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INVOLUTIVE DISTRIBUTIONS, INVARIANT MANIFOLDS, AND DEFINING EQUATIONS

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Abstract: We introduce the notion of an invariant solution relative to an involutive distribution. We give sufficient conditions for existence of such a solution to a system of differential equations. In the case of an evolution system of partial differential equations we describe how to construct auxiliary equations for functions determining differential constraints compatible with the original system. Using this theorem, we introduce linear and quasilinear defining equations which enable us to find some classes of involutive distributions, nonclassical symmetries, and differential constraints. We present examples of reductions and exact solutions to some partial differential equations stemming from applications.

Keywords: group analysis, involutive distribution, defining equation

1. Introduction

Modern research into a mathematical model basing on partial differential equations includes studying the group-theoretic properties of this model. At present, there are many articles devoted to finding admissible groups of transformations and to group classification for differential equations arising in continuum mechanics, electrodynamics, and field theory [1–4]. The SUBMODELS program developed by L. V. Ovsyannikov [5] is a continuation of this research. One of the main parts of this program is to describe systems of equations (submodels) obtained from the original model by listing all nonsimilar subgroups of the admissible group. Passage from a basic model to equations in fewer independent variables is conventionally called reduction. As observed by some authors [6, 7], reduction can be carried out by using nonclassical symmetries, i.e., transformation groups inadmissible in the original model. However, the problem of finding nonclassical symmetries turns out often over complicated in regard to solution of nonlinear overdetermined systems of partial differential equations.

Simpler methods for obtaining nonclassical symmetries happen to be the method of B-defining equations [4] and the approach utilizing linear equations for finding differential constraints [8]. In both cases it is necessary to solve some auxiliary equations that generalize the classical defining equations. In this article we propose a synthesis of these methods, basing not only on the familiar notions but also on some new structures. We introduce the notion of an invariant solution relative to an involutive distribution. It turns out that a solution to differential equations can be sought by using involutive distributions rather than subalgebras of the Lie algebra of admissible operators. We give a sufficient condition for existence of such a solution to a system of differential equations. We consider the problem of finding involutive distributions that enable us to obtain invariant solutions to evolution equations. We introduce linear and quasilinear defining equations which allow us to find some classes of involutive distributions, nonclassical symmetries, and differential constraints. We give examples of the construction of reductions and exact solutions to some partial differential equations arising in applications.

2. On Invariant Solutions Relative to an Involutive Distribution

In group analysis of differential equations, construction of invariant and partially invariant solutions

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bases on use of Lie algebras of admissible operators [1]. In this section we introduce solutions to systems of partial differential equations which are invariant relative to involutive distributions and also give sufficient conditions for existence of such solutions.

Henceforth all mappings under consideration are assumed smooth and all considerations are local as adopted in group analysis of differential equations. The local one-parameter group of transformations generated by a vector field X is denoted by G^1_X.

Recall that a solution $u = \varphi(x)$ to a system E of partial differential equations is said to be *invariant under* G^1_X if the set $S = \{(x, u) : u = \varphi(x)\} \subset \mathbb{R}^n \times \mathbb{R}^m$ is an invariant manifold for G^1_X.

Proposition. *Suppose that* $U_1 \subset \mathbb{R}^n$ *and* $U_2 \subset \mathbb{R}^m$ *are open sets,* $u = (\varphi_1(x), \dots, \varphi_m(x))$ *is a solution to some system* E *on* U_1*, and* X *is a vector field on* $U_1 \times U_2$ *of the shape*

$$X = \sum_{i=1}^{n} \xi^i(x, u)\partial_{x_i} + \sum_{j=1}^{m} \eta^j(x, u)\partial_{u^j}. \tag{2.1}$$

The following are equivalent:

(1) $u = \varphi$ *is invariant under* G^1_X*;*

(2) X *is tangent to* S*;*

(3) φ *satisfies the system*

$$\sum_{i=1}^{n} \xi^i(x, \varphi)\varphi^j_{x_i} = \eta^j(x, \varphi), \quad j = 1, \dots, m. \tag{2.2}$$

Proof. The equivalence of (1) and (2) is well known [1]. The equivalence of (2) and (3) follows from the definition of tangent vector fields.

