Several Complex Variables

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Contents

Chapter 1

Holomorphic Functions in Several Variables

1.1 Basic notions and formulas in several variables

Let $\mathbb{C}^n$ denote the n-dimensional *complex* Euclidean space. Let us denote the coordinates of $\mathbb{C}^n$ by $z = (z_1, z_2, ..., z_n)$ and $x = (x_1, x_2, ..., x_n)$ and $(y_1, y_2, ..., y_n)$ denote coordinates in $\mathbb{R}^n$. We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by using the relation $z_j = x_j + iy_j$ for all $1 \leq j \leq n$. Similar to the one variable complex case, we denote the complex conjugate of a complex variable z_j by $\bar{z}_j = x_j - iy_j$. We call z *holomorphic coordinates* and $\bar{z}$ *antiholomorphic coordinates*.

Definition 1.1.1. For $\rho = (\rho_1, \rho_2, ...\rho_n)$ where $\rho_j > 0$ and $a \in \mathbb{C}^n$, define a polydisc

$$\Delta_\rho(a) := \{z \in \mathbb{C}^n : |z_j - a_j| < \rho_j \text{ for } j = 1, 2, ..., n\}.$$

where a is called the *center* and ρ the polyradius or radius of the polydisc $\Delta_\rho(a)$.

If ρ is a number, then

$$\Delta_\rho(a) := \{z \in \mathbb{C}^n : |z_j - a_j| < \rho \text{ for } j = 1, 2, ..., n\}.$$

If n = 2, then a polydisc is called a bidisc.

Simliar to the one-variable case, there is also the unit *polydisc* in several variables:

$$\mathbb{D}^n = \mathbb{D} \times \mathbb{D} \times \cdots \times \mathbb{D} := \Delta_1(0) = \{z \in \mathbb{C}^n : |z_j| < 1 \text{ for } j = 1, 2, ..., n\}.$$

Recall the Euclidean inner product on $\mathbb{C}^n$:

$$\langle z, w \rangle := z_1\bar{w}_1 + z_2\bar{w}_2 + \cdots + z_n\bar{w}_n.$$

Definition 1.1.2. The above Euclidean inner product gives rise to the standard Euclidean norm on $\mathbb{C}^n$:

$$||z|| := \sqrt{\langle z, z \rangle} = \sqrt{z_1^2 + z_2^2 + \cdots + z_n^2}.$$

This agrees with the standard Euclidean norm on $\mathbb{R}^{2n}$.

Now we can define balls on $\mathbb{C}^n$:

$$B_\rho(a) := \{z \in \mathbb{C}^n : ||z - a|| < \rho\},$$

and unit ball as:

$$\mathbb{B}_n := B_1(0) = \{z \in \mathbb{C}^n : ||z|| < 1\}.$$

Definition 1.1.3 (Separate Holomorphicity)**.** Given a function f on an open set $U \subseteq \mathbb{C}^n$. We say that the function f is separately holomorphic (or analytic) on U in each variable if for each $1 \leq i \leq n$, f is holomorphic on U in z_i.

Definition 1.1.4 (Holomorphic/Analytic)**.** Given a function f on an open set $U \subseteq \mathbb{C}^n$. We say that the function f is holomorphic (or analytic) on U if given any a in U, and for all z in the neighbourhood contained in U of a :

$$f(z) = \sum_{i_1,\ldots,i_n=0}^{\infty} c_{i_1,\ldots i_n}(z_1 - a_1)^{i_1} \ldots (z_n - a_n)^{i_n}.$$

Alternatively, due to Cauchy's Integral formula, we say that a function f is holomorphic on an open set $U \subseteq \mathbb{C}^n$. If the limit:

$$\lim_{z \to z_0} \frac{f(z) - f(a)}{z - a}$$

exists for all $a \in U$.

Theorem 1.1.1 (Cauchy's Integral Formula for $\mathbb{C}^n$)**.** *Let $U \subseteq \mathbb{C}^n$ be an open set containing the closed polydisc $\overline{\Delta}_r(a)$. Let $f : U \to \mathbb{C}$ be separately holomorphic in each variable z_i in U, then:*

$$f(z) = (\frac{1}{2\pi i})^n \int_{\partial\Delta_{r_n}(a_n)} \cdots \int_{\partial\Delta_{r_1}(a_1)} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)} d\zeta_1 \ldots d\zeta_n.$$

for each $z = (z_1, ..., z_n) \in \overline{\Delta}_r(a)$ where each $\partial\Delta_{r_i}(a_i)$ is positively oriented.

Proof. The proof of the above theorem is the repeated application of the 1-variable Cauchy Integral Formula to each of the complex variables. □

Theorem 1.1.2 (Identity Theorem for $\mathbb{C}^n$)**.** *Let $U \subseteq \mathbb{C}^n$ be a domain (connected open set) and let $f : U \to \mathbb{C}$ be a holomorphic function. Suppose f is identitically 0 on a nonempty open set $V \subseteq U$. Then we have, $f \equiv 0$ on U.*

Proof. We are going to exploit the connectedness of the set U. First let

$$Z = \{a \in U | f^{(n)}(a) = 0 \text{ for all nonnegative integer } n\}.$$

We note that $V \subseteq Z$ so that Z is not empty. We claim that Z is open and closed. Firstly, Z is closed as f is holomorphic on U so that each $f^{(n)}$ is continuous, therefore, taking limits, we see that the any limit point lies back in Z. Secondly, from the openness of Z, we take any $a \in Z$ and look at power series expansion of f on the polydisc $\Delta_r(a) \subseteq U$. Since each $f^{(n)}(a) = 0$, the power series expansion on the aforementioned polydisc is identically zero so that $\Delta_r(a) \subseteq Z$. Thus, Z is open.

Since we have shown that Z is both open and closed, and that it is also not empty in a connected set, we must have that $Z = U$ as the only nonempty clopen set is the whole set itself. Therefore such an f is identically 0 everywhere on U. $\square$

1.2 Hartog's Theorem

Lemma 1.2.1 (Osgood's Lemma). *Let f be a function on an open set $U \subseteq \mathbb{C}^n$, which is bounded on each compact subset of U, and separately holomorphic in each variable on U, then, for each compact polydisc $\overline{\Delta}_s(a) \subset U$, there is a power series of the form*

$$\sum_{i_1,\ldots,i_n=0}^{\infty} c_{i_1,\ldots,i_n}(z_1 - a_1)^i_1 \ldots (z_n - a_n)^i_n$$

which converges uniformly to f on $\overline{\Delta}_s(a)$.

Proof. Fix polyradius $s = (s_1, s_2, ..., s_n)$. Since $\overline{\Delta}_s(a)$ is compact in U, we can find polyradius $r = (r_1, \ldots, r_n)$ with $r_i > s_i$ such that $\overline{\Delta}_s(a) \subset \overline{\Delta}_r(a) \subset U$. We also note that, by hypothesis, since $\overline{\Delta}_r(a)$ is compact, it is also bounded. Starting with the Cauchy Integral Formula in $\mathbb{C}^n$:

$$f(z) = (\frac{1}{2\pi i})^n \int_{\partial\Delta_{r_n}(a_n)} \cdots \int_{\partial\Delta_{r_1}(a_1)} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1)\ldots(\zeta_n - z_n)} d\zeta_1 \ldots d\zeta_n.$$

We substitute in the series expansion (we made use of the separate holomorphicity of $\frac{1}{\zeta_i - z_i}$ here):

$$\frac{1}{(\zeta_1 - z_1)\ldots(\zeta_n - z_n)} = \sum_{i_1,\ldots,i_n=0}^{\infty} \frac{(z_1 - a_1)^{i_1} \ldots (z_n - a_n)^{i^n}}{(\zeta_1 - a_1)^{i_1+1} \ldots (\zeta_n - a_n)^{i_n+1}}$$

We then bring interchange the order of summation with each of the integrals. We are able to to this as the convergence is uniform in each variable. Thus we get

$$f(z) = \sum_{i_1,\ldots,i_n=0}^{\infty} c_{i_1,\ldots,i_n}(z_1 - a_1)^{i_1} \ldots (z_n - a_n)^{i_n}$$

uniformly convergent on $\overline{\Delta}_s(a)$ as required where

$$c_{i_1,\ldots,i_n} = (\frac{1}{2\pi i})^n \int_{\partial\Delta_{r_n}(a_n)} \cdots \int_{\partial\Delta_{r_1}(a_1)} \frac{f(\zeta_1, \ldots, \zeta_n)\, d\zeta_1 \ldots d\zeta_n}{(\zeta_1 - a_1)^{i_1+1} \ldots (\zeta_n - a_n)^{i_n+1}}$$

□

Lemma 1.2.2 (Hartogs Lemma). *Let f be holomorphic in $\Delta_r(0)$ and let*

$$f(z) = \sum_k f_k(z') z_n^k$$

be the power series expansion of f in the variable z_n, where the f_k's are holomorphic in $\Delta_{r'}(0)$. If there is a number $c > r_n$ such that this series converges in $\overline{\Delta}_c(0)$, for each $z' \in \Delta_{r'}(0)$, then it converges uniformly on each compact subset of $\Delta_{r'}(0) \times \Delta_c(0)$. In other words, f extends to be holomorphic on the larger polydisc.

The proof of the above involves concepts from measure theory which is out of the scope of this thesis.

Theorem 1.2.1 (Hartogs Theorem)**.** *Let $U \subseteq \mathbb{C}^n$ be an open set. If $f : U \to \mathbb{C}$ is a separately holomorphic function in each variable on U, then f is holomorphic on U.*

Proof. We proceed by induction on the dimension n. For the case $n = 1$, this is trivially true. (Holomorphicity and separate holomorphicity are the same since there is only 1 variable). Suppose that the statement is true for some $n-1 > 0$, we want to show that it is also true for n.

Fix any point $a \in U$ and let $\overline{\Delta}_r(a) \subseteq U$ be a closed polydisc. Denote $z = (z', z_n) \in C^n$ and $\overline{\Delta}_r(a) = \overline{\Delta}(a', r') \times \overline{\Delta}(a_n, r_n)$.

Consider the following sets:

$$X_k = \{z_n \in \overline{\Delta}_{r_n/2(a_n)} : |f(z', z_n)| \leq k, \ \forall z' \in \overline{\Delta}_{r'}(a')\}$$

where $k \in \mathbb{N}$. For each of these k, X_k is closed as $f(z', z_n)$ is continuous in z_n for fixed z'. By induction hypothesis, $f(z', z_n)$ is also continuous in z' and hence bounded on the closed polydisc $\overline{\Delta}_{r'}(a')$ for each fixed $z_n \in \overline{\Delta}_{r_n/2}(a_n)$. Therefore, we have $\overline{\Delta}_{r_n/2}(a_n) \subseteq \bigcup_{k=1}^{\infty} X_k$. Thus, $\overline{\Delta}_{r_n/2}(a_n) = \bigcup_{k=1}^{\infty} X_k$ by definition of each X_k. Since each X_k is closed and the space is complete, we know it is of second category by Baire's Category Theorem, and thus, there exists $k \in \mathbb{N}$ such that $int(X_k)$ is non empty. In other words, X_k contains, a neighbourhood $\Delta_\delta(b_n)$ for some point $b_n \in \Delta_{r_n/2}(a_n)$.

Next, f is separately holomorphic and uniformly bounded in the polydisc $\Delta'_r(a') \times \overline{\Delta}_\delta(b_n)$. By Osgood's lemma, f is holomorphic on $\Delta'_r(a') \times \Delta_\delta(b_n)$. Additionally, f has a power series expansion about (a', b_n) which converges uniformly on compact subsets of the aforementioned polydisc.

Finally, we let $s_n < \frac{r_n}{2}$ so that $\Delta_{s_n}(b_n) \subseteq \Delta_{r_n}(a_n)$. By the above paragraph, we have that $f(z', z_n)$ is holomorphic in z_n on $\Delta_{s_n}(b_n)$ for each $z' \in \Delta_{r'}(a')$. Therefore, its power series expansion about (a', b_n) converges as a power series in z_n on $\Delta_{s_n}(b_n)$ for each fixed $z' \in \Delta_{r'}(a')$. By Hartogs' lemma, f is holomorphic on the whole of $\Delta_{r'}(a') \times \Delta_{s_n}(b_n)$. This holds for any $a \in U$. This completes the proof. □

1.3 Derivatives

Let $f = u + iv$ be a function, the complex conjugate of f is then $\overline{f = u - iv}$ denoted by the map $z \mapsto \overline{f(z)}$. If f is holomorphic, then $\bar{f}$ is called an antiholomorphic function. However it is **not** the case z are $\bar{z}$ are independednt variables. When we look deeper, we see that a complex function that is not holomorphic but smooth is a function of real variables x and y with $z = x + iy$. An antiholomorphic function is one that does not depend on z, but only on $\bar{z}$. By the definition of Wirtinger operators, we have

$$\frac{\partial \bar{f}}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u - iv) = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{i}{2}\frac{\partial v}{\partial x} - \frac{i}{2}\frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial v}{\partial y} = 0, \qquad \text{for all } j = 1, \ldots, n$$

where the last equality holds as f is holomorphic and thus obey the Cauchy Riemann equations, the real and imaginary parts cancels out. Similarly we have

$$\frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}\frac{\partial u}{\partial x} + \frac{i}{2}\frac{\partial v}{\partial x} + \frac{i}{2}\frac{\partial u}{\partial y} - \frac{1}{2}\frac{\partial v}{\partial y} = 0, \qquad \text{for all } j = 1, \ldots, n$$

by a similar reasoning.

Next we prove a useful equality.

$$\frac{\partial \bar{f}}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u - iv) = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{i}{2}\frac{\partial v}{\partial x} + \frac{i}{2}\frac{\partial u}{\partial y} + \frac{1}{2}\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x} = \overline{\left(\frac{\partial f}{\partial z_j}\right)},$$

where the above holds for all $j = 1, ..., n$

In order to simplify our calculations down the line. We derive a chain rule in terms of Wirtinger derivatives rather than relying on derivatives in x and y.

Proposition 1.3.1. *(Complex chain rule). Suppose $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ are open sets and suppose $f : U \to V$, and $g : V \to C$ are (real) differentiable functions (mappings). Write the variables as $z = (z_1, \ldots, z_n) \in U \subset \mathbb{C}^n$ and $w = (w_1, \ldots, w_m) \in V \subset \mathbb{C}^m$. Then for $j = 1, \ldots, n$*

$$\frac{\partial}{\partial z_j}[g \circ f] = \sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial w_\ell}\frac{\partial f_\ell}{\partial z_j} + \frac{\partial g}{\partial \bar{w}_\ell}\frac{\partial f_\ell}{\partial z_j}\right), \frac{\partial}{\partial \bar{z}_j}[g \circ f] = \sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial w_\ell}\frac{\partial f_\ell}{\partial \bar{z}_j} + \frac{\partial g}{\partial \bar{w}_\ell}\frac{\partial \bar{f}_\ell}{\partial \bar{z}_j}\right).$$

Proof. Write $f = u + iv, z = x + iy, w = s + it, f$ is a function of z, and g is a function of w. The composition plugs in f for w, and so it plugs in u for s, and v for t.

