MFE 409: Financial Risk Management
Assignment 7 Solution

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Abstract

This assignment involves analyzing and pricing credit risk instruments using Credit Default Swaps (CDS). In Question 1, we are tasked with bootstrapping a piecewise constant hazard rate (default intensity) curve from observed CDS spreads for maturities of 3, 5, and 10 years. These hazard rates are then used to evaluate the survival probabilities and, ultimately, price a 7-year bond. This document presents a rigorous breakdown of the bootstrapping process using Python and interprets the results in financial terms.

Contents

Question 1: Bootstrapping a CDS Curve

1.1 Bootstrapping Hazard Rates

In this part, we recover the piecewise constant hazard rate curve from the CDS spreads given on Slide 17 of the lecture slides. The inputs are:

- CDS spread for 3-year maturity: 50 basis points (0.005)
- CDS spread for 5-year maturity: 60 basis points (0.006)
- CDS spread for 10-year maturity: 100 basis points (0.010)
- Recovery rate $R = 60\%$
- Risk-free rate $r = 0\%$

We aim to determine three constant hazard rates:

- λ_1 for the interval [0, 3]
- λ_2 for the interval (3, 5]
- λ_3 for the interval (5, 10]

Python Code Used

The following figure (Figure 1) shows the Python code used to implement the bootstrapping of hazard rates using the 'brentq' root-finding method from 'scipy.optimize'.

```python
import numpy as np
from scipy.optimize import brentq

cds_spreads = {
    3: 0.0050,  # 50 bps
    5: 0.0060,  # 60 bps
    10: 0.0100  # 100 bps
}
R = 0.60  # Recovery rate
payment_interval = 0.5  # Semiannual

def survival_probability(t, hazard_rates):
    λ1, λ2, λ3 = hazard_rates
    if t <= 3:
        return np.exp(-λ1 * t)
    elif t <= 5:
        return np.exp(-λ1 * 3 - λ2 * (t - 3))
    else:
        return np.exp(-λ1 * 3 - λ2 * 2 - λ3 * (t - 5))

def calculate_legs(T, hazard_rates):
    times = np.arange(payment_interval, T + payment_interval, payment_interval)
    prem_leg = sum(payment_interval * survival_probability(t, hazard_rates) for t in times)
    prot_leg = sum(
        survival_probability(t - payment_interval, hazard_rates) - survival_probability(t, hazard_rates)
        for t in times
    )
    return prem_leg, prot_leg

def bootstrap_hazard_rate(prev_lambdas, T, market_spread):
    def objective(new_lambda):
        trial_lambdas = prev_lambdas + [new_lambda]
        while len(trial_lambdas) < 3:
            trial_lambdas.append(1e-6)
        prem_leg, prot_leg = calculate_legs(T, trial_lambdas)
        model_spread = (1 - R) * prot_leg / prem_leg
        return model_spread - market_spread
    return brentq(objective, 1e-6, 1.0)

bootstrapped_hazard_rates = []
for T, spread in cds_spreads.items():
    λ_new = bootstrap_hazard_rate(bootstrapped_hazard_rates, T, spread)
    bootstrapped_hazard_rates.append(λ_new)

print("Bootstrapped hazard rates:")
for i, (interval, rate) in enumerate(zip([(0, 3), (3, 5), (5, 10)], bootstrapped_hazard_rates)):
    print(f"λ{i+1} for years {interval}: {rate*100:.6f}%")
```

Figure 1: Python code to bootstrap hazard rates from CDS spreads

Step-by-Step Explanation

The core idea behind bootstrapping CDS spreads is to use observed market spreads to recover implied default intensities over different maturity intervals. We assume that default follows a Poisson process with a piecewise constant intensity.

We define the survival probability function $S(t)$ using the following logic:

$$S(t) = \begin{cases} e^{-\lambda_1 t} & \text{if } t \leq 3 \\ e^{-\lambda_1 \cdot 3 - \lambda_2 (t-3)} & \text{if } 3 < t \leq 5 \\ e^{-\lambda_1 \cdot 3 - \lambda_2 \cdot 2 - \lambda_3 (t-5)} & \text{if } t > 5 \end{cases}$$

For each maturity, we:

1. Calculate the expected present value of the **premium leg** of the CDS, which is the sum of semiannual payments weighted by the survival probabilities.
2. Calculate the expected present value of the **protection leg**, which reflects the expected loss from default, weighted by the probability of default in each time slice.
3. Use root-finding to match the model CDS spread with the market spread to solve for each λ_i.

