PS2_406534669

May 17, 2025

1 Advanced Computational Methods Project 2

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1.1.1 Problem 1

[1]:
```
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns

CURRENT_DATE_STR = "2025-05-17"

def configure_plot_aesthetics():
    """
    Sets global plotting aesthetics for a professional and consistent look.
    Utilizes seaborn for enhanced visual appeal.
    """
    sns.set_theme(style="whitegrid", palette="viridis")
    plt.rcParams.update({
        'font.size': 12,
        'axes.titlesize': 16,
        'axes.labelsize': 14,
        'xtick.labelsize': 12,
        'ytick.labelsize': 12,
        'legend.fontsize': 12,
        'figure.titlesize': 18
    })

class OptionPricingError(Exception):
    """Custom exception for errors during option pricing calculation."""
    pass

INITIAL_ASSET_PRICE = 180.00
STRIKE_PRICE = 180.00
TIME_TO_MATURITY_YEARS = 0.5
RISK_FREE_INTEREST_RATE = 0.055
VOLATILITY = 0.25
```

```python
NUMBER_OF_STEPS_SEQUENCE = [20, 40, 80, 100, 200, 500]

def calculate_american_put_cox_ross_rubinstein(
    initial_asset_price: float,
    strike_price: float,
    time_to_maturity: float,
    risk_free_rate: float,
    asset_volatility: float,
    num_time_steps: int
) -> float:
    """
    Calculates the price of an American put option using the␣
  ↪Cox-Ross-Rubinstein (CRR)
    binomial tree model.

    The CRR model defines the up and down factors based on volatility and
    a risk-neutral probability.

    Parameters
    ----------
    initial_asset_price : float
        The current market price of the underlying asset (S0).
    strike_price : float
        The price at which the option holder can sell the asset (K).
    time_to_maturity : float
        The lifespan of the option, expressed in years (T).
    risk_free_rate : float
        The annualized continuously compounded risk-free interest rate (r).
    asset_volatility : float
        The annualized volatility of the underlying asset's returns (sigma).
    num_time_steps : int
        The number of discrete time steps in the binomial tree (n).

    Returns
    -------
    float
        The estimated price of the American put option.

    Raises
    ------
    OptionPricingError
        If `num_time_steps` is not a positive integer or other calculation␣
  ↪errors occur.
    ValueError
        If inputs lead to invalid mathematical operations (e.g., negative sqrt).
    """
    if not isinstance(num_time_steps, int) or num_time_steps <= 0:
```

```
        raise OptionPricingError("Number of time steps (num_time_steps) must be␣
↪a positive integer.")

    try:
        delta_t = time_to_maturity / num_time_steps

        c_factor = 0.5 * (np.exp(-risk_free_rate * delta_t) +
                          np.exp((risk_free_rate + asset_volatility**2) *␣
↪delta_t))

        sqrt_term = c_factor**2 - 1
        if sqrt_term < 0:
            raise ValueError(
                "Invalid parameters leading to a negative value under square␣
↪root "
                "for 'd_factor' calculation. Check parameters like delta_t and␣
↪volatility."
            )

        down_factor = c_factor - np.sqrt(sqrt_term)

        if down_factor == 0:
            raise ValueError("Down factor (d_factor) is zero, leading to␣
↪division by zero for up_factor.")

        up_factor = 1.0 / down_factor

        prob_up = (np.exp(risk_free_rate * delta_t) - down_factor) / (up_factor␣
↪- down_factor)
        prob_down = 1.0 - prob_up

        if not (0 <= prob_up <= 1):
            raise ValueError(
                f"Risk-neutral probability 'prob_up' ({prob_up:.4f}) is outside␣
↪the valid range [0, 1]."
            )

        asset_price_lattice = np.zeros((num_time_steps + 1, num_time_steps + 1))
        option_value_lattice = np.zeros((num_time_steps + 1, num_time_steps +␣
↪1))

        for i_time_step in range(num_time_steps + 1):
            for j_state_node in range(i_time_step + 1):
                asset_price_lattice[j_state_node, i_time_step] =␣
↪initial_asset_price * \
```

```
                    (up_factor**(i_time_step - j_state_node)) *␣
↪(down_factor**j_state_node)

        option_value_lattice[:, num_time_steps] = np.maximum(
            strike_price - asset_price_lattice[:, num_time_steps], 0
        )

        for i_time_step in range(num_time_steps - 1, -1, -1):
            for j_state_node in range(i_time_step + 1):
                value_if_held = np.exp(-risk_free_rate * delta_t) * \
                    (prob_up * option_value_lattice[j_state_node, i_time_step +␣
↪1] +
                     prob_down * option_value_lattice[j_state_node + 1,␣
↪i_time_step + 1])
                value_if_exercised = strike_price -␣
↪asset_price_lattice[j_state_node, i_time_step]
                option_value_lattice[j_state_node, i_time_step] =␣
↪max(value_if_exercised, value_if_held)

        return option_value_lattice[0, 0]

    except FloatingPointError as e:
        raise OptionPricingError(f"A floating point error occurred during CRR␣
↪calculation: {e}")
    except ValueError as e:
        raise OptionPricingError(f"A value error occurred: {e}")


def calculate_american_put_jarrow_rudd(
    initial_asset_price: float,
    strike_price: float,
    time_to_maturity: float,
    risk_free_rate: float,
    asset_volatility: float,
    num_time_steps: int
) -> float:
    """
    Calculates the price of an American put option using the Jarrow-Rudd (JR)
    binomial tree model, also known as the equal-probability model.

    The JR model defines up and down movements such that the risk-neutral
    probability of an up move is 0.5.

    Parameters
    ----------
    initial_asset_price : float
```

```
        The current market price of the underlying asset (S0).
    strike_price : float
        The price at which the option holder can sell the asset (K).
    time_to_maturity : float
        The lifespan of the option, expressed in years (T).
    risk_free_rate : float
        The annualized continuously compounded risk-free interest rate (r).
    asset_volatility : float
        The annualized volatility of the underlying asset's returns (sigma).
    num_time_steps : int
        The number of discrete time steps in the binomial tree (n).

    Returns
    -------
    float
        The estimated price of the American put option.

    Raises
    ------
    OptionPricingError
        If `num_time_steps` is not a positive integer or other calculation␣
↪errors occur.
    ValueError
        If inputs lead to invalid mathematical operations.
    """
    if not isinstance(num_time_steps, int) or num_time_steps <= 0:
        raise OptionPricingError("Number of time steps (num_time_steps) must be␣
↪a positive integer.")

    try:
        delta_t = time_to_maturity / num_time_steps

        up_factor = np.exp(
            (risk_free_rate - 0.5 * asset_volatility**2) * delta_t +
            asset_volatility * np.sqrt(delta_t)
        )
        down_factor = np.exp(
            (risk_free_rate - 0.5 * asset_volatility**2) * delta_t -
            asset_volatility * np.sqrt(delta_t)
        )
        prob_up = 0.5
        prob_down = 0.5

        asset_price_lattice = np.zeros((num_time_steps + 1, num_time_steps + 1))
        option_value_lattice = np.zeros((num_time_steps + 1, num_time_steps +␣
↪1))
```

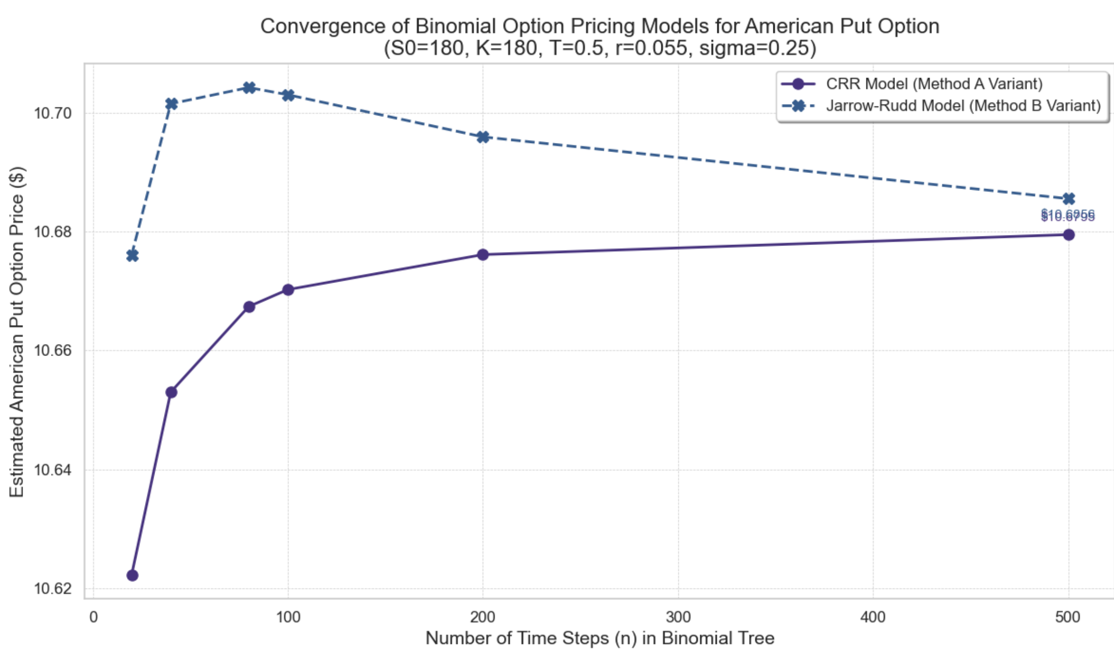
```
        for i_time_step in range(num_time_steps + 1):
            for j_state_node in range(i_time_step + 1):
                asset_price_lattice[j_state_node, i_time_step] = initial_asset_price * \
                    (up_factor**(i_time_step - j_state_node)) * (down_factor**j_state_node)

        option_value_lattice[:, num_time_steps] = np.maximum(
            strike_price - asset_price_lattice[:, num_time_steps], 0
        )

        for i_time_step in range(num_time_steps - 1, -1, -1):
            for j_state_node in range(i_time_step + 1):
                value_if_held = np.exp(-risk_free_rate * delta_t) * \
                    (prob_up * option_value_lattice[j_state_node, i_time_step + 1] +
                     prob_down * option_value_lattice[j_state_node + 1, i_time_step + 1])
                value_if_exercised = strike_price - asset_price_lattice[j_state_node, i_time_step]
                option_value_lattice[j_state_node, i_time_step] = max(value_if_exercised, value_if_held)

        return option_value_lattice[0, 0]

    except FloatingPointError as e:
        raise OptionPricingError(f"A floating point error occurred during JR calculation: {e}")
    except ValueError as e:
        raise OptionPricingError(f"A value error occurred: {e}")


def run_convergence_analysis_and_plot():
    """
    Performs a convergence analysis for American put option pricing using two
    binomial methods (CRR and JR) for a range of time steps.
    Results are then plotted to visualize convergence.
    """
    configure_plot_aesthetics()

    option_prices_crr = []
    option_prices_jr = []

    print("--- American Put Option Price Convergence Analysis ---")
    print(f"Date of Analysis: {CURRENT_DATE_STR}")
    print(f"Initial Asset Price (S0): ${INITIAL_ASSET_PRICE:.2f}")
```

```
    print(f"Strike Price (K): ${STRIKE_PRICE:.2f}")
    print(f"Time to Maturity (T): {TIME_TO_MATURITY_YEARS} years")
    print(f"Risk-Free Rate (r): {RISK_FREE_INTEREST_RATE*100:.2f}%")
    print(f"Volatility (sigma): {VOLATILITY*100:.2f}%\n")
    print("Calculating option prices for different numbers of time steps (n):")
    print("-" * 60)
    print(f"{'N Steps':<10} | {'CRR Price ($)':<20} | {'JR Price ($)':<20}")
    print("-" * 60)

    price_crr_val = np.nan
    price_jr_val = np.nan

    for num_steps in NUMBER_OF_STEPS_SEQUENCE:
        try:
            price_crr_val = calculate_american_put_cox_ross_rubinstein(
                INITIAL_ASSET_PRICE, STRIKE_PRICE, TIME_TO_MATURITY_YEARS,
                RISK_FREE_INTEREST_RATE, VOLATILITY, num_steps
            )
            option_prices_crr.append(price_crr_val)
        except OptionPricingError as e:
            print(f"Error (CRR, N={num_steps}): {e}")
            option_prices_crr.append(np.nan)
            price_crr_val = np.nan


        try:
            price_jr_val = calculate_american_put_jarrow_rudd(
                INITIAL_ASSET_PRICE, STRIKE_PRICE, TIME_TO_MATURITY_YEARS,
                RISK_FREE_INTEREST_RATE, VOLATILITY, num_steps
            )
            option_prices_jr.append(price_jr_val)
        except OptionPricingError as e:
            print(f"Error (JR, N={num_steps}): {e}")
            option_prices_jr.append(np.nan)
            price_jr_val = np.nan

        if not (np.isnan(price_crr_val) or np.isnan(price_jr_val)):
             print(f"{num_steps:<10} | {price_crr_val:<20.4f} | {price_jr_val:<20.4f}")
        elif not np.isnan(price_crr_val):
             print(f"{num_steps:<10} | {price_crr_val:<20.4f} | {'Calculation Error':<20}")
        elif not np.isnan(price_jr_val):
             print(f"{num_steps:<10} | {'Calculation Error':<20} | {price_jr_val:<20.4f}")
        else:
```

```
            print(f"{num_steps:<10} | {'Calculation Error':<20} |␣
↪{'Calculation Error':<20}")


    print("-" * 60)

    plt.figure(figsize=(12, 7))

    plt.plot(
        NUMBER_OF_STEPS_SEQUENCE, option_prices_crr,
        marker='o', linestyle='-', linewidth=2, markersize=8,
        label='CRR Model (Method A Variant)'
    )
    plt.plot(
        NUMBER_OF_STEPS_SEQUENCE, option_prices_jr,
        marker='X', linestyle='--', linewidth=2, markersize=8,
        label='Jarrow-Rudd Model (Method B Variant)'
    )

    plt.xlabel("Number of Time Steps (n) in Binomial Tree")
    plt.ylabel("Estimated American Put Option Price ($)")
    plt.title(
        "Convergence of Binomial Option Pricing Models for American Put␣
↪Option\n"
        f"(S0={INITIAL_ASSET_PRICE:.0f}, K={STRIKE_PRICE:.0f},␣
↪T={TIME_TO_MATURITY_YEARS}, "
        f"r={RISK_FREE_INTEREST_RATE:.3f}, sigma={VOLATILITY:.2f})"
    )
    plt.legend(loc='best', frameon=True, shadow=True)
    plt.grid(True, which="both", linestyle="--", linewidth=0.5)

    if option_prices_crr and not np.isnan(option_prices_crr[-1]):
        plt.annotate(f"${option_prices_crr[-1]:.4f}",
                     (NUMBER_OF_STEPS_SEQUENCE[-1], option_prices_crr[-1]),
                     textcoords="offset points", xytext=(0,10), ha='center',
                     fontsize=10, color=sns.color_palette("viridis")[0])
    if option_prices_jr and not np.isnan(option_prices_jr[-1]):
        plt.annotate(f"${option_prices_jr[-1]:.4f}",
                     (NUMBER_OF_STEPS_SEQUENCE[-1], option_prices_jr[-1]),
                     textcoords="offset points", xytext=(0,-15), ha='center',
                     fontsize=10, color=sns.color_palette("viridis")[1])

    plt.tight_layout()
    plt.show()


if __name__ == "__main__":
```

```
run_convergence_analysis_and_plot()
```

```
--- American Put Option Price Convergence Analysis ---
Date of Analysis: 2025-05-17
Initial Asset Price (S0): $180.00
Strike Price (K): $180.00
Time to Maturity (T): 0.5 years
Risk-Free Rate (r): 5.50%
Volatility (sigma): 25.00%

Calculating option prices for different numbers of time steps (n):
------------------------------------------------------------
N Steps     | CRR Price ($)        | JR Price ($)
------------------------------------------------------------
20          | 10.6223              | 10.6761
40          | 10.6530              | 10.7015
80          | 10.6674              | 10.7043
100         | 10.6702              | 10.7030
200         | 10.6761              | 10.6960
500         | 10.6795              | 10.6856
------------------------------------------------------------
```