Using the standard chain rule,

$$\begin{aligned}
\frac{\partial}{\partial z_j}[g \circ f] &= \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right)[g \circ f] \\
&= \frac{1}{2}\sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial s_\ell}\frac{\partial u_\ell}{\partial x_j} + \frac{\partial g}{\partial t_\ell}\frac{\partial v_\ell}{\partial x_j} - i\left(\frac{\partial g}{\partial s_j}\frac{\partial u_\ell}{\partial y_j} + \frac{\partial g}{\partial t_\ell}\frac{\partial v_\ell}{\partial y_j}\right)\right) \\
&= \sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial s_\ell}\frac{1}{2}\left(\frac{\partial u_\ell}{\partial x_j} - i\frac{\partial u_\ell}{\partial y_j}\right) + \frac{\partial g}{\partial t_\ell}\frac{1}{2}\left(\frac{\partial v_\ell}{\partial x_j} - i\frac{\partial v_\ell}{\partial y_j}\right)\right) \\
&= \sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial s_\ell}\frac{\partial u_\ell}{\partial z_j} + \frac{\partial g}{\partial t_\ell}\frac{\partial v_\ell}{\partial z_j}\right).
\end{aligned}$$

For $\ell = 1, \ldots, m$,

$$\frac{\partial}{\partial s_\ell} = \frac{\partial}{\partial w_\ell} + \frac{\partial}{\partial \bar{w}_\ell}, \quad \frac{\partial}{\partial t_\ell} = i\left(\frac{\partial}{\partial w_\ell} - \frac{\partial}{\partial \bar{w}_\ell}\right).$$

Continuing:

$$\begin{aligned}
\frac{\partial}{\partial z_j}[g \circ f] &= \sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial s_\ell}\frac{\partial u_\ell}{\partial z_j} + \frac{\partial g}{\partial t_\ell}\frac{\partial v_\ell}{\partial z_j}\right) \\
&= \sum_{\ell=1}^{m}\left(\left(\frac{\partial g}{\partial w_\ell}\frac{\partial u_\ell}{\partial z_j} + \frac{\partial g}{\partial \bar{w}_\ell}\frac{\partial u_\ell}{\partial z_j}\right) + i\left(\frac{\partial g}{\partial w_\ell}\frac{\partial v_\ell}{\partial z_j} - \frac{\partial g}{\partial \bar{w}_\ell}\frac{\partial v_\ell}{\partial z_j}\right)\right) \\
&= \sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial w_\ell}\left(\frac{\partial u_\ell}{\partial z_j} + i\frac{\partial v_\ell}{\partial z_j}\right) + \frac{\partial g}{\partial \bar{w}_\ell}\left(\frac{\partial u_\ell}{\partial z_j} - i\frac{\partial v_\ell}{\partial z_j}\right)\right) \\
&= \sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial w_\ell}\frac{\partial f_\ell}{\partial z_j} + \frac{\partial g}{\partial \bar{w}_\ell}\frac{\partial \bar{f}_\ell}{\partial z_j}\right).
\end{aligned}$$

The $\bar{z}$ derivative works similarly. □

Definition 1.3.1. Let $U \subset \mathbb{C}^n$ be an open set. A mapping $f : U \to \mathbb{C}^m$ is said to be holomorphic if each component is holomorphic. That is, if $f = (f_1, \ldots, f_m)$ then each f_j is a holomorphic function.

As in one variable, the composition of holomorphic functions (mappings) is holomorphic.

Theorem 1.3.1. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open sets and suppose $f : U \to V$ and $g : V \to \mathbb{C}^k$ are both holomorphic. Then the composition g o f is holomorphic.*

Proof. The proof is almost trivial by chain rule. Again let g be a function of $w \in V$ and f be a function of $z \in U$. For any $j = 1, \ldots, n$ and any $p = 1, \ldots, k$, we compute

$$\frac{\partial}{\partial \bar{z}_j}[g_p \circ f] = \sum_{\ell=1}^{m}\left(\frac{\partial g_p}{\partial w_\ell}\frac{\partial f_j}{\partial \bar{z}_j} + \frac{\partial g_p}{\partial \bar{w}_\ell}\frac{\partial}{\partial \bar{z}_j}\bar{f}_\ell\right) = 0,$$

where the last equality holds as $\frac{\partial f_j}{\partial \bar{z}_j} = 0$ and $\frac{\partial g_p}{\partial \bar{w}_\ell} = 0$ as f and g are both holomorphic respectively. □

For holomorphic functions the complex chain rule is simpler and in fact looks like the usual vector calculus rule. calculus rule. Suppose again U $\subset \mathbb{C}^n$ and V $\subset \mathbb{C}^m$ are open sets and $f : U \to V$, and $g : V \to \mathbb{C}$ are holomorphic. Again name the variables $z = (z_1, \ldots, z_n) \in U \subset \mathbb{C}^n$ and $w = (w_1, \ldots, w_m) \in V \subset \mathbb{C}^m$. Then the formula

$$\frac{\partial}{\partial z_j}[g \circ f] = \sum_{\ell=1}^{m}\left(\frac{\partial g}{\partial w_\ell}\frac{\partial f_\ell}{\partial z_j} + \frac{\partial g}{\partial \bar{w}_\ell}\frac{\partial \bar{f}_j}{\partial z_j}\right),$$

simplifies to

$$\frac{\partial}{\partial z_j}[g \circ f] = \sum_{\ell=1}^{m} \left(\frac{\partial g}{\partial w_\ell} \frac{\partial f_\ell}{\partial z_j} \right)$$

as for the z_j derivative, the $\bar{w}_j$ derivative of g is zero and the z_j derivative of $\bar{f}_\ell$ is also zero as both f and g are holomorphic.

For a $U \subset \mathbb{C}^n$, a holomorphic mapping $f : U \to \mathbb{C}^m$, and a point $p \in U$, define the holomorphic derivative, sometimes called the Jacobian matrix,

$$Df(p) \overset{\text{def}}{=} \left[\frac{\partial f_j}{\partial z_k}(p) \right]_{jk}.$$

The notation $f'(p) = Df(p)$ is also used.

Using the holomorphic chain rule above, as in the theory of real functions, the derivative of the composition is the composition of derivatives (multiplied as matrices).

Proposition 1.3.2. *(Chain rule for holomorphic mappings). Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open sets. Suppose $f : U \to V$ and $g : V \to \mathbb{C}^k$ are both holomorphic, and $p \in U$. Then*

$$D(g \circ f)(p) = Dg(f(p))Df(p).$$

In shorthand, we often simply write $D(g \circ f) = DgDf$.

Proof. □

Suppose $U \subset \mathbb{C}^n, p \in U$, and $f : U \to \mathbb{C}^m$ is a differentiable function at p. since $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$, the mapping f takes $U \subset \mathbb{R}^{2n}$ to $\mathbb{R}^{2m}$. The normal vector calculus Jacobian at p of this mapping (a $2m \times 2n$ real matrix) is called the real Jacobian, and we write it as $D_{\mathbb{R}} f(p)$.

Proposition 1.3.3. *Let $U \subset \mathbb{C}^n$ be an open set, $p \in U$, and $f : U \to \mathbb{C}^n$ be holomorphic. Then*

$$|\det Df(p)|^2 = \det D_{\mathbb{R}} f(p).$$

The expression det $Df(p)$ is called the Jacobian determinant and clearly it is important to know if we are talking about the holomorphic Jacobian determinant or the standard real Jacobian determinant $\det D_{\mathbb{R}} f(p)$. *Recall from vector calculus that if the real Jacobian determinant det $D_{\mathbb{R}} f(p)$ of a smooth mapping is positive, then the mapping preserves orientation. In particular, the proposition says that holomorphic mappings preserve orientation.*

Proof. The real mapping using our identification is $(\operatorname{Re} f_1, \operatorname{Im} f_1, \ldots, \operatorname{Re} f_n, \operatorname{Im} f_n)$ as a function of $(x_1, y_1, \ldots, x_n, y_n)$. The statement is about the two Jacobians at p which are the derivatives at p. Hence, we can assume that $p = 0$ and f is complex linear, $f(z) = Az$ for some $n \times n$ matrix A. It is just a statement about matrices. The matrix A is the (holomorphic) Jacobian matrix of f. Let B be the real Jacobian matrix of f.

We change the basis of B to be $(z, \bar{z})$ using $z = x + iy$ and $\bar{z} = x - iy$ on both the image and the domain. The change of basis is some invertible complex matrix M such that $M^{-1}BM$ (the real Jacobian matrix B in these new coordinates) is a matrix of the derivatives of $\left(f_1, \ldots, f_n, \bar{f}_1, \ldots, \bar{f}_n\right)$ in terms of $(z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n)$. In other words,

$$M^{-1}BM = \begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix}.$$

Thus

$$\det(B) = \det\left(M^{-1}MB\right) = \det\left(M^{-1}BM\right) = \det(A)\det(\bar{A}) = \det(A)\overline{\det(A)} = |\det(A)|^2$$

$\square$

The regular implicit function theorem and the chain rule give that the implicit function theorem holds in the holomorphic setting. The main thing to check is to check that the solution given by the standard implicit function theorem is holomorphic, which follows by the chain rule.

Theorem 1.3.2. *Let $U \subset \mathbb{C}^n \times \mathbb{C}^m$ be an open set, let $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$ be our coordinates, and let $f : U \to \mathbb{C}^m$ be a holomorphic mapping. Let $(z^0, w^0) \in U$ be a point such that $f\left(z^0, w^0\right) = 0$ and such that the $m \times m$ matrix*

$$\left[\frac{\partial f_j}{\partial w_k}\left(z^0, w^0\right)\right]_{jk}$$

is invertible. Then there exists an open set $V \subset \mathbb{C}^n$ with $z^0 \in V$, open set $W \subset \mathbb{C}^m$ with $w^0 \in W$, $V \times W \subset U$, and a holomorphic mapping $g : V \to W$, with $g\left(z^0\right) = w^0$ such that for every $z \in V$, the point g(z) is the unique point in W such that

$$f(z, g(z)) = 0.$$

Chapter 2

Convexity and Pseudoconvexity

2.1 Domains of Holomorphy and Holomorphic Extensions

Definition 2.1.1 (Domain of Holomorphy)**.** Let $\Omega \subset \mathbb{C}^n$ be a domain (open connected). The set Ω is a domain of holomorphy if there do not exist nonempty open sets U and V, with $U \subset \Omega \cap V$. $V \not\subset \Omega$ and V connected, such that for every $f \in \mathcal{O}(\Omega)$ there exists $g \in \mathcal{O}(V)$ with $f(z) = g(z)$ for all $z \in U$.

Intuitively the definition above means that a domain Ω is not a domain of holomorphy if there is a function which can be extended across somewhere along the boundary of Ω. In other words, Ω is a domain of holomorphy if every holomorphic function on Ω cannot be extended across anywhere along the boundary.

In $\mathbb{C}$ there was no real meaning to look at domains at holomorphy as every domain in $\mathbb{C}$ is a domain of holomorphy as shown in the example below.

Example 2.1.1. In $\mathbb{C}$, every domain is a domain of holomorphy.

First, take any domain $\Omega \subset \mathbb{C}$. Suppose for a contradiction that U and V exists as in the defintion. Take any $z_0 \in \partial\Omega \cap V$. Consider the holomorphic function on Ω: $f(z) = \frac{1}{z-z_0}$. The limit $\lim_{z\to z_0} \frac{1}{z-z_0}$ does not exists and thus f is not continuous there, let alone holomorphic. Thus such a g does not exists and $\mathbb{C}$ is a domain of holomorphy.

Example 2.1.2. A polydisc and unit ball in $\mathbb{C}^n$ are domains of holomorphy. The same trick in the previous example works in this case as well. By Hartog's theorem provided in the previous chapter.

Proposition 2.1.1. .

(i) A finite intersection of domains of holomorphy is a domain of holomorphy

(ii) If $\Omega_j \subset \mathbb{C}^n$ for $j \in J$ (an arbitrary index set). Then either $\bigcap_{j\in J} \Omega_j$ is empty or every connected component of it is a domain of holomorphy.

(iii) If $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^k$ are domains of holomorphy, their products $U \times V \subset \mathbb{C}^{m+k}$ is also a domain of holomorphy.

Proof. The proof of (i) and (ii) is similar so only the proof of (ii) is shown. The case for $\bigcap_{j\in J} \Omega_j$ is trivial. Assume that $\bigcap_{j\in J} \Omega_j$ is not empty. Take any connected component Ω of it. Assume for a contradiction that Ω is not a domain of holomorphy. That is, there exists nonempty, open U, V with $U \subset \Omega \cap V, V \not\subset \Omega$ and V connected. Now, it follows that $U \subset \Omega \cap V \subset \Omega_j \cap V$ for all $j \in J$. If $V \subset \Omega_j$ for all $j \in J$, then $V \subset \bigcap_{j\in J} \Omega_j$ and V is a connected component. We know that connected components are disjoint, so from the assumption that $\varnothing \neq U \subset \Omega \cap V$ which implies $V \subset \Omega$ contradicting one of the initial assumptions that $V \not\subset \Omega$. Therefore Ω is a domain of holomorphy.

For the proof of (iii), we know that $U \times V$ is also open and connected. It remains to show that $U \times V$ is indeed a domain of holomorphy. We first assume for a contradiction that it is not. Then there exists open non-empty W, X with $W \subset (U \times V) \cap X$, such that for all $f \in \mathcal{U} \times \mathcal{V}$ there exists a $g \in \mathcal{X}$ where $f = g$ on W. Since W and X is open, under the subspace topology we can write $W = W_1 \times W_2$ and $X = X_1 \times X_2$ with $W_1, X_1 \subset \mathbb{C}^m$ and $W_2, X_2 \subset \mathbb{C}^k$. Thus, we have $W_1 \times W_2 \subset (U \times V) \cap (X_1 \times X_2)$. This however implies W_1 and X_1 exists for U (similarly for V) where the domain of holomorphy condition is violated. Therefore, it must be the case that $U \times V$ is indeed a domain of holomorphy. $\square$

Definition 2.1.2 (Geometric Convexity)**.** A set S is called *geometrically convex* if $tx + (1-t)y \in S$ for all $x, y \in S$ and $t \in [0, 1]$

Proposition 2.1.2. *A geometrically convex domain Ω in $\mathbb{C}^n$ is a domain of holomorphy.*

We note that that converse of this proposition is not true in general. Therefore we will study a weaker notion of convexity called pseudoconvexity later in this chapter. For now we will give a counter example to the converse of the above proposition.

Example 2.1.3. The set $\mathbb{D}\backslash\{0\} = \{z \in \mathbb{C} : |z| < 1, z \neq 0\}$ is a domain of holomorphy but it is not convex. To see this, we note that any domain in $\mathbb{C}$ is a domain of holomorphy. However it is not convex as $tx + (1-t)y = 0.5 * (-0.5) + (1 - 0.5) * 0.5 = 0$ which does not belong in $\mathbb{D}\backslash\{0\}$

In the above example, we used $\mathbb{C}$ where $n = 1$. We will provide examples where $n \geq 2$ later on.