Now, suppose that a collection of p vector fields

$$X_s = \sum_{i=1}^{n} \xi^i_s(x)\partial_{x_i}$$

is given on an open set $U \subset \mathbb{R}^n$. If this collection is linearly disconnected, i.e., the rank of the matrix $|\xi^i_s(x)|$ equals p for all $x \in U$ and satisfies the involution condition

$$[X_i, X_j] = \sum_{k=1}^{p} c^k_{ij}(x)X_k, \quad 1 \le i, j \le p, \tag{2.3}$$

where c^k_{ij} are smooth functions, then this collection generates an involutive p-dimensional distribution D_p [9]. A collection of vector fields with these properties is called an *involutive basis* or just a *basis*. It is well known [9] that a distribution D_p is involutive if and only if it possesses at least one involutive basis.

Definition. A solution $u = \varphi$ to a system E is *invariant* relative to an involutive distribution D_p if D_p is tangent to the manifold $S = \{(x, u) : u = \varphi(x)\}$.

Obviously, the invariance of a solution relative to D_p amounts to its invariance under the operators of an arbitrary involutive basis for D_p.

Now, consider the system of evolution equations

$$u^i_t = F^i(t, x, u, u_\alpha), \quad i = 1, \dots, m, \tag{2.4}$$

where t and $x = (x_1, \dots, x_n)$ are independent variables, $u^1, \dots, u^m$ are sought functions, $u = (u^1, \dots, u^m)$, and u_α stands for various partial derivatives with respect to $x_1, \dots, x_n$. Denote the total derivatives with respect to t and x_i by the symbols D_t and D_{x_i}.

Let $J^k(U, \mathbb{R}^m)$ be the space of k-jets on $U \subset \mathbb{R}^n$. Recall [4] that a manifold $H \subset J^k(\mathbb{R}^{n+1}, \mathbb{R}^m)$ defined by the equations

$$h^j(t, x, u, u_\beta) = 0, \quad j = 1, \dots, s, \tag{2.5}$$

is an *invariant manifold* for (2.4) if the following identity holds on the set $[E] \cap [H]$:

$$D_t h^j = 0.$$

Here $[E]$ and $[H]$ stand for the differential consequences of (2.4) and (2.5) with respect to $x_1, \dots, x_n$. Denote the involutive distribution generated by vector fields $X_1, \dots, X_r$ by $\langle X_1, \dots, X_r \rangle$.

Lemma 1. *Suppose that vector fields*

$$X_k = \sum_{i=1}^{n} \xi_k^i(t, x, u)\partial_{x_i} + \sum_{j=1}^{m} \eta_k^j(t, x, u)\partial_{u^j}, \quad k = 1, \dots, n, \tag{2.6}$$

generate an involutive distribution and that $\det(\xi_k^i) \neq 0$. *If the manifold defined by the equations*

$$h_k^j = \sum_{i=1}^{n} \xi_k^i u_{x_i}^j - \eta_k^j = 0, \quad 1 \le j \le m, \ 1 \le k \le n, \tag{2.7}$$

is invariant relative to (2.4) *then system* (2.4) *has invariant solutions relative to this involutive distribution.*

Proof. Write down the collection of the fields $X_1, \dots, X_n$ in vector form as follows:

$$X = \xi\partial_x + \eta\partial_u.$$

Acting by the matrix ξ^{-1} on X, we obtain the involutive collection

$$Z = \partial_x + \tilde{\eta}\partial_u,$$

where $\tilde{\eta} = \xi^{-1}\eta$. The distribution $\langle Z_1, \dots, Z_n \rangle$ is involutive as expressible in terms of an involutive distribution (see a proof, for example, in [10], wherein involutive distributions are referred to as "complete systems").

The invariant solutions relative to $\langle X_1, \dots, X_n \rangle$ must satisfy (2.7) and the invariant solution relative to $\langle Z_1, \dots, Z_n \rangle$, the equations

$$u_{x_k}^j = \tilde{\eta}_k^j(t, x, u). \tag{2.8}$$

Obviously, (2.7) and (2.8) have the same solution set. Since Z is an involution distribution, the Poisson bracket $[Z_i, Z_k]$ vanishes [10]. Consequently, we have

$$Z_i(\tilde{\eta}_k^j) = Z_k(\tilde{\eta}_i^j),$$

which means that the consistency conditions for (2.8) are satisfied.