1.1.2 Question 2

[2]:
```
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns

CURRENT_DATE_STR = "2025-05-17"

def configure_plot_aesthetics():
    """
    Sets global plotting aesthetics for a professional and consistent look.
    Utilizes seaborn for enhanced visual appeal.
    """
    sns.set_theme(style="whitegrid", palette="mako")
    plt.rcParams.update({
        'font.size': 12,
        'axes.titlesize': 16,
        'axes.labelsize': 14,
        'xtick.labelsize': 12,
        'ytick.labelsize': 12,
        'legend.fontsize': 12,
        'figure.titlesize': 18,
        'lines.markersize': 7,
        'lines.linewidth': 1.8
    })

class OptionPricingError(Exception):
    """Custom exception for errors during option pricing or related␣
 ↪calculations."""
    pass

STRIKE_PRICE = 180.00
RISK_FREE_INTEREST_RATE = 0.055
VOLATILITY = 0.25
TIME_TO_MATURITY_YEARS = 0.5
NUM_TIME_STEPS_FOR_DELTA = 100

ASSET_PRICE_RANGE_FOR_DELTA = np.arange(170.0, 192.0, 2.0)


def calculate_american_put_node_values_crr(
    initial_asset_price: float,
    strike_price: float,
    time_to_maturity: float,
    risk_free_rate: float,
    asset_volatility: float,
    num_time_steps: int
```

```python
) -> tuple[float, float, float, float, float]:
    """
    Calculates the American put option price at t=0 and key node values at t=1
    using the Cox-Ross-Rubinstein (CRR) binomial tree model.

    This version of CRR uses u = exp(sigma*sqrt(dt)) and d = 1/u.
    It returns values necessary for calculating option Greeks like Delta.

    Parameters
    ----------
    initial_asset_price : float
        The current market price of the underlying asset (S0).
    strike_price : float
        The price at which the option holder can sell the asset (K).
    time_to_maturity : float
        The lifespan of the option, expressed in years (T).
    risk_free_rate : float
        The annualized continuously compounded risk-free interest rate (r).
    asset_volatility : float
        The annualized volatility of the underlying asset's returns (sigma).
    num_time_steps : int
        The number of discrete time steps in the binomial tree (n).
        Must be at least 1 for Delta calculation.

    Returns
    -------
    tuple[float, float, float, float, float]
        A tuple containing:
        - option_price_t0: Estimated option price at time 0.
        - option_value_node_up_t1: Option value at t=1 after an up move.
        - option_value_node_down_t1: Option value at t=1 after a down move.
        - asset_price_node_up_t1: Asset price at t=1 after an up move.
        - asset_price_node_down_t1: Asset price at t=1 after a down move.

    Raises
    ------
    OptionPricingError
        If `num_time_steps` is less than 1 or if other calculation errors occur
        (e.g., invalid risk-neutral probability).
    """
    if not isinstance(num_time_steps, int) or num_time_steps < 1:
        raise OptionPricingError("Number of time steps (num_time_steps) must be␣
↪at least 1 for Delta calculation.")

    try:
        delta_t = time_to_maturity / num_time_steps
```

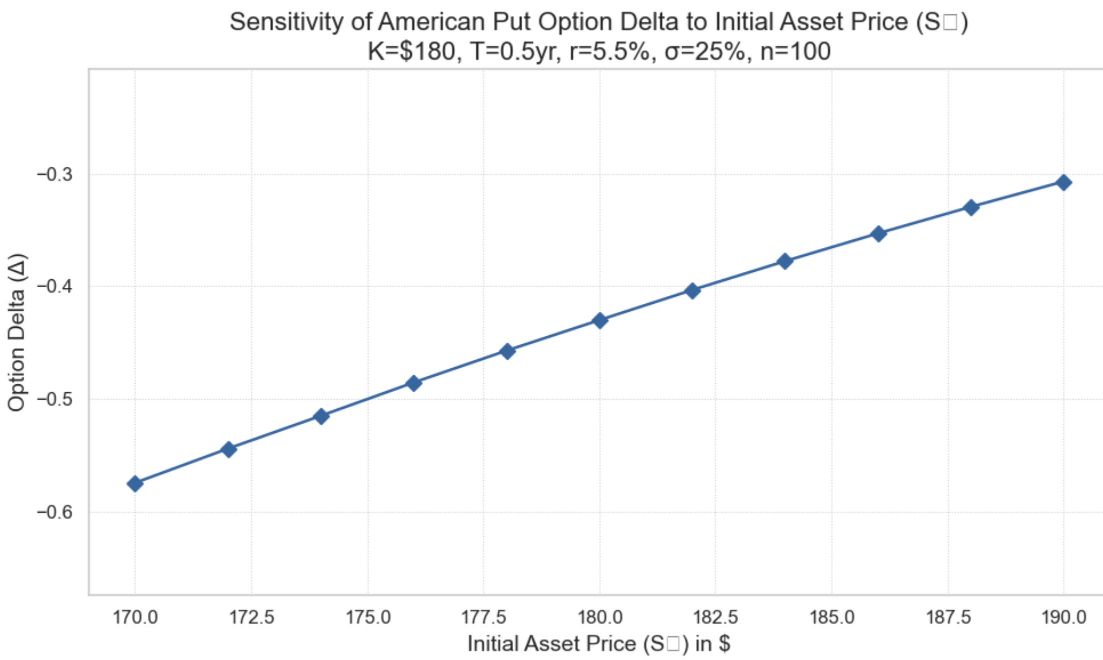
```
        up_factor = np.exp(asset_volatility * np.sqrt(delta_t))
        if up_factor == 1.0: # Avoids division by zero if sigma or dt is zero
             raise ValueError("Up factor is 1.0, leading to u-d = 0. Check volatility and time step.")
        down_factor = 1.0 / up_factor

        risk_neutral_prob_up = (np.exp(risk_free_rate * delta_t) - down_factor) / (up_factor - down_factor)
        risk_neutral_prob_down = 1.0 - risk_neutral_prob_up

        if not (0 <= risk_neutral_prob_up <= 1):
            raise ValueError(
                f"Risk-neutral probability 'prob_up' ({risk_neutral_prob_up:.4f}) is outside the valid range [0, 1]."
            )

        asset_price_lattice = np.zeros((num_time_steps + 1, num_time_steps + 1))
        option_value_lattice = np.zeros((num_time_steps + 1, num_time_steps + 1))

        for i_time in range(num_time_steps + 1):
            for j_nodes in range(i_time + 1):
                asset_price_lattice[j_nodes, i_time] = initial_asset_price * \
                    (up_factor**(i_time - j_nodes)) * (down_factor**j_nodes)

        option_value_lattice[:, num_time_steps] = np.maximum(
            strike_price - asset_price_lattice[:, num_time_steps], 0.0
        )

        for i_time in range(num_time_steps - 1, -1, -1):
            for j_nodes in range(i_time + 1):
                value_if_held = np.exp(-risk_free_rate * delta_t) * \
                    (risk_neutral_prob_up * option_value_lattice[j_nodes, i_time + 1] +
                     risk_neutral_prob_down * option_value_lattice[j_nodes + 1, i_time + 1])
                value_if_exercised = strike_price - asset_price_lattice[j_nodes, i_time]
                option_value_lattice[j_nodes, i_time] = np.maximum(value_if_exercised, value_if_held)

        option_price_t0 = option_value_lattice[0, 0]

        if num_time_steps >= 1:
            option_value_node_up_t1 = option_value_lattice[0, 1]
            option_value_node_down_t1 = option_value_lattice[1, 1]
```

```
            asset_price_node_up_t1 = asset_price_lattice[0, 1]
            asset_price_node_down_t1 = asset_price_lattice[1, 1]
        else: # Should not happen due to initial check, but as a fallback
            option_value_node_up_t1 = np.nan
            option_value_node_down_t1 = np.nan
            asset_price_node_up_t1 = np.nan
            asset_price_node_down_t1 = np.nan

        return (
            option_price_t0,
            option_value_node_up_t1,
            option_value_node_down_t1,
            asset_price_node_up_t1,
            asset_price_node_down_t1
        )

    except FloatingPointError as e:
        raise OptionPricingError(f"A floating point error occurred during CRR␣
↪node value calculation: {e}")
    except ValueError as e:
        raise OptionPricingError(f"A value error occurred during CRR node value␣
↪calculation: {e}")


def perform_delta_sensitivity_analysis():
    """
    Calculates and plots the Delta of an American put option for a range of
    initial asset prices (S0).
    """
    configure_plot_aesthetics()
    calculated_deltas = []

    print("--- American Put Option Delta Sensitivity Analysis ---")
    print(f"Analysis Date: {CURRENT_DATE_STR}")
    print(f"Strike Price (K): ${STRIKE_PRICE:.2f}")
    print(f"Risk-Free Rate (r): {RISK_FREE_INTEREST_RATE*100:.2f}%")
    print(f"Volatility (sigma): {VOLATILITY*100:.2f}%")
    print(f"Time to Maturity (T): {TIME_TO_MATURITY_YEARS} years")
    print(f"Number of Time Steps for Binomial Tree (n):␣
↪{NUM_TIME_STEPS_FOR_DELTA}\n")
    print("Calculating Delta for various initial asset prices (S0):")
    print("-" * 50)
    print(f"{'S0 ($)':<15} | {'Delta':<15}")
    print("-" * 50)

    for current_asset_price in ASSET_PRICE_RANGE_FOR_DELTA:
        try:
```

```
            _, opt_val_up, opt_val_down, asset_price_up, asset_price_down = \
                calculate_american_put_node_values_crr(
                    current_asset_price, STRIKE_PRICE, TIME_TO_MATURITY_YEARS,
                    RISK_FREE_INTEREST_RATE, VOLATILITY, NUM_TIME_STEPS_FOR_DELTA
                )

            if (asset_price_up - asset_price_down) == 0:
                delta_value = np.nan # Avoid division by zero
                print(f"{current_asset_price:<15.2f} | {'Undefined (Su=Sd)':<15}")
            else:
                delta_value = (opt_val_up - opt_val_down) / (asset_price_up - asset_price_down)
                print(f"{current_asset_price:<15.2f} | {delta_value:<15.4f}")
            calculated_deltas.append(delta_value)

        except OptionPricingError as e:
            print(f"Error calculating Delta for S0=${current_asset_price:.2f}: {e}")
            calculated_deltas.append(np.nan)
    print("-" * 50)

    plt.figure(figsize=(10, 6))
    plt.plot(
        ASSET_PRICE_RANGE_FOR_DELTA,
        calculated_deltas,
        marker='D',
        linestyle='-',
        color=sns.color_palette("mako")[2]
    )

    title_str = (
        f"Sensitivity of American Put Option Delta to Initial Asset Price (S )\n"
        f"K=${STRIKE_PRICE:.0f}, T={TIME_TO_MATURITY_YEARS}yr, r={RISK_FREE_INTEREST_RATE*100:.1f}%, "
        f" ={VOLATILITY*100:.0f}%, n={NUM_TIME_STEPS_FOR_DELTA}"
    )
    plt.title(title_str)
    plt.xlabel("Initial Asset Price (S ) in $")
    plt.ylabel("Option Delta (Δ)")
    plt.grid(True, which="both", linestyle=":", linewidth=0.7)

    plt.axhline(0, color='black', lw=0.75, linestyle='--')
    plt.axhline(-1, color='black', lw=0.75, linestyle='--')
```

```
    plt.ylim(min(calculated_deltas)-0.1 if calculated_deltas and not all(np.
 ↪isnan(d) for d in calculated_deltas) else -1.1,
             max(calculated_deltas)+0.1 if calculated_deltas and not all(np.
 ↪isnan(d) for d in calculated_deltas) else 0.1)


    plt.tight_layout()
    plt.show()

if __name__ == "__main__":
    perform_delta_sensitivity_analysis()
```

```
--- American Put Option Delta Sensitivity Analysis ---
Analysis Date: 2025-05-17
Strike Price (K): $180.00
Risk-Free Rate (r): 5.50%
Volatility (sigma): 25.00%
Time to Maturity (T): 0.5 years
Number of Time Steps for Binomial Tree (n): 100

Calculating Delta for various initial asset prices (S0):
--------------------------------------------------
S0 ($)          | Delta
--------------------------------------------------
170.00          | -0.5747
172.00          | -0.5443
174.00          | -0.5151
176.00          | -0.4855
178.00          | -0.4572
180.00          | -0.4300
182.00          | -0.4032
184.00          | -0.3776
186.00          | -0.3529
188.00          | -0.3293
190.00          | -0.3068
--------------------------------------------------

C:\Users\vikal\AppData\Local\Temp\ipykernel_31224\2274194656.py:217:
UserWarning: Glyph 8320 (\N{SUBSCRIPT ZERO}) missing from font(s) Arial.
  plt.tight_layout()
C:\ProgramData\anaconda3\Lib\site-packages\IPython\core\pylabtools.py:170:
UserWarning: Glyph 8320 (\N{SUBSCRIPT ZERO}) missing from font(s) Arial.
  fig.canvas.print_figure(bytes_io, **kw)
```