Definition 2.1.3 (Hartogs figure)**.** Let $(z, w) = (z_1, \ldots, z_m, w_1, \ldots, w_k) \in \mathbb{C}^m \times \mathbb{C}^k$ be the coordinates. For two numbers $0 < a, b < 1$ define $H \subset \mathbb{D}^{m+k}$ by

$$H = \{(z, w) \in \mathbb{D}^{m+k} : |z_j| > a \; for \; j = 1, \ldots, m\} \cup \{(z, w) \in \mathbb{D}^{m+k} : |w_j| < b \; for \; j = 1, \ldots, k\}.$$

Theorem 2.1.1. *Let H be defined as above. If $f \in \mathcal{O}(H)$, then f extends holomorphically to $\mathbb{D}^{m+k}$. In particular, H is not a domain of holomorphy.*

Proof. Our goal is to extend f holomorphically to $\mathbb{D}^{m+k}$ we will proceed by providing an explicit construction of a F on the undefined parts of f, agrees with f on their intersection and together this will form a function on $\mathbb{D}^{m+k}$. Pick $c \in (a, 1)$. Denote

$$\Gamma = \{z \in \mathbb{D}^m : |z_j| = c \; for \; j = 1, \ldots, m\},$$

which is the distinguished boundary of $c\mathbb{D}^m$, where $c\mathbb{D}^m$ is a polydisc centred at 0 of radius c in $\mathbb{D}^m$. Now we define the function F on $c\mathbb{D}^m \times \mathbb{D}^k$:

$$F(z, w) = \frac{1}{(2\pi i)^m} \int_\Gamma \frac{f(\xi, w)}{\xi - z} d\xi.$$

This is clearly well defined as $|z_j| < c$ and $|\xi_j| = c$ so the denominator is always non-zero. Also, $w \in \mathbb{D}^k$ then $(\xi, w) \in H$. F is holomorphic in w by the fact that we are able to differentiate under the integral sign and noting that f is indeed holomorphic in W on H. We also note that F is holomorphic in z as the denominator is not zero on $c\mathbb{D}^m$ and is a holomorphic function.

Given a fixed w with $|w_j| < b$ for all j, by Cauchy integral formula , f and F agrees for all $z \in c\mathbb{D}^m$. Thus $f = F$ on $c\mathbb{D}^m \times b\mathbb{D}^k$, and so also agrees on $(c\mathbb{D}^m \times b\mathbb{D}^k) \cap H$. With this F and the given f we have a function on the entire $\mathbb{D}^{m+k}$. $\square$

Corollary 2.1.1. *Let U be an open set with $U \subset \mathbb{C}^n$, and $n \geq 2$. Then every $f \in \mathcal{O}(U \backslash \{p\})$ extends holomorphically to U.*

Proof. Without loss of generality, by translating and scaling by orthogonal matrices(these operations are holomorphic), we may assume that $p = (\frac{3}{4}, 0, \ldots, 0)$ and the unit polydisc is contained in U. Let $m = 1$ and $k = n - 1$ so that $\mathbb{C}^n = \mathbb{C} \times \mathbb{C}^{n-1}$ and let $a = b = \frac{1}{2}$ we do this so that we can fit a Hartogs figure in U. Then $H \subset U$ and $p \in \mathbb{D}^n \backslash H$. By the above theorem, we have that f extends holomorphically to the whole polydisc, specifically its extension is defined at p. $\square$

Actually the corollary above tells us more. Specifically it explains why holomorphic functions in more than one variable have no isolated zeroes (or poles). Let $U \subset \mathbb{C}^n$ open and with $n \geq 2$ and let f be a holomorphic function with a zero only at p. Then $1/f$ would be holomorphic at $U \backslash \{p\}$. By the corollary, this says that we would be able to extend f to include it to be defined at p. However this would be a contradiction as $1/f$ tends to ∞ at p, so it is not even continuous there.

2.2 Tangent Vectors, the Hessian, and convexity

In this section, we shall explore the basic ideas of the usual geometrical convexity, before moving on to weaken certain notions of this convexity to form the idea of a so called pseudoconvexity and show that the 2 concepts pseudoconvexity and domain of holomorphy are indeed equivalent. We first discuss the idea of classic convexity locally on a smooth boundary.

Definition 2.2.1. A set $M \subset \mathbb{R}^n$ is a real C^k-smooth hypersurface if at each point $p \in M$, there exists a k-times continuously differentiable function $r : V \to \mathbb{R}$ defined in a neighbourhood V of p with nonvanishing derivative such that $M \cap V = \{x \in V : r(x) = 0\}$. We call this r the defining function (at p).

Definition 2.2.2. An open set (or domain) U with C^k-smooth boundary is a set where ∂U is a C^k-smooth hypersurface, and for every $p \in \partial U$ there is a defining function r such that $r < 0$ for points in U and $r > 0$ for points not in U.

When we say that a function is smooth, we mean that the function is infinitely differentiable.

The definition provided above is actually that of an embedded hypersurface. The topology on the set M above is that of the subset topology.

To simplify things, we will only deal with smooth functions and hypersurfaces.

The same definition applies for $\mathbb{C}^n$ by treating $\mathbb{C}^n$ as $\mathbb{R}^{2n}$.

Definition 2.2.3. For a point $p \in \mathbb{R}^n$, the set of *tangent vectors* $T_p\mathbb{R}^n$ is given by

$$T_p\mathbb{R}^n = \operatorname{span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x_1}\bigg|_p, \ldots, \frac{\partial}{\partial x_n}\bigg|_p \right\}.$$

That is, a vector $X_p \in T_p\mathbb{R}^n$ is an object of the form

$$X_p = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}\bigg|_p,$$

for real numbers a_j. For computations, X_p could be represented by an n-vector $a = (a_1, \ldots, a_n)$. However if $p \neq q$, then $T_p\mathbb{R}^n$ and $T_q\mathbb{R}^n$ are two distinct spaces.

Proposition 2.2.1. *The tangent space T_pM is independent of which defining function we take. In other words if r and $\tilde{r}$ are defining functions for M at p, then $\sum_j a_j \frac{\partial r}{\partial x_j}\Big|_p = 0$ if and only if $\sum_j a_j \frac{\partial \tilde{r}}{\partial x_j}\Big|_p = 0$.*

Definition 2.2.4. The disjoint union

$$T\mathbb{R}^n = \bigcup_{p \in \mathbb{R}^n} T_p\mathbb{R}^n$$

is called the *tangent bundle*. There is a natural identification $\mathbb{R}^n \times \mathbb{R}^n \cong T\mathbb{R}^n$:

$$(p, a) \in \mathbb{R}^n \times \mathbb{R}^n \mapsto \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}\Big|_p \in T\mathbb{R}^n.$$

Definition 2.2.5. Suppose $U \subset R^n$ is an open set with smooth boundary, and r is a defining function for ∂U at $p \in \partial U$ such that $r < 0$ on U. If

$$\sum_{j=1,\ell=1}^{n} a_j a_\ell \frac{\partial^2}{\partial x_j \partial x_\ell}\Big|_p \geq 0 \text{ for all } X_p = \sum_{j=1}^{n} a_j \frac{\partial}{\partial x_j}\Big|_p \in T_p\partial U,$$

then U is said to be *convex* at p. If the inequality above is strict for all nonzero $X_p \in T_p\partial U$ then U is *strongly convex* at p. Say that a set U is convex if it is convex at every p in ∂U. If U is bounded, we say that U is strongly convex if it is strongly convex at every p in ∂U.

The Hessian of the matrix r is denoted as

$$H = \left[\frac{\partial^2 r}{\partial x_j \partial x_\ell}\Big|_p\right]_{j\ell}.$$

In other words, in matrix notation, for a set U to be convex at p we need the following condition:

$$a^t H a \geq 0, \quad \text{for all } a \in \mathbb{R}^n \text{ satisfying } \nabla r|_p \cdot a = 0$$

Proposition 2.2.2. *The definition of convexity is independent of the defining function.*

Example 2.2.1. The unit open disk U in $\mathbb{R}^2$ is strongly convex.

Let (x, y) be the coordinates and choose the defining function $r(x, y) = x^2 + y^2 - 1$ so that $r < 0$ in the U and $r > 0$ outside the boundary and $r = 0$ on ∂U. We use the gradient $\nabla r = (2x, 2y)$ to find a vector tangent to the curve defined by r (it will be a line since tangent space here is 1D). We use the vector $(y, -x)$ as $(2x, 2y) \cdot (y, -x) = 0$ for all $x, y \in \partial U$. The Hessian matrix is found to be

$$H_r = \begin{bmatrix} \frac{\partial^2 r}{\partial x^2} & \frac{\partial^2 r}{\partial x \partial y} \\ \frac{\partial^2 r}{\partial y \partial x} & \frac{\partial^2 r}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Taking the vector $(y, -x)$ and evaluating $a^T H a$ we get

$$\begin{bmatrix} y & -x \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} y \\ -x \end{bmatrix} = 2y^2 + 2x^2 = 2 > 0.$$

The above relation holds for all vectors $(y, -x)$ so that we know that U is strongly convex at all p in ∂U. Therefore U is strongly convex.

Example 2.2.2. The set $V := \{(x, y) \in \mathbb{R}^2 : x^4 + y^4 < 1\}$ is convex but not strongly convex at certain points.

Choose $r = x^4 + y^4 - 1$, this satisfies the requirements of the defining function. $\nabla r = (x^3, y^3)$ and the tangent vector will be of the form $(y^3, -x^3)$. Evaluating, we get

$$\begin{bmatrix} y^3 & -x^3 \end{bmatrix} \begin{bmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{bmatrix} \begin{bmatrix} y^3 \\ -x^3 \end{bmatrix} = 12x^2y^6 + 12x^6y^2 = 12x^2y^2(x^4 + y^4) = 12x^2y^2 \geq 0.$$

We have $12x^2y^2 = 0$ whenever $x = 0$ or $y = 0$ (but not both 0).

Therefore U is not convex at $(1, 0), (-1, 0), (0, 1), (0, -1)$.

In the parts below we will use a less standard big-o notation. That is a smooth function is $O(\ell)$ at a point p if the derivatives of order $0, 1, 2, \ldots, \ell - 1$ vanish at p.

Lemma 2.2.1. *Suppose $M \subset R^n$ is a smooth hypersurface, and $p \in M$. Then after a rotation and translation, p is the origin, and near the origin M is given by*

$$y = \varphi(x),$$

where $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ are our coordinates and φ is a smooth function that is $O(2)$ at the origin, i.e. $\varphi(0) = 0$ and $d\varphi(0) = 0$.

Furthermore, if M is the boundary of an open set U with smooth boundary and $r < 0$ on U, then the rotation can be chosen so that $y > \varphi(x)$ for points (x, y) in U.

Proof. First let r be the defining function at p. Denote $v = \nabla r|_p$. We can translate and rotate M so that $p = 0$ and $v = (0, \ldots, v_n)$ with $v_n < 0$. Using $(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R}$ as our coordinates, we have $\nabla r|_0 = v$ and $\frac{\partial r}{\partial y}(0) = v_n \neq 0$. By the implicit function theorem, there exists a smooth function φ defined in a neighbourhood of the origin with $r(x, \varphi(x)) = 0$ and the set $\{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y = \varphi(x)\}$ are all the solutions to $r = 0$ near the origin.

Next we proceed to show that $d\varphi(0) = 0$. From the above we have $r(x, \varphi(x)) = 0$ for x near the origin. Differentiating, we get that for every $j = 1, \ldots, n - 1$, we have

$$0 = \frac{\partial}{\partial x_j} [r(x, \varphi(x)] = \left(\sum_{\ell=1}^{n-1} \frac{\partial r}{\partial x_\ell} \frac{\partial x_\ell}{\partial x_j} \right) + \frac{\partial r}{\partial y} \frac{\partial \varphi}{\partial x_j} = \frac{\partial r}{\partial x_j} + \frac{\partial r}{\partial y} \frac{\partial \varphi}{\partial x_j}.$$

Since $\frac{\partial x_\ell}{\partial x_j} = 0$ for $\ell \neq j$. Now since $\frac{\partial r}{\partial x_j}(0, 0) = 0$ for all $j = 1, \ldots, n - 1$ and $\frac{\partial r}{\partial y}(0, 0) = v_n \neq 0$ by our rotation earlier, we get $\frac{\partial \varphi}{\partial x_j} = 0$ for all $j = 1, \ldots, n - 1$ so that the jacobian $d\varphi(0, 0) = 0$.

Finally, we will show that the rotation indeed gives us $y > \varphi(x)$ for $(x, y) \in U$. Since $r < 0$ on U, it suffices to check that $r < 0$ for $(0, y)$ if y is small, but this is implied by $\frac{\partial r}{\partial y}(0, 0) = v_n < 0$. □

2.3 Holomorphic vectors, the Levi form, and pseudoconvexity

Since we can identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ via the relation $z = x + iy$, we get $T_p\mathbb{C}^n = T_p\mathbb{R}^{2n}$. Instead of using the real span, we can use the complex span and thus we get the *complexified tangent space*

$$\mathbb{C} \otimes T_p\mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial x_1}\Big|_p, \frac{\partial}{\partial y_1}\Big|_p, \cdots, \frac{\partial}{\partial x_n}\Big|_p, \frac{\partial}{\partial y_n}\Big|_p \right\}.$$

We achieve this by replace all the real coefficients with complex ones. The space $\mathbb{C}\otimes T_p\mathbb{C}^n$ is a $2n$-dimensional complex vector space. Both $\frac{\partial}{\partial z_j}\Big|_p$ and $\frac{\partial}{\partial \bar{z}_j}\Big|_p$ are in $\mathbb{C}\otimes T_p\mathbb{C}^n$ and in fact:

$$\mathbb{C} \otimes T_p\mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}\Big|_p, \frac{\partial}{\partial \bar{z}_1}, \cdots, \frac{\partial}{\partial z_n}\Big|_p, \frac{\partial}{\partial \bar{z}_n}\Big|_p \right\}.$$

The relation above can be derived a change of basis given by the Wirtinger derivatives. Define

$$T_p^{(1,0)}\mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_n} \right\}, \; T_p^{(0,1)}\mathbb{C}^n = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_1}, \frac{\partial}{\partial \bar{z}_2}, \cdots, \frac{\partial}{\partial \bar{z}_n} \right\}$$

$$\text{and } \mathbb{C} \otimes T_p\mathbb{C} = T_p^{(1,0)}\mathbb{C}^n \oplus T_p^{(0,1)}\mathbb{C}^n.$$

A holomorphic function is one that vanishes on $T_p^{(0,1)}\mathbb{C}^n$.

We shall first explore the effects of a holomorphic map $f : \mathbb{C}^n \to \mathbb{C}^m$ on the space above. We denote the real derivative of f at p as $D_{\mathbb{R}}f(p) : T_p\mathbb{C}^n \to T_{f(p)}\mathbb{C}^m$. This can be represented as a $2m \times 2n$ Jacobian matrix.

Proposition 2.3.1. *Let $f : U \subset \mathbb{C}^n \to \mathbb{C}^m$ be a holomorphic mapping with $p \in U$. Suppose $D_{\mathbb{R}}f(p) : T_p\mathbb{C}^n \to T_{f(p)}\mathbb{C}^m$ is the real derivative of f at p. We natrually extend the derivative to $D_{\mathbb{C}}f(p) : \mathbb{C} \otimes T_p\mathbb{C}^n \to \mathbb{C} \otimes T_{f(p)}\mathbb{C}^m$. Then*

$$D_{\mathbb{C}}f(p)(T_p^{(1,0)}\mathbb{C}^n) \subset T_{f(p)}^{(1,0)}\mathbb{C}^m \quad \text{and} \quad D_{\mathbb{C}}f(p)(T_p^{(0,1)}\mathbb{C}^n) \subset T_{f(p)}^{(0,1)}\mathbb{C}^m.$$

In other words, the holomorphic vectors at p get mapped into holomorphic vectors at $f(p)$ and similarly for the antiholomorphic vectors.