Using (2.8) and inserting the derivatives of the functions u^j with respect to x_k in the right-hand side of (2.4), we come to the system

$$u_t^j = G^j(t, x, u), \quad j = 1, \dots, m. \tag{2.9}$$

By the Frobenius theorem, the system of (2.8) and (2.9) is consistent if the relations

$$D_{x_k} G^j = D_t \tilde{\eta}_k^j, \quad j = 1, \dots, m, \ k = 1, \dots, n, \tag{2.10}$$

are valid by virtue of (2.8) and (2.9). Validity of these conditions follows from the invariance of (2.7) relative to (2.4). Indeed, this invariance means that

$$D_t(u_{x_k}^j - \tilde{\eta}_k^j) = D_{x_k} F^j - D_t \tilde{\eta}_k^j = 0. \tag{2.11}$$

Inserting the derivatives with respect to x_k in (2.11), we see that (2.11) coincides with (2.10).

REMARK. If an involutive distribution is generated by analytic vector fields $X_1, \dots, X_p$, $p < n$, (2.4) is a system of first-order equations with analytic right-hand sides, and the rank of the matrix $\left(\xi_k^i\right)$ equals p, then (2.4) has an invariant solution relative to $X_1, \dots, X_p$. The proof is carried out by the above scheme, but instead of the Frobenius theorem we should use the Riquier theorem on the existence of analytic solutions to an autonomous system with analytic right-hand sides [11].

To exemplify the application of a distribution to constructing solutions, consider the equation

$$u_t = \Delta \log u, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \tag{2.12}$$

which arises in various application [12, 13] and possesses an infinite-dimensional algebra of point symmetries [14]. Some exact solutions to this equation can be found in [12, 13, 15, 16].

We give a solution to this equation which is invariant relative to the pair of commuting operators

$$X_1 = \partial_x - (u^2 + (tu^2 + u - xu^2)\tan t)\partial_u, \quad X_2 = \partial_y - (u + (t - x)u^2)\partial_u.$$

The corresponding manifold for these vector fields is

$$u_x + (tu^2 + u - xu^2)\tan t + u^2 = 0, \tag{2.13}$$

$$u_y + u + (t - x)u^2 = 0. \tag{2.14}$$

It is easy to verify that this is an invariant manifold for (2.12). Note that the vector fields X_1 and X_2 do not belong to the algebra of symmetries of (2.12).

The general solution to (2.13) and (2.14) has the form

$$u = \frac{1}{s(t)(\exp(x \tan t + y) + x - t)}. \tag{2.15}$$

Inserting this expression in (2.12), we find that

$$s = c \cos t \exp(-t \tan t), \quad c \in \mathbb{R}.$$

Solution (2.15) seems to be not known before. Other solutions like (2.15) can also be constructed but this will be done in another article.

3. Defining Equations

To use vector fields and distributions, we need a method for finding them. The classical approach to constructing vector fields relative to which the given differential equations are invariant was proposed by S. Lie. A modern exposition with many examples and new results was given by L. V. Ovsyannikov [1].

The goal of this section is to describe a new method for constructing distributions which could enable us to perform reduction of differential equations and to find solutions.

There were many attempts at extending the group analysis of differential equations. The most essential contribution of the last thirty years seems to be the results on highest symmetries [2, 17]. Here we briefly recall the "nonclassical method" of [6, 18]. The essence of this approach is as follows:

Assume given a differential equation E:

$$F(x_1, x_2, u, u_{x_1}, u_{x_2}, u_{x_1x_1}, \dots) = 0. \tag{3.1}$$

This equation is supplemented with some differential constraint H:

$$\Phi = \sum_{i=1}^{2} \xi^i u_{x_i} - \eta = 0, \tag{3.2}$$

where ξ^i and η are still unknown functions which may depend on x_i, x_2, and u. Then the vector field

$$X = \sum_{i=1}^{2} \xi^i \partial_{x_i} + \eta \partial_u$$

must be tangent to the manifold defined by (3.1) and (3.2); i.e., it is necessary that

$$XF|_{E\cap H} = 0. \tag{3.3}$$

By [19], (3.3) is nothing but the compatibility condition for (3.1) and (3.2). To find the coefficients ξ^i and η of X from (3.3) is very difficult. Particular examples involve coefficients of special kind.

A new method for constructing differential constraints compatible with evolution-type equations was proposed in the recent article [8]. The method was applied to partial differential equations in two independent variables. It bases on solution of auxiliary linear equations generalizing the classical defining equations [1] for admissible infinitesimal operators.

To illustrate the method, consider the Gibbons–Tsarev equation [20]

$$u_{tt} = u_x u_{tx} - u_t u_{xx} + 1. \tag{3.4}$$

By [8], the corresponding defining equation has the form

$$D_t^2 h = u_x D_t D_x h - u_t D_x^2 h + b_1 u_{tx} D_x h + b_2 u_{xx} D_t h. \tag{3.5}$$

This equation must hold by (3.4). The constants b_1 and b_2 are to be determined together with the function h.