[3]:

```python
import numpy as np
import matplotlib.pyplot as plt
import seaborn as sns
import datetime

CURRENT_DATETIME_STR = datetime.datetime.now().strftime("%Y-%m-%d %H:%M:%S")

def configure_plot_aesthetics():
    """
    Sets global plotting aesthetics for a professional and consistent look.
    Utilizes seaborn for enhanced visual appeal.
    """
    sns.set_theme(style="whitegrid", palette="crest_r")
    plt.rcParams.update({
        'font.size': 12,
        'axes.titlesize': 16,
        'axes.labelsize': 14,
        'xtick.labelsize': 12,
        'ytick.labelsize': 12,
        'legend.fontsize': 12,
        'figure.titlesize': 18,
        'lines.markersize': 7,
        'lines.linewidth': 1.8
    })
```

```
class OptionPricingError(Exception):
    """Custom exception for errors during option pricing or related␣
↪calculations."""
    pass

STRIKE_PRICE = 180.00
RISK_FREE_INTEREST_RATE = 0.055
VOLATILITY = 0.25
FIXED_INITIAL_ASSET_PRICE = 180.00
FIXED_NUM_TIME_STEPS = 100

TIME_TO_MATURITY_ANALYSIS_RANGE = np.arange(0.003, 0.183, 0.003)


def perform_delta_vs_time_to_maturity_analysis():
    """
    Calculates and plots the Delta of an American put option for a range of
    times to maturity (T), keeping other parameters constant.
    """
    configure_plot_aesthetics()
    calculated_deltas_over_time = []

    print("--- American Put Option Delta vs. Time to Maturity Analysis ---")
    print(f"Analysis Datetime: {CURRENT_DATETIME_STR}")
    print(f"Fixed Initial Asset Price (S0): ${FIXED_INITIAL_ASSET_PRICE:.2f}")
    print(f"Strike Price (K): ${STRIKE_PRICE:.2f}")
    print(f"Risk-Free Rate (r): {RISK_FREE_INTEREST_RATE*100:.2f}%")
    print(f"Volatility (sigma): {VOLATILITY*100:.2f}%")
    print(f"Number of Time Steps for Binomial Tree (n):␣
↪{FIXED_NUM_TIME_STEPS}\n")
    print("Calculating Delta for various times to maturity (T):")
    print("-" * 60)
    print(f"{'Time to Maturity (T)':<25} | {'Delta':<15}")
    print("-" * 60)

    for current_time_to_maturity in TIME_TO_MATURITY_ANALYSIS_RANGE:
        try:
            _, opt_val_up, opt_val_down, asset_price_up, asset_price_down = \
                calculate_american_put_node_values_crr(
                    FIXED_INITIAL_ASSET_PRICE, STRIKE_PRICE,␣
↪current_time_to_maturity,
                    RISK_FREE_INTEREST_RATE, VOLATILITY, FIXED_NUM_TIME_STEPS
                )

            denominator = asset_price_up - asset_price_down
            if denominator == 0:
                delta_value = np.nan
```

```
                print(f"{current_time_to_maturity:<25.4f} | {'Undefined (Su=Sd)':<15}")
            else:
                delta_value = (opt_val_up - opt_val_down) / denominator
                print(f"{current_time_to_maturity:<25.4f} | {delta_value:<15.4f}")
            calculated_deltas_over_time.append(delta_value)

        except OptionPricingError as e:
            print(f"Error for T={current_time_to_maturity:.4f}: {e}")
            calculated_deltas_over_time.append(np.nan)
    print("-" * 60)

    plt.figure(figsize=(10, 6))
    plt.plot(
        TIME_TO_MATURITY_ANALYSIS_RANGE,
        calculated_deltas_over_time,
        marker='^',
        linestyle='-',
        color=sns.color_palette("crest_r")[2]
    )

    title_str = (
        f"American Put Option Delta vs. Time to Maturity (T)\n"
        f"S =${FIXED_INITIAL_ASSET_PRICE:.0f}, K=${STRIKE_PRICE:.0f}, "
        f"r={RISK_FREE_INTEREST_RATE*100:.1f}%,  ={VOLATILITY*100:.0f}%, n={FIXED_NUM_TIME_STEPS}"
    )
    plt.title(title_str)
    plt.xlabel("Time to Maturity (T) in Years")
    plt.ylabel("Option Delta (Δ)")
    plt.grid(True, which="both", linestyle=":", linewidth=0.7)

    plt.axhline(0, color='black', lw=0.75, linestyle='--')
    plt.axhline(-1, color='black', lw=0.75, linestyle='--')

    valid_deltas = [d for d in calculated_deltas_over_time if not np.isnan(d)]
    if valid_deltas:
        min_y = min(valid_deltas)
        max_y = max(valid_deltas)
        padding = (max_y - min_y) * 0.1 if (max_y - min_y) > 0 else 0.1
        plt.ylim(min_y - padding -0.05, max_y + padding + 0.05) # Adjusted padding for better view near -1 or 0
    else: # Fallback if all are NaN
        plt.ylim(-1.1, 0.1)
```

```python
    plt.tight_layout()
    plt.show()

if __name__ == "__main__":
    perform_delta_vs_time_to_maturity_analysis()
```

```
--- American Put Option Delta vs. Time to Maturity Analysis ---
Analysis Datetime: 2025-05-17 14:35:02
Fixed Initial Asset Price (S0): $180.00
Strike Price (K): $180.00
Risk-Free Rate (r): 5.50%
Volatility (sigma): 25.00%
Number of Time Steps for Binomial Tree (n): 100

Calculating Delta for various times to maturity (T):
------------------------------------------------------------
Time to Maturity (T)      | Delta
------------------------------------------------------------
0.0030                    | -0.4935
0.0060                    | -0.4909
0.0090                    | -0.4890
0.0120                    | -0.4874
0.0150                    | -0.4859
0.0180                    | -0.4846
0.0210                    | -0.4835
0.0240                    | -0.4824
0.0270                    | -0.4814
0.0300                    | -0.4804
0.0330                    | -0.4795
0.0360                    | -0.4787
0.0390                    | -0.4779
0.0420                    | -0.4771
0.0450                    | -0.4763
0.0480                    | -0.4756
0.0510                    | -0.4749
0.0540                    | -0.4742
0.0570                    | -0.4736
0.0600                    | -0.4729
0.0630                    | -0.4723
0.0660                    | -0.4717
0.0690                    | -0.4711
0.0720                    | -0.4705
0.0750                    | -0.4700
0.0780                    | -0.4694
0.0810                    | -0.4689
0.0840                    | -0.4683
0.0870                    | -0.4678
0.0900                    | -0.4673
```

```
0.0930                    | -0.4668
0.0960                    | -0.4663
0.0990                    | -0.4659
0.1020                    | -0.4654
0.1050                    | -0.4649
0.1080                    | -0.4645
0.1110                    | -0.4640
0.1140                    | -0.4636
0.1170                    | -0.4631
0.1200                    | -0.4627
0.1230                    | -0.4623
0.1260                    | -0.4619
0.1290                    | -0.4615
0.1320                    | -0.4611
0.1350                    | -0.4607
0.1380                    | -0.4603
0.1410                    | -0.4599
0.1440                    | -0.4595
0.1470                    | -0.4591
0.1500                    | -0.4587
0.1530                    | -0.4584
0.1560                    | -0.4580
0.1590                    | -0.4576
0.1620                    | -0.4573
0.1650                    | -0.4569
0.1680                    | -0.4566
0.1710                    | -0.4562
0.1740                    | -0.4559
0.1770                    | -0.4555
0.1800                    | -0.4552
------------------------------------------------------------

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dummy symbol.
Font 'default' does not have a glyph for '\u2080' [U+2080], substituting with a
dummy symbol.
```
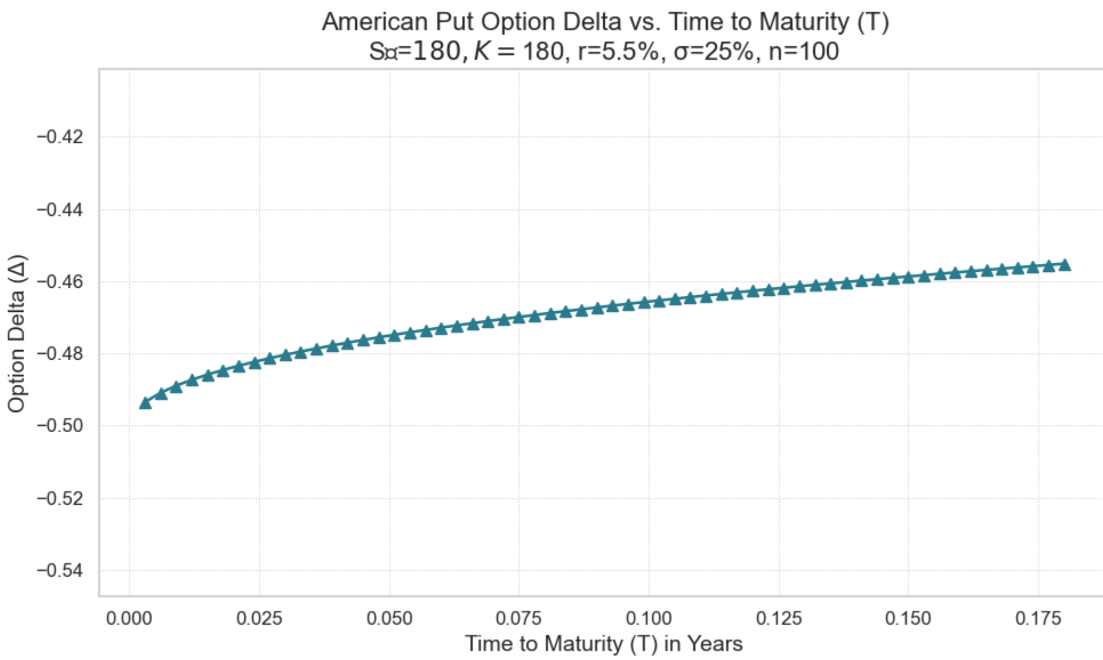



[4]:

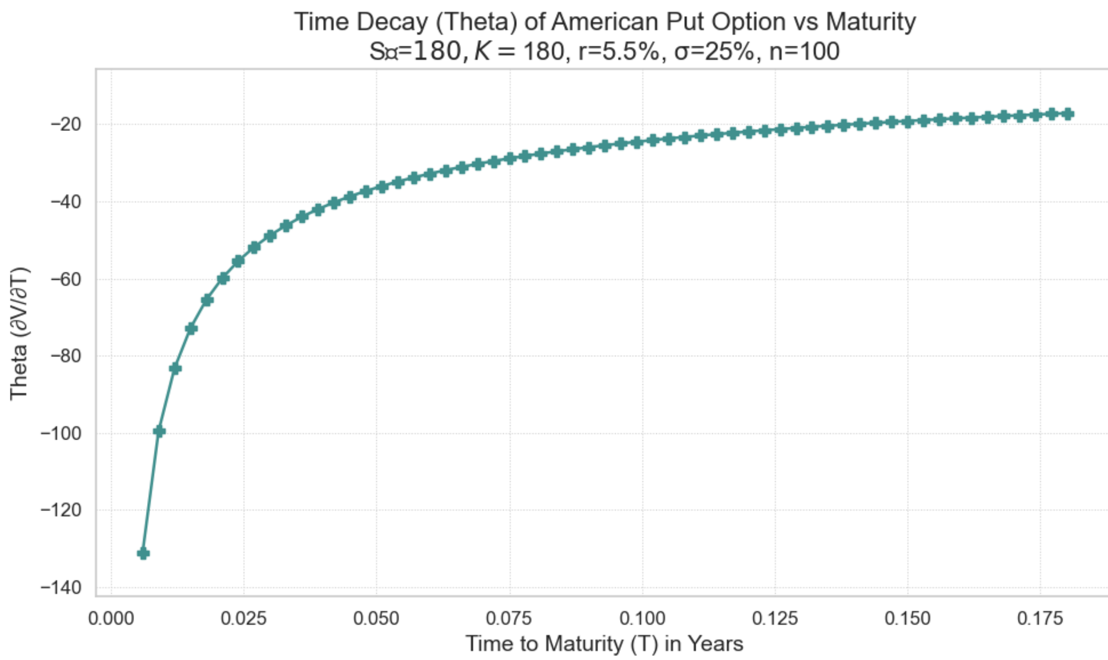
```
# ---- Theta Sensitivity Analysis ----
THETA_ANALYSIS_TIMESTAMP = CURRENT_DATE_STR
BASE_PRICE_THETA = 180.00
STEP_COUNT_THETA = 100
THETA_TIME_VALUES = np.arange(0.006, 0.183, 0.003)
TIME_SHIFT_EPSILON = 0.003

def estimate_theta_via_finite_difference(s0, k, t, r, sigma, n, shift_dt=0.003):
    """
    Estimates Theta by computing option prices at T and T - shift_dt.
    Returns the rate of time decay (Theta) using backward difference.
    """
    try:
        price_now, *_ = calculate_american_put_node_values_crr(s0, k, t, r,␣
↪sigma, n)
        price_before, *_ = calculate_american_put_node_values_crr(s0, k, t -␣
↪shift_dt, r, sigma, n)
        return (price_before - price_now) / shift_dt
    except OptionPricingError as err:
        return np.nan

def run_theta_vs_maturity_analysis():
    """
    Computes and plots the Theta of an American Put option
    against different times to maturity, using CRR model.
```

```
    """
    configure_plot_aesthetics()
    theta_outputs = []

    print("=== American Put Option Theta vs Time to Maturity ===")
    print(f"Run Date: {THETA_ANALYSIS_TIMESTAMP}")
    print(f"S0 = ${BASE_PRICE_THETA:.2f}, K = ${STRIKE_PRICE:.2f}, r =␣
↪{RISK_FREE_INTEREST_RATE:.2%}, "
          f"sigma = {VOLATILITY:.2%}, Steps = {STEP_COUNT_THETA}")
    print("-" * 55)
    print(f"{'T (Years)':<20} | {'Theta':<15}")
    print("-" * 55)

    for ttm in THETA_TIME_VALUES:
        theta_val = estimate_theta_via_finite_difference(
            BASE_PRICE_THETA, STRIKE_PRICE, ttm,
            RISK_FREE_INTEREST_RATE, VOLATILITY,
            STEP_COUNT_THETA, TIME_SHIFT_EPSILON
        )
        theta_outputs.append(theta_val)
        theta_display = f"{theta_val:.5f}" if not np.isnan(theta_val) else␣
↪"Calculation Error"
        print(f"{ttm:<20.4f} | {theta_display:<15}")

    print("-" * 55)

    plt.figure(figsize=(10, 6))
    plt.plot(
        THETA_TIME_VALUES, theta_outputs,
        marker='P', linestyle='-', color=sns.color_palette("crest_r")[3]
    )
    plt.title(
        f"Time Decay (Theta) of American Put Option vs Maturity\n"
        f"S =${BASE_PRICE_THETA:.0f}, K=${STRIKE_PRICE:.0f}, "
        f"r={RISK_FREE_INTEREST_RATE*100:.1f}%,  ={VOLATILITY*100:.0f}%,␣
↪n={STEP_COUNT_THETA}"
    )
    plt.xlabel("Time to Maturity (T) in Years")
    plt.ylabel("Theta ( V/ T)")
    plt.grid(True, linestyle=":", linewidth=0.7)
    plt.axhline(0, color='black', linestyle='--', linewidth=0.8)

    finite_values = [theta for theta in theta_outputs if not np.isnan(theta)]
    if finite_values:
        min_theta = min(finite_values)
        max_theta = max(finite_values)
```

```
        pad = 0.1 * (max_theta - min_theta) if (max_theta - min_theta) > 0 else␣
↪0.05
        plt.ylim(min_theta - pad, max_theta + pad)
    else:
        plt.ylim(-1.1, 0.1)

    plt.tight_layout()
    plt.show()

if __name__ == "__main__":
    run_theta_vs_maturity_analysis()
```

```
=== American Put Option Theta vs Time to Maturity ===
Run Date: 2025-05-17
S0 = $180.00, K = $180.00, r = 5.50%, sigma = 25.00%, Steps = 100
-------------------------------------------------------
T (Years)            | Theta
-------------------------------------------------------
0.0060               | -131.08931
0.0090               | -99.60484
0.0120               | -83.31529
0.0150               | -72.91071
0.0180               | -65.51749
0.0210               | -59.91867
0.0240               | -55.48708
0.0270               | -51.86401
0.0300               | -48.83428
0.0330               | -46.25056
0.0360               | -44.01388
0.0390               | -42.05231
0.0420               | -40.31010
0.0450               | -38.75373
0.0480               | -37.35219
0.0510               | -36.08131
0.0540               | -34.91808
0.0570               | -33.85056
0.0600               | -32.86900
0.0630               | -31.95857
0.0660               | -31.11164
0.0690               | -30.32067
0.0720               | -29.58677
0.0750               | -28.89770
0.0780               | -28.24871
0.0810               | -27.63503
0.0840               | -27.05590
0.0870               | -26.50817
0.0900               | -25.98581
0.0930               | -25.49073
```

```
0.0960               | -25.02497
0.0990               | -24.57592
0.1020               | -24.14624
0.1050               | -23.73636
0.1080               | -23.34627
0.1110               | -22.96863
0.1140               | -22.61363
0.1170               | -22.26763
0.1200               | -21.93439
0.1230               | -21.61505
0.1260               | -21.30691
0.1290               | -21.00926
0.1320               | -20.72315
0.1350               | -20.44649
0.1380               | -20.17818
0.1410               | -19.92048
0.1440               | -19.66927
0.1470               | -19.42641
0.1500               | -19.19013
0.1530               | -18.96185
0.1560               | -18.74414
0.1590               | -18.52756
0.1620               | -18.31786
0.1650               | -18.11376
0.1680               | -17.91514
0.1710               | -17.72129
0.1740               | -17.53194
0.1770               | -17.34757
0.1800               | -17.17301
-------------------------------------------------------

Font 'default' does not have a glyph for '\u2080' [U+2080], substituting with a
dummy symbol.
Font 'default' does not have a glyph for '\u2080' [U+2080], substituting with a
dummy symbol.
```

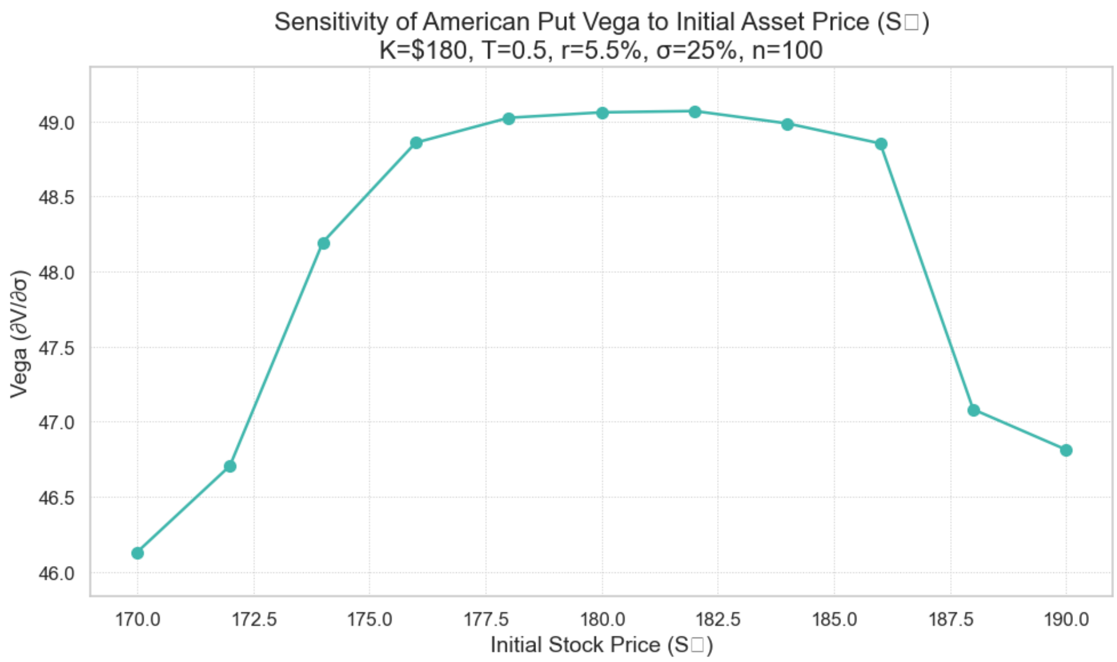


[5]:
```python
# ---- Vega Sensitivity Analysis ----
VEGA_ANALYSIS_DATE = CURRENT_DATE_STR
FIXED_STRIKE_VEGA = STRIKE_PRICE
FIXED_RATE_VEGA = RISK_FREE_INTEREST_RATE
FIXED_MATURITY_VEGA = TIME_TO_MATURITY_YEARS
FIXED_STEPS_VEGA = 100
VOL_SHIFT_EPSILON = 0.01
SO_SPAN_VEGA = np.arange(170, 192, 2)

def estimate_vega_crr_binomial(
    s0: float, k: float, t: float, r: float, vol: float, n: int, d_sigma: float = 0.01
) -> float:
    """
    Estimates Vega using central finite difference on volatility ().
    """
    try:
        price_up, *_ = calculate_american_put_node_values_crr(s0, k, t, r, vol + d_sigma, n)
        price_down, *_ = calculate_american_put_node_values_crr(s0, k, t, r, vol - d_sigma, n)
        return (price_up - price_down) / (2 * d_sigma)
    except OptionPricingError:
        return np.nan
```

```python
def perform_vega_vs_spot_analysis():
    """
    Evaluates and plots Vega of an American Put with respect to varying S .
    """
    configure_plot_aesthetics()
    vega_values = []

    print("=== American Put Vega Sensitivity to Spot Price ===")
    print(f"Analysis Date: {VEGA_ANALYSIS_DATE}")
    print(f"Strike = ${FIXED_STRIKE_VEGA:.2f}, r = {FIXED_RATE_VEGA:.2%}, T = {FIXED_MATURITY_VEGA}, "
          f"  = {VOLATILITY:.2%}, Steps = {FIXED_STEPS_VEGA}")
    print(f"Volatility shift for Vega   {VOL_SHIFT_EPSILON}\n")
    print("-" * 50)
    print(f"{'S  ($)':<15} | {'Vega':<15}")
    print("-" * 50)

    for spot in S0_SPAN_VEGA:
        vega_val = estimate_vega_crr_binomial(
            s0=spot, k=FIXED_STRIKE_VEGA, t=FIXED_MATURITY_VEGA,
            r=FIXED_RATE_VEGA, vol=VOLATILITY, n=FIXED_STEPS_VEGA,
            d_sigma=VOL_SHIFT_EPSILON
        )
        vega_values.append(vega_val)
        display_val = f"{vega_val:.5f}" if not np.isnan(vega_val) else "Calc Error"
        print(f"{spot:<15.2f} | {display_val:<15}")
    print("-" * 50)

    plt.figure(figsize=(10, 6))
    plt.plot(
        S0_SPAN_VEGA, vega_values,
        marker='o', linestyle='-', color=sns.color_palette("mako")[4]
    )
    plt.title(
        f"Sensitivity of American Put Vega to Initial Asset Price (S )\n"
        f"K=${FIXED_STRIKE_VEGA:.0f}, T={FIXED_MATURITY_VEGA}, "
        f"r={FIXED_RATE_VEGA*100:.1f}%,  ={VOLATILITY*100:.0f}%, n={FIXED_STEPS_VEGA}"
    )
    plt.xlabel("Initial Stock Price (S )")
    plt.ylabel("Vega ( V/ )")
    plt.grid(True, linestyle=":", linewidth=0.7)
    plt.axhline(0, color='black', linestyle='--', linewidth=0.8)

    finite_vegas = [v for v in vega_values if not np.isnan(v)]
    if finite_vegas:
```

```
        buffer = 0.1 * (max(finite_vegas) - min(finite_vegas)) if␣
↪max(finite_vegas) != min(finite_vegas) else 0.05
        plt.ylim(min(finite_vegas) - buffer, max(finite_vegas) + buffer)
    else:
        plt.ylim(-1.0, 1.0)

    plt.tight_layout()
    plt.show()

if __name__ == "__main__":
    perform_vega_vs_spot_analysis()
```

```
=== American Put Vega Sensitivity to Spot Price ===
Analysis Date: 2025-05-17
Strike = $180.00, r = 5.50%, T = 0.5,   = 25.00%, Steps = 100
Volatility shift for Vega   0.01

--------------------------------------------------
S  ($)          | Vega
--------------------------------------------------
170.00          | 46.13131
172.00          | 46.70613
174.00          | 48.19371
176.00          | 48.85663
178.00          | 49.02224
180.00          | 49.05830
182.00          | 49.06724
184.00          | 48.98469
186.00          | 48.85165
188.00          | 47.08231
190.00          | 46.81402
--------------------------------------------------

C:\Users\vikal\AppData\Local\Temp\ipykernel_31224\980761553.py:72: UserWarning:
Glyph 8320 (\N{SUBSCRIPT ZERO}) missing from font(s) Arial.
  plt.tight_layout()
```



