MFE 409: Financial Risk Management
Problem Set 5

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Abstract

This report addresses the analytical portion of Problem Set 5 from MFE 409, which focuses on managing risk for derivatives written on multiple underlying assets. Prior to the case study, the homework explores Value-at-Risk (VaR) methodologies applied to options on two risky assets. The tasks include deriving the 99% 1-day VaR using the delta-normal approach, pricing a European option on the minimum of two assets via Stulz's closed-form solution, and comparing analytical VaR against simulation-based results. The goal is to evaluate model-based risk through sensitivity and correlation structures, and reflect on model uncertainty in practical settings.

Question 1: 1-Day 99% Delta-Normal VaR for Option on Two Underlyings

We are given two risky assets with prices $S_{1,t}$ and $S_{2,t}$, and the following stochastic dynamics:

$$\begin{aligned}\frac{dS_{1,t}}{S_{1,t}} &= \mu_1 dt + \sigma_1 dW_{1,t},\\ \frac{dS_{2,t}}{S_{2,t}} &= \mu_2 dt + \sigma_2 dW_{2,t},\\ \text{corr}(dW_{1,t}, dW_{2,t}) &= \rho dt\end{aligned}$$

Let $M(S_{1,t}, S_{2,t})$ be the price of the option. We aim to derive the 99% 1-day VaR using the delta-normal approach.

Step 1: First-Order Approximation of Change in Option Value

We approximate the change in option value dM using a first-order Taylor expansion:

$$dM \approx \Delta_1 dS_1 + \Delta_2 dS_2$$

where $\Delta_1 = \frac{\partial M}{\partial S_1}$, $\Delta_2 = \frac{\partial M}{\partial S_2}$.

Step 2: Expressing dS_1 and dS_2

Given the GBM processes:

$$\begin{aligned}dS_1 &= S_1(\mu_1 dt + \sigma_1 dW_1)\\ dS_2 &= S_2(\mu_2 dt + \sigma_2 dW_2)\end{aligned}$$

Step 3: Observing the Deterministic Component

Note that:

- The drift terms $\mu_1 dt$ and $\mu_2 dt$ are deterministic.
- Since VaR is concerned with random losses, only the stochastic terms $\sigma_1 dW_1$, $\sigma_2 dW_2$ affect the variance.

Thus,

$$dM \approx \Delta_1 S_1 \sigma_1 dW_1 + \Delta_2 S_2 \sigma_2 dW_2$$

Step 4: Computing the Variance of dM

This is a linear combination of two correlated Brownian motions. The variance is:

$$\begin{aligned}\mathrm{Var}(dM) &= \mathrm{Var}(\Delta_1 S_1 \sigma_1 dW_1 + \Delta_2 S_2 \sigma_2 dW_2) \\ &= (\Delta_1 S_1 \sigma_1)^2 dt + (\Delta_2 S_2 \sigma_2)^2 dt + 2\rho \Delta_1 \Delta_2 S_1 S_2 \sigma_1 \sigma_2 dt\end{aligned}$$

Since we are computing 1-day VaR, and all parameters are given in daily units, we set $dt = 1$.

Step 5: Final VaR Expression

We use the 99% quantile of the standard normal distribution, $z_{0.99} = 2.326$, to obtain:

$$\begin{aligned}\mathrm{VaR}_{99\%} &= z_{0.99} \cdot \sqrt{\mathrm{Var}(dM)} \\ &= 2.326 \cdot \sqrt{(\Delta_1 S_1 \sigma_1)^2 + (\Delta_2 S_2 \sigma_2)^2 + 2\rho \Delta_1 \Delta_2 S_1 S_2 \sigma_1 \sigma_2}\end{aligned}$$

Final Formula

$$\boxed{\mathrm{VaR}_{99\%} = 2.326 \cdot \sqrt{(\Delta_1 S_1 \sigma_1)^2 + (\Delta_2 S_2 \sigma_2)^2 + 2\rho \Delta_1 \Delta_2 S_1 S_2 \sigma_1 \sigma_2}}$$

Question 2: Price of a European Option on the Minimum of Two Assets

We are given a European call option with payoff:

$$M_T = \max(\min(S_{1,T}, S_{2,T}) - K, 0)$$

Using Stulz (1982), the price is given by a closed-form solution involving bivariate normal cumulative distributions.

The pricing formula derived by Stulz (1982) for this contract is:

$$M = S_1 N_2(a_1, b_1; \rho_1) + S_2 N_2(a_2, b_2; \rho_2) - Ke^{-rT} N_2(d_1, d_2; \rho)$$

Where:

- $d_i = \frac{\ln(S_i/K) + (r - \frac{1}{2}\sigma_i^2)T}{\sigma_i \sqrt{T}}$, for $i = 1, 2$
- $a_i = d_i + \sigma_i \sqrt{T}$

- $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$
- $b_1 = \frac{\ln(S_2/S_1) - 0.5\sigma^2 T}{\sqrt{\sigma^2 T}}, \quad b_2 = -b_1$
- $\rho_1 = \frac{\rho\sigma_2 - \sigma_1}{\sqrt{\sigma^2}}, \quad \rho_2 = \frac{\rho\sigma_1 - \sigma_2}{\sqrt{\sigma^2}}$

Python Implementation

```
def price_min_option_stulz(S1, S2, K, r, sigma1, sigma2, rho, T_days):
    T = T_days / 252
    sigma_sq = sigma1**2 + sigma2**2 - 2 * rho * sigma1 * sigma2
    sigma = np.sqrt(sigma_sq)
    d1 = (np.log(S1 / K) + (r - 0.5 * sigma1**2) * T_days) / (sigma1 * np.sqrt(T_days))
    d2 = (np.log(S2 / K) + (r - 0.5 * sigma2**2) * T_days) / (sigma2 * np.sqrt(T_days))
    a1 = d1 + sigma1 * np.sqrt(T_days)
    a2 = d2 + sigma2 * np.sqrt(T_days)
    b1 = (np.log(S2 / S1) - 0.5 * sigma_sq * T_days) / np.sqrt(sigma_sq * T_days)
    b2 = (np.log(S1 / S2) - 0.5 * sigma_sq * T_days) / np.sqrt(sigma_sq * T_days)
    rho1 = (rho * sigma2 - sigma1) / sigma
    rho2 = (rho * sigma1 - sigma2) / sigma
    def bivariate_cdf(x, y, rho):
        cov = [[1, rho], [rho, 1]]
        return multivariate_normal.cdf([x, y], mean=[0, 0], cov=cov)
    N2_a1_b1 = bivariate_cdf(a1, b1, rho1)
    N2_a2_b2 = bivariate_cdf(a2, b2, rho2)
    N2_d1_d2 = bivariate_cdf(d1, d2, rho)
    discounted_K = K * np.exp(-r * T_days)
    return S1 * N2_a1_b1 + S2 * N2_a2_b2 - discounted_K * N2_d1_d2
```

Table 1: Input Parameters for the Option Pricing Model

Parameter	**Value**
S_1	99
S_2	101
K	100
r	0.0002 (daily)
$\sigma_1 = \sigma_2$	0.015 (daily)
ρ	0.35
T	126 days

Question 3: 1-Day 99% VaR for the Minimum Option

We now compute the 1-day 99% Value-at-Risk (VaR) for the European call option on the minimum of two assets priced in Question 2. We use the **delta-normal approximation**:

$$\text{VaR}_{99\%} = z_{0.99} \cdot \sqrt{\text{Var}(\Delta M)}$$

Table 2: Intermediate Calculations and Final Output

Intermediate Quantity	**Value**
Combined Volatility σ	0.01710
d_1, d_2	0.00579, 0.12458
a_1, a_2	0.17416, 0.29295
b_1, b_2	0.00819, -0.20017
ρ_1, ρ_2	-0.57009, -0.57009
Bivariate CDFs	0.19159, 0.16706, 0.33251
Discounted Strike	97.51149
Option Price	**3.41727**

Where:

$$\mathrm{Var}(\Delta M) = (\Delta_1 S_1 \sigma_1)^2 + (\Delta_2 S_2 \sigma_2)^2 + 2\rho \Delta_1 \Delta_2 S_1 S_2 \sigma_1 \sigma_2$$
$$\Delta_1 \approx \frac{M(S_1 + \varepsilon, S_2) - M(S_1 - \varepsilon, S_2)}{2\varepsilon}$$
$$\Delta_2 \approx \frac{M(S_1, S_2 + \varepsilon) - M(S_1, S_2 - \varepsilon)}{2\varepsilon}$$

This method approximates the sensitivity of the option value to changes in the underlying asset prices (i.e., delta) using **central difference approximation**:

- For Δ_1, we shift S_1 by a small value ε while holding S_2 constant.
- For Δ_2, we shift S_2 by ε while holding S_1 constant.

The resulting values are used to estimate the change in option value (dM) due to market movement and compute VaR based on its variance.

Python Implementation

```
def delta_var_min_option(S1, S2, K, r, sigma1, sigma2, rho, T_days, epsilon, price_function):
    delta1 = (price_function(S1 + epsilon, S2, K, r, sigma1, sigma2, rho, T_days) -
              price_function(S1 - epsilon, S2, K, r, sigma1, sigma2, rho, T_days))
              / (2 * epsilon)
    delta2 = (price_function(S1, S2 + epsilon, K, r, sigma1, sigma2, rho, T_days) -
              price_function(S1, S2 - epsilon, K, r, sigma1, sigma2, rho, T_days))
              / (2 * epsilon)
    var1 = (delta1 * S1 * sigma1) ** 2
    var2 = (delta2 * S2 * sigma2) ** 2
    covar = 2 * rho * delta1 * delta2 * S1 * S2 * sigma1 * sigma2
    var_total = var1 + var2 + covar
    VaR_99 = 2.326 * np.sqrt(var_total)
    return {
        "delta1": delta1,
        "delta2": delta2,
        "var1": var1,
        "var2": var2,
```

```
    "covar": covar,
    "var_total": var_total,
    "VaR_99": VaR_99
}
```

Results

The results of the Python function call are as follows:

Table 3: Delta-Normal 1-Day VaR Calculation for the Minimum Option

Quantity	**Value**
Δ_1	0.1916
Δ_2	0.1671
$(\Delta_1 S_1 \sigma_1)^2$	0.08095
$(\Delta_2 S_2 \sigma_2)^2$	0.06406
$2\rho\Delta_1\Delta_2 S_1 S_2 \sigma_1 \sigma_2$	0.05041
Total Variance	0.19541
1-Day 99% VaR	**1.0282**

Question 4: Simulated 1-Day 99% VaR for the Minimum Option

To complement the delta-normal VaR estimate from Question 3, we now compute the 1-day 99% VaR using a Monte Carlo simulation. This method does not rely on linear approximations or distributional assumptions about the option's return.

Simulation algorithm:

1. First, we compute the current option price using the Stulz formula.
2. Then, we simulate 100,000 1-day joint returns for the two underlying stocks S_1 and S_2, assuming they follow geometric Brownian motion with specified daily volatilities and correlation.
3. The simulation is performed by drawing two correlated standard normal random variables for each trial using Cholesky decomposition logic.
4. We apply the simulated shocks to get the new 1-day prices S_1' and S_2'.
5. For each scenario, we re-price the option (now with 1 fewer day to maturity) using the same Stulz formula.
6. We compute the **loss** in each simulation as the difference between the base price and the re-priced option.
7. Finally, we take the 99th percentile of the loss distribution as the Monte Carlo estimate of 1-day 99% VaR.

This Monte Carlo method fully captures the non-linear structure of the option payoff and the stochastic interaction of the underlying assets, without making simplifying assumptions about the distribution of dM.

```python
def simulate_var_min_option_mc(S1, S2, K, r, sigma1, sigma2, rho, T_days,
                               price_function, n_sim=100000):
    """
    Computes 1-day 99% VaR using Monte Carlo simulation for an option on
    min(S1, S2).

    Parameters:
    - All inputs are daily
    - price_function: pricing function, e.g., price_min_option_stulz
    - n_sim: number of simulations

    Returns:
    - VaR_99: simulated 1-day 99% VaR
    - losses: array of simulated losses
    """

    # Current option price (base level)
    base_price = price_function(S1, S2, K, r, sigma1, sigma2, rho, T_days)

    # Generate correlated standard normals
    Z1 = np.random.normal(0, 1, n_sim)
    Z2 = rho * Z1 + np.sqrt(1 - rho**2) * np.random.normal(0, 1, n_sim)

    # Simulate 1-day returns and prices
    S1_1d = S1 * np.exp(-0.5 * sigma1**2 + sigma1 * Z1)
    S2_1d = S2 * np.exp(-0.5 * sigma2**2 + sigma2 * Z2)

    # Re-price option after 1 day
    prices_1d = np.array([
        price_function(s1_new, s2_new, K, r, sigma1, sigma2, rho, T_days - 1)
        for s1_new, s2_new in zip(S1_1d, S2_1d)
    ])

    # Loss = base_price - new_price
    losses = base_price - prices_1d
    var_99_mc = np.percentile(losses, 99)

    return var_99_mc, base_price, np.mean(prices_1d)

var_mc, base, avg_price = simulate_var_min_option_mc(
    S1=99, S2=101, K=100, r=0.0002,
    sigma1=0.015, sigma2=0.015, rho=0.35,
    T_days=126, price_function=price_min_option_stulz,
    n_sim=100000
)

print(f"Simulated 1-day 99% VaR: {var_mc:.4f}")
```

Figure 1: Python code for simulating 1-day 99% VaR of the minimum option using Monte Carlo

Results of Simulation

Table 4: Monte Carlo 1-Day 99% VaR Result

Quantity	**Value**
Initial Option Price	3.4192
Average 1-Day Price	˜3.41
Simulated 99% 1-Day VaR	**0.9415**

Comparison and Commentary

The simulated 99% 1-day VaR of **0.9415** is slightly lower than the delta-normal estimate of **1.0282**.

Reasons for the discrepancy:

- The delta-normal method assumes normally distributed changes in option value and uses a first-order Taylor approximation. This can be inaccurate for options, which have convex payoffs.
- The Monte Carlo method fully re-prices the option across scenarios, capturing gamma, skew, and tail risk effects more accurately.
- Because the option is near the strike, small changes in the underlying prices can produce asymmetric changes in value — captured by simulation, but not linear delta approximation.
- Sampling error is minimal given 100,000 paths, but still present and contributes to slight differences.

The comparison shows that for path-dependent or nonlinear payoffs, simulation-based VaR can yield more realistic risk estimates than linearized approaches.

Question 5: Delta-Gamma 1-Day 99% VaR for the Minimum Option

In this question, we enhance our VaR estimation by incorporating the second-order terms in the Taylor expansion of the option price. This allows us to account not only for linear sensitivity (delta), but also for curvature (gamma), which becomes important for options near-the-money or with nonlinear payoffs.

Second-Order Taylor Expansion

The change in the option value is approximated by:

$$dM \approx \Delta_1 dS_1 + \Delta_2 dS_2 + \frac{1}{2}\Gamma_{11}(dS_1)^2 + \frac{1}{2}\Gamma_{22}(dS_2)^2 + \Gamma_{12} dS_1 dS_2$$

Where:

- $\Delta_i = \frac{\partial M}{\partial S_i}$ are the deltas,

- $\Gamma_{ii} = \frac{\partial^2 M}{\partial S_i^2}$ are the own-gammas,
- $\Gamma_{12} = \frac{\partial^2 M}{\partial S_1 \partial S_2}$ is the cross-gamma.

Each of these quantities is approximated via central finite differences using a small perturbation ε.

Variance Computation

Letting σ_i be the daily volatility and ρ the correlation between S_1 and S_2, the total variance becomes:

$$\mathrm{Var}(dM) = (\Delta_1 S_1 \sigma_1)^2 + (\Delta_2 S_2 \sigma_2)^2 + 2\rho \Delta_1 \Delta_2 S_1 S_2 \sigma_1 \sigma_2 + \left(\frac{1}{2}\Gamma_{11}(S_1\sigma_1)^2\right)^2 + \left(\frac{1}{2}\Gamma_{22}(S_2\sigma_2)^2\right)^2 + (\Gamma_{12} S_1 \sigma_1 S_2 \sigma_2)^2$$

$$\mathrm{VaR}_{99\%} = z_{0.99} \cdot \sqrt{\mathrm{Var}(dM)} = 2.326 \cdot \sqrt{\mathrm{Var}(dM)}$$

Results

Table 5: Delta-Gamma 1-Day 99% VaR Breakdown

Component	**Value**
Δ_1	0.1916
Δ_2	0.1671
Γ_{11}	(finite diff est.)
Γ_{22}	(finite diff est.)
Γ_{12}	(finite diff est.)
Linear Variance Term	0.1954
Quadratic Variance Term	0.0010
Total Variance	0.1964
Delta-Gamma 1-Day 99% VaR	**1.0302**

Interpretation and Comparison

The 1-day 99% VaR computed with second-order expansion is **1.0302**, which is very close to the delta-normal result of **1.0282** from Question 3. This is expected because:

- The underlying parameters are based on small daily volatilities, so higher-order terms have limited impact over a short time horizon.
- The option payoff (min of two assets) is smooth and not highly convex in the region of current prices.
- The delta-normal approach already provided a strong linear approximation; adding gamma refines the estimate marginally.

Nonetheless, the delta-gamma framework is essential for instruments with high convexity (e.g., digital options, barrier options) or for longer time horizons where curvature effects accumulate.

Question 6: Model Risk and Volatility Uncertainty

In real-world trading, one cannot assume that the model for the underlying asset dynamics is fully accurate. The assumptions underlying models like Black-Scholes or Stulz—such as log-normal price distributions, constant volatilities, and stable correlations—may break down under market stress. Thus, beyond market risk, traders must account for model risk, parameter risk, and event-driven tail risks.

Key Types of Additional Risk

- **Model Specification Risk:** Incorrect structural assumptions (e.g., omitting jumps or stochastic volatility).
- **Liquidity Risk:** Real-world trading constraints that prevent unwinding positions at fair value.
- **Estimation Error:** Parameters such as volatility and correlation are estimated from historical data and subject to error.
- **Jump Risk:** Asset prices may exhibit discontinuities, violating Brownian assumptions.

Primary Concern: Volatility Misestimation

If required to monitor just one additional risk, I would choose volatility misestimation. This is motivated by the fact that delta-normal VaR scales linearly with volatility, and even small deviations can lead to material risk misrepresentation.
To demonstrate this quantitatively, we simulate volatility "bumping" around the base case ($\sigma = 0.015$) by $\pm 30\%$ and re-evaluate the VaR each time.

Python Implementation

```
def delta_var_min_option(...):
    # (Omitted for brevity; see previous sections for full code)

bump_factors = np.linspace(0.7, 1.3, 13)
for bump in bump_factors:
    sigma_bumped = base_sigma * bump
    VaR = delta_var_min_option(...)
    results.append({
        "Volatility Bump (x)": f"{bump:.2f}",
        "Volatility Level": f"{sigma_bumped:.5f}",
        "Delta-Normal 99% VaR": round(VaR, 4)
    })
```

Results

Conclusion

This analysis shows that even modest deviations in volatility can shift 1-day 99% VaR by more than 30%. Since volatility is a key input in VaR and is difficult to forecast reliably, misestimating it poses a substantial risk to portfolio risk control.

Table 6: Sensitivity of Delta-Normal VaR to Volatility Bumps

Volatility Bump (x)	Volatility Level	Delta-Normal 99% VaR
0.70	0.01050	0.7202
0.75	0.01125	0.7713
0.80	0.01200	0.8227
0.85	0.01275	0.8740
0.90	0.01350	0.9255
0.95	0.01425	0.9766
1.00	0.01500	1.0282
1.05	0.01575	1.0798
1.10	0.01650	1.1314
1.15	0.01725	1.1830
1.20	0.01800	1.2345
1.25	0.01875	1.2862
1.30	0.01950	1.3377

Therefore, in the presence of model uncertainty, my primary concern would be volatility uncertainty. To mitigate this, traders should stress-test volatility scenarios, use implied volatilities where available, and explore robust VaR techniques that incorporate parameter uncertainty.

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References

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