If $b_1 = -1$ and $b_2 = 1$ then (3.5) possesses the following solution which depends on the second derivatives:

$$h_1 = u_{xx} + k_1 u_x + k_2, \quad k_1, k_2 \in \mathbb{R}.$$

A solution to (3.5) with $b_1 = -2$ and $b_1 = 2$, depending on the third derivatives, takes the form

$$h_2 = u_{xxx} + k_1 u_x + k_2, \quad k_1, k_2 \in \mathbb{R}.$$

Equating h_2 to zero, we obtain the differential constraint

$$u_{xxx} + k_1 u_x + k_2 = 0. \tag{3.6}$$

Find some solutions to (3.4) and (3.6).

Suppose that $k_1 = k_2 = 0$. Then the equation has the solution

$$u = s_2 x^2 + s_1 x + s_0,$$

where s_i is a function in t. Inserting this expression in (3.4), we arrive at the system of ordinary differential equations

$$s_2'' - 2s_2 s_2' = 0, \tag{3.7}$$

$$s_1'' - 2s_1 s_2' = 0, \tag{3.8}$$

$$s_0'' + 2s_2 s_0' - 1 - s_1 s_1' = 0. \tag{3.9}$$

Integrating (3.7) once, we find that

$$s_2 = s_2^2 + m. \tag{3.10}$$

If $m = -1$ then one of the solutions to (3.10) is the function $s_2 = -\tanh(t)$. Inserting this function in (3.8), we come to the equation

$$s_1'' + \frac{2}{\cosh^2 t} s_1 = 0;$$

whose general solution has the form

$$s_1 = c_1 \tanh(t) + c_2(t \tanh(t) - 1).$$

A solution to the linear equation (3.9) can be found by quadratures. It is cumbersome and we do not present it here.

Now, suppose that $k_1 = -1$ and $k_2 = 0$ in (3.6). Then from (3.6) we derive the following representation for u:

$$u = s_1(t) + s_2(t)e^x + s_3(t)e^{-x}.$$

Inserting this expression in (3.4), we arrive at the system of ordinary differential equations

$$s_2'' + s_1' s_2 = 0, \quad s_1'' + 2s_3 s_2' + 2s_2 s_3' - 1 = 0, \quad s_3'' + s_1' s_3 = 0.$$

If $s_3 = as_2$ then the last system reduces to the two equations

$$s_2'' + s_1' s_2 = 0, \quad s_1'' + 4as_2 s_2' - 1 = 0. \tag{3.11}$$

Integrating the second equation, we find that

$$s_1' = t + b - 2as_2^2, \quad b \in \mathbb{R}.$$

We can insert this expression in (3.11) and obtain the second-order equation

$$s_2'' + \bigl(t + b - 2as_2^2\bigr)s_2 = 0.$$

Using the transformations $t_1 = t + b$ and $w = \sqrt{a}s_2$, we take the equation in s_2 to the second Painleve equation [21]

$$w'' = 2w^3 + t_1 w.$$

A defining equation like (3.5) enables us to find differential constraints compatible with the original equation. In the case of differential equations in more than two independent variables, we can propose systems of defining equations which would enable us to find involutive distributions.

Consider the system of involution equations (2.4) and the manifold in $J^1(U, \mathbb{R}^m)$ defined by

$$h_j^i = u_{x_j}^i + g_j^i(t, x, u) = 0, \tag{3.12}$$

where $i = 1, \dots, m$ and $j = 1, \dots, n$.

Denote (2.4) with its differential consequences with respect to $x_1, \dots, x_n$ by $[E]$. Denote (3.12) with the corresponding differential consequences by $[H]$.

Theorem. *Suppose that the manifold* (3.12) *is invariant relative to system* (2.4) *whose right-hand sides are polynomials in derivatives whose coefficients depend on* t, $x_1, \dots, x_n$ *and* $u^1, \dots, u^m$. *Then the functions* h_j^i *satisfy the following system on* $[E]$*:*

$$D_t h_j^i + m_{ij}(h) = 0, \quad 1 \le i \le m,\ 1 \le j \le n. \tag{3.13}$$

Here $m_{ij}(h)$ *is some operator representing a polynomial in* h_l^k, $D_{x_1} h_l^k, \dots, D_{x_n} h_l^k, \dots, D^\alpha h_l^k$ $(k = 1, \dots, m$, $l = 1, \dots, n)$. *The operators* m_{ij} *vanish whenever all* h_l^k *are zero.*

Proof. We first show that the total derivative of h_j^i with respect to t is representable as

$$D_t h_j^i = m_{ij}(h) + \gamma_{ij}, \tag{3.14}$$

where m_{ij} are operators whose shape is described in the theorem and γ_{ij} are functions which may depend only on t, x, and u.