1.1.3 Question 3

[6]:
```
def compute_trinomial_american_put(
    s0: float = 180.0,
    k: float = 180.0,
    t: float = 0.5,
    r: float = 0.055,
    sigma: float = 0.25,
    steps: int = 100
) -> float:
    """
    Computes the price of an American put option using a recombining trinomial␣
 ↪tree.

    This implementation follows a lognormal trinomial model where the asset␣
 ↪price can
    move up, down, or stay the same in each time step. The model accounts for␣
 ↪early
    exercise, making it suitable for pricing American-style options.

    Parameters
    ----------
    s0 : float
        Current price of the underlying asset.
    k : float
```

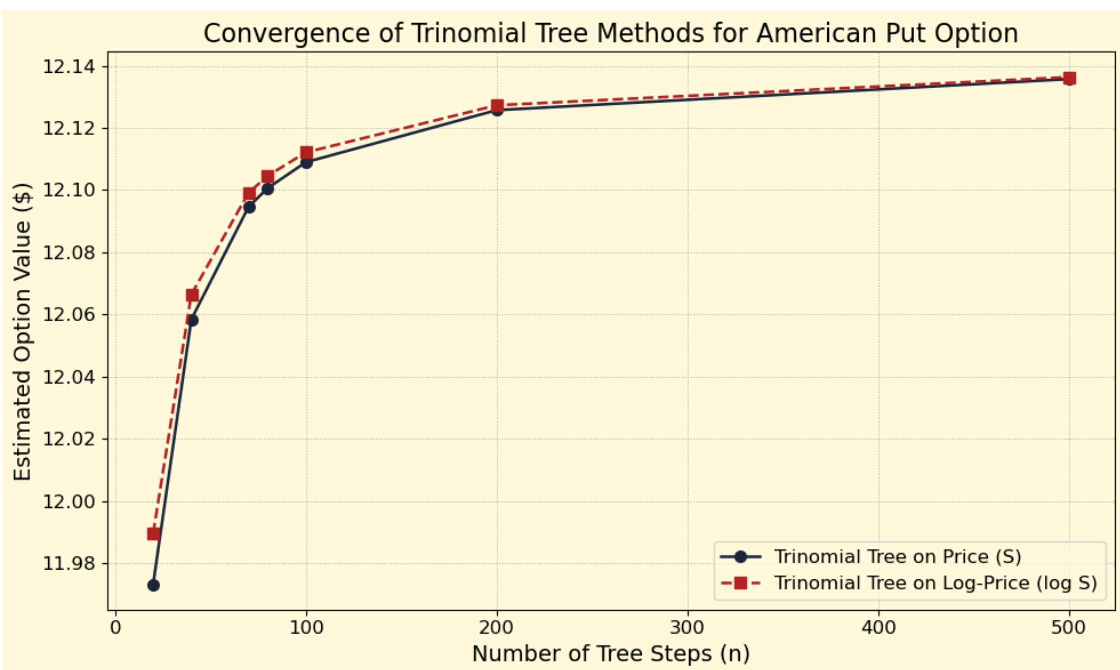
```python
        Strike price of the option.
    t : float
        Time to maturity in years.
    r : float
        Continuously compounded annual risk-free rate.
    sigma : float
        Annual volatility of the underlying asset.
    steps : int
        Number of time intervals in the trinomial tree.

    Returns
    -------
    float
        The estimated price of the American put option at time 0.
    """
    dt = t / steps
    df = np.exp(-r * dt)
    d = np.exp(-sigma * np.sqrt(3 * dt))
    u = 1 / d

    alpha = r * dt
    beta = sigma**2 * dt
    gamma = alpha**2

    pu = (alpha * (1 - d) + beta + gamma) / ((u - d) * (u - 1))
    pd = (alpha * (1 - u) + beta + gamma) / ((u - d) * (1 - d))
    pm = 1.0 - pu - pd

    tree = np.zeros((2 * steps + 1, steps + 1))

    for i in range(2 * steps + 1):
        m = i - steps
        s_terminal = s0 * (u ** max(m, 0)) * (d ** max(-m, 0))
        tree[i, steps] = max(k - s_terminal, 0.0)

    for j in range(steps - 1, -1, -1):
        for i in range(j + steps + 1):
            cont = df * (
                pu * tree[i - 1, j + 1] +
                pm * tree[i, j + 1] +
                pd * tree[i + 1, j + 1]
            )
            m = i - steps
            s_curr = s0 * (u ** max(m, 0)) * (d ** max(-m, 0))
            tree[i, j] = max(cont, k - s_curr)

    return tree[steps, 0]
```

```
[7]: def compute_trinomial_logprice_american_put(
    s0: float = 180.0,
    k: float = 180.0,
    t: float = 0.5,
    r: float = 0.055,
    sigma: float = 0.25,
    steps: int = 100
) -> float:
    """
    Computes the price of an American put option using a trinomial tree
    constructed in the log-price domain (X = log(S)).

    This implementation is based on a recombining log-space tree with
    risk-neutral probabilities derived from the Taylor approximation
    of the SDE under the log-normal model. Early exercise is allowed.

    Parameters
    ----------
    s0 : float
        Current asset price.
    k : float
        Strike price of the option.
    t : float
        Time to expiration in years.
    r : float
        Continuously compounded risk-free interest rate.
    sigma : float
        Volatility of the underlying asset.
    steps : int
        Number of discrete steps in the trinomial tree.

    Returns
    -------
    float
        Estimated price of the American put option.
    """
    dt = t / steps
    dx = sigma * np.sqrt(3 * dt)
    disc = np.exp(-r * dt)
    mu = r - 0.5 * sigma**2

    a = sigma**2 * dt + (mu**2) * dt**2
    b = mu * dt

    pu = 0.5 * (a / dx**2 + b / dx)
    pd = 0.5 * (a / dx**2 - b / dx)
    pm = 1.0 - pu - pd
```

```
    value_matrix = np.zeros((2 * steps + 1, steps + 1))

    for i in range(2 * steps + 1):
        displacement = i - steps
        x_terminal = np.log(s0) + displacement * dx
        s_terminal = np.exp(x_terminal)
        value_matrix[i, steps] = max(k - s_terminal, 0.0)

    for j in range(steps - 1, -1, -1):
        for i in range(j + steps + 1):
            continuation = disc * (
                pu * value_matrix[i - 1, j + 1] +
                pm * value_matrix[i, j + 1] +
                pd * value_matrix[i + 1, j + 1]
            )
            x_curr = np.log(s0) + (i - steps) * dx
            s_curr = np.exp(x_curr)
            value_matrix[i, j] = max(continuation, k - s_curr)

    return value_matrix[steps, 0]
```

[9]:
```
def plot_trinomial_model_convergence():
    """
    Evaluates and visualizes the convergence behavior of two trinomial tree␣
↪approaches
    for pricing American put options: one using direct price modeling, and the␣
↪other
    using log-price modeling.
    """
    plt.style.use("default")
    plt.rcParams.update({
        'font.size': 12,
        'axes.titlesize': 16,
        'axes.labelsize': 14,
        'xtick.labelsize': 12,
        'ytick.labelsize': 12,
        'legend.fontsize': 12,
        'figure.titlesize': 18,
        'lines.markersize': 7,
        'lines.linewidth': 1.8,
        'figure.facecolor': '#FFF9DB',
        'axes.facecolor': '#FFF9DB',
        'savefig.facecolor': '#FFF9DB'
    })

    step_sequence = [20, 40, 70, 80, 100, 200, 500]
```

```
    results_price_tree = [compute_trinomial_american_put(steps=n) for n in␣
↪step_sequence]
    results_logprice_tree = [compute_trinomial_logprice_american_put(steps=n)␣
↪for n in step_sequence]

    plt.figure(figsize=(10, 6))
    plt.plot(
        step_sequence, results_price_tree,
        marker='o', linestyle='-', color='#1B263B',
        label='Trinomial Tree on Price (S)'
    )
    plt.plot(
        step_sequence, results_logprice_tree,
        marker='s', linestyle='--', color='#B22222',
        label='Trinomial Tree on Log-Price (log S)'
    )

    plt.title("Convergence of Trinomial Tree Methods for American Put Option")
    plt.xlabel("Number of Tree Steps (n)")
    plt.ylabel("Estimated Option Value ($)")
    plt.grid(True, linestyle=':', linewidth=0.7)
    plt.legend()
    plt.tight_layout()
    plt.show()

if __name__ == "__main__":
    plot_trinomial_model_convergence()
```



1.1.4 Question 4

[10]:
```
import numpy as np
import scipy.stats as stats
from numpy.polynomial.laguerre import lagval
from numpy.polynomial.hermite import hermval
import matplotlib.pyplot as plt
import pandas as pd

def simulate_log_paths(s0, r, sigma, t, paths, steps):
    """
    Simulates antithetic log-normal asset paths using geometric Brownian motion.
    """
    dt = t / steps
    z = np.random.randn(paths // 2, steps)
    z = np.concatenate([z, -z], axis=0)
    increments = (r - 0.5 * sigma**2) * dt + sigma * np.sqrt(dt) * z
    log_paths = np.cumsum(increments, axis=1)
    final_paths = s0 * np.exp(log_paths)
    return np.concatenate([np.full((paths, 1), s0), final_paths], axis=1)

def construct_basis_matrix(s, method, degree):
    """
    Constructs basis function matrix using the chosen orthogonal polynomial␣
↪type.
    """
    x = s.reshape(-1, 1)
    if method == 'laguerre':
        x_scaled = x / 100
        return np.column_stack([lagval(x_scaled.flatten(), [0]*i + [1]) for i␣
↪in range(degree)])
    elif method == 'hermite':
        x_scaled = (x - np.mean(x)) / np.std(x)
        return np.column_stack([hermval(x_scaled.flatten(), [0]*i + [1]) for i␣
↪in range(degree)])
    elif method == 'monomial':
        return np.column_stack([x.flatten()**i for i in range(degree)])
    else:
        raise ValueError("Invalid basis type selected.")

def price_american_put_lsmc(s0, k, r, sigma, t, num_paths, poly_degree,␣
↪basis_type):
    """
```

```
    Prices an American put option using the Longstaff-Schwartz Monte Carlo␣
↪method.
    """
    steps = int(np.sqrt(num_paths))
    dt = t / steps
    paths = simulate_log_paths(s0, r, sigma, t, num_paths, steps)
    cashflows = np.maximum(k - paths[:, -1], 0)

    for step in range(steps - 1, 0, -1):
        itm = (k - paths[:, step]) > 0
        if not np.any(itm):
            continue
        s_itm = paths[itm, step]
        disc_cf = cashflows[itm] * np.exp(-r * dt)
        basis = construct_basis_matrix(s_itm, basis_type, poly_degree)
        coeffs, *_ = np.linalg.lstsq(basis, disc_cf, rcond=None)
        cont_val = construct_basis_matrix(s_itm, basis_type, poly_degree) @␣
↪coeffs
        exercise_val = k - s_itm
        cashflows[itm] = np.where(exercise_val > cont_val, exercise_val,␣
↪cashflows[itm] * np.exp(-r * dt))

    return np.mean(cashflows * np.exp(-r * dt))

s0 = 180
k = 180
r = 0.055
sigma = 0.25
n_sim = 100_000
degree_values = [2, 3, 4, 5]
maturities = [0.5, 1.5]
basis_families = ['laguerre', 'hermite', 'monomial']

output_table = {}

for basis in basis_families:
    output_table[basis] = {}
    for maturity in maturities:
        output_table[basis][maturity] = []
        for degree in degree_values:
            price = price_american_put_lsmc(s0, k, r, sigma, maturity, n_sim,␣
↪degree, basis)
            output_table[basis][maturity].append(price)

df_output = pd.DataFrame({
    (basis.capitalize(), f"T={maturity}"): output_table[basis][maturity]
    for basis in basis_families
```

```
    for maturity in maturities
}, index=[f"k={d}" for d in degree_values])

print(df_output)
```

