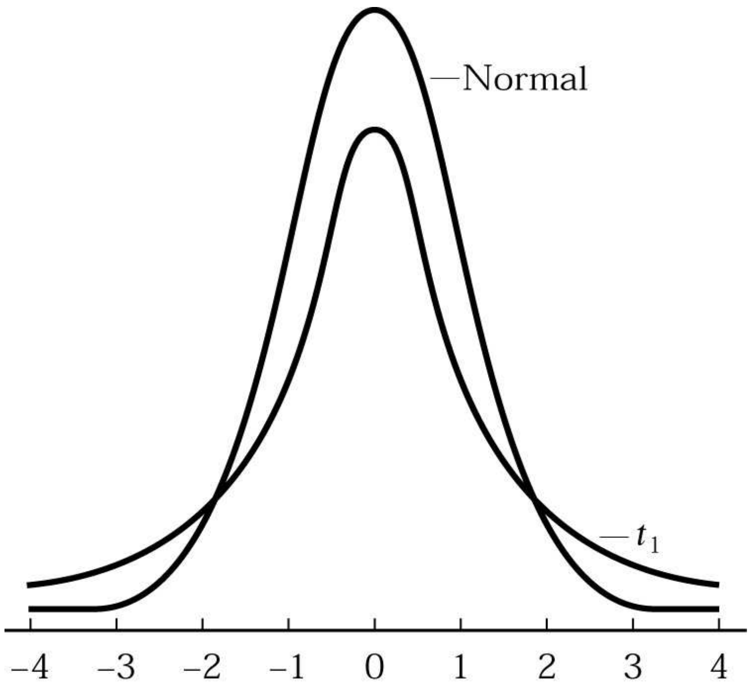


Table A.4 in Appendix A. (t-distribution)

How construct the 95% confidence interval?

$$P\left(-t_{(n-1,\alpha/2)} \le \overline{X} \le t_{(n-1,\alpha/2)}\right) = 0.95$$

$$\Rightarrow P\left(-t_{(n-1,\alpha/2)} \le \frac{\overline{X}-\mu}{\sqrt{s^2/n}} \le t_{(n-1,\alpha/2)}\right) = (1-\alpha)\times 100\%$$

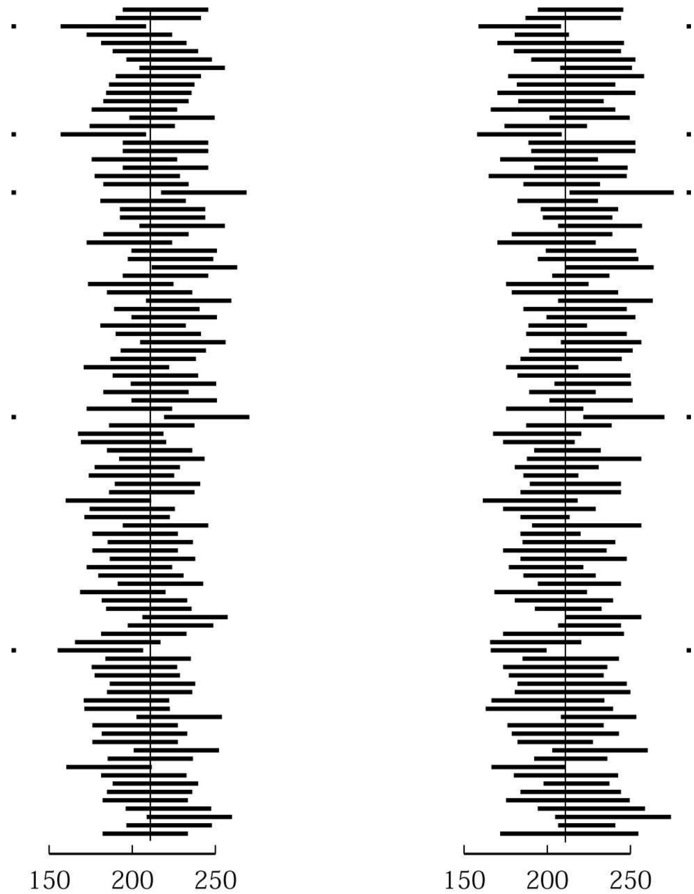
$$\Rightarrow P\left(-t_{(n-1,\alpha/2)} \times \sqrt{s^2/n} \le \overline{X}-\mu \le t_{(n-1,\alpha/2)} \times \sqrt{s^2/n}\right) = (1-\alpha)\times 100\%$$

$$\Rightarrow P\left(-t_{(n-1,\alpha/2)} \times \sqrt{s^2/n} - \overline{X} \le -\mu \le t_{(n-1,\alpha/2)} \times \sqrt{s^2/n} - \overline{X}\right) = (1-\alpha)\times 100\%$$

$$\Rightarrow P\left(\overline{X} - t_{(n-1,\alpha/2)} \times \sqrt{s^2/n} \le \mu \le \overline{X} + t_{(n-1,\alpha/2)} \times \sqrt{s^2/n}\right) = (1-\alpha)\times 100\%$$

- ✓ n=10, s=7.13, $\bar{x}=37.2$, finding the 95% confidence limits for μ.
- ✓ n=10, σ=7.13, $\bar{x}=37.2$, finding the 95% confidence limits for μ.

Figure 9.3



9.5 Review Exercises

1, 2, 4, 5, 6, 8, 9, 11