Furthermore, if f is a biholomorphism, then $D_{\mathbb{C}}f(p)$ restricted to $T_p^{(1,0)}\mathbb{C}^n$ is a vector space isomorphism. A similar case holds for $T_p^{(0,1)}\mathbb{C}^n$.

Proof. We use the change of basis and its inverse (since B is a change of basis, we take conjugate transpose to get its inverse)

$$B = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \text{ and } B^{-1} = \begin{bmatrix} 1 & i \\ 1 & -i \end{bmatrix}.$$

This is a change of basis for $\mathbb{C}^1$. In general for $\mathbb{C}^n$ using the identification $\mathbb{R}^{2n} \ni (x_1, y_1, \dots, x_n, y_n) \hookrightarrow (z_1, \bar{z}_1, \dots, z_n\bar{z}_n) \in \mathbb{C}^{2n}$ with $z_j = x_j + y_j$ for $1 \leq j \leq n$. We have the following as a change of basis

$$B = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & \dots & 0 \\ -i/2 & i/2 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1/2 & 1/2 & \dots & 0 \\ 0 & 0 & -i/2 & i/2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & 1/2 & 1/2 \\ 0 & 0 & \dots & \dots & -i/2 & i/2 \end{bmatrix}$$

where the diagonals of the matrix are blocks of $\begin{bmatrix} 1/2 & 1/2 \\ -i/2 & i/2 \end{bmatrix}$ are repeated n times.

For the jacobian $D_{\mathbb{R}}f(p)$ in terms of basis (x, y) we have

$$D_{\mathbb{R}}f(p) = \begin{bmatrix} a_{11} & -b_{11} & a_{12} & -b_{12} & \dots \\ b_{11} & a_{11} & b_{12} & a_{12} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

with $a_{jk} = \frac{\partial u_j}{\partial x_k}$ and $b_{jk} = \frac{\partial v_j}{\partial y_k}$ where $f_j = u_j + iv_j$. The above matrix is due to the fact that f itself is holomorphic and it satisfies the Cauchy-Riemann equations i.e. $\frac{\partial u_j}{\partial x_k} = \frac{\partial v_j}{\partial y_k}$ and $\frac{\partial u_j}{\partial y_k} = -\frac{\partial v_j}{\partial x_k}$ for each $1 \leq j \leq m$ and $1 \leq k \leq n$.

Changing basis we get the following

$$B^{-1}D_{\mathbb{R}}f(p)B = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & 0 & \frac{\partial f_1}{\partial z_2} & 0 & \dots & \frac{\partial f_1}{\partial z_n} & 0 \\ 0 & \overline{\frac{\partial f_1}{\partial z_1}} & 0 & \overline{\frac{\partial f_1}{\partial z_2}} & \dots & 0 & \overline{\frac{\partial f_1}{\partial z_n}} \\ \frac{\partial f_2}{\partial z_1} & 0 & \ddots & & & \vdots & \vdots \\ 0 & \overline{\frac{\partial f_2}{\partial z_1}} & & & & & \\ \vdots & \vdots & & & & & \\ \frac{\partial f_m}{\partial z_1} & 0 & \dots & & & \frac{\partial f_m}{\partial z_n} & 0 \\ 0 & \overline{\frac{\partial f_m}{\partial z_1}} & \dots & & & 0 & \overline{\frac{\partial f_m}{\partial z_n}} \end{bmatrix}.$$

We look at the unit vectors in coordinates $(z_1, \bar{z}_1, \dots, z_n, \bar{z}_n)$. For example the unit (holomorphic) vector $(1, 0, \dots, 0)$ gets transformed into a vector with entries only in the holomorphic part. Particularly it is a linear combination of holomorphic vectors

$\frac{\partial}{\partial z_1}, \ldots \frac{\partial}{\partial z_n}$, in vector form, the unit vector transformation is

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \to \begin{bmatrix} \frac{\partial f_1}{\partial z_1} \\ 0 \\ \frac{\partial f_2}{\partial z_1} \\ 0 \\ \vdots \\ \frac{\partial f_m}{\partial z_1} \\ 0 \end{bmatrix},$$

where the LHS is a $2n \times 1$ matrix and the RHS is a $2m \times 1$ matrix.

Similarly, antiholomorphic unit vectors get transformed into antiholomorphic vectors as well. Therefore holomorphic and antiholomorphic vectors get transformed back into both holomorphic and antiholomorphic vectors respectively in the image of the tangent space which is the result as desired.

For the furthermore part, since f is biholomorphic is assumed, the differential $D_{\mathbb{C}}f^{-1}(f(p))$ is well defined and exists, so that $f$is indeed a vector space isomorphism when restricted to $T_p^{(1,0)}\mathbb{C}^n$.

□

When referring to holomorphic functions and holomorphic vectors, when we say derivative of f, we mean the holomorphic part of the derivative

$$Df(p) : T_p^{(1,0)}\mathbb{C}^n \to T_{f(p)}^{(1,0)}\mathbb{C}^m.$$

In other words, $Df(p)$ is the restriction of $D_{\mathbb{C}}f(p)$ to $T_p^{(1,0)}\mathbb{C}^n$. Let z be the coordinates in $\mathbb{C}^n$ and w be the coordinates on $\mathbb{C}^m$. Using the bases $\{\frac{\partial}{\partial z_1}|_p, \ldots, \frac{\partial}{\partial z_n}|_p\}$ on $\mathbb{C}^n$ and $\{\frac{\partial}{\partial w_1}|_{f(p)}, \ldots, \frac{\partial}{\partial w_m}|_{f(p)}\}$ on $\mathbb{C}^m$. The holomorphic derivative of $f : \mathbb{C}^n \to \mathbb{C}^m$ is represented as the $m \times n$ Jacobian matrix

$$\left[\frac{\partial f_j}{\partial z_k}\Big|_p \right]_{jk}.$$

Similarly we can define the tangent bundles

$$\mathbb{C} \otimes T\mathbb{C}^n, T^{(1,0)}\mathbb{C}^n, T^{(0,1)}\mathbb{C}^n$$

by taking disjoint unions about every point p. We can also define vector fields in these bundles.

Let us describe $\mathbb{C} \otimes T_pM$ for a real smooth hypersurface $M \subset \mathbb{C}^n$. Let r be a real-valued defining function of M at p. A vector $X_p \in \mathbb{C} \otimes T_p\mathbb{C}^n$ is in $\mathbb{C} \otimes T_pM$ whenever $X_pr = 0$. That is,

$$X_p = \sum_{j=1}^{n} \left(a_j \frac{\partial}{\partial z_j}\Big|_p + b_j \frac{\partial}{\partial \bar{z}_j}\Big|_p \right) \in \mathbb{C} \otimes T_pM \quad \text{whenever} \quad \sum_{j=1}^{n} \left(a_j \frac{\partial r}{\partial z_j}\Big|_p + b_j \frac{\partial r}{\partial \bar{z}_j}\Big|_p \right) = 0$$

Therefore, $\mathbb{C} \otimes T_pM$ is a $(2n-1)$ -dimensional complex vector space. We decompose $\mathbb{C} \otimes T_pM$ as

$$\mathbb{C} \otimes T_pM = T_p^{(1,0)}M \oplus T_p^{(0,1)}M \oplus B_p$$

where

$$T_p^{(1,0)}M \overset{\text{def}}{=} (\mathbb{C} \otimes T_pM) \cap \left(T_p^{(1,0)}\mathbb{C}^n\right), \quad \text{and} \quad T_p^{(0,1)}M \overset{\text{def}}{=} (\mathbb{C} \otimes T_pM) \cap \left(T_p^{(0,1)}\mathbb{C}^n\right).$$

The B_p is just the "leftover" and must be included so that the dimensions works out. The space T_pM is a real vector space; $\mathbb{C} \otimes T_pM, T_p^{(1,0)}M, T_p^{(0,1)}M$, and B_p are complex vector spaces. To see that these give vector bundles, we must first show that their dimensions do not vary from point to point. The easiest way to see this fact is to write down convenient local coordinates. First, let us examine what a biholomorphic map does to the holomorphic and antiholomorphic vectors. A biholomorphic map f is a diffeomorphism. And if a real hypersurface M is defined by a function r near p, then the image $f(M)$ is also a real hypersurface is given by the defining function $r \circ f^{-1}$ near $f(p)$.

Proposition 2.3.2. *Suppose $M \subset \mathbb{C}^n$ is a smooth real hypersurface, $p \in M$, and $U \subset \mathbb{C}^n$ is an open set such that $M \subset U$. Let $f : U \to \mathbb{C}^n$ be a holomorphic mapping such that $Df(p)$ is invertible (i.e. f is biholomorphic near p). Let $D_{\mathbb{C}}f(p)$ be the complexified real derivative. Then*

$$D_{\mathbb{C}}f(p)(T_p^{(1,0)}M) = T_{f(p)}^{(0,1)}f(M),\ D_{\mathbb{C}}f(p)(T_p^{(0,1)}M) = T_{f(p)}^{(0,1)}f(M).$$

That is, the spaces are isomorphic as complex vector spaces.

The proposition is local, if U is only a neighbourhood of p, replace M with $M \cap U$.

Proof. By the previous proposition we have

$$D_{\mathbb{C}}f(p)(T_p^{(1,0)}\mathbb{C}^n) = T_{f(p)}^{(1,0)}\mathbb{C}^n,\ D_{\mathbb{C}}f(p)(T_p^{(0,1)}\mathbb{C}^n) = T_{f(p)}^{(0,1)}\mathbb{C}^n,$$

$$D_{\mathbb{C}}f(p)(\mathbb{C} \otimes T_pM) = \mathbb{C} \otimes T_{f(p)}f(M).$$

Then we must have that $D_{\mathbb{C}}f(p)$ takes $T_p^{(1,0)}M$ to $T_{f(p)}^{(1,0)}f(M)$ and $T_p^{(0,1)}M$ to $T_{f(p)}^{(0,1)}f(M)$. $\square$

Proposition 2.3.3. *Let $M \subset \mathbb{C}^n$ be a smooth real hypersurface, $p \in M$. After a translation and rotation by unitary matrices, p is the origin, and near the origin, M is written in the variables $(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C}$ as*

$$\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w),$$

with $\varphi(0) = 0$ and $d\varphi(0) = 0$. Consequently

$$T_0^{(1,0)}M = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial z_1}\Big|_0, \dots, \frac{\partial}{\partial z_{n-1}}\Big|_0 \right\},\ T_0^{(0,1)}M = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial \bar{z}_1}\Big|_0, \dots, \frac{\partial}{\partial \bar{z}_{n-1}}\Big|_0 \right\},$$

$$B_0 = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial}{\partial(\operatorname{Re} w)}\Big|_0 \right\}.$$

In particular, $\dim_{\mathbb{C}} T_p^{(1,0)} M = \dim_{\mathbb{C}} T_p^{(0,1)} M = n-1$ *and* $\dim_{\mathbb{C}} B_p = 1$.

Also, if M is the boundary of a open set U with smooth boundary, the rotation can be chosen so that $\operatorname{Im} w > \varphi(z, \bar{z}, \operatorname{Re} w)$ *on* U.

Proof. We proceed in a similar fashion as lemma 2.2.1. First translate M such that $p = 0$ and rotate M so that ∇r is in the $-\frac{\partial}{\partial \operatorname{Im} w}|_0$. direction via a unitary matrix. $\varphi(0) = 0$ and $d\varphi(0) = 0$ follows by the lemma. Since translation and rotation via unitary matrix are holomorphic, we have by the previous proposition that the holomorphic and antiholomorphic vectors at p are transformed back to holomorphic and antiholomorphic vectors respectively at $f(p)$. For the also part, we note that the rotation is such that $\frac{\partial}{\partial(\operatorname{Im} w)}|_0$ is normal to M at the origin and that M has dimension $2n-1$ so that the remaining part must be a span of vectors of partial derivatives with respect to $\operatorname{Re} w$. $\square$

Definition 2.3.1. Suppose $U \subset \mathbb{C}^n$ is an open set with smooth boundary, and r is a defining function for ∂U at $p \in \partial U$ such that $r < 0$ on U. If

$$\sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_p \geq 0 \text{ for all } X_p = \sum_{j=1}^{n} a_j \frac{\partial}{\partial z_j}\Big|_p \in T_p^{(1,0)} \partial U,$$

then U is said to be pseudoconvex at p (or *Levi pseudoconvex*). If the inequality above is strict for all nonzero $X_p \in T_p^{(1,0)} \partial U$, then U is said to be *strongly pseudoconvex*. If U is pseudoconvex, but not strongly pseudoconvex at p, then we say that U is *weakly pseudoconvex*.

A domain U is *pseudoconvex* if it is pseudoconvex at all $p \in \partial U$. For a bounded U, we say that U is *strongly pseudoconvex* if it is strongly pseudoconvex at all $p \in \partial U$.

For $X_p \in T_p^{(1,0)} \partial U$, the sesquilinear form

$$\mathcal{L}(X_p, X_p) = \sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_p$$

is called the *Levi form* at p. In other words, U is pseudoconvex (resp. strongly pseudoconvex) at $p \in \partial U$ if the Levi form is positive semidefinite (resp. positive definite) at p. The Levi form can be defined for any real hypersurface M, although one has to decide which side of M is the "inside".

The matrix

$$\left[\frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_p \right]_{j\ell}$$

is called the complex Hessian of r at p. We note that this is not the full Hessian of r at p. It is in fact the lower left or the transpose of the upper right block of the full Hessian of r at p.

Note that a change of variable is a complex linear matrix A that acts on the Hessian as $A \oplus \bar{A}$, in other words in matrix form $\begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix}$. Writing the full Hessian of r as $\begin{bmatrix} X & L^t \\ L & \bar{X} \end{bmatrix}$, where L is the complex Hessian. Then the effect of a complex linear change of basis A is as follows

$$\begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix}^t \begin{bmatrix} X & L^t \\ L & \bar{X} \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix} = \begin{bmatrix} A^t X A & (A^* L A)^t \\ A^* L A & \overline{A^t X A} \end{bmatrix}.$$

We see that L is transformed under *-congruence and by Sylvester's Law of Inertia, the number of positive, negative and zero eigenvalues of L and $A^* L A$ are the same. Therefore we can conclude that a change of variables preserves pseudoconvexity.

Proposition 2.3.4. *If a function $r : \mathbb{C}^n \to \mathbb{R}$ is real-valued, then the complex Hessian of r is Hermitian.*

Proof. Given the complex Hessian of r:

$$\left[\frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell} \right]_{j\ell},$$

taking transpose we get:

$$\left[\frac{\partial^2 r}{\partial \bar{z}_\ell \partial z_j} \right]_{j\ell},$$

conjugating, we get:

$$\left[\frac{\partial^2 \bar{r}}{\partial z_\ell \partial \bar{z}_j} \right]_{j\ell}.$$

Since r is 2 times continuously differentiable, the partial derivatives $\frac{\partial}{\partial z}$'s and $\frac{\partial}{\partial \bar{z}}$'s commute and noting that since r is real valued $r = \bar{r}$, we get:

$$\left[\frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell} \right]_{j\ell}$$

as required. □

The above proposition tells us that given a defining function r, its complex Hessian is Hermitian, in other words, the eigenvalues of the complex Hessian is indeed real and therefore the notion of pseudoconvexity makes sense.

Proposition 2.3.5. *Let $U \subset \mathbb{C}^n$ be an open set with smooth boundary and $p \in \partial U$. The inertia of the Levi form at p does not depend on the defining function at p. In particular, the definition of pseudoconvexity and strong pseudoconvexity is independent of the defining function.*

Proof. Given another defining function s, we note that there exists a smooth function $g > 0$ such that $s = gr$. Partial differentiating with respect to x_j (or y_j) for any $1 \le j \le n$. We get

$$\frac{\partial s}{\partial x_j} = \frac{\partial g}{\partial x_j} r + g \frac{\partial r}{\partial x_j}.$$

However we note that since the partial derivatives are evaluated at the point p where $r = 0$, we have the equality

$$\frac{\partial s}{\partial x_j} = g \frac{\partial r}{\partial x_j}.$$

Since $g > 0$ the values on the LHS and RHS of the above have the same sign. We repeat the process by differentiating again and realising $r = 0$ at p,

$$\frac{\partial^2 s}{\partial x_k x_j} = g \frac{\partial^2 r}{\partial x_k x_j}.$$

Therefore the signs on LHS and RHS are the same and therefore the complex Hermitian of r and s differ by a multiple constant $g(p)$ so that the number of positive, negative and zero eigenvalues are preserved. □

Theorem 2.3.1. *Suppose $U, U' \subset C^n$ are open sets with smooth boundary, $p \in \partial U, V \subset \mathbb{C}^n$ a neighbourhood of $p, q \in \partial U', V' \subset \mathbb{C}^n$ a neighbourhood of q, and $f : V \to V'$ a biholomorphic map with $f(p) = q$, such that $f(U \cap V) = U' \cap V'$.*

Then the inertia of the Levi form of U at p is the same as the inertia of the Levi form of U' at q. Particularly, U is pseudoconvex at p if and only if U' is pseudoconvex q. Similarly, U is strongly pseudoconvex at p if and only if U' is strongly pseudoconvex at q.

Proof. If $f(U \cap V) = U' \cap V'$ then $f(\partial U \cap V) = \partial U' \cap V'$. Also observe that if r is a defining function for U' at q, then $r \circ f$ is a defining function for U at p. Now, we examine the mixed derivatives:

$$\begin{aligned}
\frac{\partial^2 (r \circ f)}{\partial \bar{z}_j \partial z_k}(z, \bar{z}) &= \frac{\partial}{\partial \bar{z}_j} \sum_{\ell=1}^{n} \left(\frac{\partial r}{\partial \zeta_\ell}(f(z), \bar{f}(\bar{z})) \frac{\partial f_\ell}{\partial z_k}(z) \right) \\
&= \sum_{\ell,m=1}^{n} \frac{\partial^2 r}{\partial \bar{\zeta}_m \partial \zeta_\ell}(f(z), \bar{f}(\bar{z})) \frac{\partial \bar{f}_m}{\partial \bar{z}_j}(\bar{z}) \frac{\partial f_\ell}{\partial z_k}(z) + \sum_{\ell=1}^{n} \frac{\partial r}{\partial \zeta_\ell}(f(z), \bar{f}(\bar{z})) \frac{\partial^2 f_\ell}{\partial \bar{z}_j \partial z_k}(z) \\
&= \sum_{\ell,m=1}^{n} \frac{\partial^2 r}{\partial \bar{\zeta}_m \partial \zeta_\ell} \frac{\partial \bar{f}_m}{\partial \bar{z}_j} \frac{\partial f_\ell}{\partial z_k}.
\end{aligned}$$

We get the last expression by noting that the right most expression in the second last line is zero as f is a holomorphic function and hence it's derivative is also holomorphic and hence the antiholmorphic derivative is zero. Let D be the holomorphic derivative matrix of f at z and D^* the conjugate transpose. Then the complex Hessian of $r \circ f$ is

D^*HD where H is the complex Hessian of r. By Sylvester's law of inertia, the number of positive, zero and negative eigenvalues of H and D^*HD is the same.

Now we're only looking at a subspace of H namely $T = T^{(1,0)}_{f(p)}M$ where M be the smooth hypersurface given by $r = 0$ and therefore $f^{-1}(M)$ is the smooth hypersurface given by $r \circ f = 0$. The same still applies as D here is invertible and the inertia of H restricted to DT is the same as the inertia of D^*HD restricted to T. We have shown in a previous proposition that $D = Df(p)$ maps $T^{(1,0)}_p f^-1(M)$ to $T^{(1,0)}_{f(p)}$ and this map is isomorphic. Therefore, H is positive (semi)definite on $T^{(1,0)}_{f(p)}M$ if and only if D^*HD is positive (semi)definite $T^{(1,0)}_p f^{-1}(M)$. $\square$

Lemma 2.3.1. *Let M be a smooth real hypersurface in $\mathbb{C}^n$ and $p \in M$. Then there exists a local biholomorphic change of coordinates taking p to the origin and M to the hypersurface given by*

$$\operatorname{Im} w = \sum_{j=1}^{\alpha} |z_j|^2 + \sum_{j=\alpha+1}^{\alpha+\beta} |z_j|^2 + E(z, \bar{z}, \operatorname{Re} w),$$

where E is $O(3)$ at the origin. Here α is the number of positive eigenvalues of the Levi form at p, β, is the number of negative eigenvalues and $\alpha + \beta \leq n - 1$.

Proof. We invoke proposition 2.3.3., that is we do a change of coordinates so that the hypersurface M is defined by $\operatorname{Im} w = \varphi(z, \bar{z}, \operatorname{Re} w)$, with φ being $O(2)$. First we apply Taylor's theorem to φ up to the second order and collate the terms in this way:

$$\varphi(z, \bar{z}, \operatorname{Re} w) = q(z, \bar{z}) + (\operatorname{Re} w)(Lz + \bar{L}z) + a(\operatorname{Re} w)^2 + O(3),$$

with q being quadratic in z and $\bar{z}$, $L : \mathbb{C}^{n-1} \to \mathbb{C}$ is linear and $a \in \mathbb{R}$ since $\varphi(z, \bar{z}, \operatorname{Re} w) = \operatorname{Im} w$ is a real-valued function. If it turns out that $L \neq 0$, then we do a linear change of coordinates in z so that $Lz = z_1$. Thus we can say that $Lz = \epsilon z_1$ with $\epsilon = 0$ or $\epsilon = 1$.

Second, we proceed with another change of coordinates that is $w = w' + bw'^2 + cw'z_1$, notice that this does not change z in anyway. We are going to ignore the $q(z, \bar{z})$ and $O(3)$ term for now and concentrate on the following terms and only considering them up to the second order:

$$
\begin{aligned}
-\operatorname{Im} w + \epsilon(\operatorname{Re} w)\,(z_1 + \bar{z}_1) + a(\operatorname{Re} w)^2 &= -\frac{w-\bar{w}}{2i} + \epsilon\frac{w+\bar{w}}{2}\,(z_1+\bar{z}_1) + a\left(\frac{w+\bar{w}}{2}\right)^2 \\
&= -\frac{w' + bw'^2 + cw'z_1 - \bar{w}' - \bar{b}\bar{w}'^2 - \bar{c}\bar{w}'\bar{z}_1}{2i} \\
&\quad + \epsilon\frac{w' + bw'^2 + cw'z_1 + \bar{w}' + \bar{b}\bar{w}'^2 + \bar{c}\bar{w}'\bar{z}_1}{2}\,(z_1+\bar{z}_1) \\
&\quad + a\frac{\left(w' + bw'^2 + cw'z_1 + \bar{w}' + \bar{b}\bar{w}'^2 + \bar{c}\bar{w}'\bar{z}_1\right)^2}{4} \\
&= -\frac{w'-\bar{w}'}{2i} \\
&\quad + \frac{\left((\epsilon i - c)w' + \epsilon i\bar{w}'\right)z_1 + \left((\epsilon i + \bar{c})\bar{w}' + \epsilon i w'\right)\bar{z}_1}{2i} \\
&\quad + \frac{(ia-2b)w'^2 + (ia+2\bar{b})\bar{w}'^2 + 2iaw'\bar{w}'}{4i} + O(3).
\end{aligned}
$$

At this point we are not able to eliminate all the quadratic terms in φ. However, we can remove the dependence on $\operatorname{Re} w'$ by picking $b = ia$ and $c = 2i\epsilon$. Now we substitute $q(z,\bar{z})$ and the other $O(3)$ term back to get the following:

$$
\begin{aligned}
-\operatorname{Im} w + \varphi(z,\bar{z},\operatorname{Re} w) &= -\operatorname{Im} w + q(z,\bar{z}) + \epsilon(\operatorname{Re} w)\,(z_1+\bar{z}_1) + a(\operatorname{Re} w)^2 + O(3) \\
&= -\frac{w'-\bar{w}'}{2i} + q(z,\bar{z}) - \epsilon i\frac{w'-\bar{w}'}{2i}\,(z_1-\bar{z}_1) + a\left(\frac{w'-\bar{w}'}{2i}\right)^2 + O(3) \\
&= -\operatorname{Im} w' + q(z,\bar{z}) - \epsilon i\,(\operatorname{Im} w')\,(z_1-\bar{z}_1) + a\,(\operatorname{Im} w')^2 + O(3).
\end{aligned}
$$

The expression that we end up with is the defining equation in terms of (z, w'). However we run into the problem that it is not longer written as a graph of $\operatorname{Im} w'$ over the rest of the terms. To resolve this, we use proposition 2.3.3. to see that $\operatorname{Im} w'$ is $O(2)$ and thus $-i\epsilon(\operatorname{Im} w')(z_1 - \bar{z}_1) + a(\operatorname{Im} w')^2$ is $O(3)$ and thus writing W as a graph, we obtain

$$\operatorname{Im} w' = q(z,\bar{z}) + E(z,\bar{z},\operatorname{Re} w'),$$

where E is $O(3)$.

Next we can proceed to write the quadratic polynomial q as

$$q(z,\bar{z}) = \sum_{j,k=1}^{n-1} a_{jk}z_jz_k + b_{jk}\bar{z}_j\bar{z}_k + c_{jk}\bar{z}_jz_k,$$

q is real-valued due to the fact $\operatorname{Im} w'$ and $\operatorname{Im} w$ are real valued. Thus we can derive $a_{jk} = \overline{b_{jk}}$ and $c_{jk} = \overline{c_{kj}}$. In other words, the matrix $[c_{jk}]$ is Hermitian.

Now for a final change of coordinates once again fixing the z's, and let

$$w' = w'' + i\sum_{j,k=1}^{n-1} a_{jk}z_jz_k.$$

Particularly,

$$\operatorname{Im} w' = \operatorname{Im} w'' + \operatorname{Im}(i \sum_{j,k=1}^{n-1} a_{jk} z_j z_k = \operatorname{Im} w'' + \sum_{j,k=1}^{n-1} (a_{jk} z_j z_k + b_{jk} \bar{z}_j \bar{z}_k),$$

due to the fact that $a_{jk} = \overline{b_{jk}}$.

Substituting this back into $\operatorname{Im} w' = q(z, \bar{z}) + E(z, \bar{z}, \operatorname{Re} w')$ and solving for $\operatorname{Im} w''$ cancels the holomorphic and antiholomorphic terms in q, and the remaining E is $O(3)$. With the final change of coordinates, we can assume that q takes the form

$$q(z, \bar{z}) = \sum_{j,k=1}^{n-1} c_{jk} z_j \bar{z}_k.$$

In other words, q is a sesquilinear form. Since q is real-valued, we must have that the matrix $C = [c_{jk}]$ is Hermitian. Using linear algebra notation we can write q as $q(z, \bar{z}) = z^* C z$, with z now being a column vector. If we let T be a linear transformation on the z variables, say $z' = Tz$, we get that $z'^* C' = (Tz)^* C T z = z * (T * CT) z$. In other words, we can normalize C up to $*-$congruence. We make use of the fact that a Hermitian matrix is $*-$congruent to a diagonal matrix with only ± 1's and 0's using Sylvester's law of inertia. Thus this leads us to the conclusion as stated in the lemma. □

Theorem 2.3.2 (Tomato can principle)**.** *Suppose $U \subset C^n$ is an open set with smooth boundary and at some point $p \in \partial U$ the Levi form has a negative eigenvalue. Then every holomorphic function on U extends to a neighbourhood of p. In particular, U is not a domain of holomorphy.*

Pseudoconvexity at the point p means that all eigenvalues of the Levi form are nonnegative due to Sylvester's law of inertia. Therefore, this theorem shows that if an open set with smooth boundary is not pseudoconvex at a point. Therefore the above theorem tells us that if a open set with smooth boundary is not pseudoconvex at some point, the it is not a domain of holomorphy. The contrapositive of this means that a domain of holomorphy implies that a open set with smooth boundary is pseudoconvex at every point on the boundary.

Proof. The way we show this to invoke theorem 2.1.1. but first, we have to fit a Hartogs figure inside U, then we find that there exist an extension past the boundary of U by theorem 2.1.1. and we will done. Therefore we first show how to fit the Hartogs figure.

Firstly, since there is only 1 negative eigenvalue and $n-1$ positive eigenvalues, by invoking lemma 2.3.1 and proposition 2.3.3, we are able to change variables so that $p = 0$, and near p, U is given by

$$\operatorname{Im} w > -|z_1|^2 + \sum_{j=2}^{n-1} \epsilon_j |z_j|^2 + E(z_1, z', \bar{z}_1, \bar{z}', \operatorname{Re} w)$$

where $z' = (z_2, \dots, z_{n-1})\,, \epsilon_j = -1, 0, 1,$ and E is $O(3)$. We are able to embed an analytic disc using the map $\xi \overset{\varphi}{\mapsto} (\lambda\xi, 0, 0, \dots, 0)$ for some small $\lambda > 0$. We can see that $\varphi(0) = 0 \in \partial U$. For $\xi \neq 0$ near the origin

$$-\lambda^2|\xi|^2 + \sum_{j=2}^{n-1} \epsilon_j |0|^2 + E(\lambda\xi, 0, \lambda\bar{\xi}, 0, 0) = -\lambda^2|\xi|^2 + E(\lambda\xi, 0, \lambda\bar{\xi}, 0, 0) < 0.$$

The last inequality is due to the second derivative in one variable, namely ξ. We differentiate the expression with respect to $|\xi|$ just before the inequality "$<$", since E is $O(3)$ E still vanishes when letting $\xi = 0$ after differentiating 2 times. We are left with $-\lambda^2$ which is less than 0. Therefore, the function has a strict maximum at $\xi = 0$. The local maximum for the function above has a value of 0. Thus for any $\xi \neq 0$, the function above must be less than 0. Thus, for $\xi \neq 0$ near the origin, $\varphi(\xi) \in U$. If we let λ be small enough, we have $\varphi(\overline{\mathbb{D}} \backslash \{0\}) \subset U$.

Since $\partial\mathbb{D}$ is compact and φ is continous, we know that $\varphi(\partial\mathbb{D})$ is also compact. Since $\varphi(\partial\mathbb{D})$ is compact, for small $s > 0$, the closed disc given by

$$\xi \mapsto \varphi(\lambda\xi, 0, 0, \dots, 0, is) \subset U.$$

In other words, the disc lies inside U in its entirety for small positive $\operatorname{Im} w$. Fix this small $s > 0$. Let $\epsilon > 0$ be small and $\epsilon < s$. We are now going to define the Hartogs figure as follows:

$$\begin{aligned} &H = \{(z, w) : \lambda - \epsilon < |z_1| < \lambda + \epsilon \text{ and } |z_j| < \epsilon \text{ for } j = 2, \dots, n-1, \text{ and } |w - is| < s + \epsilon\} \\ &\cup \{(z, w) : |z_1| < \lambda + \epsilon, \text{ and } |z_j| < \epsilon \text{ for } j = 2, \dots, n-1, \text{ and } |w - is| < \epsilon\}\,. \end{aligned}$$

For small enough $\epsilon > 0$, we have $H \subset U$. This is due to the fact that the set with $|z_1| = \lambda$, $z' = 0$ and $|w| \leq s$ is a subset of U for s small enough. Therefore, we can take an $\epsilon-$neighbourhood of that. Similarly, for $w = is$ the whole disc with $|z_1| \leq \lambda$ is in U. Thus, we can take an $\epsilon-$neighbourhood of that. Intuitively, we are taking a Hartogs figure in z_1 and w variables and then "expanding it" to the z' variables (z_j for $j = 2, \dots, n-1$). Figure 2.1 to help with the visualisation is given below, on the left, we only show the $|z_1|$ and $|w - is|$ axis.

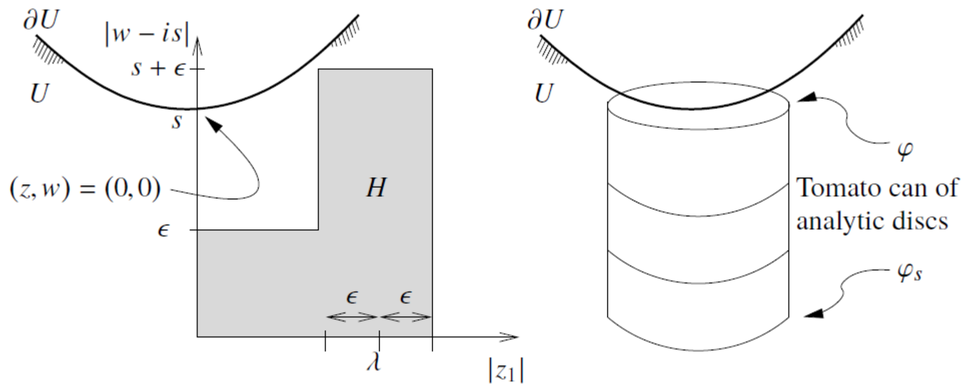

Figure 2.1: On the left we have a graph with $|z_1|$ and $|w - is|$, on the right we include the other complex variables.

The origin $(z, w) = (0, 0)$ lies in $\mathbb{D}^n$ and $H \subset \mathbb{D}^n$, therefore, every holomorphic function in U, and thus in H (since $\mathcal{O}(U) \subseteq \mathcal{O}(H)$), extends through the origin $(0, 0)$. Therefore, U is not a domain of holomorphy. $\square$

2.4 Harmonic, subharmonic, and plurisubharmonic functions

Definition 2.4.1. Let $U \subset \mathbb{R}^n$ be an open set. A C^2 -smooth function $f : U \to \mathbb{R}$ is harmonic if

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0 \quad \text{on } U.$$

A function $f : U \to \mathbb{R} \cup \{-\infty\}$ is subharmonic if it is upper-semicontinuous* and for every ball $B_r(a)$ with $\overline{B_r(a)} \subset U$, and every function g continuous on $B_r(a)$ and harmonic on $B_r(a)$, such that $f(x) \leq g(x)$ for $x \in \partial B_r(a)$, we have

$$f(x) \leq g(x), \qquad \text{for all } x \in B_r(a).$$

In other words, a subharmonic function is a function that is less than every harmonic function on every ball.

Proposition 2.4.1 (Mean-value property and sub-mean-value property)**.** *Let $U \subset \mathbb{C}$ be an open set.*

(i) A continuous function $f : U \to \mathbb{R}$ is harmonic if and only if

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta \quad \textit{whenever } \overline{\Delta_r(a)} \subset U.$$

(ii) An upper-semicontinuous function $f : U \to \mathbb{R} \cup \{-\infty\}$ is subharmonic if and only if

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta \quad \textit{whenever } \overline{\Delta_r(a)} \subset U.$$

This proposition provides a way for us to prove some theorems with relative since it is less general and removes the harmonic g function which is hard to quantify. Below is a general version in n complex dimensions.

Proposition 2.4.2 (Maximum Principle)**.** *Suppose $U \subset \mathbb{C}$ is a domain and $f : U \to \mathbb{R} \cup \{-\infty\}$ is subharmonic. If f attains a maximum in U, then f is constant.*

Proof. Suppose f attains a maximum at $a \in U$. If $\overline{\Delta_r(a)} \subset U$, then

$$f(a) \leq \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta \leq f(a).$$

Hence $f(z) = f(a)$ almost everywhere on $\partial\Delta_r(a)$. By upper-semicontinuity, $f = f(a)$ everywhere on $\partial\Delta_r(a)$. Therefore this also holds true for all r with $\overline{\Delta_r(a)} \subset U$, so $f = f(a)$ on $\Delta_r(a)$ and so the set of all points z that satisfies $f(z) = f(a)$ is open. The set where an upper-semicontinuous function attains a maximum is closed. Since U is connected, the only clopen sets are the empty set and U itself, since $f(a) = f(a)$ this set is not empty and thus it is the whole set U. ☐

Proposition 2.4.3. *Suppose $U \subset \mathbb{C}$ is an open set and $f : U \to \mathbb{R}$ is a C^2 function. The function f is subharmonic if and only if $\nabla^2 f \geq 0$.*

Proof. We only show the "if" part. Suppose that f is a C^2-smooth function on a subset $\mathbb{C} \equiv \mathbb{R}^2$ with $\nabla^2 f \geq 0$. We would like to show that f is subharmonic. Take any disc Δ such that $\overline{\Delta} \subset U$. Consider any function g continuous on $\overline{\Delta}$, harmonic on Δ, with $f \leq g$ on the boundary $\partial\Delta$. Due to the fact that $\nabla^2(f-g) = \nabla^2 f - \nabla^2 g = \nabla f \geq 0$ with last inequality given by assumption, we can assume $g = 0$ and $f \leq 0$ on the boundary $\partial\Delta$. We now consider 2 cases.

First let us suppose that $\nabla^2 f > 0$ at all points on Δ. Suppose f attains maximum in Δ at a point p. We know for a fact that $\nabla^2 f$ is the trace of the Hessian matrix, however, in order for f to have a maximum, the Hessian must have only nonpositive eigenvalues at the critical points, but this contradicts $\nabla^2 f > 0$ as the trace is precisely the sum of the eigenvalues. Therefore, there is no maximum in Δ, therefore $f \leq 0$ for all points in $\overline{\Delta}$.

For the second case, we consider $\nabla^2 f \geq 0$. Let M be the maximum of $x^2 + y^2$ on $\overline{\Delta}$. Take $f_n(x,y) = f(x,y) + \frac{1}{n}(x^2+y^2) - \frac{1}{n}M$. We can see that $\nabla^2 f_n > 0$ for all points in Δ and $f_n \leq 0$ on $\partial\Delta$, thus, $f_n \leq 0$ on the whole of $\overline{\Delta}$. Since $f_n \to f$, we get the desired result that $f \leq 0$ for all points in $\overline{\Delta}$. □

In analogy to convex functions, a C^2-smooth function f of one real variable is convex if and only if $f''(x) \geq 0$ for all x.

Proposition 2.4.4. *Suppose $U \subset \mathbb{C}$ is an open set and $f_\alpha : U \to \mathbb{R} \cup \{-\infty\}$ is a family of subharmonic functions. Let*

$$\varphi(z) = \sup_\alpha f_\alpha(z).$$

If the family is finite, then φ is subharmonic. If the family is infinite, $\varphi(z) \neq \infty$ for all z and φ is upper-semicontinuous, then φ is subharmonic.

Proof. Suppose $\overline{\Delta_r(a)} \subset U$. For any α,

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(a + re^{i\theta} d\theta \geq \frac{1}{2\pi}\int_0^{2\pi} f_\alpha(a + re^{i\theta})d\theta \geq f_\alpha(a).$$

By taking supremum on the right over all α, we get the desired result. □

Definition 2.4.2. Let $U \subset \mathbb{C}^n$ be open. A C^2 -smooth $f : U \to \mathbb{R}$ is *pluriharmonic* if for every $a, b \in \mathbb{C}^n$, the function of one variable

$$\xi \mapsto f(a + b\xi)$$

is harmonic (on the set of $\xi \in \mathbb{C}$ where $a + b\xi \in U$). That is, f is harmonic on every complex line.

A function $f : U \to \mathbb{R} \cup \{-\infty\}$ is *plurisubharmonic*, sometimes *plush* or *psh* for short, if it is upper-semicontinuous and for every $a, b \in \mathbb{C}^n$, the function of one variable

$$\xi \mapsto f(a + b\xi)$$

is subharmonic (whenever $a + b\xi \in U$).

A harmonic function of one complex variable is in some sense a generalization of an affine linear function of one real variable. In a similar spirit, as far as several complex variables are concerned, a plurisubharmonic function is a right generalization to $\mathbb{C}^n$ of an affine linear function on $\mathbb{R}^n$. In the same way plurisubharmonic functions are the correct complex variable generalizations of convex functions. A convex function of one real variable is like a subharmonic function, and a convex function of several real variables is a function that is convex when restricted to any real line.

Proposition 2.4.5. *Let $U \subset \mathbb{C}^n$ be open. A C^2-smooth $f : U \to \mathbb{R}$ is pluriharmonic if and only if*

$$\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k} = 0 \text{ on } U \text{ for all } j, k = 1, \ldots, n.$$

Proof. Write $b = (b_1, ..., b_n)$. We realise the above function in the definition (by introducing a function $g : \mathbb{C} \to \mathbb{C}^n$) as

$$\xi \overset{g}{\mapsto} a + b\xi \overset{f}{\mapsto} f(a + b\xi).$$

By chain rule applied to $f \circ g$ we have the expression

$$\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial \bar{z}_j \partial z_k} b_k.$$

We show the simpler if part first. Since $\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k} = 0$ on U for all $j, k = 1, \ldots, n$, this implies that $\sum_{j,k=1}^{n} \frac{\partial^2 f}{\partial \bar{z}_j \partial z_k} = 0$. This holds for any function g (that is any complex line in $\mathbb{C}^n$).

For the only if part, consider the function $z_k \mapsto f(0, ..., z_k, ..., 0)$ (by translation of the domain U to be centered near the origin). Then the only term remaining is the z_k term and now the sum simplifies, we have

$$\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k} \text{ on } U \text{ for all } j, k = 1, ..., n.$$

as desired as j and k is chosen arbitrarily.

□

Proposition 2.4.6. *Let $U \subset \mathbb{C}^n$ be open. A C^2- smooth $f : U \to \mathbb{R}$ is plurisubharmonic if and only if the complex Hessian matrix*

$$\left[\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k}\right]_{jk}$$

is positive semidefinite at every point.

Proof. We first show the "only if" part. We begin by supposing that the complex Hessian of f has a negative eigenvalue at a point $p \in U$, by translation, we can assume $p = 0$. Since f is real-valued, the complex Hessian $\left[\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k}\Big|_0\right]_{jk}$ is Hermitian. A complex linear change of coordinates act on the Hermitian by *-congruence and by Sylvester's law of inertia, we can diagonalize. Therefore we can safely assume that $\left[\frac{\partial^2 f}{\partial \bar{z}_j \partial z_k}\Big|_0\right]_{jk}$ is a diagonal matrix with ± 1's and 0's. If the complex Hessian has a negative eigenvalue, then one of the diagonal entries is negative. Without loss of generality, we can assume that it is the first complex coordinate, that is $\frac{\partial^2 f}{\partial \bar{z}_1 \partial z_1}\Big|_0 < 0$. The function $z_1 \mapsto f(z_1, 0, \ldots, 0)$ has a negative Laplacian and thus is not subharmonic, therefore f cannot be plurisubharmonic.

Next for the "if" part. Suppose that the complex Hessian is positive semidefinite at all points. Let $p \in U$. After an affine change of coordinates we can assume that the line $\xi \mapsto a + b\xi$ is simply setting all but the first variable to 0, in other words, $a = 0$ and $b = (1, 0, \ldots, 0)$. Since the complex Hessian is positive semidefinite, we have $\frac{\partial^2 f}{\partial \bar{z}_1 \partial z_1} \geq 0$ for all points $(z_1, 0, \ldots, 0)$. By the above proposition that g is subharmonic if and only if $\nabla^2 g \geq 0$, we have finished the proof of the "if" part. □

We would like to see if a holomorphic mapping preserves plurisubharmonicity, if that is indeed the case, it provides us a powerful way to transform our domain under a holomorphic map to something simpler and prove plurisubharmonicity in the simpler domain. Below is the proposition and proof.

Theorem 2.4.1. *Suppose $U \subset \mathbb{C}^n$ is an open set and $f : U \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic. For every $\epsilon > 0$, let $U_\epsilon \subset U$ be the set of points further than ϵ away from ∂U. Then there exists a smooth plurisubharmonic function $f_\epsilon : U_\epsilon \to \mathbb{R}$ such that $f_\epsilon(z) \geq f(z)$, and*

$$f(z) = \lim_{\epsilon \to 0} f_\epsilon(z) \quad \textit{for all } z \in U.$$

That is, f is a limit of smooth plurisubharmonic functions.

Proof. We smooth f out by convolving with mollifiers, or *approximate delta functions.* There are many different mollifers that will work, but we choose a specific one for concreteness. For $\epsilon > 0$, define

$$g(z) = \begin{cases} Ce^{-1/(1-||z||^2)} & \text{if } ||z|| < 1, \\ 0 & \text{if } ||z|| \geq 1, \end{cases} \quad \text{and} \quad g_\epsilon(z) = \frac{1}{\epsilon^{2n} g(z/\epsilon)}.$$

We are first going to show that g is smooth everywhere. We note the expression is differentiable/holomorphic in each variable z_j for $j = 1, \ldots, n$ when $||z|| < 1$, and of the form $P((1-z_j)^{-1}) * g(z)$ where P is a polynomial, by taking limits as $||z|| \to 1$, since $g(z)$ is an exponential function, it goes to 0 faster than the polynomial blows up.

Next we see that g has compact support as it is zero outside the unit ball and non-zero inside the unit ball. The support of g_ϵ is the $\epsilon-$ball. Both g and g_ϵ are non-negative. We choose C such that

$$\int_{\mathbb{C}^n} g\, dV = 1, \quad \text{and by change of variable} \int_{\mathbb{C}^n} g_\epsilon\, dV = 1.$$

Where dV is the volume measure. The function g only depends on $||z||$.

Since f is plurisubharmonic, it is upper semicontinuous by definition and thus is bounded above on compact sets. If it happens that f is not bounded below, we replace f by the function $\max\{f, \frac{-1}{\epsilon}\}$ which is also plurisubharmonic since the max function is integrable and using th. Thus, without loss of generality, we can assume that f is locally bounded.

For $z \in U_\epsilon$, we define f_ϵ as the convoltion with g_ϵ:

$$f_\epsilon(z) = (f * g_\epsilon)(z) = \int_{\mathbb{C}^n} f(w) g_\epsilon(z-w) dV(w) = \int_{\mathbb{C}^n} f(z-w) g_\epsilon(w) dV(w).$$

Where the two forms are related by a simple change of variables. On the surface, there seems to be an abuse of notation since f is only defined on U and not the whole of $\mathbb{C}^n$, however on closer inspection, we see that for $w \notin U$, $f(w)$ is not defined but $||z-w|| \geq 1$ and thus $g(z-w) = 0$ and the integral is also 0. Differentiating the first form under the integral, we see that f_ϵ is smooth.

Next we proceed to show that f_ϵ is plurisubharmonic. First we restrict to a complex line $\xi \mapsto a + b\xi$. We will show subharmonicity by using the sub-mean-value property on a circle of radius r around $\xi = 0$:

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} f_\epsilon\left(a + bre^{i\theta}\right) d\theta &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{C}^n} f\left(a + bre^{i\theta} - w\right) g_\epsilon(w) dV(w) d\theta \\
&= \int_{\mathbb{C}^n}\left(\frac{1}{2\pi}\int_0^{2\pi} f\left(a - w + bre^{i\theta}\right) d\theta\right) g_\epsilon(w) dV(w) \\
&\geq \int_{\mathbb{C}^n} f(a-w) g_\epsilon(w) dV(w) = f_\epsilon(a).
\end{aligned}$$

The inequality is due to the fact that $g_\epsilon \geq 0$ and the fact that f is plurisubharmonic on U. Therefore f is plurisubharmonic.

We move on to show that $f_\epsilon(z) \geq f(z)$ for all $z \in U_\epsilon$. Since $g_\epsilon(w)$ only depends on $|w_1|, \ldots, |w_n|$, we can write $g_\epsilon(w_1, \ldots, w_n) = g_\epsilon(|w_1|, \ldots, |w_n|)$. Without loss of generality, by translation, we can assume $z = 0$, using polar coordinates, we have the following as the integral:

$$
\begin{aligned}
f_\epsilon(0) &= \int_{\mathbb{C}^n} f(-w)g_\epsilon(|w_1|,\dots,|w_n|)dV(w) \\
&= \int_0^\epsilon \cdots \int_0^\epsilon (\int_0^{2\pi} \cdots \int_0^{2\pi} f(-r_1e^{i\theta_1},\dots,-r_ne^{i\theta_n})d\theta_1\dots d\theta_n) \\
&\qquad\qquad\qquad\qquad g_\epsilon(r_1,\dots,r_n)r_1\dots r_n dr_1\dots dr_n \\
&\geq \int_0^\epsilon \cdots \int_0^\epsilon (\int_0^{2\pi} \cdots \int_0^{2\pi} (2\pi) f(0,-r_2e^{i\theta_2}\dots,-r_ne^{i\theta_n})d\theta_1\dots d\theta_n) \\
&\qquad\qquad\qquad\qquad g_\epsilon(r_1,\dots,r_n)r_1\dots r_n dr_1\dots dr_n \\
&\geq f(0)\int_0^\epsilon \cdots \int_0^\epsilon (2\pi)^n g_\epsilon(r_1,\dots,r_n)r_1\dots r_n dr_1\dots dr_n \\
&= f(0)\int_{\mathbb{C}^n} g_\epsilon(w)dV(w) \\
&= f(0).
\end{aligned}
$$

The first equality is by definition of the convolution, the second equality is due to the fact by expanding the volume measure and noting that g_ϵ is zero outside the polydisc of radius ϵ. For the first inequality, we make use of the fact that f is plurisubharmonic and taking the complex line in the direction of $-r_1e^{i\theta_1}$, in other words, using the map $\xi \mapsto a + b\xi$ with a and b of the form $a = (a_1, 0, \dots, 0)$ and $b = (b_1, 0, \dots, 0)$ and invoking the sub-mean-value property of this map since this map must be subharmonic by definition. We have also used the fact that $g_\epsilon \geq 0$. The next inequality is a repeated application of this fact to each of the remaining $n-1$ complex. variables. The second last inequality is due to $\int_0^{2\pi} d\theta_i = 2\pi$, where $1 \leq i \leq n$ and also by collating the terms to form back the volume measure $dV(w)$.

The final step is to show $\lim_{\epsilon\to 0} f_\epsilon(z) = f(z)$. We know that for subharmonic functions, $\limsup_{\xi\to z} f(\xi) = f(z)$ and so by extension, plurisubharmonic functions satisfies this relation as well. Therefore, given $\delta > 0$, we can find an $\epsilon > 0$ such that $f(\xi) - f(z) \leq \delta$ for all $\xi \in B_\epsilon(z)$.

$$
\begin{aligned}
f_\epsilon(z) - f(z) &= \int_{B_\epsilon(0)} f(z-w)g_\epsilon(w)dV(w) - f(z)\int_{B_\epsilon(0)} g_\epsilon(w)dV(w) \\
&= \int_{B_\epsilon(0)} (f(z-w) - f(z))g_\epsilon(w)dV(w) \\
&\leq \delta \int_{B_\epsilon(0)} g_\epsilon(w)dV(w) = \delta.
\end{aligned}
$$

We used the fact the $g_\epsilon \geq 0$ and $\int_{\mathbb{C}^n} g_\epsilon(w)dV(w) = 1$. Thus, the requirement for δ is that $0 \leq f_\epsilon(z) - f(z) \leq \delta$, which satisfies the definition of a limit, therefore, $f_\epsilon(z) \to f(z)$. □

Proposition 2.4.7. *Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open. If $g : U \to V$ is holomorphic and $f : V \to \mathbb{R}$ is a plurisubharmonic function, then $f \circ g$ is plurisubharmonic.*

Proof. We first show that theorem for only when f is C^2. We would make use of proposition 2.4.6. to show plurisubharmonicity since f is C^2, and thus $f \circ g$ is also C^2, what is left to show is that the matrix

$$\left[\frac{\partial^2 (f \circ g)}{\partial \bar{z}_j \partial z_k}\right]_{jk}$$

is positive semidefinite at every point. First we write f in coordinates $(w_1, \dots, w_m)$ and g in coordinates $(z_1, \dots, z_n)$ in preparation of the application of the chain rule. We first evaluate the $\frac{\partial}{\partial z_k}$ to get

$$\frac{\partial}{\partial \bar{z}_j}\left(\sum_{\ell=1}^{m} \frac{\partial f}{\partial w_\ell}\frac{\partial g_\ell}{\partial z_k} + \frac{\partial g_\ell}{\partial \bar{w}_\ell}\frac{\partial \bar{g}_\ell}{\partial z_k}\right)$$

for each $1 \leq j, k \leq n$. We note that the term in the sum on the right is zero due to the fact that g is holomorphic so the holomorphic derivative of $\bar{g}$ is zero. Therefore we have,

$$\frac{\partial}{\partial \bar{z}_j}\left(\sum_{\ell=1}^{m} \frac{\partial f}{\partial w_\ell}\frac{\partial g_\ell}{\partial z_k}\right) = \sum_{\ell=1}^{m}\sum_{p=1}^{m}\left(\frac{\partial^2 f}{\partial \bar{w}_p \partial w_\ell}\frac{\partial g_p}{\partial \bar{z}_p}\frac{\partial g_\ell}{\partial z_k} + \frac{\partial f}{\partial w_\ell}\frac{\partial^2 g_\ell}{\partial \bar{z}_p \partial z_k}\right).$$

Similarly, since g is holomorphic, $\frac{\partial g_\ell}{\partial z_k}$ is also holomorphic and thus the term on the right of the sum is also equal zero. Thus we are left with the expression

$$\sum_{\ell=1}^{m}\sum_{p=1}^{m}\left(\frac{\partial^2 f}{\partial \bar{w}_p \partial w_\ell}\frac{\partial \bar{g}_p}{\partial \bar{z}_p}\frac{\partial g_\ell}{\partial z_k}\right)$$

for $1 \leq j, k \leq n$. We note that we can write this as a matrix

$$\bar{D}HD$$

where D is the derivative of g and H is the complex Hessian of f. Now take any vector $z \in \mathbb{C}^n$ and we want to show that $z^*\bar{D}HDz \geq 0$. We can write the following

$$z^*\bar{D}HDz = (Dz)^*H(Dz) = v^*Hv \geq 0$$

where $v \in \mathbb{C}^n$ and we get the final inequality by the fact that H itself is positive semidefinite by proposition 2.4.6 again. Therefore we have show that the matrix

$$\left[\frac{\partial^2 (f \circ g)}{\partial \bar{z}_j \partial z_k}\right]_{jk}$$

is indeed positive semidefinite and by proposition 2.4.6, it is also plurisubharmonic.

It remains to extend this to all plurisubharmonic function and not only C^2 ones. First consider any plurisubharmonic function f and $U_\epsilon \subset U$ be the set of points further than ϵ away from ∂U. By theorem 2.4.1, there exists a smooth plurisubharmonic function $f_\epsilon : U_\epsilon \to \mathbb{R}$ such that $f_\epsilon(z) \geq f(z)$, and

$$f(z) = \lim_{\epsilon \to 0} f_\epsilon(z) \quad \text{for all} \quad z \in U.$$

Since we have shown that plurisubharmonicity is preserved under holomorphic mappings for C^2 functions, it holds for (smooth) C^∞ functions as well. Now take any $a, b \in \mathbb{C}^n$ and any $B_r(a + b\xi) \subset U$ and we have

$$\begin{aligned}\xi \mapsto f(a + b\xi) &= \lim_{\epsilon \to 0} f_\epsilon(a + b\xi)\\ &\leq \lim_{\epsilon \to 0} \frac{1}{2\pi} \int_0^r f_\epsilon(a + b(\xi + re^{i\theta}))dr\\ &= \frac{1}{2\pi} \int_0^r \lim_{\epsilon \to 0} f_\epsilon(a + b(\xi + re^{i\theta}))dr\\ &= \frac{1}{2\pi} \int_0^r f(a + b(\xi + re^{i\theta}))dr.\end{aligned}$$

Where we have used the plurisubharmonicity of f_ϵ in the second inequality, for the third equality, we invoked the monotone convergence theorem to "bring" the limit under the integral sign. Thus we have shown that the map is above is indeed subharmonic. Since a, b were chosen arbitrarily, we have that f is indeed plurisubharmonic. $\square$

2.5 Hartog's Pseudoconvexity

By the above proposition, we have that particularly, if $\varphi : \mathbb{D} \to \mathbb{C}^n$ is an analytic disc and f is plurisubharmonic in a neighbourhood of $\varphi(\mathbb{D})$, then $f \circ \varphi$ is subharmonic.

Definition 2.5.1. Let $\mathcal{F}$ be a class of (extended)-real-valued functions defined on an open $U \subset \mathbb{R}^n$. If $K \subset U$, define $\widehat{K}$, the hull of K with respect to $\mathcal{F}$, as the set

$$\widehat{K} \stackrel{\text{def}}{=} \left\{ x \in U : f(x) \leq \sup_{y \in K} f(y) \text{ for all } f \in \mathcal{F} \right\}.$$

Definition 2.5.2. Let $U \subset C^n$ be open. An $f : U \to \mathbb{R}$ is an exhaustion function for U if

$$\{z \in U : f(z) < r\} \subset\subset U \text{ for every } r \in \mathbb{R}$$

where "$A \subset\subset B$" means that A is a relatively compact subset of B, that is, the closure of A or A itself is a compact subset of B A domain $U \subset \mathbb{C}^n$ is Hartogs pseudoconvex if there exists a continuous plurisubharmonic exhaustion function. The set $\{z \in U : f(z) < r\}$ is called the *sublevel set* of f, or the $r-$*sublevel set*.

Theorem 2.5.1 (Kotinuitätssatz—Continuity principle)**.** *Suppose an open set $U \subset \mathbb{C}^n$ is convex with respect to plurisubharmonic functions, then given any collection of closed analytic discs $\Delta_\alpha \subset U$ such that $\bigcup_\alpha \partial\Delta_\alpha \subset\subset U$, we have $\bigcup_\alpha \Delta_\alpha \subset\subset U$.*

There are many similar theorems called the conitinuity principle. The commonality between all these theorems is that there is family of analytic discs whose boundaries stay inside a domain, and whose conclusion has to do with extension of holomorphic functions, or with domains of holomorphy.

Proof. Let f be a plurisubharmonic function on U. If $\varphi_\alpha : \overline{\mathbb{D}} \to U$ is the holomorphic (in $\mathbb{D}$) mapping giving the closed analytic disc. By the maximum principle, f on Δ_α must be less than or equal to the supremum of f on $\partial\Delta_\alpha$, so $\overline{\Delta_\alpha}$ is the hull of $\partial\Delta_\alpha$. In other words $\bigcup_\alpha \Delta_\alpha$ is in the hull of $\bigcup_\alpha \partial\Delta_\alpha$ and therefore $\bigcup_\alpha \Delta_\alpha \subset\subset U$ by convexity. $\square$

We provide a visualisation of the failure of the continuity principle in Figure 2.2. If a domain is not convex with respect to plurisubharmonic functions, then it may be the case there are discs (denoted by straight line segments) that approach the boundary as shown in the following picture. In the diagram, the boundaries of the discs are denoted by the dark dots at the end of the segments.

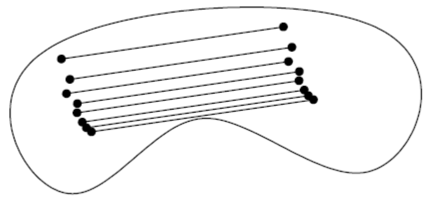

Figure 2.2: Visualisation of failure of continuity principle.

Example 2.5.1. The unit ball $\mathbb{B}_n$ is Hartogs pseduoconvex. The continuous function

$$z \mapsto -\log(1 - ||z||)$$

is an exhaustion function.

Example 2.5.2. The entire $\mathbb{C}^n$ is Hartogs pseudoconvex as $\|z\|^2$ is a continuous plurisubharmonic exhaustion function. Also, because $\|z\|^2$ is plurisubharmonic, then given any $K \subset\subset \mathbb{C}^n$, the hull $\widehat{K}$ with respect to plurisubharmonic functions must be bounded. In other words, $\mathbb{C}^n$ is convex with respect to plurisubharmonic functions.

Theorem 2.5.2 (Lebl, pg 75-76)**.** *Suppose $U \subsetneq \mathbb{C}^n$ is a domain. The following are equivalent:*

(i) $-\log \rho(z)$ is plurisubharmonic, where $\rho(z)$ is the distance from z to ∂U

(ii) U has a continuous plurisubharmonic exhaustion function, that is, U is Hartogs pseudoconvex.

(iii) U is convex with respect to plurisubharmonic functions defined on U.

Lemma 2.5.1 (Lebl, pg 77-78)**.** *A domain $U \subset \mathbb{C}^n$ is Hartogs pseudoconvex if and only if for every point $p \in \partial U$ there exists a neighbourhood W of p such that $W \cap U$ is Hartogs pseudoconvex.*

We proceed to show that Levi and Hartogs pseudoconvexity are the same thing on domains where both concepts are defined. Due to the result of the theorem, we can call a domain pseudoconvex without any ambiguity as to what concept we are referring to.

Theorem 2.5.3. *Let $U \subset \mathbb{C}^n$ be a domain with smooth boundary. Then U is a Hartogs pseudoconvex if and only if U is Levi pseudoconvex.*

Proof. We first show the "only if" part first. Suppose $U \subset \mathbb{C}^n$ is a domain with a smooth boundary that is not Levi pseudoconvex at $p \in \partial U$. Similar to the proof of the tomato can principle, we change coordinates so that $p = 0$ and U is defined by

$$\operatorname{Im} z_n > -|z_1|^2 + \sum_{j=2}^{n-1} \epsilon_j |z_j|^2 + O(3).$$

With small fixed $\lambda > 0$, the closed analytic discs defined by $\xi \in \overline{\mathbb{D}} \mapsto (\lambda\xi, 0, \ldots, 0, is)$ are in U for all small enough $s > 0$. The origin is a limit point of the insides, however, it is not a limit point of their boundaries. The contrapositive of the continuity principle then tells us that U is not convex with respect to the plurisubharmonic functions. Therefore, U is not Hartogs pseudoconvex.

Next we show the "if" part. Suppose that U is Levi pseudoconvex. Take any $p \in \partial U$. After translation and rotation by a unitary matrix, assume $p = 0$ and write the defining function r as

$$r(z, \bar{z}) = \varphi(z', \bar{z}', \operatorname{Re} z_n) - \operatorname{Im} z_n,$$

with $z' = (z_1, \ldots, z_{n-1})$. We recall what Levi pseudoconvexity means

$$\sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\bigg|_q \geq 0 \quad \text{whenever} \quad \sum_{j=1}^{n} a_j \frac{\partial r}{\partial z_j}\bigg|_q = 0,$$

for all $q \in \partial U$ near 0. Let s be a small real constant, and let $\tilde{q} = (q_1, \ldots, q_{n-1}, q_n + is)$. None of the derivatives of r depends on $\operatorname{Im} z_n$, and therefore $\frac{\partial r}{\partial z_\ell}\Big|_{\tilde{q}} = \frac{\partial r}{\partial z_\ell}\Big|_q$ and $\frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_{\tilde{q}} = \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\Big|_q$ for all j and ℓ. Therefore, the Levi pseudoconvexity condition holds for all $q \in U$ near 0. We will use r to produce a plurisubharmonic exhaustion function with a semidefinite Hessian.

Let $\nabla_z r|_q = (\frac{\partial r}{\partial z_1}|_q, \ldots, \frac{\partial r}{\partial z_n}|_q)$ denote the gradient of r in the holomorphic directions only. Given $q \in U$ near 0, decompose an arbitrary $c \in \mathbb{C}^n$ as $c = a + b$ where $a = (a_1, \ldots, a_n)$ satisfies

$$\sum_{j=1}^{n} a_j \frac{\partial r}{\partial z_j}\bigg|_q = \langle a, \overline{\nabla_z r|_q} \rangle = 0.$$

Taking the orthogonal decomposition, b is a scalar multiple of $\overline{\nabla_z r|_q}$. By Cauchy-Schwarz,

$$\left|\sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\bigg|_q\right| = \left|\sum_{j=1}^{n} b_j \frac{\partial r}{\partial z_j}\bigg|_q\right| = |\langle b, \overline{\nabla_z r|_q} \rangle| = \|b\| \|\nabla_z r|_q\|.$$

As $\nabla_z r|_0 = (0, \ldots, 0, -1/2i)$, then for q sufficiently near 0 we have that $\left\|\nabla_z r|_q\right\| \geq 1/3$, and

$$\|b\| = \frac{1}{\left\|\nabla_z r|_q\right\|} \left|\sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\bigg|_q\right| \leq 3 \left|\sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\bigg|_q\right|.$$

As $c = a + b$ is the orthogonal decomposition $\|c\| \geq \|b\|$.

The complex Hessian matrix of r is continuous, and so let $M \geq 0$ be an upper bound on its operator norm for q near the origin. Again using Cauchy-Schwarz

$$\begin{aligned}
\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\bigg|_q &= \sum_{j=1,\ell=1}^{n} \left(\bar{a}_j + \bar{b}_j\right)\left(a_\ell + b_\ell\right) \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\bigg|_q \\
&= \sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\bigg|_q \\
&\quad + \sum_{j=1,\ell=1}^{n} \bar{b}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\bigg|_q + \sum_{j=1,\ell=1}^{n} \bar{c}_j b_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\bigg|_q - \sum_{j=1,\ell=1}^{n} \bar{b}_j b_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\bigg|_q \\
&\geq \sum_{j=1,\ell=1}^{n} \bar{a}_j a_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\bigg|_q - M\|b\|\|c\| - M\|c\|\|b\| - M\|b\|^2 \\
&\geq -3M\|c\|\|b\|.
\end{aligned}$$

Together with what we know about $\|b\|$, for $q \in U$ near the origin

$$\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\bigg|_q \geq -3M\|c\|\|b\| \geq -\,3^2 M\|c\| \left|\sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\bigg|_q\right|.$$

For $z \in U$ sufficiently close to 0 define

$$f(z) = -\log(-r(z)) + A\|z\|^2,$$

where $A > 0$ is some constant we will choose later. The log is there to make f blow up as we approach the boundary. The $A\|z\|^2$ is there to add a constant diagonal matrix to the complex Hessian of f, which we hope is enough to make it positive semidefinite at all z near 0. Compute:

$$\frac{\partial^2 f}{\partial \bar{z}_j \partial z_\ell} = \frac{1}{r^2}\frac{\partial r}{\partial \bar{z}_j}\frac{\partial r}{\partial z_\ell} - \frac{1}{r}\frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell} + A\delta_j^\ell$$

where δ_j^ℓ is the Kronecker delta. Apply the complex Hessian of f to c at $q \in U$ near the origin (recall that r is negative on U and so for $q \in U, -r = |r|$) :

$$\begin{aligned}
\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 f}{\partial \bar{z}_j \partial z_\ell}\bigg|_q &= \frac{1}{r^2}\left|\sum_{\ell=1}^{n} c_\ell \frac{\partial r}{\partial z_\ell}\bigg|_q\right|^2 + \frac{1}{|r|}\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 r}{\partial \bar{z}_j \partial z_\ell}\bigg|_q + A\|c\|^2 \\
&\geq \frac{1}{r^2}\left|\sum_{\ell=1}^{n} c_\ell \frac{\partial r}{\partial z_\ell}\bigg|_q\right|^2 - \frac{3^2 M}{|r|}\|c\| \sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j}\bigg|_q \;| + A\|c\|^2.
\end{aligned}$$

As a quadratic polynomial in $\|c\|$, the right-hand side of the inequality is always non-negative if $A > 0$ and if the discriminant is negative or zero. Let us set the discriminant to zero:

$$0 = \left(\frac{3^2 M}{|r|} \left| \sum_{j=1}^{n} c_j \frac{\partial r}{\partial z_j} \Big|_q \right| \right)^2 - 4A \frac{1}{r^2} \left| \sum_{\ell=1}^{n} c_\ell \frac{\partial r}{\partial z_\ell} \Big|_q \right|^2 .$$

All the nonconstant terms go away and $A = \frac{3^4 M^2}{4}$ makes the discriminant zero. Thus for that A,

$$\sum_{j=1,\ell=1}^{n} \bar{c}_j c_\ell \frac{\partial^2 f}{\partial \bar{z}_j \partial z_\ell} \Bigg|_q \geq 0.$$

In other words, the complex Hessian of f is positive semidefinite at all points $q \in U$ near 0. The function $f(z)$ goes to infinity as z approaches ∂U. So for every $t \in \mathbb{R}$, the t -sublevel set (the set where $f(z) < t$) must be a positive distance away from ∂U near 0.

We have constructed a local continuous plurisubharmonic exhaustion function for U near p. If we intersect with a small ball B centered at p, then we get that $U \cap B$ is Hartogs pseudoconvex. This is true at all $p \in \partial U$, so U is Hartogs pseudoconvex. $\square$

2.6 Holomorphic Convexity

Definition 2.6.1. Let $U \subset \mathbb{C}^n$ be a domain. For a set $K \subset U$, define the holomorphic hull

$$\widehat{K}_U \stackrel{\text{def}}{=} \left\{ z \in U : |f(z)| \leq \sup_{w \in K} |f(w)| \text{ for all } f \in \mathcal{O}(U) \right\}.$$

A domain U is holomorphically convex if whenever $K \subset\subset U$, then $\widehat{K}_U \subset\subset U$. In other words, U is holomorphically convex if it is convex with respect to moduli of holomorphic functions on U.

Theorem 2.6.1 (Solution of the Levi problem)**.** *A domain $U \subset \mathbb{C}^n$ is holomorphically convex if and only if it is Hartogs pseudoconvex.*

We refer to [Hormander] for the proof of this theorem.

Theorem 2.6.2 (Cartan-Thullen)**.** *Let $U \subsetneq \mathbb{C}^n$ be a domain. The following are equivalent:*

(i) U is a domain of holomorphy.

(ii) For all $K \subset\subset U$, $\operatorname{dist}(K, \partial U) = \operatorname{dist}(\hat{K}_U, \partial U)$.

(iii) U is holomorphically convex.

Proof. Let us start with (i) $\Rightarrow$ (ii) .
Suppose there is a $K \subset\subset U$ with $\operatorname{dist}(K, \partial U) > \operatorname{dist}(\widehat{K}_U, \partial U)$. After possibly a rotation by a unitary, there exists a point $p \in \widehat{K}_U$ and a polydisc $\Delta = \Delta_r(0)$ with polyradius $r = (r_1, \ldots, r_n)$ such that $p + \Delta = \Delta_r(p)$ contains a point of ∂U, but

$$K + \Delta = \bigcup_{q \in K} \Delta_r(q) \subset\subset U.$$

We give a visualisation below in Figure 2.3:

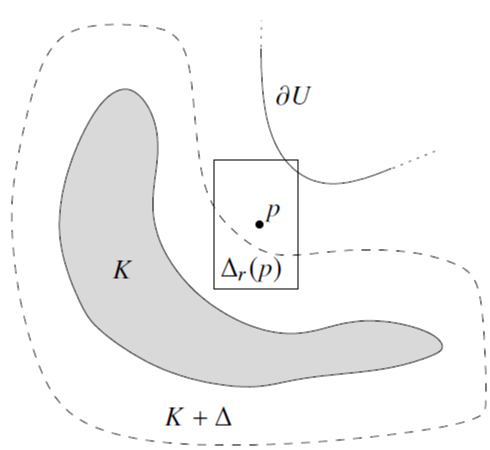

Figure 2.3: Visualization of the proof of Cartan-Thullen theorem for (i) $\to$ (ii).

If $f \in \mathcal{O}(U)$, then there is an $M > 0$ such that $|f| \leq M$ on $K + \Delta$ as that is a relatively compact set. By the Cauchy estimates for each $q \in K$ we get

$$\left| \frac{\partial^\alpha f}{\partial z^\alpha}(q) \right| \leq \frac{M \alpha!}{r^\alpha}.$$

This inequality therefore holds on $\widehat{K}_U$ and hence at p. The series

$$\sum_\alpha \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial z^\alpha}(p)(z-p)^\alpha$$

converges in $\Delta_r(p)$. Hence f extends to all of $\Delta_r(p)$ and $\Delta_r(p)$ contains points outside of U, in other words, U is not a domain of holomorphy.

The implication (ii) $\Rightarrow$ (iii) is immediate.

Finally, we prove (iii) $\Rightarrow$ (i). Suppose U is holomorphically convex. Let $p \in \partial U$. By convexity choose nested compact sets $K_{j-1} \subsetneq K_j \subset\subset U$ such that $\bigcup_j K_j = U$, and $(\hat{K_j})_U = K_j$. As the sets exhaust U, we can perhaps pass to a subsequence to ensure that there exists a sequence of points $p_j \in K_j \backslash K_{j-1}$ such that $\lim_{j\to\infty} p_j = p$ As p_j is not in the hull of K_{j-1}, there is a function $f_j \in \mathcal{O}(U)$ such that $|f_j| < 2^{-j}$ on K_{j-1}, but

$$|f_j(p_j)| > j + \left| \sum_{k=1}^{j-1} f_k(p_j) \right|.$$

For any j, the series $\sum_{k=1}^\infty f_k(z)$ converges uniformly on K_j as for all $k > j, |f_k| < 2^{-k}$ on K_j. As the K_j exhaust U, the series converges uniformly on compact subsets of U. Consequently,

$$f(z) = \sum_{k=1}^\infty f_k(z)$$

is a holomorphic function on U. We bound

$$|f(p_j)| \geq |f_j(p_j)| - \left| \sum_{k=1}^{j-1} f_k(p_j) \right| - \left| \sum_{k=j+1}^\infty f_k(p_j) \right| \geq j - \sum_{k=j+1}^\infty 2^{-k} \geq j - 1.$$

So $\lim_{j\to\infty} f(p_j) = \infty$. Clearly there cannot be any open $W \subset \mathbb{C}^n$ containing p to which f extends (see definition of domain of holomorphy). As any connected open W such that $W \backslash U \neq \emptyset$ must contain a point of ∂U, we are done. $\square$

Appendix A

Differentiable and Complex Manifolds

Definition A.0.1. A topological space X is said to be *separated* or *Hausdorff* if for all distinct points a and b in X there exists disjoint open sets U, V such that $a \in U$ and $b \in V$. We say that X satisfies the *second axiom of countability* if X admits a countable basis of open sets.

Definition A.0.2. Let X be a separated topological space satisfying the second axiom of countability. Let n be a positive integer. A holomorphic *atlas* (of dimension n) on X is defined as a family of open sets $(U_j)_{j \in J}$ which covers X, together with homeomorphisms $\phi_j : U_j \to \Omega_j$ from U_j to an open set Ω_j of $\mathbb{C}^n$, such that for all $j, k \in J$ the transition map:

$$\phi_j \circ \phi_k^- 1 : \phi_k(U_j \cap U_k) \to \phi_j(U_j \cap U_k)$$

is holomorphic.

Definition A.0.3. Two atlas on X are equivalent if their union is also at atlas. The space X equipped with an equivalence class of (holomorphic) atlas of dimension n is called a *complex manifold of dimension* n. The pair (U_j, ϕ_j) is called a (holomorphic) chart of X. The components of the map ϕ_j are called *local coordinates* of X. A complex manifold of dimension 1 is also called a *Riemann surface* or *complex curve* C.

In order to simplify the notation, we often forget the maps ϕ_j , and identify U_j with open subsets of $\mathbb{C}^n$ and we consider the standard coordinates $z = (z_1, ..., z_n)$ of $\mathbb{C}^n$ as local coordinates of X.

Appendix B

Sylvester's Law of Inertia

Definition B.0.1 (Inertia of a Matrix)**.**

Definition B.0.2 (*-congruence/star-congruence)**.**

Theorem B.0.1 (Sylvester's Law of Inertia)**.** *Let $A, B \in M_n$ be Hermitian. Then A and B are *-congruent if and only if they have the same inertia.*

Proposition B.0.1. *Given a complex matrix L that is positive (semi)definite, and $A^*LA = M$ for some invertible matrix A, then M is also positive (semi)definite*

Proof. We'll show the proof for semidefiniteness, the proof for positive definiteness is the same. Since $w^*Lw \geq 0$ for any vector w and writing $w = Av$ for any vector v, we have

$$0 \leq w^*Lw = (Av)^*L(Av) = v^*(A^*LA)v = v^*Mv,$$

thereby showing that M is indeed positive semidefinite. $\square$

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