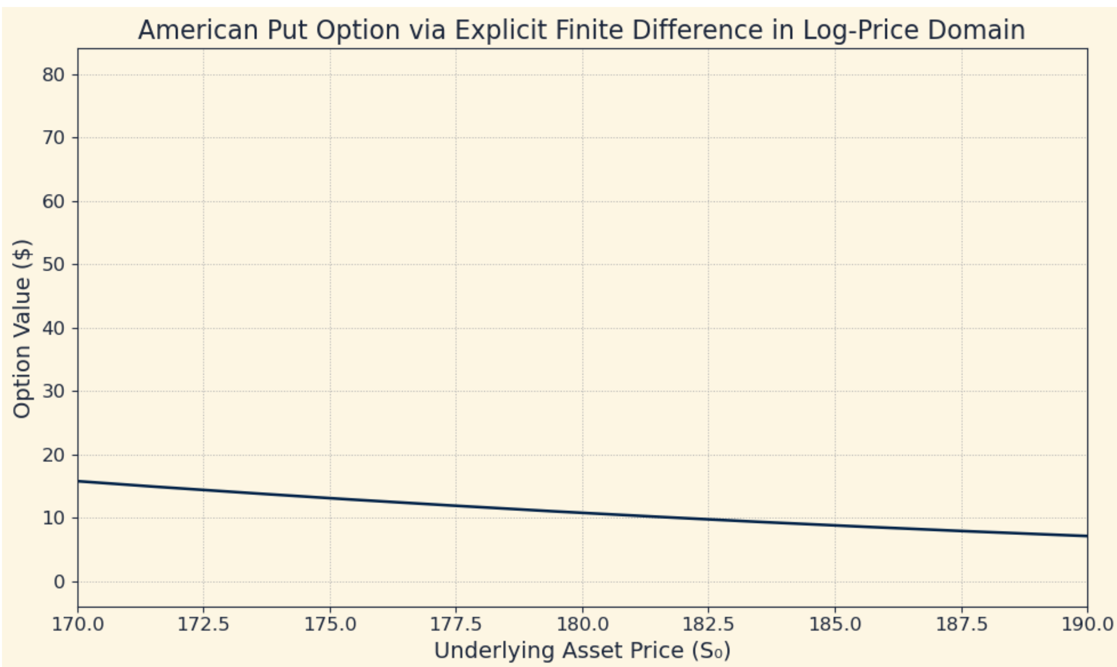
```
      Laguerre                Hermite                Monomial
         T=0.5      T=1.5       T=0.5      T=1.5       T=0.5      T=1.5
k=2  10.574035  16.239865  10.529183  16.171364  10.588040  16.157523
k=3  10.679767  16.440293  10.690132  16.465174  10.663443  16.416877
k=4  10.703827  16.536469  10.719832  16.491100  10.721126  16.512954
k=5  10.706310  16.536867  10.760994  16.536630  10.732327  16.439124
```

1.1.5 Question 5

[12]:
```
def configure_plot_aesthetics():
    """
    Applies a custom plot style using navy blue, crimson red, and eggshell␣
 ↪white.
    """
    plt.style.use("default")
    plt.rcParams.update({
        'font.size': 12,
        'axes.titlesize': 16,
        'axes.labelsize': 14,
        'xtick.labelsize': 12,
        'ytick.labelsize': 12,
        'legend.fontsize': 12,
        'figure.titlesize': 18,
        'lines.linewidth': 1.8,
        'lines.markersize': 7,
        'axes.facecolor': '#FDF6E3',
        'figure.facecolor': '#FDF6E3',
        'savefig.facecolor': '#FDF6E3',
        'axes.edgecolor': '#1B263B',
        'axes.labelcolor': '#1B263B',
        'xtick.color': '#1B263B',
        'ytick.color': '#1B263B',
        'text.color': '#1B263B'
    })

def compute_explicit_fd_logspace_put(
    strike: float = 180.0,
    volatility: float = 0.25,
    rate: float = 0.055,
    maturity: float = 0.5,
    time_step: float = 0.002,
    space_step: float = None
```

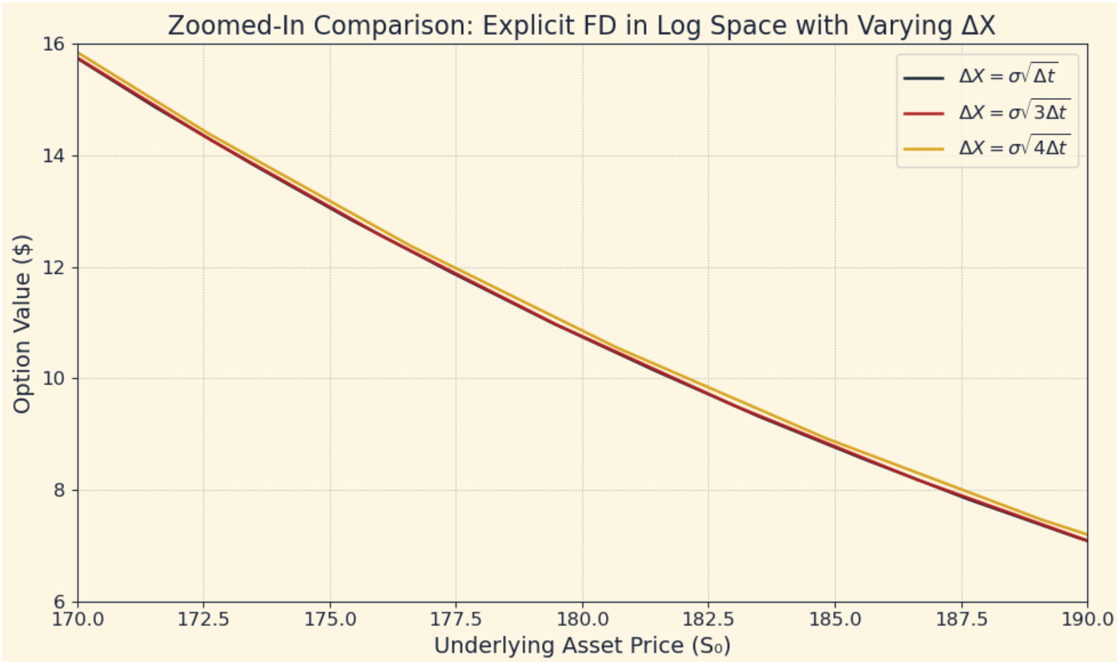
```python
):
    """
    Solves the Black-Scholes PDE for an American put option using the explicit␣
    ↪finite
    difference method in the log-price domain.
    """
    if space_step is None:
        space_step = volatility * np.sqrt(time_step)

    x_lower = np.log(100)
    x_upper = np.log(260)
    num_x = int((x_upper - x_lower) / space_step) + 1
    num_t = int(maturity / time_step)

    x_grid = np.linspace(x_lower, x_upper, num_x)
    s_grid = np.exp(x_grid)
    payoff = np.maximum(strike - s_grid, 0)

    a = 0.5 * volatility**2 / space_step**2
    b = 0.5 * (rate - 0.5 * volatility**2) / space_step

    weight_up = time_step * (a + b)
    weight_mid = 1 - time_step * (2 * a + rate)
    weight_down = time_step * (a - b)

    option_values = payoff.copy()

    for _ in range(num_t):
        updated = option_values.copy()
        for j in range(1, num_x - 1):
            val = (
                weight_up * option_values[j + 1] +
                weight_mid * option_values[j] +
                weight_down * option_values[j - 1]
            )
            intrinsic = strike - s_grid[j]
            updated[j] = max(val, intrinsic)
        updated[0] = strike - s_grid[0]
        updated[-1] = 0.0
        option_values = updated

    return s_grid, option_values

def visualize_fd_logspace_put():
    """
    Generates and visualizes the American put price curve using the
    explicit finite difference method in log space.
```

```
    """
    configure_plot_aesthetics()
    prices, values = compute_explicit_fd_logspace_put()

    plt.figure(figsize=(10, 6))
    plt.plot(prices, values, linestyle='-', marker=None, color='#002147')
    plt.xlim(170, 190)
    plt.xlabel("Underlying Asset Price (S )")
    plt.ylabel("Option Value ($)")
    plt.title("American Put Option via Explicit Finite Difference in Log-Price␣
↪Domain")
    plt.grid(True, linestyle=":", linewidth=0.7)
    plt.tight_layout()
    plt.show()

if __name__ == "__main__":
    visualize_fd_logspace_put()
```



[13]:
```
def plot_fd_dx_sensitivity():
    """
    Compares the effect of different space step sizes (ΔX) in the explicit
    finite difference solution for American put options in log-space.
    """
    configure_plot_aesthetics()
```

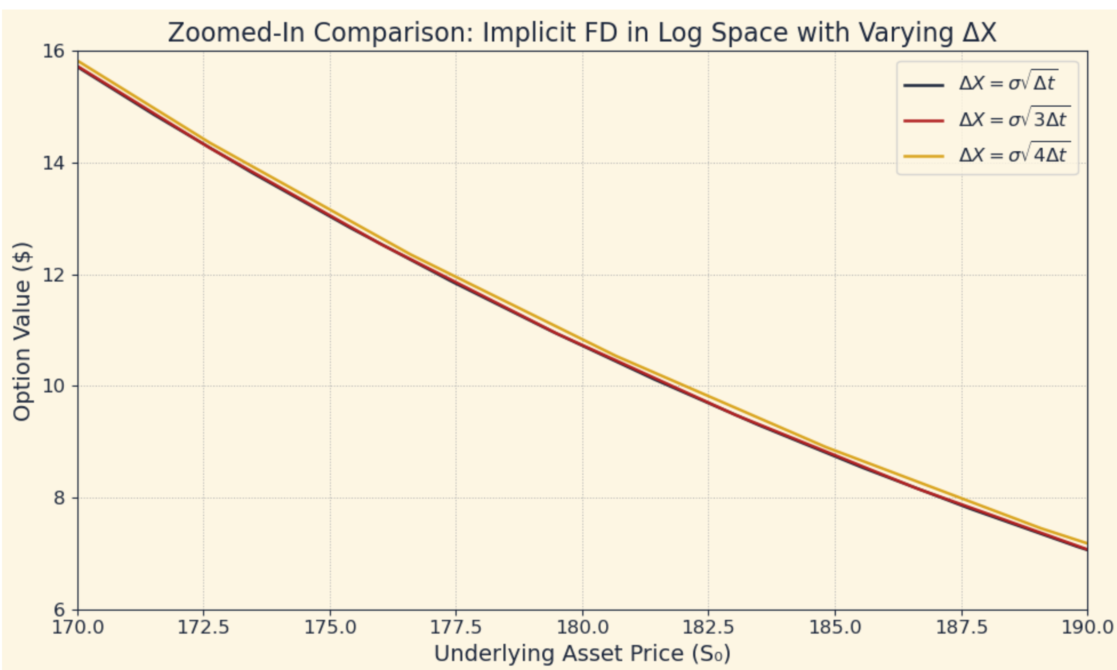
```
    vol = 0.25
    dt = 0.002

    dx_values = [
        vol * np.sqrt(dt),
        vol * np.sqrt(3 * dt),
        vol * np.sqrt(4 * dt)
    ]

    s1, v1 = compute_explicit_fd_logspace_put(volatility=vol, time_step=dt,␣
 ↪space_step=dx_values[0])
    s2, v2 = compute_explicit_fd_logspace_put(volatility=vol, time_step=dt,␣
 ↪space_step=dx_values[1])
    s3, v3 = compute_explicit_fd_logspace_put(volatility=vol, time_step=dt,␣
 ↪space_step=dx_values[2])

    plt.figure(figsize=(10, 6))
    plt.plot(s1, v1, label=r"$\Delta X = \sigma \sqrt{\Delta t}$",␣
 ↪color='#1B263B')
    plt.plot(s2, v2, label=r"$\Delta X = \sigma \sqrt{3\Delta t}$",␣
 ↪color='#B22222')
    plt.plot(s3, v3, label=r"$\Delta X = \sigma \sqrt{4\Delta t}$",␣
 ↪color='#DAA520')

    plt.xlim(170, 190)
    plt.ylim(6, 16)
    plt.xlabel("Underlying Asset Price (S )")
    plt.ylabel("Option Value ($)")
    plt.title("Zoomed-In Comparison: Explicit FD in Log Space with Varying ΔX")
    plt.grid(True, linestyle=":", linewidth=0.7)
    plt.legend()
    plt.tight_layout()
    plt.show()

if __name__ == "__main__":
    plot_fd_dx_sensitivity()
```