We bootstrap each hazard rate sequentially:

- λ_1 is obtained using the 3-year CDS spread.
- λ_2 is obtained using the 5-year CDS spread and the already-known λ_1.
- λ_3 is obtained using the 10-year CDS spread and the known λ_1, λ_2.

Bootstrapped Hazard Rates Table

The resulting hazard rates are shown below in Table 1.

Table 1: Bootstrapped Hazard Rates by Interval

Interval (Years)	**Hazard Rate λ_i**	**Interpretation**
0 – 3	0.01246	1.25% default intensity per year
3 – 5	0.01881	1.88% default intensity per year
5 – 10	0.03613	3.61% default intensity per year

These hazard rates represent the implied likelihood of default in each time segment, consistent with market CDS spreads.

1.2 Pricing the 7-Year Bond

In this part, we use the piecewise constant hazard rate curve recovered in Subsection 1.1 to price a 7-year bond. The bond pays a 3% annual coupon semiannually (i.e., \$1.50 every 6 months), has a face value of \$100, and matures in 7 years.

Mathematical Approach

Let c denote the semiannual coupon payment, and F the face value of the bond. Then the bond price P is computed as the sum of expected present values of all future payments, discounted using the survival probability $S(t)$, i.e.,

$$P = \sum_{t=0.5}^{7} [c \cdot S(t)] + F \cdot S(7)$$

where:

$$c = \frac{3\% \times 100}{2} = 1.5, \quad F = 100$$

The survival probability function $S(t)$ is defined as:

$$S(t) = \begin{cases} e^{-\lambda_1 t} & \text{if } t \leq 3 \\ e^{-\lambda_1 \cdot 3 - \lambda_2 \cdot (t-3)} & \text{if } 3 < t \leq 5 \\ e^{-\lambda_1 \cdot 3 - \lambda_2 \cdot 2 - \lambda_3 \cdot (t-5)} & \text{if } t > 5 \end{cases}$$

This accounts for the piecewise hazard structure: $\lambda_1 = 0.01246$, $\lambda_2 = 0.01881$, and $\lambda_3 = 0.03613$, previously obtained.

Python Code Used

The figure below (Figure 2) shows the Python code used to implement the bond pricing logic described above.

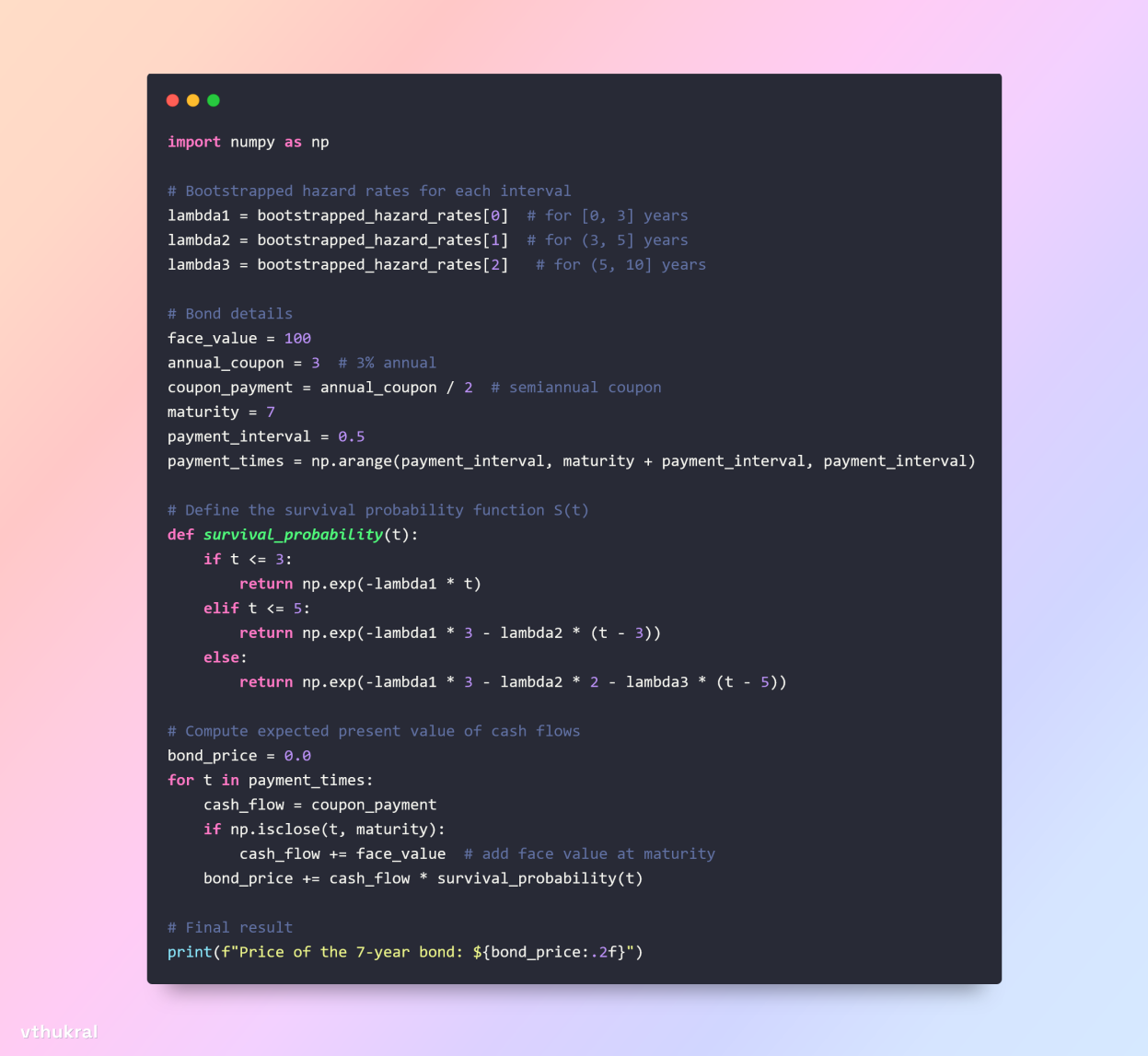

Figure 2: Python code to price a 7-year bond using bootstrapped hazard rates

Explanation of the Code

The Python code:

- Defines bond parameters like face value, coupon, and payment schedule.
- Implements the piecewise hazard function $S(t)$.
- Loops over all semiannual payment times (0.5 to 7.0 years).
- For each payment time t, it multiplies the cash flow by the survival probability and adds it to the total price.

- At maturity (t = 7), the final face value payment is also included.

This approach directly models credit risk in the cash flows, assuming a 0% risk-free rate (i.e., no time discounting beyond survival adjustment).

Final Bond Price:

$106.08

This is the fair value of the bond under the default intensity curve inferred from CDS spreads.

Question 2: Historical vs Bond-Implied Hazard Rates

Context and Objective

This question refers to Slide 19 from the lecture, titled *"Comparing Hazard Rates"*. The slide presents a comparative table of 7-year hazard rates by credit rating. Two sources are compared:

- **Historical Hazard Rates** — derived from Moody's default data (1970–2013). These are *realized*, long-run empirical estimates of the probability of default.
- **Bond-Implied Hazard Rates** — derived from observed bond prices (Merrill Lynch data from 1996–2007). These are *implied* using pricing models under risk-neutral expectations.

The objective is to interpret the table, compare the patterns, and explain why the two estimates may differ across credit ratings.

Reconstructed Table: Hazard Rates by Rating

Table 2: 7-Year Hazard Rates: Historical vs Bond-Implied

Rating	Historical Hazard Rate (%)	Bond-Implied Hazard Rate (%)
AAA	0.08	0.48
AA	0.22	0.79
A	0.74	1.39
BBB	2.55	2.41
BB	11.86	5.64
B	25.86	13.90
CCC	49.15	33.79

Interpretation and Analysis

We analyze the data and trends across ratings:

1. **Investment-Grade Credits (AAA to BBB)**
   These show a consistent pattern: **bond-implied hazard rates are higher** than historical rates. For example:
   - AAA: Implied = 0.48%, Historical = 0.08% $\rightarrow$ 6x higher
   - A: Implied = 1.39%, Historical = 0.74%

This difference arises because market prices reflect a **credit risk premium**. Investors are risk-averse and demand compensation for uncertainty — not just expected losses but also tail risks. As a result, even "safe" credits have non-zero spreads and higher implied default risk under risk-neutral pricing.

2. **Crossover Zone (BBB)**
   BBB sits at the boundary of investment and speculative grade. Interestingly, the historical and implied hazard rates are **very close**:

   $$\text{BBB: Historical} = 2.55\%, \text{ Implied} = 2.41\%$$

   This may indicate efficient pricing: the market's forward-looking estimate aligns with the long-run average. Alternatively, it could reflect that BBB bonds are frequently studied and relatively liquid, so pricing is more accurate.

3. **High-Yield/Speculative Grades (BB to CCC)**
   In this region, the historical hazard rates **exceed** bond-implied rates. For instance:

   - CCC: Historical = 49.15%, Implied = 33.79%
   - B: Historical = 25.86%, Implied = 13.90%

   Why the reversal? There are a few explanations:

   - **Liquidity effects**: Distressed bonds often trade at low volumes, and prices can understate true default risk.
   - **Structural recovery assumptions**: The market might expect higher recovery rates, reducing implied hazard.
   - **Survivorship and sample selection bias**: Historical data may overstate actual future risk due to economic regime shifts.

Conceptual Explanation: Why the Two Differ

- **Historical hazard rates** reflect empirical default probabilities under the real-world (physical) measure $\mathbb{P}$. They are backward-looking, based on frequency data over decades and credit cycles.
- **Bond-implied hazard rates** are derived under the risk-neutral measure $\mathbb{Q}$. These are forward-looking and incorporate investor expectations, risk aversion, and risk premia.
- Thus, the bond-implied hazard rate is generally:

  $$\text{Hazard}_{\mathbb{Q}} = \text{Hazard}_{\mathbb{P}} + \text{Risk Premium}$$

  — especially pronounced in safer credits where the premium dominates the small true hazard rate.
- The discrepancy reflects a core concept in credit risk modeling: **market prices reflect perceived risk, not just historical evidence**.

Conclusion

This comparison reinforces a key takeaway in credit modeling: historical data alone cannot predict current pricing or risk. While historical hazard rates help us understand average realized defaults, bond-implied hazard rates tell us what the market believes today — including risk compensation, liquidity factors, and macroeconomic sentiment.

For credit pricing and risk management, both perspectives are valuable:

- Use historical rates for stress testing and conservative planning.
- Use bond-implied rates for valuation, trading, and hedging.

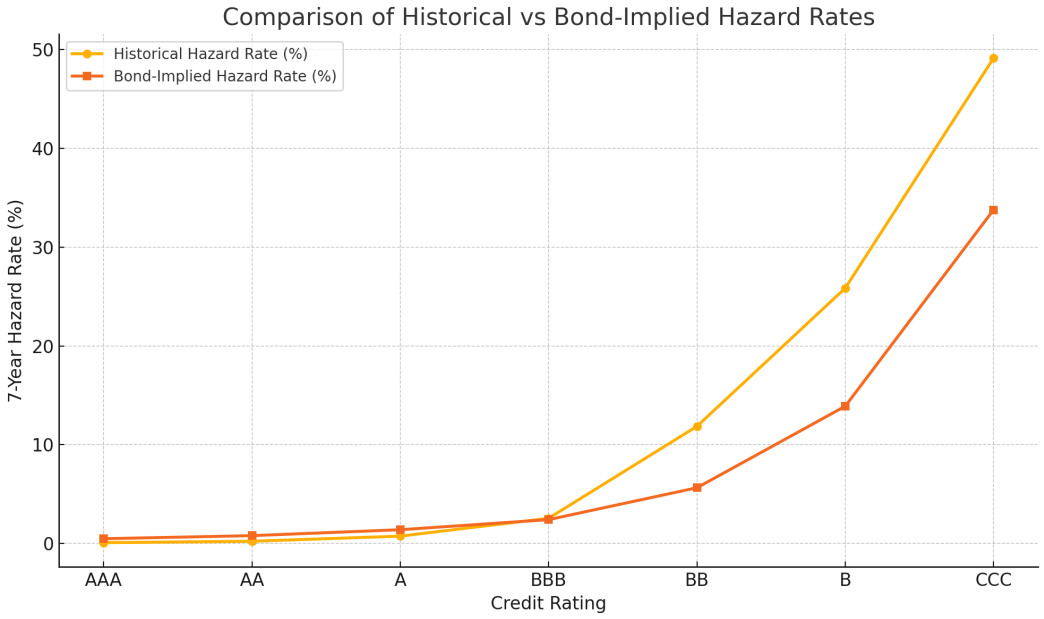

Figure 3: Comparison of Historical vs Bond-Implied 7-Year Hazard Rates

Acknowledgements

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Additionally, I would like to acknowledge the use of generative AI tools, specifically for resolving implementation doubts, generating interpretive explanations, and refining the structure and formatting of this report. These tools served as interactive companions during the problem-solving process but were used in a manner that supplemented — not substituted — my own analytical reasoning, modeling effort, and interpretation of financial theory. The critical thinking and financial logic applied throughout remain my own.

This work is the product of a rigorous academic process, supported by resources but driven by personal effort.

References

1. Haddad, V. (2025). *Lecture 6: Credit Risk*. MFE 409: Financial Risk Management, UCLA Anderson.