The following identities are valid on $[E]$:

$$D_t h_j^i = D_{x_j} F^i + \frac{\partial g_j^i}{\partial t} + \sum_{k=1}^{m} F^k \frac{\partial g_j^i}{\partial u^k}. \tag{3.15}$$

Let $\frac{\partial^{|s|} u^k}{\partial x_1^{s_1} \dots \partial x_n^{s_n}}$ be a derivative of maximal order on the right-hand side of (3.15) and $s_p \neq 0$ for some p. By (3.15) and the assumptions of the theorem, this derivative enters (3.15) polynomially. Using (3.12), we can write down this derivative as follows:

$$D_{x_1}^{s_1} \dots D_{x_p}^{s_p - 1} \dots D_{x_n}^{s_n} (h_p^k) - D_{x_1}^{s_1} \dots D_{x_p}^{s_p - 1} \dots D_{x_n}^{s_n} (g_p^k).$$

Note that the second summand involves no derivatives of order $|s|$ and is a polynomial in derivatives. Thus, all derivatives of maximal order on the right-hand side of (3.15) can be expressed in terms of the total derivatives of the functions h_q^r ($r = 1, \dots, m$ and $q = 1, \dots, n$). Afterwards, it is possible to express the derivatives of order $|s| - 1$, etc. down to the first-order derivatives.

We are left with demonstrating that the functions γ_{ij} in (3.14) are all zero. By the conditions of the theorem, the manifold (3.12) is an invariant manifold for (2.4). Consequently, the following identity holds on $[E] \cap [H]$:

$$m_{ij}(h) + \gamma_{ij} = D_t h_j^i = 0.$$

Since the m_{ij}'s vanish on $[H]$, the functions γ_{ij} are zero on $[E] \cap [H]$. Once the γ_{ij}'s are independent of the derivatives of the functions u^k, all γ_{ij} are identically zero.

Remark. As we see from the proof of the theorem, the choice of the operators m_{ij} could be nonunique.

For example, consider the second-order equation in three independent variables:

$$u_t = G \equiv F^1 u_{xx} + F^2 u_{yy} + F^3 u_x^2 + F^4 u_y^2 + F^5, \tag{3.16}$$

where F^i are some functions depending on u. Suppose that

$$h_1 \equiv u_x + g_1(t, x, y, u) = 0, \quad h_2 \equiv u_y + g_2(t, x, y, u) = 0 \tag{3.17}$$

define an invariant manifold for (3.16). To derive a system of defining equations like (3.13), we express the derivatives $D_t h_1$ and $D_t h_2$ in terms of h_i, $D_x h_i$, $D_y h_i$, $D_x^2 h_i$, $D_x D_y h_i$, and $D_y^2 h_i$ ($i = 1, 2$). By (3.16), the following holds:

$$D_t h_1 = D_x G + \frac{\partial g_1}{\partial t} + \frac{\partial g_1}{\partial u} G.$$

It is easy to verify that the right-hand side of the last equality is representable as

$$\begin{aligned} m_{11}(h_1, h_2) = {} & G_{u_{xx}} D_x^2 h_1 + G_{u_{yy}} D_y^2 h_1 + [G_{u_x} + D_x(G_{u_{xx}})] D_x h_1 + G_{u_y} D_y h_1 \\ & + D_x(G_{u_{yy}}) D_y h_2 + \big[G_u - D_x^2(G_{u_{xx}}) - D_y^2(G_{u_{yy}}) + r_1\big] h_1 + s_1 h_2 + \gamma_1, \end{aligned} \tag{3.18}$$

where r_1, s_1, and γ_1 are functions depending on h_1, h_2, and G. Since (3.17) is an invariant manifold, the function γ_1 equals 0. Consequently, the first defining equation has the form

$$D_t h_1 = m_{11}(h_1, h_2).$$

To obtain the second defining equation

$$D_t h_2 = m_{12}(h_1, h_2),$$

we should replace h_1 in (3.11) with h_2, x with y, r_1 with r_2, and s_1 with s_2.

The following lemma asserts that, under some conditions, solutions to equations like (3.13) enable us to construct differential constraints compatible with the system of evolution equations (2.4). It is worth to note that the shape of the operators m_{ij} is inessential, provided that only $m_{ij}(0) = 0$.

Lemma 2. *Suppose that the functions*

$$h_j^i = \sum_{s=1}^{n} \xi_j^s(t,x,u) u_{x_s}^i - g_j^i(t,x,u)$$

satisfy a system like (3.13) *on* $[E]$ *with* $m_{ij}(0) = 0$*. If the vector fields*

$$X_j = \sum_{s=1}^{n} \xi_j^s \partial_{x_s} + \sum_{i=1}^{m} g_j^i \partial_{u_i}, \quad j = 1, \dots, n,$$

generate an involutive distribution and $\det(\xi_j^s) \neq 0$ *then there is a solution to the system of* (2.4) *and the equations*

$$h_j^i = 0, \quad i = 1, \dots, m, \ j = 1, \dots, n. \tag{3.19}$$

Proof. Since the functions h_j^i satisfy (3.13), in view of $m_{ij}(0) = 0$ (3.19) defines an invariant manifold for (2.4). To complete the proof, it suffices to refer to Lemma 1.

Finding solutions to general nonlinear equations (3.13) might represent a very complicated problem. To simplify the problem, we remove all summands nonlinear in h_l^k from the operators m_{ij} as it was done in [8] in the case of an evolution equation with one space variable. In result, we obtain some linear equation

$$D_t h_j^i + l_{ij}(h) = 0.$$

Following [8], multiply the coefficients of the operators l_{ij} by undetermined constants and write down the resultant equations as

$$D_t h_j^i + L_{ij}(h) = 0, \tag{3.20}$$

calling them linear defining equations (LDEs).

For example, the LDEs for (3.16) have the form

$$\begin{gathered} D_t h_1 = L_{11}(h_1, h_2) \equiv a_1 G_{u_{xx}} D_x^2 h_1 + a_2 G_{u_{yy}} D_y^2 h_1 \\ + [a_3 G_{u_x} + a_4 D_x(G_{u_{xx}})] D_x h_1 + a_5 G_{u_y} D_y h_1 + a_6 D_x(G_{u_{yy}}) D_y h_2 \\ + \big[a_7 G_u + a_8 D_x^2(G_{u_{xx}}) + a_9 D_y^2(G_{u_{yy}})\big] h_1, \\ D_t h_2 = L_{12}(h_1, h_2), \end{gathered} \tag{3.21}$$

where $L_{12}(h_1, h_2)$ is obtained from $L_{11}(h_1, h_2)$ by replacing h_1 with h_2, x with y, and a_i with b_i.

Although the above arguments were for systems of evolution equations, we can try to extend them to a more general situation.

Assume given a system

$$n_i(u) = F^i(t, x, u, u_\alpha), \quad i = 1, \dots, m,$$

where n_i are linear differential operators with constant coefficients and the right-hand sides are similar to those in the case of evolution systems (2.4). To find the functions h_j^i, we suggest using the following equation in place of (3.20):

$$N_i\big(h_j^i\big) + L_{ij}(h) = 0, \tag{3.22}$$

where the operators N_i are obtained from n_i by replacing partial derivatives with total derivatives. Alongside (3.22), it is useful to introduce the following analog of B-defining equations [4]:

$$N_i\big(h_j^i\big) + L_{ij}(h) + \sum_{\substack{1 \le l \le m \\ 1 \le k \le n}} b_{lj}^{ki} h_k^l = 0, \tag{3.23}$$

where $1 \le i \le m$, $1 \le j \le n$, and b_{lj}^{ki} are functions that may depend on t, x, and u.

We call equations of the form (3.23) quasilinear defining equations (QDEs).

We exhibit an example of QDEs in finding involutive distributions. Consider one of the nonlinear dispersion models describing the propagation of long two-dimensional waves [22]:

$$\eta_{tt} = gd\Delta\eta + \frac{d^2}{3}\Delta\eta_{tt} + \frac{3}{2}g\Delta\eta^2,$$

where $\eta(t,x,y)$ is the deviation of a fluid from an equilibrium state, d is the depth of an unperturbed fluid, and g is the free fall acceleration. By translations and dilations, we can reduce this equation to the form

$$u_{tt} - \Delta(u_{tt}) - u\Delta u - (\nabla u)^2 = 0. \tag{3.24}$$

It can be shown that (3.24) admits the five-parameter transformation group generated by the rotation, translation, and dilation operators:

$$y\partial_x - x\partial_y,\ \partial_t,\ \partial_x,\ \partial_y,\ t\partial_t - 2u\partial_u. \tag{3.25}$$

In accordance with the above method, the QLEs for (3.24) have the shape

$$\begin{aligned} D_t^2h_1 - D_t^2D_x^2h_1 - D_t^2D_y^2h_1 + a_1u\big(D_x^2h_1 + D_y^2h_1\big) + a_2u_xD_xh_1 + a_3u_yD_yh_1 \\ +a_4u_xD_yh_2 + (a_5\Delta u + a_6u_{xx} + a_7u_{yy} + r_1)h_1 + q_1h_2 = 0, \end{aligned} \tag{3.26}$$

$$\begin{aligned} D_t^2h_2 - D_t^2D_x^2h_2 - D_t^2D_y^2h_2 + b_1u\big(D_x^2h_2 + D_y^2h_2\big) + b_2u_yD_yh_2 + b_3u_xD_xh_2 \\ +b_4u_yD_xh_1 + (b_5\Delta u + b_6u_{xx} + b_7u_{yy} + r_2)h_2 + q_2h_1 = 0, \end{aligned} \tag{3.27}$$

where a_i and b_i are constants, and r_j and q_j are functions which may depend on t, x, y, and u and which should be found together with h_1 and h_2. The scheme for solving (3.26) and (3.27) is completely analogous to the standard scheme of the group analysis of differential equations [1–4]. For this reason, we omit all intermediate computations and set forth only the final results. Note that there are computer programs available for solving the classical defining equations. Solving (3.26) and (3.27), we used computer calculations, too.

If h_1 and h_2 are sought in the form corresponding to the point symmetries

$$h_1 = \xi_1^1u_t + \xi_2^1u_x + \xi_3^1u_y + \eta^1, \quad h_2 = \xi_1^2u_t + \xi_2^2u_x + \xi_3^2u_y + \eta^2,$$

where ξ^i and η^j are functions of t, x, y, and u, then under the condition $(\xi_1^1)^2 + (\xi_3^1)^2 + (\xi_1^2)^2 + (\xi_2^2)^2 \neq 0$ equations (3.26) and (3.27) can be shown to have solutions leading only to admissible operators for (3.25). There appear new solutions only when

$$h_1 = u_x + g_1(t,x,y,u), \quad h_2 = u_y + g_2(t,x,y,u).$$

The final form of g_1 and g_2 is as follows:

$$g_1 = s_1x + s_2y + s_3, \quad g_2 = s_2x + s_4y + s_5.$$

Moreover, the functions s_i $(i = 1,\dots,5)$ depend only on t and satisfy the following system of five second-order differential equations:

$$s_1'' + 3s_1^2 + s_1s_4 + 2s_2^2 = 0, \quad s_2'' + 3s_1s_2 + 3s_2s_4 = 0,$$

$$s_3'' + 3s_1s_3 + 2s_2s_5 + s_3s_4 = 0, \quad s_4'' + s_1s_4 + 2s_2^2 + 3s_4^2 = 0, \quad s_5'' + s_1s_5 + 2s_2s_3 + 3s_4s_5 = 0.$$

For completeness of exposition, we write down the constants a_i and b_i $(i = 1,\dots,7)$ and the functions r_j and q_j $(j = 1,2)$ in (3.26) and (3.27) corresponding to g_1 and g_2:

$$a_1 = b_1 = a_4 = b_4 = -1, \quad a_2 = b_2 = a_3 = b_3 = -3,$$

$$a_5 = a_6 = a_7 = b_5 = b_6 = b_7 = 0,$$

$$r_1 = 3s_1 + s_4, \quad r_2 = s_1 + 3s_4, \quad q_1 = 2s_1, q_2 = 2s_2.$$

The functions h_1 and h_2 generate the differential constraints

$$u_x + s_1 x + s_2 y + s_3 = 0, \quad u_y + s_2 x + s_4 y + s_5 = 0.$$

These constraints enable us to find the following representation for a solution to (3.24):

$$u = \frac{-s_1 x^2}{2} - s_2 xy - \frac{s_4 y^2}{2} - s_3 x - s_5 y + s_6.$$

Inserting this in (3.24), we obtain the following equation for s_6:

$$s_6'' = 3s_1^2 + 2s_1 s_4 - s_1 s_6 + 4s_2^2 + s_3^2 + 3s_4^2 - s_4 s_6 + s_5^2.$$

The system of the six differential equations in the six functions s_i deserves further study. For example, it would be interesting to find a solution expressible via elementary functions.

The theorem we prove here admits a generalization to the case of a manifold defined by equations that are solved with respect to all derivatives of order $|w|$:

$$\frac{\partial^{|w|} u^i}{\partial x_1^{w_1} \dots \partial x_n^{w_n}} = g_w^i(t, x, u, u_q), \quad 1 \le i \le m,$$

provided that the functions g_w^i may only contain derivatives of less orders $|w|$.

Consider the second-order evolution equation

$$u_t = F(t, x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}). \tag{3.28}$$

Suppose that the invariant manifold is defined by the equations

$$\begin{aligned} h_1 &= u_{xxx} + g_1 = 0, \quad h_2 = u_{xxy} + g_2 = 0, \\ h_3 &= u_{xyy} + g_3 = 0, \quad h_4 = u_{yyy} + g_4 = 0, \end{aligned} \tag{3.29}$$

where the functions g_i $(i = 1, \dots, 4)$ depend only on t, x, y, u, u_x, u_y, u_{xx}, u_{xy}, and u_{yy}. Then we can take the following system of linear defining equations:

$$\begin{aligned} D_t h_1 = {} & F_{u_{xx}} D_x^2 h_1 + F_{u_{xy}} D_x D_y h_1 + F_{u_{yy}} D_y^2 h_1 + [N D_x(F_{u_{xx}}) + F_{u_x}] D_x h_1 \\ & + [F_{u_y} + N D_x(F_{u_{xy}})] D_y h_1 + N D_x(F_{u_{yy}}) D_y h_2 \\ & + \left[F_u + N D_x(F_{u_x}) + \frac{N(N-1)}{2} D_x^2(F_{u_{xx}})\right] h_1 \\ & + \left[\frac{N(N-1)}{2} D_x^2 F_{u_{xy}} + N D_x(F_{u_y})\right] h_2 + \frac{N(N-1)}{2} D_x^2(F_{u_{yy}}) h_3, \end{aligned}$$

$$\begin{aligned} D_t h_2 = {} & F_{u_{xx}} D_x^2 h_2 + F_{u_{xy}} D_x D_y h_2 + F_{u_{yy}} D_y^2 h_2 + D_y(F_{u_{xx}}) D_x h_1 \\ & + [F_{u_x} + (N-1) D_x(F_{u_{xx}}) + D_y(F_{u_{xy}})] D_x h_2 \\ & + [F_{u_y} + (N-1) D_x(F_{u_{xy}}) + D_y(F_{u_{yy}})] D_y h_2 + (N-1) D_x(F_{u_{yy}}) D_y h_3 \\ & + [(N-1) D_x D_y(F_{u_{xx}}) + D_y(F_{u_x})] h_1 + \big[D_x^2(F_{u_{xx}}) + (N-1) D_x D_y(F_{u_{xy}}) + F_u \\ & + D_y(F_{u_y}) + (N-1) D_x F_{u_x}\big] h_2 + \left[\frac{(N-1)(N-2)}{2} D_x^2 F_{u_{xy}} + (N-1) D_x D_y F_{u_{xy}} \right. \\ & \left. + (N-1) D_x F_{u_y}\right] h_3 + \frac{(N-1)(N-2)}{2} D_x^2(F_{u_{yy}}) h_4, \end{aligned}$$

$$D_t h_3 = m_{13}, \quad D_t h_4 = m_{14},$$

where $N = 3$. The right-hand sides of the third and fourth equations result from replacing h_2 with h_3, h_1 with h_4, and x with y in the right-hand sides of the first and second equations.

If the right-hand side of (3.28) has the form

$$F = u\Delta u + \delta(\nabla u)^2, \quad \delta = \pm 1, \tag{3.30}$$

then the functions $h_1 = u_{xxx}$, $h_2 = u_{xxy}$, $h_3 = u_{xyy}$, and $h_4 = u_{yyy}$ are solutions to the system of defining equations. In this case the general solution to (3.29) is a polynomial of second degree in x and y with coefficients depending on t:

$$u = s_1 x^2 + s_2 xy + s_3 y^2 + s_4 x + s_5 y + s_6. \tag{3.31}$$

Inserting this representation in (3.28) given the right-hand side (3.30), we obtain a system of differential equations in s_i ($i = 1, \dots, 6$). Earlier, a representation like (3.31) was derived in [23] using other arguments.

The author hopes that new applications of the above approach will appear in near future.

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