[14]:
```
from scipy.linalg import solve_banded

def compute_implicit_fd_logspace_put(
    strike: float = 180.0,
    volatility: float = 0.25,
    rate: float = 0.055,
    maturity: float = 0.5,
    time_step: float = 0.002,
    space_step: float = None
):
    """
    Solves the Black-Scholes PDE for an American put using an implicit finite␣
 ↪difference
    scheme in the log-price domain. Applies early exercise condition at each␣
 ↪step.
    """
    if space_step is None:
        space_step = volatility * np.sqrt(time_step)

    x_lower = np.log(100)
    x_upper = np.log(260)
    num_x = int((x_upper - x_lower) / space_step) + 1
    num_t = int(maturity / time_step)

    x_grid = np.linspace(x_lower, x_upper, num_x)
```

```
    s_grid = np.exp(x_grid)

    option = np.maximum(strike - s_grid, 0.0)

    alpha = volatility**2 / (2 * space_step**2)
    beta = (rate - 0.5 * volatility**2) / (2 * space_step)
    gamma = rate

    lower = -time_step * (alpha - beta)
    center = 1 + time_step * (2 * alpha + gamma)
    upper = -time_step * (alpha + beta)

    banded_matrix = np.zeros((3, num_x - 2))
    banded_matrix[0, 1:] = upper
    banded_matrix[1, :] = center
    banded_matrix[2, :-1] = lower

    for _ in range(num_t):
        rhs = option[1:-1].copy()
        rhs[0] -= lower * (strike - s_grid[0])
        rhs[-1] -= upper * 0.0
        solution = solve_banded((1, 1), banded_matrix, rhs)
        option[1:-1] = np.maximum(solution, strike - s_grid[1:-1])
        option[0] = strike - s_grid[0]
        option[-1] = 0.0

    return s_grid, option

def plot_implicit_fd_dx_variation():
    """
    Visualizes the impact of space step size (ΔX) on American put pricing using␣
 ↪the
    implicit finite difference scheme in log space.
    """
    configure_plot_aesthetics()
    vol = 0.25
    dt = 0.002

    dx_values = [
        vol * np.sqrt(dt),
        vol * np.sqrt(3 * dt),
        vol * np.sqrt(4 * dt)
    ]

    s1, v1 = compute_implicit_fd_logspace_put(volatility=vol, time_step=dt,␣
 ↪space_step=dx_values[0])
```

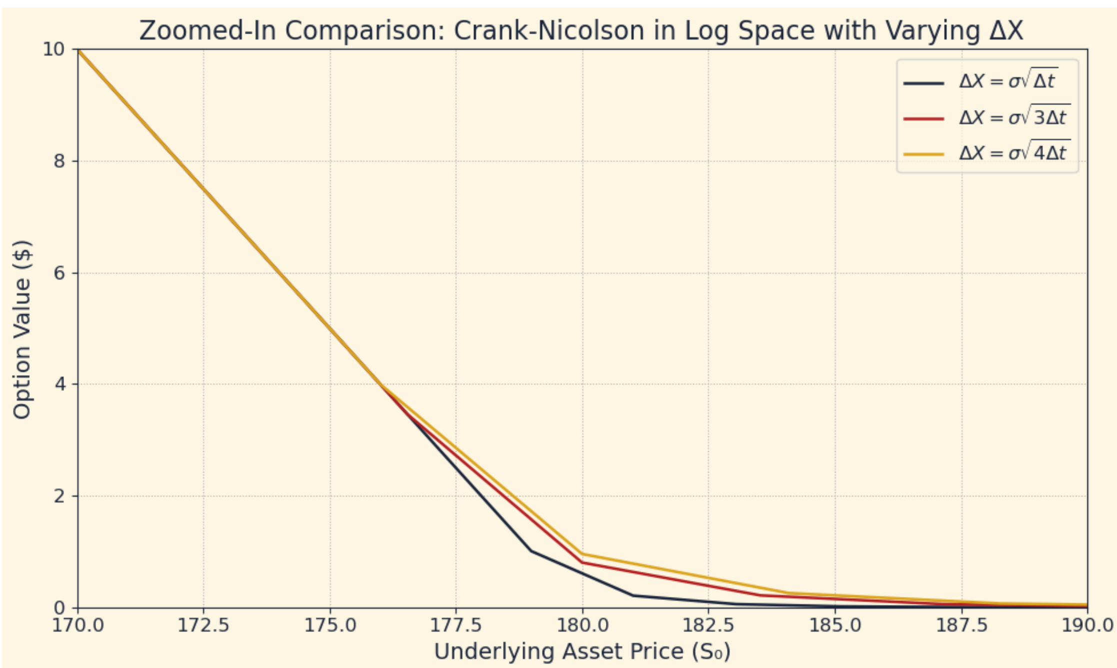
```
    s2, v2 = compute_implicit_fd_logspace_put(volatility=vol, time_step=dt, space_step=dx_values[1])
    s3, v3 = compute_implicit_fd_logspace_put(volatility=vol, time_step=dt, space_step=dx_values[2])

    plt.figure(figsize=(10, 6))
    plt.plot(s1, v1, label=r"$\Delta X = \sigma \sqrt{\Delta t}$", color='#1B263B')
    plt.plot(s2, v2, label=r"$\Delta X = \sigma \sqrt{3\Delta t}$", color='#B22222')
    plt.plot(s3, v3, label=r"$\Delta X = \sigma \sqrt{4\Delta t}$", color='#DAA520')

    plt.xlim(170, 190)
    plt.ylim(6, 16)
    plt.xlabel("Underlying Asset Price (S )")
    plt.ylabel("Option Value ($)")
    plt.title("Zoomed-In Comparison: Implicit FD in Log Space with Varying ΔX")
    plt.grid(True, linestyle=":", linewidth=0.7)
    plt.legend()
    plt.tight_layout()
    plt.show()

if __name__ == "__main__":
    plot_implicit_fd_dx_variation()
```



[15]:
```python
from scipy.linalg import solve_banded

def compute_crank_nicolson_fd_logspace_put(
    strike: float = 180.0,
    volatility: float = 0.25,
    rate: float = 0.055,
    maturity: float = 0.5,
    time_step: float = 0.002,
    space_step: float = None
):
    """
    Solves the Black-Scholes PDE for an American put option using the␣
 ↪Crank-Nicolson
    finite difference scheme in log-price space. Applies early exercise at each␣
 ↪step.
    """
    if space_step is None:
        space_step = volatility * np.sqrt(time_step)

    x_center = np.log(strike)
    x_span = 4.0
    x_min = x_center - x_span / 2
    x_max = x_center + x_span / 2

    num_x = int((x_max - x_min) / space_step) + 1
    num_t = int(maturity / time_step)

    x_vals = np.linspace(x_min, x_max, num_x)
    s_vals = np.exp(x_vals)

    option = np.maximum(strike - s_vals, 0.0)

    alpha = volatility**2 / (2 * space_step**2)
    beta = (rate - 0.5 * volatility**2) / (2 * space_step)
    gamma = rate

    a_A = 0.5 * time_step * (alpha - beta)
    b_A = 1 + time_step * (2 * alpha + gamma)
    c_A = 0.5 * time_step * (alpha + beta)

    banded_A = np.zeros((3, num_x - 2))
    banded_A[0, 1:] = -c_A
    banded_A[1, :] = b_A
    banded_A[2, :-1] = -a_A

    a_B = 0.5 * time_step * (alpha - beta)
    b_B = 1 - time_step * (2 * alpha + gamma)
```

```
    c_B = 0.5 * time_step * (alpha + beta)

    for _ in range(num_t):
        rhs = np.zeros(num_x - 2)
        for i in range(1, num_x - 1):
            rhs[i - 1] = (
                a_B * option[i - 1] +
                b_B * option[i] +
                c_B * option[i + 1]
            )
        rhs[0] -= a_A * (strike - s_vals[0])
        rhs[-1] += 0.0

        solution = solve_banded((1, 1), banded_A, rhs)
        option[1:-1] = np.maximum(solution, strike - s_vals[1:-1])
        option[0] = strike - s_vals[0]
        option[-1] = 0.0

    return s_vals, option

def plot_crank_nicolson_dx_variation():
    """
    Visualizes the sensitivity of the Crank-Nicolson log-space scheme for␣
 ↪American puts
    to changes in spatial discretization (ΔX).
    """
    configure_plot_aesthetics()
    sigma = 0.25
    dt = 0.002

    dx_list = [
        sigma * np.sqrt(dt),
        sigma * np.sqrt(3 * dt),
        sigma * np.sqrt(4 * dt)
    ]

    s1, v1 = compute_crank_nicolson_fd_logspace_put(volatility=sigma,␣
 ↪time_step=dt, space_step=dx_list[0])
    s2, v2 = compute_crank_nicolson_fd_logspace_put(volatility=sigma,␣
 ↪time_step=dt, space_step=dx_list[1])
    s3, v3 = compute_crank_nicolson_fd_logspace_put(volatility=sigma,␣
 ↪time_step=dt, space_step=dx_list[2])

    plt.figure(figsize=(10, 6))
    plt.plot(s1, v1, label=r"$\Delta X = \sigma \sqrt{\Delta t}$",␣
 ↪color="#1B263B")
```

```
    plt.plot(s2, v2, label=r"$\Delta X = \sigma \sqrt{3\Delta t}$", color="#B22222")
    plt.plot(s3, v3, label=r"$\Delta X = \sigma \sqrt{4\Delta t}$", color="#DAA520")

    plt.xlim(170, 190)
    plt.ylim(0, 10)
    plt.xlabel("Underlying Asset Price (S )")
    plt.ylabel("Option Value ($)")
    plt.title("Zoomed-In Comparison: Crank-Nicolson in Log Space with Varying ΔX")
    plt.grid(True, linestyle=":", linewidth=0.7)
    plt.legend()
    plt.tight_layout()
    plt.show()

if __name__ == "__main__":
    plot_crank_nicolson_dx_variation()
```



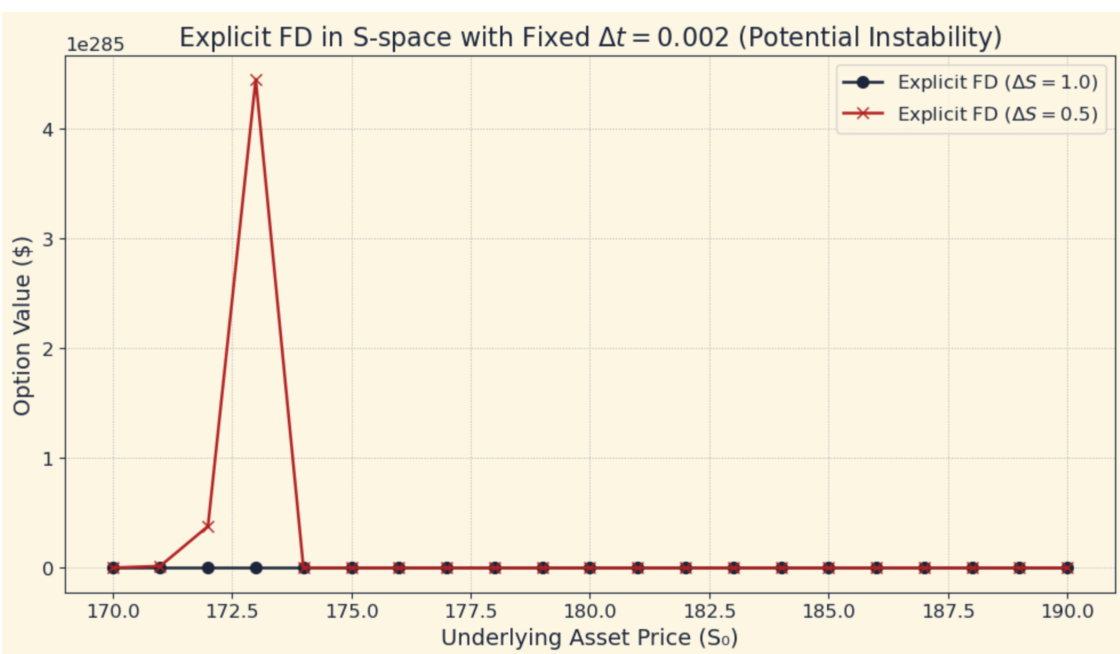
1.1.6 Problem 6

[16]:
```
def compute_explicit_fd_S_space(
    strike: float = 180.0,
    volatility: float = 0.25,
    rate: float = 0.055,
```

```
    maturity: float = 0.5,
    space_step: float = 1.0,
    time_step: float = 0.002
):
    """
    Solves the Black-Scholes PDE for an American put using an explicit finite difference
    scheme in the asset price domain (S-space), applying early exercise at each step.
    """
    s_max = 2.5 * strike
    m = int(s_max / space_step)
    n = int(maturity / time_step)

    s_vals = np.linspace(0, s_max, m + 1)
    option = np.maximum(strike - s_vals, 0.0)

    for _ in range(n):
        updated = option.copy()
        for j in range(1, m):
            s = s_vals[j]
            delta = (option[j + 1] - option[j - 1]) / (2 * space_step)
            gamma = (option[j + 1] - 2 * option[j] + option[j - 1]) / (space_step ** 2)
            theta = (
                0.5 * volatility ** 2 * s ** 2 * gamma +
                rate * s * delta -
                rate * option[j]
            )
            updated[j] = option[j] - time_step * theta
            updated[j] = max(updated[j], strike - s)
        updated[0] = strike
        updated[m] = 0.0
        option = updated

    return s_vals, option

def plot_explicit_fd_s_space_instability():
    """
    Compares the results of explicit FD in S-space for different ΔS at fixed (unstable) Δt.
    """
    configure_plot_aesthetics()
    dt = 0.002

    s1, v1 = compute_explicit_fd_S_space(space_step=1.0, time_step=dt)
    s2, v2 = compute_explicit_fd_S_space(space_step=0.5, time_step=dt)
```

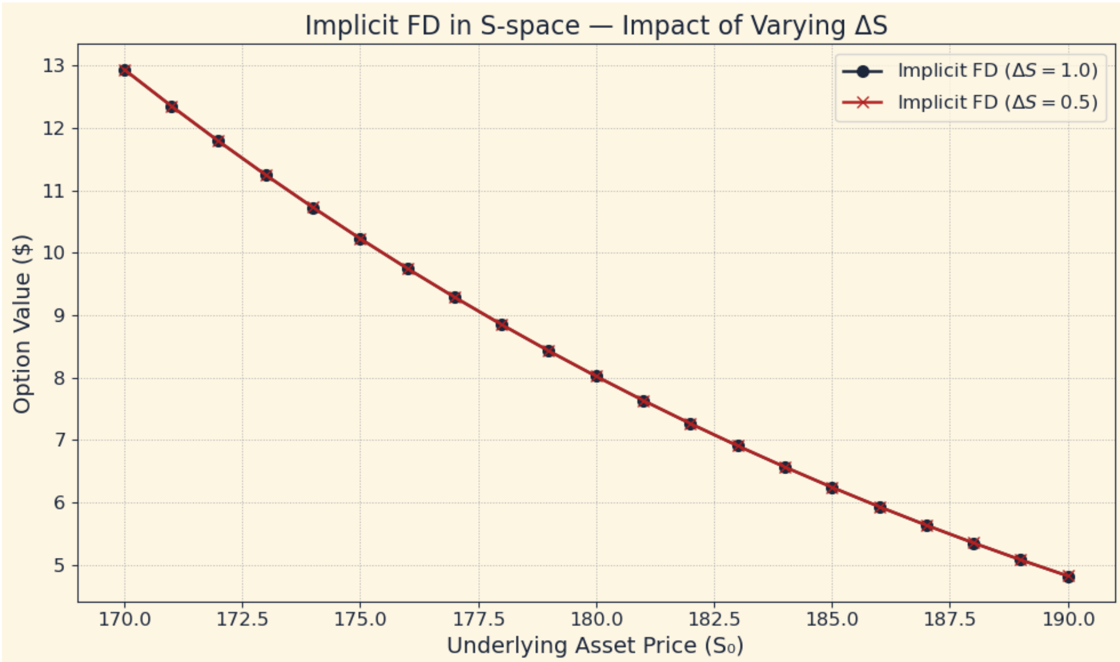
```python
    s_eval = np.arange(170, 191)
    v1_interp = np.interp(s_eval, s1, v1)
    v2_interp = np.interp(s_eval, s2, v2)

    plt.figure(figsize=(10, 6))
    plt.plot(s_eval, v1_interp, label=r"Explicit FD ($\Delta S = 1.0$)", color="#1B263B", marker='o')
    plt.plot(s_eval, v2_interp, label=r"Explicit FD ($\Delta S = 0.5$)", color="#B22222", marker='x')
    plt.xlabel("Underlying Asset Price (S )")
    plt.ylabel("Option Value ($)")
    plt.title("Explicit FD in S-space with Fixed $\Delta t = 0.002$ (Potential Instability)")
    plt.grid(True, linestyle=":", linewidth=0.7)
    plt.legend()
    plt.tight_layout()
    plt.show()

if __name__ == "__main__":
    plot_explicit_fd_s_space_instability()
```

```
<>:58: SyntaxWarning: invalid escape sequence '\D'
<>:58: SyntaxWarning: invalid escape sequence '\D'
C:\Users\vikal\AppData\Local\Temp\ipykernel_31224\4240631177.py:58:
SyntaxWarning: invalid escape sequence '\D'
  plt.title("Explicit FD in S-space with Fixed $\Delta t = 0.002$ (Potential
Instability)")
```



[17]:
```python
from scipy.linalg import solve_banded

def compute_implicit_fd_S_space(
    strike: float = 180.0,
    volatility: float = 0.25,
    rate: float = 0.055,
    maturity: float = 0.5,
    space_step: float = 1.0,
    time_step: float = 0.002
):
    """
    Solves the Black-Scholes PDE for an American put using an implicit
    finite difference scheme in asset price space (S-space).
    """
    s_max = 2.5 * strike
    m = int(s_max / space_step)
    n = int(maturity / time_step)

    s_vals = np.linspace(0, s_max, m + 1)
    option = np.maximum(strike - s_vals, 0.0)

    lower = np.zeros(m - 1)
    center = np.zeros(m - 1)
    upper = np.zeros(m - 1)

    for j in range(1, m):
        s = s_vals[j]
        a = 0.5 * volatility**2 * s**2
        b = 0.5 * rate * s

        lower[j - 1] = -time_step * (a / space_step**2 - b / space_step)
        center[j - 1] = 1 + time_step * (2 * a / space_step**2 + rate)
        upper[j - 1] = -time_step * (a / space_step**2 + b / space_step)

    banded_matrix = np.zeros((3, m - 1))
    banded_matrix[0, 1:] = upper[1:]
    banded_matrix[1, :] = center
    banded_matrix[2, :-1] = lower[:-1]

    for _ in range(n):
        rhs = option[1:m].copy()
        rhs[0] -= lower[0] * strike
        rhs[-1] -= upper[-1] * 0.0
```

```
        solution = solve_banded((1, 1), banded_matrix, rhs)
        option[1:m] = np.maximum(solution, strike - s_vals[1:m])
        option[0] = strike
        option[-1] = 0.0

    return s_vals, option

def plot_implicit_fd_s_space_comparison():
    """
    Plots American put pricing via implicit FD in S-space for two ΔS values.
    """
    configure_plot_aesthetics()
    dt = 0.002

    s1, v1 = compute_implicit_fd_S_space(space_step=1.0, time_step=dt)
    s2, v2 = compute_implicit_fd_S_space(space_step=0.5, time_step=dt)

    s_eval = np.arange(170, 191)
    v1_interp = np.interp(s_eval, s1, v1)
    v2_interp = np.interp(s_eval, s2, v2)

    plt.figure(figsize=(10, 6))
    plt.plot(s_eval, v1_interp, label=r"Implicit FD ($\Delta S = 1.0$)", color="#1B263B", marker='o')
    plt.plot(s_eval, v2_interp, label=r"Implicit FD ($\Delta S = 0.5$)", color="#B22222", marker='x')
    plt.xlabel("Underlying Asset Price (S )")
    plt.ylabel("Option Value ($)")
    plt.title("Implicit FD in S-space - Impact of Varying ΔS")
    plt.grid(True, linestyle=":", linewidth=0.7)
    plt.legend()
    plt.tight_layout()
    plt.show()

if __name__ == "__main__":
    plot_implicit_fd_s_space_comparison()
```

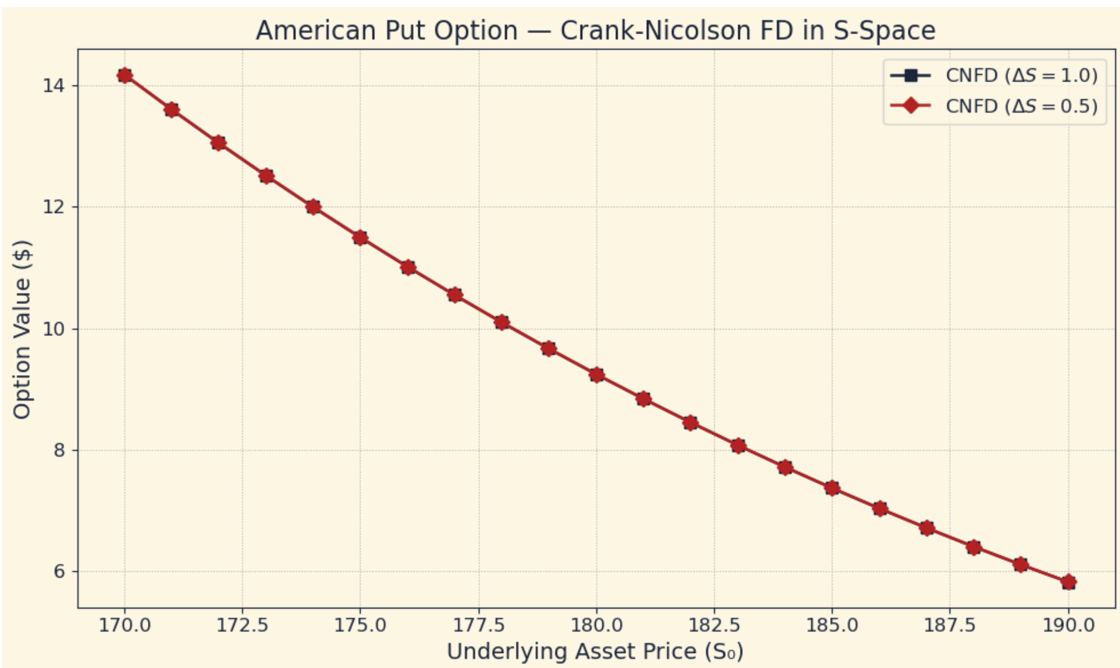


[18]:

```python
from scipy.linalg import solve_banded

def compute_crank_nicolson_fd_S_space(
    strike: float = 180.0,
    volatility: float = 0.25,
    rate: float = 0.055,
    maturity: float = 0.5,
    space_step: float = 1.0,
    time_step: float = 0.002
):
    """
    Prices an American put option using the Crank-Nicolson finite difference
    method in S-space (asset price grid), applying early exercise condition.
    """
    s_max = 2.5 * strike
    m = int(s_max / space_step)
    n = int(maturity / time_step)

    s_vals = np.linspace(0, s_max, m + 1)
    option = np.maximum(strike - s_vals, 0.0)

    lower_A = np.zeros(m - 1)
    center_A = np.zeros(m - 1)
    upper_A = np.zeros(m - 1)
```

```python
    for j in range(1, m):
        s = s_vals[j]
        a = 0.5 * volatility**2 * s**2
        b = 0.5 * rate * s

        lower_A[j - 1] = -0.5 * time_step * (a / space_step**2 - b / space_step)
        center_A[j - 1] = 1 + 0.5 * time_step * (2 * a / space_step**2 + rate)
        upper_A[j - 1] = -0.5 * time_step * (a / space_step**2 + b / space_step)

    banded_A = np.zeros((3, m - 1))
    banded_A[0, 1:] = upper_A[1:]
    banded_A[1, :] = center_A
    banded_A[2, :-1] = lower_A[:-1]

    lower_B = -lower_A
    center_B = 2 - center_A
    upper_B = -upper_A

    for _ in range(n):
        rhs = np.zeros(m - 1)
        for j in range(1, m):
            rhs[j - 1] = (
                lower_B[j - 1] * option[j - 1] +
                center_B[j - 1] * option[j] +
                upper_B[j - 1] * option[j + 1]
            )
        rhs[0] += lower_A[0] * strike
        rhs[-1] += 0.0

        solution = solve_banded((1, 1), banded_A, rhs)
        option[1:m] = np.maximum(solution, strike - s_vals[1:m])
        option[0] = strike
        option[-1] = 0.0

    return s_vals, option

def plot_crank_nicolson_fd_s_space_comparison():
    """
    Compares CNFD results for American puts in S-space across two ΔS values.
    """
    configure_plot_aesthetics()
    dt = 0.002
    s_eval = np.arange(170, 191)

    s1, v1 = compute_crank_nicolson_fd_S_space(space_step=1.0, time_step=dt)
    s2, v2 = compute_crank_nicolson_fd_S_space(space_step=0.5, time_step=dt)
```

```
    v1_interp = np.interp(s_eval, s1, v1)
    v2_interp = np.interp(s_eval, s2, v2)

    plt.figure(figsize=(10, 6))
    plt.plot(s_eval, v1_interp, label=r"CNFD ($\Delta S = 1.0$)",␣
 ↪color="#1B263B", marker='s')
    plt.plot(s_eval, v2_interp, label=r"CNFD ($\Delta S = 0.5$)",␣
 ↪color="#B22222", marker='D')
    plt.xlabel("Underlying Asset Price (S )")
    plt.ylabel("Option Value ($)")
    plt.title("American Put Option - Crank-Nicolson FD in S-Space")
    plt.grid(True, linestyle=":", linewidth=0.7)
    plt.legend()
    plt.tight_layout()
    plt.show()

if __name__ == "__main__":
    plot_crank_nicolson_fd_s_space_comparison()
```



[ ]: