Mechanics, Control, and Variational Principles.

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Preface

Our goal in this book is to explore some of the connections between control theory and geometric mechanics, in particular classical mechanics in its Lagrangian and Hamiltonian forms as well as the theory of mechanical systems subject to motion constraints. The latter turns out to have a particularly rich connection with nonlinear control theory. While an introduction to aspects of the mechanics of nonholonomically constrained systems may be found in such sources as the monograph of Naiimark and Fu'faev [1972], the control theory of such systems remains largely scattered through various research journals. Our aim is to provide a unified treatment of the control theory of mechanical systems which will incorporate material that has not yet made its way into texts and monographs.

Mechanics has traditionally described the behavior of free and interacting particle and bodies, the interaction being decribed by potential forces. It encompasses the Hamiltonian and Lagrangian and Hamiltonian pictures and in its modern form relies heavily on the tools of differential geometry (see, for example, Abraham and Marsden [1978] and Arnold [1978]).

Control theory is the theory of prescribing motion for dynamical systems rather than describing their observed behavior. These systems may or may not be mechanical in nature, and in fact traditionally, the underlying system is not assumed to be mechanical. Modern control theory began largely as linear theory, having its roots in electrical engineering and using linear algebra, complex variable theory, and functional analysis as its princpal tools. The nonlinear theory of control on the other hand relies to a large extent again on differential geometry.

Nonholonomic mechanics describes the motion of systems constrained by nonintegrable constraints, i.e. constraints on the system velocities but not its configuration. Classic examples are rolling and skating motion. Nonholonomic mechanics fits uneasily into the classical mechanics as it is not variational in nature, i.e. it is neither Lagrangian or Hamiltonian in the strict sense of the word. It has a close cousin (variational axiomatic mechanics – a term coined by Arnold [1988]) which is variational for systems subject to nonintegrable constraints, but which does not describe the motion of mechanical systems. It is important however for the theory of optimal control, as we shall describe.

There is a very close link between nonholonomic constraints and controllability of nonlinear systems. Nonholonomic constraints are given by nonintegrable distributions – i.e. taking the bracket of two vector fields in such distribution gives rise in general to a vector field not contained in this distribution. It is precisely this property that one wants in a nonlinear control systems, in order that we can drive the sytem to as large a part of the state space as possible.

It turns out that a key concept for studying the control and geometry of nonholonomic systems, as well as may other mechanical systems, is the notion of fiber bundle. The bundle not only gives us a way of organizing variables in a physically meaningful way, but gives us basic ideas on how the system behaves physically, and on how to prescribe controls. A bundle connection relates base and fiber variables in the system, and in this sense the theory of nonholonomic control sytems is, as we shall see, closely related to gauge theories.

There is also a beautiful link between optimal control of nonholonomic systems and so-called subRiemannian geometry. For a large class of physically interesting systems, the optimal control problem reduces to finding geodesics with respect to a singular (subRiemannian) metric. The geometry of such geodesic flows is exceptionally rich and provides guidance for designing control laws. While the mechanical complexity of systems with a great number of nonlinearly coupled degrees of freedom will present computational challenges, the underlying geometry will nevertheless be useful in the solution of interpolation problems.

One of the aims of this book is to illustrate the elegant mathematics behind many simple, interesting, and useful mechanical examples. Among these are the rigid body and rolling rigid body, the rolling ball on a rotating turntable, the rattleback top, the rolling penny, the satellite with momentum wheels, and the rigid body in a perfect fluid. There are clearly a number of points in common between these systems, among them the fact the rotational motion and existence of constraints, either externally imposed or dynamically generated (conserved momenta) play a key role. In a sense these notions – rotatation and constaints form heart of the book and are vital to studying both the dynamics and control of these systems. Further, one of the delights of this subject is that the behavior of these systems is that the although they may have many features in common their behavior is quite different and often quite unexpected. Why does a rattleback top rotate in only one direction? What is the behavior of a ball on a rotating turntable? Why does a tennis racket not want to spin about its middle axis? How do I role a penny to a particular point on a table, parallel to an edge, and with Lincoln's head in the upright position?

We remark finally that while we have attempted to cover quite a lot of material here, this text is very much written from the authors' perspective, and there much fascinating work in this area that we have had to omit.

Notes

- Section 3.1, 3.2 should have more on Newton's law and inertial frames. Tony, Peter, if you have a section on this from your Brockett Festshrift paper, send it along and I will plug it in, even as a placeholder.

- Peter plans on looking at 7.6.1 – dynamic optimal control

- Chapter 9 needs a lot of blending, eg, with the work of Bloch, Leonard and Marsden, etc. We also need at least some stuff on some other really key developments in stabilization etc.: eg. the work of Coron/ Sontag/ Malkin/Van der Schaft. Some of us have notes on this.

- An internet supplement should be kept in mind as the book grows too much more. It should never get beyond 500 pages. At this point, anything that goes in, something has to come out and go in the internet supplement. Good practice at prioritizing.
- We need a bit more on stability, e.g. LaSalle.

1
Introduction

We begin this book in a concrete fashion by describing some examples (for the most part these are physical examples, but not in all cases), which we shall use throughout the course of the book to illustrate the theory. These examples are simple to write down in general and to understand at an elementary level, but they are also useful for the understanding of deeper parts of the theory.

The rolling disk and ball[1] are archetypal nonholonomic systems: systems with constraints on their velocities. The free rigid body and the somewhat more complex satellite with momentum wheels are examples of free and and coupled rigid body motion respectively—the motion of bodies with nontrivial spatial extent, as opposed to the motion of point particles. The latter is illustrated by the Toda lattice, which models a set of interacting particles on the line; we shall also be interested in some associated optimal control systems.

[1] These examples have long history going back to Vierkandt, A. [1892] Über gleitende und rollende Bewegung. *Monatshefte der Math. und Phys.* **III**, 31–54 and Chaplygin, S.A. [1897] On the motion of a heavy body of revolution on a horizontal plane (in Russian). *Physics Section of the Imperial Society of Friends of Physics, Anthropology and Ethnographics, Moscow* **9**, 10–16.

The Heisenberg system[2] does not model any particular physical system exactly, but is a prototypical example for nonlinear control problems (both optimal and nonoptimal) and can be viewed as an approximation to a number of interesting physical systems; in particular, this example is basic for understanding more sophiticated optimal reorientation and locomotion problems, such as the falling cat theorem that we shall treat later. A key point about this system (and many others in this book) is that the corresponding linear theory gives little information.

1.1 The Rolling Disk

1.1.1 The Vertical Rolling Disk

Geometry and Kinematics of the Vertical Disk. We consider here a useful example of a system subject to nonholonomic constraints—a homogeneous disk rolling without slipping on a horizontal plane. In the first instance we consider the "vertical" disk, a disk, which, unphysically of course, may not tilt away from the vertical; it is not difficult to generalize the situation to the "falling" disk. It is helpful to think of a coin such as the penny, since we are concerned with orientation of the disk and the roll angle (the position of Lincoln's head, for example). See for example the treatments in Bloch, Reyhanoglu and McClamroch [1992], Bloch and Crouch [1995] and Bloch, Krishnaprasad, Marsden and Murray [1996].[3]

The configuration space for the vertical rolling disk is $Q = \mathbb{R}^2 \times S^1 \times S^1$ and is parameterized by coordinates $q = (x, y, \theta, \varphi)$, denoting the position of the contact point in the xy-plane, the rotation angle of the disk, and the orientation of the disk, respectively, as in figure 1.1.1.

The Lagrangian for the system is taken to be the kinetic energy

$$L(x, y, \theta, \phi, \dot{x}, \dot{y}, \dot{\theta}, \dot{\phi}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2 \qquad (1.1.1)$$

where m is the mass of the disk, I is the moment of inertia of the disk about the axis perpendicular to the plane of the disk and J is the moment

[2]This system was first studied by Brockett, R.W. [1981] Control theory and singular Riemannian geometry, in *New Directions in Applied Mathematics*, P.J. Hilton and G.S. Young (eds.), Springer-Verlag. See also J. Ballieul [1975], *Some Optimization Problems in Geometric Control Theory*, Thesis, Harvard University.

[3]Bloch, A.M., M. Reyhanoglu and N.H. McClamroch [1992], Control and stabilization of nonholonomic systems. *IEEE Trans. on Automatic Control* **37**, 1746–1757, Bloch, A.M., and P.E. Crouch [1995], Nonholonomic control systems on Riemmanian manifolds. *SIAM J. on Control and Optimization* **37**, 126–148, Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden, & R. Murray [1996] Nonholonomic Mechanical Systems with Symmetry. *Arch. Rat. Mech. An.*, **136**, 21–99.

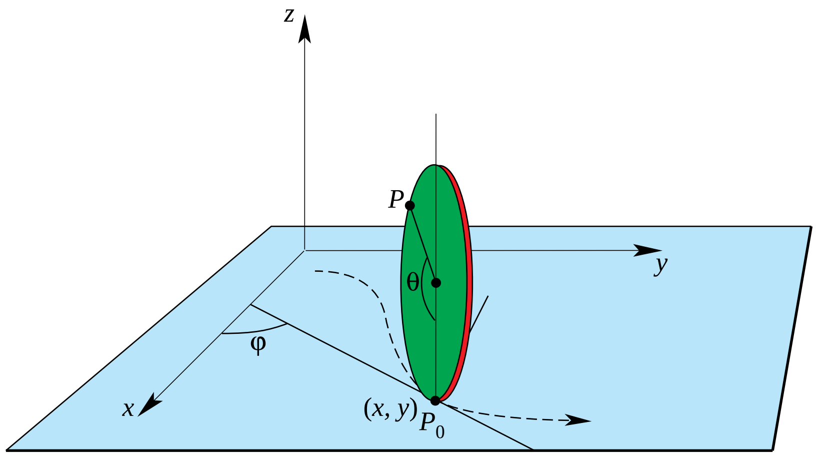

FIGURE 1.1.1. The geometry for the rolling disk.

of intertia about an axis in the plane of the disk (both axes passing through the disk's center).

If R is the radius of the disk, the nonholonomic constraints of rolling without slipping are

$$\begin{aligned} \dot{x} &= R(\cos\varphi)\dot{\theta} \\ \dot{y} &= R(\sin\varphi)\dot{\theta}, \end{aligned} \tag{1.1.2}$$

which state that the point P_0 fixed on the disk has zero velocity when it makes contact with the horizontal plane.

Dynamics of the Controlled Disk. Consider the case where we have two controls, one that can steer the disk and another that determines the roll torque. One of the things we will develop later is the general theory that allows one to formulate the equations of motion in the presence of control forces. While there are many ways of formulating these, as we shall see, one way adds the forces to the right hand side of the Euler-Lagrange equations for the given Lagrangian as well as Lagrange multipliers to enforce the constraints. The resulting equations are called the ***Lagrange d'Alembert equations***.

In our case, L is cyclic in the configuration variables $q = (x, y, \theta, \phi)$, and so these equations become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = u_1 X_1 + u_2 X_2 + \lambda_1 W_1 + \lambda_2 W_2, \tag{1.1.3}$$

where

$$\frac{\partial L}{\partial \dot{q}} = (m\dot{x}, m\dot{y}, I\dot{\theta}, J\dot{\varphi})^T,$$

$$X_1 = (0,0,1,0)^T, X_2 = (0,0,0,1)^T,$$

and

$$W_1 = (1,0,-R\cos\varphi,0), W_2 = (0,1,-R\sin\varphi,0)^T,$$

together with the constraint equations (1.1.2). Here the superscript T denotes transpose, u_1 and u_2 are control functions and the λ_i are Lagrange multipliers, chosen to ensure satisfaction of the constraints.

We may now eliminate the multilpliers; solving for λ_1 and λ_2 from (1.1.3), we get

$$\begin{aligned}\lambda_1 &= m\frac{d}{dt}(R\cos\varphi\dot{\theta})\\ \lambda_2 &= m\frac{d}{dt}(R\sin\varphi\dot{\theta})\,.\end{aligned}$$

Substitution of these expressions into the last two components of (1.1.3) gives the dynamic equations

$$\begin{aligned}(I+mR^2)\ddot{\theta} &= u_1\\ J\ddot{\varphi} &= u_2\,,\end{aligned} \tag{1.1.4}$$

plus the constraints

$$\begin{aligned}\dot{x} &= R(\cos\varphi)\dot{\theta}\\ \dot{y} &= R(\sin\varphi)\dot{\theta}\,.\end{aligned}$$

The *free equations*, in which we set $u_1 = u_2 = 0$, are easily integrated. The first two equations in (1.1.3) show that $\dot{\theta}$ and $\dot{\varphi}$ are constants; calling these constants Ω and ω respectively, we have

$$\begin{aligned}\theta &= \Omega t + \theta_0\\ \varphi &= \omega t + \phi_0\\ x &= \frac{\Omega}{\omega}R\sin(\omega T + \varphi_0) + x_0\\ y &= -\frac{\Omega}{\omega}R\cos(\omega t + \varphi_0) + y_0\,.\end{aligned}$$

Consider next the controlled case, in which the controls u_1, u_2 need not be zero. Call the variables θ and ϕ "base" or "controlled" variables and the variables x and y "fiber" variables. The distinction is that while θ and φ are controlled directly, the variables x and y are controlled indirectly via the constraints.[4]

[4]The notation "base" and "fiber" comes from the fact that the configuration space Q splits naturally into the base and fiber of a trivial fiber bundle, as we shall see later.

It is clear that the base variables are controllable in any sense we can imagine. One may ask whether the full system is controllable. Indeed it is, in a precise sense as we shall show later, by virtue of the nonholonomic nature of the constraints.

The Kinematic Controlled Disk. It is also useful to define the so-called "kinematic" controlled rolling disk. In this case we imagine we have direct control over velocities rather than forces and, accordingly, we consider the most general first order system satisfying the constraints or lying in the "constraint distribution".

This system is

$$\dot{q} = u_1 \overline{X}_1 + u_2 \overline{X}_2 \tag{1.1.5}$$

where $\overline{X}_1 = (\cos\varphi, \sin\varphi, 1, 0)^T$ and $\overline{X}_2 = (0, 0, 0, 1)^T$.

In fact, $\overline{X}_1$ and $\overline{X}_2$ comprise a maximal set of independent vector fields on Q satisfying the constraints. As we shall see it is most instructive to analyze both the control and optimal control of such systems.

The Nonholonomic and the Variational Systems. It is interesting to compare the dynamic equations (1.1.3), which can be shown to be consistent with Newton's second law $F = ma$ in the presense of reaction forces—see Vershik and Gershkovich [1988], Bloch and Crouch [1997], Jalnapurkar [1998],[5] with the corresponding variational system. The distinction between these two possible formulations has a long and distinguished history going back to the review article of Korteweg [1898] and is discussed in a more modern context in Arnold [1988]. The upshot of the distinction is that the equations (1.1.3) are the right dynamical equations while the corresponding variational problem is asking a different question, one of optimal control.

Perhaps it is surprising, at least at first, that these two procedures give different equations — what is the difference in the two procedures? The answer is that with the dynamic Lagrange d'Alembert equations, we impose constraints only on the variations, whereas in the variational problem we impose the constraints on the velocity vectors of the class of allowable curves. We will show explicitly in this example that one gets two different sets of equations.

The variational system is obtained by using Lagrange multipliers with the Lagrangian rather than Lagrange multipliers with the equations, as we

[5]Vershik, A. M. and v. Ya. Gershkovich [1988] Nonholonomic problems and the theory of distributions. *Acta Applicandae Mathematica* **12**, 181–209, Bloch, A. M. and P.E. Crouch [1997], Newton's Law and integrability of nonholonomic systems, to appear in *The SIAM Journal on Optimal Control*, Jalnapurkar [1998], Thesis, Department of Mathematics, UC Berkeley, 1999.

did earlier. Namely, we consider the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}\dot{\varphi}^2 + \mu_1(\dot{x} - R\dot{\theta}\cos\varphi) + \mu_2(\dot{x} - R\dot{\theta}\sin\varphi)\,, \tag{1.1.6}$$

where now, because of the Lagrange multipliers, we can relax the constraints and take variations over all curves. In other words, we can simply write down the Euler-Lagrange equations for this Lagrangian.

The variational equations with external forces therefore are

$$m\ddot{x} + \dot{\mu}_1 = 0 \tag{1.1.7}$$

$$m\ddot{y} + \dot{\mu}_2 = 0 \tag{1.1.8}$$

$$I\ddot{\theta} - R\frac{d}{dt}(\mu_1\cos\varphi + \mu_2\sin\varphi) = u_1 \tag{1.1.9}$$

$$J\ddot{\varphi} + R\frac{\partial d}{\partial\varphi}(\mu_1\dot{\theta}\cos\varphi + \mu_2\dot{\theta}\sin\varphi) = u_2\,. \tag{1.1.10}$$

From the constraint equations and equations (1.1.7) and (1.1.8), we have

$$\mu_1 = -mR\dot{\theta}\cos\phi + A$$
$$\mu_2 = -mR\dot{\theta}\sin\phi + B,$$

where A and B are constants.

Substituting these into equations (1.1.9) and (1.1.10), we obtain

$$(I + mR^2)\ddot{\theta} = R\dot{\varphi}(-A\sin\varphi + B\cos\varphi) + u_1 \tag{1.1.11}$$

$$J\ddot{\varphi} = R\dot{\theta}(A\sin\varphi - B\cos\varphi) + u_2\,. \tag{1.1.12}$$

These equations, together with the constraints, define the dynamics. Notice that they are different from the dynamic (Lagrange d'Alembert) equations. As we have indicated, the motion determined by these equations is not that associated with physical dynamics in general, but is a model of the type of problem that is relevant to optimal control problems, as we shall see later.

1.1.2 The Falling Rolling Disk

A more realistic disc is of course one that is allowed to fall over (deviate from the vertical). This turns out to be a very useful example to analyze. See Figure 1.1.2.

As the figure indicates, we denote the coordinates of contact of the disk in the xy-plane by (x, y) and let θ, ϕ, and ψ denote the angle between the plane of the disk and the vertical axis, the "heading angle" of the disk, and "self-rotation" angle of the disk respectively.

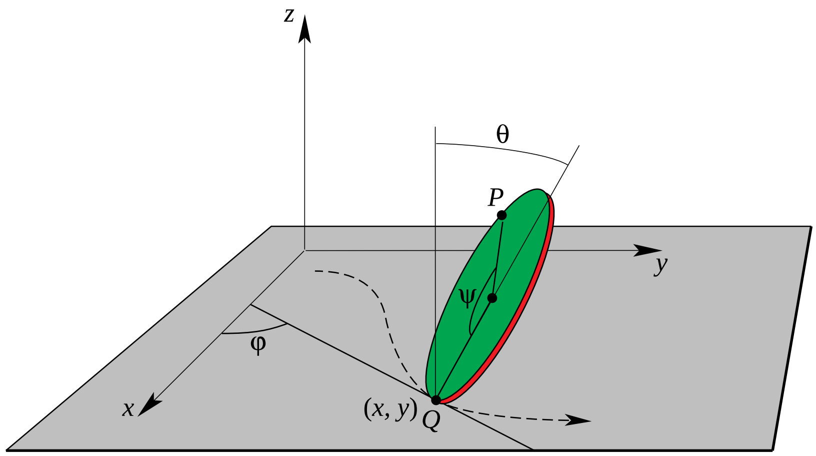

FIGURE 1.1.2. The geometry for the rolling disk.

A classical reference for the rolling disk is Vierkandt [1892] who showed that on the reduced space $\mathcal{D}/SE(2)$—the constrained velocity phase space modulo the action of the Euclidean group $SE(2)$—all orbits of the system are periodic. Modern references that treat this example are Hermans [1995] and O'Reilly [1996]. [6]

For the moment, we just give the Lagrangian and constraints, and return to this example later on. As we will eventually show, this is a system which exhibits stability but not asymptotic stability. Denote the mass, the radius, and the moments of inertia of the disk by m, R, A, B respectively. The Lagrangian is given by the kinetic minus potential energies:

$$\begin{aligned} L = {} & \frac{m}{2}\left[(\xi - R(\dot\phi\sin\theta + \dot\psi))^2 + \eta^2\sin^2\theta + (\eta\cos\theta + R\dot\theta)^2\right] \\ & + \frac{1}{2}\left[A(\dot\theta^2 + \dot\phi^2\cos^2\theta) + B(\dot\phi\sin\theta + \dot\psi)^2\right] - mgR\cos\theta, \end{aligned}$$

where $\xi = \dot x\cos\phi + \dot y\sin\phi + R\dot\psi$ and $\eta = -\dot x\sin\phi + \dot y\cos\phi$, while the constraints are given by

$$\begin{aligned} \dot x &= -\dot\psi R\cos\phi, \\ \dot y &= -\dot\psi R\sin\phi. \end{aligned}$$

Note that the constraints may also be written as $\xi = 0$, $\eta = 0$.

[6] See Vierkandt, A. [1892] Über gleitende und rollende Bewegung. *Monatshefte der Math. und Phys.* **III**, 31–54, Hermans, J. [1995a], *Rolling Rigid Bodies, with and without Symmetries*, Thesis, University of Utrecht, O'Reilly, O.M. [1996] The Dynamics of Rolling Disks and Sliding Disks. *Nonlinear Dynamics*, **10**, 287–305, Zenkov, D.V., A.M. Bloch, and J.E. Marsden [1998] The Energy Momentum Method for the Stability of Nonholonomic Systems *Dyn. Stab. of Systems.*, **13**, 123–166.

1.2 The Heisenberg System

The Heisenberg Algebra. The Heisenberg alegbra is the algebra one meets in quantum mechanics, wherein one has two operators q and p that have a nontrivial commutator, in this case a multiple of the identity. Thereby, one generates a three dimensional Lie algebra. The system studied in this section has an associated Lie algebra with a similar structure, which is the reason the system is called the Heisenberg system. There is no intended relation to quantum mechanics *per se* other than this.

In Lie algebra theory this sort of a Lie algebra is of considerable interest—one refers to it as an example of a ***central extension*** because the element that one extends by (in this case a multiple of the identity) is in the center of the algebra; that is, it commutes with all elements of the algebra.

The Heisenberg system has played a significant role as an example in both nonlinear control and nonholonomic mechanics.

The Dynamic Heisenberg System. As with the previous example, the system comes in two forms, one associated with the Lagrange d'Alembert principle and one with an optimal control problem. As in the previous examples, the equations in each case are different.

In the dynamic setting, we consider the following standard kinetic energy Lagrangian on Euclidean three space $\mathbb{R}^3$:

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

subject to the constraint

$$\dot{z} = y\dot{x} - x\dot{y}. \tag{1.2.1}$$

Controls u_1 and u_2 are given in the x and y directions. Letting $q = (x, y, z)^T$, the dynamic nonholonomic control system is[7]

$$\ddot{q} = u_1 X_1 + u_2 X_2 + \lambda W \tag{1.2.2}$$

where $X_1 = (1, 0, 0)^T$ and $X_2 = (0, 1, 0)^T$ and $W = (-y, x, 1)^T$. Eliminating λ we obtain the dynamic equations

$$\begin{aligned}
(1 + x^2 + y^2)\ddot{x} &= (1 + x^2)u_1 + xyu_2 \\
(1 + x^2 + y^2)\ddot{y} &= (1 + y^2)u_1 + xyu_1 \\
(1 + x^2 + y^2)\ddot{z} &= yu_1 - xu_2\,.
\end{aligned} \tag{1.2.3}$$

[7]This example with controls was analyzed in Bloch and Crouch [1993] Nonholonomic and vakonomic control systems on Riemannian manifolds. *Fields Institute Communications*, **1**, 25-52. A related nonholonomic system, but with slightly different constraints may be found in Rosenberg, R.M. [1977] *Analytical Dynamics of Discrete Systems.* Plenum Press, NY., Bates, L. and J. Sniatycki [1993] Nonholonomic reduction, *Reports on Math. Phys.* **32**, 99–115 and Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden, and R. Murray [1996] Nonholonomic mechanical systems with symmetry. *Arch. Rat. Mech. An.*, **136**, 21–99.

Optimal Control for the Heisenberg System. The control and optimal control of the corresponding kinematic problem has been quite important historically and we shall return to it later on in the book in connection with, for example, the falling cat problem and optimal steering problems.[8]

The system may be written as

$$\dot{q} = u_1 g_1 + u_2 g_2 \tag{1.2.4}$$

where $g_1 = (1, 0, y)^T$ and $g_2 = (0, 1, -x)^T$. As in the rolling disk example g_1 and g_2 are a maximal set of independent vector fields satisfying the constraint

$$\dot{z} = x\dot{y} - y\dot{x}\,. \tag{1.2.5}$$

Written out in full, these equations are

$$\dot{x} = u_1 \tag{1.2.6}$$
$$\dot{y} = u_2 \tag{1.2.7}$$
$$\dot{z} = xu_2 - yu_1 \tag{1.2.8}$$

One verifies that the Jacobi-Lie bracket of the vector fields g_1 and g_2 is

$$[g_1, g_2] = 2g_3$$

where $g_3 = (0, 0, 1)$. In fact, the three vector fields g_1, g_2, g_3 span all of $\mathbb{R}^3$ and, as a Lie algebra, is just the Heisenberg algebra described earlier.

By general controllability theorems that we shall discuss in Chapter 3 (Chow's theorem), one knows that one can, with suitable controls, steer trajectories between any two points in $\mathbb{R}^3$. In particular, we are interested in the following optimal steering problem (see Figure 1.2):

> **Optimal Steering Problem.** Given a number $a > 0$, find time dependent controls u_1, u_2 that steer the trajectory starting at $(0, 0, 0)$ at time $t = 0$ to the point $(0, 0, a)$ after a given time $T > 0$ and that, amongst all such controls, minimizes
>
> $$\frac{1}{2}\int_0^T (u_1^2 + u_2^2)\, dt\,.$$

FIGURE 1.2.1. An optimal steering problem.

An equivalent formulation is the following: minimize the integral

$$\frac{1}{2}\int_0^T (\dot{x}^2 + \dot{y}2)\, dt$$

[8]As we mentioned earlier, this example was introduced in Brockett [1981]

amongst all curves $x(t)$ joining $x(0) = (0,0,0)$ to $x(T) = (0,0,a)$ that satisfy the constraint

$$\dot{z} = y\dot{x} - x\dot{y}\,.$$

As before, any solution must satisfy the Euler-Lagrange equations for the Lagrangian with a Lagrange multiplier inserted:

$$L\left(x, \dot{x}, y, \dot{y}, z, \dot{z}, \lambda, \dot{\lambda}\right) = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) + \lambda\left(\dot{z} - y\dot{x} + x\dot{y}\right)$$

The corresponding Euler-Lagrange equations are given by

$$\ddot{x} - 2\lambda\dot{y} = 0 \tag{1.2.9}$$
$$\ddot{y} + 2\lambda\dot{x} = 0 \tag{1.2.10}$$
$$\dot{\lambda} = 0 \tag{1.2.11}$$

From the third equation, λ is a constant, and the first two equations state that *the particle $(x(t), y(t))$ moves in the plane in a constant magnetic field (pointing in the z direction, with charge proportional to the constant λ.* For more on these ideas see the chapter on optimal control.

Some remarks are in order here:

1. The fact that this optimal steering problem gives rise to an interesting mechanical system is not an accident; we shall see this in much more generality later.

2. Since particles in constant magnetic fields move in circles with constant speed, they have a sinusoidal time dependence, and hence so do the controls. This has led to the "steering by sinusoids" approach in many nonholonomic steering problems (see for example Murray and Sastry [1993] [9]).

The equations (1.2.9) and (1.2.10) are linear first order equations in the velocities and are readily solved:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} \cos(2\lambda t) & \sin(2\lambda t) \\ -\sin(2\lambda t) & \cos(2\lambda t) \end{bmatrix} \begin{bmatrix} \dot{x}(0) \\ \dot{y}(0) \end{bmatrix} \tag{1.2.12}$$

Integrating once more and using the initial conditions $x(0) = 0, y(0) = 0$ gives:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{2\lambda} \begin{bmatrix} \cos(2\lambda t) - 1 & \sin(2\lambda t) \\ -\sin(2\lambda t) & \cos(2\lambda t) - 1 \end{bmatrix} \begin{bmatrix} -\dot{y}(0) \\ \dot{x}(0) \end{bmatrix} \tag{1.2.13}$$

[9] Murray and Sastry [1993], Nonholonomic motion planning: steering using sinusoids. *IEEE Trans on Automatic Control* **38**, 700-716

The other boundary condition $x(T) = 0, y(T) = 0$ gives

$$\lambda = \frac{n\pi}{T}.$$

Using this information, we find z by integration: from $\dot{z} = x\dot{y} - y\dot{x}$ and the preceding expressions, we get

$$\dot{z}(t) = \frac{1}{2\lambda}\left[-\dot{x}(0)^2 - \dot{y}(0)^2 + \cos(2\lambda t)(\dot{x}(0)^2 + \dot{y}(0)^2)\right].$$

Integration from 0 to T and using $z(0) = 0$ gives

$$z(T) = \frac{T}{2\lambda}\left[-\dot{x}(0)^2 - \dot{y}(0)^2\right].$$

Thus, to achieve the boundary condition $z(T) = a$ one must choose

$$\dot{x}(0)^2 + \dot{y}(0)^2 = -\frac{2\pi na}{T^2}.$$

One also finds that

$$\begin{aligned}\frac{1}{2}\int_0^T \left[\dot{x}(t)^2 + \dot{y}(t)^2\right] dt &= \frac{1}{2}\int_0^T \left[\dot{x}(0)^2 + \dot{y}(0)^2\right] dt \\ &= \frac{T}{2}\left[\dot{x}(0)^2 + \dot{y}(0)^2\right] \\ &= -\frac{\pi na}{T}\end{aligned}$$

so that the minimum is achieved when $n = -1$.

Summary: The solution of the optimal control problem is given by choosing initial conditions such that $\dot{x}(0)^2 + \dot{y}(0)^2 = 2\pi a/T^2$ and with the trajectory in the xy plane given by the circle

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \frac{1}{2\lambda}\begin{bmatrix} \cos(2\pi t/T) - 1 & -\sin(2\pi t/T) \\ \sin(2\pi t/T) & \cos(2\pi t/T) - 1 \end{bmatrix}\begin{bmatrix} -\dot{y}(0) \\ \dot{x}(0) \end{bmatrix} \tag{1.2.14}$$

and with z given by

$$z(t) = \frac{ta}{T} - ta^2 \sin\left(\frac{2\pi t}{T}\right).$$

Notice that any such solution can be rotated about the z axis to obtain another one.

1.3 The Toda Lattice

An important and beautiful mechanical system that describes the interaction of particles on the line (i.e. in one dimension) is the Toda lattice.

We shall describe here the nonperiodic finite Toda lattice as analyzed by Moser [1975][10]

The model consists of n- particles moving freely on the x–axis and interacting under an exponential potential. Denote the position of the kth particle by x_k. The Hamiltonian is given by

$$H(x,y) = \frac{1}{2}\sum_{k=1}^{n} y_k^2 + \sum_{k-1}^{n-1} e^{(x_k - x_{k+1})}.$$

The associated Hamiltonian equations are

$$\dot{x}_k = \frac{\partial H}{\partial y_k} = y_k \tag{1.3.1}$$

$$\dot{y}_k = -\frac{\partial H}{\partial x_k} = e^{x_{k-1}-x_k} - e^{x_k - x_{k-1}}, \tag{1.3.2}$$

where we use the convention $e^{x_0 - x_1} = e^{x_n - x_{n+1}} = 0$, which corresponds to formally setting $x_0 = -\infty$ and $x_{n+1} = +\infty$.

This system of equations has an extraordinarily rich structure. Part of this is revealed by Flaschka's ([1974] [11]) observation that a change of variables enables one to write the equation in Lax pair form. To achieve this, set

$$a_k = \frac{1}{2} e^{(x_k - x_{k+1})/2} \quad b_k = -\frac{1}{2} y_k\,. \tag{1.3.3}$$

The equations of motion then become

$$\dot{a}_k = a_k(b_{k+1} - b_k) \quad , k = 1, \cdots, n-1 \tag{1.3.4}$$

$$\dot{b}_k = 2(a_k^2 - a_{k-1}^2) \quad , k = 1, \cdots, n \tag{1.3.5}$$

with the boundary conditions $a_0 = a_n = 0$. This system may be written in the matrix form

$$\frac{d}{dt} L = [B, L] = BL - LB, \tag{1.3.6}$$

where

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & 0 \\ & & \ddots & & \\ & & & b_{n-1} & a_{n-1} \\ 0 & & & a_{n-1} & b_n \end{pmatrix} \quad B = \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 \\ -a_1 & 0 & a_2 & \cdots & 0 \\ & & \ddots & & \\ & & & 0 & a_{n-1} \\ 0 & & & a_{n-1} & 0 \end{pmatrix}$$

[10] Moser, J. [1975] Finitely many mass points on the under the influence of an exponential potential – an integrable system. *Springer Lecture Notes in Physics* **38**, 467-497, Moser, J. [1975b] Three integrable Hamiltonian systems connected with isospectral deformations. *Adv. Math.* **16**, 197–220.

[11] Flaschka, H. [1974] The Toda Lattice. *Phys. Rev. B* **9**, 1924-25.

If $O(t)$ is the orthogonal matrix solving the equation

$$\frac{d}{dt}O = BO\,, \quad O(0) = \text{ Identity}$$

then from (1.3.6), we have

$$\frac{d}{dt}(O^{-1}LO) = 0\,.$$

Thus, $O^{-1}LO = L(0)$, i.e. $L(t)$ is related to $L(0)$ by a similarity transformation and thus the eigenvalues of L, which are real and distinct, are preserved along the flow. This is enough to show that in fact this system is explicitly solvable or integrable.

There is, however, much more structure in this example. For instance, if N is the matrix $\mathrm{diag}[1, 2, \cdots, n]$ the Toda flow (1.3.6) may be written in the form[12]

$$\dot{L} = [L, [L, N]]\,. \tag{1.3.7}$$

This shows that the flow is also gradient (on a level set of its integrals). This is explicitly exhibited by writing the equation in the so-called double bracket form of Brockett [1988].[13]

1.4 The Rigid Body

1.4.1 The Free Rigid Body

A key system in mechanics is the free rigid body. There are many excellent treatments of this topic – see for example Whittaker Arnold [1978], and Marsden and Ratiu [1994]. [14] We restrict ourselves here to some essentials,

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[12]See Bloch A. M. [1990] Steepest descent, linear programming and Hamiltonian flows. *Contemp. Math. A.M.S.* **114**, 77-88, and Bloch, Brockett and Ratiu [1992]; Bloch, A.M., R.W. Brockett, and T.S. Ratiu [1992] Completely integrable gradient flows. *Comm. Math. Phys.* **147**, 57–74.

[13]Brockett, R.W. [1988] Dynamical systems that sort lists and solve linear programming problems. *Proc. IEEE* **27**, 799–803 and *Linear Algebra and its Appl.* **146**, (1991), 79–91.

[14]Whittaker, E.T. [1937] *A treatise on the analytical dynamics of particles and rigid bodies*, Cambridge University Press, 4th Ed. (reprinted by Dover 1944, and Cambridge University 1988.), Arnold, V. I. [1978] *Mathematical Methods of Classical Mechanics* Graduate Texts in Math. **60**, Springer Verlag. (Second Edition, 1989), Marsden, J.E. and T.S. Ratiu [1994] *Introduction to Mechanics and Symmetry.* Texts in Applied Mathematics, **17**, Springer-Verlag.

although we shall return to it in detail in the context of nonholonomic mechanics and optimal control.

The configuration space of a rigid body moving freely in space is $\mathbb{R}^3 \times SO(3)$ – describing the position of a coordinate frame fixed in the body and the orientation of the frame say, the orientation of the frame given by a element of of $SO(3)$, i.e., an orthogonal 3 by 3 matrix with determinant 1. Since the the three components of translational momentum are conserved, the body behaves as if it were rotating freely about its center of mass.[15]

Hence the phase space for the body may be taken to be $TSO(3)$ – the tangent bundle of $SO(3)$ – with points representing the position and velocity of the body, or in the Hamiltonian context we may choose the phase space to be the cotangent bundle $T^*SO(3)$, with points representing the position and momentum of the body. (This example may be equally well formulated for the group $SO(n)$ or indeed any compact Lie group.)

If I is the moment of inertia tensor computed with respect to a body fixed frame, which, in a *principal* body frame, we may represent by the diagonal matrix $\mathrm{diag}(I_1, I_2, I_3)$, the Lagrangian of the body is given by the kinetic energy, namely

$$L = \frac{1}{2}\Omega \cdot I\Omega, \tag{1.4.1}$$

where Ω is the vector of angular velocities computed with respect to the axes fixed in the body.

The Euler-Lagrange equations of motion may be written as

$$\dot{A} = A\Omega \tag{1.4.2}$$

$$I\dot{\Omega} = I\Omega \times \Omega\,, \tag{1.4.3}$$

where $A \in SO(3)$. Writing

$$\hat{\Omega} \equiv \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix},$$

the dynamics may be rewritten

$$I\dot{\hat{\Omega}} = [I\hat{\Omega}, \hat{\Omega}] \tag{1.4.4}$$

or, in terms of the angular momentum $\hat{M} = I\hat{\Omega}$

$$\dot{\hat{M}} = [\hat{M}, \hat{\Omega}]\,. \tag{1.4.5}$$

[15]This is not the case with other systems, such as a rigid body moving in a fluid; even though the system is translation invariant, its "center of mass" need not move on a straight line, so the configuration space must be taken to be the full Euclidean group.

1.4.2 Example: The Rolling Ball

We consider here the controlled rolling inhomogeneous ball on the plane, the kinematics of which were discussed in Brockett and Dai [1991][16], establishing the completely nonholonomic nature of the constraint distribution H. (A distribution is completely nonholonomic if the span of the iterated brackets of the vector fields lying in it has dimension equal to the dimension of the underlying manifold – see Chapter 4 for a full explanation.) The dynamics of the uncontrolled system are described for example in McMillan [1936] (see also Bloch, Krishnaprasad, Marsden and Murray [1996], Jurdjevic [1993], Koon and Marsden [1997] and Krishnaprasad and Dayawansa [1992]). [17] We will use the coordinates x, y for the linear horizontal displacement and $P \in SO(3)$ for the angular displacement of the ball. Thus P gives the orientation of the ball with respect to inertial axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ fixed in the plane, where the $\mathbf{e}_i$ are the standard basis vectors aligned with the $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ axes respectively. See figure 1.4.1

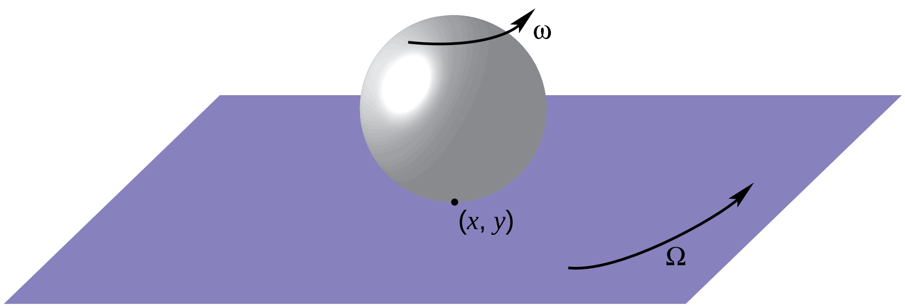

FIGURE 1.4.1. The rolling ball.

Let the ball have radius a and mass m and let $\omega \in \mathbb{R}^3$ denote the angular velocity of the ball with respect to inertial axes. In particular, the ball may spin freely about the z-axis and the z-component of angular momentum is conserved. If J denotes the inertia tensor of the ball with respect to the body axes (i.e. fixed in the body), then $\mathbb{J} = P^T J P$ denotes the inertia tensor of the ball with respect to the inertial axes (i.e. fixed in space) and

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[16]Brockett, R.W. & L. Dai [1992] Nonholonomic kinematics and the role of elliptic functions in constructive controllability, in *Nonholonomic Motion Planning*, eds. Z. Li & J. F. Canny , Kluwer, 1–22, 1993.

[17]McMillan, W. D. [1936] *Dynamics of Rigid Bodies*, Duncan MacMillan Rowles, U.K., Bloch, A.M., P.S. Krishnaprasad, J.E. Marsden and R. Murray [1996] Nonholonomic Mechanical Systems with Symmetry. *Arch. Rat. Mech. An.*, **136**, 21–99, Jurdjevic, V. [1993] The geometry of the plate-ball problem. *Arch. Rat. Mech. An.* **124**, 305–328. Krishnaprasad, P.S., W. Dayawansa & R. Yang [1992] The geometry of nonholonomic constraints. (*Preprint, University of Maryland*, Koon, W.S. and J.E. Marsden [1997] The Hamiltonian and Lagrangian Approaches to the Dynamics of Nonholonomic Systems, *Reports on Math Phys (to appear).*

$\mathbb{J}\omega$ is the angular momentum of the ball with respect to the inertial axes. The conservation law alluded to above is expressed as

$$\mathbf{e}_3^T \mathbb{J}\omega = c\,. \tag{1.4.6}$$

The nonholonomic constraints may be expressed as

$$\begin{aligned} a\mathbf{e}_2^T\omega + \dot{x} &= 0 \\ a\mathbf{e}_1^T\omega - \dot{y} &= 0\,. \end{aligned} \tag{1.4.7}$$

We may express the kinematics for the rotating ball as $\dot{P} = \hat{\Omega}P$ where $\Omega = P\omega$ is the angular velocity in the body frame.

Appending the constraints via Lagrange multipliers we obtain the equations of motion

$$\begin{aligned} &\dot{\hat{\Omega}}P - (\widehat{J^{-1}\hat{\Omega}J\Omega})P \\ &\quad = \lambda_1(\widehat{J^{-1}P\mathbf{e}_1})P + \lambda_2(\widehat{J^{-1}P\mathbf{e}_2})P \\ &m\ddot{x} = \lambda_2 + u_1 \\ &m\ddot{y} = -\lambda_1 + u_2\,. \end{aligned} \tag{1.4.8}$$

As mentioned before, the details of the derivation of these equations will be discussed later.

Using inertial coordinates $\omega = P^T\Omega$, the system becomes

$$\begin{aligned} \dot{\omega} &= \mathbb{J}^{-1}\hat{\omega}\mathbb{J}\omega + \lambda_1\mathbb{J}^{-1}\mathbf{e}_1 + \lambda_2\mathbb{J}^{-1}\mathbf{e}_2 \\ m\ddot{x} &= \lambda_2 + u_1 \\ m\ddot{y} &= -\lambda_1 + u_2 \\ \dot{P} &= P\hat{\omega}\,. \end{aligned} \tag{1.4.9}$$

Also, from the constraints and the constants of motion we obtain the following expression for ω:

$$\omega = \dot{x}(\alpha_2\mathbf{e}_3 - \mathbf{e}_2) + \dot{y}(\mathbf{e}_1 - \alpha_1\mathbf{e}_3) + \alpha_3\mathbf{e}_3$$

where

$$\alpha_1 = \frac{\mathbf{e}_3^T\mathbb{J}\mathbf{e}_1}{a\mathbf{e}_3^T\mathbb{J}\mathbf{e}_3}\,, \quad \alpha_2 = \frac{\mathbf{e}_3^T\mathbb{J}\mathbf{e}_2}{a\mathbf{e}_3^T\mathbb{J}\mathbf{e}_3}\,, \quad \alpha_3 = \frac{c}{\mathbf{e}_3^T\mathbb{J}\mathbf{e}_3}\,. \tag{1.4.10}$$

Then the equations become

$$\begin{aligned} m\ddot{x} &= \lambda_2 + u_1 \\ m\ddot{y} &= -\lambda_1 + u_2 \\ \dot{P} &= P(\dot{x}(\alpha_2\mathbf{e}_3 - \mathbf{e}_2) + \widehat{\dot{y}(\mathbf{e}_1} - \alpha_1\mathbf{e}_3) + \alpha_3\mathbf{e}_3) \end{aligned} \tag{1.4.11}$$

One can now eliminate the multipliers using the first three equations of 1.4.9 and the constraints. The resulting expressions are a little complicated in the general case (although can be found in straighforward fashion), but become pleasingly simple in the case of a homogeneous ball where say $J = mk^2$ (k is called the radius of gyration in the classical literature).

In the latter case the equations of motion for ω_1 and ω_2 become simply

$$\begin{aligned} mk^2\dot{\omega}_1 &= a\lambda_1 \\ mk^2\dot{\omega}_2 &= a\lambda_2\,. \end{aligned} \tag{1.4.12}$$

Rewriting these equations in terms of x and y using the multipliers, and substituting the resulting expressions for the λ_i into the equations of motion for x and y yields the equations

$$\begin{aligned} m\ddot{x} &= \frac{a^2}{a^2+k^2}u_1 \\ m\ddot{y} &= \frac{a^2}{a^2+k^2}u_2\,. \end{aligned} \tag{1.4.13}$$

.

A similar elimination argument works in the general case.

Note that the homogeous ball moves under the action of external forces like a point mass located at its center but with force reduced by the ratio $a^2/a^2 + k^2$ – see also the following subsection.

1.4.3 A Homogeneous Ball on a Rotating Plate

A useful example is a model of a homogeneous ball on a rotating plate (see Neimark and Fufaev [1972] and Yang [1992] for the affine case and, for example, Bloch and Crouch [1992], Brockett and Dai [1992] and Jurdjevic [1993] for the linear case). As we mentioned earlier, Chaplygin [1897b,1903] studied the motion of a *nonhomogeneous* rolling ball.

Let the plane rotate with constant angular velocity $\tilde{\Omega}$ about the z-axis. The configuration space of the sphere is $Q = \mathbb{R}^2 \times SO(3)$, parameterized by $(x, y, R), R \in SO(3)$, all measured with respect to the inertial frame. Let $\omega = (\omega_x, \omega_y, \omega_z)$ be the angular velocity vector of the sphere measured also with respect to the inertial frame, let m be the mass of the sphere, mk^2 its inertia about any axis, and let a be its radius.

The Lagrangian of the system is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}mk^2({\omega_x}^2 + {\omega_y}^2 + {\omega_z}^2) \tag{1.4.14}$$

with the affine nonholonomic constraints

$$\left.\begin{aligned} \dot{x} - a\omega_y &= -\tilde{\Omega}y \\ \dot{y} + a\omega_x &= \tilde{\Omega}x. \end{aligned}\right\} \tag{1.4.15}$$

Note that the Lagrangian here is a metric on Q which is bi-invariant on $SO(3)$ as the ball is homogeneous. Note also that $\mathbb{R}^2 \times SO(3)$ is a principal bundle over $\mathbb{R}^2$ with respect to the right $SO(3)$ action on Q given by

$$(x, y, R) \mapsto (x, y, RS) \tag{1.4.16}$$

for $S \in SO(3)$. The action is on the *right* since the symmetry is a material symmetry.

The equations of motion can be shown to reduce to

$$\left.\begin{aligned} \ddot{x} + \frac{k^2\tilde{\Omega}}{a^2+k^2}\dot{y} = 0 \\ \ddot{y} - \frac{k^2\tilde{\Omega}}{a^2+k^2}\dot{x} = 0. \end{aligned}\right\} \tag{1.4.17}$$

These equations are easily integrated to show that ball simply oscillates on the plate between two circles rather than flying off as one might expect.

Set

$$\alpha = \frac{k^2\tilde{\Omega}}{a^2+k^2}\,.$$

Then one can see the equations are equivalent to

$$\dddot{x} + \alpha^2\dot{x} = 0 \tag{1.4.18}$$

$$\dddot{y} + \alpha^2\dot{y} = 0\,. \tag{1.4.19}$$

Hence

$$x = A\cos\alpha t + B\sin\alpha t + C$$

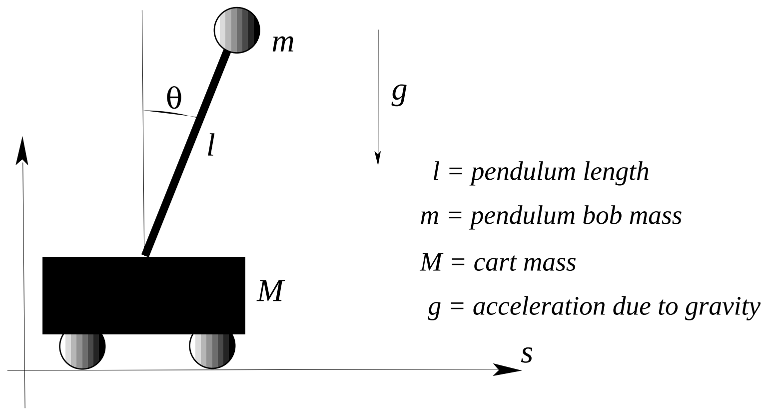
for constants A, B, C depending on the initial data and similarly for y.

1.4.4 The inverted pendulum on a cart

A useful classical systems for testing control theortic ideas is the inverted pendulum on a cart – the goal being to stabilize the pendulum about the vertical using a force acting on the cart. In this book we will use this system to illustrate stabilization using the energy methods as discussed in Bloch, Marsden and Sánchez de Alvarez [1997] and Bloch, Marsden and Leonard [1997]. Here we just write down the equations of motion.

First, we compute the Lagrangian for the cart-pendulum system. Let s denote the position of the cart on the s-axis and let θ denote the angle of the pendulum with the upright vertical, as in the figure.

Here, the configuration space is $Q = G \times S = \mathbb{R} \times S^1$ with the first factor being the cart position s, and the second factor being the pendulum angle, θ. The velocity phase space, TQ has coordinates $(s, \theta, \dot{s}, \dot{\theta})$.



The velocity of the cart relative to the lab frame is $\dot{s}$, while the velocity of the pendulum relative to the lab frame is the vector

$$v_{\text{pend}} = (\dot{s} + l\cos\theta\,\dot{\theta}, -l\sin\theta\,\dot{\theta}). \tag{1.4.20}$$

The system kinetic energy is just the sum of the kinetic energies of the cart and the pendulum:

$$K((s,\theta,\dot{s},\dot{\theta}) = \frac{1}{2}(\dot{s},\dot{\theta})\begin{pmatrix} M+m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{pmatrix}\begin{pmatrix} \dot{s} \\ \dot{\theta} \end{pmatrix}. \tag{1.4.21}$$

The Lagrangian is the kinetic minus potential energy, so we get

$$L(s,\theta,\dot{s},\dot{\theta}) = K(s,\theta,\dot{s},\dot{\theta}) - V(\theta), \tag{1.4.22}$$

where the potential energy is $V = mgl\cos\theta$.

Note that there is a symmetry group G of the pendulum-cart system – that of translation in the s variable so $G = \mathbb{R}$. We do not destroy this symmetry when doing stabilization in θ.

For convenience we rewrite the Lagrangian as

$$L(s,\theta,\dot{s},\dot{\theta}) = \frac{1}{2}(\alpha\dot{\theta}^2 + 2\beta\cos\theta\dot{s}\dot{\theta} + \gamma\dot{s}^2) + Dcos\theta\,, \tag{1.4.23}$$

where $\alpha = ml^2, \beta = ml, \gamma = M+m$ and $D = -mgl$ are constants. Note that $\alpha\gamma - \beta^2 > 0$.

The momentum conjugate to s is $p_s = \gamma\dot{s} + \beta\cos\theta\dot{\theta}$ and the momentum conjugate to θ is $p_\theta = \alpha\dot{\theta} + \beta\cos\theta\dot{s}$.

The relative equilibrium defined by $\theta = 0, \dot{\theta} = 0$ and $\dot{s} = 0$ is unstable since $D < 0$.

The equations of motion of the cart pendulum system with a control force u acting on the cart (and no direct forces acting on the pendulum) are, since s is a cyclic variable (i.e. L is independent of s),

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{s}} = u$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0\,,$$

i.e.,

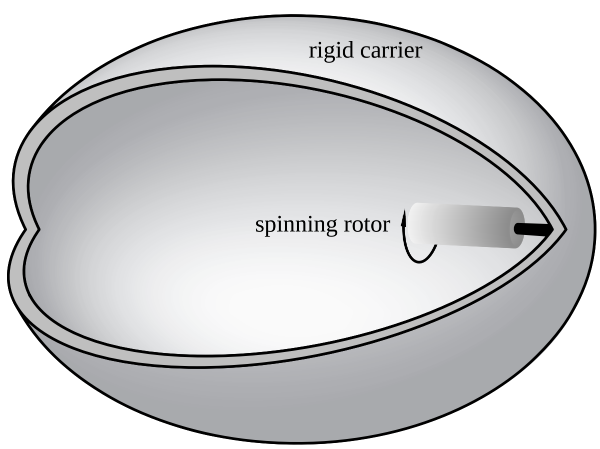
$$\frac{d}{dt}p_s = \frac{d}{dt}(\gamma\dot{s} + \beta\cos\theta\dot{\theta}) = u$$

$$\frac{d}{dt}p_\theta + \beta\sin\theta\dot{s}\dot{\theta} + D\sin\theta =$$
$$\frac{d}{dt}(\alpha\dot{\theta} + \beta\cos\theta\dot{s}) + \beta\sin\theta\dot{s}\dot{\theta} + D\sin\theta = 0\,. \tag{1.4.24}$$

1.4.5 The Rotating Pendulum

1.4.6 Rigid Body with Rotors

Following Krishnaprasad [1985] and Bloch, Krishnaprasad, Marsden and Sánchez de Alvarez [1992] and Bloch, Leonard and Marsden [1997], we consider a rigid body with a rotor aligned along the third principal axis of the body as in the figure. The rotor spins under the influence of a torque u acting on the rotor. The configuration space is $Q = SO(3) \times S^1$, with the first factor being the spacecraft attitude and the second factor being the rotor angle. The Lagrangian is total kinetic energy of the system, (rigid carrier plus rotor), with no potential energy.



Again this system will be used to illustrate energy method is analyzing stabilization and stability.

The Lagrangian for this system is

$$L = \frac{1}{2}(\lambda_1\Omega_1^2 + \lambda_2\Omega_2^2 + I_3\Omega_3^2 + J_3(\Omega_3 + \dot{\alpha})^2) \tag{1.4.25}$$

where $I_1 > I_2 > I_3$ are the rigid body moments of inertia,$J_1 = J_2$ and J_3 are the rotor moments of inertia, $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ is the body angular velocity vector of the carrier and α is the relative angle of the rotor.

The body angular momenta are determined by the Legendre transform to be

$$\begin{aligned}
\Pi_1 &= \lambda_1\Omega_1 \\
\Pi_2 &= \lambda_2\Omega_2 \\
\Pi_3 &= \lambda_3\Omega_3 + J_3\dot{\alpha} \\
l_3 &= J_3(\Omega_3 + \dot{\alpha}),
\end{aligned}$$

$\lambda_i = I_i + J_i$. The momentum conjugate to α is l_3.

The equations of motion with a control torque u acting on the rotor are

$$\begin{aligned}
\lambda_1\dot{\Omega}_1 &= \lambda_2\Omega_2\Omega_3 - (\lambda_3\Omega_3 + J_3\dot{\alpha})\Omega_2 \\
\lambda_2\dot{\Omega}_2 &= -\lambda_1\Omega_1\Omega_3 + (\lambda_3\Omega_3 + J_3\dot{\alpha})\Omega_1 \\
\lambda_3\dot{\Omega}_3 + J_3\ddot{\alpha} &= (\lambda_1 - \lambda_2)\Omega_1\Omega_2 \\
\dot{l}_3 &= u\,.
\end{aligned} \tag{1.4.26}$$

The equations may also be written in Hamiltonian form:

$$\begin{aligned}
\dot{\Pi}_1 &= \left(\frac{1}{I_3} - \frac{1}{\lambda_2}\right)\Pi_2\Pi_3 - \frac{l_3\Pi_2}{I_3} \\
\dot{\Pi}_2 &= \left(\frac{1}{\lambda_1} - \frac{1}{I_3}\right)\Pi_1\Pi_3 + \frac{l_3\Pi_1}{I_3} \\
\dot{\Pi}_3 &= \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right)\Pi_1\Pi_2 \doteq a_3\Pi_1\Pi_2 \\
\dot{l}_3 &= u.
\end{aligned}$$

Here, $\lambda_i = I_i + J_i$.

2
Mathematical preliminaries

In this chapter we discuss the following topics:

1. Vector fields, flows and differential equations
2. Differentiable manifolds
3. Riemannian and subRiemannian Geometry
4. Hamiltonian Systems and Symplectic Geometry
5. Lie Groups.
6. Distributions on Manifolds
7. Fiber Bundles, Connections, and Gauge Theory

Of course this is a vast array of topics and we do not pretend to give all the needed background for the reader to learn these things from scratch. However, we hope that this summary will be helpful to set the notation, fill in some gaps the reader may have and to provide a guide to the literature for needed background and proofs.

2.1 Vector fields, flows and differential equations

This section introduces vector fields on Euclidean space and the flows they determine. This topic puts together and globalizes two basic ideas learned in undergraduate mathematics: vector fields and differential equations.

A Basic Example. An example that illustrates many of the concepts of dynamical systems is the ball in a rotating hoop. Refer to figure 2.1.1.

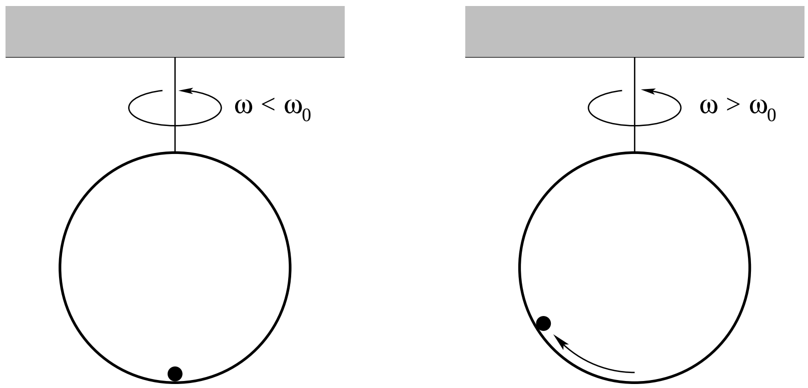


FIGURE 2.1.1. The ball in the hoop system; the equilibrium is stable for $\omega < \omega_c$ and unstable for $\omega > \omega_c$.

This system consists of a rigid hoop that hangs from the ceiling with a small ball resting in the bottom of the hoop. The hoop rotates with frequency ω about a vertical axis through its center (Figure2.1.1(left)).

Consider varying ω, keeping the other parameters fixed. For small values of ω, the ball stays at the bottom of the hoop and that position is stable. Accept this in an intuitive sense for the moment; one has to eventually define this concept carefully. However, when ω reaches the critical value ω_0, this point becomes unstable and the ball rolls up the side of the hoop to a new position $x(\omega)$, which is stable. The ball may roll to the left or to the right, depending, perhaps upon the side of the vertical axis to which it was initially leaning. (see Figure 2.1.1(right)). The position at the bottom of the hoop is still a fixed point, but it has become *unstable*. The solutions to the initial value problem governing the ball's motion are unique for all values of ω. For $\omega > \omega_0$, we cannot predict which way the ball will roll.

Using principles of mechanics which we shall review later, one can show that the equations for this system are given by

$$mR^2\ddot{\theta} = mR^2\omega^2 \sin\theta\cos\theta - mgR\sin\theta - \nu R\dot{\theta},$$

where R is the radius of the hoop, θ is the angle from the bottom vertical, m is the mass of the ball, g is the acceleration due to gravity, and ν is a coefficient of friction.[1] To analyze this system, we use a *phase plane*

[1]This does not represent a realistic friction law, but is an ad hoc one for illustration only; even for this simple problem friction laws are controversial, depending on the

analysis; that is, we write the equation as a system:

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= \frac{g}{R}(\alpha \cos x - 1)\sin x - \beta y\end{aligned}$$

where $\alpha = R\omega^2/g$ and $\beta = \nu/m$. This system of equations produces for each initial point in the xy-plane, a unique trajectory. That is, given a point (x_0, y_0) there is a unique solution $(x(t), y(t))$ of the equation that equals (x_0, y_0) at $t = 0$. This statement is proved by using general existence and uniqueness theory that we shall discuss later. When we draw these curves in the plane, we get figures like those shown in Figure 2.1.2.

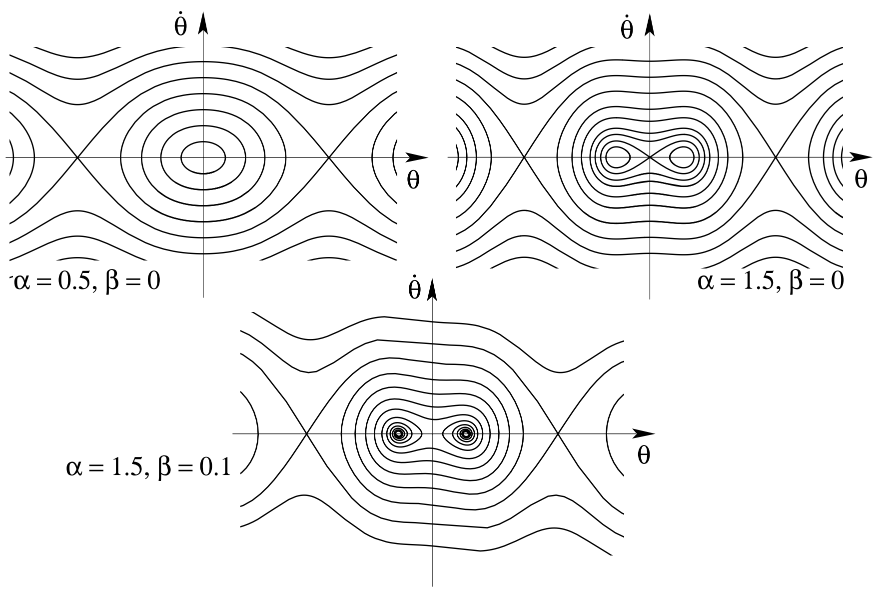


FIGURE 2.1.2. The phase portrait for the ball in the hoop before and after the onset of instability for the case $g/R = 1$.

The above example has ***system parameters*** such as g, R, and ω. In many problems one takes the point of view, as we have done in the preceding discussion, of looking at how the phase portrait of the system changes as the parameters change. Sometimes these parameters are called ***control parameters*** since one imagines changing them. However, this is still a *passive point of view* since we imagine sitting back and watching the dynamics

exact nature of the mechanical system. If one were to suppose Coulomb friction one would make the tangential force proportional to the normal force.

unfold for each value of the parameters. In control theory we take a more active point of view, and try to directly intervene with the dynamics to achieve a desired end. For example, we might imagine manipulating ω as a *function of time* to make the ball move in a desired way.

Dynamical Systems. More generally than in the above example, in Euclidean space $\mathbb{R}^n$ whose points are denoted by $x = (x^1, \ldots, x^n)$, we are concerned with a system of the form

$$\dot{x} = F(x) \tag{2.1.1}$$

which, in components, reads

$$\dot{x}^i = F^i(x^1, \ldots, x^n), \qquad i = 1, \ldots, n$$

where F is an n-component function of the n variables $x^1, \ldots, x^n$. Sometimes the function, or *vector field* F depends on time or on other parameters (such as the mass, angular velocity etc in the example) and keeping track of this explicitly is important as we shall see momentarily. For general dynamical systems one needs some theory to develop properties of solutions; roughly, we draw curves in $\mathbb{R}^n$ emanating from initial conditions, just as we

Equilibrium points, stability and bifurcation. Equilibrium points are points where the right hand side of the system vanish. Mathematically, we say that the original stable fixed point has become *unstable* and has split into two *stable* fixed points. One can use some basic stability theory that we shall develop to show that $\omega_0 = \sqrt{g/R}$. See Figure2.1.2. This is one of the simplest situations in which *symmetric problems can have non-symmetric solutions* and in which *there can be multiple stable equilibria*, so there is non-uniqueness of equilibria (even though the solution of the initial value problem is unique).

This example shows that in some systems the phase portrait can change as certain parameters are changed. Changes in the qualitative nature of phase portraits as parameters are varied are called ***bifurcations***. Consequently, the corresponding parameters are often called ***bifurcation parameters***. These changes can be simple such as the formation of new fixed points—these are called ***static bifurcations*** to ***dynamic bifurcations*** such as the formation of ***periodic orbits***, that is, an orbit $x(t)$ with the property that $x(t + T) = x(t)$ for some T and all t, or more complex dynamical structures.

An important bifurcation called the Hopf bifurcation, or more properly, the Poincaré-Andronov-Hopf bifurcation occurs in this example. This is a dynamic bifurcation in which, roughly speaking, a periodic orbit rather than another fixed point is formed when an equilibrium looses stability. An everyday example of a Hopf bifurcation we all encounter is ***flutter***. For example, when venetian blinds flutter in the wind or a television antenna "sings" in the wind, there is probably a Hopf bifurcation occurring. A related example that is physically easy to understand is flow through a hose: considers a straight vertical rubber tube conveying fluid. The lower end is a nozzle from which the fluid escapes. This is called a ***follower-load*** problem since the water exerts a force on the free end of the tube which follows the movement of the tube. Those with any experience in a garden will not be surprised by the fact that the hose will begin to oscillate if the water velocity is high enough.

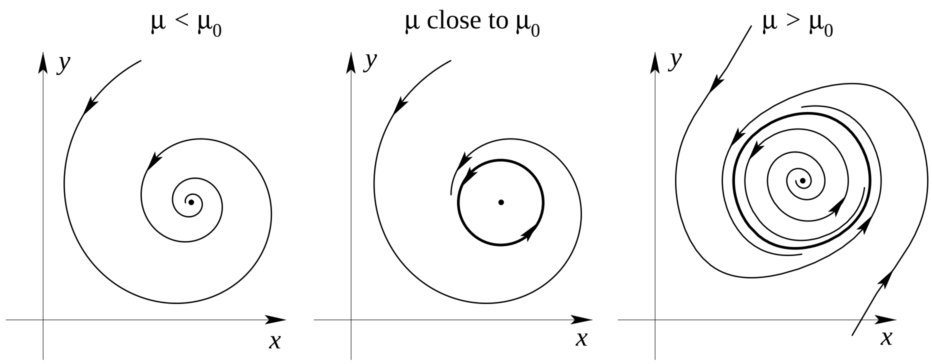


FIGURE 2.1.3. A periodic orbit appears for μ close to μ_0.

Vector fields. With the above example as motivation, we can begin the more formal treatment of vector fields and their associated differential equations. Of course we will eventually add the concept of controls to these vector fields, but we need to understand the notion of vector field itself first.

Definition 2.1.1. *Let $r \geq 0$ be an integer. A C^r **vector field** on $\mathbb{R}^n$ is a mapping $X : U \to \mathbb{R}^n$ of class C^r from an open set $U \subset \mathbb{R}^n$ to $\mathbb{R}^n$. The set of all C^r vector fields on U is denoted by $\mathfrak{X}^r(U)$ and the C^∞ vector fields by $\mathfrak{X}^\infty(U)$ or $\mathfrak{X}(U)$.*

We think of a vector field as assignning to each point $x \in U$ a vector $X(x)$ based (*i.e.,* bound) at that same point.

Example: Newton's law of gravitation. Here the set U is $\mathbb{R}^3$ minus the origin and the vector field is defined by

$$\mathbf{F}(x, y, x) = -\frac{mMG}{r^3}\mathbf{r},$$

where m is the mass of a test body, M is the mass of the central body, G is the constant of gravitation, $\mathbf{r}$ is the vector from the origin to (x, y, z), and $r = (x^2 + y^2 + z^2)^{1/2}$; see Fig. 2.1.4.

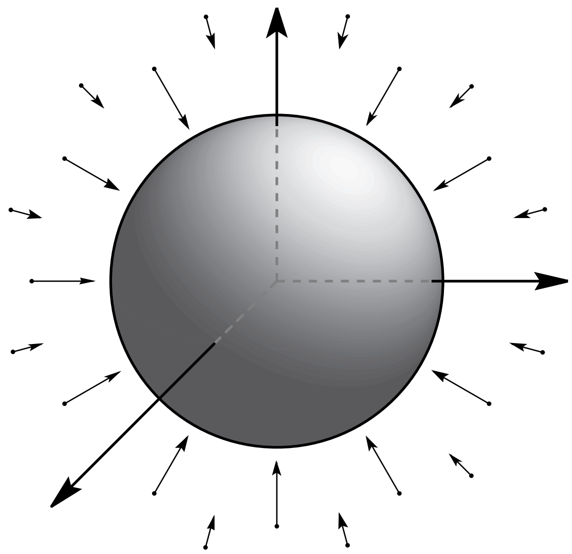


FIGURE 2.1.4. The gravitational force field.

Evolution operators. Consider a general physical system that is capable of assuming various "states" described by points in a set Z. For example, Z might be $\mathbb{R}^3 \times \mathbb{R}^3$ and a state might be the position and momentum $(\mathbf{q}, \mathbf{p})$ of a particle. As time passes, the state evolves. If the state is $z_0 \in Z$ at time t_0 and this changes to z at a later time t, we set

$$F_{t,t_0}(z_0) = z$$

and call F_{t,t_0} the ***evolution operator***; it maps a state at time t_0 to what the state would be at time t; *i.e.,* after time $t - t_0$ has elapsed. "Determinism" is expressed by the law

$$F_{t_2,t_1} \circ F_{t_1,t_0} = F_{t_2,t_0} \quad F_{t,t} = \text{ identity},$$

sometimes called the ***Chapman-Kolmogorov law***.

The evolution laws are called ***time independent*** when F_{t,t_0} depends only on $t - t_0$; *i.e.,*

$$F_{t,t_0} = F_{s,s_0} \quad \text{if} \quad t - t_0 = s - s_0.$$

Setting $F_t = F_{t,0}$, the preceding law becomes the ***group property***:

$$F_\tau \circ F_t = F_{\tau+t}, \quad F_0 = \text{ identity}.$$

We call such an F_t a ***flow*** and F_{t,t_0} a ***time-dependent flow***, or an evolution operator. If the system is defined only for $t \geq 0$, we speak of a ***semi-flow***.

It is usually not F_{t,t_0} that is given, but rather the ***laws of motion***. In other words, differential equations are given that we must solve to find the flow. In general, Z is a manifold (a generalization of a smooth surface), but we confine ourselves here to the case that $Z = U$ is an open set in some Euclidean space $\mathbb{R}^n$. These equations of motion have the form

$$\frac{dx}{dt} = X(x), \quad x(0) = x_0$$

where X is a (possibly time-dependent) vector field on U.

Example: Newton's Second Law. The motion of a particle of mass m under the influence of the gravitational force field is determined by Newton's second law:

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F};$$

i.e., by the ordinary differentatial equations

$$\begin{aligned} m\frac{d^2x}{dt^2} &= -\frac{mMGx}{r^3}; \\ m\frac{d^2y}{dt^2} &= -\frac{mMGy}{r^3}; \\ m\frac{d^2z}{dt^2} &= -\frac{mMGz}{r^3}; \end{aligned}$$

Letting $\mathbf{q} = (x, y, z)$ denote the position and $\mathbf{p} = m(d\mathbf{r}/dt)$ the momentum, these equations become

$$\frac{d\mathbf{q}}{dt} = \frac{\mathbf{p}}{m}; \quad \frac{d\mathbf{p}}{dt} = \mathbf{F}(\mathbf{q})$$

The phase space here is the open set $U = (\mathbb{R}^3 \backslash \{\mathbf{0}\}) \times \mathbb{R}^3$. The right-hand side of the preceding equations define a vector field by

$$X(\mathbf{q}, \mathbf{p}) = ((\mathbf{q}, \mathbf{p}), (\mathbf{p}/m, \mathbf{F}(\mathbf{q}))).$$

In courses on mechanics or differential equations, it is shown how to integrate these equations explicitly, producing trajectories, which are planar conic sections. These trajectories comprise the flow of the vector field.

Relative to a chosen set of Euclidean coordinates, we can identify a vector field X with an n-component vector function $(X^1(x), \ldots, X^n(x))$, the components of X.

Definition 2.1.2. *Let $U \subset \mathbb{R}^n$ be an open set and $X \in \mathfrak{X}^r(U)$ a vector field on U. An* ***integral curve*** *of X with initial condition x_0 is a differentiable curve c defined on some open interval $I \subset \mathbb{R}$ containing 0 such that $c(0) = x_0$ and $c'(t) = X(c(t))$ for each $t \in I$.*

Clearly c is an integral curve of X when the following system of ordinary differential equations is satisfied:

$$\begin{aligned}
\frac{dc^1}{dt}(t) &= X^1(c^1(t), \ldots, c^n(t)) \\
\vdots \qquad & \qquad \vdots \\
\frac{dc^n}{dt}(t) &= X^n(c^1(t), \ldots, c^n(t))
\end{aligned}$$

We shall often write $x(t) = c(t)$, an admitted abuse of notation. The preceding system of equations are called ***autonomous***, when X is time independent. If X were time dependent, time t would appear explicitly on the right-hand side. As we have already seen, the preceding system of equations includes equations of higher order by the usual reduction to first-order systems.

Existence and uniqueness theorems. One of the basic theorems concerning the existence and uniqueness of solutions is the following.

Theorem 2.1.3 (Local Existence, Uniqueness, and Smoothness). *Suppose $U \subset \mathbb{R}^n$ is open and that X is a C^r vector field on U for some $r \geq 1$. For each $x_0 \in U$, there is a curve $c : I \to U$ with $c(0) = x_0$ such that $c'(t) = X(c(t))$ for all $t \in I$. Any two such curves are equal on the intersection of their domains. Furthermore, there is a neighborhood U_0 of the point $x_0 \in U$, a real number $a > 0$, and a C^r mapping $F : U_0 \times I \to U$, where I is the open interval $]-a, a[$, such that the curve $c_u : I \to U$, defined by $c_u(t) = F(u, t)$ is a curve satisfying $c_u(0) = u$ and the differential equations $c_u'(t) = X(c_u(t))$ for all $t \in I$.*

This theorem has many variants. We refer to Coddington and Levinston [1955], and Hartman [1982] and Abraham, Marsden and Ratiu [1988] for proofs.[2]

For example, with just continuity of X one can get existence (the Peano existence theorem) without uniqueness. The equation in one dimension given by $\dot{x} = \sqrt{x}, x(0) = 0$ has the two C^1 solutions $x_1(t) = 0$ and $x_2(t)$ which is defined to be 0 for $t \leq 0$ and $x_2(t) = t^2/4$ for $t > 0$. This shows that one can indeed have existence without uniqueness for continuous vector

[2]Coddington, E.A. and N. Levinson [1955] *Theory of Ordinary Differential Equations.* McGraw-Hill, New York, Hartman, P. [1982] *Ordinary Differential Equations,* Wiley (1964), Second Edition, Birkjäuser, Abraham, R., J.E. Marsden, and T.S. Ratiu [1988] *Manifolds, Tensor Analysis, and Applications.* Second Edition, Applied Mathematical Sciences **75**, Springer-Verlag.This last reference also has a proof based directly on the implicit function theorem applied in suitable function spaces. This proof has a technical advantage: it works easily for other types of differentiability assumptions on X or on F_t, such as Hölder or Sobolev differentiability; this result is due to Ebin, D.G. and J.E. Marsden [1970] Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. Math.* **92**, 102–163.

fields. However, the normal proof starts with a Lipschitz assumption on the vector field and proceeds to show existence and uniqueness in this case by showing that there is a unique solution to the integral equation

$$x(t) = x_0 + \int_0^t X(x(s))\, ds.$$

This is normally done by using the contraction mapping principle on a suitable space of curves or by showing that the sequence of curves given by ***Picard iteration*** converges: let $x_0(t) = x_0$ and define inductively

$$x_{n+1}(t) = x_0 + \int_{t_0}^t X(x_n(s))\, ds.$$

The existence and uniqueness theory also holds if X depends explicitly on t and/or on a parameter ρ, is jointly continuous in (t, ρ, x), and is Lipschitz or class C^r in x uniformly in t and ρ.

Dependence on initial conditions and parameters. The following inequality is of basic importance in not only existence and uniqueness theorems, but also in making estimates on solutions.

Theorem 2.1.4 (Gronwall's Inequality). *Let $f, g : [a, b[\to \mathbb{R}$ be continuous and nonnegative. Suppose there is a constant $A \geq 0$ such that for all t satisfying $a \leq t \leq b$,*

$$f(t) \leq A + \int_a^t f(s)\, g(s)\, ds.$$

Then

$$f(t) \leq A \exp\left(\int_a^t g(s)\, ds \right) \quad \text{for all} \quad t \in [a, b[.$$

We refer to the preceding references for the proof. This lemma is the one of the key ingredients in showing that the solutions depend in a Lipshitz or smooth way on initial conditions. Specifically, let $F_t(x_0)$ denote the solution (= integral curve) of $x'(t) = X(x(t)), x(0) = x_0$. Then for Lipschitz vector fields, $F_t(x)$ depends in a continuous and indeed Lipschitz, manner on the initial condition x and is jointly continuous in (t, x). Again, the same result holds if X depends explicitly on t and on a parameter ρ is jointly continuous in (t, ρ, x), and is Lipschitz in x uniformly in t and ρ. We let $F^\rho_{t,\lambda}(x)$ be the unique integral curve $x(t)$ satisfying $x'(t) = X(x(t), t, \rho)$ and $x(\lambda) = x$. Then $F^\rho_{t,t_0}(x)$ is jointly continuous in the variables (t_0, t, ρ, x), and is Lipschitz in x, uniformly in (t_0, t, ρ).

Additional work along these same lines shows that F_t is C^r if X is. Again, there is an analogous result for the evolution operator $F^\rho_{t,t_0}(x)$ for a time-dependent vector field $X(x, t, \rho)$, which depends on extra parameters ρ in some other Euclidean space, say $\mathbb{R}^m$. If X is C^r, then $F^\rho_{t,t_0}(x)$ is C^r in all variables and is C^{r+1} in t and t_0.

Suspenstion Trick. The variable ρ can be easily dealt with by suspending X to a new vector field obtained by appending the trivial differential equation $\rho' = 0$; this defines a vector field on $\mathbb{R}^n \times \mathbb{R}^m$ and the basic existence and uniqueness theorem may be applied to it. The flow on $\mathbb{R}^n \times \mathbb{R}^m$ is just $F_t(x, \rho) = (F_t^\rho(x), \rho)$.

An interesting result called the *rectification theorem*[3] shows that near a point x_0 satisfying $X(x_0) \neq 0$, the flow can be transformed by a change of variables so that the integral curves become straight lines moving with unit speed. This shows that, in effect, nothing interesting happens with flows away from equilibrium points *as long as one looks at the flow only locally and for short time.*

The mapping F gives a locally unique integral curve c_u for each $u \in U_0$, and for each $t \in I, F_t = F|(U_0 \times \{t\})$ maps U_0 to some other set. It is convenient to think of each point u being allowed to "flow for time t" along the integral curve c_u (see Fig. 2.1.5). This is a picture of a U_0 "flowing," and the system (U_0, a, F) is a local flow of X, or ***flow box***.

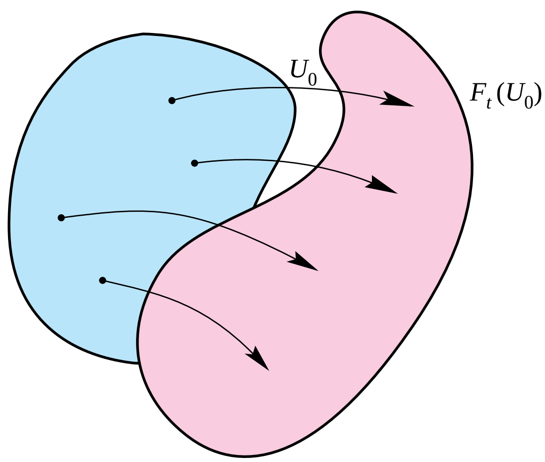


FIGURE 2.1.5. The flow of a vector field

Some global questions. The first global issue concerns uniqueness.

Proposition 2.1.5 (Global Uniqueness). *Suppose c_1 and c_2 are two integral curves of X in U and that for some time t_0, $c_1(t_0) = c_2(t_0)$. Then $c_1 = c_2$ on the intersection of their domains.*

[3]The proof can be found in Arnold, V. I. [1983] *Geometrical Methods in the Theory of Ordinary Differential Equations.* Springer Verlag, and Abraham, R., J.E. Marsden, and T.S. Ratiu [1988] *Manifolds, Tensor Analysis, and Applications.* Second Edition, Applied Mathematical Sciences **75**, Springer-Verlag.

Other global issues center on considering the flow of a vector field as a whole, extended as far as possible in the t-variable.

Definition 2.1.6. *Given an open set U and a vector field X on U, let $\mathcal{D}_X \subset U \times \mathbb{R}$ be the set of $(x, t) \in U \times \mathbb{R}$ such that there is an integral curve $c : I \to U$ of X with $c(0) = x$ with $t \in I$. The vector field X is* ***complete*** *if $\mathcal{D}_X = U \times \mathbb{R}$. A point $x \in U$ is called σ-**complete**, where $\sigma = +, -$, or $\pm$, if $\mathcal{D}_X \cap (\{x\} \times \mathbb{R})$ contains all (x, t) for $t > 0, < 0$, or $t \in \mathbb{R}$, respectively. Let $T^+(x)$ (resp. $T^-(x)$) denote the sup (resp. inf) of the times of existence of the integral curves through x; $T^+(x)$ resp. $T^-(x)$ is called the* ***positive (negative) lifetime of*** *x.*

Thus, X is complete iff each integral curve can be extended so that its domain becomes $]-\infty, \infty[$; *i.e.*, $T^+(x) = \infty$ and $T^-(x) = -\infty$ for all $x \in U$.

Example

A. For $U = \mathbb{R}^2$, let X be the constant vector field $X(x, y) = (0, 1)$. Then X is complete since the integral curve of X through (x, y) is $t \mapsto (x, y + t)$. ♦

B. On $U = \mathbb{R}^2 \backslash \{0\}$, the same vector field is not complete since the integral curve of X through $(0, -1)$ cannot be extended beyond $t = 1$; in fact as $t \to 1$ this integral curve tends to the point $(0, 0)$. Thus $T^+(0, -1) = 1$, while $T^-(0, -1) = -\infty$. ♦

C. On $\mathbb{R}$ consider the vector field $X(x) = 1 + x^2$. This is not complete since the integral curve c with $c(0) = 0$ is $c(\theta) = \tan\theta$ and thus it cannot be continuously extended beyond $-\pi/2$ and $\pi/2$; *i.e.*, $T^{\pm}(0) = \pm\pi/2$. ♦

Proposition 2.1.7. *Let $U \subset \mathbb{R}^n$ be open and $X \in \mathfrak{X}^r(M), r \geq 1$. Then*

i *$\mathcal{D}_X \supset U \times \{0\}$;*

ii *$\mathcal{D}_X$ is open in $U \times \mathbb{R}$;*

iii *there is a unique C^r mapping $F_X : \mathcal{D}_X \to U$ such that the mapping $t \mapsto F_X(x, t)$ is an integral curve at x for all $x \in U$;*

iv *for $(x, t) \in \mathcal{D}_X, (F_X(x, t), s) \in \mathcal{D}_X$ iff $(m, t + s) \in \mathcal{D}_X$; in this case*

$$F_X(x, t + s) = F_X(F_X(x, t), s).$$

Definition 2.1.8. *Let $U \subset \mathbb{R}^n$ be open and $X \in \mathfrak{X}^r(U), r \geq 1$. Then the mapping F_X is called the* ***integral*** *of X, and the curve $t \mapsto F_X(x, t)$ is called the* ***maximal integral curve*** *of X at x. In case X is complete, F_X is called the* ***flow*** *of X.*

Thus, if X is complete with flow F, then the set $\{F_t \mid t \in \mathbb{R}\}$ is a group of diffeomorphisms on U, sometimes called a ***one-parameter group of diffeomorphisms***. Since $F_n = (F_1)^n$ (the n-th power), the notation F^t is sometimes convenient and is used where we use F_t. For incomplete flows, **iv** says that $F_t \circ F_s = F_{t+s}$ wherever it is defined. Note that $F_t(x)$ is defined for $t \in]T^-(x), T^+(x)[$. The reader should write out similar definitions for the time-dependent case and note that the lifetimes depend on the starting time t_0.

A useful criterion for completeness is the following:

Proposition 2.1.9. *Let X be a C^r vector field on an open subset U of $\mathbb{R}^n$, where $r \geq 1$. Let $c(t)$ be a maximal integral curve of X such that for every finite open interval $]a,b[$ in the domain $]T^-(c(0)), T^+(c(0))[$ of $c, c(]a,b[)$ lies in a compact subset of U. Then c is defined for all $t \in \mathbb{R}$. If $U = \mathbb{R}^n$, this holds provided $c(t)$ lies in a bounded set.*

For example, this is used to prove:

Corollary 2.1.10. *A C^r vector field on an open set U with compact support contained in U is complete.*

Completeness corresponds to well-defined dynamics persisting eternally. In some circumstances (shock waves in fluids and solids, singularities in general relativity, etc.) one has to live with incompleteness or overcome it in some other way.

Example A. Let X be a C^r vector field, $r \geq 1$, on the open set $U \subset \mathbb{R}^n$ admitting a ***first integral***, *i.e.,* a function $f : U \to \mathbb{R}$ such that $X[f] = 0$. If all level sets $f^{-1}(r), r \in \mathbb{R}$ are compact, X is complete. Indeed, each integral curve lies on a level set of f so that the result follows by the preceding Proposition. ♦

Example B. Suppose

$$X(x) = A \cdot x + B(x),$$

where A is a linear operator of $\mathbb{R}^n$ to itself and B is ***sublinear***; *i.e.,* $B : \mathbb{R}^n \to \mathbb{R}^n$ is C^r with $r \geq 1$ and satisfies $\|B(x)\| \leq K\|x\| + L$ for constants K and L. We shall show that X is complete. Let $x(t)$ be an integral curve of X on the bounded interval $[0, T]$. Then

$$x(t) = x(0) + \int_0^t (A \cdot x(s) + B(x(s)))\, ds$$

Hence

$$\|x(t)\| \leq \|x(0)\| \int_0^t (\|A\| + K)\|x(s)\|\, ds + Lt.$$

By Gronwall's inequality,

$$\|x(t)\| \leq (LT + \|x(0)\|)e^{(\|A\|+K)t}.$$

Hence $x(t)$ remains bounded on bounded t-intervals, so the result follows by Proposition 2.1.9. ♦

Example C. Newton's equations for a moving particle of mass m in a potential field in $\mathbb{R}^n$ are given by $\ddot{\mathbf{q}}(t) = -(1/m)\nabla V(\mathbf{q}(t))$, for $V : \mathbb{R}^n \to \mathbb{R}$ a smooth function. We shall prove that *if there are constants* $a, b \in \mathbb{R}, b \geq 0$ *such that* $(1/m)V(\mathbf{q}) \geq a - b\|\mathbf{q}\|^2$, *then every solution exists for all time.* To show this, rewrite the second order equations as a first order system $\dot{\mathbf{q}} = (1/m)\mathbf{p}, \dot{\mathbf{p}} = -\nabla V(\mathbf{q})$ and note that the energy

$$E(\mathbf{q}, \mathbf{p}) = (1/2m)\|\mathbf{p}\|^2 + V(\mathbf{q})$$

is a first integral. Thus, for any solution $(\mathbf{q}(t), \mathbf{p}(t))$ we have $\beta = E(\mathbf{q}(t), \mathbf{p}(t)) = E(\mathbf{q}(0), \mathbf{p}(0)) \geq V(\mathbf{q}(0))$. We can assume $\beta > V(\mathbf{q}(0))$, *i.e.,* $\mathbf{p}(0) \neq 0$, for if $\mathbf{p}(t) \equiv 0$, then the conclusion is trivially satisifed; thus there exists a t_0 for which $\mathbf{p}(t_0) \neq 0$ and by time translation we can assume that $t_0 = 0$. Thus we have

$$\begin{aligned}
\|\mathbf{q}(t)\| &\leq \|\mathbf{q}(t) - \mathbf{q}(0)\| + \|\mathbf{q}(0)\| \leq \|\mathbf{q}(0)\| + \int_0^t \|\dot{\mathbf{q}}(s)\|\, ds \\
&= \|\mathbf{q}(0)\| + \int_0^t \sqrt{2\left[\beta - \frac{1}{m}V(\mathbf{q}(s))\right]}\, ds \\
&\leq \|\mathbf{q}(0)\| + \int_0^t \sqrt{2(\beta - a + b\|\mathbf{q}(s)\|^2)}\, ds
\end{aligned}$$

or in differential form

$$\frac{d}{dt}\|\mathbf{q}(t)\| \leq \sqrt{2(\beta - a + b\|\mathbf{q}(t)\|^2)}$$

whence

$$t \leq \int_{\|\mathbf{q}(0)\|}^{\|\mathbf{q}(t)\|} \frac{du}{\sqrt{2(\beta - a + bu^2)}} \tag{2.1.2}$$

Now let $r(t)$ be the solution of the differential equation

$$\frac{d^2r(t)}{dt^2} = -\frac{d}{dr}(a - br^2)(t) = 2br(t),$$

which, as a second order equation with constant coefficients, has solutions for all time for any initial conditions. Choose

$$r(0) = \|\mathbf{q}(0)\|, [\dot{r}(0)]^2 = 2(\beta - a + b\|\mathbf{q}(0)\|^2)$$

and let $r(t)$ be the corresponding solution. Since

$$\frac{d}{dt}\left(\frac{1}{2}\dot{r}(t)^2 + a - br(t)^2\right) = 0,$$

it follows that $(1/2)\dot{r}(t)^2 + a - br(t)2 = (1/2)\dot{r}(0)^2 + a - br(0)^2 = \beta$, *i.e.*,

$$\frac{dr(t)}{dt} = \sqrt{2(\beta - a + br(t)^2)}$$

whence

$$t = \int_{\|\mathrm{q}(0)\|}^{r(t)} \frac{du}{\sqrt{s(\beta - \alpha + \beta u^2}} \tag{2.1.3}$$

Comparing these two expressions and taking into account that the integrand is > 0, it follows that for any finite time interval for which $\mathbf{q}(t)$ is defined, we have $\|\mathbf{q}(t)\| \leq r(t)$, *i.e.*, $\mathbf{q}(t)$ remains in a compact set for finite t-intervals. But then $\dot{\mathbf{q}}(t)$ also lies in a compact set since $\|\dot{\mathbf{q}}(t)\| \leq 2(\beta - a + b\|\mathbf{q}(s)\|^2)$. Thus by 2.1.9, the solution curve $(\mathbf{q}(t), \mathbf{p}(t))$ is defined for any $t \geq 0$. However, since $(\mathbf{q}(-t), \mathbf{p}(-t))$ is the value at t of the integral curve with initial conditions $(-\mathbf{q}(0), -\mathbf{p}(0))$, it follows that the solution also exists for all $t \leq 0$.

The following counterexample shows that the condition $V(\mathbf{q}) \geq a - b\|\mathbf{q}\|^2$ cannot be relaxed much further. Take $n = 1$ and $V(q) = -\varepsilon^2 q^{2+(4/\varepsilon)}/8, \varepsilon > 0$. Then the equation

$$\ddot{q} = \varepsilon(\varepsilon + 2)q^{1+(4/\varepsilon)}/4$$

has the solution $q(t) = 1/(t-1)^{\varepsilon/2}$, which cannot be extended beyond $t = 1$. ♦

The following is proved by a study of the local existence theory; we state it for completeness only.

Proposition 2.1.11. *Let X be a C^r vector field on $U, r \geq 1, x_0 \in U$, and $T^+(x_0)(T^-(x_0))$ the positive (negative) lifetime of x_0. Then for each $\varepsilon > 0$, there exists a neighborhood V of x_0 such that for all $x \in V, T^+(x) > T^+(x_0) - \varepsilon$ (respectively, $T^-(x_0) < T^-(x_0) + \varepsilon$). [One says that $T^+(x_0)$ is a lower semi-continuous function of x.]*

Corollary 2.1.12. *Let X_t be a C^r time-dependent vector field on $U, r \geq 1$, and let x_0 be an **equilibrium** of X_t, i.e., $X_t(x_0) = 0$, for all t. Then for any T there exists a neighborhood V of x_0 such that any $x \in V$ has integral curve existing for time $t \in [-T, T]$.*

Linear equations. Finally we note that the flows of linear equations

$$\dot{x} = Ax$$

where A is an $n \times n$ matrix are given by

$$F_t(x) = e^{tA}$$

where the exponential is defined, for example, by a power series

$$e^{tA} = I + tA + \frac{1}{2}t^2A^2 + \frac{1}{3!}t^3A^3 + \dots .$$

One of course has to show that this series converges and is differentiable in t and that the derivative is given by Ae^{tA}, but this is learned in courses on real analysis. One also learns how to carry out the exponentiation in linear algebra by bringing A into a canonical form. We shall discuss how to do some of these cases in the next sections.

What about stability theory? Should we put anyting in at this point or leave this to later?

2.2 Differentiable Manifolds

Modern analytical mechanics and nonlinear control theory are most naturally discussed in the mathematical language of differential geometry. The present chapter is meant to serve as an introduction to the elements of geometry which we shall use in the remainder of the book. Since this is not primarily a text on geometry, there is a great deal that must be left out.[4]

Studying the motion of physical systems leads immediately to the study of rates of change of position and velocity—i.e. to calculus. Differentiable manifolds provide the most natural setting in which to study calculus. Roughly speaking, differentiable manifolds are topological spaces which locally look like Euclidean space, but which may be globally quite different from Euclidean space. Since taking a derivative involves only a local computation—carried out in a neighborhood of the point of interest, it would appear that derivatives should be computable on any topological

[4] Some references are Abraham, R., J.E. Marsden, and T.S. Ratiu [1988] *Manifolds, Tensor Analysis, and Applications.* Second Edition, Applied Mathematical Sciences **75**, Springer-Verlag, Warner, F.W. [1983] *Foundations of Differentiable Manifolds and Lie Groups.* Graduate Texts in Math. **94** Springer–Verlag, Auslander, L. and R.E. MacKenzie [1977]. *Introduction to Differentiable Manifolds*, Dover Publications, New York, Boothby, W.M. [1992] *An Introduction to Differentiable Manifolds and Riemannian Geometry*, publisher? and Dubrovin, B.A., A.T. Fomenko, and S.P. Novikov [1984] *Modern Geometry – Methods and Applications. Part II. The Geometry and Topology of Manifolds.* Graduate Texts in Math. **104**, Springer–Verlag.

space that is infinitesimally indistinguishable from Euclidean space. This is indeed the case. What makes differentiable manifolds most important in the study of analytical mechanics, however, is the global features and their implications for the large scale behavior of trajectories of physical systems.

With these remarks in mind, we begin with a definition of manifold which relates these objects to Euclidean space in small neighborhoods of each point. Questions about important global features of differentiable manifolds will be discussed in subsequent sections.

Definition 2.2.1. *An n-dimensional* ***differentiable manifold*** *M is a set of points together with a finite or countably infinite set of subsets $U_\alpha \subset M$ and 1-1-mappings $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ such that*

1. *$\bigcup_\alpha U_\alpha = M$.*
2. *For each non-empty intersection $U_\alpha \cap U_\beta$, $\phi_i(U_\alpha \cap U_\beta)$ is an open subset of $\mathbb{R}^n$, and the 1-1-mapping $\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ is a smooth function.*
3. *The family $\{U_\alpha, \phi_\alpha\}$ is maximal with respect to conditions 1 and 2.*

The situation is illustrated in Figure 2.2.1.

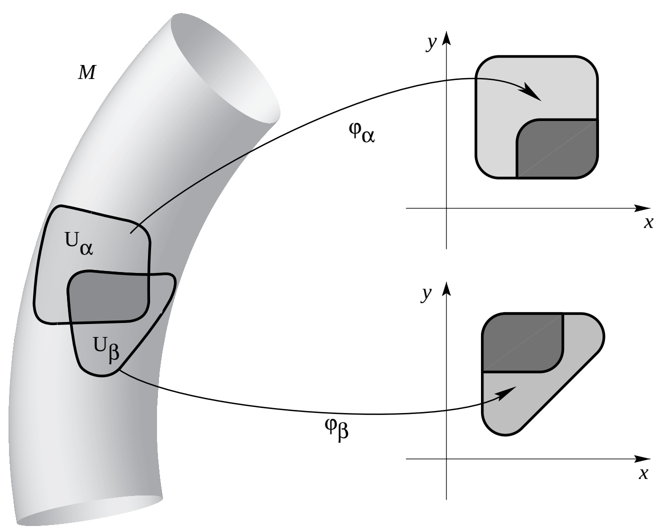


FIGURE 2.2.1. Coordinate charts on a manifold.

A number of remarks are in order. First, the sets U_α in the definition are called ***coordinate neighborhoods*** or ***coordinate charts***. The mappings ϕ_α are called ***coordinate functions*** or ***local coordinates***. A collection of charts satisfying 1 and 2 is called an ***atlas***. The notion of a C^k-differentiable manifold (resp. analytic) is defined similarly wherein the

coordinate transformations $\phi_\alpha \circ \phi_\beta^{-1}$ are only required to have continous partial derivatives of all orders up to k (resp. be analytic). We remark that condition is included merely to make the definition of manifold independent of a choice of atlas. A set of charts satisfying 1 and 2 can always be extended to a maximal set and in practise 1 and 2 define the manifold.

A ***neighborhood*** of a point x in a manifold M is the image under a map $\varphi : U \to M$ of a neighborhood of the representation of x in a chart U. Neighborhoods define open sets and one checks that the open sets in M define a topology. *Usually we assume without explicit mention that the topology is Hausdorff*: two different points x, x' in M have nonintersecting neighborhoods.

A useful viewpoint is to think of M as a set covered by a collection of coordinate charts with local coordinates $(x^1, \dots, x^n)$ with the property that all mutual changes of coordinates are smooth maps.

Differentiable manifolds as level sets in $\mathbb{R}^n$. A typical and important way that manifolds arise is as follows. Let $p_1, p_2, \dots, p_m : \mathbb{R}^n \to \mathbb{R}$. The zero (or level) set (also called a locus)

$$M = \{x \mid p_i(x) = 0, \, i = 1, \dots, m\}$$

is called a ***differentiable variety*** in $\mathbb{R}^n$. If the $n \times m$ matrix

$$\left(\frac{\partial p_1}{\partial x} \vdots \cdots \vdots \frac{\partial p_m}{\partial x} \right)$$

has rank ρ at each point $x \in M$, then M admits the structure of a differentiable manifold of dimension $n - \rho$. The idea of the proof is that under our rank assumption an argument using the implicit function theorem shows that an $n - \rho$-dimensional coordinate chart may be defined in a neighborhood of each point on M. In this situation, we say that the level set is a ***submanifold*** of $\mathbb{R}^n$.

Exercises

◇ **2.2-1.** Using this criterion, show that the level set $x_1^2 + \cdots + x_n^2 - 1 = 0$ is a differentiable manifold of dimension $n - 1$.

◇ **2.2-2.** Show that the variety $\{(x, y) \mid x^2(x + 1) - y^2 = 0\}$ in $\mathbb{R}^2$ is *not* a differentiable manifold.

Matrix groups. Let $\mathbb{R}^{n \times n}$ be the set of $n \times n$ matrices with entries in $\mathbb{R}$, and let $GL(n, \mathbb{R})$ denote the set of all $n \times n$ invertible matrices with entries in $\mathbb{R}$. Clearly, $Gl(n, \mathbb{R})$ is a group, called the ***general linear group***. Let $A \in Gl(n, \mathbb{R})$ be symmetric. Consider the subset

$$\mathcal{S} = \{X \in Gl(n, \mathbb{R}) \mid X \cdot A \cdot X^T = A\}.$$

It is easy to see that if $X \in \mathcal{S}$, then $X^{-1} \in \mathcal{S}$, and if $X, Y \in \mathcal{S}$ then the product $X \cdot Y \in \mathcal{S}$. Hence, *$\mathcal{S}$ is a subgroup of $Gl(n, \mathbb{R})$*.

We can also show that *$\mathcal{S}$ is a submanifold of* $\mathbb{R}^{n\times n}$. Indeed, $\mathcal{S}$ is the zero locus of the mapping $X \mapsto X \cdot A \cdot X^T - A$. Let $X \in \mathcal{S}$, and let δX be an arbitrary element of $\mathbb{R}^{n\times n}$. Then

$$\begin{aligned}(X + \delta X) \cdot A \cdot (X + \delta X)^T - A = \\ X \cdot A \cdot X^T - A + \delta X \cdot A \cdot X + X \cdot A \delta X^T + o(\delta X)^2.\end{aligned}$$

We can conclude that $\mathcal{S}$ is a submanifold of $\mathbb{R}^{n\times n}$ if we can show that the linearization of the locus map, namely the linear mapping L defined by $\delta X \mapsto \delta X \cdot A \cdot X + X \cdot A \delta X^T$ of $\mathbb{R}^{n\times n}$ to itself has constant rank for all $X \in \mathcal{S}$. We see that the image of L lies in the subspace of $n \times n$ symmetric matrices. Hence, rank $L \ \leq (n(n-1))/2$. But given X and any symmetric matrix S we can find δX such that $\delta X \cdot A \cdot X + X \cdot A \delta X^T = S$. This shows that rank $L \ = (n(n-1))/2$, independent of X. In summary, we have established the following fact.

Proposition 2.2.2. *Let $A \in Gl(n, \mathbb{R})$ be symmetric. Then the subgroup $\mathcal{S}$ of $Gl(n, \mathbb{R})$ given by*

$$\mathcal{S} = \{X \in Gl(n, \mathbb{R}) \mid X \cdot A \cdot X^T = A\}$$

is a submanifold of $\mathbb{R}^{n\times n}$ *of dimension $n(n-1)/2$.*

The orthogonal group. Of special interest in mechanics is the case $A = I$. Here $\mathcal{S}$ specializes to $O(n)$, the group of $n \times n$ orthogonal matrices. It is both a subgroup of $Gl(n, \mathbb{R})$ and a submanifold of the vector space $\mathbb{R}^{n\times n}$. $Gl(n, \mathbb{R})$ is an open, dense subset of $\mathbb{R}^{n\times n}$ which inherits the topology and manifold structure from $\mathbb{R}^{n\times n}$. Thus, $O(n)$ (or any $\mathcal{S}$ defined as above) is both a subgroup and a submanifold of $Gl(n, \mathbb{R})$. Subgroups of $Gl(n, \mathbb{R})$ which are also submanifolds are called ***matrix Lie groups***. We shall discuss Lie groups more abstractly later on.

Tangent vectors. Two curves $t \mapsto c_1(t)$ and $t \mapsto c_2(t)$ in an n-manifold M are called ***equivalent at*** $x \in M$ if

$$c_1(0) = c_2(0) = x \quad \text{and} \quad (\varphi \circ c_1)'(0) = (\varphi \circ c_2)'(0)$$

in some chart φ, where the prime denotes the derivative with respect to the curve parameter. It is easy to check that this definition is chart independent. A ***tangent vector*** v to a manifold M at a point $x \in M$ is an equivalence class of curves at x. One proves that the set of tangent vectors to M at x forms a vector space. It is denoted T_xM and is called the ***tangent space*** to M at $x \in M$. Given a curve $c(t)$, we denote by $c'(s)$ the tangent vector at $c(s)$ defined by the equivalence class of $t \mapsto c(s + t)$ at $t = 0$.

Let U be a chart of an atlas for the manifold M with coordinates $(x^1, \ldots, x^n)$. The ***components*** of the tangent vector v to the curve $t \mapsto (\varphi \circ c)(t)$ are the numbers $v^1, \ldots, v^n$ defined by

$$v^i = \frac{d}{dt}(\varphi \circ c)^i \bigg|_{t=0},$$

where $i = 1, \ldots, n$. The ***tangent bundle*** of M, denoted by TM, is the differentiable manifold whose underlying set is the disjoint union of the tangent spaces to M at the points $x \in M$, that is,

$$TM = \bigcup_{x \in M} T_x M.$$

Thus, a point of TM is a vector v that is tangent to M at some point $x \in M$. To define the differentiable structure on TM, we need to specify how to construct local coordinates on TM. To do this, let $x^1, \ldots, x^n$ be local coordinates on M and let $v^1, \ldots, v^n$ be components of a tangent vector in this coordinate system. Then the $2n$ numbers $x^1, \ldots, x^n, v^1, \ldots, v^n$ give a local coordinate system on TM. Notice that $\dim TM = 2 \dim M$.

The ***natural projection*** is the map $\tau_M : TM \to M$ that takes a tangent vector v to the point $x \in M$ at which the vector v is attached (that is, $v \in T_x M$). The inverse image $\tau_M^{-1}(x)$ of a point $x \in M$ under the natural projection τ_M is the tangent space $T_x M$. This space is called the ***fiber*** of the tangent bundle over the point $x \in M$.

Tangent spaces to level sets. Let $M = \{x \mid p_i(x) = 0, i = 1, \ldots, m\}$ be a differentiable variety in $\mathbb{R}^n$. For each $x \in M$,

$$T_x M = \left\{ v \in \mathbb{R}^n \;\middle|\; \frac{\partial p_i}{\partial x}(x) \cdot v = 0 \right\}$$

is called the ***tangent space to*** M ***at*** x. Clearly $T_x M$ is a vector space. If the rank condition holds so that M is a differentiable manifold, then this definition may be shown to be equivalent to the one given earlier. For example, the reader may show that the tangent spaces to spheres are what they should intuitively be, as in Figure 2.2

Tangent spaces to matrix groups. Let $A \in Gl(n, \mathbb{R})$ be a symmetric matrix. We wish to explicitly describe the tangent space at a typical point of the group $\mathcal{S} = \{X \in Gl(n, \mathbb{R}) : X^T A X = A\}$. Given our definition, it is clear that the tangent space $T_X \mathcal{S}$ is a subspace of the space of $n \times n$ matrices, $\mathbb{R}^{n \times n}$. Let $V \in \mathbb{R}^{n \times n}$. Then $V \in T_X \mathcal{S}$ precisely when it is tangent to a curve in the group:

$$\frac{d}{d\epsilon}_{|_{\epsilon=0}} \left[(X + \epsilon V)^T A (X + \epsilon V) - A \right] = 0.$$

This condition is equivalent to $V^T A X + X^T A V = 0$.

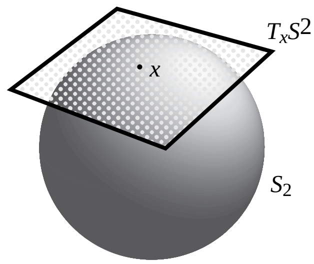


FIGURE 2.2.2. The tangent space to a sphere.

This shows that if $X \in \mathcal{S}$, then

$$T_X\mathcal{S} = \{V \in \mathbb{R}^{n\times n} \mid V^TAX + X^TAV = 0\}.$$

Note that the $n \times n$ identity matrix $I \in \mathcal{S}$, and show that for any pair of matrices $V_1, V_2 \in T_I\mathcal{S}$ we have $V_1V_2 - V_2V_1 \in T_I\mathcal{S}$.

Differentiable maps. Let E and F be vector spaces and let $f : U \subset E \to V \subset F$, where U and V are open sets, be of class C^{r+1}. We define the ***tangent map*** of f, Tf (sometimes denoted f_* by

$$Tf : TU = U \times E \to TV = V \times F$$

where

$$Tf(u, e) = (f(u), Df(u), e).e \in E. \tag{2.2.1}$$

This notion from calculus may be generalized to the context of manifolds as follows. Let $f : M \to N$ be a map of a manifold M to a manifold N. We call f ***differentiable*** (or C^k) if in local coordinates on M and N it is given by differentiable (or C^k) functions. The ***derivative*** of a differentiable map $f : M \to N$ at a point $x \in M$ is defined to be the linear map

$$T_xf : T_xM \to T_{f(x)}N$$

constructed in the following way. For $v \in T_xM$, choose a curve $c :]-\epsilon, \epsilon[\to M$ with $c(0) = x$, and velocity vector $dc/dt\,|_{t=0} = v$. Then $T_xf \cdot v$ is the velocity vector at $t = 0$ of the curve $f \circ c : \mathbb{R} \to N$, that is,

$$T_xf \cdot v = \left.\frac{d}{dt}f(c(t))\right|_{t=0}.$$

The vector $T_xf \cdot v$ does not depend on the curve c but only on the vector v. If M and N are manifolds and $f : M \to N$ is of class C^{r+1}, then $Tf : TM \to TN$ is a mapping of class C^r. Note that

$$\left.\frac{dc}{dt}\right|_{t=0} = T_0c \cdot 1.$$

Vector fields and flows. Let us now interpret what we did with vector fields and flows in the previous section in the context of manifolds. A ***vector field*** X on a manifold M is a map $X : M \to TM$ that assigns a vector $X(x)$ at the point $x \in M$; that is, $\tau_M \circ X = \text{identity}$. An ***integral curve*** of X with initial condition x_0 at $t = 0$ is a (differentiable) map $c :]a, b[\to M$ such that $]a, b[$ is an open interval containing $0, c(0) = x_0$ and

$$c'(t) = X(c(t))$$

for all $t \in]a, b[$. In formal presentations we usually suppress the domain of definition, even though this is technically important. The ***flow*** of X is the collection of maps

$$\varphi_t : M \to M$$

such that $t \mapsto \varphi_t(x)$ is the integral curve of X with initial condition x. Existence and uniqueness theorems from ordinary differential equations, as reviewed in the last section guarantee φ is smooth in x and t (where defined) if X is. From uniqueness, we get the ***flow property***

$$\varphi_{t+s} = \varphi_t \circ \varphi_s$$

along with the initial conditions $\varphi_0 = \text{identity}$. The flow property generalizes the situation where $M = V$ is a *linear* space, $X(x) = Ax$ for a (bounded) *linear* operator A, and where

$$\varphi_t(x) = e^{tA}x$$

to the *nonlinear* case.

Differentials and covectors. If $f : M \to \mathbb{R}$ is a smooth function, we can differentiate it at any point $x \in M$ to obtain a map $T_x f : T_x M \to T_{f(x)}\mathbb{R}$. Identifying the tangent space of $\mathbb{R}$ at any point with itself (a process we usually do in any vector space), we get a linear map $\mathbf{d}f(x) : T_x M \to \mathbb{R}$. That is, $\mathbf{d}f(x) \in T_x^* M$, the dual of the vector space $T_x M$.

In coordinates, the ***directional derivatives*** defined by $\mathbf{d}f(x) \cdot v$, where $v \in T_x M$, are given by

$$\mathbf{d}f(x) \cdot v = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} v^i.$$

We will employ the ***summation convention*** and drop the summation sign when there are repeated indices. We also call $\mathbf{d}f$ the ***differential*** of f.

One can show that specifying the directional derivatives completely determines a vector, and so we can identify a basis of $T_x M$ using the operators $\partial/\partial x^i$. We write

$$(e_1, \ldots, e_n) = \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n} \right)$$

for this basis so that $v = v^i \partial/\partial x^i$.

If we replace each vector space T_xM with its dual T_x^*M, we obtain a new $2n$-manifold called the ***cotangent bundle*** and denoted T^*M. The dual basis to $\partial/\partial x^i$ is denoted dx^i. Thus, relative to a choice of local coordinates we get the basic formula

$$\mathbf{d}f(x) = \frac{\partial f}{\partial x^i} dx^i$$

for any smooth function $f : M \to \mathbb{R}$.

Exercises

◇ **2.2-3.** If $\varphi_t : S^2 \to S^2$ rotates points on S^2 about a fixed axis through an angle t, show that φ_t is the flow of a certain vector field on S^2.

◇ **2.2-4.** Let $f : S^2 \to \mathbb{R}$ be defined by $f(x, y, z) = z$. Compute $\mathbf{d}f$ relative to spherical coordinates (θ, φ).

2.3 Differential Forms

We next review some of the basic definitions, properties, and operations on differential forms, without proofs (see Abraham, Marsden, and Ratiu [1988] and references therein). *The main idea of differential forms is to provide a generalization of the basic operations of vector calculus,* div*,* grad*, and* curl*, and the integral theorems of Green, Gauss, and Stokes to manifolds of arbitrary dimension.*

Basic definitions. A ***2-form*** Ω on a manifold M is a function $\Omega(x) : T_xM \times T_xM \to \mathbb{R}$ that assigns to each point $x \in M$ a skew-symmetric bilinear form on the tangent space T_xM to M at x. More generally, a ***k-form*** α (sometimes called a ***differential form of degree*** k) on a manifold M is a function $\alpha(x) : T_xM \times \ldots \times T_xM$ (there are k factors) $\to \mathbb{R}$ that assigns to each point $x \in M$ a skew-symmetric k-multilinear map on the tangent space T_xM to M at x. Without the skew-symmetry assumption, α would be called a $(0, k)$-***tensor***. A map $\alpha : V \times \ldots \times V$ (there are k factors) $\to \mathbb{R}$ is ***multilinear*** when it is linear in each of its factors, that is,

$$\alpha(v_1, \ldots, av_j + bv_j', \ldots, v_k) = a\alpha(v_1, \ldots, v_j, \ldots, v_k) + b\alpha(v_1, \ldots, v_j', \ldots, v_k)$$

for all j with $1 \leq j \leq k$. A k-multilinear map $\alpha : V \times \ldots \times V \to \mathbb{R}$ is ***skew*** (or ***alternating***) when it changes sign whenever two of its arguments are interchanged, that is, for all $v_1, \ldots, v_k \in V$,

$$\alpha(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\alpha(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k).$$

Let $x^1, \ldots, x^n$ denote coordinates on M, let

$$\{e_1, \ldots, e_n\} = \{\partial/\partial x^1, \ldots, \partial/\partial x^n\}$$

be the corresponding basis for T_xM, and let $\{e^1, \ldots, e^n\} = \{dx^1, \ldots, dx^n\}$ be the dual basis for T_x^*M. Then at each $x \in M$, we can write a 2-form as

$$\Omega_x(v, w) = \Omega_{ij}(x)v^i w^j, \quad \text{where} \quad \Omega_{ij}(x) = \Omega_x\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right),$$

and more generally a k-form can be written

$$\alpha_x(v_1, \ldots, v_k) = \alpha_{i_1 \ldots i_k}(x) v_1^{i_1} \ldots v_k^{i_k},$$

where there is a sum on $i_1, \ldots, i_k$ and where

$$\alpha_{i_1 \ldots i_k}(x) = \alpha_x\left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}}\right),$$

and where $v_i = v_i^j \partial/\partial x^j$, with a sum on j.

Tensor and wedge products. If α is a $(0,k)$-tensor on a manifold M, and β is a $(0,l)$-tensor, their ***tensor product*** $\alpha \otimes \beta$ is the $(0, k+l)$-tensor on M defined by

$$(\alpha \otimes \beta)_x(v_1, \ldots, v_{k+l}) = \alpha_x(v_1, \ldots, v_k)\beta_x(v_{k+1}, \ldots, v_{k+l}) \qquad (2.3.1)$$

at each point $x \in M$.

If t is a $(0,p)$-tensor, define the ***alternation operator*** $\mathbf{A}$ acting on t by

$$\mathbf{A}(t)(v_1, \ldots, v_p) = \frac{1}{p!} \sum_{\pi \in S_p} \operatorname{sgn}(\pi) t(v_{\pi(1)}, \ldots, v_{\pi(p)}), \qquad (2.3.2)$$

where $\operatorname{sgn}(\pi)$ is the ***sign*** of the permutation π:

$$\operatorname{sgn}(\pi) = \begin{cases} +1 & \text{if } \pi \text{ is even,} \\ -1 & \text{if } \pi \text{ is odd,} \end{cases} \qquad (2.3.3)$$

and S_p is the group of all permutations of the numbers $1, 2, \ldots, p$. The operator $\mathbf{A}$ therefore skew-symmetrizes p-multilinear maps.

If α is a k-form and β is an l-form on M, their ***wedge product*** $\alpha \wedge \beta$ is the $(k+l)$-form on M defined by[5]

$$\alpha \wedge \beta = \frac{(k+l)!}{k!\, l!} \mathbf{A}(\alpha \otimes \beta). \qquad (2.3.4)$$

[5] The numerical factor in (2.3.4) agrees with the convention of Abraham and Marsden [1978], Abraham, Marsden, and Ratiu [1988], and Spivak [1976], but *not* that of Arnold [1989], Guillemin and Pollack [1974], or Kobayashi and Nomizu [1963]; it is the Bourbaki [1971] convention.

For example, if α and β are one-forms,

$$(\alpha \wedge \beta)(v_1, v_2) = \alpha(v_1)\beta(v_2) - \alpha(v_2)\beta(v_1)$$

while if α is a 2-form and β is a 1-form,

$$(\alpha \wedge \beta)(v_1, v_2, v_3) = \alpha(v_1, v_2)\beta(v_3) + \alpha(v_3, v_1)\beta(v_2) + \alpha(v_2, v_3)\beta(v_1).$$

We state the following without proof:

Proposition 2.3.1. *The wedge product has the following properties:*

i *$\alpha \wedge \beta$ is **associative**: $\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma$.*

ii *$\alpha \wedge \beta$ is **bilinear** in α, β :*

$$\begin{aligned} (a\alpha_1 + b\alpha_2) \wedge \beta &= a(\alpha_1 \wedge \beta) + b(\alpha_2 \wedge \beta), \\ \alpha \wedge (c\beta_1 + d\beta_2) &= c(\alpha \wedge \beta_1) + d(\alpha \wedge \beta_2). \end{aligned}$$

iii *$\alpha \wedge \beta$ is **anticommutative**: $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$, where α is a k-form and β is an l-form.*

In terms of the dual basis dx^i, any k-form can be written locally as

$$\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where the sum is over all i_j satisfying $i_1 < \cdots < i_k$.

Pull back and push forward. Let $\varphi : M \to N$ be a C^∞ map from the manifold M to the manifold N and α be a k-form on N. Define the ***pull back*** $\varphi^* \alpha$ of α by φ to be the k-form on M given by

$$(\varphi^* \alpha)_x(v_1, \dots, v_k) = \alpha_{\varphi(x)}(T_x\varphi \cdot v_1, \dots, T_x\varphi \cdot v_k). \tag{2.3.5}$$

If φ is a diffeomorphism, the ***push forward*** φ_* is defined by $\varphi_* = (\varphi^{-1})^*$.

Here is another basic property.

Proposition 2.3.2. *The pull back of a wedge product is the wedge product of the pull backs:*

$$\varphi^*(\alpha \wedge \beta) = \varphi^* \alpha \wedge \varphi^* \beta. \tag{2.3.6}$$

Interior products and exterior derivatives. Let α be a k-form on a manifold M and X a vector field. The ***interior product*** $\mathbf{i}_X \alpha$ (sometimes called the contraction of X and α, and written $\mathbf{i}(X)\alpha$) is defined by

$$(\mathbf{i}_X \alpha)_x(v_2, \dots, v_k) = \alpha_x(X(x), v_2, \dots, v_k). \tag{2.3.7}$$

Proposition 2.3.3. *Let α be a k-form and β an l-form on a manifold M. Then*

$$\mathbf{i}_X(\alpha \wedge \beta) = (\mathbf{i}_X\alpha) \wedge \beta + (-1)^k \alpha \wedge (\mathbf{i}_X\beta). \tag{2.3.8}$$

The ***exterior derivative*** $\mathbf{d}\alpha$ of a k-form α on a manifold M is the $(k+1)$-form on M determined by the following proposition:

Proposition 2.3.4. *There is a unique mapping $\mathbf{d}$ from k-forms on M to $(k+1)$-forms on M such that:*

i *If α is a 0- form ($k = 0$), that is, $\alpha = f \in C^\infty(M)$, then $\mathbf{d}f$ is the one-form which is the differential of f.*

ii *$\mathbf{d}\alpha$ is **linear** in α, that is, for all real numbers c_1 and c_2,*

$$\mathbf{d}(c_1\alpha_1 + c_2\alpha_2) = c_1\mathbf{d}\alpha_1 + c_2\mathbf{d}\alpha_2.$$

iii *$\mathbf{d}\alpha$ satisfies the **product rule**, that is,*

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta,$$

where α is a k-form and, β is an l-form.

iv *$\mathbf{d}^2 = 0$, that is, $\mathbf{d}(\mathbf{d}\alpha) = 0$ for any k-form α.*

v *$\mathbf{d}$ is a **local operator**, that is, $\mathbf{d}\alpha(x)$ only depends on α restricted to any open neighborhood of x; in fact, if U is open in M, then*

$$\mathbf{d}(\alpha|U) = (\mathbf{d}\alpha)|U.$$

If α is a k-form given in coordinates by

$$\alpha = \alpha_{i_1 \dots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad (\text{sum on } i_1 < \cdots < i_k),$$

then the coordinate expression for the exterior derivative is

$$\mathbf{d}\alpha = \frac{\partial \alpha_{i_1 \dots i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \quad (\text{sum on all } j \text{ and } i_1 < \cdots < i_k). \tag{2.3.9}$$

Formula (2.3.9) can be taken as the definition of the exterior derivative, provided one shows that (2.3.9) has the above-described properties and, correspondingly, is independent of the choice of coordinates.

Next is a useful proposition that, in essence, rests on the chain rule:

Proposition 2.3.5. *Exterior differentiation commutes with pull back, that is,*

$$\mathbf{d}(\varphi^*\alpha) = \varphi^*(\mathbf{d}\alpha), \tag{2.3.10}$$

where α is a k-form on a manifold N and φ is a smooth map from a manifold M to N.

A k-form α is called ***closed*** if $\mathbf{d}\alpha = 0$ and ***exact*** if there is a $(k-1)$-form β such that $\alpha = \mathbf{d}\beta$. By Proposition 2.3.4 every exact form is closed. The exercises give an example of a closed nonexact one-form.

Proposition 2.3.6 (Poincaré Lemma). *A closed form is locally exact, that is, if* $\mathbf{d}\alpha = 0$ *there is a neighborhood about each point on which* $\alpha = \mathbf{d}\beta$.

The proof is given in the exercises.

Vector calculus. The table below entitled "Vector calculus and differential forms" summarizes how forms are related to the usual operations of vector calculus. We now elaborate on a few items in this table. In item 4, note that

$$\mathbf{d}f = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = (\mathrm{grad} f)^\flat = (\nabla f)^\flat$$

which is equivalent to $\nabla f = (\mathbf{d}f)^\sharp$.

The Hodge star operator on $\mathbb{R}^3$ maps k-forms to $(3-k)$-forms and is uniquely determined by linearity and the properties in item 2.[6]

In item 5, if we let $F = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$, so $F^\flat = F_1\,dx + F_2\,dy + F_3\,dz$, then,

$$\begin{aligned}
\mathbf{d}(F^\flat) &= \mathbf{d}F_1 \wedge dx + F_1\mathbf{d}(dx) + \mathbf{d}F_2 \wedge dy + F_2\mathbf{d}(dy) + \mathbf{d}F_3 \wedge dz \\
&\qquad + F_3\mathbf{d}(dz) \\
&= \left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz\right) \wedge dx \\
&\qquad + \left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz\right) \wedge dy \\
&\qquad + \left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz\right) \wedge dz \\
&= -\frac{\partial F_1}{\partial y}dx \wedge dy + \frac{\partial F_1}{\partial z}dz \wedge dx + \frac{\partial F_2}{\partial x}dx \wedge dy - \frac{\partial F_2}{\partial z}dy \wedge dz \\
&\qquad - \frac{\partial F_3}{\partial x}dz \wedge dx + \frac{\partial F_3}{\partial y}dy \wedge dz \\
&= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx \wedge dy + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dz \wedge dx \\
&\qquad + \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz.
\end{aligned}$$

[6] This operator can be defined on general Riemannian manifolds; see Abraham, Marsden, and Ratiu [1988].

Hence, using item 2,

$$
\begin{aligned}
*(\mathbf{d}(F^\flat)) &= \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dz + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) dy \\
&\quad + \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dx, \\
(*(\mathbf{d}(F^\flat)))^\sharp &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \mathbf{e}_1 + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \mathbf{e}_2 \\
&\quad + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathbf{e}_3 \\
&= \operatorname{curl} F = \nabla \times F.
\end{aligned}
$$

With reference to item 6, let $F = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + F_3\mathbf{e}_3$, so $F^\flat = F_1\,dx + F_2\,dy + F_3\,dz$. Thus $*(F^\flat) = F_1\,dy \wedge dz + F_2(-dx \wedge dz) + F_3\,dx \wedge dy$, and so

$$
\begin{aligned}
\mathbf{d}(*(F^\flat)) &= \mathbf{d}F_1 \wedge dy \wedge dz - \mathbf{d}F_2 \wedge dx \wedge dz + \mathbf{d}F_3 \wedge dx \wedge dy \\
&= \left(\frac{\partial F_1}{\partial x}dx + \frac{\partial F_1}{\partial y}dy + \frac{\partial F_1}{\partial z}dz\right) \wedge dy \wedge dz \\
&\quad - \left(\frac{\partial F_2}{\partial x}dx + \frac{\partial F_2}{\partial y}dy + \frac{\partial F_2}{\partial z}dz\right) \wedge dx \wedge dz \\
&\quad + \left(\frac{\partial F_3}{\partial x}dx + \frac{\partial F_3}{\partial y}dy + \frac{\partial F_3}{\partial z}dz\right) \wedge dx \wedge dy \\
&= \frac{\partial F_1}{\partial x}dx \wedge dy \wedge dz + \frac{\partial F_2}{\partial y}dx \wedge dy \wedge dz + \frac{\partial F_3}{\partial z}dx \wedge dy \wedge dz \\
&= \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx \wedge dy \wedge dz = (\operatorname{div}\ F)\, dx \wedge dy \wedge dz.
\end{aligned}
$$

Therefore $*(\mathbf{d}(*(F^\flat))) = \operatorname{div}\ F = \nabla \cdot F$.

The definition and properties of vector-valued forms are direct extensions of these for usual forms on vector spaces and manifolds. One can think of a vector-valued form as an array of usual forms.

Vector Calculus and Differential Forms

1. **Sharp and Flat** (Using standard coordinates in $\mathbb{R}^3$)
   (a) $v^\flat = v^1\,dx + v^2\,dy + v^3\,dz$ = one-form corresponding to the vector $v = v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3$.
   (b) $\alpha^\sharp = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$ = vector corresponding to the one-form $\alpha = \alpha_1\,dx + \alpha_2\,dy + \alpha_3\,dz$.
2. **Hodge Star Operator**
   (a) $*1 = dx \wedge dy \wedge dz$.

(b) $*dx = dy \wedge dz$, $*dy = -dx \wedge dz$, $*dz = dx \wedge dy$,
$*(dy \wedge dz) = dx$, $*(dx \wedge dz) = -dy$, $*(dx \wedge dy) = dz$.

(c) $*(dx \wedge dy \wedge dz) = 1$.

3. **Cross Product and Dot Product**

(a) $v \times w = [*(v^\flat \wedge w^\flat)]^\sharp$.

(b) $(v \cdot w) dx \wedge dy \wedge dz = v^\flat \wedge *(w^\flat)$.

4. **Gradient** $\nabla f = \mathrm{grad} f = (\mathbf{d} f)^\sharp$.

5. **Curl** $\nabla \times F = \mathrm{curl}\, F = [*(\mathbf{d}F^\flat)]^\sharp$.

6. **Divergence** $\nabla \cdot F = \mathrm{div}\ F = *\mathbf{d}(*F^\flat)$.

Exercises

◇ **2.3-1.** Let $\varphi : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $\varphi(x, y, z) = (x + z, xy)$. For $\alpha = e^v\, du + u\, dv \in \Omega^1(\mathbb{R}^2)$ and $\beta = u\, du \wedge dv$ compute $\alpha \wedge \beta, \varphi^*\alpha, \varphi^*\beta$, and $\varphi^*\alpha \wedge \varphi^*\beta$.

◇ **2.3-2.** Given

$$\alpha = y^2\, dx \wedge dz + \sin(xy)\, dx \wedge dy + e^x\, dy \wedge dz \in \Omega^2(\mathbb{R}^3)$$

and

$$X = 3\partial/\partial x + \cos z \partial/\partial y - x^2 \partial/\partial z \in \mathfrak{X}(\mathbb{R}^3),$$

compute $\mathbf{d}\alpha$ and $\mathbf{i}_X\alpha$.

◇ **2.3-3.** **(a)** Denote by $\Lambda^k(\mathbb{R}^n)$ the vector space of all skew-symmetric k-linear maps on $\mathbb{R}^n$. Prove that this space has dimension $n!/k!\,(n-k)!$ by showing that a basis is given by $\{e^{i_1} \wedge \cdots \wedge e^{i_k} \mid i_1 < \ldots < i_k\}$ where $\{e_1, \ldots, e_n\}$ is a basis of $\mathbb{R}^n$ and $\{e^1, \ldots, e^n\}$ is its dual basis, that is, $e^i(e_j) = \delta^i_j$.

(b) If $\mu \in \Lambda^n(\mathbb{R}^n)$ is nonzero, prove that the map $v \in \mathbb{R}^n \mapsto \mathbf{i}_v\mu \in \Lambda^{n-1}(\mathbb{R}^n)$ is an isomorphism.

(c) If M is a smooth n-manifold and $\mu \in \Omega^n(M)$ is nowhere vanishing (in which case it is called a volume form), show that the map $X \in \mathfrak{X}(M) \mapsto \mathbf{i}_X\mu \in \Omega^{n-1}(M)$ is a module isomorphism over $\mathcal{F}(M)$.

◇ **2.3-4.** Let $\alpha = \alpha_i\, dx^i$ be a closed one-form in a ball around the origin in $\mathbb{R}^n$. Show that $\alpha = \mathbf{d}f$ for

$$f(x^1, \ldots, x^n) = \int_0^1 \alpha_j(tx^1, \ldots, tx^n) x^j\, dt.$$

◇ **2.3-5.** **(a)** Let U be an open ball around the origin in $\mathbb{R}^n$ and $\alpha \in \Omega^k(U)$ a closed form. Verify that $\alpha = \mathbf{d}\beta$ where

$$\begin{aligned}&\beta(x^1, \ldots, x^n)\\&\quad = \left(\int_0^1 t^{k-1}\alpha_{ji_1\ldots i_{k-1}}(tx^1, \ldots, tx^n)x^j\, dt\right) dx^{i_1} \wedge \ldots \wedge dx^{i_{k-1}},\end{aligned}$$

and where the sum is over $i_1 < \cdots < i_{k-1}$. Here, $\alpha = \alpha_{j_1\ldots j_k}\, dx^{j_1} \wedge \ldots \wedge dx^{j_k}$, where $j_1 < \cdots < j_k$ and where α is extended to be skew-symmetric in its lower indices.

(b) Deduce the Poincaré lemma from **(a).**

2.4 Lie derivatives

The *dynamic definition* of the Lie derivative is as follows. Let α be a k-form and let X be a vector field with flow φ_t. The ***Lie derivative*** of α along X is given by

$$\pounds_X\alpha = \lim_{t\to 0}\frac{1}{t}[(\varphi_t^*\alpha) - \alpha] = \left.\frac{d}{dt}\varphi_t^*\alpha\right|_{t=0}. \tag{2.4.1}$$

This definition together with properties of pull-backs yields the following.

Theorem 2.4.1 (Lie Derivative Theorem).

$$\frac{d}{dt}\varphi_t^*\alpha = \varphi_t^*\pounds_X\alpha. \tag{2.4.2}$$

This formula holds also for *time-dependent* vector fields.

If f is a real-valued function on a manifold M and X is a vector field on M, the ***Lie derivative of f along X*** is the ***directional derivative***

$$\pounds_X f = X[f] := \mathbf{d}f \cdot X. \tag{2.4.3}$$

In coordinates on M,

$$\pounds_X f = X^i\frac{\partial f}{\partial x^i}. \tag{2.4.4}$$

If Y is a vector field on a manifold N and $\varphi : M \to N$ is a diffeomorphism, the ***pull back*** φ^*Y is a vector field on M defined by

$$(\varphi^*Y)(x) = T_x\varphi^{-1} \circ Y \circ \varphi(x). \tag{2.4.5}$$

Two vector fields X on M and Y on N are said to be ***φ-related*** if

$$T\varphi \circ X = Y \circ \varphi. \tag{2.4.6}$$

Clearly, if $\varphi : M \to N$ is a diffeomorphism and Y is a vector field on N, φ^*Y and Y are φ-related. For a diffeomorphism φ, the ***push forward*** is defined, as for forms, by $\varphi_* = (\varphi^{-1})^*$.

Jacobi–Lie brackets. If M is finite dimensional and C^∞ then the set of vector fields on M coincides with the set of derivations on $\mathcal{F}(M)$. The same result is true for C^k manifolds and vector fields if $k \geq 2$. [7] If M is C^∞ and smooth, then the derivation $f \mapsto X[Y[f]] - Y[X[f]]$, where $X[f] = \mathbf{d}f \cdot X$, determines a unique vector field denoted by $[X, Y]$ and called the ***Jacobi-Lie bracket*** of X and Y. Defining $\pounds_X Y = [X, Y]$ gives the ***Lie derivative*** of Y along X. Then the Lie derivative theorem holds with α replaced by Y.

In coordinates,

$$(\pounds_X Y)^j = X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} = (X \cdot \nabla) Y^i - (Y \cdot \nabla) X^j, \tag{2.4.7}$$

and in general, where we identify X, Y with their local representatives

$$[X, Y] = \mathbf{D}Y \cdot X - \mathbf{D}X \cdot Y. \tag{2.4.8}$$

The Lie bracket has the following geometric meaning: see e.g. Brockett [1978]. Let the differential equations describing the vector fields X and Y be given by

$$\dot{x} = f \tag{2.4.9}$$

$$\dot{x} = g\,, \tag{2.4.10}$$

where the components of f and g are X_i and Y_i respectively. The local integral curves of f through x_0 may be then be written as $(\exp tf)x_0$ and similarly for g. Suppose now we follow the integral curve of X from the point x_0 for t units of time then Y for t units, then $-X$ for t units and finally $-Y$ for t units. We arrive then at the point $(\exp{-tg})(\exp{-tf})(\exp tg)(\exp tf)x_0$, Using the local second order expansion

$$x(t) = x_0 + tf(x_0) + t^2/2 \frac{\partial f}{\partial x} f(x_0) + O(t^3) \tag{2.4.11}$$

shows that $x(t)$, after following the sequence of paths described above, is given by

$$x(t) = x_0 + (t^2/2)[g, f]x_0 + O(t^3)\,,$$

where

$$[g, f]^i = [Y, X]^i\,.$$

The reader may verify this expression as an exercise.

[7] This property is false for infinite-dimensional manifolds; see Abraham, Marsden, Ratiu [1988].

Thus, if the Lie bracket of X and Y is not a linear combination of X and Y, it represents a "new" direction in which one can move on the manifold, by following a path such as the one describe above.

In general if a set of vector fields X_i is such that there exist constants γ_{ijk} such that

$$[X_i, X_j] = \gamma_{ijk} X_k(x)$$

the set is said to be ***involutive***.

Algebraic definiton of the Lie derivative. The *algebraic approach* to the Lie derivative on forms or tensors proceeds as follows. Extend the definition of the Lie derivative from functions and vector fields to differential forms, by requiring that the Lie derivative is a derivation; for example, for one-forms α, write

$$\pounds_X \langle \alpha, Y \rangle = \langle \pounds_X \alpha, Y \rangle + \langle \alpha, \pounds_X Y \rangle \,, \tag{2.4.12}$$

where X, Y are vector fields and $\langle \alpha, Y \rangle = \alpha(Y)$. More generally,

$$\pounds_X (\alpha(Y_1, \dots, Y_k)) = (\pounds_X \alpha)(Y_1, \dots, Y_k) + \sum_{i=1}^{k} \alpha(Y_1, \dots, \pounds_X Y_i, \dots, Y_k), \tag{2.4.13}$$

where $X, Y_1, \dots, Y_k$ are vector fields and α is a k-form.

Proposition 2.4.2. *The dynamic and algebraic definitions of the Lie derivative of a differential k-form are equivalent.*

Cartan's Magic Formula. A very important formula for the Lie derivative is given by the following.

Theorem 2.4.3. *For X a vector field and α a k-form on a manifold M, we have*

$$\pounds_X \alpha = \mathbf{d}\mathbf{i}_X \alpha + \mathbf{i}_X \mathbf{d}\alpha. \tag{2.4.14}$$

This is proved by a lengthy but straightforward calculation.

Another property of the Lie derivative is the following: if $\varphi : M \to N$ is a diffeomorphism,

$$\varphi^* \pounds_Y \beta = \pounds_{\varphi^* Y} \varphi^* \beta$$

for $Y \in \mathfrak{X}(N), \beta \in \Omega^k(M)$. More generally, if $X \in \mathfrak{X}(M)$ and $Y \in \mathfrak{X}(N)$ are ψ related, that is, $T\psi \circ X = Y \circ \psi$ for $\psi : M \to N$ a smooth map, then $\pounds_X \psi^* \beta = \psi^* \pounds_Y \beta$ for all $\beta \in \Omega^k(N)$.

Volumes forms and divergence. An n-manifold M is said to be ***orientable*** if there is a nowhere vanishing n-form μ on it; μ is called a ***volume form*** and it is a basis of $\Omega^n(M)$ over $\mathcal{F}(M)$. Two volume forms μ_1 and μ_2 on M are said to define the same ***orientation*** if there is an $f \in \mathcal{F}(M)$, with $f > 0$ and such that $\mu_2 = f\mu_1$. Connected orientable manifolds admit precisely two orientations. A basis $\{v_1, \ldots v_n\}$ of T_mM is said to be ***positively oriented*** relative to the volume form μ on M if $\mu(m)(v_1, \ldots, v_n) > 0$. Note that the volume forms defining the same orientation form a convex cone in $\Omega^n(M)$, that is, if $a > 0$ and μ is a volume form, then $a\mu$ is again a volume form and if $t \in [0,1]$ and μ_1, μ_2 are volume forms, then $t\mu_1 + (1-t)\mu_2$ is again a volume form. The first property is obvious. To prove the second, let $m \in M$ and let $\{\sigma_1, \ldots \sigma_n\}$ be a positively oriented basis of T_mM relative to the orientation defined by μ_1, or equivalently (by hypothesis) by μ_2. Then $\mu_1(m)(v_1, \ldots, v_n) > 0, \mu_2(m)(v_1, \ldots, v_n) > 0$ so that their convex combination is again strictly positive.

If $\mu \in \Omega^n(M)$ is a volume form, since $\pounds_X\mu \in \Omega^n(M)$ there is a function, called the ***divergence*** of X relative to μ and denoted $\operatorname{div}_\mu(X)$ or simply $\operatorname{div}(X)$, such that

$$\pounds_X\mu = \operatorname{div}_\mu(X)\mu. \tag{2.4.15}$$

From the dynamic approach to Lie derivatives it follows that $\operatorname{div}_\mu(X) = 0$ iff $F_t^*\mu = \mu$, where F_t is the flow of X. This condition says that F_t is ***volume preserving***. If $\varphi : M \to M$, since $\varphi^*\mu \in \Omega^n(M)$ there is a function, called the ***Jacobian*** of φ and denoted $J_\mu(\varphi)$ or simply $J(\varphi)$, such that

$$\varphi^*\mu = J_\mu(\varphi)\mu. \tag{2.4.16}$$

Thus, *φ is volume preserving iff $J_\mu(\varphi) = 1$. The inverse function theorem shows that φ is a local diffeomorphism iff $J_\mu(\varphi) \neq 0$ on M.*

There are a number of valuable identities relating the Lie derivative, the exterior derivative and the interior product. For example, if Θ is a one form and X and Y are vector fields, identity 6 in the following table gives

$$\mathbf{d}\Theta(X,Y) = X[\Theta(Y)] - Y[\Theta(X)] - \Theta([X,Y]). \tag{2.4.17}$$

Identities for Vector Fields and Forms

1. Vector fields on M with the bracket $[X,Y]$ form a ***Lie algebra***; that is, $[X,Y]$ is real bilinear, skew-symmetric, and ***Jacobi's identity*** holds:

$$[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0.$$

Locally,

$$[X, Y] = \mathbf{D}Y \cdot X - \mathbf{D}X \cdot Y = (X \cdot \nabla)Y - (Y \cdot \nabla)X$$

and on functions,

$$[X, Y][f] = X[Y[f]] - Y[X[f]].$$

2. For diffeomorphisms φ and ψ,

$$\varphi_*[X, Y] = [\varphi_* X, \varphi_* Y] \quad \text{and} \quad (\varphi \circ \psi)_* X = \varphi_* \psi_* X.$$

3. The forms on a manifold comprise a real associative algebra with $\wedge$ as multiplication. Furthermore, $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$ for k and l-forms α and β, respectively.

4. For maps φ and ψ,

$$\varphi^*(\alpha \wedge \beta) = \varphi^* \alpha \wedge \varphi^* \beta \quad \text{and} \quad (\varphi \circ \psi)^* \alpha = \psi^* \varphi^* \alpha.$$

5. $\mathbf{d}$ is a real linear map on forms, $\mathbf{dd}\alpha = 0$, and

$$\mathbf{d}(\alpha \wedge \beta) = \mathbf{d}\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{d}\beta$$

for α a k-form.

6. For α a k-form and $X_0, \ldots, X_k$ vector fields,

$$\begin{aligned}(\mathbf{d}\alpha)(X_0, \ldots, X_k) &= \sum_{i=0}^{k} (-1)^i X_i[\alpha(X_0, \ldots, \hat{X}_i, \ldots, X_k)] \\ &\quad + \sum_{0 \le i < j \le k} (-1)^{i+j} \alpha([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_k)\end{aligned}$$

where $\hat{X}_i$ means that X_i is omitted. Locally,

$$\mathbf{d}\alpha(x)(v_0, \ldots, v_k) = \sum_{i=0}^{k} (-1)^i \mathbf{D}\alpha(x) \cdot v_i(v_0, \ldots, \hat{v}_i, \ldots, v_k).$$

7. For a map φ, $\qquad \varphi^* \mathbf{d}\alpha = \mathbf{d}\varphi^* \alpha.$

8. **Poincaré Lemma.** If $\mathbf{d}\alpha = 0$, then α is locally exact; that is, there is a neighborhood U about each point on which $\alpha = \mathbf{d}\beta$. The same result holds globally on a contractible manifold.

9. $\mathbf{i}_X\alpha$ is real bilinear in X, α and for $h : M \to \mathbb{R}$,

$$\mathbf{i}_{hX}\alpha = h\mathbf{i}_X\alpha = \mathbf{i}_X h\alpha.$$

Also, $\mathbf{i}_X\mathbf{i}_X\alpha = 0$ and

$$\mathbf{i}_X(\alpha \wedge \beta) = \mathbf{i}_X\alpha \wedge \beta + (-1)^k \alpha \wedge \mathbf{i}_X\beta$$

for α a k-form.

10. For a diffeomorphism φ,

$$\varphi^*(\mathbf{i}_X\alpha) = \mathbf{i}_{\varphi^* X}(\varphi^*\alpha);$$

if $f : M \to N$ is a mapping and Y is f-related to X, that is, $Tf \circ X = Y \circ f$, then

$$\mathbf{i}_Y f^*\alpha = f^*\mathbf{i}_X\alpha.$$

11. $\pounds_X\alpha$ is real bilinear in X, α and

$$\pounds_X(\alpha \wedge \beta) = \pounds_X\alpha \wedge \beta + \alpha \wedge \pounds_X\beta.$$

12. **Cartan's Magic Formula**

$$\pounds_X\alpha = \mathbf{d}\mathbf{i}_X\alpha + \mathbf{i}_X\mathbf{d}\alpha.$$

13. For a diffeomorphism φ,

$$\varphi^*\pounds_X\alpha = \pounds_{\varphi^* X}\varphi^*\alpha;$$

if $f : M \to N$ is a mapping and Y is f-related to X, then

$$\pounds_Y f^*\alpha = f^*\pounds_X\alpha.$$

14. $$(\pounds_X\alpha)(X_1, \ldots, X_k) = X[\alpha(X_1, \ldots, X_k)] - \sum_{i=0}^{k} \alpha(X_1, \ldots, [X, X_i], \ldots, X_k).$$

Locally,

$$(\pounds_X\alpha)(x) \cdot (v_1, \ldots, v_k) = (\mathbf{D}\alpha_x \cdot X(x))(v_1, \ldots, v_k) + \sum_{i=0}^{k} \alpha_x(v_1, \ldots, \mathbf{D}X_x \cdot v_i, \ldots, v_k).$$

15. The following identities hold:

    (a) $\pounds_{fX}\alpha = f\pounds_X\alpha,\ \pounds_X f\alpha = f\pounds_X\alpha + \mathbf{d}f \wedge \mathbf{i}_X\alpha$;

    (b) $\pounds_{[X,Y]}\alpha = \pounds_X\pounds_Y\alpha - \pounds_Y\pounds_X\alpha$;

    (c) $\mathbf{i}_{[X,Y]}\alpha = \pounds_X\mathbf{i}_Y\alpha - \mathbf{i}_Y\pounds_X\alpha$;

    (d) $\pounds_X\mathbf{d}\alpha = \mathbf{d}\pounds_X\alpha$; and

    (e) $\pounds_X\mathbf{i}_X\alpha = \mathbf{i}_X\pounds_X\alpha$.

16. If M is a finite-dimensional manifold, $X = X^l\partial/\partial x^l$, and $\alpha = \alpha_{i_1\ldots i_k}dx^{i_1}\wedge \cdots \wedge dx^{i_k}$, where $i_1 < \cdots < i_k$, then the following formulas hold:

$$
\begin{aligned}
\mathbf{d}\alpha &= \left(\frac{\partial\alpha_{i_1\ldots i_k}}{\partial x^l}\right)dx^l \wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k},\\
\mathbf{i}_X\alpha &= X^l\alpha_{li_2\ldots i_k}dx^{i_2}\wedge\cdots\wedge dx^{i_k},\\
\pounds_X\alpha &= X^l\left(\frac{\partial\alpha_{i_1\ldots i_k}}{\partial x^l}\right)dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
&\qquad + \ \alpha_{li_2\ldots i_k}\left(\frac{\partial X^l}{\partial x^{i_1}}\right)dx^{i_1}\wedge dx^{i_2}\wedge\ldots\wedge dx^{i_k} + \ldots .
\end{aligned}
$$

Exercises

◇ **2.4-1.** Let M be an n-manifold, $\omega \in \Omega^n(M)$ a volume form, $X, Y \in \mathfrak{X}(M)$, and $f, g : M \to \mathbb{R}$ smooth functions such that $f(m) \neq 0$ for all m. Prove the following identities:

(a) $\operatorname{div}_{f\omega}(X) = \operatorname{div}_\omega(X) + X[f]/f$;

(b) $\operatorname{div}_\omega(gX) = g\ \operatorname{div}_\omega(X) + X[g]$; and

(c) $\operatorname{div}_\omega([X,Y]) = X[\operatorname{div}_\omega(Y)] - Y[\operatorname{div}_\omega(X)]$.

◇ **2.4-2.** Show that the partial differential equation

$$\frac{\partial f}{\partial t} = \sum_{i=1}^n X^i(x^1,\ldots,x^n)\frac{\partial f}{\partial x^i}$$

with initial condition $f(x,0) = g(x)$ has the solution $f(x,t) = g(F_t(x))$, where F_t is the flow of the vector field $(X^1,\ldots,X^n)$ in $\mathbb{R}^n$ whose flow is assumed to exist for all time. Show that the solution is *unique*. Generalize this exercise to the equation

$$\frac{\partial f}{\partial t} = X[f]$$

for X a vector field on a manifold M.

◇ **2.4-3.** Show that if M and N are orientable manifolds, so is $M \times N$.

2.5 Stokes' Theorem, Riemannian Manifolds and Distributions

The basic idea of the definition of the integral of an n-form ω on an oriented n-manifold M is to pick a covering by coordinate charts and to sum up the ordinary integrals of $f(x^1, \ldots, x^n)\, dx^1 \cdots dx^n$, where

$$\omega = f(x^1, \ldots, x^n)\, dx^1 \wedge \cdots \wedge dx^n$$

is the local representative of ω, being careful not to count overlaps twice. The change of variables formula guarantees that the result denoted by $\int_M \omega$, is well defined.

If one has an oriented manifold with boundary, then the boundary, ∂M, inherits a compatible orientation. This proceeds in a way that generalizes the relation between the orientation of a surface and its boundary in the classical Stokes' theorem in $\mathbb{R}^3$.

Theorem 2.5.1 (Stokes' Theorem). *Suppose that M is a compact, oriented k-dimensional manifold with boundary ∂M. Let α be a smooth $(k-1)$-form on M. Then*

$$\int_M \mathbf{d}\alpha = \int_{\partial M} \alpha. \tag{2.5.1}$$

Special cases of Stokes' theorem are as follows:

The integral theorems of calculus. Stokes' theorem generalizes and synthesizes the classical theorems:

(a) Fundamental Theorem of Calculus.

$$\int_b^a f'(x)\, dx = f(b) - f(a). \tag{2.5.2}$$

(b) Green's Theorem. For a region $\Omega \subset \mathbb{R}^2$:

$$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\, dy = \int_{\partial\Omega} P\, dx + Q\, dy. \tag{2.5.3}$$

(c) Divergence Theorem. For a region $\Omega \subset \mathbb{R}^3$:

$$\iiint_\Omega \operatorname{div} \mathbf{F}\, dV = \iint_{\partial\Omega} \mathbf{F} \cdot n\, dA. \tag{2.5.4}$$

(d) Classical Stokes' Theorem. For a surface $S \subset \mathbb{R}^3$:

$$\begin{aligned}
&\iint_S \left\{ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz \right. \\
&\qquad \left. + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy \right\} \\
&= \iint_S \mathbf{n} \cdot \operatorname{curl} \mathbf{F}\, dA \\
&= \int_{\partial S} P\, dx + Q\, dy + R\, dz,
\end{aligned} \tag{2.5.5}$$

where $\mathbf{F} = (P, Q, R)$.

Notice that the Poincaré lemma generalizes the vector calculus theorems in $\mathbb{R}^3$ saying that if $\operatorname{curl} \mathbf{F} = 0$, then $\mathbf{F} = \nabla f$ and if $\operatorname{div} \mathbf{F} = 0$, then $\mathbf{F} = \nabla \times \mathbf{G}$. Recall that it states: *If α is closed, then locally α is exact; that is, if $\mathbf{d}\alpha = 0$, then locally $\alpha = \mathbf{d}\beta$ for some β.*

Change of variables. Another basic result in integration theory is the global change of variables formula.

Theorem 2.5.2 (Change of Variables). *Let M and N be oriented n-manifolds and let $F : M \to N$ be an orientation-preserving diffeomorphism. If α is an n-form on N (with, say, compact support), then*

$$\int_M F^* \alpha = \int_N \alpha.$$

Riemannian Manifolds. A differentiable manifold with a positive definite quadratic form $\langle \cdot, \cdot \rangle$ on every tangent space TM_x is called a Riemannian manifold. The quadratic form $\langle\, ,\rangle$, often denoted $g(\, ,)$ is called a ***Riemmannian metric***.

In local coordinates $\dot{q}_i$ the length of a vector v is then given by

$$g(v, v) = \sum_{i,j}^{n} g_{ij}(q) \dot{q}_i \dot{q}_j \,, \; g_{ij} = g_{ji} \,.$$

Let f be a smooth function on M. The ***gradient flow*** of $f, \operatorname{grad} f$ is defined by

$$df(v) = \langle \operatorname{grad} f, v \rangle$$

for any $v \in TM$.

Frobenius' theorem. A basic result called ***Frobenius' theorem*** plays a critical role in control theory and we shall have much to say about it later in the book. For now we just state it briefly as it is normally regarded as part of the theory of differentiable manifolds. The theory of distributions plays a key role in both the theory of nonholonomic systems and nonlinear control theory. Two useful references (from the control-theortic) point of view are Sussmann [1973] and Isidori [1985].

Definition 2.5.3. *A **smooth distribution** on a manifold M is the assigment to each point $x \in M$ of the subspace spanned by the values at x of a set of smooth vector fields on M; ie, it is a "smooth assignment of a subspace to the tangent space at each point, also called a **vector subbundle**. We denote the distribution by Δ and the subspace at $x \in M$ by $\Delta_x \subset T_xM$.*

A distribution is said to be ***involutive*** if for any two vector fields X, Y on M with values in Δ, $[X, Y]$ is also a vector field with values in Δ. The subbundle Δ is said to be ***integrable*** if for each point $x \in M$ there is a local submanifold of M containing x such that its tangent bundle equals Δ restricted to this submanifold. See Figure 2.5.1.

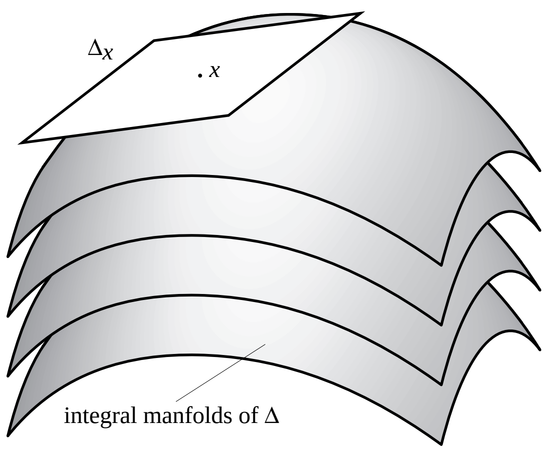


FIGURE 2.5.1. The integral manifolds of a distribution.

If Δ is integrable, the local integral manifolds can be extended to get, through each $x \in M$, a maximal integral manifold, which is an immersed submanifold of M. The collection of all maximal integral manifolds through all points of M forms a ***foliation***.

Theorem 2.5.4 (Frobenius theorem). *Involutivity of Δ is equivalent to the integrability of Δ, which in turn is equivalent to the existence of a foliation on M whose tangent bundle equals Δ.*

Given a set of smooth vector fields $X_1, \cdots, X_d$ on M we denote the distribution defined by their span by

$$\Delta = \text{span}\{X_1, \cdots, X_d\}.$$

The distribution at any point is denoted by Δ_x. A distribution Δ on M is said to be **nonsingular** on M if there exists an integer d such that $\dim(\Delta_x) = d$ for all $x \in M$. A point $x \in M$ is said to be a regular point is there exists a neighbohood U of x such that Δ is nonsingular on U. Otherwise the point is said to singular.

Exercises

◇ **2.5-1.** Let Ω be a closed bounded region in $\mathbb{R}^2$. Use Green's theorem to show that the area of Ω equals the line integral

$$\frac{1}{2}\int_{\partial\Omega}(x\,dy - y\,dx).$$

◇ **2.5-2.** On $\mathbb{R}^2\backslash\{(0,0)\}$ consider the one-form

$$\alpha = (x\,dy - y\,dx)/(x^2 + y^2).$$

(a) Show that this form is closed.

(b) Using the angle θ as a variable on S^1, compute $i^*\alpha$, where $i : S^1 \to \mathbb{R}^2$ is the standard embedding.

(c) Show that α is not exact.

◇ **2.5-3.** **The magnetic monopole.** Let $\mathbf{B} = g\mathbf{r}/r^3$ be a vector field on Euclidean three-space minus the origin where $r = \|\mathbf{r}\|$. Show that $\mathbf{B}$ cannot be written as the curl of something.

2.6 Lie groups, Spatial Relationships, and the Euclidean Group

Lie groups arise in discussing conservation laws for mechanical and control systems and in the anaysis of systems with some underlying symmetry. There is a huge literature on the subject. Useful referernces include Abraham and Marsden [1978], Marsden and Ratiu [1994] and Sattinger and Weaver [1986].

Definition 2.6.1. *A Lie group is a smooth manifold G which is a group and for which the group operations of multiplication, $(g, h) \to gh$ for $g, h \in G$, and inversion, $g \to g^{-1}$ are smooth (i.e. analytic).*

Before giving a brief description of some of the theory of Lie groups we mention some important examples:

1. The group of linear isomorphisms of $\mathbb{R}^n$ to itself. This is a Lie group of dimension n^2 general linear group and denoted by $GL(n, \mathbb{R})$.

Definition 2.6.2. *A **matrix Lie group** is a set of invertible $n \times n$ matrices which is closed under matrix multiplication and which is a submanifold of $\mathbb{R}^{n \times n}$.*

One could equivalently define a matrix Lie group to be a (topologically) closed subgroup of $GL(n, \mathbb{R})$. All of the Lie groups we have mentioned so far, and indeed all the Lie groups discussed in this book will be matrix Lie groups. Lie groups are frequently studied in conjunction with *Lie algebras* which are associated with the tangent spaces of Lie groups as we now describe. To begin with, we state a generalization of the result established in Exercise 2.2.3.

Proposition 2.6.3. *Let $\mathcal{G}$ be a matrix Lie group, and let $A, B \in T_I\mathcal{G}$ (the tangent space to $\mathcal{G}$ at the identity element). Then $AB - BA \in T_I\mathcal{G}$.*

Our proof makes use of the following Lemma.

Lemma 2.6.4. *Let R be an arbitrary element of a matrix Lie group $\mathcal{G}$, and let $B \in T_I\mathcal{G}$. Then $RBR^{-1} \in T_I\mathcal{G}$.*

Proof. Let $R_B(t)$ be a curve in $\mathcal{G}$ such that $R_B(0) = I$ and $R'_B(0) = B$. Then $S(t) = RR_B(t)R^{-1} \in \mathcal{G}$ for all t, and $S(0) = I$. Hence $S(0) \in T_I\mathcal{G}$, proving the lemma. ■

Proof of Proposition. Let $R_A(s)$ be a curve in G such that $R_A(0) = I$ and $R'_A(0) = A$. The by Lemma, $S(t) = R_A(t)BR_A(t)^{-1} \in T_I\mathcal{G}$. Hence $S'(t) \in T_I\mathcal{G}$ and in particular $S'(0) = AB - BA \in T_I\mathcal{G}$. ■

Definition 2.6.5. *For any pair of $n \times n$ matrices A, B, we define the **matrix Lie bracket** $[A, B] = AB - BA$.*

Proposition 2.6.6. *The matrix Lie bracket operation has the following two properties:*

*(i) For any $n \times n$ matrices A and B, $[B, A] = -[A, B]$. (Property of **skew-symmetry**)*

(ii) For and $n \times n$ matrices A, B, and C,

$$[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$$

*(This is known as the **Jacobi identity**.)*

The proof of this proposition involves a straightforward calculation and is left to the reader.

Definition 2.6.7. *A (matrix)* ***Lie algebra*** $\mathfrak{g}$ *is a set of $n \times n$ matrices which is a vector space with respect to the usual operations of matrix addition and multiplication by real numbers (scalars), and which is closed under the matrix Lie bracket operation $[\cdot, \cdot]$.*

Proposition 2.6.8. *For any matrix Lie group, $\mathcal{G}$, the tangent space at the indentity $T_I\mathcal{G}$ is a Lie algebra.*

Proof. This is an immediate consquence of the fact that $T_I\mathcal{G}$ is a vector space and Proposition 2. ■

It is a remarkable and remarkably useful fact that a great deal of the structure of a Lie group may be inferred from studying the Lie algebra. Before discussing important general relationships between Lie groups and Lie algebras, we describe several examples which play an important role in mechanics and control.

The special orthogonal group. The set of all elements of $O(n)$ having determinant $= 1$ is a subgroup called the *special orthogonal group*. This is denoted by $SO(n)$. Because any $X \in O(n)$ satisfies $XX^T = I$, it follows that $\det X = \pm 1$. We could also characterize $SO(n)$ as the connected component of the identity element in $O(n)$. Thus, $T_ISO(n) = T_IO(n)$. From this observation and the calculation carried out in Example 2.2.2, $T_ISO(n)$ is just the set of $n \times n$ skew-symmetric matrices. Throughout this book, we reserve the notation $\mathfrak{so}(n)$ for the Lie algebra $T_ISO(n)$.

The symplectic group. Suppose $n = 2\ell$ and consider the non-singular skew-symmetric matrix $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ where $I = n \times n$ identity matrix. It is an exercise left to the reader to verify that

$$Sp(\ell) = \{X \in Gl(2\ell) \,:\, XJX^T = J\}$$

is a group. It is called the *symplectic group*. Again referring to Example 2.2.2, we find that this matrix Lie algebra $T_ISp(\ell)$ is the set of $n \times n$ matrices satisfying $JY^T + YJ = 0$. Throughout the remainder of the book, we denote this Lie algebra by $\mathfrak{sp}(\ell)$.

Let $Y \in \mathfrak{sp}(\ell)$ be partitioned into $\ell \times \ell$ blocks, $Y = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Write down a complete set of equations involving A, B, C, and D which must be satisfied if $Y \in \mathfrak{sp}(\ell)$. Deduce that the dimension of $\mathfrak{sp}(\ell)$ as a real vector space is $2\ell^2 + \ell = n(n+1)/2$, and consequently $\dim Sp(\ell) = 2\ell^2 + \ell$.

The Heisenberg group. Consider the set of all 3×3 matrices of the form

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where x, y, and z are real numbers. It is straightforward to show that this is a group, and since it is a submanifold of the set of all 3×3 matrices, it is a Lie group. Call it $\mathcal{H}$. The corresponding Lie algebra may be written down from the definition. Specifically,

$$X_1(t) = \begin{pmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad X_2(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

$$X_3(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

are three curves in $\mathcal{H}$ which pass through the identity when $t = 0$. The derivatives $X_i'(0)$ are elements of the Lie algebra: $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ respectively. Since three parameters are used to specify $\mathcal{H}$, $\mathcal{H}$ is three dimensional. A, B, and C are linearly independent and span the Lie algebra. The commutation relations for the Lie brackets of these three basis elements are: $[A, B] = C$, $[A, C] = 0$, and $[B, C] = 0$. This Lie algebra is call the *Heisenberg algebra*.

The Euclidean group. Consider the Lie group of all 4×4 matrices of the form $\begin{pmatrix} R & v \\ 0 & 1 \end{pmatrix}$ where $R \in SO(3)$ and $v \in \mathbb{R}^3$. This group is usually denoted by $SE(3)$ and is called the *special Euclidean group*. The corresponding Lie algebra, $se(3)$, is six dimensional and is spanned by

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The special Euclidean group is of central interest in mechanics since it describes the set of rigid motions and coordinate transformations of 3-space. More specifically, suppose there are two coordinate frames A and B located in space such that the origin of the B-frame has A-frame coordinates $v = (v_1, v_2, v_3)^T$ and such that the unit vectors in the principal B-frame cooridnate directions are $(r_{11}, r_{21}, r_{31})^T$, $(r_{12}, r_{22}, r_{32})^T$, and $(r_{13}, r_{23}, r_{33})^T$ with respect to A-frame coordinates. The *rigid motion* which moves the A-frame into coindcidence with the B-frame is specified by the rotation

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

followed by the translation $v = (v_1, v_2, v_3)^T$. Thus a point with A-frame coordinates $x = (x_1, x_2, x_3)$ is moved under the rigid motion to a new location whose A-frame coordinates are $Rx + v$.

The group $SE(3)$ is also associated with the set of *rigid coordinate transformations* of $\mathbb{R}^3$. Suppose a point Q is located in space and has A-frame coordinates $(x_1^A, x_2^A, x_3^A)^T$ and B-frame coordinates $(x_1^B, x_2^B, x_3^B)^T$. The relationship between these coordinate descriptions is given by

$$x^A = Rx^B + v.$$

Exercises

◇ **2.6-1.**

A point P in $\mathbb{R}^3$ undergoes a rigid motion associated with $\begin{pmatrix} R_1 & v_1 \\ 0 & 1 \end{pmatrix}$ followed by a rigid motion associated with $\begin{pmatrix} R_2 & v_2 \\ 0 & 1 \end{pmatrix}$. What matrix element of $SE(3)$ is assoicated with the composite of these motions in the given order.

◇ **2.6-2.** A coordinate frame B is located with respect to a coordinate frame A as follows. B is initially cooincident with A, but is displaced by the rigid motion associated with $\begin{pmatrix} R_1 & v_1 \\ 0 & 1 \end{pmatrix}$ and is then subsequently further displaced by $\begin{pmatrix} R_2 & v_2 \\ 0 & 1 \end{pmatrix}$. What matrix element of $SE(3)$ is associated with the coordinate transformation from the A frame to the B frame? (I.e., what matrix element of $SE(3)$ is used to describe A-frame coordinates of a point in terms of thhe B-frame coordinates of the same point?)

Let $\mathcal{G}$ be a matrix Lie group and let $\mathfrak{g} = T_I\mathcal{G}$ be the corresponding Lie algebra. The dimensions of the differentiable manifold $\mathcal{G}$ and the vector space $\mathfrak{g}$ are of course the same, and there must be a one-one local correspondence between a neighborhood of 0 in $\mathfrak{g}$ and a neighborhood of the identity element I in $\mathcal{G}$. One explicit local correspondence is provided by the exponential mapping $\exp : \mathfrak{g} \to \mathcal{G}$ which we now describe.

Let $A \in \mathbb{R}^{n\times n}$ (= the space of $n \times n$ matrices). We define $\exp(A)$ by the series

$$I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots . \tag{2.6.1}$$

Proposition 2.6.9. *The series (2.6.1) is absolutely convergent.*

Proof. The ij-th entry in the n-th term of this matrix series is bounded in absolute value by $(n-1)\bar{a}^n/n!$ where $\bar{a} = \max_{ij}\{|a_{ij}|\}$. Hence, the ij-th element in each term in the series is bounded in absolute value by the corresponding term in the absolutely convergent series $e^{\bar{a}n} = 1 + \bar{a}n + \frac{1}{2}\bar{a}^2n^2 + \cdots$. Hence each entry in the series of matrices converges absolutely, proving the proposition. ■

Proposition 2.6.10. *Let $\mathcal{G}$ be a matrix Lie group with corresponding Lie algebra $\mathfrak{g}$. If $A \in \mathfrak{g}$, then $\exp(At) \in \mathcal{G}$ for all real numbers t.*

Exercise

◇ **2.6-3.** Suppose the $n\times n$ matrices A and M satisfy $AM + MA^T = 0$. Show that $\exp(At)M\exp(A^Tt) = M$ for all t. This direct calculation shows that for $A \in \mathfrak{so}(n)$ or $A \in \mathfrak{sp}(\ell)$, we have $\exp(At) \in SO(n)$ or $\exp(At) \in Sp(\ell)$ respectively.

2.7 Fiber Bundles, Connections, and Gauge Theory

We assume the reader is familar with the elementary theory of smooth manifolds. The remainder of the treatment will also be brief. For more comprehensive treatments, one may consult one of the standard texts.

2.7.1 Fiber Bundles

Fiber bundles provide a basic geometric structure for the understanding of many mechanical and control problems, in particular for nonholonomic problems. Good references include Abraham and Marsden [1978] and Steenrod [1951] and Schutz [1980].

A fiber bundle essentially consists of a given space (the base) together with another space (the fiber) attached at each point, plus some compatibility and smoothness conditions.

More formally we have the following:

Definition 2.7.1. *A fiber bundle is a space Q for which the following are given: another space B called the base manifold, a projection $\pi : Q \to B$*

with fibers $\pi^{-1}(b), b \in B$ homeomorphic to a space F, a structure group G of homeomorphisms of F into itself and a covering of B by open sets U_j, satisfying

i) the bundle is locally trivial, i.e. $\pi^{-1}(U_j)$ is homeoemorphic to the product space $U_j \times F$ and ii) if h_j is the map giving the homeomorphism on the set U_j, for any $x \in U_j \cap U_k$ $h_j(h_k^{-1}(x))$ is an element of the structure group G

If the fibers of the bundle are homeomorphic to the structure group we call the bundle a *principal bundle*.

If the fibers of the bundle are homeomorphic to a vector space we call the bundle a *vector bundle*.

Connections. An important additional structure on a bundle is an **connection** or **Ehresmannn connection**, see for example Kobayashi and Nomizu [1963], Marsden, Montgomery and Ratiu [1990] or Bloch, Krishnaprasad, Marsden and Ratiu [1996]. We follow the treatment in the latter here.

However, before we give the precise mathematical definitions we will give a somewhat intuitive discussion of the nature of and need for connections. A nice reference in this regard is the book by Burke [1985].

Suppose we have a bundle and consider (locally) a section of the bundle: i.e. a choice at of a point in the fiber over each point in the base. Let us call such a choice a “field.”

The idea is to single out fields which are “constant”. For vector fields on the plane for example, it is clear what these are. For vector fields on a manifold or far an arbitrary bundle, we have to specify this notion. Such fields are called “horizontal” and are also key to defining a notion of derivative – or rate of change of a vector field. A connection is used to single out horizontal fields, and is chosen to have other desirable structure, such as linearity. For example, the sum of two constant fields should still be constant. As we shall see below, we can specify horizontality by taking a class of fields that are the kernal of a suitable form.

More formally:

Let Q be a differentiable manifold.

Consider a bundle with projection map π and as usual let $T_q\pi$ denote its tangent map at any point. We call the kernel of $T_q\pi$ at any point the vertical space and denote it by V_q.

Definition 2.7.2. *An Ehresmann connection A is a vertical valued one form on Q that satisfies*

1. $A_q : T_qQ \to V_q$ is a linear map for each point $q \in Q$

2. A is a projection: $A(v_q) = v_q$ for all $v_q \in V_q$.

The key property of the connection is the following: if we denote by H_q or hor_q the kernel of A_q and call it the horizontal space, the tangent space to Q is the direct sum of the V_q and H_q, i.e. we can split the tangent space to

Q into horizontal and vertical parts. For example, we can project a tangent vector onto its vertical part using the connection. Note that the vertical space at Q is tangent to the fiber over q.

Later on when we discuss nonholonomic systems we shall choose the connection so that the constraint distribution is the horizontal space of the connection.

Now define the bundle coordinates $q^i = (r^\alpha, s^a)$ for the base and fiber. The coordinate representation of the projection $\pi_{Q,R}$ is just projection onto the factor r and the connection A can be represented locally by a vector valued differential form ω^a:

$$A = \omega^a \frac{\partial}{s^a} \qquad \omega^a(q) = ds^a + A^a_\alpha(r,s) dr^\alpha .$$

We can see this as follows:
Let

$$v_q = \sum_\beta \dot{r}^\beta \frac{\partial}{\partial r^\beta} + \sum_b \dot{s}^b \frac{\partial}{\partial s^b}$$

be an element of T_qQ. Then

$$\omega_a(v_q) = \dot{s}^a + A^a_\alpha \dot{r}^\alpha$$

and

$$A(v_q) = (\dot{s}^a + A^a_\alpha \dot{r}^\alpha) \frac{\partial}{\partial s^a} .$$

This clearly demonstrates that A is a projection since when A acts again only ds^a results in a nonzero term and this has coefficient unity.

Note that we use a different notation, namely ω^a, for the local coordinate representation of the connection A for three reasons. First, it is common in the literature to use ω to stand for constraint one forms. Second, in the preceding formula, it is standard to define the components of the connection A by A^a_α as shown, reflecting the fact that the connection is a projection; to distinguish this use of indices on A from the use of indices on the constraint one forms, it is convenient to use a different letter. Third, we want to regard ω^a as (coordinate dependent) differential forms, as opposed to A which is a vertical valued form; again, a different letter emphasizes this fact. Note in particular, that the exterior derivative of A is not defined, but we can (locally) take the exterior derivative of ω^a. In fact, this will give an easy way to compute the curvature as we shall see.

Given an Ehresmann connection A, a point $q \in Q$ and a vector $v_r \in T_rR$ tangent to the base at a point $r = \pi_{Q,R}(q) \in R$, we can define the horizontal lift of v_r to be the unique vector v_r^h in H_q that projects to v_r under $T_q\pi_{Q,R}$. If we have a vector $X_q \in T_qQ$, we shall also write its horizontal part as

$$\operatorname{hor} X_q = X_q - A(q) \cdot X_q .$$

In coordinates, the vertical projection is the map

$$(\dot{r}^\alpha, \dot{s}^a) \mapsto (0, \dot{s}^a + A^a_\alpha(r,s)\dot{r}^\alpha)$$

while the horizontal projection is the map

$$(\dot{r}^\alpha, \dot{s}^a) \mapsto (\dot{r}^\alpha, -A^a_\alpha(r,s)\dot{r}^\alpha).$$

Next, we recall the basic notion of curvature.

Definition 2.7.3. *The* ***curvature*** *of A is the vertical valued two form B on Q defined by its action on two vector fields X and Y on Q by*

$$B(X,Y) = -A([\operatorname{hor} X, \operatorname{hor} Y])$$

where the bracket on the right hand side is the Jacobi-Lie bracket of vector fields obtained by extending the stated vectors to vector fields.

Notice that this definition shows that the curvature exactly measures the failure of the constraint distribution to be an integrable bundle.

A useful standard identity for the exterior derivative $\mathbf{d}\alpha$ of a one form α (which could be vector space valued) on a manifold M acting on two vector fields X, Y is

$$(\mathbf{d}\alpha)(X,Y) = X[\alpha(Y)] - Y[\alpha(X)] - \alpha([X,Y]).$$

This identity shows that *in coordinates*, one can evaluate the curvature by writing the connection as a form ω^a in coordinates, computing its exterior derivative (component by component) and restricting the result to horizontal vectors, that is, to the constraint distribution. In other words,

$$B(X,Y) = d\omega^a(\operatorname{hor} X, \operatorname{hor} Y)\frac{\partial}{\partial s^a},$$

so that the local expression for curvature is given by

$$B(X,Y)^a = B^a_{\alpha,\beta} X^\alpha Y^\beta \tag{2.7.1}$$

where the coefficients $B^a_{\alpha,\beta}$ are given by

$$B^b_{\alpha\beta} = \left(\frac{\partial A^b_\alpha}{r^\beta} - \frac{\partial A^b_\beta}{r^\alpha} + A^a_\alpha \frac{\partial A^b_\beta}{s^a} - A^a_\beta \frac{\partial A^b_\alpha}{s^a} \right). \tag{2.7.2}$$

2.7.2 Connections on $T\mathbb{R}^1$

The idea of a connection can be illustrated by considering the simplest possible example: a connection on the bundle $T\mathbb{R}^1$ with coordinates $(x, \dot{x})$. We may define the horizontal space to be the kernal of the form

$$d\dot{x} + A(x, \dot{x})dx.$$

More specifically we can choose a linear connection

$$d\dot{x} + a(x)\dot{x}dx.$$

(More on linear connections below.) Here A is the $\mathbb{R}$-valued form

$$(d\dot{x} + a(x)\dot{x}dx)\frac{\partial}{\partial \dot{x}}.$$

Elements of T_qQ are of the form

$$v_q = \dot{x}\frac{\partial}{\partial x} + \ddot{x}\frac{\partial}{\partial \dot{x}}$$

and their projection onto the vertical space is

$$A(v_q) = (\ddot{x} + a(x)\dot{x}^2)\frac{\partial}{\partial \dot{x}}\,.$$

The kernal of A, i.e. the horizontal vectors, is the span of

$$\frac{\partial}{\partial x} - a(x)\dot{x}\frac{\partial}{\partial \dot{x}}\,.$$

Note that the standard choice is $a(x) = 0$, i.e. the standard horizontal space is the span of the vectors $\frac{\partial}{\partial x}$.

Note that we can also use the above argument to define geodesic curves. Suppose we have a curve $x(t)$ such that its tangent vector is parallel transported along the curve, i.e. v_q along the curve is always horizontal, or $A(v_q)$ is zero.

From above this means

$$\ddot{x} + a(x)\dot{x}^2 = 0\,.$$

For $a(x) = 0$ this reduces $\ddot{x} = 0$, the equation of motion for a free particle on the line. We have given the generalization of this equation for arbitrary connections.

2.7.3 Linear connections, affine connections and geodesics

In this section we consider how Ehresmann connections specialize to linear connections and affine connections in the tangent bundle, and we shall derive the geodesic equations. (As above a good related reference for some of these ideas, but with a rather different approach, is Burke [1985].)

As above we use bundle coordinates $r^\alpha, s^a)$ and we specify the connection by the one forms

$$\omega^a(q) = ds^a + A^a_\alpha(r,s)dr^\alpha$$

and the action of A on a tangent vector $v_q = (\dot{r}^\alpha, \dot{s}^a)$ is given by

$$A(v_q) = (\dot{s}^a + A^a_\alpha \dot{r}^\alpha)\frac{\partial}{\partial s^\alpha}\,. \tag{2.7.3}$$

For a linear connections we require that the sum of two (local) horizontal sections is horizontal, i.e. if $(r^\alpha, s^a(r))$ and $(r^\alpha, \hat{s}^a(r))$ are horizontal then so should be $(r^\alpha, s^a(r) + \hat{s}^a(r).)$

Thus if we have

$$\dot{s}^a + A^a_\alpha \dot{r}^\alpha = \dot{\hat{s}}^a + A^a_\alpha \dot{r}^\alpha = 0$$

we require

$$\dot{s}^a + \dot{\hat{s}}^a + A^a_\alpha \dot{r}^\alpha = 0\,.$$

This (plus scaling) requires the connection coefficients be of the form

$$A^a_\alpha(r,s) = \Gamma^a_{\alpha b}(r)s^b\,. \tag{2.7.4}$$

If the bundle is the tangent bundle these are called the components of the affine connection in the tangent bundle.

In the tangent bundle $s^a = \dot{r}^a$. We define geodesic motion along a curve $r(t)$ as being that for which the tangent vector is parallel transported along the curve, i.e. v_q along the curve is always horizontal, or $A(v_q)$ is zero, as in the example.

From 2.7.3 this condition is

$$\ddot{r}^a + \Gamma^a_{bc}\dot{r}^b\dot{r}^b = 0\,. \tag{2.7.5}$$

This is the equation of geodesic motion.

We can also determine this equation as follows:

In the tangent bundle we can specify a linear connection by its action on vector fields, or by a map from vector fields (X, Y) to the vector field $\nabla_X Y$ which satisfiees smooth functions f and g and a vector fields X, Y, Z

(i) $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$

(ii) $\nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z$

(iii) $\nabla_X(fY) = f\nabla_X Y + (df\cdot X)Y$

where $df \cdot X$ is the directional derivative of f along X, or Lie derivative.

Given a basis of vector fields $\frac{\partial}{\partial r_j}$ we can represent ∇ by

$$\nabla_{\frac{\partial}{\partial r_i}}\frac{\partial}{\partial r_j} = \Gamma^k_{ij}\frac{\partial}{\partial r_k}\,. \tag{2.7.6}$$

The geodesic equations above then may be written

$$\nabla_{\dot{r}}\dot{r} = 0\,. \tag{2.7.7}$$

We can see this by a simple computation:

$$\begin{aligned}
\nabla_{\dot{r}^i \frac{\partial}{\partial r_i}} \dot{r}^j \frac{\partial}{\partial r_j} &= \dot{r}^i \nabla_{\frac{\partial}{\partial r_i}} \dot{r}^j \frac{\partial}{\partial r_j} \\
&= \dot{r}^i \frac{\partial}{\partial r_i} r^j \frac{\partial}{\partial r_j} + \dot{r}^i \dot{r}^j \Gamma^k_{ij} \frac{\partial}{\partial r_k} \\
&= (\dot{r}^j + \Gamma^j_{ik} \dot{r}^i \dot{r}^k) \frac{\partial}{\partial r_j}\,.
\end{aligned}$$

Sometimes we will write

$$\nabla_{\dot{r}}\dot{r} = \frac{D^2 r}{dt^2}\,. \tag{2.7.8}$$

Here $\frac{Dr}{dt}$ denotes the covariant derivative. A slightly more formal treatment of some of this material will be given below.

2.7.4 Principal Connections

We now consider the special case of Principal Connections. We start with a free and proper group action of a Lie group on a manifold Q and construct the projection map $\pi : Q \to Q/G$; this setup is also referred to as a ***principal bundle***. The kernel $\ker T_q\pi$ (the tangent space to the group orbit through q) is called the vertical space of the bundle at the point q and is denoted by ver_q.

Definition 2.7.4. *A **principal connection** on the principle bundle $\pi : Q \to Q/G$ is a map (referred to as the connection form) $\mathcal{A} : TQ \to \mathfrak{g}$ that is linear on each tangent space (i.e., $\mathcal{A}$ is a $\mathfrak{g}$-valued one form) and is such that*

1. *$\mathcal{A}(\xi_Q(q)) = \xi$ for all $\xi \in \mathfrak{g}$ and $q \in Q$, and*
2. *$\mathcal{A}$ is equivariant:*

$$\mathcal{A}(T_q\Phi_g(v_q)) = \mathrm{Ad}_g \mathcal{A}(v_q)$$

for all $v_q \in T_qQ$ and $g \in G$, where Φ_g denotes the given action of G on Q and where Ad denotes the adjoint action of G on $\mathfrak{g}$.

The *horizontal space* of the connection at a point $q \in Q$ is the linear space

$$\mathrm{hor}_q = \{v_q \in T_qQ \mid \mathcal{A}(v_q) = 0\}\,.$$

Thus, at any point, we have the decomposition

$$T_qQ = \mathrm{hor}_q \oplus \mathrm{ver}_q.$$

Often one finds connections defined by specifying the horizontal spaces (complementary to the vertical spaces) at each point and requiring that they transform correctly under the group action. In particular, notice that a connection is uniquely determined by the specification of its horizontal spaces, a fact that we will use later on. We will denote the projections onto the horizontal and vertical spaces relative to the above decomposition using the same notation; thus, for $v_q \in T_qQ$, we write

$$v_q = \mathrm{hor}_q v_q + \mathrm{ver}_q v_q.$$

The projection onto the vertical part is given by

$$\mathrm{ver}_q v_q = (\mathcal{A}(v_q))_Q(q)$$

and the projection to the horizontal part is thus

$$\mathrm{hor}_q v_q = v_q - (\mathcal{A}(v_q))_Q(q).$$

The projection map at each point defines an isomorphism from the horizontal space to the tangent space to the base; its inverse is called the *horizontal lift.* Using the uniqueness theory of ODE's one finds that a curve in the base passing through a point $\pi(q)$ can be lifted uniquely to a horizontal curve through q in Q (*i.e.,* a curve whose tangent vector at any point is a horizontal vector).

Since we have a splitting, we can also regard a principal connection as a special type of Ehresmann connection. However, Ehresmann connections are regarded as vertical valued forms whereas principal connections are regarded as Lie algebra valued. Thus, the Ehresmann connection A and the connection one form $\mathcal{A}$ are different and we will distinguish them; they are related in this case by

$$A(v_q) = (\mathcal{A}(v_q))_Q(q).$$

The general notions of curvature and other properties which hold for general Ehresmann connections specialize to the case of principal connections. As in the general case, given any vector field X on the base space (in this case, the shape space), using the horizontal lift, there is a unique vector field X^h that is horizontal and that is π-related to X; that is, at each point q, we have

$$T_q\pi \cdot X^h(q) = X(\pi(q))$$

and the vertical part is zero:

$$(\mathcal{A}(X^h_q))_Q(q) = 0.$$

It is well known (see, for example, Abraham, Marsden and Ratiu [1988]) that the relation of being π-related is bracket preserving; in our case, this means that

$$\operatorname{hor}[X^h, Y^h] = [X, Y]^h,$$

where X and Y are vector fields on the base.

Definition 2.7.5. *The* ***covariant exterior derivative*** $\mathbf{D}$ *of a Lie algebra valued one form α is defined by applying the ordinary exterior derivative* $\mathbf{d}$ *to the horizontal parts of vectors:*

$$\mathbf{D}\alpha(X, Y) = \mathbf{d}\alpha(\operatorname{hor} X, \operatorname{hor} Y).$$

The ***curvature*** *of a connection $\mathcal{A}$ is its covariant exterior derivative and it is denoted by $\mathcal{B}$.*

Thus, $\mathcal{B}$ is the Lie algebra valued two form given by

$$\mathcal{B}(X, Y) = \mathbf{d}\mathcal{A}(\operatorname{hor} X, \operatorname{hor} Y).$$

Using the identity

$$(\mathbf{d}\alpha)(X, Y) = X[\alpha(Y)] - Y[\alpha(X)] - \alpha([X, Y]).$$

together with the definition of horizontal shows that for two vector fields X and Y on Q, we have

$$\mathcal{B}(X, Y) = -\mathcal{A}([\operatorname{hor} X, \operatorname{hor} Y])$$

where the bracket on the right hand side is the Jacobi-Lie bracket of vector fields. The Cartan structure equations say that if X and Y are vector fields that are invariant under the group action, then

$$\mathcal{B}(X, Y) = \mathbf{d}\mathcal{A}(X, Y) - [\mathcal{A}(X), \mathcal{A}(Y)]$$

where the bracket on the right hand side is the Lie algebra bracket. This follows readily from the definitions, the fact that $[\xi_Q, \eta_Q] = -[\xi, \eta]_Q$, the first property in the definition of a connection, and writing $\operatorname{hor} X = X - \operatorname{ver} X$ and similarly for Y, in the preceding formula for the curvature.

Next, we give some useful local formulas for the curvature. To do this, we pick a local trivialization of the bundle; that is, locally in the base, we write $Q = Q/G \times G$ where the action of G is given by left translation on the second factor. We choose coordinates r^α on the first factor and a basis e_a of the Lie algebra $\mathfrak{g}$ of G. We write coordinates of an element ξ relative to this basis as ξ^a. Let tangent vectors in this local trivialization at the point (r, g) be denoted (u, w). We will write the action of $\mathcal{A}$ on this vector simply as $\mathcal{A}(u, w)$. Using this notation, we can write the connection form in this local trivialization as

$$\mathcal{A}(u, w) = \mathrm{Ad}_g(w_b + \mathcal{A}_{\mathrm{loc}}(r) \cdot u), \tag{2.7.9}$$

where w_b is the left translation of w to the identity (that is, the expression of w in "body coordinates"). The preceding equation defines the expression $\mathcal{A}_{\mathrm{loc}}(r)$. We define the connection components by writing

$$\mathcal{A}_{\mathrm{loc}}(r) \cdot u = \mathcal{A}^a_\alpha u^\alpha e_a.$$

Similarly, the curvature can be written in a local representation as

$$\mathcal{B}((u_1, w_1), (u_2, w_2)) = \mathrm{Ad}_g(\mathcal{B}_{\mathrm{loc}}(r) \cdot (u_1, u_2)),$$

which again serves to define the expression $\mathcal{B}_{\mathrm{loc}}(r)$. We can also define the coordinate form for the local expression of the curvature by writing

$$\mathcal{B}_{\mathrm{loc}}(r) \cdot (u_1, u_2) = \mathcal{B}^a_{\alpha\beta} u_1^\alpha u_2^\beta e_a.$$

Then one has the formula

$$\mathcal{B}^b_{\alpha\beta} = \left(\frac{\partial \mathcal{A}^b_\beta}{r^\alpha} - \frac{\partial \mathcal{A}^b_\alpha}{r^\beta} - C^b_{ac} \mathcal{A}^a_\alpha \mathcal{A}^c_\beta \right),$$

where C^b_{ac} are the structure constants of the Lie algebra defined by

$$[e_a, e_c] = C^b_{ac} e_b.$$

2.7.5 Parallel Translation and Holonomy Groups

Let P be a principal bundle with a connection and C a piecewise differentiable curve in its base space M with beginning point p and endpoint q. Suppose x is a point on the fiber over p. Then there is unique curve C_x^* in P starting at x such that $\pi(C_x^*) = C$ and each tangent vector to C is horizontal. C_x^* is said to be a *lift* of C that starts at x and the map that takes x to the lift of q, the endpoint of the lifted curve, is said to be *parallel translation.*

Now suppose C is a closed curve starting at p. Parallel translation then maps the point x in the fiber over p to itself, to a point xa say, $a \in G$. Thus each closed curve at p and fiber point x determines an element of G and the set of all such elements forms a subgroup of G called the *holonomy group* of the connection with reference point x.

2.7.6 Affine Connections and Covariant Derivatives

Now let M be a differential manifold and let P be the bundle of tangent n-frames over M, i.e. the bundle whose fiber consists of n independent tangent vectors at each point. Then P is a principal bundle with structure group $Gl(n, \mathbb{R})$ and is the principal bundle associated with the tangent vector bundle of M. A connection in this bundle is called an *affine connection.*

An affine connection on M gives a parallel displacement of the tangent vector space of M as follows:

Let $C = p_t(0 \leq t \leq 1)$ be a curve in M and let $C^* = x_t$ be a lift to P. The parallel displacment of the tangent frame x_0 at p_0 along the curve is x_t and defines a parallel displacement of $T_{p_0}(M)$ to $T_{p_y}(M)$ along C which is independent of the choice of lift.

Now suppose that we have a vector Y_t in $T_{p_t}M$ such that the map from t to Y_t is differentiable, i.e. Y_t is a differentiable vector field along the curve. Let $\phi_{t,h}$ denote the parallel displacement of $T_{p_t}(M)$ onto $T_{p_{t+h}}(M)$ along C. Then the vector field

$$Y' = \lim_{h \to 0} 1/h(\phi_{t,h}^{-1}(Y_{t+h}) - Y_t)$$

is called the *covariant* derivative of Y along C. Y_t is parallel along C, i.e. $Y_t = \phi_{0,t}(Y_0)$ if and only if Y_t' is identically zero.

A geodesic curve is a curve whose tangent vectors are parallel along the curve.

Now let X and Y be given vector fields on M and let $C = p_t$, $(-\epsilon \leq 0 \leq \epsilon)$ be an integral curve of X through p_0 and let ϕ_t be the parallel displacement along C. Then the *covariant derivative* of Y in the direction of X is given by

$$(\nabla_X Y)_{p_0} = \lim_{t \to 0} 1/t(\phi_t^{-1}(Y_{p_t}) - Y_{p_0}) \,.$$

$\nabla_X Y$ is a vector field on Y such that the mapping $(X, Y) \to \nabla_X Y$ satisfies: a)$\nabla_X Y$ is linear with respect to X and Y, b) $\nabla_{fX} Y = f \cdot \nabla_X Y$ and c) $\nabla_X (fY) = f \cdot \nabla_X Y + (Xf) \cdot Y$, for f any differentiable function on M. In fact given any mapping satisfying these conditions there exists a unique affine connection whose covariant derivative is the given mapping. The covariant derivative can be naturally extended to tensor fields of arbitrary type.

For X, Y and Z arbitrary vector fields on M the *curvature tensor* R and the *torsion tensor* are defined by

$$R(X, Y)(Z) = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}(Z) \,,$$

and

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] \,.$$

Now let $(x^1, \cdots x^n)$ be local coordinates on M and let X_i be the vector fields $\partial/\partial x^i$. Then for an affine connection the covariant derivative may be expressed as

$$\nabla_{X_j}(X_k) = \sum_i \Gamma^i_{jk} X_i \,.$$

2.7.7 Riemannian Connections

Now suppose M is endowed with a Riemannian metric g. This means we can define orthonormal bases of $T_p(M)$ at each $p \in M$ and can define a subbundle P' of P whose fibers are orthonormal bases and which has structure group $O(n)$. This subbundle is said to be a reduced bundle of P.

There exists a unique affine connection on M, called the *Riemmanian connection,*, such that $\nabla g = 0$ and the torsion tensor T vanishes. An affine connection called a metric connection if $\nabla g = 0$.

If the metric is given by $g = \sum g_{ij} dx^i dx^j$ the connection coefficients, which are called *Christoffel symbols*, are given by

$$\Gamma^i_{jk} = \frac{1}{2} \sum_l g^{il} \{ \frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \} \, .$$

3
Basic concepts in mechanics

3.1 First Principles - Coordinates and Kinematics

The most basic goal of analytical mechanics is to provide a formalism for describing motion. This is conveniently done in terms of a set of *generalized coordinates*. This is a set of variables whose values uniquely specify the location in 3-space of each physical point on the mechanism. Aset og generalized coordinates is minimal in the sense that no set of fewer variables suffices to determine the locations of all points on the mechanism. The number of variables in a set of generalized coordinates for a mechanical system is called the number of *degrees of freedom* of the system.

3.2 First Principles - Conservation of Energy, Work, Virtual Work

Analytical mechanics seeks to provide a formal description of the motion of mechanical systems which is as simple as possible. Over many centuries, people who studied the motions of physical systems have developed guiding principles or "laws" which govern the movement of all systems. An example is the principle of *conservation of energy*. This principle states that in any "closed" mechanical system, the total energy is conserved. In the physical world, the principle always holds, although it may be difficult in specific cases to be explicit in specifying the extent of the closed system. For systems about which it is possible to be explicit, however, the principle may be used to guide us in the development of mathematical models of motion.

Simple ideas along this line, which will be generalized to provide the foundation of most of the models studied in this book, may be illustrated using the simple kinematic chain illustrated in Figure 1. Here there are drawn two copies of the same mechanism. In the first, the motion of a typical point, P, is described in terms of coordinate variables (θ_1, θ_2), where θ_2 is the relative angle between the two links in the chain. In Figure **??**(b), the motion of the typical point P is described in terms of coordinate variables (ϕ_1, ϕ_2), which are the (absolute) angles of the links with respect to the vertical direction. Other choices of coordinate variables are, of course, possible. In any case, the coordinate variables serve the purpose of describing the location of typical points of the mechanism with respect to some fixed frame of reference—which we may refer to as the *inertial frame*.

Specifically, in this case, the inertial frame is chosen to its origin at the hinge point of the upper link. The y-axis is directed parallel and opposite to the gravitational field, and the x-axis is chosen so as to give the coordinate frame the standard orientation. Suppose the point P is located on the second link, as depicted. If this has coordinates (x_ℓ, y_ℓ) with respect to the local frame, then the coordinates with respect to the inertial frame are given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r_1 \sin\theta_1 + x_\ell \sin(\theta_1 + \theta_2) + y_\ell \cos(\theta_1 + \theta_2) \\ -r_1 \cos\theta_1 - x_\ell \cos(\theta_1 + \theta_2) + y_\ell \sin(\theta_1 + \theta_2) \end{bmatrix} \tag{3.2.1}$$

or equivalently by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r_1 \sin\phi_1 + x_\ell \sin\phi_2 + y_\ell \cos\phi_2 \\ -r_1 \cos\phi_1 - x_\ell \cos\phi_2 + y_\ell \sin\phi_2 \end{bmatrix}. \tag{3.2.2}$$

The mappings $(\theta_1, \theta_2) \mapsto (x, y)$ and $(\phi_1, \phi_2) \mapsto (x, y)$ are examples of *kinematics functions*. They associate values of the coordinate variables (θ_1, θ_2) (respectively (ϕ_1, ϕ_2)) to inertial coordinates of the point P. For every point on the kinematic chain mechanism of Figure **??**, there is a kinematics function $f(\cdot, \cdot, P)$ for which the formula

$$\begin{bmatrix} x \\ y \end{bmatrix} = f(\theta_1, \theta_2, P)$$

associates inertial coordinates (x, y) to values of the joint angle coordinates. When kinematics functions are as simple to write down explicitly as in the present example, they provide a powerful tool for describing motion since the inertial coordinates of any point are completely determined by (θ_1, θ_2).

An interesting question in mechanics is: when do the values of a finite set of configuration variable (such as (θ_1, θ_2) or (ϕ_1, ϕ_2)) determine the configuration of an entire mechanism (by means of a set of kinematics functions

$f(\cdot, P)$)? As we have seen (**shall see**), there are simple examples of systems wherein a physically interesting finite set of configuration variables $(\theta_1(\cdot), \ldots, \theta_n(\cdot))$ (Think of the wheels on a car!) determines the configuration of the entire mechanism *only* if we know the entire time history of the $\theta_j(\cdot)$'s over the interval of interest. Such systems will be called *nonholonomic*. One of the aims of this book is to develop a theory of mechanics and control of both holonomic and nonholonomic systems.

It will be useful to describe a general procedure for constructing kinematics functions for open single-strand spatial kinematic chains. . . .

3.3 Newtonian, Lagrangian, and Hamiltonian Mechanics

Newtonian mechanics starts from the premise that in every physical system the forces due to the acceleration of each particle exactly balance all other forces acting on the particle. If $\vec{x}(t)$ denotes the coordinates of a particle of mass m at time t with respect to some fixed (but arbitrary) coordinate frame, then the motion of the particle is elegantly and simply described by the differential equation

$$m\ddot{\vec{x}}(t) = F(t). \tag{3.3.1}$$

Since this equation completely describes the motion of the particle, the motion of any assemblage of particles could in principle be described by solving (3.3.1) for each particle in the system. Since this is not feasible for most systems of physical interest, we turn our attention the study of *generalized coordinates*, which are minimal systems of variables in terms of which the motion of the entire system can be described.

Consider, for example, a planar kinematic chain, anchored at one end as depicted in Figure (3.3.1). **figure missing**

FIGURE 3.3.1. A planar kinematic chain.

Within each link there is affixed a coordinate system, and a "base" coordinate frame is prescribed which serves as a reference for the motion of the entire system. In each frame, the x and y axes are as depicted, and the z-axis is assumed to be perpendicular to the plane of the page. Let a typical physical point on the k-th link have coordinates (x_k, y_k, z_k) with respect to the k-th link frame. The coordinates with respect to the base frame are then given by

$$\begin{bmatrix} x(\theta_1,\ldots,\theta_k;x_k,y_k,z_k) \\ y(\theta_1,\ldots,\theta_k;x_k,y_k,z_k) \\ z(\theta_1,\ldots,\theta_k;x_k,y_k,z_k) \end{bmatrix} = \sum_{j=1}^{k-1} \begin{bmatrix} r_j\cos(\theta_{1\ldots j}) \\ r_j\sin(\theta_{1\ldots j}) \\ 0 \end{bmatrix} + \begin{bmatrix} x_k\cos(\theta_{1\ldots k}) - y_k\sin(\theta_{1\ldots k}) \\ x_k\sin(\theta_{1\ldots k}) + y_k\cos(\theta_{1\ldots k}) \\ z_k \end{bmatrix}. \tag{3.3.2}$$

As the mechanism undergoes any motion, both the velocity $\dot{\vec{x}}$ and the acceleration $\ddot{\vec{x}}$ with repsect to the base frame may be computed explicitly using this formula.

The motion of every infinitessimal piece of this chain is governed by Newton's law, which we may write as

$$m(\vec{x})\ddot{\vec{x}}\,dV = \vec{F}_x\,dV \tag{3.3.3}$$

where $m(\vec{x})$ is the mass density, $\vec{F}_x$ is the force density, and dV is the volume element defined on the link. Although the motion of the k-th link in the chain is completely described in terms of $\theta_1,\ldots,\theta_k$, for notational convenience, and to pursue the study of more general systems we shall write $\vec{x} = \vec{x}(\theta_1,\ldots,\theta_n;x_k,y_k,z_k)$. We wish to study the dynamics of our chain purely in terms of the θ_i's, and our goal will be to use the basic force balance law (3.3.3) to develop an equivalent system of equations in terms of the θ_i's. Begin by multiplying both sides of (3.3.3) by the $n\times 3$ matrix $(\frac{\partial\vec{x}}{\partial\theta})^T$:

$$m(x)(\frac{\partial\vec{x}}{\partial\theta})^T\ddot{\vec{x}}\,dV = (\frac{\partial\vec{x}}{\partial\theta})^T\vec{F}_x\,dV. \tag{3.3.4}$$

We examine $(\frac{\partial\vec{x}}{\partial\theta})^T\ddot{\vec{x}}$ and show how this may be integrated over the volume occupied by the entire mechanism to give a quantity on the left hand side of (3.3.4) which depends only on the variables $\theta, \dot{\theta}$, and $\ddot{\theta}$. The following lemma is useful in this connection.

Lemma 3.3.1.

$$\frac{\partial\vec{x}}{\partial\theta} = \frac{\partial\dot{\vec{x}}}{\partial\dot{\theta}}$$

Proof.

$$\begin{aligned} \frac{\partial\dot{\vec{x}}}{\partial\dot{\theta}_i} &= \frac{\partial}{\partial\dot{\theta}_i}\dot{\vec{x}} && (3.3.5) \\ &= \frac{\partial}{\partial\dot{\theta}_i}(\frac{\partial\vec{x}}{\partial\theta_1}\dot{\theta}_1 + \cdots + \frac{\partial\vec{x}}{\partial\theta_n}\dot{\theta}_n) && (3.3.6) \\ &= \frac{\partial\vec{x}}{\partial\theta_i}. && (3.3.7) \end{aligned}$$

■

Now

$$(\frac{\partial \vec{x}}{\partial \theta})^T \ddot{\vec{x}} = \frac{d}{dt}\left(\frac{\partial \vec{x}}{\partial \theta}^T \dot{\vec{x}}\right) - \left[\frac{d}{dt}\left(\frac{\partial \vec{x}}{\partial \theta}\right)^T\right]\dot{\vec{x}}.$$

By the lemma, this may be rewritten as

$$\frac{d}{dt}\left(\frac{\partial \dot{\vec{x}}}{\partial \dot{\theta}}^T \dot{\vec{x}}\right) - \left[\frac{d}{dt}\left(\frac{\partial \vec{x}}{\partial \theta}\right)^T\right]\dot{\vec{x}}. \tag{3.3.8}$$

The first term may be further rewritten as

$$\frac{d}{dt}\frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}\|\dot{\vec{x}}\|^2\right).$$

We next show that the second term may be written $-\frac{\partial}{\partial \theta}\left(\frac{1}{2}\|\dot{\vec{x}}\|^2\right)$. This follows from the equality

$$\frac{d}{dt}\frac{\partial \vec{x}}{\partial \theta_i} = \sum_{j=1}^{n} \frac{\partial^2 \vec{x}}{\partial \theta_j \partial \theta_i}\dot{\theta}_j \tag{3.3.9}$$

$$= \frac{\partial}{\partial \theta_i}\sum_{j=1}^{n} \frac{\partial \vec{x}}{\partial \theta_j}\dot{\theta}_j \tag{3.3.10}$$

$$= \frac{\partial \dot{\vec{x}}}{\partial \theta_i}. \tag{3.3.11}$$

The left-hand side of (3.3.4) may thus be written as

$$m(\vec{x})\left[\frac{d}{dt}\frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}\|\dot{\vec{x}}\|^2\right) - \frac{\partial}{\partial \theta}\left(\frac{1}{2}\|\dot{\vec{x}}\|^2\right)\right] dV.$$

If this quantity is integrated over the spatial extent of the mechanism, we obtain

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta}, \tag{3.3.12}$$

where $T = T(\theta, \dot{\theta}) = (1/2)\int m(\vec{x})\|\dot{\vec{x}}\|^2\, dV$ is the total kinetic energy of the mechanism's motion.

We also wish to integrate the right hand side of (3.3.4) over the volume occupied by the mechanism. Recall that $(\frac{\partial \vec{x}}{\partial \theta})^T \vec{F_x}$ is the component of generalized force acting in the direction of the generalized coordinate θ which is due to the force $\vec{F_x}$ acting at $\vec{x} = \vec{x}(\theta)$. Hence, integrating (3.3.4) over the mechanism results in the total net generalized force being applied in the direction θ. Call this quantity $\vec{F_\theta}$. We summarize these calculations by writing the dynamic equations for the mechanism in terms of the variables θ:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} = \vec{F}_\theta. \tag{3.3.13}$$

We now set out some of the notation and terminology used in subsequent chapters. The reader is referred to one of the standard books, such as Abraham and Marsden [1978], Arnold [1989], Guillemin and Sternberg [1984] and Marsden and Ratiu [1992] for proofs omitted here.

3.4 Symplectic and Poisson Manifolds

Definition 3.4.1. *Let P be a manifold and let $\mathcal{F}(P)$ denote the set of smooth real-valued functions on P. Consider a given bracket operation denoted*

$$\{\,,\}:\mathcal{F}(P)\times\mathcal{F}(P)\to\mathcal{F}(P).$$

The pair $(P,\{\,,\})$ is called a ***Poisson manifold*** *if $\{\,,\}$ satisfies*

(PB1)	***bilinearity***	*$\{f,g\}$ is bilinear in f and g.*
(PB2)	***anticommutativity***	*$\{f,g\}=-\{g,f\}$.*
(PB3)	***Jacobi's identity***	*$\{\{f,g\},h\}+\{\{h,f\},g\}+\{\{g,h\},f\}=0$.*
(PB4)	***Leibniz' rule***	*$\{fg,h\}=f\{g,h\}+g\{f,h\}$.*

Conditions (PB1)–(PB3) make $(\mathcal{F}(P),\{\,,\})$ into a Lie algebra. If $(P,\{\,,\})$ is a Poisson manifold, then because of (PB1) and (PB4), there is a tensor B on P, assigning to each $z\in P$ a linear map $B(z): T_z^*P\to T_zP$ such that

$$\{f,g\}(z)=\langle B(z)\cdot \mathbf{d}f(z),\mathbf{d}g(z)\rangle. \tag{3.4.1}$$

Here, $\langle\,,\rangle$ denotes the natural pairing between vectors and covectors. Because of (PB2), $B(z)$ is antisymmetric. Letting $z^I, I=1,\ldots,M$ denote coordinates on P, (3.4.1) becomes

$$\{f,g\}=B^{IJ}\frac{\partial f}{\partial z^I}\frac{\partial g}{\partial z^J}. \tag{3.4.2}$$

Antisymmetry means $B^{IJ}=-B^{JI}$ and Jacobi's identity reads

$$B^{LI}\frac{\partial B^{JK}}{\partial z^L}+B^{LJ}\frac{\partial B^{KI}}{\partial z^L}+B^{LK}\frac{\partial B^{IJ}}{\partial z^L}=0. \tag{3.4.3}$$

Definition 3.4.2. *Let $(P_1,\{\,,\}_1)$ and $(P_2,\{\,,\}_2)$ be Poisson manifolds. A mapping $\varphi:P_1\to P_2$ is called* ***Poisson*** *if for all $f,h\in\mathcal{F}(P_2)$, we have*

$$\{f,h\}_2\circ\varphi=\{f\circ\varphi,h\circ\varphi\}_1. \tag{3.4.4}$$

Definition 3.4.3. *Let P be a manifold and Ω a 2-form on P. The pair (P, Ω) is called a **symplectic manifold** if Ω satisfies*

(S1) $\mathbf{d}\Omega = 0$ *(i.e., Ω is closed) and*

(S2) Ω *is nondegenerate.*

Definition 3.4.4. *Let (P, Ω) be a symplectic manifold and let $f \in \mathcal{F}(P)$. Let X_f be the unique vector field on P satisfying*

$$\Omega_z(X_f(z), v) = \mathbf{d}f(z) \cdot v \quad \textit{for all} \quad v \in T_zP. \tag{3.4.5}$$

*We call X_f the **Hamiltonian vector field** of f. **Hamilton's equations** are the differential equations on P given by*

$$\dot{z} = X_f(z). \tag{3.4.6}$$

*If (P, Ω) is a symplectic manifold, define the **Poisson bracket operation** $\{\cdot, \cdot\} : \mathcal{F}(P) \times \mathcal{F}(P) \to \mathcal{F}(P)$ by*

$$\{f, g\} = \Omega(X_f, X_g). \tag{3.4.7}$$

The construction (3.4.7) makes $(P, \{\,,\})$ into a Poisson manifold. In other words,

Proposition 3.4.5. *Every symplectic manifold is Poisson.*

The converse is not true; for example the zero bracket makes any manifold Poisson. In §2.4 we shall see some non-trivial examples of Poisson brackets that are not symplectic, such as Lie-Poisson structures on duals of Lie algebras.

Hamiltonian vector fields are defined on Poisson manifolds as follows.

Definition 3.4.6. *Let $(P, \{\,,\})$ be a Poisson manifold and let $f \in \mathcal{F}(P)$. Define X_f to be the unique vector field on P satisfying*

$$X_f[k] := \langle \mathbf{d}k, X_f \rangle = \{k, f\} \quad \textit{for all} \quad k \in \mathcal{F}(P).$$

*We call X_f the **Hamiltonian vector field** of f.*

A check of the definitions shows that in the symplectic case, the Definitions 2.1.4 and 2.1.6 of Hamiltonian vector fields coincide. If $(P, \{\,,\})$ is a Poisson manifold, there are therefore three equivalent ways to write Hamilton's equations for $H \in \mathcal{F}(P)$:

i $\dot{z} = X_H(z)$

ii $\dot{f} = \mathbf{d}f(z) \cdot X_H(z)$ for all $f \in \mathcal{F}(P)$, and

iii $\dot{f} = \{f, H\}$ for all $f \in \mathcal{F}(P)$.

3.5 The Flow of a Hamiltonian Vector Field

Hamilton's equations described in the abstract setting of the last section are very general. They include not only what one normally thinks of as Hamilton's canonical equations in classical mechanics, but Schrödinger's equation in quantum mechanics as well. Despite this generality, the theory has a rich structure.

Let $H \in \mathcal{F}(P)$ where P is a Poisson manifold. Let φ_t be the flow of Hamilton's equations; thus, $\varphi_t(z)$ is the integral curve of $\dot{z} = X_H(z)$ starting at z. (If the flow is not complete, restrict attention to its domain of definition.) There are two basic facts about Hamiltonian flows (ignoring functional analytic technicalities in the infinite dimensional case — see Chernoff and Marsden [1974]).

Proposition 3.5.1. *The following hold for Hamiltonian systems on Poisson manifolds:*

i *each φ_t is a Poisson map*

ii $H \circ \varphi_t = H$ *(conservation of energy).*

The first part of this proposition is true even if H is a time dependent Hamiltonian, while the second part is true only when H is independent of time.

3.6 Cotangent Bundles

Let Q be a given manifold (usually the configuration space of a mechanical system) and T^*Q be its cotangent bundle. Coordinates q^i on Q induce coordinates (q^i, p_j) on T^*Q, called the ***canonical cotangent coordinates*** of T^*Q.

Proposition 3.6.1. *There is a unique* 1*-form* Θ *on* T^*Q *such that in any choice of canonical cotangent coordinates,*

$$\Theta = p_i dq^i; \tag{3.6.1}$$

Θ *is called the* ***canonical 1-form****. We define the* ***canonical 2-form*** Ω *by*

$$\Omega = -\mathbf{d}\Theta = dq^i \wedge dp_i \quad (\textit{a sum on } i \textit{ is understood}). \tag{3.6.2}$$

In infinite dimensions, one needs to use an intrinsic definition of Θ, and there are many such; one of these is the identity $\beta^*\Theta = \beta$ for $\beta : Q \to T^*Q$ any one form. Another is

$$\Theta(w_{\alpha_q}) = \langle \alpha_q, T\pi_Q \cdot w_{\alpha_q} \rangle,$$

where $\alpha_q \in T_q^*Q, w_{\alpha_q} \in T_{\alpha_q}(T^*Q)$ and where $\pi_Q : T^*Q \to Q$ is the cotangent bundle projection.

Proposition 3.6.2. (T^*Q, Ω) *is a symplectic manifold.*

In canonical coordinates the Poisson brackets on T^*Q have the classical form

$$\{f, g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}, \tag{3.6.3}$$

where summation on repeated indices is understood.

Theorem 3.6.3. **(Darboux' Theorem)** *Every symplectic manifold locally looks like T^*Q; in other words, on every finite dimensional symplectic manifold, there are local coordinates in which Ω has the form* (3.6.2)*.*

(See Marsden [1981] and Olver [1988] for a discussion of the infinite dimensional case.)

Hamilton's equations in these canonical coordinates have the classical form

$$\dot{q}^i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q^i}$$

as one can readily check.

The local structure of Poisson manifolds is more complex than the symplectic case. However, Kirillov [1976a] has shown that every Poisson manifold is the union of ***symplectic leaves***; to compute the bracket of two functions in P, one does it ***leaf-wise***. In other words, to calculate the bracket of f and g at $z \in P$, select the symplectic leaf S_z through z, and evaluate the bracket of $f|S_z$ and $g|S_z$ at z. We shall see a specific case of this picture shortly.

3.7 Lagrangian Mechanics

Let Q be a manifold and TQ its tangent bundle. Coordinates q^i on Q induce coordinates $(q^i, \dot{q}^i)$ on TQ, called ***tangent coordinates***. A mapping $L : TQ \to \mathbb{R}$ is called a ***Lagrangian***. Often we choose L to be $L = K - V$ where $K(v) = \frac{1}{2}\langle v, v\rangle$ is the ***kinetic energy*** associated to a given Riemannian metric and where $V : Q \to \mathbb{R}$ is the ***potential energy***.

Definition 3.7.1. ***Hamilton's principle*** *singles out particular curves $q(t)$ by the condition*

$$\delta \int_b^a L(q(t), \dot{q}(t))dt = 0, \tag{3.7.1}$$

where the variation is over smooth curves in Q with fixed endpoints.

It is interesting to note that (3.7.1) is unchanged if we replace the integrand by $L(q, \dot{q}) - \frac{d}{dt} S(q,t)$ for any function $S(q,t)$. This reflects the ***gauge invariance*** of classical mechanics and is closely related to Hamilton-Jacobi theory. We shall return to this point in Chapter 9. It is also interesting to note that if one keeps track of the boundary conditions in Hamilton's principle, they essentially *define* the canonical one form, $p_i dq^i$. This turns out to be a useful remark in more complex field theories.

If one prefers, the action principle states that the map I defined by $I(q(\cdot)) = \int_a^b L(q(t), \dot{q}(t))dt$ from the space of curves with prescribed endpoints in Q to $\mathbb{R}$ has a critical point at the curve in question. In any case, a basic and elementary result of the calculus of variations, whose proof was sketched in §1.2, is:

Proposition 3.7.2. *The principle of critical action for a curve $q(t)$ is equivalent to the condition that $q(t)$ satisfies the **Euler-Lagrange equations***

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0. \tag{3.7.2}$$

Definition 3.7.3. *Let L be a Lagrangian on TQ and let $\mathbb{F}L : TQ \to T^*Q$ be defined (in coordinates) by*

$$(q^i, \dot{q}^j) \mapsto (q^i, p_j)$$

*where $p_j = \partial L / \partial \dot{q}^j$. We call $\mathbb{F}L$ the **fiber derivative**. (Intrinsically, $\mathbb{F}L$ differentiates L in the fiber direction.)*

*A Lagrangian L is called **hyperregular** if $\mathbb{F}L$ is a diffeomorphism. If L is a hyperregular Lagrangian, we define the corresponding **Hamiltonian** by*

$$H(q^i, p_j) = p_i \dot{q}^i - L.$$

*The change of data from L on TQ to H on T^*Q is called the **Legendre transform**.*

One checks that the Euler-Lagrange equations for L are equivalent to Hamilton's equations for H.

In a relativistic context one finds that the two conditions $p_j = \partial L / \partial \dot{q}^j$ and $H = p_i \dot{q}^i - L$, defining the Legendre transform, fit together as the spatial and temporal components of a single object. Suffice it to say that the formalism developed here is useful in the context of relativistic fields.

3.8 Lie-Poisson Structures and the Rigid Body

Not every Poisson manifold is symplectic. For example, a large class of non-symplectic Poisson manifolds is the class of Lie-Poisson manifolds, which we now define. Let G be a Lie group and $\mathfrak{g} = T_eG$ its Lie algebra with $[\,,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ the associated Lie bracket.

Proposition 3.8.1. *The dual space* $\mathfrak{g}^*$ *is a Poisson manifold with either of the two brackets*

$$\{f, k\}_\pm(\mu) = \pm \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta k}{\delta \mu}\right]\right\rangle . \tag{3.8.1}$$

Here $\mathfrak{g}$ is identified with $\mathfrak{g}^{**}$ in the sense that $\delta f/\delta\mu \in \mathfrak{g}$ is defined by $\langle \nu, \delta f/\delta\mu\rangle = \mathbf{D}f(\mu)\cdot\nu$ for $\nu \in \mathfrak{g}^*$, where $\mathbf{D}$ denotes the derivative. (In the infinite dimensional case one needs to worry about the existence of $\delta f/\delta\mu$; in this context, methods like the Hahn-Banach theorem are not always appropriate!) The notation $\delta f/\delta\mu$ is used to conform to the functional derivative notation in classical field theory. In coordinates, $(\xi^1, \ldots, \xi^m)$ on $\mathfrak{g}$ and corresponding dual coordinates $(\mu_1, \ldots, \mu_m)$ on $\mathfrak{g}^*$, the ***Lie-Poisson bracket*** (3.8.1) is

$$\{f, k\}_\pm(\mu) = \pm\mu_a C^a_{bc} \frac{\partial f}{\partial \mu_b}\frac{\partial k}{\partial \mu_c}; \tag{3.8.2}$$

here C^a_{bc} are the ***structure constants*** of $\mathfrak{g}$ defined by $[e_a, e_b] = C^c_{ab}e_c$, where $(e_1, \ldots, e_m)$ is the coordinate basis of $\mathfrak{g}$ and where, for $\xi \in \mathfrak{g}$, we write $\xi = \xi^a e_a$, and for $\mu \in \mathfrak{g}^*, \mu = \mu_a e^a$, where $(e^1, \ldots, e^m)$ is the dual basis. Formula (3.8.2) appears explicitly in Lie [1890], §75.

Which sign to take in (3.8.2) is determined by understanding ***Lie-Poisson reduction***, which can be summarized as follows. Let

$$\lambda : T^*G \to \mathfrak{g}^* \quad \text{be defined by} \quad p_g \mapsto (T_eL_g)^*p_g \in T^*_eG \cong \mathfrak{g}^* \tag{3.8.3}$$

and

$$\rho : T^*G \to \mathfrak{g}^* \quad \text{be defined by} \quad p_g \mapsto (T_eR_g)^*p_g \in T^*_eG \cong \mathfrak{g}^*. \tag{3.8.4}$$

Then λ *is a Poisson map if one takes the* $-$ *Lie-Poisson structure on* $\mathfrak{g}^*$ *and* ρ *is a Poisson map if one takes the* $+$ *Lie-Poisson structure on* $\mathfrak{g}^*$.

Every left invariant Hamiltonian and Hamiltonian vector field is mapped by λ to a Hamiltonian and Hamiltonian vector field on $\mathfrak{g}^*$. There is a similar statement for right invariant systems on T^*G. One says that the original system on T^*G has been ***reduced*** to $\mathfrak{g}^*$. The reason λ and ρ are both Poisson maps is perhaps best understood by observing that they are both equivariant momentum maps generated by the action of G on itself by right and left translations, respectively. We take up this topic in §2.7.

We saw in Chapter 1 that the ***Euler equations*** of motion for rigid body dynamics are given by

$$\dot{\Pi} = \Pi \times \Omega, \tag{3.8.5}$$

where $\Pi = \mathbb{I}\Omega$ is the body angular momentum and Ω is the body angular velocity. Euler's equations are Hamiltonian relative to a Lie-Poisson structure. To see this, take $G = SO(3)$ to be the configuration space. Then $\mathfrak{g} \cong (\mathbb{R}^3, \times)$ and we identify $\mathfrak{g} \cong \mathfrak{g}^*$. The corresponding Lie-Poisson structure on $\mathbb{R}^3$ is given by

$$\{f, k\}(\Pi) = -\Pi \cdot (\nabla f \times \nabla k). \tag{3.8.6}$$

For the rigid body one chooses the minus sign in the Lie-Poisson bracket. This is because the rigid body Lagrangian (and hence Hamiltonian) is left invariant and so its dynamics pushes to $\mathfrak{g}^*$ by the map λ in (3.8.3).

Starting with the kinetic energy Hamiltonian derived in Chapter 1, we directly obtain the formula $H(\Pi) = \frac{1}{2}\Pi \cdot (\mathbb{I}^{-1}\Pi)$, the kinetic energy of the rigid body. One verifies from the chain rule and properties of the triple product that:

Proposition 3.8.2. *Euler's equations are equivalent to the following equation for all $f \in \mathcal{F}(\mathbb{R}^3)$:*

$$\dot{f} = \{f, H\}. \tag{3.8.7}$$

Definition 3.8.3. *Let $(P, \{\,,\})$ be a Poisson manifold. A function $C \in \mathcal{F}(P)$ satisfying*

$$\{C, f\} = 0 \quad \textit{for all} \quad f \in \mathcal{F}(P) \tag{3.8.8}$$

*is called a **Casimir function**.*

A crucial difference between symplectic manifolds and Poisson manifolds is this: On symplectic manifolds, the only Casimir functions are the constant functions (assuming P is connected). On the other hand, on Poisson manifolds there is often a large supply of Casimir functions. In the case of the rigid body, every function $C : \mathbb{R}^3 \to \mathbb{R}$ of the form

$$C(\Pi) = \Phi(\|\Pi\|^2) \tag{3.8.9}$$

where $\Phi : \mathbb{R} \to \mathbb{R}$ is a differentiable function, is a Casimir function, as we noted in Chapter 1. Casimir functions are constants of the motion for *any* Hamiltonian since $\dot{C} = \{C, H\} = 0$ for any H. In particular, for the rigid body, $\|\Pi\|^2$ is a constant of the motion — this is the invariant sphere we saw in Chapter 1.

There is an intimate relation between Casimirs and symmetry generated conserved quantities, or ***momentum maps***, which we study in §2.7.

The maps λ and ρ induce Poisson isomorphisms between $(T^*G)/G$ and $\mathfrak{g}^*$ (with the $-$ and $+$ brackets respectively) and this is a special instance of Poisson reduction, as we will see in §2.8. The following result is one useful way of formulating the general relation between T^*G and $\mathfrak{g}^*$. We treat the left invariant case to be specific. Of course, the right invariant case is similar.

Theorem 3.8.4. *Let G be a Lie group and $H : T^*G \to \mathbb{R}$ be a left invariant Hamiltonian. Let $h : \mathfrak{g}^* \to \mathbb{R}$ be the restriction of H to the identity. For a curve $p(t) \in T^*_{g(t)}G$, let $\mu(t) = (T^*_{g(t)}L) \cdot p(t) = \lambda(p(t))$ be the induced curve in $\mathfrak{g}^*$. Assume that $\dot{g} = \partial H/\partial p \in T_gG$. Then the following are equivalent:*

- **i** *$p(t)$ is an integral curve of X_H; i.e., Hamilton's equations on T^*G hold,*
- **ii** *for any $F \in \mathcal{F}(T^*G)$, $\dot{F} = \{F, H\}$, where $\{\,,\}$ is the canonical bracket on T^*G*
- **iii** *$\mu(t)$ satisfies the **Lie-Poisson equations***

$$\frac{d\mu}{dt} = \mathrm{ad}^*_{\delta h/\delta\mu}\mu \tag{3.8.10}$$

where $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ is defined by $\mathrm{ad}_\xi \eta = [\xi, \eta]$ and ad^*_ξ is its dual, i.e.,

$$\dot{\mu}_a = C^d_{ba}\frac{\delta h}{\delta \mu_b}\mu_d \tag{3.8.11}$$

- **iv** *for any $f \in \mathcal{F}(\mathfrak{g}^*)$, we have*

$$\dot{f} = \{f, h\}_- \tag{3.8.12}$$

where $\{\,,\}_-$ is the minus Lie-Poisson bracket.

We now make some remarks about the proof. First of all, the equivalence of i and ii is general for any cotangent bundle, as we have already noted. Next, the equivalence of ii and iv follows directly from the fact that λ is a Poisson map (as we have mentioned, this follows from the fact that λ is a momentum map; see Proposition 2.7.6 below) and $H = h \circ \lambda$. Finally, we establish the equivalence of iii and iv. Indeed, $\dot{f} = \{f, h\}_-$ means

$$\begin{aligned}\left\langle \dot{\mu}, \frac{\delta f}{\delta\mu}\right\rangle &= -\left\langle \mu, \left[\frac{\delta f}{\delta\mu}, \frac{\delta h}{\delta\mu}\right]\right\rangle \\ &= \left\langle \mu, \mathrm{ad}_{\delta h/\delta\mu}\frac{\delta f}{\delta\mu}\right\rangle \\ &= \left\langle \mathrm{ad}^*_{\delta h/\delta\mu}\mu, \frac{\delta f}{\delta\mu}\right\rangle.\end{aligned}$$

Since f is arbitrary, this is equivalent to iii. ■

3.9 The Euler-Poincaré Equations

In §1.3 we saw that for the rigid body, there is an analogue of the above theorem on $SO(3)$ and $\mathfrak{so}(3)$ using the Euler-Lagrange equations and the variational principle as a starting point. We now generalize this to an arbitrary Lie group and make the direct link with the Lie-Poisson equations.

Theorem 3.9.1. *Let G be a Lie group and $L : TG \to \mathbb{R}$ a left invariant Lagrangian. Let $l : \mathfrak{g} \to \mathbb{R}$ be its restriction to the identity. For a curve $g(t) \in G$, let*

$$\xi(t) = g(t)^{-1} \cdot \dot{g}(t); \quad i.e., \quad \xi(t) = T_{g(t)} L_{g(t)^{-1}} \dot{g}(t).$$

Then the following are equivalent

i *$g(t)$ satisfies the Euler-Lagrange equations for L on G,*

ii *the variational principle*

$$\delta \int L(g(t), \dot{g}(t)) dt = 0 \tag{3.9.1}$$

holds, for variations with fixed endpoints,

iii *the **Euler-Poincaré equations** hold:*

$$\frac{d}{dt} \frac{\delta l}{\delta \xi} = \mathrm{ad}_{\xi}^{*} \frac{\delta l}{\delta \xi}, \tag{3.9.2}$$

iv *the variational principle*

$$\delta \int l(\xi(t)) dt = 0 \tag{3.9.3}$$

holds on $\mathfrak{g}$, using variations of the form

$$\delta \xi = \dot{\eta} + [\xi, \eta], \tag{3.9.4}$$

where η vanishes at the endpoints.

Let us discuss the main ideas of the proof. First of all, the equivalence of i and ii holds on the tangent bundle of any configuration manifold Q, as we have seen, secondly, ii and iv are equivalent. To see this, one needs to compute the variations $\delta \xi$ induced on $\xi = g^{-1} \dot{g} = T L_{g^{-1}} \dot{g}$ by a variation of g. To calculate this, we need to differentiate $g^{-1} \dot{g}$ in the direction of a variation δg. If $\delta g = dg/d\epsilon$ at $\epsilon = 0$, where g is extended to a curve g_ϵ, then,

$$\delta \xi = \frac{d}{d\epsilon} \left(g^{-1} \frac{d}{dt} g \right) \bigg|_{\epsilon = 0}$$

while if $\eta = g^{-1}\delta g$, then

$$\dot{\eta} = \frac{d}{dt}\left(g^{-1}\frac{d}{d\epsilon}g\right)\bigg|_{\epsilon=0}.$$

The difference $\delta\xi - \dot{\eta}$ is the commutator, $[\xi, \eta]$. This argument is fine for matrix groups, but takes a little more work to make precise for general Lie groups. See Bloch, Krishnaprasad, Ratiu and Marsden [1994b] for the general case. Thus, ii and iv are equivalent.

To complete the proof, we show the equivalence of iii and iv. Indeed, using the definitions and integrating by parts,

$$\begin{aligned}\delta \int l(\xi) dt &= \int \frac{\delta l}{\delta \xi} \delta\xi \, dt \\ &= \int \frac{\delta l}{\delta \xi}(\dot{\eta} + \mathrm{ad}_\xi \eta) dt \\ &= \int \left[-\frac{d}{dt}\left(\frac{\delta l}{\delta \xi}\right) + \mathrm{ad}_\xi^* \frac{\delta l}{\delta \xi}\right] \eta \, dt\end{aligned}$$

so the result follows. ■

Generalizing what we saw directly in the rigid body, one can check directly from the Euler- Poincaré equations that conservation of spatial angular momentum holds:

$$\frac{d}{dt}\pi = 0 \tag{3.9.5}$$

where π is defined by

$$\pi = \mathrm{Ad}_g^* \frac{\partial l}{\partial \xi}. \tag{3.9.6}$$

Since the Euler-Lagrange and Hamilton equations on TQ and T^*Q are equivalent, it follows that the Lie-Poisson and Euler-Poincaré equations are also equivalent. To see this directly, we make the following Legendre transformation from $\mathfrak{g}$ to $\mathfrak{g}^*$:

$$\mu = \frac{\delta l}{\delta \xi}, \quad h(\mu) = \langle \mu, \xi \rangle - l(\xi).$$

Note that

$$\frac{\delta h}{\delta \mu} = \xi + \left\langle \mu, \frac{\delta \xi}{\delta \mu} \right\rangle - \left\langle \frac{\delta l}{\delta \xi}, \frac{\delta \xi}{\delta \mu} \right\rangle = \xi$$

and so it is now clear that (2.5.10) and (2.6.2) are equivalent.

3.10 Momentum Maps

Let G be a Lie group and P be a Poisson manifold, such that G acts on P by Poisson maps (in this case the action is called a ***Poisson action***). Denote the corresponding infinitesimal action of $\mathfrak{g}$ on P by $\xi \mapsto \xi_P$, a map of $\mathfrak{g}$ to $\mathfrak{X}(P)$, the space of vector fields on P. We write the action of $g \in G$ on $z \in P$ as simply gz; the vector field ξ_P is obtained at z by differentiating gz with respect to g in the direction ξ at $g = e$. Explicitly,

$$\xi_P(z) = \frac{d}{d\epsilon}[\exp(\epsilon\xi) \cdot z]\bigg|_{\epsilon=0}.$$

Definition 3.10.1. *A map $\mathbf{J} : P \to \mathfrak{g}^*$ is called a **momentum map** if $X_{\langle \mathrm{J},\xi\rangle} = \xi_P$ for each $\xi \in \mathfrak{g}$, where $\langle \mathbf{J}, \xi\rangle(z) = \langle \mathbf{J}(z), \xi\rangle$.*

Theorem 3.10.2. **(Noether's Theorem)** *If H is a G invariant Hamiltonian on P, then $\mathbf{J}$ is conserved on the trajectories of the Hamiltonian vector field X_H.*

Proof Differentiating the invariance condition $H(gz) = H(z)$ with respect to $g \in G$ for fixed $z \in P$, we get $\mathbf{d}H(z) \cdot \xi_P(z) = 0$ and so $\{H, \langle \mathbf{J}, \xi\rangle\} = 0$ which by antisymmetry gives $\mathbf{d}\langle \mathbf{J}, \xi\rangle \cdot X_H = 0$ and so $\langle \mathbf{J}, \xi\rangle$ is conserved on the trajectories of X_H for every ξ in G. ■

Turning to the construction of momentum maps, let Q be a manifold and let G act on Q. This action induces an action of G on T^*Q by cotangent lift — that is, we take the transpose inverse of the tangent lift. The action of G on T^*Q is always symplectic and therefore Poisson.

Theorem 3.10.3. *A momentum map for a cotangent lifted action is given by*

$$\mathbf{J} : T^*Q \to \mathfrak{g}^* \quad \textit{defined by} \quad \langle \mathbf{J}, \xi\rangle(p_q) = \langle p_q, \xi_Q(q)\rangle. \tag{3.10.1}$$

In canonical coordinates, we write $p_q = (q^i, p_j)$ and define the ***action functions*** $K^i_{\ a}$ by $(\xi_Q)^i = K^i_{\ a}(q)\xi^a$. Then

$$\langle \mathbf{J}, \xi\rangle(p_q) = p_i K^i_{\ a}(q)\xi^a \tag{3.10.2}$$

and therefore

$$J_a = p_i K^i_{\ a}(q). \tag{3.10.3}$$

Recall that by differentiating the conjugation operation $h \mapsto ghg^{-1}$ at the identity, one gets the ***adjoint action*** of G on $\mathfrak{g}$. Taking its dual produces the ***coadjoint action*** of G on $\mathfrak{g}^*$.

Proposition 3.10.4. *The momentum map for cotangent lifted actions is **equivariant**, i.e., the diagram in* Figure 2.7.1 *commutes.*

$$
\begin{tikzcd}
T^*Q \arrow[r, "\mathbf{J}"] \arrow[d, "\substack{G\text{-action}\\ \text{on } T^*Q}"'] & \mathfrak{g}^* \arrow[d, "\substack{\text{coadjoint}\\ \text{action}}"] \\
T^*Q \arrow[r, "\mathbf{J}"'] & \mathfrak{g}^*
\end{tikzcd}
$$

FIGURE 3.10.1. Equivariance of the momentum map.

Proposition 3.10.5. *Equivariance implies infinitesimal equivariance, which can be stated as the* ***classical commutation relations****:*

$$\{\langle \mathbf{J}, \xi \rangle, \langle J, \eta \rangle\} = \langle \mathbf{J}, [\xi, \eta] \rangle.$$

Proposition 3.10.6. *If* $\mathbf{J}$ *is infinitesimally equivariant, then* $\mathbf{J} : P \to \mathfrak{g}^*$ *is a Poisson map. If* $\mathbf{J}$ *is generated by a left* (*respectively right*) *action then we use the* $+$ (*respectively* $-$) *Lie-Poisson structure on* $\mathfrak{g}^*$.

The above development concerns momentum maps using the Hamiltonian point of view. However, one can also consider them from the Lagrangian point of view. In this context, we consider a Lie group G acting on a configuration manifold Q and lift this action to the tangent bundle TQ using the tangent operation. Given a G-invariant Lagrangian $L : TQ \to \mathbb{R}$, the corresponding momentum map is obtained by replacing the momentum p_q in (3.10.1) with the fiber derivative $\mathbb{F}L(v_q)$. Thus, $\mathbf{J} : TQ \to \mathfrak{g}^*$ is given by

$$\langle \mathbf{J}(v_q), \xi \rangle = \langle \mathbb{F}L(v_q), \xi_Q(q) \rangle \tag{3.10.4}$$

or, in coordinates,

$$J_a = \frac{\partial L}{\partial \dot{q}^i} K_a^i, \tag{3.10.5}$$

where the action coefficients K_a^i are defined as before by writing $\xi_Q(q^i) = K_a^i \xi^a \partial / \partial q^i$.

Proposition 3.10.7. *For a solution of the Euler-Lagrange equations* (*even if the Lagrangian is degenerate*), $\mathbf{J}$ *is constant in time.*

Proof In case L is a regular Lagrangian, this follows from its Hamiltonian counterpart. It is useful to check it directly using Hamilton's principle (which is the way it was originally done by Noether). To do this, choose any function $\phi(t, s)$ of two variables such that the conditions $\phi(a, s) = \phi(b, s) = \phi(t, 0) = 0$ hold. Since L is G-invariant, for each Lie algebra element $\xi \in \mathfrak{g}$, the expression

$$\int_a^b L(\exp(\phi(t,s)\xi)q, \exp(\phi(t,s)\xi)\dot{q}))\, dt \tag{3.10.6}$$

is independent of s. Differentiating this expression with respect to s at $s = 0$ and setting $\phi' = \partial\phi/\partial s$ taken at $s = 0$, gives

$$0 = \int_a^b \left(\frac{\partial L}{\partial q^i} \xi_Q^i \phi' + \frac{\partial L}{\partial \dot{q}^i} (T\xi_Q \cdot \dot{q})^i \phi' \right) dt. \tag{3.10.7}$$

Now we consider the variation $q(t,s) = \exp(\phi(t,s)\xi) \cdot q(t)$. The corresponding infinitesimal variation is given by

$$\delta q(t) = \phi'(t)\xi_Q(q(t)).$$

By Hamilton's principle, we have

$$0 = \int_a^b \left(\frac{\partial L}{\partial q^i} \delta q^i + \frac{\partial L}{\partial \dot{q}^i} \dot{\delta q}^i \right) dt. \tag{3.10.8}$$

Note that

$$\dot{\delta q} = \dot{\phi}' \xi_Q + \phi' (T\xi_Q \cdot \dot{q})$$

and subtract (3.10.8) from (3.10.7) to give

$$0 = \int_a^b \frac{\partial L}{\partial \dot{q}^i} (\xi_Q)^i \dot{\phi}'\, dt \tag{3.10.9}$$

$$= \int_a^b \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}^i} \xi_Q^i \right) \phi'\, dt. \tag{3.10.10}$$

Since ϕ' is arbitrary, except for endpoint conditions, it follows that the integrand vanishes, and so the time derivative of the momentum map is zero and so the proposition is proved. ■

3.11 Symplectic and Poisson Reduction

We have already seen how to use variational principles to reduce the Euler-Lagrange equations. On the Hamiltonian side, there are three levels of reduction of decreasing generality, that of Poisson reduction, symplectic reduction, and cotangent bundle reduction. Let us first consider Poisson reduction.

For ***Poisson reduction*** we start with a Poisson manifold P and let the Lie group G act on P by Poisson maps. Assuming P/G is a smooth manifold, endow it with the unique Poisson structure on such that the canonical projection $\pi : P \to P/G$ is a Poisson map. We can specify the Poisson structure on P/G explicitly as follows. For f and $k : P/G \to \mathbb{R}$, let $F = f \circ \pi$ and $K = k \circ \pi$, so F and K are f and k thought of as G-invariant functions on P. Then $\{f, k\}_{P/G}$ is defined by

$$\{f, k\}_{P/G} \circ \pi = \{F, K\}_P. \tag{3.11.1}$$

To show that $\{f, k\}_{P/G}$ is well defined, one has to prove that $\{F, K\}_P$ is G-invariant. This follows from the fact that F and K are G-invariant and the group action of G on P consists of Poisson maps.

For $P = T^*G$ we get a very important special case.

Theorem 3.11.1. **(Lie-Poisson Reduction)** *Let $P = T^*G$ and assume that G acts on P by the cotangent lift of left translations. If one endows $\mathfrak{g}^*$ with the minus Lie-Poisson bracket, then $P/G \cong \mathfrak{g}^*$.*

For ***symplectic reduction*** we begin with a symplectic manifold (P, Ω). Let G be a Lie group acting by symplectic maps on P; in this case the action is called a ***symplectic action***. Let $\mathbf{J}$ be an equivariant momentum map for this action and H a G-invariant Hamiltonian on P. Let $G_\mu = \{g \in G \mid g \cdot \mu = \mu\}$ be the isotropy subgroup (symmetry subgroup) at $\mu \in \mathfrak{g}^*$. As a consequence of equivariance, G_μ leaves $\mathbf{J}^{-1}(\mu)$ invariant. Assume for simplicity that μ is a regular value of $\mathbf{J}$, so that $\mathbf{J}^{-1}(\mu)$ is a smooth manifold (see §2.8 below) and that G_μ acts freely and properly on $\mathbf{J}^{-1}(\mu)$, so that $\mathbf{J}^{-1}(\mu)/G_\mu =: P_\mu$ is a smooth manifold. Let $i_\mu : \mathbf{J}^{-1}(\mu) \to P$ denote the inclusion map and let $\pi_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ denote the projection. Note that

$$\dim P_\mu = \dim P - \dim G - \dim G_\mu. \tag{3.11.2}$$

Building on classical work of Jacobi, Liouville, Arnold and Smale, we have the following basic result of Meyer [1973] and Marsden and Weinstein [1974].

Theorem 3.11.2. **(Reduction Theorem)** *There is a unique symplectic structure Ω_μ on P_μ satisfying*

$$i_\mu^* \Omega = \pi_\mu^* \Omega_\mu. \tag{3.11.3}$$

Given a G-invariant Hamiltonian H on P, define the reduced Hamiltonian $H_\mu : P_\mu \to \mathbb{R}$ by $H = H_\mu \circ \pi_\mu$. Then the trajectories of X_H project to those of X_{H_μ}. An important problem is how to reconstruct trajectories of X_H from trajectories of X_{H_μ}. Schematically, we have the situation in Figure 2.8.1.

As we shall see later, the reconstruction process is where the holonomy and "geometric phase" ideas enter. In fact, we shall put a connection on the bundle $\pi_\mu : \mathbf{J}^{-1}(\mu) \to P_\mu$ and it is through this process that one encounters the gauge theory point of view of mechanics.

$$
\begin{tikzcd}
P \arrow[dd, bend right=60, "\text{reduction}"'] \\
\mathbf{J}^{-1}(\mu) \arrow[u, "i_\mu"'] \arrow[d, "\pi_\mu"'] \\
P_\mu \arrow[uu, bend right=60, "\text{reconstruction}"']
\end{tikzcd}
$$

FIGURE 3.11.1. Reduction to P_μ and reconstruction back to P.

Let $\mathcal{O}_\mu$ denote the coadjoint orbit through μ. As a special case of the symplectic reduction theorem, we get

Corollary 3.11.3. $(T^*G)_\mu \cong \mathcal{O}_\mu$.

The symplectic structure inherited on $\mathcal{O}_\mu$ is called the ***(Lie-Kostant-Kirillov) orbit symplectic structure***. This structure is compatible with the Lie-Poisson structure on $\mathfrak{g}^*$ in the sense that the bracket of two functions on $\mathcal{O}_\mu$ equals that obtained by extending them arbitrarily to $\mathfrak{g}^*$, taking the Lie-Poisson bracket on $\mathfrak{g}^*$ and then restricting to $\mathcal{O}_\mu$.

Example 1 $G = SO(3), \mathfrak{g}^* = so(3)^* \cong \mathbb{R}^3$. In this case the coadjoint action is the usual action of $SO(3)$ on $\mathbb{R}^3$. This is because of the orthogonality of the elements of G. The set of orbits consists of spheres and a single point. The reduction process confirms that all orbits are symplectic manifolds. One calculates that the symplectic structure on the spheres is a multiple of the area element. ⧫

Example 2 ***Jacobi-Liouville theorem*** Let $G = \mathbb{T}^k$ be the k-torus and assume G acts on a symplectic manifold P. In this case the components of $\mathbf{J}$ are in involution and $\dim P_\mu = \dim P - 2k$, so $2k$ variables are eliminated. As we shall see, reconstruction allows one to reassemble the solution trajectories on P by quadratures in this abelian case. ⧫

Example 3 ***Jacobi-Deprit elimination of the node*** Let $G = SO(3)$ act on P. In the classical case of Jacobi, $P = T^*\mathbb{R}^3$ and in the generalization of Deprit [1983] one considers the phase space of n particles in $\mathbb{R}^3$. We just point out here that the reduced space P_μ has dimension $\dim P - 3 - 1 = \dim P - 4$ since $G_\mu = S^1$ (if $\mu \neq 0$) in this case. ⧫

The ***orbit reduction theorem*** of Marle [1976] and Kazhdan, Kostant and Sternberg [1978] states that P_μ may be alternatively constructed as

$$P_{\mathcal{O}} = \mathbf{J}^{-1}(\mathcal{O})/G, \tag{3.11.4}$$

where $\mathcal{O} \subset \mathfrak{g}^*$ is the coadjoint orbit through μ. As above we assume we are away from singular points (see §2.8 below). The spaces P_μ and $P_{\mathcal{O}}$ are isomorphic by using the inclusion map $l_\mu : \mathbf{J}^{-1}(\mu) \to \mathbf{J}^{-1}(\mathcal{O})$ and taking equivalence classes to induce a symplectic isomorphism $L_\mu : P_\mu \to P_{\mathcal{O}}$. The symplectic structure $\Omega_{\mathcal{O}}$ on $P_{\mathcal{O}}$ is uniquely determined by

$$j_{\mathcal{O}}^* \Omega = \pi_{\mathcal{O}}^* \Omega_{\mathcal{O}} + \mathbf{J}_{\mathcal{O}}^* \omega_{\mathcal{O}} \tag{3.11.5}$$

where $j_{\mathcal{O}} : \mathbf{J}^{-1}(\mathcal{O}) \to P$ is the inclusion, $\pi_{\mathcal{O}} : \mathbf{J}^{-1}(\mathcal{O}) \to P_{\mathcal{O}}$ is the projection, and where $\mathbf{J}_{\mathcal{O}} = \mathbf{J}|\mathbf{J}^{-1}(\mathcal{O}) : \mathbf{J}^{-1}(\mathcal{O}) \to \mathcal{O}$ and $\omega_{\mathcal{O}}$ is the orbit symplectic form. In terms of the Poisson structure, $\mathbf{J}^{-1}(\mathcal{O})/G$ has the bracket structure inherited from P/G; in fact, $\mathbf{J}^{-1}(\mathcal{O})/G$ *is a symplectic leaf* in P/G. Thus, we get the picture in Figure 2.8.2.

$$\begin{tikzcd}
\mathbf{J}^{-1}(\mu) \arrow[d, "/G_\mu"'] & \subset & \mathbf{J}^{-1}(\mathcal{O}) \arrow[d, "/G"'] & \subset & P \arrow[d, "/G"'] \\
P_\mu & \cong & P_{\mathcal{O}} & \subset & P/G
\end{tikzcd}$$

FIGURE 3.11.2. Orbit reduction gives another realization of P_μ.

Kirillov has shown that *every* Poisson manifold P is the union of symplectic leaves, although the preceding construction explicitly realizes these symplectic leaves in this case by the reduction construction. A special case is the foliation of the dual $\mathfrak{g}^*$ of any Lie algebra $\mathfrak{g}$ into its symplectic leaves, namely the coadjoint orbits. For example $SO(3)$ is the union of spheres plus the origin, each of which is a symplectic manifold. Notice that the drop in dimension from $T^*SO(3)$ to $\mathcal{O}$ is from 6 to 2, a drop of 4, as in general $SO(3)$ reduction. An exception is the singular point, the origin, where the drop in dimension is larger. We turn to these singular points next.

3.12 Singularities and Symmetry

Proposition 3.12.1. *Let (P, Ω) be a symplectic manifold, let G act on P by Poisson mappings, and let $\mathbf{J} : P \to \mathfrak{g}^*$ be a momentum map for this action ($\mathbf{J}$ need not be equivariant). Let G_z denote the symmetry group of $z \in P$ defined by $G_z = \{g \in G \mid gz = z\}$ and let $\mathfrak{g}_z$ be its Lie algebra, so $\mathfrak{g}_z = \{\zeta \in \mathfrak{g} \mid \zeta_P(z) = 0\}$. Then z is a regular value of $\mathbf{J}$ if and only if $\mathfrak{g}_z$ is trivial; i.e., $\mathfrak{g}_z = \{0\}$, or G_z is discrete.*

Proof The point z is regular when the range of the linear map $\mathbf{DJ}(z)$ is all of $\mathfrak{g}^*$. However, $\zeta \in \mathfrak{g}$ is orthogonal to the range (in the sense of the $\mathfrak{g}, \mathfrak{g}^*$ pairing) if and only if for all $v \in T_z P$,

$$\langle \zeta, \mathbf{DJ}(z) \cdot v \rangle = 0$$

i.e.,

$$\mathbf{d}\langle \mathbf{J}, \zeta \rangle (z) \cdot v = 0$$

or

$$\Omega(X_{\langle \mathbf{J}, \zeta \rangle}(z), v) = 0$$

or

$$\Omega(\zeta_P(z), v) = 0.$$

As Ω is nondegenerate, ζ is orthogonal to the range iff $\zeta_P(z) = 0$. ■

The above proposition is due to Smale [1970]. It is the starting point of a large literature on singularities in the momentum map and singular reduction. Arms, Marsden and Moncrief [1981] show, under some reasonable hypotheses, that the level sets $\mathbf{J}^{-1}(0)$ have *quadratic* singularities. As we shall see in the next chapter, there is a general *shifting construction* that enables one to effectively reduce $\mathbf{J}^{-1}(\mu)$ to the case $\mathbf{J}^{-1}(0)$. In the finite dimensional case, this result can be deduced from the equivariant Darboux theorem, but in the infinite dimensional case, things are much more subtle. In fact, the infinite dimensional results were motivated by, and apply to, the singularities in the solution space of relativistic field theories such as gravity and the Yang-Mills equations (see Fischer, Marsden and Moncrief [1980], Arms, Marsden and Moncrief [1981, 1982] and Arms [1981]). The ***convexity theorem*** states that the image of the momentum map of a torus action is a convex polyhedron in $\mathfrak{g}^*$; the boundary of the polyhedron is the image of the singular (symmetric) points in P; the more symmetric the point, the more singular the boundary point. These results are due to Atiyah [1982] and Guillemin and Sternberg [1984] based on earlier convexity results of Kostant and the Shur-Horne theorem on eigenvalues of symmetric matrices. The literature on these topics and its relation to other areas of mathematics is vast. See, for example, Goldman and Millison [1990], Sjamaar [1990], Bloch, Flaschka and Ratiu [1990], Sjamaar and Lerman [1991] and Lu and Ratiu [1991].

3.13 A Particle in a Magnetic Field

During cotangent bundle reduction considered in the next chapter, we shall have to add terms to the symplectic form called "magnetic terms". To explain this terminology, we consider a particle in a magnetic field.

Let B be a closed two-form on $\mathbb{R}^3$ and $\mathbf{B} = B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k}$ the associated divergence free vector field, *i.e.,* $\mathbf{i}_\mathbf{B}(dx \wedge dy \wedge dz) = B$, or

$$B = B_x dy \wedge dz - B_y dx \wedge dz + B_z dx \wedge dy.$$

Thinking of $\mathbf{B}$ as a magnetic field, the equations of motion for a particle with charge e and mass m are given by the ***Lorentz force law***:

$$m\frac{d\mathbf{v}}{dt} = \frac{e}{c}\mathbf{v} \times \mathbf{B} \tag{3.13.1}$$

where $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$. On $\mathbb{R}^3 \times \mathbb{R}^3$ *i.e.,* on $(\mathbf{x}, \mathbf{v})$-space, consider the symplectic form

$$\Omega_B = m(dx \wedge d\dot{x} + dy \wedge d\dot{y} + dz \wedge d\dot{z}) - \frac{e}{c}B. \tag{3.13.2}$$

For the Hamiltonian, take the kinetic energy:

$$H = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{3.13.3}$$

writing $X_H(u, v, w) = (u, v, w, (\dot{u}, \dot{v}, \dot{w}))$, the condition defining X_H, namely $\mathbf{i}_{X_H}\Omega_B = \mathbf{d}H$ is

$$\begin{aligned} &m(u d\dot{x} - \dot{u} dx + v d\dot{y} - \dot{v} dy + w d\dot{z} - \dot{w} dz) \\ &\qquad - \frac{e}{c}[B_x v dz - B_x w dy - B_y u dz + B_y w dx + B_z u dy - B_z v dx] \\ &\quad = m(\dot{x} d\dot{x} + \dot{y} d\dot{y} + \dot{z} d\dot{z}) \end{aligned} \tag{3.13.4}$$

which is equivalent to $u = \dot{x}, v = \dot{y}, w = \dot{z}, m\dot{u} = e(B_z v - B_y w)/c, m\dot{v} = e(B_x w - B_z u)/c$, and $m\dot{w} = e(B_y u - B_x v)/c$, *i.e.,* to

$$\begin{aligned} m\ddot{x} &= \frac{e}{c}(B_z\dot{y} - B_y\dot{z}) \\ m\ddot{y} &= \frac{e}{c}(B_x\dot{z} - B_z\dot{x}) \\ m\ddot{z} &= \frac{e}{c}(B_y\dot{x} - B_x\dot{y}) \end{aligned} \tag{3.13.5}$$

which is the same as (3.13.1). Thus the *equations of motion for a particle in a magnetic field are Hamiltonian, with energy equal to the kinetic energy and with the symplectic form* Ω_B.

If $B = \mathbf{d}A$; *i.e.,* $\mathbf{B} = -\nabla \times \mathbf{A}$, where A is a one-form and $\mathbf{A}$ is the associated vector field, then the map $(\mathbf{x}, \mathbf{v}) \mapsto (\mathbf{x}, \mathbf{p})$ where $\mathbf{p} = m\mathbf{v} + e\mathbf{A}/c$ pulls back the *canonical* form to Ω_B, as is easily checked. *Thus, Equations* (3.13.1) *are also Hamiltonian relative to the canonical bracket on* $(\mathbf{x}, \mathbf{p})$*-space with the Hamiltonian*

$$H_\mathbf{A} = \frac{1}{2m}\|\mathbf{p} - \frac{e}{c}\mathbf{A}\|^2. \tag{3.13.6}$$

Even in Euclidean space, not every magnetic field can be written as $\mathbf{B} = \nabla \times \mathbf{A}$. For example, the field of a magnetic monopole of strength $g \neq 0$, namely

$$\mathbf{B}(\mathbf{r}) = g \frac{\mathbf{r}}{\|\mathbf{r}\|^3} \tag{3.13.7}$$

cannot be written this way since the flux of $\mathbf{B}$ through the unit sphere is $4\pi g$, yet Stokes' theorem applied to the two hemispheres would give zero. Thus, one might think that the Hamiltonian formulation involving only B (*i.e.,* using Ω_B and H) is preferable. However, one can recover the magnetic potential A by regarding A as a connection on a nontrivial bundle over $\mathbb{R}^3 \backslash \{0\}$. The bundle over the sphere S^2 is in fact the same ***Hopf fibration*** $S^3 \to S^2$ that we encountered in §1.6. This same construction can be carried out using reduction. For a readable account of some aspects of this situation, see Yang [1980]. For an interesting example of Weinstein in which this monopole comes up, see Marsden [1981], p. 34.

When one studies the motion of a colored (rather than a charged) particle in a Yang-Mills field, one finds a beautiful generalization of this construction and related ideas using the theory of principal bundles; see Sternberg [1977], Weinstein [1978] and Montgomery [1985]. In the study of centrifugal and Coriolis forces one discovers some structures analogous to those here (see Marsden and Ratiu [1999] for more information).

3.14 The Mechanical Connection

An important connection for analyzing the dynamics and control of mechanical systems is the so-called mechanical connection – see e.g. Marsden and Scheurle [1993], Marsden [1992].

We assume we have a configuration manifold Q and Lagrangian $L : TQ \to \mathbb{R}$. Let G be a Lie group with Lie algebra $\mathfrak{g}$, assume G acts on Q, and lift the action to TQ via the tangent mapping. We assume that G acts freely and properly on Q. We assume also we have a metric $\langle\!\langle\, , \rangle\!\rangle$ on Q that is invariant under the group action.

For each $q \in Q$ define the the ***locked inertia tensor*** to be the map $\mathbb{R} : \mathfrak{g} \to \mathfrak{g}^*$ defined by

$$\langle \mathbb{I}\eta, \zeta \rangle = \langle\!\langle \eta_Q(q), \zeta_Q(q) \rangle\!\rangle \ . \tag{3.14.1}$$

Here η_Q denotes as before the infinitesimal generator of the action of G on Q.

Locally, if

$$[\xi_Q(q)]^i = K^i_a(q)\xi^a \tag{3.14.2}$$

relative to the coordinates q^i of Q and a basis $e_a, a = 1, \cdots m$ of $\mathfrak{g}$ (K^i_a are called the action coefficients), then

$$\mathbb{I}_{ab} = g_{ij} K^i_a K^j_b \tag{3.14.3}$$

Now the momentum map in this context, $\mathbf{J} : TM \to \mathfrak{g}^*$, is defined by

$$\langle \mathbf{J}(q,v), \xi \rangle = \langle\langle \xi_Q(q), v \rangle\rangle \ , \tag{3.14.4}$$

Then

Definition 3.14.1. *We define the* ***mechanical connection*** *on the principal bundle $q \to Q/G$ to be the map $A : TQ \to \mathfrak{g}$ given by*

$$A(q,v) = \mathbb{I}(q)^{-1} \left(\mathbf{J}(q,v) \right) \ , \tag{3.14.5}$$

that is, A is the map that assigns to each (q,v) the corresponding angular velocity of the locked system.

In coordinates

$$A^a = \mathbb{I}^{ab} g_{ij} K^i_b v^j \ . \tag{3.14.6}$$

One can check that is A is G-equivariant and $(\xi_Q(q)) = \xi$.

The horizontal space of the connection is given by

$$\mathrm{hor}_q = \{(q,v) | \mathbf{J} = 0\} \subset T_q Q \, . \tag{3.14.7}$$

The vertical space of vectors that are mapped to zero under the projection $Q \to S = Q/G$ is given by

$$\mathrm{ver}_q = \{\xi_Q(q) | \xi \in \mathfrak{g}\} \, . \tag{3.14.8}$$

The horizontal-vertical decomposition of a vector $(q,v) \in T_q Q$ is given by

$$v = \mathrm{hor}_q v + \mathrm{ver}_q v \tag{3.14.9}$$

where

$$\mathrm{ver}_q v = [A(q,v)]_Q(q) \qquad \text{and} \qquad \mathrm{hor}_q v = v - \mathrm{ver}_q v \, . \tag{3.14.10}$$

3.14.1 Example – the nonlinear pendulum

As example for illustrating the mechanical connection we recall the inverted pendulum on a cart. We consider here the uncontrolled case.

Recall the the Lagrangian may be written

$$L = \frac{1}{2} \left(\alpha \dot\theta^2 - 2\beta \cos\theta \dot\theta \dot s + \gamma \dot s^2 + D \cos\theta \right) \, . \tag{3.14.11}$$

In this case the Lagrangian is cyclic in the variable s or invariant under the linear R^1 action $s \to s + a$.

The infintesimal generator of this action is thus given by

$$\xi_Q = \frac{d}{dt}|_{t=0} (s + at, \theta) = (a, 0) . \tag{3.14.12}$$

In this case using the mechanical metric induced by the Lagrangian we have

$$J(q, v) = \langle \mathbf{J}(q, v), a \rangle = \langle FL(q, v), (0, a) \rangle \tag{3.14.13}$$

and hence

$$J = \frac{\partial L}{\partial \dot{s}} = \gamma \dot{s} - \beta \cos \theta \dot{\theta} . \tag{3.14.14}$$

The locked inertia tensor $\mathbb{I}(q)$ is given here by

$$\langle \mathbb{I}(q) a, a \rangle = \langle\langle (0, a), (0, a) \rangle\rangle \tag{3.14.15}$$

and hence

$$\mathbb{I}(q) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha & -\beta \cos \theta \\ -\beta \cos \theta & \gamma \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \gamma . \tag{3.14.16}$$

Hence the action of the mechanical connection on a tangent vector v at the point q is given by

$$A(q, v) = \frac{1}{\gamma} \left(\gamma \dot{s} - \beta \cos \theta \dot{\phi} \right) = \left(\dot{s} - \frac{\beta}{\gamma} \cos \theta \dot{\phi} \right) \tag{3.14.17}$$

The vertical horizontal decomposition of a vector v is given by

$$\begin{aligned} \mathrm{ver}_q v &= [A(q, v)]_Q(q) = \left(0, \dot{s} - \frac{\beta}{\gamma} \cos \theta \dot{\phi} \right) \\ \mathrm{hor}_q v &= v - \mathrm{ver}_q v = \left(\dot{\phi}, \frac{\beta}{\gamma} \cos \phi \dot{\phi} \right) \end{aligned} \tag{3.14.18}$$

Note that the horizontal component of v is clearly zero under the action of A.

4
An Introduction to Geometric Control Theory

4.1 Nonlinear Control Systems

There are many texts on linear control theory, and a number of introductions to nonlinear control theory and in particular its differential geometric formulation which is important for this book. We mention the books by Jurdjevic [1997], Isidori [1995], Nijmeijer and van der Schaft [1990], Sontag [1990] and Brockett [1972].[1] The first three deal with the differential geometric approach, Sontag's book gives a mathematical treatment of both linear theory and various aspects of nonlinear theory, whil Brockett's book gives an approach to linear theory. We refer the reader to these books for a detailed treatment of nonlinear control theory as well a for a more exhaustive list of references. There are of course many other good sources as well and we shall refer to them as needed.

[1]Isidori, Alberto Nonlinear control systems. Third edition. Communications and Control Engineering Series. Springer-Verlag, Berlin, 1995; Nijmeijer, Henk; van der Schaft, Arjan Nonlinear dynamical control systems. Springer-Verlag, New York, 1990; Sontag, Eduardo D. Mathematical control theory. Deterministic finite-dimensional systems. Texts in Applied Mathematics, 6. Springer-Verlag, New York, 1990; Brockett, R. [1970] *Finite Dimensional Linear Systems.* Wiley; Jurdjevic, Velimir Geometric control theory. Cambridge Studies in Advanced Mathematics, 52. Cambridge University Press, Cambridge, 1997.

Definition 4.1.1. *A* ***finite dimensional nonlinear control system*** *on a smooth n-manifold is a differential equation of the form*

$$\dot{x} = f(x, u) \tag{4.1.1}$$

where $x = (x_1, \cdots, x_n)$ are local coordinates on M, $u(t)$ is a time dependent map from the nonnegative reals, $\mathbb{R}^{1+}$ to $\mathbb{R}^m$, and f and u lie in appropriate class of functions. Usually f is taken to be C^∞ (smooth) or C^ω (analytic), but less often suffices. u may be smooth or continuous, but may be also be discontinuous and even just measurable. M is said to be the ***state space*** *of the system.*

It is possible to generalize this definition further (see e.g. Brockett [1973]), but the above is usually sufficient for our purposes.

A enormous amount can of course be said about such systems. We restrict ourselves here to the concepts of accessibility, controllability, and stabilizablity and feeback linearizability.

We also restrict ourselves here to considering so-called ***affine nonlinear control systems*** which have the form

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i \,, \tag{4.1.2}$$

where f and the g_i, $i = 1, \ldots, m$ are smooth vector fields on M. f is usually called the **drift** vector field and the g_i are called the control vector fields.

4.1.1 Controllability and Accesibility

Definition 4.1.2. *The system 4.1.2 is said to be* ***controllable*** *if for any two points x_0 and x_f in M there exists a control $u(t)$ such that the system 4.1.2 with initial condition x_0 reaches the point x_f in some finite time T.*

This is a basic concept in control theory and much work has been done on deriving sufficient conditions for it. However, proving controllability is not easy in general. A related property, called ***local accessibility*** is often much easier to prove.

To define accessbility we first need the notion of reachable set. The reachable set from a given point at time T is essentially the set of points that may be reached by the system by travelling on trajectories from the initial point.

More precisely:

Definition 4.1.3. *Given $x_0 \in M$ we define $R(x_0, T)$ to be the set of all $x \in M$ for which there exists a control u such that there is a trajectory of (4.1.2) with $x(0) = x_0$ and $x(T) = x$. The* ***reachable set at time*** *T is defined to be*

$$R_T(x_0) = \cup_{t \le T} R(x_0, t) \,. \tag{4.1.3}$$

We also define:

Definition 4.1.4. *The **accessiblity algebra** $\mathcal{C}$ of (4.1.2) is the smallest Lie algebra of vector fields on M that contains the vector fields f and $g_1, \cdots, g_m$.*

Note that the accessbility algebra is just the span of all possible Lie brackets of f and the g_i.

Definition 4.1.5. *We define the accessbility distribution C of 4.1.2 to be the distribution generated by the vector fields in $\mathcal{C}$, i,e, $C(x)$ is the span of the vector fields X in $\mathcal{C}$ at x.*

We then have (see e.g., Nijmeijer and van der Schaft [1990] or Sontag [1990])

Theorem 4.1.6. *Consider the system (4.1.2). If dim $C(x_0) = n$ (i.e. the accessibilty algebra spans the tangent space to M at x_0) then for any $T > 0$, the set $R_T(x_0)$ has nonempty interior.*

If the rank condition in the preceding theorem holds we say the system is ***accessible at*** x_0 or that the ***accessibility rank condition*** holds.

Note that while this spanning condition is an intuitively reasonable condition the resulting theorem is quite weak and is far from applying controllability.

In certain special cases the accessibility rank condition does imply controllability however.

Theorem 4.1.7. *If dim $C(x) = n$ everywhere on M and either*

(i) $f = 0$, or

(ii) f is divergence free and M is compact

then (4.1.2) is controllable everywhere.

The idea behind this result is that one cannot move "backwards" along the drift directions and hence a spanning condition inolving the drift vector field does not guarentee controllability. However, if the drift is divergence free and the underlying manifold is compact Poincaré recurrance (see e.g. Arnold [1978]) ensure a drift "backward" eventually.

Exercise: Show that the Heisenberg system

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= vx - uy \end{aligned} \tag{4.1.4}$$

is accessible everywhere and hence controllable everywhere in $\mathbb{R}^3$.

Another case of interest where controllabiltiy implies accesibility is a linear system of the form:

$$\dot{x} = Ax + \sum_{i=1}^{m} b_i u_i = Ax + Bu \tag{4.1.5}$$

where $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^n \times \mathbb{R}^n$ and $B \in \mathbb{R}^n \times \mathbb{R}^m$ are constant matrices, b_i being the columns of B.

The Lie bracket of the drift vector field Ax with b_i is $-Ab_i$. Bracketing the latter constant field with Ax and so on, tells is that $\mathcal{C}$ is spanned by $Ax, b_i, Ab_i, \cdots A^{n-1}b_i, \qquad i = 1, \cdots m.$

This gives the standard controllability rank condition:

Theorem 4.1.8. *The system 4.1.5 is controllable if and only if*

$$\text{rank}[B, AB, \cdots, A^{n-1}B] = n\,. \tag{4.1.6}$$

Note that in this case accessibilty is equivalent to controllability but that the drift vector field is involved.

4.1.2 Stability and Stabilizability

In this section we summarize briefly some notions of stability and stabilizability.

Definition 4.1.9. *Let x_0 be an equilibrium of the system of differential equations $\dot{x} = f(x)$. The point x_0 is said to be **nonlinearly** or **Lyapunov stable** if for any neigborhood U of x_O there exists a neigborhood $V \subset U$ of x_0 such that any trajectory $x(t)$ of the system with initial point in V remains in U for all time. If in addition $x(t) \to x_0$ as $t \to \infty$, x_0 is said to be **aymptotically stable**.*

Definition 4.1.10. *Let x_0 be an equilibrium of the system of differential equations $\dot{x} = f(x)$. x_0 is said to be **spectrally stable** if all the eigenvalues of the linearization of f at x_0 lie in the left half plane.*

A theorem of Liapunov states that spectral stability implies asymptotic stability.

Definition 4.1.11. *Let x_0 be an equilibrium of the control system $\dot{x} = f(x, u)$. The system is said to be **nonlinearly (aymptotically) stablizable** at x_0 if a feedback control $u(x)$ can be found which renders the system nonlinearly (asympotically) stable.*

We remark that in much of the control literature stabilizability is taken to mean asymptotic stabilizability. However in this book we will distinguish between the two.

4.2 Stabilization Techniques

Here we discuss some stabilization techniques in the literature that are of relevence to the material in this book. Good references for classical material on this subject are Sontag [1990] and Nijmeijer and van der Schaft [1990]. See also Brockett [1983].

4.2.1 Linearization

The simplest result on stabilization for nonlinear systems is based on linearization. Consider the nonlinear control system

$$\dot{x} = f(x, u)\,, \tag{4.2.1}$$

with $x \in \mathbb{R}^n$ (for simplicity) and $u \in \mathbb{R}^m$, and its linearization about (x_0, u_0),

$$\dot{x} = Ax + Bu \tag{4.2.2}$$

where $A = \frac{\partial f}{\partial x}(x_0, u_0)$ and $B = \frac{\partial f}{\partial u}(x_0, u_0)$.

Then if $\mathcal{R}$ is the reachable subspace of the linearized system, i.e. the range of $[B, AB, \cdots, A^{n-1}]$, we may find a change of basis such that the system takes the form

$$\frac{d}{dt}\begin{bmatrix} x_u \\ x_l \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}\begin{bmatrix} x_u \\ x_l \end{bmatrix} + \begin{bmatrix} B_u \\ 0 \end{bmatrix} u\,, \tag{4.2.3}$$

where Range $[B, AB, \cdots, A^{n-1}]$ has dimension x_u.

If the eigenvalues of A_{22} has all eigenvalues in the left half plane a feedback may be found which stabilizes the system asymptotically to the origin. (For further details see e.g. Nijmeijer and Van der Schaft [1990]).

4.2.2 Necessary Conditions

A beautiful general theorem on necessary conditions for feedback stabilization of nonlinear systems was given by Brockett [1983].

Theorem 4.2.1 (Brockett). *Consider the nonlinear system $\dot{x} = f(x, u)$ with $f(x_0, 0) = 0$ and $f(\cdot, \cdot)$ continuously differentiable in a neighborhood of $(x_0, 0)$. Necessary conditions for the existence of a continuously differentiable control law for asymptotically stabilizing $(x_0, 0)$ are:*

(i) *The linearized system has no uncontrollable modes associated with eigenvalues with postive real part.*

(ii) *There exists a neighborhood N of $(x_0, 0)$ such that for each $\xi \in N$ there exists a control $u_\xi(t)$ defined for all $t > 0$ which drives the solution of $\dot{x} = f(x, u_\xi)$ from the point $x = \xi$ at $t = 0$ to $x = x_0$ at $t = \infty$.*

(iii) *The mapping $\gamma : N \times \mathbb{R}^m \to R^n$, N a neighborhood of the origin, defined by $\gamma : (x, u) \to f(x, u)$ should be onto an open set of the origin.*

Proof. Part (i) has been proved above. (ii) holds since if a system is asymptotically stabilizable clearly there solutions near the origin must approch it.

To prove (iii) consider the closed loop system

$$\dot{x} = f(x, u(x)) \equiv a(x) \tag{4.2.4}$$

and suppose that x_0 is locally aymptotically stable. Then for sufficiently small r the map from the ball of radius r about x_0 into S^{n-1} given by

$$x \to \frac{a(x)}{||a(x)||} \tag{4.2.5}$$

has degree $n - 1$ (see e.g. Milnor [1965]). Since this degree is nonzero the map is actually onto and hence the result. ■

A stronger necessary was condition was given by Coron [1989] but we do not discuss this here.

4.2.3 Lyapunov Methods for asymptotic stability

Another case of interest is that where the free system is Lyapunov stable, and the controls may be used to make the system as ymptotically stable. Issues of this sort are of key importance in the recent work Bloch, Leonard and Marsden [1997], [1998] for example and we shall return to them later. In that case the Hamiltonian structure of the system is used for energy shaping and then a suitable dissipation is added.

For the moment we just summarize the kind of general argument that one can find for example in Nijmeijer and van der Schaft [1990].

Consider again the affine control system

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x)\,. \tag{4.2.6}$$

Suppose that the free systems has equilibrium x_0 and suppose there exists a Lyapunov function $V(x)$ for the free system about x_0 in some neighborhood U of x_0.

One then considers feedback of the following form:

$$u_i(x) = -\mathcal{L}_{g_i} V(x)\,. \tag{4.2.7}$$

Then x_0 is an equilibrium point for the closed loop system and V remains a Lyapunov function for the the closed loop system The question is now whether the system is in addition asympotically stable. For this we can use a LaSalle invariance principle argument.

Consider the set

$$W = \{x \in U \mid \mathcal{L}_f V(x) = 0, \, \mathcal{L}_{g_i} V(x) = 0, \, i = 1, \cdots m\}. \tag{4.2.8}$$

Let W_0 be the largest invariant set in W under the closed loop dynamics. We observe that x_0 is in W since V is a Lypaunov function. If W_0 is identically equal to $\{x_0\}$ then x_0 is a locally an asympototically stable equilibrium point by LaSalle. Notice also that any trajectory of the closed loop system in W is also a trajectory of the free dynamics. Hence by LaSalle the system is locally asymptotically stable about x_0 if the largest invariant subset of the free dynamics in W equals x_0. If in addition $dV(x) \neq 0$ for $x \in U/x_0$ this condition is also necessary since, by the definition of W, along any trajectory in W of the closed loop system and hence of the free system the Lyapunov function is constant and a nontrivial trajectory can therefore not approach x_0.

Of course one would like to apply this reasoning without knowing the free dynamics. Some conditions for this are discussed in Nijmiejer and van der Schaft. The intuitive idea of course is that one would like W to be as small as possible. Locally a sufficient condition for stability is simply that the distribution

$$\operatorname{span}\{f(x), \operatorname{ad}_f^k g_i(x), \, i = 1 \cdots m, , \text{ for all } k \geq 0\} \tag{4.2.9}$$

should have rank n at x_0. But this is equivalent to the linearized system being stabilizable.

4.2.4 The center manifold

Another important tool in the analysis of the stabilization of nonlinear control systems is center manifold theory. Given that there exists a center manifold for the system and the remaining dynamnics is stable one can concentrate on the stabilizing the center manifold dynamics. A nice application of this technique to the rigid body dynamics is given in the work of Aeyels (see e.g. Aeyels [1985]) and the discussion in Nijmeijer and van der Schaft [1990]. We do not give any details here but will return to the center manifold when we consider nonholnomic systems.

4.3 Hamiltonian and Lagrangian Control Systems

This extension to the notion of Hamiltonian and Lagrangian systems to the setting of control was formally proposed in Brockett [1976] and extended and formalized by Willems [], van der Schaft [1983, 1986] and others. The book Nijmeijer and van der Schaft [1990] gives a nice summary of many of the ideas.

We begin with the Lagrangian side. The simplest form of Lagrangian control system is a Lagrangian system with external forces: in local coordinates we have

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= u_i, \;\; i = 1, \cdots, m \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= 0, \;\; i = m+1, \cdots n \cdots \end{aligned} \tag{4.3.1}$$

More generally we have the system

$$\frac{d}{dt}\left(\frac{\partial L(q, \dot{q}, u)}{\partial \dot{q}_i}\right) - \frac{\partial L(q, \dot{q}, u)}{\partial q_i} = 0 \tag{4.3.2}$$

for $q \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

The latter system is a nontrivial generalization of the former, for example if the control input is a velocity rather than a force (see Brockett [1976], Nijmeijer and van der Schaft [1990]).

Similarly one can define a Hamiltonian control system. Locally one has:

$$\begin{aligned} \dot{q}_i &= \frac{\partial H(q, p, u)}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H(q, p, u)}{\partial q_i} \end{aligned} \tag{4.3.3}$$

for $q, p \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

One can easily generalize this to a Hamiltonian control system on a symplectic or Poisson manifold.

Let M be a Poisson manifold and $H_0, H_1, \cdots, H_m$ be smooth functions on M. Then an ***affine Hamiltonian control system*** on M is given by

$$\dot{x} = X_{H_0}(x) + \sum_{i=1}^{m} X_{H_j}(x) u_i \tag{4.3.4}$$

where $x \in M$ and as usual X_{H_j} is the Hamiltonian vector field corresponding to H_j.

We shall return to systems of this type. Good discussion are given in Krishnaprasad [1985] and Sanchez de Alvarez [1986].

5
Nonholonomic Mechanics

One of the points of closest contact between control theory and mechanics is the theory of nonholonomic mechanical systems. One reason for this is that nonintegrability is essential to both: nonintegrable constraints are the essence of nonholonomic systems, while a nonintegrable distribution of control vector fields is the key to controllabiltiy of nonlinear systems.

Nonintegrable constraints – systems with constraints on the velocity which are not derivable from positions consraints arise in such mechanical situations as that of rolling contact (wheels) or sliding contact (a skate). However, such constraints also occur in less obvious ways. For example, one may view angular momentum constraints, which are really integrals of motion and are integrable constraints on the phase space (functions of position and momentum) as nonintegrable constraints on the configuration space. This point of view is very helpful for contollability. Further, many first order control systems may be simply viewed as controlled distributions lying in the kernal of a nonintegrable constraint. A classic example of the latter is the Heisenberg system.

5.1 Equations of Motion

We follow for the moment here Bloch, Krishnaprasad, Marsden and Murray [1996]. We begin by deriving the equations of motion for a nonholonomic uncontrolled system from the the Lagrange-D'Alembert principle, although this is by no means the only way of deriving the equations. We shall come back to this point.

5.1.1 The Lagrange d'Alembert Principle

The starting point is a configuration space Q and a distribution $\mathcal{D}$ that describes the kinematic constraints of interest. Thus, $\mathcal{D}$ is a collection of linear subspaces denoted $\mathcal{D}_q \subset T_qQ$, one for each $q \in Q$. A curve $q(t) \in Q$ will be said to *satisfy the constraints* if $\dot{q}(t) \in \mathcal{D}_{q(t)}$ for all t. This distribution will, in general, be nonintegrable; *i.e.,* the constraints are, in general, nonholonomic. One of our goals below will be to model the constraints in terms of Ehresmann connections (see Cardin and Favretti [1994] and Marle [1994] for some related ideas).

The above setup describes *linear* constraints; for *affine* constraints, for example, a ball on a rotating turntable (where the rotational velocity of the turntable represents the affine part of the constraints), one assumes that there is a given vector field γ on Q and the constraints are written $\dot{q}(t) - \gamma(q(t)) \in \mathcal{D}_{q(t)}$.

Consider a Lagrangian $L : TQ \to \mathbb{R}$. In coordinates $q^i, i = 1, \ldots, n$, on Q with induced coordinates $(q^i, \dot{q}^i)$ for the tangent bundle, we write $L(q^i, \dot{q}^i)$. Following standard practice, we assume that the equations of motion are given by the by the so-called principle of Lagrange d'Alembert (see, for example, Rosenberg [1977] for a discussion).

Definition 5.1.1. *The **Lagrange d'Alembert equations of motion** for the system are those determined by*

$$\delta \int_a^b L(q^i, \dot{q}^i)\, dt = 0, \tag{5.1.1}$$

where we choose variations $\delta q(t)$ of the curve $q(t)$ that satisfy $\delta q(t) \in \mathcal{D}_q(t)$ for each $t, a \leq t \leq b$.

This principle is supplemented by the condition that the curve itself satisfies the constraints. In such a principle, we follow standard procedure and take the variation *before* imposing the constraints; that is, we *do not* impose the constraints on the family of curves defining the variation. The usual arguments in the calculus of variations show that this constrained variational principle is equivalent to the equations

$$-\delta L = \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} \right) \delta q^i = 0, \tag{5.1.2}$$

for all variations δq such that $\delta q \in \mathcal{D}_q$ at each point of the underlying curve $q(t)$.

To explore the structure of these equations in more detail, consider a mechanical system evolving on a configuration space Q with a given Lagrangian $L : TQ \to \mathbb{R}$ and let $\{\omega^a\}$ be a set of p independent one forms whose vanishing describe the constraints on the system. The constraints in general are nonintegrable. Choose a local coordinate chart and a local basis for the constraints such that

$$\omega^a(q) = ds^a + A^a_\alpha(r,s)dr^\alpha \qquad a = 1,\dots,m, \tag{5.1.3}$$

where $q = (r,s) \in \mathbb{R}^{n-m} \times \mathbb{R}^m$.

The equations of motion for the system are given by (5.1.2) where we choose variations $\delta q(t)$ that satisfy the condition $\omega^a(q)\cdot\delta q = 0$, *i.e.*, where the variation $\delta q = (\delta r, \delta s)$ satisfies $\delta s^a + A^a_\alpha \delta r^\alpha = 0$. Substitution into (5.1.2) gives

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot r^\alpha} - \frac{\partial L}{\partial r^\alpha}\right) = A^a_\alpha \left(\frac{d}{dt}\frac{\partial L}{\partial \dot s^a} - \frac{\partial L}{\partial s^a}\right), \qquad \alpha = 1,\dots,n-m. \tag{5.1.4}$$

Equation (5.1.4) combined with the constraint equations

$$\dot s^a = -A^a_\alpha \dot r^\alpha \qquad a = 1,\dots,m \tag{5.1.5}$$

gives a complete description of the equations of motion of the system.

We now define the "constrained" Lagrangian by substituting the constraints (5.1.5) into the Lagrangian:

$$L_c(r^\alpha, s^a, \dot r^\alpha) = L(r^\alpha, s^a, \dot r^\alpha, -A^a_\alpha(r,s)\dot r^\alpha).$$

The equations of motion can be written in terms of the constrained Lagrangian in the following way, as a direct coordinate calculation shows:

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot r^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a} = -\frac{\partial L}{\partial \dot s^b} B^b_{\alpha\beta}\dot r^\beta, \tag{5.1.6}$$

where

$$B^b_{\alpha\beta} = \left(\frac{\partial A^b_\alpha}{\partial r^\beta} - \frac{\partial A^b_\beta}{\partial r^\alpha} + A^a_\alpha \frac{\partial A^b_\beta}{\partial s^a} - A^a_\beta \frac{\partial A^b_\alpha}{\partial s^a}\right). \tag{5.1.7}$$

Letting $d\omega^b$ be the exterior derivative of ω^b, another computation (see Remark 4 below) shows that

$$d\omega^b(\dot q, \cdot) = B^b_{\alpha\beta}\dot r^\alpha dr^\beta$$

and hence the equations of motion have the form

$$-\delta L_c = \left(\frac{d}{dt}\frac{\partial L_c}{\partial \dot r^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a}\right)\delta r^a = -\frac{\partial L}{\partial \dot s^b} d\omega^b(\dot q, \delta r).$$

This form of the equations isolates the effects of the constraints, and shows that in the case where the constraints are integrable ($d\omega = 0$) then the correct equations of motion are obtained by substituting the constraints into the Lagrangian and setting the variation of L_c to zero. However in the non-integrable case the constraints generate extra forces which must be taken into account.

5.1.2 Nonholonomic equations of motion with Lagrange multipliers

We can obtain the nonholonomic equations of motion with Lagrange multipliers from the Lagrange D'Alembert principle as follows see e.g. Neimark and Fufaev [1972]:

Recall that the Lagrange-D'Alembert principle gives us:

$$(\frac{d}{dt}\frac{\partial}{\partial \dot{q}^i}L - \frac{\partial}{\partial q^i}L)\delta q^i = 0 \;\;, i = 1 \cdots n \,. \tag{5.1.8}$$

for variations $\delta q^i \in \mathcal{D}_Q$, i.e. for variations in the constraint distribution, i.e. for variations satisfying

$$\sum_{i=1}^{n} a_i^j \delta q^i = 0\,, \qquad j = 1 \cdots m\,. \tag{5.1.9}$$

But this implies that

$$\sum_{j=1}^{m}\sum_{i=1}^{n} \lambda_j a_i^j \delta q^i = 0\,. \tag{5.1.10}$$

Hence we can append this to the Lagrange-D'Alembert equations to obtain

$$\sum_{i=1}^{n}(\frac{d}{dt}\frac{\partial}{\partial \dot{q}^i}L - \frac{\partial}{\partial q^i}L - \sum_{j=1}^{m}\lambda_j a_i^j)\delta q^i = 0\,. \tag{5.1.11}$$

Now, assuming our constraints are independent implies that one of the m by m minors of the matrix a_i^j must be nonzero. Let us assume it to be that given by letting the indices run from 1 to m.

We can now assume that the variations $\delta q^{m+1} \cdots \delta q^n$ are arbitrary since the constraint equations 5.1.9 may then be satisfied by a choice of resulting definite values for variations $\delta q^1, \cdots, \delta q^m$.

We can now choose values of the Lagrange multipliers such that the expression in brackets in 5.1.11 vanishes for each dependent variation $\delta q^1, \cdots, \delta q^m$. This entails solving a linear systems of algebraic equations in the λ_i which is solvable by virtue of our assumptions on the a_i^j.

Once this is done however equation 5.1.11 becomes

$$\sum_{i=m+1}^{n}(\frac{d}{dt}\frac{\partial}{\partial \dot{q}^i}L - \frac{\partial}{\partial q^i}L - \sum_{j=1}^{m}\lambda_j a_i^j)\delta q^i = 0\,. \tag{5.1.12}$$

where the variations are independent. Hence each term in the brackets must also vanish independently. Putting the observations for the dependent and independent variable together gives us the set of n equations

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}^i}L - \frac{\partial}{\partial q^i}L = \sum_{j=1}^{m}\lambda_j a_i^j\,. \tag{5.1.13}$$

5.2 An Invariant Approach to Nonholonomic Mechanics

In this section we consider an invariant approach to Lagrangian mechanics in general and nonholonomic mechanics in particular. We follow here the approach of Vershik [] (see also Wang []). A general invariant approach to Lagrangian mechanics is also discussed for example in Marsden and Ratiu [1994]. We remark that the approach here is somewhat more abstract than other parts of the book and may be omitted without loss of contintuity.

5.2.1 Lagrangian Mechanics

We consider firstly a general invariant formulation of Lagrangian mechanics without constraints.

Let TQ be the tangent bundle of Q, an n-dimensional manifold, with its canonical projection and let it be locally co-ordinitized by $(q, \dot{q}) \sim (q, v)$. Here q, v represent n-vectors. We consider here notions of verticality and horizontality with respect to the trivial connection on TQ, i.e. combinations of vectors $\frac{\partial}{\partial q_i}$ are horizontal and combinations of vectors $\frac{\partial}{\partial v_i}$ are vertical. (It is important to bear this in mind when discussing constraints below, so as not to confuse this discussion with the connection given by the constraints discussed elsewhere.)

Since Lagrangian equations are second order we also need the second tangent bundle $T(TQ)$ with projection $d\pi : T(TQ) \to TQ$ locally co-ordinitized by $(q, v, v, \dot{v})$. In fact we want vector fields of the type

$$\dot{q} \quad = \quad f(q, \dot{q}) \tag{5.2.1}$$

or

$$\begin{aligned} &\dot{q} = v \\ &\dot{v} = f(q, v). \end{aligned} \tag{5.2.2}$$

Written as a vector field a second order equation takes the form

$$X_{q,v} = X_s = v \, \frac{\partial}{\partial q} + f \, \frac{\partial}{\partial v} \, , \tag{5.2.3}$$

(with the obvious summation notation).

Invariantly we may write this as $d\pi X_s = v$ – its projection onto the tangent space to Q is just v. We call such a vector field X_s a "special" vector field.

Now one can formulate Lagrangian mechanics in an invariant fashion by defining fields on TQ as follows:

Let T_qQ and $T_{q,v}TQ$ define the tangent space to Q at q and TQ at (q, v) respectively.

One can set up a canonical map from T_qQ to $T_{q,v}TQ$ which takes the fiber over q (locally vectors of the form $a\frac{\partial}{\partial q}$) into the fiber over $T_{q,v}TQ$ (vectors of the form $a\frac{\partial}{\partial q} + b\frac{\partial}{\partial v}$). Locally this map is defined by

$$\gamma_{q,v}\left(a\frac{\partial}{\partial q}\right) \;=\; a\frac{\partial}{\partial v}. \tag{5.2.4}$$

We now define a tensor field on TQ, called the principal tensor field, by

$$\tau_{q,v} = \gamma_{q,v}\; d\pi_{q,v}. \tag{5.2.5}$$

So locally we have

$$\begin{aligned}\tau_{q,v}\left(a\frac{\partial}{\partial q} + b\frac{\partial}{\partial v}\right) &= \gamma_{q,v}\; d\pi_{q,v}\left(a\frac{\partial}{\partial q} + b\frac{\partial}{\partial v}\right)\\ &= \gamma_{q,v}\left(a\frac{\partial}{\partial q}\right) = a\frac{\partial}{\partial v}.\end{aligned} \tag{5.2.6}$$

The dual tensor field τ^* acts locally on forms as follows

$$\tau^*(adq + bdv) = bdq. \tag{5.2.7}$$

This can be easily checked: we have

$$\begin{aligned}&\left\langle \tau^*(adq + bdv), c\frac{\partial}{\partial q} + d\frac{\partial}{\partial v}\right\rangle\\ &= \left\langle adq + bdv, \tau\left(c\frac{\partial}{\partial q} + d\frac{\partial}{\partial v}\right)\right\rangle\\ &= \left\langle adq + bdv, c\frac{\partial}{\partial v}\right\rangle = bc.\end{aligned} \tag{5.2.8}$$

Hence

$$\tau^*(adq + bdv) = bdq. \tag{5.2.9}$$

Now we define another key concept — the *fundamental* vector field on TQ: this is a field with coordinates $\Phi_{q,v} = \gamma_{q,v} v\frac{\partial}{\partial q} = v_i\;\frac{\partial}{\partial v_i}$. Clearly a vector field X is special if and only if $\tau X = \Phi$, since in local coordinates a special vector field is of the form

$$X_{q,v} \;= v_i\frac{\partial}{\partial q^i} + \cdots\;. \tag{5.2.10}$$

Now let L be the Lagrangian — as usual a smooth function on TQ. We note that locally

$$\tau^*(dL) = \tau^*\left(\frac{\partial L}{\partial q_i}dq_i + \frac{\partial L}{\partial q^i}\; d\dot{q}^i\right) = \frac{\partial L}{\partial \dot{q}^i}\; dq_i \tag{5.2.11}$$

— clearly a horizontal 1-form since it annihilates vertical vector fields $a\frac{\partial}{\partial v}$ (Vershik calls this form the impulse field of the Lagrangian.) One can identify horizontal 1-forms in Lagrangian mechanics as forces – see later).

We define the Lagrangian 2-form to be

$$\begin{aligned}\Omega_L &= d(\tau^* dL)\\ &= d\left(\frac{\partial L}{\partial \dot{q}^i}\, dq_i\right)\\ &= \frac{\partial^2 L}{\partial q_j \partial v_i}\, dq_j \wedge dq_i + \frac{\partial^2 L}{\partial v_j \partial v_i}\, dq_j \wedge dv_i .\end{aligned} \tag{5.2.12}$$

Recalling the Legendre transformation $p_i = \frac{\partial L}{\partial q_i}$ we see that the image of Ω_L under the Legendre transformation is the canonical symplectic form on the cotangent bundle. Similarly the Hamiltonian (energy) is

$$H_L = dL(\Phi) - L. \tag{5.2.13}$$

Note that in local coordinates

$$dL(\Phi) = dL\left(v_i \frac{\partial}{\partial v_i}\right) = \frac{\partial L}{\partial v_i} v_i = p_i v_i\,, \tag{5.2.14}$$

(with the summation convention).

Now we can formulate the Lagrange d'Alemert principle as follows:

Definition 5.2.1 (The Lagrange D'Alembert Principle). *The vector field Y describing the mechanical trajectories of motion is given by*

$$\Omega_L(X, Y) = dH_L(Y) + \omega(Y) \tag{5.2.15}$$

where ω is the 1-form describing the exterior forces and X is a special vector field.

In the absence of exterior forces we recover the Lagrange equations as follows. Let

$$X = v\frac{\partial}{\partial q} + \dot{v}\frac{\partial}{\partial v} Y = a\frac{\partial}{\partial q} + b\frac{\partial}{\partial v}. \tag{5.2.16}$$

Then

$$\Omega_L(X, Y) = \frac{\partial^2 L}{\partial q_j \partial v_i}(v_j a_i - a_j v_i) + \frac{\partial^2 L}{\partial v_j \partial v_i}(v_j b_i - a_j v_i) \tag{5.2.17}$$

$$\begin{aligned} dH_L(Y) &= d\ \left(\frac{\partial L}{\partial v_i} v_i - L\right)(Y)\\ &= \Big(\frac{\partial L}{\partial v_i} dv_i + \frac{\partial^2 L}{\partial q_j \partial v_i} v_i dq^j + \frac{\partial^2 L}{\partial v_i \partial v_j} v_i dv_j\\ &\quad - \frac{\partial L}{\partial q_i} dq_j - \frac{\partial L}{\partial v_i} dv_i\Big)(Y)\\ &= \frac{\partial L}{\partial v_i} b_i + \frac{\partial^2 L}{\partial q_j \partial v_j} v_i a_j + \frac{\partial^2 L}{\partial v_j \partial v_i} v_i b_j - \frac{\partial L}{\partial q_i} a_i - \frac{\partial L}{\partial v_i} b_i \end{aligned} \tag{5.2.18}$$

Equating coefficients of a_i and b_i which are arbitrary we get:

$$\frac{\partial^2 L}{\partial q_j \partial v_j}\, v_i - \frac{\partial^2 L}{\partial q_i \partial v_j}\, v_j - \frac{\partial^2 L}{\partial v_j \partial v_i}\, v_i = -\frac{\partial L}{\partial q_i} + \frac{\partial^2 L}{\partial q_j \partial v_i}\, v_j$$
$$\frac{\partial^2 L}{\partial v_i \partial v_j}\, v_j = \frac{\partial L}{\partial v_i} + \frac{\partial^2 L}{\partial v_j \partial v_j}\, v_j - \frac{\partial L}{\partial v_i}$$

The second equation is an identity while the first equation is just

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) = \frac{\partial L}{\partial q_i}. \tag{5.2.19}$$

as required.

5.2.2 Constrained Dynamics

We now consider the invariant formulation of dynamics with constraints.

With Vershik we assume a slightly more general form of the constraints – we assume that they define a distribution on TQ, i.e. they define at each point a subspace of $TTQ(q, \dot{q})$. This fits naturally with the definition of 2nd order systems, although it is more general than needed. Hence we can define the constraints to be 1-forms on TQ (a codistribution of TQ) and they thus take the form

$$\theta_i \;=\; \Sigma_{k=1}^{n} a_{ik}(q) dq_k + \Sigma_{k=1}^{n} b_{ik}(q) dv_k, \quad i = 1 \cdots m. \tag{5.2.20}$$

A constrained dynamical system is then a special (2nd order) vector field compatible with these constraints. The case of interest to us is when the θ_i are differentials of functions on TQ i.e. of the form $df(q, v)$ – in particular the case $f(q, v) = \sum_k a_{ik}(q) v_k$ so

$$\theta_i = \sum_k a_{ik} dv_k + \sum_{k,j} \frac{\partial a_{ik}}{\partial q_j} v_k dq_j \;. \tag{5.2.21}$$

We now want to define reaction forces — the forces that keep the system on the constant manifold and to show that the in the "ideal" case such forces do no work.

Definition 5.2.2. *Given constraints θ_i as in (5.2.20) we define $\tau^*\theta_i (= \sum b_{ik} dq_k)$ to be the reaction forces.*

Note that in the case (5.2.21) this is just

$$\tau^*\theta_i = \sum_k a_{ik} dq_k. \tag{5.2.22}$$

Definition 5.2.3. *A set of of constraints is said to be admissible if* $Dim\text{span}\, \tau^*\theta_i = Dim\text{span}\, \theta_i$.

(This is equivalent to saying that the codistribution given by the θ_i has no horizontal covectors at any point, since the kernal of τ^* is horizontal 1-forms. Another way to think of this: since horizontal vectors are linear combinations of vectors $\frac{\partial}{\partial q_i}$, the vectors $a_i = (a_{i1}, \cdots, a_{in})$ must be linear combination of the vectors $b_i = (b_{i1}, \cdots, b_{in})$ so that if one has some vectors in the span of the $\frac{\partial}{\partial q_i}$ there is also a component in the span of the $\frac{\partial}{\partial v_i}$.

Definition 5.2.4. *A constraint is said to be ideal if it annihilates the fundamental vector field*
Φ *(*$= \sum_i v_i \frac{\partial}{\partial v_i}$ *locally).*

Then we have

Theorem 5.2.5. *If a set of constraints is admissable there exist special vector fields satisfying the constraints.*

Proof. Recall that the special vector fields are those satisfying $\tau X = \Phi$. Therefore we need to ask if the system

$$\tau X = \Phi \quad \theta_i(X) = 0, \quad i = 1 \cdots m \tag{5.2.23}$$

is solvable.

Let $X = \sum_i v_i \frac{\partial}{\partial q_i} + f_i \frac{\partial}{\partial v_i}$. Since the constraints are admissable, by the argument above the vectors a_i are in the span of the vectors b_i. Now we require

$$\theta_i(X) = \sum_i b_{ik} f_k + \sum_i a_{ik} v_k = 0. \tag{5.2.24}$$

Thus we wish to know whether the system of linear equations

$$\sum_i b_{ik} f_k = -\sum_i aikv_k$$

in the f_i can be solved. But this is clear by the admissability argument. In fact, since there are m conditions on the f_i there are at least $n - m$ independent special vector fields. ∎

Now we say a force does no work if $\int F \cdot\ dq = 0$ along any curve in the configuration space.

Theorem 5.2.6. *If a constraint is ideal the corresponding constraint forces do no work (i.e. they are virtual forces.)*

Proof. Let ξ be a closed curve in Q, i.e. a map from S^1 to Q and let $\tilde{\xi}$ be its lift to TQ.

Then

$$\int_{\tilde{\xi}} \tau^* \theta_i = \int_{S^1} \left\langle \tau^* \theta_i , \dot{\xi} \right\rangle = \int_{S^1} \left\langle \theta_i , \tau \dot{\xi} \right\rangle$$

where $<,>$ is the natural pairing between forms and their duals.

But if $\xi = q(t)$ and $\tilde{\xi} = (q(t), \dot{q}(t))$ then

$$\dot{\xi} = \dot{q} \frac{\partial}{\partial q} + \ddot{q} \frac{\partial}{\partial v}. \tag{5.2.25}$$

Hence $\tau(\dot{\xi}) = \dot{q} \, \frac{\partial}{\partial v} = v \, \frac{\partial}{\partial v} = \Phi$. Thus we get

$$\int_{S^1} \langle \theta_i , \Phi \rangle \;=\; 0 \tag{5.2.26}$$

by ideality.

The usual arguments imply this integral is zero along any curve. ■

Now we can show

Theorem 5.2.7. *Let $\theta_i, \ i = 1 \dots m$ define a constraint distribution on TQ and let L be a nondegenerate Lagrangian with positive definite Hessian $\frac{\partial^2 L}{\partial v_i \partial v_j}$. Then there exists a special vector field X satisfying the Lagrange d'Alembert principle*

$$\Omega_L(X) = (dH_L + \omega)(X) \, , \tag{5.2.27}$$

where ω is the constraint force that ensures $\theta^i(X) = 0, \ i = 1, \dots, m$. Locally the equations of motion are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial q_i} \right) \;-\; \frac{\partial L}{\partial q_i} \;=\; \sum \lambda_i \theta_i . \tag{5.2.28}$$

Proof. Since Ω_L is nondegenerate there is a map form 1-forms to vector fields π_L such that

$$\Omega_L \left(\pi_L(\rho), Y \right) = \rho(Y) \tag{5.2.29}$$

for Y an arbitrary tangent vector and ρ an arbitrary 1-form. Then

$$X = \pi_L(dH_L + \omega) \tag{5.2.30}$$

and we require

$$\left\langle \theta_i , \pi_L(dH_L) + \pi_L(\omega) \right\rangle = 0 \qquad i = 1, \dots, m. \tag{5.2.31}$$

Now ω, by definition of the constraint force, needs to be a linear combination of $\tau^* \theta_i$, i.e. of the form $\sum_i \lambda_i a_{ik} dq_k$.That is, we have

$$\left\langle \theta_i \, , \pi_L(dH_L) \right\rangle = - \left\langle \theta_i \, , \sum_i \lambda_i \pi_L \tau^* \theta_i \right\rangle \;\; i = 1, \dots, m. \tag{5.2.32}$$

Since the constraints are admissable, τ^* preserves the dimension of the span of θ_i, but since π_L is positive definite the restriction of π_L to the span of the $\tau^*\theta_i$ is also positive definite and thus nondegenerate. Since the operator $\pi_L\tau^*$ is thus of full rank, we can solve the system for λ_i.

The local form of the equations follows from the general Lagrange theory above and of course agree with the coordinate form derived elsewhere. ■

5.3 Projected Connections and Newton's law for Nonholonomic Systems

In this section we follow up on the formulation of the equations of motion of nonholonomic systems with forces or controls discussed in the previous section and indicate how to write such system as second order forced systems (i.e. satisfying Newton's law) on the constraint subbundle. This follows as approach due to Vershik and Fadeev [1981], but see also Montgomery [1990], Yang [] and Bloch and Crouch [1995, 97].

Let us rewrite our forced nonholonomic system 6.5.6 on the Riemannian manifold M as

$$\frac{D^2q}{dt^2} = \sum_{i=1}^{m} \lambda_i W_i + F \tag{5.3.1}$$

where F is an arbitrary external force field and the system is subject to the independent constraints

$$\omega_i(\dot{q}) = < W_i, \dot{q} > = 0\,, \tag{5.3.2}$$

where the W_i are vector fields on M.

The constraint distribution N is given by

$$N_p = \{X_p : \omega_i(X) = 0,\ 1 \le i \le m\} \tag{5.3.3}$$

for X a vector field on M and p a point on M.

We now use the independence of the one forms ω_i to eliminate the multipliers: Let $a_{ij} = \omega_i(W_j),\ 1 \le i, j \le m$. Then the matrix with entries a_{ij} is invertible on M.

Differentiating the constraints gives

$$\frac{D\omega_i}{\partial t}(\dot{q}) + \omega_i\Big(\sum_j \lambda_j W_j + F\Big) = 0,\ 1 \le i \le m\,. \tag{5.3.4}$$

Hence the multipliers have the explicit form:

$$\lambda_k = -\sum_j a_{kj}^{-1}\left(\frac{D\omega_j}{\partial t}(\dot{q}) + \omega_j(F)\right),\ 1 \le k \le m\,. \tag{5.3.5}$$

Hence the equations of motion may be rewritten as

$$\frac{D^2q}{dt^2} + \sum_{k,j} W_k a_{kj}^{-1} \frac{D\omega_j}{\partial t} = F - \sum_{k,j} W_k a_{kj}^{-1} \omega_j(F)$$
$$\omega_i(\dot q) = 0 \;\; 1 \le i \le m \tag{5.3.6}$$

Now let X e any vector field on M.

Definition 5.3.1. *Let*

$$\pi_N(X) = X - \sum_{ki} W_k a_{ki}^{-1} \omega_i(X)\,. \tag{5.3.7}$$

We note that $\pi_N(X)$ is a projection of X onto the constraint distribution N.

Now apply this projection to $\frac{D^2q}{dt^2}$ and observe that differentiating the constraints gives

$$\omega_i\left(\frac{D^2q}{dt^2}\right) + \frac{D\omega_i}{\partial t}(\dot q) = 0\,.$$

Thus

$$\pi_N\left(\frac{D^2q}{dt^2}\right) = \frac{D^2q}{dt^2} + \sum_{i,k} W_k a_{ki}^{-1} \frac{D\omega_i}{\partial t}(\dot q)\,. \tag{5.3.8}$$

Hence equation 5.3.6 may be written

$$\pi_N\left(\frac{D^2q}{dt^2}\right) = \pi_N(F),\;\; \dot q \in N\,. \tag{5.3.9}$$

We now make the following natural definition:

Definition 5.3.2. *If $\overline{\nabla}$ is any connection on M with corresponding covariant derivative $\frac{\overline{D}}{\partial t}$ then the second order system*

$$\frac{\overline{D}^2 q}{dt^2} = \overline{F},\;\; q \in M\,. \tag{5.3.10}$$

is said to be a system on M satisfying Newton's law with forces $\overline{F}$.

Thus our nonholonomic systemis a projection of the Newton law system $\frac{D^2q}{dt^2} = F$ to N.

However, as Vershik and Fadeev pointed out, by redefining the connection, the system may be written as a system obeying Newton's law on N directly.

We define a new connection on M;

$$\nabla'_X Y = \nabla_X Y + \sum_{i=1}^{m} W_i a_{ik}^{-1} (\nabla_X \omega_k)(Y)\,, \tag{5.3.11}$$

for X, Y vector fields on M. This in turn defines a covariant derivative $\frac{D'}{\partial t}$. From 5.3.11 we note that

$$\omega_i(\nabla'_X Y) = \omega_i(\nabla_X Y) + (\nabla_x \omega_i)(Y) = X(\omega_i(Y))\,. \tag{5.3.12}$$

Thus is X and Y are vector fields on N so is $\nabla'_X Y$ and thus $\nabla'|_N$ defines a connection on the subbundle N of TM and the nonholonomic system 5.3.1 may be viewed as a system of Newton law type on N:

$$\frac{D'^2 q}{dt^2} = \pi_N(F),\ \dot{q} \in N\,. \tag{5.3.13}$$

Thus the nonholonomic system is a systems of Newton law type with respect to a modified connection. This connection is not metric however. In Bloch and Crouch [1997] we explore extensions of this system to a non-metric Newton law system on the whole of M.

5.4 Systems with Symmetry

We now add symmetry to our nonholonomic system. We will begin with some general remarks about symmetry.

5.4.1 Group Actions and Invariance

We remind the reader of a few of the concepts discussed ealier. As before we refer the reader to Marsden & Ratiu [1994], Chapter 9 for further details. Assume that we are given a Lie group G and an action of G on Q. The action of G will be denoted $q \mapsto gq = \Phi_g(q)$. The group orbit through a point q, which is always an (immersed) submanifold, is denoted

$$\mathrm{Orb}(q) := \{gq \mid g \in G\}.$$

When there is danger of confusion about which group is meant, we write the orbit as $\mathrm{Orb}_G(q)$.

Let $\mathfrak{g}$ denote the Lie algebra of the Lie group G . For an element $\xi \in \mathfrak{g}$, we write ξ_Q, a vector field on Q for the corresponding infinitesimal generator; recall that this is obtained by differentiating the flow $\Phi_{\exp(t\xi)}$ with respect to t at $t = 0$. The tangent space to the group orbit through a point q is given by the set of infinitesimal generators at that point:

$$T_q(\mathrm{Orb}(q)) = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\}.$$

Throughout this paper we make the assumption that the action of G on Q is free (none of the maps Φ_g has any fixed points) and proper (the map $(q, g) \mapsto gq$ is proper; *i.e.,* that is, the inverse images of compact sets are compact). The case of nonfree actions is very important and the investigation of the associated singularities needs to be carried out, but that topic is not the subject of the present paper.

The quotient space $M = Q/G$, whose points are the group orbits, is called ***shape space***. It is known that if the group action is free and proper then shape space is a smooth manifold and the projection map $\pi : Q \to Q/G$ is a smooth surjective map with a surjective derivative $T_q\pi$ at each point. We will denote the projection map by $\pi_{Q,G}$ if there is any danger of confusion. The kernel of the linear map $T_q\pi$ is the set of infinitesimal generators of the group action at the point q, *i.e.*,

$$\ker T_q\pi = \{\xi_Q(q) \mid \xi \in \mathfrak{g}\},$$

so these are also the tangent spaces to the group orbits.

We now introduce some assumptions concerning the relation between the given group action, the Lagrangian, and the constraint distribution.

Definition 5.4.1.

(L1) *We say that the Lagrangian is* ***invariant*** *under the group action if L is invariant under the induced action of G on TQ.*

(L2) *We say that the Lagrangian is* ***infinitesimally invariant*** *if for any Lie algebra element $\xi \in \mathfrak{g}$ we have $dL \circ \dot{\xi}_Q = 0$ where, for a vector field X on Q, $\dot{X}$ denotes the vector field on TQ naturally induced by it (if F_t is the flow of X then the flow of $\dot{X}$ is TF_t).*

(S1) *We say that the distribution $\mathcal{D}$ is* ***invariant*** *if the subspace $\mathcal{D}_q \subset T_qQ$ is mapped by the tangent of the group action to the subspace $\mathcal{D}_{gq} \subset T_{gq}Q$.*

(S2) *An Ehresmann connection A on Q (that has $\mathcal{D}$ as its horizontal distribution) is* ***invariant*** *under G if the group action preserves the bundle structure associated with the connection (in particular, it maps vertical spaces to vertical spaces) and if, as a map from TQ to the vertical bundle, A is G-equivariant.*

(S3) *A Lie algebra element ξ is said to act* ***horizontally*** *if $\xi_Q(q) \in \mathcal{D}_q$ for all $q \in Q$.*

Some relationships between these conditions are as follows: condition (L1) implies (L2), as is obtained by differentiating the invariance condition. It is also clear that condition (S2) implies the condition (S1) since the invariance of the connection A implies that the group action maps its kernel to itself. Condition (S1) may be stated as follows:

$$T_q\Phi_g \cdot \mathcal{D}_q = \mathcal{D}_{gq} \tag{5.4.1}$$

In the case of affine constraints, we will explicitly state when we need the assumption that the vector field γ be invariant under the action.

To help explain condition (S1), we will rewrite it in infinitesimal form. Let $\mathfrak{X}_{\mathcal{D}}$ be the space of sections X of the distribution $\mathcal{D}$; that is, the space of vector fields X that take values in $\mathcal{D}$. The condition (S1) implies that for each $X \in \mathfrak{X}_{\mathcal{D}}$, we have $\Phi_g^* X \in \mathfrak{X}_{\mathcal{D}}$. Here, $\Phi_g^* X$ denotes the pull back of the vector field X under the map Φ_g. Differentiation of this condition with respect to g proves the following result.

Proposition 5.4.2. *Assume that condition* (S1) *holds and let X be a section of $\mathcal{D}$. Then, for each Lie algebra element ξ, we have*

$$[\xi_Q, X] \in \mathfrak{X}_{\mathcal{D}} \tag{5.4.2}$$

which we also write as

$$[\xi_Q, \mathfrak{X}_{\mathcal{D}}] \subset \mathfrak{X}_{\mathcal{D}}.$$

We now have

Proposition 5.4.3. *Under assumptions (L1) and (S1), we can form the **reduced velocity phase space** TQ/G and the **constrained reduced velocity phase space** $\mathcal{D}/G$. The Lagrangian L induces well defined functions, the **reduced Lagrangian***

$$l : TQ/G \to \mathbb{R}$$

*satisfying $L = l \circ \pi_{TQ}$ where $\pi_{TQ} : TQ \to TQ/G$ is the projection, and the **constrained reduced Lagrangian***

$$l_c : \mathcal{D}/G \to \mathbb{R},$$

*which satisfies $L|\mathcal{D} = l_c \circ \pi_{\mathcal{D}}$ where $\pi_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}/G$ is the projection. Also, the Lagrange d'Alembert equations induce well defined **reduced Lagrange d'Alembert equations** on $\mathcal{D}/G$. That is, the vector field on the manifold $\mathcal{D}$ determined by the Lagrange d'Alembert equations (including the constraints) is G-invariant, and so defines a reduced vector field on the quotient manifold $\mathcal{D}/G$.*

This proposition follows from general symmetry considerations. For example, to get the constrained reduced Lagrangian l_c we restrict the given Lagrangian to the distribution $\mathcal{D}$ and then use its invariance to pass to the quotient. The problem of constrained Lagrangian reduction is the detailed determination of these reduced structures and will be dealt with later.

5.5 The Momentum Equation

In this section we use the Lagrange d'Alembert principle to derive an equation for a generalized momentum as a consequence of the symmetries. Under the hypotheses that the action of some Lie algebra element is horizontal (that is, the infinitesimal generator is automatically in the constraint distribution), this yields a conservation law in the usual sense. We refer the reader to our earlier section ? for the classical momentum map.

As we shall see, the momentum equation does not directly involve the choice of an Ehresmann connection to describe the distribution $\mathcal{D}$, but the choice of such a connection will be useful for the coordinate versions.

We have already mentioned that simple physical systems that have symmetries do not have associated conservation laws, namely the wobblestone and the snakeboard. It is also easy to see why this is not generally the case from the equations of motion. The simplest situation would be the case of cyclic variables. Recall that the equations of motion have the form

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^\alpha} - \frac{\partial L_c}{\partial r^\alpha} + A^a_\alpha \frac{\partial L_c}{\partial s^a} = -\frac{\partial L}{\partial \dot{s}^b} B^b_{\alpha\beta}\dot{r}^\beta.$$

If this has a cyclic variable, say r^1, this would mean that all the quantities $L_c, L, B^b_{\alpha\beta}$ would be independent of r^1. This is equivalent to saying that there is a translational symmetry in the r^1 direction. Let us also suppose, as is often the case, that the s variables are also cyclic. Then the above equation for the momentum $p_1 = \partial L_c/\partial \dot{r}^1$ becomes

$$\frac{d}{dt}p_1 = -\frac{\partial L}{\partial \dot{s}^b} B^b_{1\beta}\dot{r}^\beta.$$

This fails to be a conservation law in general. Note that the right hand side is linear in $\dot{r}$ (the first term is linear in p_r) and the equation does not depend on r^1 itself. This is a very special case of the momentum equation that we shall develop in this chapter. Even for systems like the snakeboard, the symmetry group is not abelian, so the above analysis for cyclic variables fails to capture the full story. In particular, the momentum eqution is not of the preceding form in that example and thus it must be generalized.

5.5.1 The Derivation of the Momentum Equation

We now derive a generalized momentum map for nonholonomic systems. The number of equations obtained will equal the dimension of the intersection of the orbit with the given constraints. As we will see, this result will give conservation laws as a particular case.

To formulate our result, some additional ideas and notation will be useful. As the examples show, in general the tangent space to the group orbit through q intersects the constraint distribution at q nontrivially. It will be helpful to give this intersection a name.

Definition 5.5.1. *The intersection of the tangent space to the group orbit through the point $q \in Q$ and the constraint distribution at this point is denoted $\mathcal{S}_q$, as in Figure* **??**, *and we let the union of these spaces over*

$q \in Q$ be denoted $\mathcal{S}$. Thus,

$$\mathcal{S}_q = \mathcal{D}_q \cap T_q(\mathrm{Orb}(q)).$$

Definition 5.5.2. *Define, for each $q \in Q$, the vector subspace $\mathfrak{g}^q$ to be the set of Lie algebra elements in $\mathfrak{g}$ whose infinitesimal generators evaluated at q lie in $\mathcal{S}_q$:*

$$\mathfrak{g}^q = \{\xi \in \mathfrak{g} : \xi_Q(q) \in S_q\}$$

The corresponding bundle over Q whose fiber at the point q is given by $\mathfrak{g}^q$, is denoted $\mathfrak{g}^{\mathcal{D}}$.

Consider a section of the vector bundle $\mathcal{S}$ over Q; *i.e.,* a mapping that takes q to an element of $\mathcal{S}_q = \mathcal{D}_q \cap T_q(\mathrm{Orb}(q))$. Assuming that the action is free, a section of $\mathcal{S}$ can be uniquely represented as ξ^q_Q and defines a section ξ^q of the bundle $\mathfrak{g}^{\mathcal{D}}$. For example, one can construct the section by orthogonally projecting (using the kinetic energy metric) $\xi_Q(q)$ to the subspace $\mathcal{S}_q$. However, as we shall see, in later examples, it is often easy to choose a section by inspection.

Next, we choose the variation analogously to what we chose in the case of the standard Noether theorem above, namely, $q(t,s) = \exp(\phi(t,s)\xi^{q(t)}) \cdot q(t)$. The corresponding infinitesimal variation is given by $\delta q(t) = \phi'(t)\xi^q_Q(q(t))$. Letting $\partial\xi^q$ denote the derivative of ξ^q with respect to q, we have

$$\dot{\delta q} = \dot{\phi}'\xi^{q(t)}_Q + \phi'\left[(T\xi^{q(t)}_Q \cdot \dot{q}) + (\partial\xi^{q(t)} \cdot \dot{q})_Q\right].$$

In this equation, the term $T\xi^{q(t)}_Q$ is computed by taking the derivative of the vector field $\xi^{q(t)}_Q$ with $q(t)$ held fixed. By construction, the variation δq satisfies the constraints and the curve $q(t)$ satisfies the Lagrange d'Alembert equations, so that the following variational equation holds:

$$0 = \int_a^b \left(\frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\dot{\delta q}^i\right) dt. \tag{5.5.1}$$

In addition, the invariance identity from above holds using ξ^q:

$$0 = \int_a^b \left(\frac{\partial L}{\partial q^i}(\xi^{q(t)}_Q)^i\phi' + \frac{\partial L}{\partial \dot{q}^i}(T\xi^{q(t)}_Q \cdot \dot{q})^i\phi'\right) dt. \tag{5.5.2}$$

Subtracting equations (5.5.1) and (5.5.2) and using the arbitrariness of ϕ' and integration by parts shows that

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i}(\xi^{q(t)})^i_Q = \frac{\partial L}{\partial \dot{q}^i}\left[\frac{d}{dt}(\xi^{q(t)})\right]^i_Q.$$

The quantity whose rate of change is involved here is the nonholonomic version of the momentum map in geometric mechanics.

Definition 5.5.3. *The **nonholonomic momentum map** J^{nhc} is the bundle map taking TQ to the bundle $(\mathfrak{g}^{\mathcal{D}})^*$ whose fiber over the point q is the dual of the vector space $\mathfrak{g}^q$ that is defined by*

$$\langle J^{\mathrm{nhc}}(v_q), \xi \rangle = \frac{\partial L}{\partial \dot{q}^i} (\xi_Q)^i .$$

where $\xi \in \mathfrak{g}^q$. Intrinsically, this reads

$$\langle J^{\mathrm{nhc}}(v_q), \xi \rangle = \langle \mathbb{F}L(v_q), \xi_Q \rangle ,$$

where $\mathbb{F}L$ is the fiber derivative of L and where $\xi \in \mathfrak{g}^q$. For notational convenience, especially when the variable v_q is suppressed, we will often write the left hand side of this equation as $J^{\mathrm{nhc}}(\xi)$.

Notice that the nonholonomic momentum map may be viewed as giving just some of the components of the ordinary momentum map, namely along those symmetry directions that are consistent with the constraints.

We summarize these results in the following theorem.

Theorem 5.5.4. *Assume that condition (L2) of definition 5.4.1 holds (which is implied by (L1)) and that ξ^q is a section of the bundle $\mathfrak{g}^{\mathcal{D}}$. Then any solution of the Lagrange d'Alembert equations for a nonholonomic system must satisfy, in addition to the given kinematic constraints, the **momentum equation**:*

$$\frac{d}{dt} \left(J^{\mathrm{nhc}}(\xi^{q(t)}) \right) = \frac{\partial L}{\partial \dot{q}^i} \left[\frac{d}{dt} (\xi^{q(t)}) \right]_Q^i . \tag{5.5.3}$$

When the momentum map is paired with a section in this way, we will just refer to it as the momentum. The following is a direct corollary of this result.

Corollary 5.5.5. *If ξ is a horizontal symmetry (see (S3) above), then the following conservation law holds:*

$$\frac{d}{dt} J^{\mathrm{nhc}}(\xi) = 0 \tag{5.5.4}$$

A somewhat restricted version of the momentum equation was given by Koslov & Kolesnikov [1978] and the corollary was given by Arnold [1988, page 82] (see Bloch & Crouch [1992, 1994] for the controlled case).

Remarks

1. The right hand side of the momentum equation (5.5.3) can be written in more intrinsic notation as

$$\left\langle \mathbb{F}L(\dot{q}(t)), \left(\frac{d}{dt} \xi^{q(t)} \right)_Q \right\rangle .$$

2. In the theorem and the corollary, we do not need to assume that the distribution itself is G-invariant; that is, we do not need to assume condition (S1). In particular, as we shall see in the examples, one can get conservation laws in some cases in which the distribution is not invariant.

3. The validity of the form of the momentum equation is not affected by any "internal forces", that is, any control forces on shape space. Indeed, such forces would be invariant under the action of the Lie group G and so would be annihilated by the variations taken to prove the above result.

4. The momentum equation still holds in the presence of affine constraints. We do *not* need to assume that the affine vector field defining the affine constraints is invariant under the group. However, this vector field may appear in the final momentum equation (or conservation law) because the constraints may be used to rewrite the resulting equation. We will see this explicitly in the example of the ball on a rotating table.

5. Assuming that the distribution is invariant (hypothesis (S1)), the nonholonomic momentum map as a bundle map is equivariant with respect to the action of the group G on the tangent bundle TQ and on the bundle $(\mathfrak{g}^{\mathcal{D}})^*$. In fact, since the distribution is invariant and using the general identity $(\mathrm{Ad}_g \xi)_Q = \Phi^*_{g^{-1}} \xi_Q$, valid for any group action, we see that the space $\mathfrak{g}^g$ is mapped to $\mathfrak{g}^{gq}$ by the map Ad_g, and so in this sense, the adjoint action acts in a well defined manner on the bundle $\mathfrak{g}^{\mathcal{D}}$. By taking its dual, we see that the coadjoint action is well defined on $(\mathfrak{g}^{\mathcal{D}})^*$. In this setting, equivariance of the nonholonomic momentum map follows as in the usual proof (see, for example, Marsden & Ratiu [1994], Chapter 11).

6. One can find an *invariant* momentum if the section is chosen such that
$$(\mathrm{Ad}_{g^{-1}} \xi^{g \cdot q})_Q = \xi^q_Q.$$
This can always be done in the case of trivial bundles; one chooses any ξ^q at the identity in the group variable and translates it around by using the action to get a ξ^q at all points. This direction of reasoning (initiated by remarks of Ostrowski, Lewis, Burdick and Murray) is discussed in §5.5.3. As we will see later, this point of view is useful in the case of the snakeboard.

7. The form of the momentum equation in this section is valid for any curve $q(t)$ that satisfies the Lagrange d'Alembert principle; we do *not* require that the constraints be satisfied for this curve. The version of

the momentum equation given in the next section and later in §**??** will explicitly require that the constraints are satisfied. Of course, in examples we always will impose the constraints, so this is really a comment about the logical structure of the various versions of the equation.

8. In some interesting cases, one can get conservation laws *without* having horizontal symmetries, as required in the preceding corollary. These are cases in which, for reasons other than horizontality, the right hand side of the momentum equation vanishes. This may be an important observation for the investigation of completely integrable nonholonomic systems. A specific case in which this occurs is the vertical rolling disk discussed below.

5.5.2 The Momentum Equation in a Moving Basis

There are several ways of rewriting the momentum equation that are useful; the examples will show that each of them can reveal interesting aspects of the system under consideration. This subsection develops the first of these coordinate formulas, which is in some sense the most naive, but also the most direct. The next subsection will develop a form that is suitable for a local trivialization of the bundle $Q \to Q/G$. Later on, when the nonholonomic connection is introduced, we shall come back to both of these forms and rewrite them in a more sophisticated but also more revealing way.

Introduce coordinates $q^1, \ldots q^n$ in the neighborhood of a given point q_0 in Q. At the point q_0, introduce a basis

$$\{e_1, e_2, \ldots, e_m, e_{m+1}, \ldots, e_k\}$$

of the Lie algebra such that the first m elements form a basis of $\mathfrak{g}^{q_0}$. Thus, $k = \dim \mathfrak{g}$ and $m = \dim \mathfrak{g}^q$, which, by assumption, is locally constant. We can introduce a similar basis

$$\{e_1(q), e_2(q), \ldots, e_m(q), e_{m+1}(q), \ldots, e_k(q)\}$$

at neighboring points q. For example, one can choose an orthonormal basis (in either the locked inertia metric or relative to a Killing form) that varies smoothly with q. We introduce a change of basis matrix by writing

$$e_b(q) = \sum_{a=1}^{k} \psi_b^a(q) e_a$$

for $b = 1, \ldots, k$. Here, the change of basis matrix $\psi_b^a(q)$ is an invertible $k \times k$ matrix. Relative to the dual basis, we write the components of the nonholonomic momentum map as J_b. By definition,

$$J_b = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}^i} [e_b(q)]_Q^i .$$

Using this notation, the momentum equation, with the choice of section given by

$$\xi^{q(t)} = e_b(q(t)), \quad 1 \le b \le m$$

reads as follows:

$$\frac{d}{dt} J_b = \sum_{i=1}^{n} \left(\frac{\partial L}{\partial \dot{q}^i} \left[\frac{d}{dt} e_b(q(t)) \right]_Q^i \right). \tag{5.5.5}$$

Next, we define Christoffel-like symbols by

$$\Gamma_{bl}^c = \sum_{a=1}^{k} (\psi^{-1})_a^c \frac{\partial \psi_b^a}{\partial q^l} \tag{5.5.6}$$

where the matrix $(\psi^{-1})_a^d$ denotes the inverse of the matrix ψ_b^a. Observe that

$$\frac{d}{dt} e_b(q(t)) = \sum_{c=1}^{k} \sum_{l=1}^{n} \Gamma_{bl}^c \dot{q}^l e_c(q(t)), \tag{5.5.7}$$

which implies that

$$\left[\frac{d}{dt} e_b(q(t)) \right]_Q^i = \sum_{c=1}^{k} \sum_{l=1}^{n} \Gamma_{bl}^c \dot{q}^l [e_c(q(t))]_Q^i. \tag{5.5.8}$$

Thus, we can write the momentum equation as

$$\frac{d}{dt} J_b = \sum_{c=1}^{k} \sum_{i,l=1}^{n} \frac{\partial L}{\partial \dot{q}^i} \Gamma_{bl}^c \dot{q}^l [e_c(q(t))]_Q^i. \tag{5.5.9}$$

Introducing the shorthand notation $e_c^i := [e_c(q(t))]_Q^i$, the momentum equation reads

$$\frac{d}{dt} J_b = \sum_{c=1}^{k} \sum_{i,l=1}^{n} \frac{\partial L}{\partial \dot{q}^i} \Gamma_{bl}^c \dot{q}^l e_c^i \tag{5.5.10}$$

Breaking the summation over c into two ranges and using the definition

$$J_c = \frac{\partial L}{\partial \dot{q}^i} e_c^i \qquad 1 \le c \le m$$

gives the following form of the momentum equation.

Proposition 5.5.6 (Momentum equation in a moving basis). *The momentum equation in the above coordinate notation reads*

$$\frac{d}{dt} J_b = \sum_{c=1}^{m} \sum_{l=1}^{n} \Gamma_{bl}^c J_c \dot{q}^l + \sum_{c=m+1}^{k} \sum_{i,l=1}^{n} \frac{\partial L}{\partial \dot{q}^i} \Gamma_{bl}^c \dot{q}^l e_c^i. \tag{5.5.11}$$

Assuming that the Lagrangian is of the form kinetic minus potential energy, the second term on the right hand side of this equation vanishes if the orbit and the constraint distribution are orthogonal; that is, if we can choose the basis so that the vectors $[e_c(q(t))]_Q$ for $c \geq m+1$ are orthogonal to the constraint distribution. In this case, the momentum equation has the form of an equation of parallel transport along the curve $q(t)$. The connection involved is the natural one associated with the the bundle $(\mathfrak{g}^{\mathcal{D}})^*$ over Q, using a chosen decomposition of $\mathfrak{g}$, such as the orthogonal one. In the general case, the momentum equation is an equality between the covariant derivative of the nonholonomic momentum and the last term on the right hand side of the preceding equation. In the next section, we shall write the momentum equation in a body frame, which will be important for understanding how to decouple the momentum equation from the group variables.

5.5.3 The Momentum Equation in Body Representation

Next, we develop an alternative coordinate formula for the momentum equation that is adapted to a choice of local trivialization. Thus, let a local trivialization be chosen on the principal bundle $\pi : Q \to Q/G$, with the local representation having coordinates denoted (r, g). Let r have components denoted r^α as before, being coordinates on the base Q/G and let g be group variables for the fiber, G. In such a representation, the action of G is the left action of G on the second factor. We calculate the nonholonomic momentum map using well known ideas (see, for example, Marsden & Ratiu [1994], Chapter 12), as follows. Let $v_q = (r, g, \dot{r}, \dot{g})$ be a tangent vector at the point $q = (r, g)$, $\eta \in \mathfrak{g}^q$ and let $\xi = g^{-1}\dot{g}$, *i.e.*, $\xi = T_g L_{g^{-1}} \dot{g}$. Since L is G-invariant, we can define a new function l by writing

$$L(r, g, \dot{r}, \dot{g}) = l(r, \dot{r}, \xi).$$

Use of the chain rule shows that

$$\frac{\partial L}{\partial \dot{g}} = T_g^* L_{g^{-1}} \frac{\partial l}{\partial \xi},$$

and so

$$\begin{aligned}
\left\langle J^{\mathrm{nhc}}(v_q), \eta \right\rangle &= \langle \mathbb{F}L(r, g, \dot{r}, \dot{g}), \eta_Q(r, g) \rangle \\
&= \left\langle \frac{\partial L}{\partial \dot{g}}, (0, TR_g \cdot \eta) \right\rangle \\
&= \left\langle \frac{\partial l}{\partial \xi}, \mathrm{Ad}_{g^{-1}} \eta \right\rangle .
\end{aligned}$$

The preceding equation shows that we can write the momentum map in a local trivialization by making use of the Ad mapping in much the same way as we did with the connection and the local formulas in the principal kinematic case. We define $J^{\rm nhc}_{\rm loc} : TQ/G \to (\mathfrak{g}^{\mathcal{D}})^*$ in a local trivialization by

$$\left\langle J^{\rm nhc}_{\rm loc}(r, \dot r, \xi), \eta \right\rangle = \left\langle \frac{\partial l}{\partial \xi}, \eta \right\rangle .$$

Thus, as with the previous local forms, $J^{\rm nhc}$ and its version in a local trivialization are related by the Ad map; precisely,

$$J^{\rm nhc}(r, g, \dot r, \dot g) = \mathrm{Ad}^*_{g^{-1}} J^{\rm nhc}_{\rm loc}(r, \dot r, \xi).$$

Secondly, choose a q-dependent basis $e_a(q)$ for the Lie algebra such that the first m elements span the subspace $\mathfrak{g}^q$. In a local trivialization, this is done in a very simple way. First, one chooses, for each r, such a basis at the identity element $g = \mathrm{Id}$, say

$$e_1(r), e_2(r), \ldots , e_m(r), e_{m+1}(r), \ldots , e_k(r).$$

For example, this could be a basis such that the corresponding generators are orthonormal in the kinetic energy metric. (Keep in mind that the subspaces $\mathcal{D}_q$ and $T_q\mathrm{Orb}$ need not be orthogonal but here we are choosing a basis corresponding only to the subspace $T_q\mathrm{Orb}$.) Define the ***body fixed basis*** by

$$e_a(r, g) = \mathrm{Ad}_g \cdot e_a(r);$$

then the first m elements will indeed span the subspace $\mathfrak{g}^q$ provided the distribution is invariant (condition (S1)). Thus, in this basis we have

$$\left\langle J^{\rm nhc}(r, g, \dot r, \dot g), e_b(r, g) \right\rangle = \left\langle \frac{\partial l}{\partial \xi}, e_b(r) \right\rangle := p_b, \tag{5.5.12}$$

which defines p_b, a function of r, $\dot r$ and ξ. We are deliberately introducing the new notation p for the momentum in body representation to signal its special role. Note that in this body representation, the functions p_b are *invariant* rather than equivariant, as is usually the case with the momentum map. The time derivative of p_b may be evaluated using the momentum equation (5.5.3). This gives

$$\begin{aligned} \frac{d}{dt} p_b &= \frac{\partial L}{\partial \dot q^i} \left[\frac{d}{dt} e_b(r, g) \right]^i_Q \\ &= \left\langle (T_g L_{g^{-1}})^* \frac{\partial l}{\partial \xi}, \left[\frac{d}{dt} \left(\mathrm{Ad}_g \cdot e_b(r) \right) \right]_Q \right\rangle \\ &= \left\langle \frac{\partial l}{\partial \xi}, [\xi, e_b] + \frac{\partial e_b}{\partial r^\alpha} \dot r^\alpha \right\rangle . \end{aligned}$$

We summarize the conclusion drawn from this calculation as follows.

Proposition 5.5.7 (Momentum equation in body representation). *The momentum equation in body representation on the principal bundle $Q \to Q/G$ is given by*

$$\frac{d}{dt} p_b = \left\langle \frac{\partial l}{\partial \xi}, [\xi, e_b] + \frac{\partial e_b}{\partial r^\alpha} \dot{r}^\alpha \right\rangle . \tag{5.5.13}$$

Moreover, the momentum equation in this representation is independent of, that is, decouples from, the group variables g.

In this representation, the variable ξ is related to the group variable g by $\xi = g^{-1}\dot{g}$. In particular, in this representation, reconstruction of the group variable g can be done by means of the equation

$$\dot{g} = g\xi \tag{5.5.14}$$

On the other hand, this variable $\xi = g^{-1}\dot{g}$, as in the case of the reduced Euler-Poincaré equations, *is not the vertical part of the velocity vector $\dot{q}$ relative to the nonholonomic connection to be constructed in the next section.* The vertical part is related to the variable ξ by a velocity shift and this velocity shift will make the reconstruction equation look affine, as in the case of the snakeboard (see Lewis, Ostrowski, Murray & Burdick [1994]). In that example, the decoupling of the momentum equation from the group variables played a useful role. We also recall (as in the example of the rigid body with rotors discussed in Marsden & Scheurle [1993]) that it is often the shifted velocity and not ξ that diagonalizes the kinetic energy, so this shift is fundamental for a number of reasons. As we shall see later, the same ideas in this section, combined with the calculations of Marsden & Scheurle [1993] will show how to calculate the fully reduced equations.

In the above local trivialization form of the momentum equation, we may write the terms $(\partial e_b/\partial r^\alpha)\dot{r}^\alpha$ in terms of a connection, as we did in deriving the momentum equation in a moving basis.

Other noteworthy features of this form of the momentum equation are the following direct consequences of the preceding proposition.

Corollary 5.5.8.

1. *If e_b, $b = 1, \ldots, m$ are independent of r, then the momentum equation in body representation is equivalent to the Euler-Poincaré equations projected to the subspace $\mathfrak{g}^q$.*

2. *If $\mathfrak{g}$ is abelian, then the momentum equations reduce to*

$$\frac{d}{dt} p_b = \left\langle \frac{\partial l}{\partial \xi}, \frac{\partial e_b}{\partial r^\alpha} \dot{r}^\alpha \right\rangle . \tag{5.5.15}$$

3. *If $\mathfrak{g}$ is abelian, or more generally, if the bracket of an element of $\mathfrak{g}^q$ with one in $\mathfrak{g}$ is annihilated by $\partial l/\partial \xi$, and if $e_b, b = 1, \ldots, m$, are independent of r, then the quantities $p_b, b = 1, \ldots, m$ are constants of motion.*

Regarding the first item, see MARSDEN & RATIU [1994] for a discussion of the Euler-Poincaré equations; see also section?. In this case, the spatial form of the momentum is conserved, just as in the case of systems with holonomic constraints. The last case occurs for the vertical rolling penny.

6

Controllability and Stabilizability for Mechanical and Nonholonomic Systems

6.1 Controllability, Accessibility and Stabilizability for Nonholonomic Mechanical Systems

In this section we consider a class of nonholonomic control systems and various control and stabilizability properties, following the work of Bloch, Reyhanoglu and McClamroch [1992]. Related work in robotics includes Li and Canny [1990], Li and Montgomery [1990], Murray and Sastry [1992], Krishnaprasad and Yang [1991].

We consider here the class of mechanical (Lagrangian) nonholonomic control systems described by the equations

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}^i}L - \frac{\partial}{\partial q^i}L = \sum_{j=1}^{m} \lambda_j a_i^j + \sum_{j=1}^{l} b_i^j u_j \,, \tag{6.1.1}$$

$$\sum_{i=1}^{n} a_i^j \dot{q}^i = 0 \qquad j = 1 \cdots m \,. \tag{6.1.2}$$

As before we may in general assume we have a Lagrangian on the tangent bundle to an arbitrary configuration space, Lagrangian $L : TQ \to \mathbb{R}$. In coordinates $q^i, i = 1, \ldots, n$, on Q with induced coordinates $(q^i, \dot{q}^i)$ for the tangent bundle, we have $L(q^i, \dot{q}^i)$. All computations here will be local however and for the moment we will assumme $Q = \mathbb{R}^n$.

Here L is taken to be the mechanical Lagrangian

$$L = \frac{1}{2}\sum_{i,j=1}^{n} \dot{q}^i \dot{q}^j - V(q)\,. \tag{6.1.3}$$

Hence equation 6.1.1 takes the explicit form

$$\begin{aligned} &g_{ij}\ddot{q}^j + \frac{\partial g_{ij}}{\partial q^k}\dot{q}^k\dot{q}^j - \frac{1}{2}\frac{\partial g_{jk}}{\partial q^i}\dot{q}^j\dot{q}^k + \frac{\partial V}{\partial q^k} \\ &= \sum_{j=1}^{m}\lambda_j a_i^j + \sum_{j=1}^{r} b_i^j u_j\,. \end{aligned} \tag{6.1.4}$$

For convenience below we shall sometimes rewrite equation 6.1.5 as

$$g_{ij}\ddot{q}^j + f_i(q,\dot{q}) = \sum_{j=1}^{m}\lambda_j a_i^j + \sum_{j=1}^{l} b_i^j u_j\,. \tag{6.1.5}$$

All functions are assumed to be smooth. We shall make some assumptions on the controls later on.

As in the previous chapter we shall assume that the contraints may be rewritten as:

$$\dot{s}^a + A_\alpha^a(r,s)\dot{r}^\alpha = 0 \qquad a = 1,\ldots,m, \tag{6.1.6}$$

where $q = (r,s) \in \mathbb{R}^{n-m}\times\mathbb{R}^m$.

Denote as usual the distribution defined by the constraints at q by $\mathcal{D}_q$.

Definition 6.1.1. *(See Vershik and Gershkovitch [1988].) Consider the following nondecreasing sequence of locally defined distributions*

$$N_1 = \mathcal{D}_q\ ,$$

$$N_k = N_{k-1} + span\{[X,Y] \mid X\in N_1,\ Y\in N_{k-1}\}\ .$$

There exists an integer k^* such that

$$N_k = N_{k^*}$$

for all $k > k^*$. If dim $N_{k^*} = n$ and $k^* > 1$ then the constraints (2) are called completely nonholonomic and the smallest (finite) number k^* is called the degree of nonholonomy.

We assume here that the constraint equations are completely nonholonomic everywhere with nonholonomy degree k^*. Note that for this to hold $n-m$ must be strictly greater than one.

Under our assumptions the constraints define a $(2n - m)$-dimensional smooth submanifold

$$\mathbf{M} = \{(q, \dot{q}) \mid a_i^j \dot{q}^i = 0\,, \qquad j = 1 \cdots m\} \tag{6.1.7}$$

of the phase space. This manifold $\mathbf{M}$ plays a critical role in the concept of solutions and the formulation of control and stabilization problems.

We note first that equations 6.1.1 and 6.1.2 provide a well posed set of equations in the following sense:

Definition 6.1.2. *A pair of vector functions $(q(t), \lambda(t))$ defined on an interval $[0, T)$ is a solution of the initial value problem defined by equations (6.1.1), (6.1.2) and the initial data $(q_0, \dot{q}_0)$ if $q(t)$ is at least twice differentiable, $\lambda(t)$ is integrable, the vector functions $(q(t), \lambda(t))$ satisfy the differential-algebraic equations (1)-(2) almost everywhere on their domain of definition and the initial conditions satisfy $(q(0), \dot{q}(0)) = (q_0, \dot{q}_0)$.*

Theorem 6.1.3. *Assume that the control input function $u : [0, T) \to R^l$ is a given bounded and measurable function for some $T > 0$. If the initial data satisfy $(q_0, \dot{q}_0) \in \mathbf{M}$, then there exists a unique solution (at least locally defined) of the initial value problem corresponding to equations (6.1.1), (6.1.2) which satisfies $(q(t), \dot{q}(t)) \in \mathbf{M}$ for each t for which the solution is defined.*

We subsequently use the notation $(Q(t, q_0, \dot{q}_0), \Lambda(t, q_0, \dot{q}_0))$ to denote the solution of equations (6.1.1), (6.1.2) at time $t \geq 0$ corresponding to the initial conditions $(q_0, \dot{q}_0)$. Thus for each initial condition $(q_0, \dot{q}_0) \in \mathbf{M}$ and each bounded, measurable input function $u : [0, T) \to R^r$, $(Q(t, q_0, \dot{q}_0), \dot{Q}(t, q_0, \dot{q}_0)) \in \mathbf{M}$ holds for all $t \geq 0$ where the solution is defined.

We say a solution is an equilibrium solution if it is a constant solution; note that if (q^e, λ^e) is an equilibrium solution we refer to q^e as an equilibrium configuration. The following result should be clear.

Theorem 6.1.4. *Suppose that $u(t) = 0, \quad t \geq 0$. The set of equilibrium configurations of equations (1)-(2) is given by*

$$\{q^i \mid \frac{\partial V(q, 0)}{\partial q^i} - a_i^j \lambda_j = 0\,, i = 1, \cdots, n, \textit{ for some } \lambda \in R^m\}\ ,$$

We remark that generically we obtain an equilibrium ***manifold*** with dimension at least m. On the other hand, for certain cases, there may not be even a single equilibrium configuration (e.g. the dynamics of a ball on an inclined plane). However, with our controllability assumptions below we can always introduce an equilibrium manifold of dimension at least m by appropriate choice of input. This if course all in the smooth category.

6.2 Stabilization

6.2.1 Smooth Stabilization to Manifolds

It turns out that the best one can acheive in the way of smooth stabilization is stabilization to an equilibrium manifold See Bloch, Reyhanoglu and McClamroch [1992], Montgomery [1993?]).

We want to consider feedback control of the form $u_i = U_i(q, \dot{q})$ where $U : \mathbf{M} \to R^r$; the corresponding closed loop is described by

$$g_{ij}\ddot{q}^j + f_i(q, \dot{q}) = \sum_{j=1}^{m} \lambda_j a_i^j + \sum_{j=1}^{l} b_i^j U_j(q, \dot{q})\,. \tag{6.2.1}$$

$$\sum_{i=1}^{n} a_i^j \dot{q}^i = 0 \qquad j = 1 \cdots m\,. \tag{6.2.2}$$

The set of equilibrium configurations of equations 6.2.1, 6.2.2 is given by

$$\{q^i \mid \frac{\partial V(q,0)}{\partial q^i} - a_i^j \lambda_j = \sum_{j=1}^{l} b_i^j U_j(q,0)\,, i = 1, \cdots, n, \text{ for some } \lambda \in R^m\}\ ,$$

which is a smooth submanifold of the configuration space.

We now introduce a suitable stability definition for the closed loop system.

Definition 6.2.1. *Assume that $u_i = U_i(q, \dot{q})$. Let $\mathbf{M}_s = \{(q, \dot{q}) | \dot{q} = 0\}$ be an embedded submanifold of $\mathbf{M}$. Then $\mathbf{M}_s$ is locally stable if for any neighborhood $\mathbf{U} \supset \mathbf{M}_s$ there is a neighborhood $\mathbf{V}$ of $\mathbf{M}_s$ with $\mathbf{U} \supset \mathbf{V} \supset \mathbf{M}_s$ such that if $(q_0, \dot{q}_0) \in \mathbf{V} \cap \mathbf{M}$ then the solution of equations 6.2.1,6.2.2 satisfies $(Q(t, q_0, \dot{q}_0), \dot{Q}(t, q_0, \dot{q}_0)) \in \mathbf{U} \cap \mathbf{M}$ for all $t \geq 0$. If, in addition, $(Q(t, q_0, \dot{q}_0), \dot{Q}(t, q_0, \dot{q}_0)) \to (q_s, 0)$ as $t \to \infty$ for some $(q_s, 0) \in \mathbf{M}_s$ then we say that $\mathbf{M}_s$ is locally asymptotically stable equilibrium manifold of equations 6.2.1, 6.2.2.*

Note that if $(Q(t, q_0, \dot{q}_0), \dot{Q}(t, q_0, \dot{q}_0)) \to (q_s, 0)$ as $t \to \infty$ for some $(q_s, 0) \in \mathbf{M}_s$, it follows that there is $\lambda_s \in R^m$ such that $\Lambda(t, q_0, \dot{q}_0) \to \lambda_s$ as $t \to \infty$.

The usual definition of local stability corresponds to the case that $\mathbf{M}_s$ is a single equilibrium solution; the more general case is required in the present paper.

The existence of a feedback function so that a certain equilibrium manifold is asymptotically stable is of particular interest; hence we introduce the following

Definition 6.2.2. *The system defined by equations 6.1.1, 6.1.2 is said to be locally asymptotically stabilizable to a smooth equilibrium manifold $\mathbf{M}_s$*

in $\mathbf{M}$ *if there exists a feedback function* $U : \mathbf{M} \to R^l$ *such that, for the associated closed loop equations 6.2.1, 6.2.2,* $\mathbf{M}_s$ *is locally asymptotically stable.*

If there exists such a feedback function which is smooth on $\mathbf{M}$ then we say that equations 6.1.1, 6.1.2 are smoothly asymptotically stabilizable to $\mathbf{M}_s$; of course it is possible (and we subsequently show that it is generic in certain cases) that equations 6.1.1, 6.1.2 might be asymptotically stabilizable to $\mathbf{M}_s$ but not smoothly (even continuous) asymptotically stabilizable to $\mathbf{M}_s$.

6.3 Normal Form Equations

In this section we describe how to obtain a reduced normal form for the controlled nonholonomic equations.

We recall that the reduced state space is $2n - m$ dimensional. The state of the system can be specified by the n-vector of configuration variables and an $(n - m)$-vector of velocity variables. Let $q = (r, s)$ be the partition of the configuration variables corresponding to the constraints introduced previously.

Define the $n \times n - m$ matrix C as follows:

$$\begin{aligned} C^i_\alpha &= \delta^i_\alpha \,, \qquad i = 1, \cdots n - m \\ &= -A^i_\alpha \,, \qquad i = n - m + 1, \cdots n \end{aligned} \tag{6.3.1}$$

Then

$$\dot{q}^i = C^i_\alpha \dot{r}^\alpha \tag{6.3.2}$$

Taking time derivatives yields

$$\ddot{q}^i = C^i_\alpha(q)\ddot{r}^\alpha + \dot{C}^i_\alpha(q)\dot{r}^\alpha \,.$$

Substituting this into equation 6.1.1 and multiplying both sides of the resulting equation by $C^T(q)$ gives

$$C^{iT}_\alpha(q)g_{ij}(q)C^j_\beta(q)\ddot{r}^\beta = C^{iT}_\alpha(q)[b^j_i(q)u_j - f_i(q, C\dot{r}) - g_{ij}(q)\dot{C}^j_\alpha(q)\dot{r}^\alpha] \,. \tag{6.3.3}$$

Note that $C^{iT}_\alpha(q)g_{ij}(q)C^j_\beta(q)$ is an (n-m)x(n-m) symmetric positive definite matrix function.

We also assume that $l = n - m$ (for simplicity) and that the matrix product $C^{iT}_\alpha(q)b^j_i(q)$ is locally invertible. Consequently for any $u \in R^l$ there is unique $v \in R^{n-m}$ which satisfies

$$C^{iT}_\alpha(q)g_{ij}(q)C^j_\beta(q)v^\beta = C^{iT}_\alpha(q)[b^j_i(q)u_j - f_i(q, C\dot{r}) - g_{ij}(q)\dot{C}^j_\alpha(q)\dot{r}^\alpha] \,. \tag{6.3.4}$$

This assumption guarantees that the reduced configuration variables satisfy the linear equations

$$\ddot{q}^\alpha = v^\alpha \ .$$

Define the following state variables

$$x_1^\alpha = r \ ,$$

$$x_2^a = q_2^a \ ,$$

$$x_3^\alpha = \dot{r}^\alpha \ .$$

Then the normal form equations are given by

$$\dot{x}_1 = x_3 \ , \tag{6.3.5}$$

$$\dot{x}_2 = -A(x_1, x_2)x_3 \ , \tag{6.3.6}$$

$$\dot{x}_3 = v \ . \tag{6.3.7}$$

Equations 6.3.5, 6.3.6, 6.3.7 define a drift vector field $f(x) = (x_3, -A(x_1, x_2)x_3, 0)$ and control vector fields $g_i(x) = (0, 0, e_i)$, where e_i is the i'th standard basis vector in R^{n-m}, $i = 1, \cdots, n-m$, according to the standard control system form

$$\dot{x} = f(x) + \sum_{i=1}^{n-m} g_i(x)v_i \ . \tag{6.3.8}$$

We consider local properties of equations 6.3.5, 6.3.6, 6.3.7, near an equilibrium solution $(x_1^e, x_2^e, 0)$.

Note that the normal form equations 6.3.5, 6.3.6, 6.3.7 are a special case of the normal form equations studied by Byrnes and Isidori [1988]. In particular, the zero dynamics equation of 6.3.5, 6.3.6, 6.3.7, corresponding to the output x_1, is given by

$$\dot{x}_2 = 0 \ ,$$

and it is not locally asymptotically stable. The fact that the zero dynamics is a linear system with all zero eigenvalues, means that equations 6.3.5, 6.3.6, 6.3.7 are critically minimum phase at the equilibrium; this has important implications in terms of local asymptotic stabilizability of the original equations 6.1.1, 6.1.2.

6.4 Stabilization to an Equilibrium Manifold

In this section we study the problem of stabilization of equations 6.1.1,6.1.2 to a smooth equilibrium submanifold of $\mathbf{M}$ defined by

$$\mathbf{N}_e = \{(q, \dot{q}) \mid \dot{q} = 0 \ , \ s(q) = 0\}$$

where $w(q)$ is a smooth $n - m$ vector function. We show that, with appropriate assumptions, there exists a smooth feedback such that the closed loop is locally asymptotically stable to $\mathbf{N}_e$.

The smooth stabilization problem is the problem of giving conditions so that there exists a smooth feedback function $U : \mathbf{M} \to R^l$ such that $\mathbf{N}_e$ is locally asymptotically stable. Of course, we are interested not only in demonstrating that such a smooth feedback exists but also in indicating how such an asymptotically stabilizing smooth feedback can be constructed.

Note that in this section we consider nonholonomic control systems whose normal form equations satisfy the property that if $r(t)$ and $\dot{r}(t)$ are exponentially decaying functions, then the solution to

$$\dot{s} = -A(r(t), s)\dot{r}(t)$$

is bounded (all the physical examples of nonholonomic systems, of which we are aware, satisfy this assumption).

Note also that the first and second time derivatives of $w(q)$ are given by

$$\dot{w} = \frac{\partial w(q)}{\partial q} C(q)\dot{r} \ ,$$

$$\ddot{w} = \frac{\partial}{\partial q}(\frac{\partial w(q)}{\partial q} C(q)\dot{s})C(q)\dot{r} + \frac{\partial w(q)}{\partial q} C(q)v \ .$$

Theorem 6.4.1. *Assume that the above solution property holds. Then the nonholonomic control system, defined by equations 6.1.1, 6.1.2 is locally asymptotically stabilizable to*

$$\mathbf{N}_e = \{(q, \dot{q}) \mid \dot{q} = 0 \, , \ w(q) = 0\} \ , \qquad (6.4.1)$$

using smooth feedback, if the transversality condition

$$det(\frac{\partial w(q)}{\partial r}) \ det(\frac{\partial w(q)}{\partial q} C(q)) \neq 0 \qquad (6.4.2)$$

is satisfied.

Proof. It is sufficient to analyze the system in the normal form 6.3.5, 6.3.6, 6.3.7. By the transversality condition, the change of coordinates from $(r, s, \dot{r})$ to $(w, s, \dot{w})$ is a diffeomorphism.

Let

$$v = -(\frac{\partial w(q)}{\partial q}C(q))^{-1}[\frac{\partial}{\partial q}(\frac{\partial w(q)}{\partial q}C(q)\dot{r})C(q)\dot{r} + K_1\frac{\partial w(q)}{\partial q}C(q)\dot{r} + K_2 w(q)] \ ,$$

where K_1 and K_2 are symmetric positive definite $(n-m)\times(n-m)$ constant matrices. Then, obviously

$$\ddot{w} + K_1\dot{w} + K_2 w = 0$$

is asymptotically stable so that $(w, \dot{w}) \to 0$ as $t \to \infty$. The remaining system variables satisfy equation 6.3.5 of the normal form equations (with $x_2 = s$), and, by our assumption on the on the constraint matrix A, these variables remain bounded for all time. Thus $(q(t), \dot{q}(t)) \to \mathbf{N}_e$ as $t \to \infty$. ■

6.4.1 Nonsmooth Stabilization

The results in the previous section demonstrate that smooth feedback can be used to asymptotically stabilize certain smooth manifolds $\mathbf{N}_e$ in $\mathbf{M}$, where the dimension of $\mathbf{N}_e$ is equal to the number m of independent constraints. These results do not guarantee smooth asymptotic stabilization to a single equilibrium solution if $m \geq 1$.

In fact, there is no C^1 feedback which can asymptotically stabilize the closed loop system to a single equilibrium solution. For suppose that there is a C^1 feedback which asymptotically stabilizes, for example, the origin. Then it follows that there is an equilibrium manifold of dimension m containing the origin; that is, the origin is not isolated, which contradicts the assumption that it is asymptotically stable. We state this formally.

Theorem 6.4.2. *Let $m \geq 1$ and let $(q^e, 0)$ denote an equilibrium solution in $\mathbf{M}$. The nonholonomic control system, defined by equations 6.1.1, 6.1.2 is not asymptotically stabilizable using C^1 feedback to $(q^e, 0)$.*

Proof. A necessary condition for the existence of a C^1 asymptotically stabilizing feedback law for system 6.3.5, 6.3.6, 6.3.7 is that the image of the mapping

$$(x_1,\, x_2,\, x_3,\, v) \longmapsto (x_3,\ -A(x_1, x_2)x_3,\ v)$$

contains some neighborhood of zero (see Brockett [1983]). No points of the form

$$\begin{pmatrix} 0 \\ \epsilon \\ \alpha \end{pmatrix} \ , \ \epsilon \neq 0 \ \text{ and } \ \alpha \in R^{n-m} \text{ arbitrary } \ ,$$

are in its image: it follows that Brockett's necessary condition is not satisfied. Hence system 6.3.5, 6.3.6, 6.3.7 cannot be asymptotically stabilized to $(r^e, s^e, 0)$ by a C^1 feedback law. Consequently, the nonholonomic control system, defined by equations (6.1.1), (6.1.2) is not C^1 asymptotically stabilizable to $(q^e, 0)$. ∎

We remark that even C^0 (continuous) feedback (which results in existence of unique trajectories) is ruled out since Brockett's necessary condition is not satisfied (Zabcyzk [1989]).

A corollary of Theorem 4 is that a single equilibrium solution of 6.1.1, 6.1.2 cannot be asymptotically stabilized using linear feedback nor can it be asymptotically stabilized using feedback linearization or any other control design approach that uses smooth feedback. Below we indicate how a single equilibrium can be asymptotically stabilized by use of piecewise analytic feedback. This is the by no means the only approach however to stabilization. We discuss other approaches later, and as we discussed earlier, there is a large literature on this subject.

We first demonstrate that the normal form equations 6.3.5, 6.3.6, 6.3.7 and hence the nonholonomic control system defined by equations 6.1.1, 6.1.2 satisfies certain strong local controllability properties. In particular, we show that the system is strongly accessible and that the system is small time locally controllable at any equilibrium. These results provide a theoretical basis for the use of inherently nonlinear control strategies and suggest constructive procedures for the desired control strategies.

Theorem 6.4.3. *Let $m \geq 1$ and let $(q^e, 0)$ denote an equilibrium solution in $\mathbf{M}$. The nonholonomic control system, defined by equations 6.1.1, 6.1.2 is strongly accessible at $(q^e, 0)$.*

Proof. It suffices to prove that system 6.3.5, 6.3.6, 6.3.7 is strongly accessible at the origin. Let I denote the set $\{1, \cdots, n-m\}$. The drift and control vector fields can be expressed as

$$f = \sum_{j=1}^{n-m} x_{3,j} \tau_j \ ,$$

$$g_i = \frac{\partial}{\partial x_{3,i}} \ , \ i \in I \ ,$$

where

$$\tau_j = \frac{\partial}{\partial x_{1,j}} - \sum_{i=1}^{n-m} A_i^j(x_1, x_2) \frac{\partial}{\partial x_{2,i}} \ , \ j \in I$$

are considered as vector fields on the (x_1, x_2, x_3) state space. It can be verified that

$$[g_{i_1}, f] = \tau_{i_1} \ , \ \ i_1 \in I \ ;$$

$$[g_{i_2},[f,[g_{i_1},f]]] = [\tau_{i_2},\tau_{i_1}],\ i_1,i_2 \in I\ ;$$

$$\vdots$$

$$[g_{i_{k^*}},[f,\cdots,[g_{i_2},[f,[g_{i_1},f]]]\cdots]] = [\tau_{i_{k^*}},\cdots,[\tau_{i_2},\tau_{i_1}]\cdots],\ i_k \in I, 1 \leq k \leq k^*\ ,$$

hold, where k^* denotes the nonholonomy degree. Let

$$\mathcal{G} = span\{g_i, i \in I\}\ ,$$

$$\mathcal{H} = span\{[g_{i_1},f],\ \cdots,[g_{i_{k^*}},[f,\cdots,[g_{i_2},[f,[g_{i_1},f]]]\cdots]];\ i_k \in I, 1 \leq k \leq k^*\ \}\ .$$

Note that $dim\mathcal{G}(0) = n - m$ and $dim\mathcal{H}(0) = n$ since the distribution defined by the constraints is completely nonholonomic; moreover $dim\{\mathcal{G}(0) \cap \mathcal{H}(0)\} = 0$. It follows that the strong accessibility distribution

$$\mathcal{L}_0 = span\{X\ :\ X \in \mathcal{G} \cup \mathcal{H}\}$$

has dimension $2n - m$ at the origin. Hence the strong accessibility rank condition(Sussmann and Jurdjevich [1972]) is satisfied at the origin. Thus system 6.3.5, 6.3.6, 6.3.7 is strongly accessible at the origin Hence the nonholonomic control system 6.1.1, 6.1.2 is strongly accessible at $(q^e, 0)$. ■

Theorem 6.4.4. *Let $m \geq 1$ and let $(q^e, 0)$ denote an equilibrium solution in $\mathbf{M}$. The nonholonomic control system, defined by equations 6.1.1, 6.1.2 is small time locally controllable at $(q^e, 0)$.*

Proof. It suffices to prove that system 6.3.5, 6.3.6, 6.3.7 is small time locally controllable to the origin.

The proof involves the notion of the degree of a bracket. To make this notion well defined we consider, as in Sussmann [1987], a Lie algebra of indeterminates and an associated evaluation map (on vector fields) as follows:

Let $\mathbf{X} = (X_0, \cdots, X_{n-m})$ be a finite sequence of indeterminates. Let $A(\mathbf{X})$ denote the free associative algebra over R generated by the X_j, let $L(\mathbf{X})$ denote the Lie subalgebra of $A(\mathbf{X})$ generated by $X_0, \cdots, X_{n-m}$ and let $Br(\mathbf{X})$ be the smallest subset of $L(\mathbf{X})$ that contains $X_0, \cdots, X_{n-m}$ and is closed under bracketing.

Now consider the vector fields $f, g_1, \cdots, g_{n-m}$ on the manifold $\mathbf{M}$. Each $f, g_1, \cdots,$
g_{n-m} is a member of $D(\mathbf{M})$, the algebra of all partial differential operators on $C^\infty(\mathbf{M})$, the space of C^∞ real-valued functions on $\mathbf{M}$. Now let $g_0 = f$, and let $\mathbf{g} = (g_0, \cdots, g_{n-m})$ and define the evaluation map

$$Ev(\mathbf{g})\ :\ A(\mathbf{X}) \rightarrow D(\mathbf{M})$$

obtained by substituting the g_j for the X_j, i.e.

$$Ev(\mathbf{g})(\sum_I a_I X_I) = \sum_I a_I g_I$$

where $g_I = g_{i_1} g_{i_2} \cdots g_{i_k},\ I = (i_1, \cdots, i_k)$. Note that the kernel of $Ev(\mathbf{g})\ :\ A(\mathbf{X}) \to A(\mathbf{g})$ is the set of all algebraic identities satisfied by the g_i while the kernel of $Ev(\mathbf{g})\ :\ L(\mathbf{X}) \to L(\mathbf{g})$ is the set of Lie algebraic identities satisfied by g_i.

Now, let B be a bracket in $Br(\mathbf{X})$. We define the degree of a bracket to be $\delta(B) = \sum_{i=0}^{n-m} \delta^i(B)$, where $\delta^0(B),\ \delta^1(B), \cdots,\ \delta^{n-m}(B)$ denote the number of times $X_0, \cdots,\ X_{n-m}$, respectively, occur in B. The bracket B is called "bad" if $\delta^0(B)$ is odd and $\delta^i(B)$ is even for each $i, i = 1, \cdots, n-m$. The theorem of Sussmann tells us the system is STLC at the origin if it satisfies the accessibility rank condition; and if B is "bad" there exist brackets $C_1, \cdots, C_k$ of lower degree in $Br(\mathbf{X})$ such that

$$Ev_0(\mathbf{g})(\beta(B)) = \sum_{i=1}^{k} \xi_i Ev_0(\mathbf{g})(C_i)$$

where Ev_0 denotes the evaluation map at the origin and $(\xi_1, \cdots, \xi_k) \in R^k$. Here $\beta(B)$ is the symmetrization operator, $\beta(B) = \sum_{\pi \in S_{n-m}} \bar{\pi}(B)$, where $\pi \in S_{n-m}$, the group of permutations of $\{1, \cdots, n-m\}$ and for $\pi \in S_{n-m}$, $\bar{\pi}$ is the automorphism of $L(\mathbf{X})$ which fixes X_0 and sends X_i to $X_{\pi(i)}$.

By Theorem 5, the system is accessible at the origin.

The brackets in $\mathcal{G}$ are obviously "good" (not of the type defined as "bad") and $\delta^0(h) = \sum_{j=1}^{n-m} \delta^j(h)\ \forall h \in \mathcal{H}$; thus $\delta(h)$ is even for all h in $\mathcal{H}$, i.e $\mathcal{H}$ contains "good" brackets only. It follows that the tangent space $T_0\mathbf{M}$ to $\mathbf{M}$ at the origin is spanned by the brackets that are all "good". Next we show that the brackets that might be "bad" vanish at the origin. First note that f vanishes at the origin. Let B denote a bracket satisfying $\delta(B) > 1$. If B is a "bad" bracket then necessarily $\delta^0(B) \neq \sum_{j=1}^{n-m} \delta^j(B)$, i.e. $\delta(B)$ must be odd. It can be verified that if $\delta^0(B) < \sum_{j=1}^{n-m} \delta^j(B)$ then B is identically zero and if $\delta^0(B) > \sum_{j=1}^{n-m} \delta^j(B)$ then B is of the form $\sum_{i=1}^{n-m} r_i(x_3) Y_i(x_1, x_2)$, for some vector fields $Y_i(x_1, x_2),\ i \in I$, where $r_i(x_3),\ i \in I$, are homogeneous functions of degree $(\delta^0(B) - \sum_{j=1}^{n-m} \delta^j(B))$ in x_3; thus B vanishes at the origin. Consequently the Sussmann condition is satisfied and the system is small time locally controllable. ∎

Piecewise Analytic Stabilizing Controllers

We consider now stabilization for so-called "controlled nonholonomic Caplygin systems".

We first describe the class of controlled nonholonomic Caplygin systems. If the functions used in defining equations 6.1.1, 6.1.2 do not depend explicitly on the configuration variables s, so that the system is locally described by

$$g_{ij}(r)\ddot{q}^j + f_i(r,\dot{q}) = \sum_{j=1}^{m} \lambda_j a_i^j(r) + \sum_{j=1}^{l} b_i^j(r)u_j \,. \tag{6.4.3}$$

$$\dot{s}^a + A_\alpha^a(r)\dot{r}^\alpha = 0 \qquad a = 1,\ldots,m\,, \tag{6.4.4}$$

then the uncontrolled system is called a "nonholonomic Caplygin system" by Neimark and Fufaev [1972]. In terms of the Lagrangian formalism for the problem this corresponds to the Lagrangian of the free problem being cyclic in (i.e. independent of) the variables q_2 while the constraints are also independent of q_2. The cyclic property is an expression of symmetries in the problem as we have discussed. More generally, if a system can be expressed in the form 6.4.3, 6.4.4 using feedback, then we refer to it as a "controlled nonholonomic Caplygin system".

For the nonholonomic Caplygin system described by equations (16)-(17), equation 6.3.3 becomes

$$C_\alpha^{iT}(r)g_{ij}(r)C_\beta^j(r)\ddot{r}^\beta = C_\alpha^{iT}(r)[b_i^j(r)u_j - f_i(r,C\dot{r}) - g_{ij}(r)\dot{C}_\alpha^j(r)\dot{r}^\alpha] \,. \tag{6.4.5}$$

which is an equation in the variables $(r,\dot{r})$ only. As a consequence,the r variables coordinatize a reduced configuration space for the system 6.4.3, 6.4.4. This reduced configuration space is also referred to as the "base space" (or "shape space") of the system. The term shape space arises from the theory of coupled mechanical systems, where it refers to the internal degrees of freedom of the system.

As we did earlier, we assume that $r = n - m$ and that the matrix product $C_\alpha^{iT}(q)b_i^j(q)$is locally invertible; this assumption is not restrictive. Consequently, it can be shown that the normal form equations for the system 6.4.3, 6.4.4 are given

$$\dot{x}_1 = x_3 \ , \tag{6.4.6}$$

$$\dot{x}_2 = -A(x_1)x_3 \ , \tag{6.4.7}$$

$$\dot{x}_3 = v \ , \tag{6.4.8}$$

where $x_1 = r,\ x_2 = \dot{r},\ x_3 = s$ and v satisfies

$$C_\alpha^{iT}(r)g_{ij}(r)C_\beta^j(r)v^\beta = C_\alpha^{iT}(r)[b_i^j(r)u_j - f_i(r,C\dot{r}) - g_{ij}(r)\dot{C}_\alpha^j(r)\dot{r}^\alpha] \,. \tag{6.4.9}$$

Again we shall make use of these normal form equations to obtain control results.

Clearly, there is no continuous feedback which asymptotically stabilizes a single equilibrium. However, the controllability properties possessed by the system guarantee the existence of a piecewise analytic feedback (see Sussmann [1979]). We now describe the ideas that are employed to construct such a feedback which does achieve the desired local asymptotic stabilization of a single equilibrium solution. These ideas are based on the use of holonomy (geometric phase) which has proved useful in a variety of kinematics and dynamics problems (see e.g. Krishnaprasad [1991], Shapere and Wilczek [1988], and Marsden, Montgomery and Ratiu [1990]. The key observation here is that the holonomy, the extent to which a closed path in the base space fails to be closed in the configuration space, depends only on the path traversed in the base space and not on the time history of traversal of the path. Related ideas have been used for a class of path planning problems, based on kinematic relations, in Li and Canny [1990], Li and Montgomery [1990], and Krishnaprasad and Yang [1991].

For simplicity, we consider control strategies which transfer any initial configuration and velocity(sufficiently close to the origin) to the zero configuration with zero velocity. The proposed control strategy initially transfers the given initial configuration and velocity to the origin of the $(q_1, \dot{q}_1)$ base phase space. The main point then is to determine a closed path in the q_1 base space that achieves the desired holonomy. We show that the indicated assumptions guarantee that this holonomy construction can be made and that (necessarily piecewise analytic) feedback can be determined which accomplishes the desired control objective.

Let $x^0 = (x_1^0, x_2^0, x_3^0)$ denote an initial state. We now describe two steps involved in construction of a control strategy which transfers the initial state to the origin.

Step 1: *Bring the system to the origin of the* (x_1, x_3) *base phase space, i.e. find a control which transfers the initial state* (x_1^0, x_2^0, x_3^0) *to* $(0, x_2^1, 0)$ *in a finite time, for some* x_2^1.

Step 2: *Traverse a closed path (or a series of closed paths) in the* x_1 *base space to produce a desired holonomy in the* (x_1, x_2) *configuration space, i.e. find a control which transfers* $(0, x_2^1, 0)$ *to* $(0, 0, 0)$.

The desired holonomy condition is given by

$$x_2^1 = \oint_\gamma A(x_1)\ dx_1\ , \tag{6.4.10}$$

where γ denote a closed path traversed in the base space. The holonomy is reflected in the fact that traversing a closed path in the base space yields a nonclosed path in the full configuration space. Note that here, for notational simplicity in presenting the main idea, we assume that the desired holonomy can be obtained by a single closed path. In general, more than one loop may be required to produce the desired holonomy; for such cases γ can be viewed as concatenation of a series of closed paths.

Under the weak assumptions mentioned previously, explicit procedures can be given for each of the above two steps. Step 1 is classical; it is step 2, involving the holonomy, that requires special consideration. Explicit characterization of a closed path γ which satisfies the desired holonomy equation (23) can be given for several specific examples (see below). However, some problems may require a general computational approach. An algorithm based on Lie algebraic methods as in Lafferriare and Sussmann [1991] can be employed to aproximately characterize the required closed path. Suppose the closed path γ which satisfies the desired holonomy condition is chosen. Then a feedback algorithm which realizes the closed path in the base space can be constructed since the base space equations represent n-m decoupled double integrators.

This general construction procedure provides a strategy for transferring an arbitrary initial state of equations 6.4.6, 6.4.7, 6.4.8 to the origin. Implementation of this control strategy in a (necessarily piecewise analytic) feedback form can be accomplished as follows.

Let $a = (a_1, \cdots, a_{n-m})$ and $b = (b_1, \cdots, b_{n-m})$ denote displacement vectors in the x_1 base space and let $\gamma(a,b)$ denote the closed path (in the base space) formed by the line segments from $x_1 = 0$ to $x_1 = a$, from $x_1 = a$ to $x_1 = a+b$, from $x_1 = a+b$ to $x_1 = b$, and from $x_1 = b$ to $x_1 = 0$. Then the holonomy of the parameterized family

$$\{\gamma(a,b) | a, b \in R^{n-m}\}$$

is determined by the holonomy function $\gamma(a,b) \longmapsto \alpha(a,b)$ given as

$$\alpha(a,b) = -\oint_{\gamma(a,b)} A(x_1)\ dx_1\ .$$

Now let π_s denote the projection map $\pi_s : (x_1, x_2, x_3) \longmapsto (x_1, x_3)$. In order to construct a feedback control algorithm to accomplish the above two steps, we first define a feedback function $V^{x_1^*}(\pi_s x)$ which satisfies: for any $\pi_s x(t_0)$ there is $t_1 \geq t_0$ such that the unique solution of

$$\dot{x}_1 = x_3\ ,$$

$$\dot{x}_3 = V^{x_1^*}(\pi_s x)\ ,$$

satisfies $\pi_s x(t_1) = (x_1^*, 0)$. Note that the feedback function is parameterized by the vector x_1^*. Moreover, for each x_1^*, there exists such a feedback function. One such feedback function $V^{x_1^*}(\pi_s x) = (V_1^{x_1^*}(\pi_s x), \cdots, V_{n-m}^{x_1^*}(\pi_s x))$ is given as

$$V_i^{x^*}(\pi_s x) = \begin{cases} -k_i\ sign(x_{1,i} - x_{1,i}^* + x_{3,i}|x_{3,i}|/2k_i) & ,\ (x_{1,i}, x_{3,i}) \neq (x_{1,i}^*, 0)\ , \\ 0 & ,\ (x_{1,i}, x_{3,i}) = (x_{1,i}^*, 0)\ , \end{cases}$$

where $k_i, \ i = 1, \cdots, n-m$, are positive constants chosen such that the resulting motion, when projected to the base space, constitute a straight line connecting $x_1(t_0)$ to $x_1(t_1) = x_1^*$.

We specify the control algorithm, with values denoted by v^*, according to the following construction, where x denotes the "current state" :

Control algorithm for v^* :

Step 0: Choose (a^*, b^*) to achieve the desired holonomy.

Step 1: Set $v^* = V^{a^*}(\pi_s x)$, until $\pi_s x = (a^*, 0)$; then go to Step 2;

Step 2: Set $v^* = V^{a^*+b^*}(\pi_s x)$, until $\pi_s x = (a^* + b^*, 0)$; then go to Step 3;

Step 3: Set $v^* = V^{b^*}(\pi_s x)$, until $\pi_s x = (b^*, 0)$; then go to Step 4;

Step 4: Set $v^* = V^0(\pi_s x)$, until $\pi_s x = (0, 0)$; then go to Step 0;

We here assumed that the desired holonomy can be obtained by a single closed path. Clearly the above algorithm can be modified to account for cases for which more than one closed path is required to satisfy the desired holonomy.

Note that the control algorithm is constructed by appropriate switchings between members of the parameterized family of feedback functions. On each cycle of the algorithm the particular functions selected depend on the closed path parameters a^*, b^*, computed in Step 0, to correct for errors in x_2.

The control algorithm can be initialized in different ways. The most natural is to begin with Step 4 since v^* in that step does not depend on the closed path parameters; however, many other initializations of the control algorithm are possible.

Justification that the constructed control algorithm asymptotically stabilizes the origin follows as a consequence of the construction procedure: that switching between feedback functions guarantees that the proper closed path (or a sequence of closed paths) is traversed in the base space so that the origin $(0, 0, 0)$ is necessarily reached in a finite time. This construction of a stabilizing feedback algorithm represents an alternative to the approach by Hermes [1980], which is based on Lie algebraic properties.

It is important to emphasize that the above construction is based on the a priori selection of simply parametrized closed paths in the base space. The above selection simplifies the tracking problem in the base space, but other path selections could be made and they would, of course, lead to a different feedback strategy from that proposed above.

We remark that the technique presented in this section can be generalized to some systems which are not Caplygin. For instance, this generalization is tractible to systems for which equation (20) takes the form

$$\dot{x}_2 = \rho(x_2) A(x_1)$$

where $\rho(x_2)$ denotes certain Lie group representation (see e.g. Marsden, Montgomery and Ratiu [1990]). The holonomy of a closed path for such systems is given as a path ordered exponential rather than a path integral.

Examples

Control of Knife Edge Using Steering and Pushing Inputs : We first consider the control of a knife edge moving in point contact on a plane surface.. Let x and y denote the coordinates of the point of contact of the knife edge on the plane and let ϕ denote the heading angle of the knife edge, measured from the x-axis. Then the equations of motion, with all numerical constants set to unity, are given by

$$\ddot{x} = \lambda \sin\phi + u_1 \cos\phi \ , \tag{6.4.11}$$

$$\ddot{y} = -\lambda \cos\phi + u_1 \sin\phi \ , \tag{6.4.12}$$

$$\ddot{\phi} = u_2 \ , \tag{6.4.13}$$

where u_1 denotes the control force in the direction defined by the heading angle, u_2 denotes the control torque about the vertical axis through the point of contact; the components of the force of constraint arise from the scalar nonholonomic constraint

$$\dot{x} \sin\phi - \dot{y} \cos\phi = 0 \tag{6.4.14}$$

which has nonholonomy degree two at any configuration. It is clear that the constraint manifold is a five-dimensional manifold and is defined by

$$\mathbf{M} = \{(\phi, x, y, \dot{\phi}, \dot{x}, \dot{y}) | \dot{x} \sin\phi - \dot{y} \cos\phi = 0\}$$

and any configuration is an equilibrium if the controls are zero.

Define the variables

$$x_1 = x \cos\phi + y \sin\phi \ ,$$

$$x_2 = \phi \ ,$$

$$x_3 = -x \sin\phi + y \cos\phi \ ,$$

$$x_4 = \dot{x} \cos\phi + \dot{y} \sin\phi - \dot{\phi}(x \sin\phi - y \cos\phi) \ ,$$

$$x_5 = \dot{\phi} \ ,$$

so that the reduced differential equations are given by

$$\dot{x_1} = x_4 \ ,$$

$$\dot{x_2} = x_5 \ ,$$

$$\dot{x_3} = -x_1 x_5 \ ,$$

$$\dot{x_4} = u_1 + u_2 x_3 - x_1 x_5^2 \ ,$$

$$\dot{x_5} = u_2 \ .$$

We have:

Proposition 6.4.5. *Let $x^e = (x_1^e, x_2^e, x_3^e, 0, 0)$ denote an equilibrium solution of the reduced differential equations corresponding to $u = 0$. The knife edge dynamics described by equations (24)-(27) have the following properties:*

1. *There is a smooth feedback which asymptotically stabilizes the closed loop to any smooth one dimensional equilibrium manifold in $\mathbf{M}$ which satisfies the transversality condition.*
2. *There is no smooth feedback which asymptotically stabilizes x^e.*
3. *The system is strongly accessible at x^e since the space spanned by the vectors*

$$g_1, \ g_2, \ [g_1, f], \ [g_2, f], \ [g_2, [f, [g_1, f]]]$$

has dimension 5 at x^e.

4. *The system is small time locally controllable at x^e since the brackets satisfy sufficient conditions for small time local controllability.*

Note that the base variables are (x_1, x_2). Consider a parameterized rectangular closed path γ in the base space with four corner points

$$(0,0), \ (x_1, 0), \ (x_1, x_2), \ (0, x_2) \ ,$$

i.e. $a = (x_1, 0)$ and $b = (0, x_2)$ following the notation introduced in the general development. By evaluating the holonomy integral in closed form for this case, the desired holonomy equation is

$$x_3^1 = x_1 x_2 \ .$$

This equation can be explicitly solved to determine a closed path $\gamma^* = \gamma(a^*, b^*)$ which achieves the desired holonomy. One solution can be given as follows

$$a^* = (\sqrt{|x_3^1|} sign x_3^1, 0),\ b^* = (0, \sqrt{|x_3^1|}) \ .$$

Note that the previously described feedback algorithm can be used to asymptotically stabilize the knife edge to the origin.

Control of Rolling Wheel : As a second example, we consider again the control of a vertical wheel rolling without slipping on a plane surface. As before, Let x and y denote the coordinates of the point of contact of the wheel on the plane, let ϕ denote the heading angle of the wheel, measured from the x- axis and let θ denote the rotation angle of the wheel due to rolling, measured from a fixed reference. Then the equations of motion, with all numerical constants set to unity, are given by

$$\ddot{x} = \lambda_1 \ , \tag{6.4.15}$$

$$\ddot{y} = \lambda_2 \ , \tag{6.4.16}$$

$$\ddot{\theta} = -\lambda_1 \cos\phi - \lambda_2 \sin\phi + u_1 \ , \tag{6.4.17}$$

$$\ddot{\phi} = u_2 \ , \tag{6.4.18}$$

where u_1 denotes the control torque about the rolling axis of the wheel and u_2 denotes the control torque about the vertical axis through the point of contact; the components of the force of constraint arise from the two nonholonomic constraints

$$\dot{x} = \dot{\theta} \cos\phi \ , \tag{6.4.19}$$

$$\dot{y} = \dot{\theta} \sin\phi \ , \tag{6.4.20}$$

which have nonholonomy degree three at any configuration. The constraint manifold is a six-dimensional manifold and is given by

$$\mathbf{M} = \{(\theta, \phi, x, y, \dot{\theta}, \dot{\phi}, \dot{x}, \dot{y}) | \dot{x} = \dot{\theta}\cos\phi,\ \dot{y} = \dot{\theta}\sin\phi\}$$

and any configuration is an equilibrium if the controls are zero.

Define the variables

$$x_1 = \theta,\ x_2 = \phi,\ x_3 = x,\ x_4 = y,\ x_5 = \dot{\theta},\ x_6 = \dot{\phi} \ ,$$

so that the reduced differential equations are given by

$$\dot{x_1} = x_5 \ ,$$

$$\dot{x_2} = x_6 \ ,$$

$$\dot{x_3} = x_5 \cos x_2 \ ,$$

$$\dot{x_4} = x_5 \sin x_2 \ ,$$

$$\dot{x_5} = \frac{1}{2} u_1 \ ,$$

$$\dot{x_6} = u_2 \ .$$

We have a result similar to that in the previous example:

Proposition 6.4.6. *Let $x^e = (x_1^e, x_2^e, x_3^e, x_4^e, 0, 0)$ denote an equilibrium solution of the reduced differential equations corresponding to $u = 0$. The rolling wheel dynamics have the following properties:*

1. *There is a smooth feedback which asymptotically stabilizes the closed loop to any smooth two dimensional equilibrium manifold in* **M** *which satisfies the transversality condition.*
2. *There is no smooth feedback which asymptotically stabilizes x^e.*
3. *The system is strongly accessible at x^e since the space spanned by the vectors*

   $$g_1, \ g_2, \ [g_1, f], \ [g_2, f], \ [g_2, [f, [g_1, f]]], \ [g_2, [f, [g_1, [f, [g_2, f]]]]]$$

   has dimension 6 at x^e.
4. *The system is small time locally controllable at x^e since the brackets satisfy sufficient conditions for small time local controllability.*

Note that the base variables are (x_1, x_2). Consider a parameterized rectangular closed path γ in the base space with four corner points

$$(0, 0), \ (x_1, 0), \ (x_1, x_2), \ (0, x_2) \ .$$

By evaluating the holonomy integral in closed form for this case, we find the desired holonomy equations are

$$x_3^1 = x_1(\cos x_2 - 1) \ ,$$

$$x_4^1 = x_1 \sin x_2 \ .$$

These equations can be explicitly solved to determine a closed path (or a concatenation of closed paths) γ^* which achieves the desired holonomy.

One solution can be given as follows: if $x_3^1 \neq 0$ then γ^* is the closed path specified by

$$a^* = -((x_3^1)^2 + (x_4^1)^2)/2x_3^1, 0)\ , b^* = (0, -\sin^{-1}(2x_3^1 x_4^1/((x_3^1)^2 + (x_4^1)^2))\ ;$$

and if $x_3^1 = 0$ then γ^* is a concatenation of two closed paths specified by

$$a^* = (0.5x_4^1, 0),\ b^* = (0, 0.5\pi)\ ,$$

$$a^{**} = (-0.5x_4^1, 0),\ b^{**} = (0, -0.5\pi)\ .$$

Note that the previously described feedback algrithm can be used (with the modification indicated in the general development) to asymptotically stabilize the rolling wheel to the origin.

6.5 Controlled Nonholonomic Systems on Manifolds

We consider here the formulation of controlled classical nonholonomic systems on Riemannian manifolds. Such a formulation is useful for considering systems with nontrivial geometry, such as the rolling ball. We shall distinguish again here between variational nonholonomic systems and nonholonomic systems. The latter is normally used for describing mechanical systems and here we formalize the introduction of controls for such systems.

6.5.1 Holonomic Control Systems

Consider firstly the holonomic or unconstrained case:

Let $(M, \langle , \rangle)$ be an n-dimensional Riemannian manifold, with metric $g(\quad,\quad) = \langle\quad,\quad\rangle$. Denote the norm of a tangent vector X at the point p by $\|X_p\| = \langle X_p, X_p\rangle^{\frac{1}{2}}$. The geodesic flow on M is then given by

$$\frac{D^2 q}{dt^2} = 0 \tag{6.5.1}$$

where $\frac{Dq}{dt}$ denotes the covariant derivative. This flow minimizes the integral $\int_0^1 \left\|\frac{Dq}{dt}\right\|^2 dt$ along parametrized paths.

We define a *controlled holonomic system* to be a system of the form

$$\frac{D^2 q}{dt^2} = \sum_{i=1}^{N} u_i X_i \tag{6.5.2}$$

where $\{X_i\}$ is an arbitrary set of control vector fields, the u_i are functions of time, and $N \leq n$. (Note that in this paper we do not consider systems evolving under the influence of a potential.)

It is natural to pose the following optimal control problem in the case $\{X_i\}$ is an orthonormal set.

$$\begin{aligned} \min_u \int_0^T \|u\|^2 dt &= \min_u \int_0^T \sum_{i=1}^N u_i^2 dt \\ &= \min_q \int_0^T \left\| \frac{D^2 q}{dt^2} \right\|^2 dt \end{aligned} \tag{6.5.3}$$

subject to (2) and satisfying $q(0) = q_0$, $q(T) = q_T$. Such higher order variational problems (see Griffiths [1983]) were considered in the context of interpolation problems by Noakes et. al. [1989] and by Crouch and Silva-Leite [1991].

We remark that these problems are different from the optimal control problems

$$\min_u \int \|u\|^2 ds$$

subject to

$$\frac{Dq}{dt} = \sum_{i=1}^N u_i X_i. \tag{6.5.4}$$

However they do have the same singular nature for $N < n$ as considered by Brockett in the celebrated paper Brockett [1982]. Indeed, later on, we will interpret the latter systems as kinematic *nonholonomic* systems.

6.5.2 Nonholonomic Systems

We now consider the formulation of controlled nonholonomic systems. We can think of the two classes: – variational nonholonomic systems (see e.g. Arnold [1988] and Vershik and Gershkovich [1988]) and classical (or mechanical) nonholonomic systems. In either case, the constraints are given by 1-forms ω_k, $1 \leq k \leq m$, which define a (smooth) *distribution* H on M. (H is the set of subspaces of TM annihilated by the ω_k at each $x \in T_x M$.)

Variational nonholonomic systems are obtained in our context from the variational problem

$$\min_q \int_0^1 \left\| \frac{Dq}{dt} \right\|^2$$

subject to

$$\omega_k \left(\frac{Dq}{dt} \right) = 0$$

$1 \leq k \leq m$.

Classical nonholonomic systems are not obtained from a variational principle (at least in the usual sense – see Remark 3 below) but from D'Alembert's principle, as discussed earlier. The equations are

$$\frac{D^2q}{dt^2} = \sum_{k=1}^{m} \lambda_i W_i \tag{6.5.5}$$

subject to $\omega_k\left(\frac{Dq}{dt}\right) = \left\langle W_k, \frac{Dq}{dt}\right\rangle = 0$, $1 \leq k \leq m$ where $\omega_k(X) = \langle W_k, X\rangle$ and the λ_i are Lagrange multipliers.

We now define a *controlled nonholonomic mechanical system* to be a system of the form

$$\frac{D^2q}{dt^2} = \sum_{i=1}^{m} \lambda_i W_i + \sum_{i=1}^{N} u_i X_i \tag{6.5.6}$$

subject to

$$\left\langle W_k, \frac{Dq}{dt}\right\rangle = 0 \qquad 1 \leq k \leq m\,, \tag{6.5.7}$$

where the $u_i(t)$ are controls and the X_i are arbitrary smooth (control) vector fields. In fact, the X_i are not as arbitrary as they appear, as the remark below shows.

Remark 1. We now consider which general force fields F are compatible with the nonholonomic constraints i.e., which $F(q,\dot{q})$ are allowed in the system

$$\frac{D^2q}{dt^2} = F$$

subject to

$$\left\langle W_k, \frac{Dq}{dt}\right\rangle = 0\,, \qquad 1 \leq k \leq m\,. \tag{6.5.8}$$

Differentiating the constraints gives

$$\langle W_k, F\rangle + \left\langle \frac{DW_k}{dt}\,, \frac{Dq}{dt}\right\rangle = 0\,, \qquad 1 \leq k \leq m$$

or, if ∇ is the Riemannian connection on $(M, \langle , \rangle)$,

$$\langle W_k, F\rangle + \langle \nabla_{\dot{q}} W_k, \dot{q}\rangle = 0\,, \qquad 1 \leq k \leq m \tag{6.5.9}$$

These are the conditions that the force field must satisfy to be compatible with the constraints. The Lagrange multipliers ensure the forces satisfy the above relations. This also shows that number of independent force fields is less than or equal to $n - m$.

Remark 2. We define coresponding the kinematic system corresponding to be $\frac{Dq}{dt} = \sum_{i=1}^{N} u_i \widehat{X}_i$, $\widehat{X}_i \in H$. For controllability analysis, the assumption is made that H is completely nonholonomic (see Vershik and Gershkovich [1988]). (We refer again to Brockett [1982]). We analyze such systems elsewhere.

Remark 3. We note that while the nonholonomic system under discussion is not variational in the Lagrangian sense, it can be seen as the solution of an instantaneous variational problem,

$$\min_{\frac{D^2 q}{dt^2}} \frac{1}{2} \left\| \frac{D^2 q}{dt^2} - \sum_{i=1}^{N} u_i X_i(q) \right\|$$

subject to

$$\left\langle W_k, \frac{Dq}{dt} \right\rangle = 0\,, \qquad 1 \leq k \leq m\,. \tag{6.5.10}$$

This yields the nonholonomic equations as well as the following equations for λ_k:

$$\sum_{j=1}^{m} \langle W_k, W_j \rangle \lambda_j = -\left\langle \frac{DW_k}{dt}\,,\ \frac{D_q}{dt} \right\rangle - \sum_{i=1}^{N} u_i \langle W_k, X_i \rangle$$
$$1 \leq k \leq m\,. \tag{6.5.11}$$

6.6 Symmetries and Conservation Laws

Symmetries in mechanics give rise to constants of the motion (see for example Abraham and Marsden [1978]). In the Riemannian context isometries are generated by Killing vector fields. (Recall that Z is a Killing vector field if $\langle \nabla_Y Z, Y \rangle = 0$ for all vector fields Y – see e.g., Crouch [1981].) Further, a sufficient condition for $\left\langle Z, \frac{Dq}{dt} \right\rangle$ to be a constant of motion for the geodesic flow is that Z is a Killing vector field. For controlled nonholonomic systems we have

Theorem 6.6.1. *Sufficient conditions for* $\left\langle Z, \frac{Dq}{dt} \right\rangle$ *to be a constant of motion for the controlled nonholonomic system 6.5.6 are*

(i) $Z \in H$

(ii) $Z_q \in Span\{X_1, \ldots, X_N\}^{\perp}$

(iii) Z *is a Killing vector field.*

Proof. $\frac{d}{dt}\left\langle Z, \frac{Dq}{dt}\right\rangle = \left\langle \frac{DZ}{dt}, \frac{Dq}{dt}\right\rangle + \left\langle Z, \frac{D^2q}{dt^2}\right\rangle$. The first term is zero by (iii) and the second is zero by 6.5.6, (i) and (ii).

Note that when $M = \mathbb{R}^n$ and the metric g is independent of x_i, then $\frac{\partial}{\partial x_i}$ is a Killing vector field. ■

6.6.1 Reduction

We discuss here an approach to reduction for nonholonomic control systems.

It is often convenient to introduce a bundle structure in M,

$$\begin{array}{c} M \\ \Big\downarrow \pi \\ B \end{array}$$

with fiber F, $\mathrm{Dim}B = r$ and $\mathrm{Dim}F = n - r$. This structure must be compatible with the constraints in the sense that $\pi_* H_x = T_{\pi(x)}B$, $\forall x \in M$ (thus $\mathrm{Dim}H = n - m \geq \mathrm{Dim}B = r$). Our aim is to reduce the dynamical system 6.5.6, 6.5.7 so that evolution on the fiber is given by a first order equation. To do this we introduce two further assumptions. Either

(1) $\mathrm{Dim}\, H = \mathrm{Dim}\, B$, i.e., $n = m + r$. In this case $\widehat{H} = H$ clearly defines a horizontal distribution on the bundle, or,

(2) $\mathrm{Dim}\, H - \mathrm{Dim}\, B = n - m - r = s > 0$, and there exist s linearly independent vector fields $Z_1, \ldots, Z_s$ which satisfy conditions (i)-(iii) of Lemma 1. In particular

$$\langle Z_i, \frac{Dq}{dt}\rangle = c_i \tag{6.6.1}$$

are constants of the motion for 6.5.6, 6.5.7.

We define a distribution $\widehat{H}_0$ on M by setting

$$\begin{aligned} X \in \widehat{H}_0 \quad \text{if} \quad & \langle W_k, X\rangle = 0\,, \qquad 1 \leq k \leq m\,, \\ & \langle Z_k, X\rangle = 0\,, \qquad 1 \leq k \leq s\,. \end{aligned} \tag{6.6.2}$$

Condition (i) of Lemma 1 ensures that $\widehat{H}_0$ is r-dimensional. We also require that $\widehat{H}_0$ defines a horizontal distribution of the bundle, i.e.,

$$T_xF \cap \widehat{H}_0 = \{0\} \qquad \forall x \in M\,. \tag{6.6.3}$$

We further define the r-dimensional affine connection $\widehat{H}$ on M by setting

$$\begin{aligned} x \in \widehat{H} \quad \text{if} \quad & \langle W_k, X\rangle = 0\,, \qquad 1 \leq k \leq m \\ & \langle Z_k, X\rangle = c_k \qquad 1 \leq k \leq s\,. \end{aligned} \tag{6.6.4}$$

In either case 1) or 2) we have, as a direct sum of affine subspaces, $\widehat{H}_x \oplus T_xF = T_xM$, and so any vector field on M can be decomposed uniquely into components $Y_x = Y_x^H + Y_x^F$ with $Y_x^H \in \widehat{H}_x$ and $Y_x^F \in F_x$. With this structure we can now decompose the velocity:

$$\frac{Dq}{dt} = \dot{q}^H + \dot{q}^F\,, \quad \dot{q}^H \in \widehat{H}_q\,, \quad \dot{q}^F \in T_qF \tag{6.6.5}$$

Using 6.5.7 we obtain:

$$\begin{aligned} \langle W_k, \dot{q}^F\rangle &= -\langle W_k, \dot{q}^H\rangle\,, \qquad 1 \le k \le m\,, \\ \langle Z_k, \dot{q}^F\rangle &= \langle Z_k, \dot{q}^H\rangle + c_k\,, \quad 1 \le k \le s\,. \end{aligned} \tag{6.6.6}$$

Our rank conditions on Z_k and W_k allow us to solve these equations giving

$$\dot{q}^F = f_F(q, \dot{q}^H)\,. \tag{6.6.7}$$

Equations 6.5.6 now define the second order set of equations

$$\frac{D\dot{q}^H}{dt} = f_H(q, \dot{q}, u)\,. \tag{6.6.8}$$

We have now provided a reduction from the $2n$ first order equations 6.5.6 to $n + r$ first order equations.

Now locally we can write $\dot{q}^H = \frac{D}{dt}q^B$ for some trajectory $q^B(t) \in B$. In some cases we may be able to rewrite the equations globally in terms of a trajectory $q(t) = (q^B(t), q^F(t))$, $q^B \in B$, $q^F \in F$.

The class of Caplygin control systems introduced in Bloch et. al. [1992] and dicussed in an earlier section corresponds to the case where $H = \widehat{H}$ and there exists a trivial bundle structure $M = \mathbb{R}^n$ with the Euclidean structure. In this case, since $\frac{\partial}{\partial q_i^F}$ is Killing and symmetries correspond to invariance with respect to q_i^F, a global prescription can be given, in which the equations become

$$\begin{aligned} \dot{q}^F &= f_F(q^B, \dot{q}^B) \\ \frac{D\dot{q}^B}{dt} &= f_H(q^B, \dot{q}^B, u)\,, \end{aligned} \tag{6.6.9}$$

and, in fact, f_F is affine in $\dot{q}^B$.

In certain related cases, such as when M is a principal bundle with its structure group G being a group of isometries and $H = \widehat{H}$ is G-invariant, we can also obtain a global reduction where (19) is a set of second order ode's on B. (The uncontrolled class of these systems are studied in detail by Koiller [1988] as nonabelian Caplygin systems. See also recent work by Dayawansa, Krishnaprasad and Yang [1992] and Marsden and Scheurle [1991].)

Example: The Rolling Ball. We consider the controlled rolling ball on the plane, the kinematics of which were discussed in Brockett and Dai [1991], establishing the completely nonholonomic nature of the constraint distribution H. The dynamics of the uncontrolled system are described for example in McMillan [1936].

We will use the coordinates x, y for the linear horizontal displacement and $P \in SO(3)$ for the angular displacement of the ball. Thus P gives the orientation of the ball with respect to inertial axes $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ fixed in the plane, where the $\mathbf{e}_i$ are the standard basis vectors aligned with the $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ axes respectively.

Let $\underline{\omega} \in \mathbb{R}^3$ denote the angular velocity of the ball with respect to inertial axes. In particular, the ball may spin freely about the z-axis and the z-component of angular momentum is conserved. If J denotes the inertia tensor of the ball with respect to the body axes, then $\mathbb{J} = P^T J P$ denotes the inertia tensor of the ball with respect to the inertial axes and $\mathbb{J}\underline{\omega}$ is the angular momentum of the ball with respect to the inertial axes. The conservation law alluded to above is expressed as

$$\mathbf{e}_3^T \mathbb{J}\underline{\omega} = c\,. \tag{6.6.10}$$

The nonholonomic constraints are expressed as (see e.g., Brockett and Dai [1991])

$$\begin{aligned} \mathbf{e}_2^T \underline{\omega} + \dot{x} &= 0 \\ \mathbf{e}_1^T \underline{\omega} - \dot{y} &= 0\,. \end{aligned} \tag{6.6.11}$$

The kinematics for the rotating ball we express as $\dot{P} = S(\underline{\nu})P$ where $\underline{\nu} = P\underline{\omega}$ is the angular velocity in the body frame (see Crouch [1981] for the definition of the skew-symmetric matrix $S(\underline{\nu})$).

We now wish to write down equation (6) for this example using the global coordinates $q = (x, y, P)$. The metric on $M = SO(3) \times \mathbb{R}^2$ is defined by

$$\begin{aligned} \langle (S(\underline{\nu}_1)P, \dot{x}_1, \dot{y})\ ,\ &(S(\underline{\nu}_2)P, \dot{x}_2, \dot{y}_2)\rangle \\ &= \nu_1^T J \nu_2 + \dot{x}_1\dot{x}_2 + \dot{y}_1\dot{y}_2\,. \end{aligned} \tag{6.6.12}$$

An explicit expression for the Riemannian connection on $SO(3)$ may be found by consulting Arnold [1978], p. 327. Equation (6) becomes

$$\begin{aligned} S(\dot{\underline{\nu}})P &- S(J^{-1}S(\underline{\nu})J\underline{\nu})P \\ &= \lambda_1 S(J^{-1}P\mathbf{e}_1)P + \lambda_2 S(J^{-1}P\mathbf{e}_2)P \\ \ddot{x} &= \lambda_1 + u_1 \\ \ddot{y} &= -\lambda_1 + u_2\,. \end{aligned} \tag{6.6.13}$$

It is convenient to re-express this system using inertial coordinates $\underline{\omega} = P^T \underline{\nu}$, in which case we obtain

$$\begin{aligned}
\dot{\underline{\omega}} &= \mathbb{J}^{-1} S(\underline{\omega}) \mathbb{J} \underline{\omega} + \lambda_1 \mathbb{J}^{-1} \mathbf{e}_1 + \lambda_2 \mathbb{J}^{-1} \mathbf{e}_2 \\
\ddot{x} &= \lambda_2 + u_1 \\
\ddot{y} &= -\lambda_1 + u_2 \\
\dot{P} &= P S(\underline{\omega}) .
\end{aligned} \tag{6.6.14}$$

From the equations it is easy to see that indeed (21) is a constant of the motion. In fact Z in formula (12) is given by $Z = S(P\mathbf{e}_3)P$. It is easy to check conditions (i) and (ii) of Lemma 1, and, with more effort, one can verify condition (iii). Expressions for the multipliers λ_i are obtained by the analysis of Remark 3, Section 2.

In this example, there are two admissible reductions, as described in Section 4. The first takes $B = SO(3)$, the fibre $F = \mathbb{R}^2$, and the reduction of system (25) is obtained simply by substituting the two constraints (22) for the second order equations. This corresponds to the case $H = \widehat{H}$ of Section 4.

For the second reduction we take $B = \mathbb{R}^2$, $F = SO(3)$, and we now employ the constraints (22) and the constants of motion (21) to obtain the following expression for $\underline{\omega}$:

$$\underline{\omega} = \dot{x}(\alpha_1 \mathbf{e}_3 - \mathbf{e}_2) + \dot{y}(\mathbf{e}_1 - \alpha_2 \mathbf{e}_3) + \alpha_3 \mathbf{e}_3$$

where

$$\alpha_1 = \frac{\mathbf{e}_3^T \mathbb{J} \mathbf{e}_1}{\mathbf{e}_3^T \mathbb{J} \mathbf{e}_3}, \quad \alpha_2 = \frac{\mathbf{e}_3^T \mathbb{J} \mathbf{e}_2}{\mathbf{e}_3^T \mathbb{J} \mathbf{e}_3}, \quad \alpha_3 = \frac{c}{\mathbf{e}_3^T \mathbb{J} \mathbf{e}_3} . \tag{6.6.15}$$

The reduced equations become, after substituting for the multipliers,

$$\begin{aligned}
\ddot{x} &= \lambda_2 + u_1 \\
\ddot{y} &= -\lambda_1 + u_2 \\
\dot{P} &= P S(\dot{x}(\alpha_1 \mathbf{e}_3 - \mathbf{e}_2) + \dot{y}(\mathbf{e}_1 - \alpha_2 \mathbf{e}_3) + \alpha_3 \mathbf{e}_3)
\end{aligned} \tag{6.6.16}$$

In this case $\widehat{H}$ is obtained through the second definition in Section 4, with $s = 1$. Note also that here we obtain seven first order ODE's rather than the eight obtained from the previous reduction.

6.7 A Discontinuous Energy Method for a Class of Nonholonomic System

As we have seen, there has a been a great deal of research on the problem of stabilizing systems which fail a necessary condition for the existence of smooth, or even continuous, feedback (see Brockett [1983]). One of the reasons for the interest in such systems is that nonholonomic systems fall into this class. (See for, example, Murray and Sastry [1993], Bloch, McClamroch and Reyhanoglu [1992], and references therein.) Various approaches have been taken to the stabilization problem for such systems, focusing mainly on the development of either smooth dynamic feedback or nonsmooth feedback. An important paper regarding the former approach is Coron [1992]. See also Pomet [1992] and M'Closkey and Murray [1993]. Kolmanovsky and McClamroch [1993], for example, used a discontinuous, discrete-time approach, while Brockett [1993] used a stochastic approach. Other important work includes that of Liu and Sussmann [1991], as well as work of Samson, Sordalen, Walsh, Bushnell and others. Another interesting problem for such systems in the problem of tracking which was also analyzed in Brockett [1993].

6.7.1 Stabilizing the Heisenberg System

Here we follow Bloch and Drakunov [1994], [1996] in considering a a sliding mode approach to the stabilization and tracking problem for the so-called nonholonomic integrator or Heisenberg system (so called because the underlying Lie algebra of control vector fields is isomorphic to the Heisenberg algebra.) Firstly we provide a feedback which will globally asymptotically stabilize the origin. The idea is to use the natural algebraic structure of the system together with ideas from sliding mode theory (see Utkin [1978], DeCarlo et al. [1988], Drakunov and Utkin [1992]). We announced the main result on stabilization in Bloch and Drakunov [1994]. Related recent work includes the following. Khennouf and Canudas de Wit [1995] presented the alternative control scheme (6.7.24), (6.7.25), discussed below, and in Canudas de Wit and Knennouf [1995], they considered robustness issues. Hespanha [1995] and Morse [1995] discussed these control ideas in terms of logic based switching. Sliding mode control for differentially flat nonholonomic systems is discussed in Sira-Ramirez [1995]. We discussed the tracking problem in Bloch and Drakunov [1995], while a general approach to tracking via sliding modes is described in Guldner and Utkin [1995].

The nonholonomic integrator is merely the simplest example of an important class of nonlinear controllable systems of the $\dot{x} = B(x)u$, $B(x)$ an $n \times m$ matrix, $m < n$, introduced in the fundamental paper Brockett [1981], and which are prototypical examples of systems where smooth feedback fails and which have a natural controllability condition. In a related paper Bloch and Drakunov [1996] discuss the stabilization of such general systems via a discontinous (but not sliding mode) algorithm. Our algorithm for the general case involves switching between different Lyapunov functions.

6.7.2 Stabilization of the Nonholonomic Integrator in Sliding Mode

We consider the system (see Brockett [1981]):

$$\dot{x} = u \tag{6.7.1}$$
$$\dot{y} = v \tag{6.7.2}$$
$$\dot{z} = xv - yu. \tag{6.7.3}$$

As mentioned earlier it is a prototype for more complex controllable but not smoothly stabilizable systems.

We note also that this system can be obtained by a change of variables from a systems in co-called "chained form" (see e.g. Sira-Ramirez [1995] and references therein.)

The problem of stabilizing 6.7.1-6.7.2, even locally, is not a trivial task, since, as can be easily seen, the linearization in the vicinity of the origin gives the noncontrollable system

$$\dot{x} = u$$
$$\dot{y} = v$$
$$\dot{z} = 0.$$

In fact, as was proved by Brockett [1983], the system 6.7.1-6.7.3 cannot be stabilized by **any** smooth feedback control law. As discussed in the introduction, later approaches using time-periodic feedback and a randomized feedback were developed.

In this paper we present time invariant laws solving the stated problem. By nature, they are discontinuous and lead to sliding along manifolds of reduced dimensionality in the state space.

The main difficulty here is the fact that stabilization of x and y leads to zero right hand side of 6.7.3 and, therefore, the variable z cannot be steered to zero. That simple observation implies that to stabilize the system one needs to make z converge "faster", than x and y.

We suggest using the following control law

$$u = -\alpha x + \beta\, y\; sign(z) \tag{6.7.4}$$
$$v = -\alpha y - \beta\, x\; sign(z), \tag{6.7.5}$$

where α, and β are positive constants.

Let us show that there exists a set of initial conditions such that trajectories starting there converge to the origin.

Consider a Lyapunov function for (x, y)-subspace:

$$V = \frac{1}{2}(x^2 + y^2). \tag{6.7.6}$$

The time derivative of V along the trajectories of the system 6.7.3 is negative:

$$\dot{V} = -\alpha x^2 + \beta xy\ sign(z) - \alpha y^2 - \beta xy\ sign(z) = -\alpha(x^2 + y^2) = -2\alpha V. \tag{6.7.7}$$

Therefore, under the control 6.7.4, 6.7.5 the variables x and y are stabilized.

Now let us consider the variable z. Using 6.7.3 and 6.7.4, 6.7.5 we obtain:

$$\dot{z} = xv - yu = -\beta(x^2 + y^2) sign(z) = -2\beta V\ sign(z). \tag{6.7.8}$$

Since V does not depend on z and is a positive function of time, the absolute value of the variable z will decrease and will reach zero in finite time if the inequality

$$2\beta \int_0^\infty V(\tau) d\tau > |z(0)| \tag{6.7.9}$$

holds. If $z(0)$ is such that

$$2\beta \int_0^\infty V(\tau) d\tau = |z(0)|, \tag{6.7.10}$$

$z(t)$ converges to the origin in infinite time (asymptotically). Otherwise, it converges to some constant nonzero value of the same sign as $z(0)$.

If the above inequality 6.7.9 holds, the system trajectories are directed to the surface $z = 0$ and the variable $z(t)$ is stabilized at the origin in finite time. (The variables x and y, as follows from 6.7.7, always converge to the origin while within that surface.)

This phenomenon is known as *sliding mode* (see Utkin [1978], DeCarlo [1988]). The manifold $z = 0$ is a stable integral manifold of the closed loop system 6.7.1-6.7.3, 6.7.4, 6.7.5. Its characteristic feature is reachability in finite time (Drakunov and Utkin, 1992). Using a smooth control (even a control satisfying a local Lipschitz condition (in the vicinity of $\{z = 0\}$) such fast convergence cannot be achieved. On the other hand, within the sliding manifold $\{z = 0\}$ the system behavior is described in accordance with the Filippov definition for systems of differential equations with discontinuous right hand sides (Filippov, 1964).

Let us explain the version of this definition which we are using. We consider the system

$$\dot{x} = f(x), \tag{6.7.11}$$

with $f(x)$ a discontinuous function comprised of a finite number of continuous $f_k(x)$, $(k = 1, \ldots, N)$ so that

$$f(x) \equiv f_k(x) \ \text{ for } x \in \mathcal{M}_k, \tag{6.7.12}$$

where the open regions $\mathcal{M}_k$ have piecewise smooth boundaries $\partial\mathcal{M}_k$. Then according to the Filippov definition we define the right hand side of 6.7.11 within $\partial\mathcal{M}_k$ as

$$\dot{x} = \sum_{k\in I(x)} \mu_k f_k(x). \tag{6.7.13}$$

The sum is taken over the set $I(x)$ of all k such that $x \in \partial\mathcal{M}_k$ and the variables μ_k satisfy

$$\sum_{k\in I(x)} \mu_k = 1, \tag{6.7.14}$$

i.e. the right hand side belongs to the convex closure $co\{f_k(x) : k \in I(x)\}$ of the vector fields $f_k(x)$ for all $k \in I(x)$. Actually, the Filippov definition replaces the differential equation 6.7.11 by a differential inclusion

$$\dot{x} \in co\{f_k(x) : k \in I(x)\} \tag{6.7.15}$$

for the points x belonging to the boundaries $\partial\mathcal{M}_k$. If within the convex closure there exists a vector field tangent to all or some of the boundaries then there is a solution of the differential inclusion belonging to $\partial\mathcal{M}_k$ which corresponds to the sliding mode.

In the above relatively simple case, the Filippov definition provides a unique solution and implies that the system on the manifold is

$$\dot{x} = -x,$$

$$\dot{y} = -y.$$

Since from 6.7.7 it follows that

$$V(t) = V(0)e^{-2\alpha t} = \frac{1}{2}(x^2(0) + y^2(0))e^{-2\alpha t}\,, \tag{6.7.16}$$

substituting this expression in 6.7.9 and integrating we find that the condition for the system to be stabilized is

$$\frac{\beta}{2\alpha}[x^2(0) + y^2(0)] \geq |z(0)|. \tag{6.7.17}$$

The inequality

$$\frac{\beta}{2\alpha}(x^2 + y^2) < |z|. \tag{6.7.18}$$

defines a parabolic region $\mathcal{P}$ in the state space.

The above derivation can be summarized in the following theorem:

Theorem 6.7.1. *If the initial conditions for the system 6.7.1-6.7.3 belong to the complement $\mathcal{P}^c$ of the region $\mathcal{P}$ defined by 6.7.18, then the control 6.7.4, 6.7.5 stabilizes the state.*

If the initial data are such that 6.7.18 is true, i.e. the state is inside the paraboloid, we can use any control law which steers it outside. In fact, any nonzero constant control can be applied. Namely, if $u \equiv u_0 = const$, $v \equiv v_0 = const$, then

$$x(t) = u_0 t + x_0,$$

$$y(t) = v_0 t + y_0,$$

$$z(t) = t(x_0 v_0 - y_0 u_0) + z_0.$$

With such x, y and z the left hand side of 6.7.18 is quadratic with respect to time t while right hand side is linear. Hence, when the time increases the state inevitably will leave $\mathcal{P}$.

A global control law in the form of the feedback (although discontinuous) can be described as follows:

$$(u, v)^T = \begin{cases} (u_0, v_0)^T & \text{if } (x, y, z)^T \in \mathcal{P} \\ eq.6.7.4, 6.7.5 & \text{if } (x, y, z)^T \in \mathcal{P}^c \end{cases} \tag{6.7.19}$$

Theorem 6.7.2. *The closed system 6.7.1-6.7.3, 6.7.19 is globally asymptotically stable at the origin.*

Global asymptotic stability means that: (i) for all initial conditions $x(t), y(t), z(t) \to 0$, when $t \to \infty$; (ii) $\forall \varepsilon > 0$ there exists $\delta > 0$ such that $x_0^2 + y_0^2 + z_0^2 < \delta^2$ implies $x^2(t) + y^2(t) + z^2(t) < \varepsilon^2$ for any $t \geq 0$.

We have already shown above that (i) is true. (ii) follows from the fact that outside $\mathcal{P}$ and on the surface of parabola $\partial\mathcal{P}$ the state monotonically approaches the origin. For initial conditions inside $\mathcal{P}$ we have :

$$x^2(t) + y^2(t) + z^2(t) = (u_0 t + x_0)^2 + (v_0 t + y_0)^2 + [(x_0 v_0 - y_0 u_0)t + z_0]^2. \tag{6.7.20}$$

The maximum of the expression 6.7.20 is achieved for $t = 0$ or $t = t_f$, where t_f is the first moment of time when the state reaches $\partial\mathcal{P}$. This moment is defined by an equation

$$\frac{\beta}{2\alpha}(u_0 t_f + x_0)^2 + (v_0 t_f + y_0)^2 = |(x_0 v_0 - y_0 u_0)t_f + z_0|. \tag{6.7.21}$$

As can be easily seen from 6.7.21, for fixed u_0, v_0, the solution of this equation t_f tends to zero if x_0, y_0, z_0 tend simultaneously to zero. That proves (ii).

The parameters $\alpha > 0$, β define the size of paraboloid. When $\frac{\beta}{\alpha} \to \infty$ the parabolic region $\mathcal{P}$ limits to to the z-axis. From that point of view, to stabilize the system 6.7.1-6.7.3, it is reasonable to increase β as the state approaches the origin (if we decrease α the convergence of x and y will be slower). To realize this idea we can use a control law where α increases when x and y approach the origin

$$u = -\alpha x + \beta \frac{y}{x^2 + y^2}\ sign(z) \tag{6.7.22}$$

$$v = -\alpha y - \beta \frac{x}{x^2 + y^2}\ sign(z), \tag{6.7.23}$$

or even

$$u = -\alpha x + \beta \frac{y}{x^2 + y^2}\ z \tag{6.7.24}$$

$$v = -\alpha y - \beta \frac{x}{x^2 + y^2}\ z. \tag{6.7.25}$$

As mentioned above for a detailed anaysis in the case (6.7.24), (6.7.25), see Khennouf and Canudas de Wit [1995].

Then from 6.7.3 we have ??

Simulations of the algorithm for two types of initial conditions are shown in Fig. 1.

$$\dot{z} = -\beta\ sign(z),$$

or

$$\dot{z} = -\beta z,$$

respectively.

In both cases, the state converges to the origin from any initial conditions, except the ones belonging to the z-axis. But, in contrast to 6.7.4, 6.7.5 the control laws 6.7.22, 6.7.23 and 6.7.24, 6.7.25 are unbounded in in the neigbourhood of z-axis (on the axis it is not defined). If the initial conditions belong to this set again we can apply any nonzero constant control for an arbitrary small period of time and then switch to 6.7.22, 6.7.23 or 6.7.24, 6.7.25. A method of dealing with the boundedness problem is also described by Khennouf and Canudas de Wit. Another global (excluding $\{x = 0\} \cap \{y = 0\}$ space), but only ε-stabilizing control, may be obtained by switching α:

Let α be the following function of x and y

$$\alpha = \alpha_0\ sign(x^2 + y^2 - \varepsilon^2) \tag{6.7.26}$$

where $\alpha_0 > 0$, $\beta > 0$ are constants and let the control be

$$u = -\alpha x + \beta yz \tag{6.7.27}$$

$$v = -\alpha y - \beta xz. \tag{6.7.28}$$

Using 6.7.7 we find that from any initial conditions x and y the state reaches an ε-sphere of the x,y-space origin:

$$x^2 + y^2 = const = \varepsilon^2. \tag{6.7.29}$$

After that the equation for the variable z is

$$\dot{z} = -\beta\varepsilon^2 z. \tag{6.7.30}$$

Therefore, $z \to 0$ when $t \to \infty$, while the variables x and y stay in an ε-vicinity of the origin. Of course, in 6.7.27, 6.7.28 z can be replaced by any function $g(z)$ which guarantees asymptotic stability of the equation

$$\dot{z} = -\beta\varepsilon^2 g(z), \tag{6.7.31}$$

for example, $g(z) = sign(z)$.

Another interesting control can be obtained if in 6.7.26 we replace ε^2 by $|z|$:

$$\alpha = \begin{cases} \alpha_0, & if \;\; x^2 + y^2 > \varepsilon^2 \\ \alpha_1, & if \;\; x^2 + y^2 \le \varepsilon^2, \end{cases} \tag{6.7.32}$$

where $\alpha_0 > 0$, and $\alpha_1 \le 0$

$$\alpha = \alpha_0 \; sign(x^2 + y^2 - |z|). \tag{6.7.33}$$

and

$$u = -\alpha x + \beta y \; sign(z) \tag{6.7.34}$$

$$v = -\alpha y - \beta x \; sign(z). \tag{6.7.35}$$

In this case outside the parabolic region $x^2 + y^2 > |z|$ the asymptotic convergence of z is guaranteed. If the initial conditions are inside this region, $x^2 + y^2$ is increasing and reaches the parabola in finite time, remaining in sliding mode on the surface of parabola, where

$$\dot{z} = -\beta z. \tag{6.7.36}$$

In fact, this control forms two sliding surfaces in the state space of the closed system: $\{z = 0\}$ and $\{x^2 + y^2 = |z|\}$.

7
Optimal Control

7.1 The Maximum Principle

Here we describe briefly the maximum principle for deriving necessary conditions for optimal control problems. Optimal control problems have been typically cast in a different setting from the classical variational problems, more closely associated with mechanics. The basic difference lies in the way in which the trajectories are formulated; in the optimal control setting the trajectories are "parameterized" by the controlled vector field, while in the traditional variational setting trajectories are simply "constrained." The other basic difference is that the necessary conditions for extremals in the optimal control setting are typically expressed using a Hamiltonian formulation using the Pontryagin maximum principle, rather than the Lagrangian setting. (see Pontryagin et al. [1962]) , Lee and Markus [1976] , and Sussmann [1977]). We state a typical optimal control problem:

$$\begin{aligned} &\min_{u(\cdot)} \int_0^T h_0(x,u)dt \\ \text{subject to: } &\text{(i) } \dot{x} = f(x,u), \quad x \in M^n, \quad u \in \Omega \subset \mathbf{R}^k, \\ &\text{(ii) } x(0) = x_0, x(T) = x_T \end{aligned} \tag{7.1.1}$$

where f and $h_0 \geq 0$ are smooth and Ω is a closed subset of $\mathbf{R}^k$. To state necessary conditions dictated by the Pontryagin maximum principle, we introduce a parameterized Hamiltonian function on T^*M

$$\hat{H}(x,p,u) = p(f(x,u)) - p_0 h_0(x,u) \tag{7.1.2}$$

where $p_0 \geq 0$ a fixed positive constant, and $p \in T^*M$. We denote by $t \mapsto u^*(t)$ a curve which satisfies the following relationship along a trajectory $t \mapsto (x(t), p(t))$ in T^*M:

$$H^*(x(t), p(t), t) \stackrel{\Delta}{=} \hat{H}(x(t), p(t), u^*(t)) = \max_{u \in \Omega} \hat{H}(x(t), p(t), u). \tag{7.1.3}$$

The time varying Hamiltonian function H^* defines a time varying Hamiltonian vector field X_{H^*} on T^*M, with respect to the canonical symplectic structure on T^*M. One statement of Pontryagin's maximum principle gives necessary conditions for extremals of the problem (7.1.1) as follows: an extremal trajectory $t \mapsto x(t)$ of the problem (7.1.1), is the projection onto M of a trajectory of the flow of the vector field X_{H^*}, which satisfies the boundary condition (7.1.1)(ii), and for which $t \mapsto (p(t), p_0)$ is not identically zero on $[0, T]$. The extremal is called normal when $p_0 \neq 0$, (in which case we set $p_0 = 1$). When $p_0 = 0$ we call the extremal abnormal, corresponding to the case where the extremal is determined by constraints alone.

If the extremal control function u^* is not determined by the system (7.1.3) along the extremal trajectory, then the extremal is said to be singular, in which case further necessary conditions are needed to determine u^*. In this paper we are only interested in nonsingular situations in which $\Omega = \mathbf{R}^k$, (or indeed more general vector bundles). We also suppose that the data is sufficiently regular so that u^* is determined uniquely from the condition

$$0 \equiv \frac{\partial \hat{H}}{\partial u}(x(t), p(t), u^*(t)), \quad t \in [0, T]. \tag{7.1.4}$$

It follows that there exists a function k such that $u^*(t) = k(x(t), p(t))$. We then set

$$H(x, p) \stackrel{\Delta}{=} \hat{H}(x, p, k(x, p)). \tag{7.1.5}$$

It follows that along an extremal

$$H(x(t), p(t)) = H^*(x(t), p(t), t). \tag{7.1.6}$$

Given the assumed regularity, and condition (7.1.4), the extremal flow is also a flow of the time invariant Hamiltonian function H. Further, assuming that this function H is sufficiently regular, the extremal will also be governed by a Lagrangian system. However, it is not immediately obvious what variational problem would result in this Lagrangian system. This issue was investigated in depth in Bloch and Crouch [1994] , employing the classical results (Rund [1966] , Bliss [1930]), which relate classical constrained variational problems to Hamiltonian flows, although not optimal control problems. We outline the simplest relationship as discussed in Bloch and Crouch [1994] in the next section.

7.2 Variational and Nonvariational Nonholonomic Problems

A large part of mechanics deals with systems subject to constraints. When the constraints are holonomic or integrable, the formulation of the dynamics is fairly straightforward. When the constraints are nonholonomic however the situation is more delicate. In fact one approach to the dynamics applies to mechanical systems, while the other applies to optimal control problems. This dual picture is described in Vershik [1984] for example.

Suppose a submanifold of the tangent bundle is given as the zero set of a set of constraints on the bundle. Supppose also that we are given a Lagrangian or, more generally an objective function. Then we can proceed in the following two ways:

1) We can consider the conditional variational problem of minimizing a functional subject to the trajectories lying in the given submanifold and obtain the Euler Lagrange equations via the Lagrange method of appending the constraints to the Lagrangian via Lagrange multipliers.

2) We can project, via a suitable projection, the vector field of the unconditional problem on the whole tangent bundle at every point to the tangent space of a given submanifold.

These vector fields will not of course coincide in general, even though both are tangent to the constraint submanifold.

The first method gives us variational nonholonomic mechanics while (real) nonholonomic mechanics is obtained by a procedure of the second type. In fact, this is implemented by the Lagrange D'Alembert princple as we have seen earlier. Later we shall return to this principle and show how to carry it out in coordinate free fashion.

Nonholonomic mechanics is not variational, since while we allow all possible variations in taking the variations of the Lagrangian, the variations have to lie in the constraint distribution and are thus not independent.

Variational nonholonomic mechanics on the other hand is just equivalant to the classical Lagrange problem of minimizing a functional over a class of curves with fixed extreme points and satisfying a given set of equalities.

More precisely we have the following (see e.g. Bloch and Crouch [1994]).

Definition 7.2.1. *The Lagrange problem is given by*

$$\min_{q(\cdot)} \int_0^T L(q, \dot{q}) dt \tag{7.2.1}$$

subject to $q(0) = 0$, $q(T) = q_T$,

$$\Phi(q, \dot{q}) = 0,$$

where $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n-m}$.

One can then proceed to solve these equations by forming the modified Lagrangian

$$\Lambda(q, \dot{q}, \lambda) = L(q, \dot{q}) + \lambda \cdot \Phi(q, \dot{q}) \,, \tag{7.2.2}$$

with $\lambda \in \mathbb{R}^{n-m}$. The Euler Lagrange equations then take the form

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}}\Lambda(q, \dot{q}, \lambda) - \frac{\partial}{\partial q}\Lambda(q, \dot{q}, \lambda) = 0 \tag{7.2.3}$$

$$\Phi(q, \dot{q}) = 0 \,. \tag{7.2.4}$$

This system, while defining perfectly good dynamics, does not correspond in general to a physical mechanical problem. It is certainly variational – over a restricted class of curves satisfying $\Phi(q, \dot{q}) = 0$. The case we are particularly interested in is the case of of classical (linear in the velocity) nonholonomic constraints –

$$\omega_i(q, \dot{q}) = \sum_j a_{ij}(q)\dot{q}_j = 0 \qquad i = 1, \cdots, n - m \,. \tag{7.2.5}$$

In the case these constraints are integrable (equivalent to functions of q only) and L is physical – i.e. it is a holonomic system, this system will represent physical dynamics. In the nonholonomic case, these equations will not be physical since equations of this type to not satisfy Newton's law ($F = ma$!) while those of nonholonomic mechanics do – indeed the Lagrange D'Alembert principle ensures this.

To see this explicitly let us examine the Lagrange multiplier formulation of the equations of motion in each case.

Consider firstly the variational case: We have

Theorem 7.2.2. *The Lagrange problem 7.2.1 with constraints of the form 7.2.5 yields the equations of motion*

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}L - \frac{\partial}{\partial q_i}L + \sum_{jk}(\frac{d}{dt}\lambda_j)a_{jk} + \sum_j \lambda_j(\dot{a}_{ji} - \frac{\partial a_{jk}}{\partial q_i}\dot{q}_k) = 0 \tag{7.2.6}$$

with the constraints

$$\sum_j a_{ij}\dot{q}_j = 0 \,. \tag{7.2.7}$$

We can now contrast these equations of motion with the nonholonomic equations of motion with Lagrange multipliers obtained earlier from the Lagrange D'Alembert principle:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}L - \frac{\partial}{\partial q_i}L = \sum_{j=1}^{n-m} \lambda_j a_{ji}\delta q_i \,. \tag{7.2.8}$$

Note that if we set $\lambda_j = 0$ and $\dot{\lambda}_j = \lambda_j$ in the variational nonholonomic equations we recover the true nonholonomic equations of motion. It is precisely the omission of the $\dot{\lambda}_j$ term that destroys the variational nature of the nonholonomic equations.

7.3 Variational Nonholonomic Mechanics and Optimal Control

Variational nonholonomic problems (i.e. constrained variational problems) are equivalent to optimal control problems as the following analysis (see Bloch and Crouch [1994]) shows.

Consider again a modified Lagrangian

$$\Lambda(q, \dot{q}, \lambda) = L(q, \dot{q}) + \lambda \cdot \Phi(q, \dot{q}) \,. \tag{7.3.1}$$

with Euler Lagrange equations

$$\begin{aligned} \frac{d}{dt}\frac{\partial}{\partial \dot{q}}\Lambda(q, \dot{q}, \lambda) - \frac{\partial}{\partial q}\Lambda(q, \dot{q}, \lambda) = 0 \\ \Phi(q, \dot{q}) = 0 \,. \end{aligned} \tag{7.3.2}$$

We will write rewrite this equation in Hamiltonian form and show that the resulting equations are equivalent to the equations of motion the maximum principle for a suitable optimal control problem.

Set

$$p = \frac{\partial}{\partial \dot{q}}\Lambda(q, .\dot{q}, \lambda) \tag{7.3.3}$$

and consider this equation together with the constraints

$$\Phi(q, \dot{q}) = 0. \tag{7.3.4}$$

We wish to solve 7.3.3 and 7.3.4 for $(\dot{q}, \lambda)$.

Now assume that on an open set U the matrix

$$\begin{bmatrix} \frac{\partial^2}{\partial \dot{q}^2}\Lambda(q, \dot{q}, \lambda) & \frac{\partial}{\partial \dot{q}}\Phi(q, \dot{q})^T \\ \frac{\partial}{\partial \dot{q}}\Phi(q, \dot{q}) & 0 \end{bmatrix} \tag{7.3.5}$$

has full rank. (This generalizes the usual Legendre condition – that $\frac{\partial^2}{\partial \dot{q}} L(x, \dot{x})$ has full rank.)

Hence we can solve for $\dot{q}$ and λ:

$$\begin{aligned} \dot{x} &= \phi(q, p) \\ \lambda &= \psi(q, p) \,. \end{aligned} \tag{7.3.6}$$

We now have

Theorem 7.3.1. *(Caratheodory [1967], Rund [1966], Arnold [1988], Bloch and Crouch [1994]) Under the transformation 7.3.6 the Euler Lagrange system 7.3.2 is transformed to the Hamiltonian system*

$$\begin{aligned} \dot{q} &= \frac{\partial}{\partial p} H(q,p) \\ \dot{p} &= -\frac{\partial}{\partial q} H(q,p) \end{aligned} \tag{7.3.7}$$

where

$$H(q,p) = p \cdot \phi(q,p) - L(q, \phi(q,p))\,. \tag{7.3.8}$$

Proof. $\Phi(q, \phi(q,p)) = 0$ implies

$$\begin{aligned} \frac{\partial \Phi}{\partial q} + \frac{\partial \Phi}{\partial \dot{q}} \frac{\partial \Phi}{\partial q} &= 0 \\ \frac{\partial \Phi}{\partial \dot{q}} \frac{\partial \phi}{\partial p} &= 0 \end{aligned}$$

Hence, using 7.3.3, we have

$$\begin{aligned} \frac{\partial H}{\partial p} &= \phi + (p - \frac{\partial L}{\partial \dot{q}}) \cdot \frac{\partial \phi}{\partial p} \\ &= \dot{q} + \lambda \cdot (\frac{\partial \Phi}{\partial \dot{q}} \frac{\partial \phi}{\partial p}) \\ &= \dot{q} \end{aligned}$$

Similarly

$$\begin{aligned} \frac{\partial H}{\partial q} &= -\frac{\partial L}{\partial q} + (p - \frac{\partial L}{\partial \dot{q}}) \cdot \frac{\partial \phi}{\partial q} \\ &= -\frac{\partial L}{\partial q} + \lambda \cdot (\frac{\partial \Phi}{\partial \dot{q}} \frac{\partial \phi}{\partial q}) \\ &= -(\frac{\partial L}{\partial q} + \lambda \cdot \frac{\partial \Phi}{\partial q}) \\ &= -\frac{\partial \Lambda}{\partial q} = -\dot{p} \end{aligned}$$

■

Now:

Definition 7.3.2. *Let the Optimal Control problem be given by*

$$\min_{u(\cdot)} \int_0^T g(q,u) dt \tag{7.3.9}$$

subject to $q(0) = 0$, $q(T) = q_T$,

$$\dot{q} = f(x, u),$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$.

Then we have

Theorem 7.3.3. *The Lagrange problem and Optimal Control problem generate the same (regular) extremals trajectries provided*

(i) $\Phi(q, \dot{q}) = 0)$ *if and only if there exists a* u *such that* $\dot{q} = f(q, u)$,

(ii) $L(q, f(q, u)) = g(q, u)$, *and*

(iii) The optimal control u^* *is uniquely determined by the condition*

$$\frac{\partial \hat{H}}{\partial u}(q, p, u^*) = 0 \tag{7.3.10}$$

where

$$\frac{\partial^2 \hat{H}}{\partial u^2}(q, p, u^*)$$

is full rank and

$$H(q, p, u) = p \cdot f(q, u) - g(q, u) \tag{7.3.11}$$

is the Hamiltonian function given by the maximum principle.

Proof. By (iii) we may use the equation

$$p \cdot \frac{\partial f}{\partial u}(q, u^*) - \frac{\partial g}{\partial u}(q, u^*)$$

to deduce that there exists a function r such that $u^* = r(q, p)$.

The extremal trajectories are now generated by the Hamiltonian

$$\overline{H}(q, p) = \hat{H}(q, p, r(x, p)) = p \cdot f(q, r(q, p)) - g(q, r(q, p)) \,. \tag{7.3.12}$$

Then the result follows and we have

$$\begin{aligned} \overline{H}(q, p) &= H(q, p) \\ f(q, r(q, p)) &= \phi(q, p) \\ g(q, r(q, p)) &= L(q, \phi(q, p)) \end{aligned}$$

■

7.3.1 Two Examples of Variational Nonholonomic Mechanics

In this subsection we give two simple examples of the theory above, consisting of the rolling penny (or unicycle) and the "Heisenberg" control system (see Bloch and Crouch [1993]).

Example 1: Rolling Penny or Unicycle We consider again a penny rolling on the $x - y$ plane with rotation angle θ and heading angle ϕ, as discussed in section 1.1, but now we consider the corresponding variational system – just the variational control system considered in the introduction, minus the controls.

We have the augmented Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}\dot{\varphi}^2 + \lambda_1(\dot{x} - R\dot{\theta}\cos\varphi) + \lambda_2(\dot{x} - R\dot{\theta}\sin\varphi)\,. \tag{7.3.13}$$

The Euler Lagrange equations are then

$$m\ddot{x} + \dot{\lambda}_1 = 0 \tag{7.3.14}$$

$$m\ddot{y} + \dot{\lambda}_2 = 0 \tag{7.3.15}$$

$$I\ddot{\theta} - R\frac{d}{dt}(\lambda_1\cos\varphi + \lambda_2\sin\varphi) = 0 \tag{7.3.16}$$

$$J\ddot{\varphi} + R\frac{\partial d}{\partial\varphi}(\lambda_1\dot{\theta}\cos\varphi + \lambda_2\dot{\theta}\sin\varphi) = 0\,. \tag{7.3.17}$$

From the constraint equations and equations 7.3.14 and 7.3.15 we have as before

$$\lambda_1 = -mR\dot{\theta}\cos\phi + A$$
$$\lambda_2 = -mR\dot{\theta}\sin\phi + B$$

where A and B are constants.

Substituting into equations 7.3.16 and 7.3.17 we obtain

$$(I + mR^2)\ddot{\theta} = R\dot{\varphi}(-A\sin\varphi + B\cos\varphi) \tag{7.3.18}$$

$$J\ddot{\varphi} = R\dot{\theta}(A\sin\varphi - B\cos\varphi)\,. \tag{7.3.19}$$

$$\tag{7.3.20}$$

These equations, together with the constraints, define the dynamics.

♦

Example 2: Heisenberg System We now consider the variational nonholonomic version of the Heisenberg system described in section 1.2. The augmented Lagrangian in this case is

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \lambda(\dot{z} - y\dot{x} + x\dot{y}). \tag{7.3.21}$$

The corresponding Euler-Lagrange equations are:

$$\frac{d}{dt}(\dot{x} - \lambda y) - \lambda\dot{y} = 0 = \ddot{x} - 2\lambda\dot{y} - \dot{\lambda}y \tag{7.3.22}$$

$$\frac{d}{dt}(\dot{y} + \lambda x) + \lambda\dot{x} = 0 = \ddot{y} + 2\lambda\dot{x} + \dot{\lambda}x \tag{7.3.23}$$

$$\ddot{z} + \dot{\lambda} = 0. \tag{7.3.24}$$

From the constraint equation we obtain

$$\dot{\lambda} = -\ddot{z} = x\ddot{y} - y\ddot{x}. \tag{7.3.25}$$

Solving for $\ddot{x}$ and $\ddot{y}$ we obtain

$$\phi\dot{\lambda} + \dot{\phi}\lambda = 0$$

where $\phi = (1 + x^2 + y^2)$. Thus if $c = \mu(0)(1 + x(0)^2 + y(0)^2)$

$$\lambda\phi = c$$

and the variational nonholonomic equations become:

$$\phi\ddot{x} = \lambda(2\phi\dot{y} - \dot{\phi}y) \tag{7.3.26}$$

$$\phi\ddot{y} = -\lambda(2\phi\dot{x} - \dot{\phi}x) \tag{7.3.27}$$

$$\phi\ddot{z} = \lambda\dot{\phi}\,. \tag{7.3.28}$$

Note that in each case we obtain the nonholonomic equations of motion by setting the constants of integration for the multipliers equal to zero: A and B in example 1 and c in example 2.

♦

7.4 Optimal Control on Lie Algebras and Adjoint Orbits

An interesting class of optimal control problems are those which lie naturally on adjoint orbits of compact Lie groups. Analysis of problems of this type may be found in Brockett [1994] and Bloch and Crouch [1995, 1996].

In this section we will set up the variational problem on the adjoint orbits of compact Lie groups and derive the corresponding Hamiltonian equations.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{g}_u$ its compact real form, and G_u the corresponding compact group. In this case a natural drift free control system on an orbit of G_u takes the form

$$\dot{x} = [x, u] \tag{7.4.1}$$

We remark that we formulate the problem in this generality for convenience, but the most useful case to bear is mind is the algebra $su(n)$ of skew-Hermitian matrices or the algebra $so(n)$ of skew symmetric matrices (the intersection of the compact and normal real forms of the the algebra $sl(n, \mathbb{C})$). Orbits in this case are similarity orbits under the group action.

We then consider the following generalization of the functional suggested by Brockett [1994] (we shall return to Brockett's precise problem shortly):

$$\eta(x, u) = \int_0^{t_f} 1/2||u||^2 - V(x)dt \tag{7.4.2}$$

where $|| \cdot || =< \cdot, \cdot >^{1/2}$ is the norm induced on $\mathfrak{g}_u$ by the negative of the Killing form $\kappa(\cdot, \cdot)$ on $\mathfrak{g}$ and V is a smooth function on $\mathfrak{g}_u$. The pairing between vectors x in $\mathfrak{g}$ and dual vectors p in $\mathfrak{g}^*$ may be written $< p, x >= -\kappa(x, p)$.

We have

Theorem 7.4.1. *The equations of the maximum principle for the variational problem with functional 7.4.2 subject to the dynamics 7.4.1 are*

$$\begin{aligned} \dot{x} &= [x, [p, x]] \\ \dot{p} &= [p, [p, x]] - V_x \,. \end{aligned} \tag{7.4.3}$$

Proof. The Hamiltonian is given by

$$H(x, p, u) =< p, [x, u] > -1/2||u||^2 + V(x) \,. \tag{7.4.4}$$

Hence

$$\frac{\partial H}{\partial u} = - < [x, p], \cdot > - < u, \cdot >$$

and thus the optimal control is given by

$$u^* = [p, x] \tag{7.4.5}$$

Substituting this into H we find the Hamiltonian evaluated along the optimal trajectory is given by

$$H^*(p, x) = -1/2 < x, [p, [p, x]] > +V(x) \tag{7.4.6}$$

Computing

$$\dot{x} = \left(\frac{\partial H^*}{\partial p} \right)^T$$

and

$$\dot{p} = - \left(\frac{\partial H^*}{\partial x} \right)^T$$

gives the result. ■

A particularly interesting special case of this problem is that of Brockett [1994] where we have

Corollary 7.4.2. *The equations of the maximum principle for the variational problem 7.4.2 subject to equations 7.4.1 with* $V(x) = -1/2||[x,n]||^2$ *are*

$$\begin{aligned} \dot{x} &= [x,[p,x] \\ \dot{p} &= [p,[p,x]] - [n,[n,x]] \,. \end{aligned} \tag{7.4.7}$$

The proof of the corollary follows immediately, setting $V(x) = 1/2 < x,[n,[n,x]] >$. Note that with this functional the equations lie naturally on an adjoint orbit. In addition, these equations are interesting in that the optimal flow may be related to the integrable Toda lattice equations (see below and Brockett [1994].)

These equations may be recast as the Euler Lagrange equations found by Brockett.

In order to do this we introduce the following notation (see e.g. Bloch, Brockett and Ratiu [1992]): Let x and l lie in $\mathfrak{g}_u$. Then x may be decomposed as $x = x^l + x_l$ where $x_l \in \mathrm{Ker}(\mathrm{ad}_l)$ and $x^l \in \mathrm{Im}(\mathrm{ad}_l)$ and where $\mathrm{ad}_x(y) = [x,y]$. Further, given any $l \in \mathfrak{g}_u$ we may decompose $\mathfrak{g}_u$ orthogonally relative to $-\kappa(,)$ as $\mathfrak{g}_u^l \oplus \mathfrak{g}_{ul}$ where $\mathfrak{g}_u^l = \mathrm{Im}(\mathrm{ad}_l)$ and $\mathfrak{g}_{ul} = \mathrm{Ker}(\mathrm{ad}_l)$.

Now any velocity vector is tangent to the orbit of the adjoint action and hence is of the form $\dot{x} = [x,a]$ for some $x \in \mathfrak{g}_u$. Hence $\dot{x} \in \mathrm{Im}(\mathrm{ad}_x)$. The inverse of operator ad_x which, following Brockett [1994], we will denote by ad_x^{-1}, is well defined on $\mathrm{Im}(\mathrm{ad}_x)$ and hence on $\dot{x}$.

Then we have

Proposition 7.4.3. *The equations 7.4.7 are equivalent to the Euler-Lagrange equations (Brockett [1994])*

$$\ddot{x} = [\dot{x}, \mathrm{ad}_x^{-1}(\dot{x})] + \mathrm{ad}_x^2 \mathrm{ad}_n^2(x). \tag{7.4.8}$$

Proof. Eliminate p from the two equations 7.4.7. The computation is straightforward but somewhat lengthy. ■

The proof of this proposition may of course be obtained also by the Legendre transformation. In terms of the operator ad, the integrand of 7.4.2 with the given $V(x)$ may be written as the Lagrangian (see Brockett [1994])

$$L(x,\dot{x}) = 1/2(||[\mathrm{ad}_x^{-1}\dot{x}||^2 + ||[x,n]||^2). \tag{7.4.9}$$

Then, since $1/2||\mathrm{ad}_x^{-1}\dot{x}||^2 = -1/2 < \mathrm{ad}_x^{-2}\dot{x}, \dot{x} >$ we have

$$p = \frac{\partial L}{\partial \dot{x}} = -\mathrm{ad}_x^{-2}\dot{x}.$$

From this we obtain the (optimal) Hamiltonian H^*.

We remark that the kinetic energy in 7.4.9 is given by the so-called normal metric (see e.g. Bloch, Brockett and Ratiu [1992]). This is defined as follows:

Let $\mathcal{O}$ be an adjoint orbit of $\mathfrak{g}_u$ and suppose $\xi = [x, a]$ and $\eta = [x, b]$ are tangent vectors to the orbit at x then then $g_n(\xi, \eta) =< \xi, \eta >= -\kappa(a^x, b^x)$ where a^x and b^x lie in $\mathrm{Im}(\mathrm{ad}_x)$ as defined above.

We consider now the precise sense in which the equations discussed above are Hamiltonian. The discussion here is brief – more detail may be found in Bloch, Brockett and Crouch [1997].

We have

Theorem 7.4.4. *Let ω be the standard symplectic structure on $T^*\mathfrak{g}_u$. Consider the Hamiltonian*

$$H(x, p) = 1/2||[p, x]||^2 + V(x) \tag{7.4.10}$$

where V(x) is any smooth function on $\mathfrak{g}$ and p is a momentum variable viewed as lying in $\mathfrak{g}$ by indentifying $\mathfrak{g}$ with its dual. The Hamiltonian equations of motion are

$$\begin{aligned} \dot{x} &= [x, [p, x]] \\ \dot{p} &= [p, [p, x]] - V_x. \end{aligned} \tag{7.4.11}$$

Proof. Let $\xi = (\delta x, \delta p)$ denote an arbitrary tangent vector to $T^*\mathfrak{g}_u$ and denote the Hamiltonian vector field corresponding to H by $X_H = (\zeta_x, \zeta_p)$. We need to solve for X_H from the equation $dH.\xi = \omega(X_H, \xi)$. Now

$$dH.\xi =< [p, x], [\delta p, x] > + < [p, x], [p, \delta x] > + < \frac{\partial V}{\partial x}, \delta x >$$

and

$$\omega(X_H, \xi) =< \zeta_x, \delta p > - < \zeta_p, \delta x > .$$

Equating these expressions gives the result. ■

We now have

Corollary 7.4.5. *Let $V(x) = 1/2 < [x, n], [x, n] >$ in the Hamiltonian 7.4.10. Then the Hamiltonian equations 7.4.11 yield the optimal Hamiltonian equations 7.4.7.*

Consider now the case $V = 0$. Remarkably, even though these equation are Hamiltonian with respect to the standard symplectic structure on $T^*\mathfrak{g}_u$, they are in fact Hamiltonian on the cotangent bundle of an adjoint orbit of G_u. That is, even though the Hamiltonian structure on the orbit is complicated and is not the restriction of the structure on the Lie algebra, the equations themselves do restrict. We have

Theorem 7.4.6. *For $V(x) = 0$ the equations*

$$\begin{aligned} \dot{x} &= [x, [p, x]] \\ \dot{p} &= [p, [p, x]] . \end{aligned} \tag{7.4.12}$$

are the Hamiltonian form of the geodesic equations with respect to the normal metric on an adjoint orbit of $\mathfrak{g}_u$.

Proof. From the optimal control calculation above we know that these are the equations of the geodesic flow. It remains to observe that this is the flow with respect to the normal metric. Now for $V = 0$ the Hamiltonian is just the norm of the velocity in the normal metric. We now need to check that the equations of motion are Hamiltonian with respect to this Hamiltonian and a symplectic structure on the cotangent bundle of the orbit, which may be identified with the tangent bundle. A tangent vector to the tangent bundle to the orbit at the point $(x, [x, \xi])$ is of the form

$$(x, [x, \xi], [x, \eta], [[x, \eta], \xi] + [x, \zeta]) \tag{7.4.13}$$

for $\xi, \eta, \zeta \in \mathfrak{g}_u$. Then, using a natural symplectic structure as in Thimm [1981] gives the result. The details of this standard but lengthy computation are given in Bloch, Brockett and Crouch [1995]. ■

We may also endow any orbit with the right-invariant metric

$$g_{nl}([x, a], [x, b]) = -\kappa(a^x, Jb^x) \tag{7.4.14}$$

where J is a positive self-adjoint operator on the algebra. Then we have

Corollary 7.4.7. *The geodesic equations on an adjoint orbit endowed with the left invariant metric 7.4.14 are*

$$\begin{aligned} \dot{p} &= [p, J^{-1}[p, x]] \\ \dot{x} &= [x, J^{-1}[p, x]] . \end{aligned} \tag{7.4.15}$$

We shall consider this right invariant case from the optimal control point of view in the next section.

We note also that, as expected, in the bi-invariant case with $V = 0$ it is possible to explicitly compute the solutions of 7.4.12 despite their strong coupling. This follows from

Lemma 7.4.8. $[p, x]$ *is conserved along the flow of 7.4.12.*

This is proved by a simple computation. However proving complete integrability in the Hamiltonian sense, i.e. finding a complete set of commuting integrals, is by no means easy. This is the content of Thimm [1981] and Bloch, Brockett and Crouch [1995].

7.4.1 Optimal Control on Symmetric Spaces

The equations discussed above are not only well defined on adjoint orbits but also on general symmetric spaces where the tangent vectors to the space are given in the form a suitable bracket – this includes the complex and real Grassmannians of q-planes in $n+1$-space $G_{q,n+1}(\mathbb{C})$ or $G_{q,n+1}(\mathbb{R})$ and in particular the spheres.

This may be seen as follows:

The complex Grassmannian is given by

$$U(n+1)/U(q)\times U(p), \qquad q+p=n+1,\ q\leq p \tag{7.4.16}$$

and the real Grassmannian by

$$SO(n+1)/SO(q)\times SO(p), \qquad q+p=n+1,\ q\leq p \tag{7.4.17}$$

where $U(n)$ is the unitary group and $SO(n)$ the special orthogonal group. In either case let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{m}$ be the Lie algebra decomposition corresponding to G/K. We may thus represent a point in the complex (real) Grassmannian by a matrix

$$\hat{Q}=\begin{bmatrix} 0 & Q \\ -Q^* & 0 \end{bmatrix} \tag{7.4.18}$$

in $\mathfrak{m}$ where Q is a complex (real) $p\times q$ matrix of full rank and Q^* is its Hermitian conjugate (transpose). A point in $\mathfrak{k}$ may be represented by the matrix

$$\hat{K}=\begin{bmatrix} K_1 & O \\ O & K_2 \end{bmatrix} \tag{7.4.19}$$

where $K_1\in u(p)(so(p))$ and $K_2\in u(q)(so(q))$. Define $\hat{P}$ to be a similarly partitioned matrix. Then we have

Proposition 7.4.9. *Tangent vectors to the Grassmannian may be represented by matrices of the form*

$$[\hat{Q},\hat{K}]$$

.

Proof. A curve in the Grassmannian through the point $\hat{Q}$ may be given by

$$e^{-\hat{K}t}\hat{Q}e^{\hat{K}t}.$$

Note that the given curve simply provides an orthogonal (or unitary) transformation of the rows and columns of Q.

Differentiating at $t=0$ gives the result. ■

Now, since tangent vectors are given by brackets, just as in the case of orbits, a normal metric may be defined. Repeating the proof of Theorem 3.3 gives

Proposition 7.4.10. *The geodesic equations on the real or complex Grassmannian are given by*

$$\begin{aligned} \dot{\hat{Q}} &= [\hat{Q}, [\hat{P}, \hat{Q}]] \\ \dot{\hat{P}} &= [\hat{P}, [\hat{P}, \hat{Q}]] \,. \end{aligned} \tag{7.4.20}$$

where $\hat{Q}$ is given by 7.4.18 and similarly for $\hat{P}$.

Note also that for a symmetric space $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}$ and $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ and since $\hat{Q}, \hat{P} \in \mathfrak{m}$ the equations are naturally well defined.

In fact the formalism developed here can be combined with the work of Thimm [1981] to give a very explicit proof of complete integrability of the geodesic flow on symmetric spaces such as the real and complex Grassmannians. In particular it is possible to derive explicitly a complete set of commuting flows and to prove their involutivity. This is the subject of a forthcoming paper, Bloch, Brockett and Crouch [1995].

This should be contrasted with the observation of Brockett regarding the optimal control equations for 7.4.2 with $V(x) = 1/2||[x, n]||^2$. He observed that the optimal flow under suitable conditions was given by $\dot{x} = [x, [x, n]]$ and in the particular case of x being tridiagonal one obtains the integrable Toda lattice equations in Flaschka's form (see Bloch [1990]). Integrability of optimal control problems is a rich subject (see for example Faybusovich [1988]).

As in the variational problem on adjoint orbits discussed in section 2 the double, double bracket equations on Grassmannians are solutions to a natural optimal problem. For conciseness of the exposition we will begin by considering a very general optimal control problem on symmetric spaces which we will then reduce to various special cases.

Theorem 7.4.11. *Let Q be a $p \times q$ complex matrix and let $U \in u(p)$ and $V \in u(q)$. Let J_U and J_V be constant symmetric positive definite operators on the space of complex $p \times p$ and $q \times q$ matrices respectively and let $<,>$ denote the trace inner product $< A, B >= \mathrm{Tr} A^* B$, where A^* is the Hermitian conjugate.*

Consider the optimal control problem

$$\min_{U,V} \int 1/4\{< U, J_U U > + < V, J_V V >\} dt \tag{7.4.21}$$

subject to

$$\dot{Q} = UQ - QV. \tag{7.4.22}$$

Then the optimal controls are given by

$$U = J_U^{-1}(PQ^* - QP^*)$$
$$V = J_V^{-1}(P^*Q - Q^*P)\,. \tag{7.4.23}$$

and the optimal evolution of the states Q and costates P is given by

$$\dot{Q} = J_U^{-1}(PQ^* - QP^*)Q - QJ_V^{-1}(P^*Q - Q^*P)$$
$$\dot{P} = J_U^{-1}(PQ^* - QP^*)P - PJ_V^{-1}(P^*Q - Q^*P)\,. \tag{7.4.24}$$

Proof. Form the Hamiltonian

$$H(Q,P,U,V) = < P, UQ - QV > -1/4 < U, J_U U > -1/4 < V, J_V V > \,. \tag{7.4.25}$$

To find the optimal control we differentiate the Hamiltonian with respect to U and V in the directions Y and Z respectively and set equal to zero. This yields

$$< P, YQ > -1/4 < Y, J_U U > -1/4 < U, J_U Y > = 0$$

and

$$< P, -QZ > -1/4 < Z, J_V V > -1/4 < V, J_V Z > = 0\,.$$

We have $\overline{< Y, J_U U >} = < U, J_U Y >$ and $\overline{< U, J_U Y >} = < Y, J_U U >$ and thus along the optimal trajectory

$$\overline{< P, YQ >} = < P, YQ > \,. \tag{7.4.26}$$

Now

$$< P, YQ > = 1/2\overline{< Y, PQ^* >} - 1/2 < Y, QP^* >$$

and

$$\overline{< P, YQ >} = 1/2 < Y, PQ^* > -1/2\overline{< Y, QP^* >}\,.$$

Thus, using the fact that $< P, YQ >$ is real, along the optimal trajectory we have

$$\begin{aligned} < P, YQ > &= 1/2 < P, YQ > +1/2\overline{< P, YQ >} \\ &= 1/4\overline{< Y, PQ^* - QP^* >} + 1/4 < Y, PQ^* - QP^* > \\ &= 1/4 < Y, J_U U > +1/4\overline{< Y, J_U U >}\,. \end{aligned} \tag{7.4.27}$$

Hence

$$J_U U = PQ^* - QP^*.$$

Similarly for V.

Now the equations for Q and P are given by

$$
\begin{aligned}
&< Z, \dot{Q} > = \nabla_P H(Z) =< Z, UQ - QV > \\
&< \dot{P}, Z > = -\nabla_Q H(Z) = - < P, UZ - ZV > =< UP - PV, Z(\!>\! \tag{7.4.28}
\end{aligned}
$$

and hence the result. ■

We remark that this result does not preclude the existence of conjugate points.

We have the immediate corollary:

Corollary 7.4.12. *For J_U and J_V the identity the optimal control equations for the problem 7.4.21 subject to 7.4.22 are*

$$
\begin{aligned}
\dot{Q} &= PQ^*Q + QQ^*P - 2QP^*Q \\
\dot{P} &= 2PQ^*P - QP^*P - PP^*Q\,.
\end{aligned} \tag{7.4.29}
$$

Further, we have

Corollary 7.4.13. *The equations 7.4.24 are given by the double double bracket equations*

$$
\begin{aligned}
\dot{\hat{Q}} &= [\hat{Q}, J^{-1}[\hat{P}, \hat{Q}]] \\
\dot{\hat{P}} &= [\hat{P}, J^{-1}[\hat{P}, \hat{Q}]]\,.
\end{aligned} \tag{7.4.30}
$$

where J is the operator $\mathrm{diag}(J_U, J_V)$.

The proof is a computation.

Note that the equations (7.4.30) or (7.4.24) give the geodesic equations on the complex Grassmannian (or real Grassmannian in the real case) with respect to right invariant "normal" metric.

As an example in the current setting we write explicitly the geodesic flow on the sphere S^n.

Recall (see e.g. Moser [1980]) that the geodesic motion on S^n may be written as follows:

Let $\mathbf{q} = [q_1, \cdots, q_{n+1}]^T \in \mathbb{R}^{n+1}$ with Euclidean norm $||\mathbf{q}|| = 1$ represent an element of S^n. Then the geodesic flow can be found by setting $\ddot{\mathbf{q}} = \lambda \mathbf{q}$ where λ is chosen so that $||\mathbf{q}||$ is compatible with the flow. This implies $< \mathbf{q}, \dot{\mathbf{q}} >= 0$ and $< \mathbf{q}, \ddot{\mathbf{q}} > + ||\dot{\mathbf{q}}||^2 = 0$. Thus $\lambda = -||\dot{\mathbf{q}}||^2$ and the geodesic flow is given by

$$
\ddot{\mathbf{q}} = -||\dot{\mathbf{q}}||^2 \mathbf{q}. \tag{7.4.31}
$$

Letting $\mathbf{p} = [p_1, \cdots, p_{n+1}]^T \in \mathbb{R}^{n+1}$ this may be viewed as a Hamiltonian system restricted to $||\mathbf{q}|| = 1\,, < \mathbf{q}, \mathbf{p} >= 0$. With Hamiltonian

$H = 1/2||\mathbf{q}||^2||\mathbf{p}||^2$ we get the flow

$$\dot{\mathbf{q}} = \left(\frac{\partial H}{\partial \mathbf{p}}\right)^T = \mathbf{p} \qquad \dot{\mathbf{p}} = -\left(\frac{\partial H}{\partial \mathbf{q}}\right)^T = -||\mathbf{p}||^2\mathbf{q} \tag{7.4.32}$$

In our current setting we have

Proposition 7.4.14. *Let*

$$\hat{P} = \begin{bmatrix} 0 & \cdots & 0 & p_1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & p_{n+1} \\ -p_1 & \cdots & -p_{n+1} & 0 \end{bmatrix} \qquad \hat{Q} = \begin{bmatrix} 0 & \cdots & 0 & q_1 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & q_{n+1} \\ -q_1 & \cdots & -q_{n+1} & 0 \end{bmatrix} \tag{7.4.33}$$

where we normalize $||\mathbf{q}|| = 1$ *and* $< \mathbf{q}, \mathbf{p} >= 0$. *Then the flow 7.4.12 yields the geodesic flow 7.4.33.*

The proof is a computation.

Note that this result may be obtained directly from the variational problem. In this case $Q \equiv \mathbf{q}$ is $n + 1 \times 1$ and real and similarly for $P \equiv \mathbf{p}$. U is then $n \times n$, while $V = 0$.

Then we have

Corollary 7.4.15. *The optimal control problem*

$$\min_U \int 1/4(U^T U)dt \tag{7.4.34}$$

subject to

$$\dot{\mathbf{q}} = U\mathbf{q} \tag{7.4.35}$$

where $U \in so(n)$ *restricts to the geodesic flow on the sphere.*

Proof. The optimal flow equations are then

$$\begin{aligned} \dot{\mathbf{q}} &= \mathbf{p}(\mathbf{q}^T\mathbf{q}) - (\mathbf{q}^T\mathbf{p})\mathbf{q} \\ \dot{\mathbf{p}} &= (\mathbf{q}^T\mathbf{p})\mathbf{p} - \mathbf{q}(\mathbf{p}^T\mathbf{p}) \end{aligned} \tag{7.4.36}$$

which are precisely the equations 7.4.33. It is easy to check that the time derivatives of $\mathbf{q}^T\mathbf{q}$ and $\mathbf{q}^T\mathbf{p}$ are conserved along the flow. Normalizing as in the proposition we indeed remain on the sphere. ■

This flow is completely integrable and again provides an example of an integable optimal control problem. Details of the proof of integrability can be found in Thimm [1981] or Bloch, Brockett and Crouch [1995].

We now discuss the $SO(n)$ rigid body equations in our setting. These equations may again be obtained as a special case of our general optimal control problem.

We recall the rigid body equations on $SO(3)$ (or generally on $SO(n)$ or any compact Lie group – see e.g. Ratiu [1980]) may be written (in right invariant form) as

$$\begin{aligned}\dot{Q} &= \Omega Q \\ \dot{M} &= [\Omega, M]\end{aligned} \tag{7.4.37}$$

where $Q \in SO(3)$ denotes the configuration space variables, $\Omega \in so(3)$ is the angular velocity, and $M = J\Omega = \Lambda\Omega + \Omega\Lambda$ is the angular momentum. Here J is a symmetric positive definite operator defined by the diagonal positive definite matrix Λ.

We now consider the following equations:

$$\begin{aligned}\dot{Q} &= \Omega Q \\ \dot{P} &= \Omega P\end{aligned} \tag{7.4.38}$$

where $\Omega = J^{-1}M$ and $M = PQ^T - QP^T$. We then can easily check that

Proposition 7.4.16. *The equations 7.4.38 reduce to the rigid body equations 7.4.37.*

Proof. Differentiating $M = PQ^T - QP^T$ and using the equations 7.4.38 gives the second of equations 7.4.37. ■

Conversely, given the rigid body equations 7.4.37 we may solve for the variable P in the expression

$$M = PQ^T - QP^T$$

in a neighborhood of $M = 0$. Locally

$$P = \left(e^{\sinh^{-1} M/2}\right) Q\,. \tag{7.4.39}$$

This follows from the observation that

$$M = e^{\sinh^{-1} M/2} - e^{-\sinh^{-1} M/2}\,.$$

For $so(n)$ however, sinh is many to one, so the two representations 7.4.37 and 7.4.38 are not entirely equivalent. However the form 7.4.38 of the rigid body equations does arise from the optimal control problem as follows.

We now take Q to be real, $n \times n$ and lying in $SO(n)$. V is now zero and U in $so(n)$. We have

Corollary 7.4.17. *The optimal control problem*

$$\min_U \int 1/4 < U, JU > dt \tag{7.4.40}$$

subject to

$$\dot{Q} = UQ, \tag{7.4.41}$$

where $U \in so(n)$ yields the rigid body equations 7.4.38.

The optimal controls in this case are given by

$$U = J^{-1}(PQ^T - QP^T) \tag{7.4.42}$$

Now the rigid body equations may be given as singular case of the double double bracket equations discussed earlier for the general optimal control problem.

Let

$$\hat{Q} = \begin{bmatrix} 0 & Q \\ -Q^T & 0 \end{bmatrix} \tag{7.4.43}$$

as before and similarly for $\hat{P}$. Note that these matrices now lie in $so(2n)$ and each block lies in $SO(n)$.

Corollary 7.4.18. *The generalized rigid body equations on $SO(n)$ are given by the double double bracket equations 7.4.30 in the case Q and P lie in $SO(n)$, $J_U = J$, and the operator $J_V^{-1} = 0$.*

Note that the reduced generalized rigid body equations (the dynamics) are completely integrable (see e.g. Ratiu [1980]).

7.4.2 Optimal Control and the Toda Flow

In this section we introduce an optimal control problem which yields the full Toda flow as described originally in Faybusovich [1988]. We follow here the treatment in Bloch and Crouch [1997]. This problem may be viewed as a control problem on an adjoint orbit of lower triangular matrices.

We begin by introducing some specialized notation. Let $\mathcal{G} = gl(n)$ denote the Lie algebra of $n \times n$ matrices (with corresponding Lie group $GL(n)$ of invertible $n \times n$ matrices).

Definition 7.4.19. *For $A \in gl(n)$, let $A = A_+ + A_0 + A_-$ where A_+ is the strictly upper part of A, A_0 the diagonal part, and A_- is the strictly lower part. Let*

$$\begin{aligned}
\pi_l(A) &= A_- + A_+^T + A_0 \ \textit{(lower)} \\
\pi_k(A) &= A_+ - A_+^T \ \textit{(skew)} \\
\pi_{k^\perp}(A) &= A_- + A_0 + A_-^T \ \textit{(symmetric)} \\
\pi_{l^\perp}(A) &= A_+ - A_-^T \ \textit{(strictly upper)}.
\end{aligned}$$

The reason for the above notation is as follows (see Symes [1980]):
Endow $\mathcal{G}$ with the inner product $\langle A, B\rangle_T = \mathrm{Tr} A^T B$. We observe

$$\mathcal{G} = \mathcal{L} \oplus \mathcal{K}$$

where $\mathcal{L}$ is the subalgebra of lower triangular matrices and $\mathcal{K}$ the subalgebra of skew-symmetric matrices. $\mathcal{L}$ is the Lie algebra of L, the lower triangular group and $\mathcal{K}$ is the Lie algebra of K, the group of orthogonal matrices.

Now denote by $\perp$ the perpendicular subspace under the scalar product. Then

$$\mathcal{L}^\perp = \{x : \mathrm{Tr}(x^T y) = 0, \forall y \in \mathcal{L}\}$$

is the algebra of strictly upper triangular matrices and

$$\mathcal{K}^\perp = \{x : \mathrm{Tr}(x^T z) = 0, \forall z \in \mathcal{K}\}$$

is the space of all symmetric matrices.

Hence we can write

$$\mathcal{G} \simeq \mathcal{G}^* = \mathcal{L}^\perp \oplus \mathcal{K}^\perp,$$

and we can make the identification

$$\begin{aligned} \mathcal{L}^* &\simeq \mathcal{K}^\perp \\ \mathcal{K}^* &\simeq \mathcal{L}^\perp . \end{aligned}$$

We now give some preliminary results that will be useful in the main theorem.

Lemma 7.4.20. *For any matrix $S \in \mathcal{G}$ and $L \in \mathcal{L}$*

$$\begin{aligned} \pi_{k^\perp}(L^T S) &= \pi_{k^\perp}(L^T \pi_{k^\perp} S) \\ \pi_{k^\perp}(S L^T) &= \pi_{k^\perp}(\pi_{k^\perp}(S) L^T). \end{aligned}$$

Let $F : \mathcal{G} \to \mathcal{G}$ be analytic and consider

$$f(PQ^T) = \mathrm{Tr} F(PQ^T).$$

Let

$$\mathcal{F}_A f(A)(R) = \lim_{h \to 0} \frac{f(A + hR) - f(A)}{h} = \langle R, \nabla f(A)\rangle$$

for $A, R \in gl(n)$. Hence

$$\begin{aligned} \mathcal{F}_Q f(PQ^T)(R) &= \langle PR^T, \nabla f(PQ^T)\rangle \\ &= \langle RP^T, \nabla f(PQ^T)^T\rangle \\ &= \langle R, \nabla f(PQ^T)^T P\rangle \\ \mathcal{F}_P f(PQ^T)(R) &= \langle RQ^T, \nabla f(PQ^T)\rangle \\ &= \langle R, \nabla f(PQ^T) Q\rangle . \end{aligned}$$

Hence, we have

Lemma 7.4.21. *Let $f(PQ^T) = TrF(PQ^T)$. Then the Hamiltonian flows with respect to the canonical structure on $T^*\mathcal{G}$ are given by*

$$\begin{aligned}\dot{Q} &= \nabla f(PQ^T)Q\\ \dot{P} &= -\nabla f(PQ^T)^T P,\end{aligned}$$

respectively.

The main result in Bloch and Crouch [1997] can now be stated in the following way. Consider the optimal control problem

$$\begin{aligned}&\min_U \int_0^T \tfrac{1}{2}\langle U, U\rangle_T dt\\ &\text{subject to: } \dot{X} = \pi_l(U)X,\ X(0) = X_0,\ X(T) = X_T,\\ &\text{where } X \in L,\ U \in \mathcal{G}.\end{aligned} \tag{7.4.44}$$

Theorem 7.4.22. *For the optimal control problem (7.4.44), the optimal controls are given by*

$$U = \pi_{k^\perp}(RX^T)$$

where R is the costate vector and the corresponding extremal flow is given by

$$\begin{aligned}\dot{X} &= \pi_l(\pi_{k^\perp}(RX^T))X\\ \dot{R} &= -\pi_{k^\perp}(\pi_l^T(\pi_{k^\perp}(RX^T))R).\end{aligned} \tag{7.4.45}$$

Equations (7.4.45) are Hamiltonian with respect to the canonical structure on $T^*\mathcal{G}$, with corresponding Hamiltonian function

$$H(R, X) = \tfrac{1}{2}\langle \pi_{k^\perp}(RX^T), \pi_{k^\perp}(RX^T)\rangle_T. \tag{7.4.46}$$

From this result, and Lemma 7.4.20, we observe that if $S = \pi_{k^\perp}(R)$ we may rewrite H in (7.4.46) in the form:

$$H = \tfrac{1}{2}\langle \pi_{k^\perp}(SX^T), \pi_{k^\perp}(SX^T)\rangle_T. \tag{7.4.47}$$

Using Lemma 7.4.20 once more, the Hamiltonian equations may be rewritten in the form

$$\begin{aligned}\dot{X} &= \pi_l(\pi_{k^\perp}(SX^T))X\\ \dot{S} &= -\pi_{k^\perp}(\pi_l^T(\pi_{k^\perp}(SX^T))S).\end{aligned} \tag{7.4.48}$$

Now set

$$A = \pi_{k^\perp}(SX^T). \tag{7.4.49}$$

Then the equations (7.4.48) are of the form

$$\begin{aligned}\dot{X} &= \pi_l(\nabla f(A))X \\ \dot{S} &= -\pi_{k^\perp}(\pi_l^T(\nabla f(A))S)\end{aligned}$$

where $f(A) = \frac{1}{2}\mathrm{Tr}(A^2)$.

Now we compute

$$\begin{aligned}\dot{A} &= \pi_{k^\perp}(\dot{S}X^T + S\dot{X}^T) \\ &= \pi_{k^\perp}(-\pi_{k^\perp}(\pi_l^T(\nabla f)S)X^T) \\ &\quad +\pi_{k^\perp}(SX^T\pi_l^T(\nabla f)) \\ &= -\pi_{k^\perp}(\pi_l^T(\nabla f)SX^T) \\ &\quad +\pi_{k^\perp}(SX^T\pi_l^T(\nabla f)) \\ &\quad \text{(by Lemma (7.4.20)} \\ &= \pi_{k^\perp}([A, \pi_l^T(\nabla f)]).\end{aligned}$$

Hence

$$\dot{A} = \pi_{k^\perp}([A, \pi_l^T(\nabla f)]). \tag{7.4.50}$$

But since

$$\nabla f(A) = \pi_l(\nabla f(A)) + \pi_k(\nabla f(A))$$

and $\nabla f(A) \in \mathcal{K}^\perp$ (for any invariant polynomial in A), we have

$$\nabla f(A) = \pi_l^T(\nabla f(A)) - \pi_k(\nabla f(A)).$$

Observing also that $[A, \nabla f(A)] = 0$, we obtain

$$\begin{aligned}\dot{A} &= \pi_{k^\perp}[A, \pi_k(\nabla f(A))] \\ &= [A, \pi_k(\nabla f(A))] \\ &\quad \text{(since } [\mathcal{K}^\perp, \mathcal{K}] \subset \mathcal{K}^\perp) \\ &= [A, \nabla f(A) - \pi_l \nabla f(A)] \\ &= [\pi_l(\nabla f(A)), A].\end{aligned}$$

Thus for $f(A) = \frac{1}{2}\mathrm{Tr}(A^2)$, we obtain from (7.4.48) and (7.4.49) the following set of equations which are equivalent to the extremal flow (7.4.45), and represent the (augmented) full Toda flow:

$$\begin{aligned}\dot{X} &= \pi_l(A)X \\ \dot{A} &= [A, \pi_k(A)] = [\pi_l(A), A].\end{aligned} \tag{7.4.51}$$

The equations (7.4.45) and (7.4.48) were originally derived in a completely different fashion in the work of Symes [1980]. We note the special form of the full reduced Toda flow equation (7.4.50)

$$\dot{A} = \pi_{k^\perp}([A, \pi_l^T(\nabla f)]). \tag{7.4.52}$$

We observe that this evolves naturally in a coadjoint orbit of L. We can see this by considering tangent vectors to an orbit. These are of the form $ad^*_W A$ for $W \in \mathcal{L}$ and $A \in \mathcal{L}^* \simeq \mathcal{K}^\perp$. Let $A = \langle S, \cdot \rangle_T$ for $S \in \mathcal{K}^\perp$, and let $V \in \mathcal{L}$. Then

$$\begin{aligned} ad^*_W(A)(V) &= -A(ad_W V) = -A([W,V]) \\ &= -\langle S, [W,V] \rangle_T = -\langle [W^T, S], V \rangle_T \\ &= -\langle [W^T, S], \pi_l V \rangle_T \\ &\quad \text{since } V \in \mathcal{L} \\ &= \langle \pi_{k^\perp}[S, W^T], V \rangle_T . \end{aligned}$$

Hence indeed the right hand side of (7.4.52) is a tangent vector to a coadjoint orbit. The classical tridiagonal Toda flow lies on a low rank nongeneric orbit of this type.

7.5 Kinematic/subRiemannian Optimal Control Problems

7.5.1 The kinematic subRiemannian optimal control problem

Here we consider the optimal control of singular kinematic control problems – singular in the sense that the number of controls is less than the number of states. The problem is referred to as subRiemannian in that it gives rise to a geodesic flow with respect to a singular metric (see the work of Strichartz [1986,87] and Montgomery [1990]. This problem has an interesting history in control theory (see Brockett [1973], [1981], Baillieul [1975]) and is of interest in nonholonomic mechanics because such systems represent nonholonomic control systems with velocity controls. There has been much work in this area: for example work Murray and Sastry [1990], Lafferiere and Sussmann [1991], Brockett and Dai [1982], Bloch and Crouch [1992], Bloch, Crouch and Ratiu [1994].

We now consider a specific class of optimal control problems made famous by the initial work of Brockett [1981]. We consider control systems of the form:

$$\dot{x} = \sum_{i=1}^{k} X_i u_i, \quad x \in M^n, \quad u \in \Omega \subset \mathbf{R}^k \tag{7.5.1}$$

where Ω contains an open subset which contains the origin, and the vector fields $F = \{X_1, \ldots, X_k\}$ are complete. Setting

$$D_F(x) = \left\{ X_x(x); X = \sum_{i=1}^{k} \alpha_i X_i, \alpha_i \in \mathbf{R} \right\},$$

then $x \mapsto D_F(x)$ defines a distribution on M, whose involutive closure is just the distribution D_L defined by the Lie algebra L of vector fields generated by F. (We assume that all vector fields in L are also complete). We say that D_L has the spanning property if $D_L(x) = T_xM$ for all $x \in M$. System (7.5.1) is said to have the controllability property, if whenever x_0, x_T are two points on M, there exists a piecewise constant control $u(\cdot)$ on $[0, T]$ such that the corresponding solution $x(\cdot)$ of (7.5.1) satisfies $x(0) = x_0$, $x(T) = x_T$. It is a fundamental fact of control theory (see Sussmann [1973]) that if D_L has the spanning property then (7.5.1) has the controllability property. (In the analytic case this condition is also necessary for controllability.)

Thus in case D_L has the spanning property the following optimal control problem is well posed:

$$\min_{u(\cdot)} \int_0^T \frac{1}{2} \sum_{i=1}^k u_i^2(t) dt \tag{7.5.2}$$

subject to: dynamics (7.5.1) and $x(0) = x_0$, $x(T) = x_T$.

To view this as a constrained variational problem we make some additional regularity assumptions, which are not necessary, but even when they hold produce a very rich class of problems.

Assumption

(i) For the problem defined by (7.5.1) and (7.5.2) the distribution D_L satisfies the spanning property.

(ii) The dimension of D_F is constant on M and equal to k. (Thus the vector fields $X_1 \cdots X_k$ are everywhere independent).

(iii) There exist exactly $n - k = m$ one forms on M $\omega_1 \cdots \omega_m$, such that the co-distribution

$$D_F^{\perp}(x) = \{\bar{\omega} \in T_x^*M; \quad \bar{\omega} D_F(x) = 0\}$$

is spanned by $\omega_1, \ldots, \omega_m$ everywhere. (This condition implies that M is parallelizable.)

Since D_F has constant dimension on M we may define a norm on each subspace $D_F(x)$; if $X \in D_F(x)$, and $X = \sum_{i=1}^k \alpha_i X_i(x)$ then

$$|X| = \sum_{i=1}^k \alpha_i^2.$$

This norm defines an inner product on $D_F(x)$, denoted $\langle \cdot, \cdot \rangle_x$, which can be extended to a metric on M. The optimal control problem (7.5.2) is now equivalent to the following constrained variational problem when the assumptions (i), (ii), (iii) hold:

$$\min_{x(\cdot)} \frac{1}{2} \int_0^T \langle \dot{x}, \dot{x} \rangle_x dt \tag{7.5.3}$$

$$\text{subject to: } x(\cdot) \text{ is a piecewise } C^1 \text{ curve in } M \text{ such that} \tag{7.5.4}$$
$$x(0) = x_0, \quad x(T) = x_T \quad \text{and} \quad \omega_i(x)(\dot{x}) = 0, \quad 1 \le i \le m.$$

This problem is often referred to as the sub-Riemannian geodesic problem, to distinguish it from the Riemannian geodesic problem, in which the constraints are absent. These problems were studied by Griffiths [1983] from the constrained variational viewpoint, before they gained much wider recognition, from the optimal control viewpoint by Brockett [1981]. This "singular" geodesic problem is an example of the situation described in (2.47 iv). It is the case however that in the sub-Riemannian geodesic problem, abnormal extremals play an important role, see Sussmann [1992]. See also work by Hermann [1962], and Strichartz [1983], [1989].

We note that the solution of the model problem above, for normal extremals, involves adding the constraints to the Lagrangian by Lagrange multipliers, resulting in necessary conditions in the form of constrained variational mechanics with $L(q, \dot{q}) = 1/2\langle \dot{q}, \dot{q} \rangle_q$. Perhaps unfortunately, in some of the Russian literature, Vershik and Gershkovich [1988] , for example, distributions D_F for which D_L has the spanning property, are said to be completely nonholonomic, and so the system (7.5.1) is sometimes said to be a nonholonomic control system. Clearly, there is another system which has the right to be called the non-holonomic control system generated by the Lagrangian $L = 1/2\langle \dot{q}, \dot{q} \rangle_q$ and the constraints: the corresponding dynamic control problem.

However, the term completely nonholonomic distribution is also well deserved as we briefly demonstrate. Suppose that one of the forms ω_k is exact, $\omega_k = dP$, for some function P on M. By the assumption (iii) $dP(x) \neq 0$ for all x in M, and

$$0 = \omega_k([X_i, X_j]) = dP([X_i, X_j]) = X_i(X_j(P)) - X_j(X_i(P))$$

since $dP(X_i) \equiv dP(X_j) \equiv 0$. It follows that the distribution D_L is annihilated by ω_k everywhere on M, and so cannot have the spanning property. Hence the assumption (i) implies that none of the forms ω_k, $1 \le k \le m$, are exact, and so none of the constraints may be absorbed as holonomic constraints.

The singular nature of the sub-Riemannian geodesic problem is manifested in many ways, such as the existence of distinct abnormal extremals, and the singular nature of the sub-Riemannian geodesic ball, as first investigated by Brockett [1981]. Defining a metric on M by setting

$$d(x_0, x_T) = \min_{x(\cdot)} \int_0^T |\dot{x}| dt, \dot{x} \in D_F(x), x(0) = x_0, x(T) = x_T$$
$$B_\epsilon^F(x_0) = \{\bar{x} \in M; d(\bar{x}, x_0) \leq \epsilon\}$$

then the sub-Riemannian geodesic ball $S_\epsilon^F(x_0)$ is simply the boundary of $B_\epsilon^F(x_0)$. In Brockett [1981] the Heisenberg sub-Riemannian geodesic problem was considered, which expressed as a control problem becomes:

$$\begin{aligned}
&\min_{u(\cdot)} \int_0^T \frac{1}{2}(u_1^2 + u_2^2) dt, \\
&\text{subject to: } \dot{x}_1 = u_1 \\
&\qquad\qquad\quad \dot{x}_2 = u_2 \\
&\qquad\qquad\quad \dot{x}_3 = x_1 u_2 - x_2 u_1 \\
&x(0) = x_0, \quad x(T) = x_T.
\end{aligned}$$

In this case $F = \{(\partial/\partial x_1) - x_2(\partial/\partial x_3), (\partial/\partial x_2) + x_1(\partial/\partial x_3)\}$, while $D_F^\perp$ is spanned by $\omega = x_1 dx_2 - x_2 dx_1 - dx_3$. The Lie algebra L is simply the Heisenberg Lie algebra. $S_\epsilon^F(0)$ has a singularity along the x_3 axis. This class of problem continues to invoke a great deal of interest, especially in the areas of establishing when abnormal extremals are optimal and obtaining a precise description of $S_\epsilon^D(x_0)$.

We shall return to the notion of abnormal extremals later.

We now set up the problem on a Riemannian manifold as follows, although the cases of most interest usually have more structure, such as a group structure. In the following subsections we will look at such special cases. We follow here the approach of Bloch, Crouch and Ratiu [1994].

Let M^n be a Riemannian manifold of dimension n with metric denoted by $< \cdot, \cdot >$. The corresponding Riemannian connection and covariant derivative will be denoted by ∇ and $D/\partial t$ respectively. Now assume that M is such that there exist smooth vector fields $X^1(q), \cdots, X^n(q)$ satisfying $< X^i(q), X^j(q) >= \delta_{ij}$, an orthonormal frame for $T_q M$ for all $q \in M$. This of course limits the class of manifolds we consider, but is satisfied for the main case of interest to us, M a Lie group G.

We now define the kinematic control system on M

$$\frac{dq}{dt} = \sum_{i=1}^m u_i X^i(q), \quad m < n\,. \tag{7.5.5}$$

The singular optimal control problem for 7.5.5 is defined by

$$\min_u \int_0^T \frac{1}{2} \sum_{i=1}^m u_i^2(t) dt; \quad q(0) = q_0 \quad q(T) = q_T \tag{7.5.6}$$

subject to 7.5.5.

This may be posed as a variational problem on M as follows:
Define the constraints

$$\omega_k\left(\frac{dq}{dt}\right) =< X^k, \frac{dq}{dt} >= 0, \quad m < k \leq n \tag{7.5.7}$$

and let

$$Z_t = \sum_{k=m+1}^{n} \lambda_k(t) X^k \tag{7.5.8}$$

where the λ_k are Lagrange multipliers. By the orthonormality of the X^i the optimal control problem then becomes

$$\min_q J(q) = \min_q \int_0^T \left(\frac{1}{2} < \frac{dq}{dt}, \frac{dq}{dt} > + < Z_t, \frac{dq}{dt} >\right) dt \tag{7.5.9}$$

$$< Z_t, \frac{dq}{dt} > = 0\,. \tag{7.5.10}$$

We now briefly derive the necessary conditions for the regular extremals of this variational problem following Milnor [1963] and Crouch and Silva-Leite [1991]. (For interesting recent work on abnormal extremals see for example Bryant and Hu [1993], Montgomery [1992], and Sussmann [1992].)

Firstly we have to define the variations we are going to use:

The tangent space to the space Ω of C^2 curves satisfying the boundary conditions of 7.5.6 is denoted by $T_q\Omega$. It is the space of C^1 vector fields $t \to W_t$ along $q(t)$ satisfying $W_0 = 0 = W_T$. The curve $t \to \frac{DWt}{\partial t}$ in TM is continuous. Exponentiating a vector field in $T_q\Omega$ we obtain a one-parameter variation of q:

$$\alpha : [0, T] \times (-\epsilon, \epsilon) \to M, \tag{7.5.11}$$

$$\alpha_u(t) = \alpha(t, u) = \exp_{q(t)}(uW_t) \tag{7.5.12}$$

where exp is the exponential mapping on M. Note that $\alpha_u(0) = q(0) = q_0$, $\alpha_u(T) = q(T) = q_T$, $\alpha_0(t) = q(t)$, $\frac{\partial \alpha_0(t)}{\partial u} = W_t$, $0 \leq t \leq T$.

The necessary conditions for regular extremals are obtained from

$$\frac{d}{du} J(\alpha_u)|_{u=0} = 0 \tag{7.5.13}$$

where

$$J(\alpha_u) = \int_0^T \left(\frac{1}{2}\left\langle \frac{\partial \alpha_u}{\partial t}, \frac{\partial \alpha_u}{\partial t}\right\rangle + \left\langle Z_t(\alpha_u), \frac{\partial \alpha_u}{\partial t}\right\rangle\right) dt\,. \tag{7.5.14}$$

Now

$$\begin{aligned}\left.\frac{DJ(\alpha_u)}{du}\right|_{u=0} &= \int_0^T \left(\left\langle \frac{dq}{dt}, \frac{DW_t}{\partial t} \right\rangle + \left\langle \nabla_{W_t} Z_t, \frac{dq}{dt} \right\rangle + \left\langle Z_t, \frac{D}{\partial t} W_t \right\rangle \right) dt \\ &= \int_0^T \left(- \left\langle \frac{D}{dt} V_t, W_t \right\rangle - \left\langle \frac{D}{\partial t} Z_t, W_t \right\rangle \right. \\ &\quad \left. - \langle \nabla_{Z_t} V_t, W_t \rangle + \langle [W_t, Z_t], V_t \rangle \right) dt \end{aligned} \tag{7.5.15}$$

where $V_t = \frac{dq}{dt} = \sum_{i=1}^m v_i(t) X^i(q)$. Note that in this computation we use $\nabla_W Z = \nabla_Z W + [W, Z]$ and $Z[< V, W >] =< \nabla_Z V, W > + < V, \nabla_Z W >$.

7.5.2 Necessary Conditions on a Compact Semisimple Lie Group

In this section we show that is the underlying manifold is a compact semisimple Lie group, then the subRiemmanian optimal control problem discussed in the previous section may be reduced to a computation in the Lie algebra.

Now let $M = G$, G a compact semisimple Lie group, with Lie algebra $\mathfrak{g}$, and let $<< \cdot, \cdot >>= -\frac{1}{2}\kappa(\cdot, \cdot)$ where κ is the Killing form on $\mathfrak{g}$.

Let J be a positive definite linear mapping $J : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$<< JX, Y >> = << X, JY >> \tag{7.5.16}$$

$$<< JX, X >> \geq 0 \quad (= 0 \text{ if and only if } X = 0) \tag{7.5.17}$$

Now we can define a right invariant metric on G as follows:

If $X, Y \in \mathfrak{g}$ and R_g is right translation on G by $g \in G$, then $X_g^r = X^r(g) = R_{g^*} X$ and $Y_g^r = Y^r(g) = R_{g^*} Y$ are corresponding right invariant vector fields. Now

$$< X^r(g), Y^r(g) ><< X, JY >> \tag{7.5.18}$$

defines a right-invariant metric on G. Corresponding to the right invariant metric $< \quad , \quad >$ there is a unique Riemannian connection ∇ (see e.g. Nomizu [1956] and earlier sections). ∇ defines a bilinear form on $\mathfrak{g}$:

$$(X, Y) \to \nabla_X Y = \frac{1}{2}\{[X, Y] + J^{-1}[X, JY] + J^{-1}[Y, JX]\}, \quad X, Y \in \mathfrak{g} \tag{7.5.19}$$

and the expression for ∇ on right-invariant vector fields on G is

$$(\nabla_{X^r} Y^r)(g) = (\nabla_X Y)_g^r . \tag{7.5.20}$$

We now show how to reduce the variational problem to one in the Lie algebra:

Choose an orthonormal basis e_i on $\mathfrak{g}$, $<< e_i, Je_j >>= \delta_{ij}$, and extend it to a right invariant orthonormal frame on T_gG, $X^i(g) = R_{g^*}e_i \equiv X^{ir}(g)$.

We consider again the computation of $\frac{d}{du}J(\alpha_u)\mid_{u=0}$. Suppose in $\mathfrak{g}$

$$V_t = \sum_{i=1}^{m} v_i(t)e_i, \quad \dot{V}_t = \sum_{i=1}^{m} \dot{v}_i(t)e_i \tag{7.5.21}$$

and similarly for $W_t = \sum_{i=1}^{n} w_i(t)e_i$ and Z_t. Then at the group level we have for $\frac{dg}{dt} = V_t^r$,

$$\begin{aligned}\frac{DV_t^r}{dt} &= \frac{D}{dt}\left(\sum_{i=1}^{m} v_i(t)X^i\right) \\ &= (\dot{V}_t + \nabla_{V_t}V_t)_g^r = (\dot{V}_t + J^{-1}[V_t, JV_t])_g^r\end{aligned} \tag{7.5.22}$$

by 7.5.19 and 7.5.20, and

$$\frac{DZ_t^r}{dt} = (\dot{Z}_t + \nabla_{V_t}Z_t)_g^r. \tag{7.5.23}$$

Now, by 7.5.18,

$$\begin{aligned}\frac{d}{du}J(\alpha_u)\mid_{u=0} = \int_0^T &(- << \dot{V}_t + J^{-1}[V_t, JV_t] + \dot{Z}_t + \nabla_{V t}Z_t + \nabla_{Z_t}V_t, \quad JW_t >> \\ &+ << [W_t, Z_t], JV_t >>)dt\,.\end{aligned} \tag{7.5.24}$$

Using,

$$<< [X,Y], Z >> + << Y, [X,Z] >>= 0 \tag{7.5.25}$$

the necessary conditions are thus

$$\dot{V}_t + J^{-1}[V_t, JV_t] + \dot{Z}_t + \nabla_{V_t}Z_t + \nabla_{Z_t}V_t + J^{-1}[JV_t, Z_t] = 0\,. \tag{7.5.26}$$

By 7.5.19

$$\nabla_{V_t}Z_t + \nabla_{Z_t}V_t = J^{-1}[V_t, JZ_t] + J^{-1}[Z_t, JV_t]\,.$$

Hence the necessary conditions on $\mathfrak{g}$ are

$$\dot{V}_t + J^{-1}[V_t, JZ_t] + \dot{Z}_t + J^{-1}[V_t, JV_t] = 0 \tag{7.5.27}$$

with the constraint

$$< \frac{dg}{dt}, Z_t >=<< V_t, JZ_t >>= 0\,. \tag{7.5.28}$$

Equations 7.5.27 are identical to the equations (3.7) of Brockett [1973] in the case $J = I$. We can see this as follows:

Write the system 7.5.5 as

$$\frac{dg}{dt} = \sum_{i=1}^{m} u_i B^i g, \quad m < n, \tag{7.5.29}$$

where $B_i \in \mathfrak{g}$ and the $X^i(g) = B^i g$ are thus right invariant vector fields on G. Then $V_t = \sum_{i=1}^{m} u_i(t) B^i$. Now set $L_t = V_t + Z_t$. Then equation 7.5.27 becomes

$$\dot{L}_t = J^{-1}[JL_t, V_t] \tag{7.5.30}$$

or

$$\dot{L}_t = \sum_{i=1}^{m} u_i J^{-1}[JL_t, B^i]\,. \tag{7.5.31}$$

Setting $J = I$ we recover precisely Brockett's equations (in the case of zero drift). Note also that in the case $J = I$ our equations 7.5.27 assume the symmetric form

$$\dot{V}_t + [V_t, Z_t] + \dot{Z}_t = 0\,. \tag{7.5.32}$$

Brockett obtained his equations by applying the maximum principle to Lie groups, while we have taken a direct variational approach. The approaches are of course equivalent (see also the next section).

7.5.3 The Case of Symmetric Space Structure

Suppose now that G/K is a Riemannian symmetric space, G as above, K a closed subgroup of G with Lie algebra $\mathfrak{k}$. Then $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}$ with $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, $[\mathfrak{p}, \mathfrak{k}] \subset \mathfrak{p}$, $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ and $<< \mathfrak{k}, \mathfrak{p} = 0 >>$. We now want to consider the necessary conditions 7.5.27 in this case. We shall see that they simplify in an intriguing fashion, giving us a singular case of the so-called generalized rigid body equations.

Assume that $e_1, \ldots, e_m$ is a basis for $\mathfrak{p}$ and $e_{m+1}, \ldots e_n$ is a basis for $\mathfrak{k}$. Suppose that $J : \mathfrak{p} \to \mathfrak{p}$ and $J : \mathfrak{k} \to \mathfrak{k}$. Then $<< V_t, JZ_t >>= 0$ for $Z_t = \sum_{i=m+1}^{n} \lambda_i(t) e_i \in \mathfrak{k}$ and $V_t = \sum_{k=1}^{m} v_i(t) e_i \in \mathfrak{p}$.

Since $\dot{V}_t + J^{-1}[V_t, JZ_t] \in \mathfrak{p}$ and $\dot{Z}_t + J^{-1}[V_t, JV_t] \in \mathfrak{k}$, the necessary conditions 7.5.27 become

$$\begin{aligned} \dot{V}_t &= J^{-1}[JZ_t, V_t] \\ \dot{Z}_t &= J^{-1}[JV_t, V_t] \end{aligned} \tag{7.5.33}$$

or, if we define $P_t = JV_t$ and $Q_t = JZ_t$

$$\begin{aligned} \dot{P}_t &= [Q_t, J^{-1}P_t] \\ \dot{Q}_t &= [P_t, J^{-1}P_t] \,. \end{aligned} \tag{7.5.34}$$

We will now show that equations 7.5.34 are Hamiltonian with respect to the Lie-Poisson structure on $\mathfrak{g}$.

Recall that for F, H functions on $\mathfrak{g}$, their $(-)$ Lie-Poisson bracket is given by

$$\{F, H\}(X) = - << X, [\nabla F(X), \nabla H(X)] >>, \quad X \in \mathfrak{g}, \tag{7.5.35}$$

where $dF(X) \cdot Y =<< \nabla F(X), Y >>$.

For $H(X)$ a given Hamiltonian, we thus have the Lie-Poisson equations $\dot{F}(X) = \{F, H\}(X)$. Letting $F(X) =<< A, X >>$, $A \in \mathfrak{g}$, we obtain

$$\begin{aligned} << A, \dot{X} >> &= - << X, [A, \nabla H(X)] >> \\ &= << A, [X, \nabla H(X)] >> \end{aligned} \tag{7.5.36}$$

and hence

$$\dot{X} = [X, \nabla H(X)] \,. \tag{7.5.37}$$

For $H(M) = \frac{1}{2} << M, J^{-1}M >>$, $M \in \mathfrak{g}$ and J as in the previous subsection, we obtain the generalized rigid body equations

$$\dot{M} = [M, J^{-1}M] \,. \tag{7.5.38}$$

Now for $X = P + Q \in \mathfrak{p} \oplus \mathfrak{k}$, let $H(X) = H(P) = \frac{1}{2} << P, J^{-1}P >>$, $P \in \mathfrak{p}$. Then $\nabla H(X) = J^{-1}P \in \mathfrak{p}$ and equations 7.5.37 become

$$(Q + P)^{\cdot} = [Q + P, J^{-1}P] \tag{7.5.39}$$

or

$$\begin{aligned} \dot{P} &= [Q, J^{-1}P] \\ \dot{Q} &= [P, J^{-1}P] \end{aligned} \tag{7.5.40}$$

precisely equations 7.5.34.

Thus equations 7.5.34) are Lie-Poisson with respect to the "singular" Hamiltonian $H(P)$. Summarizing then, we have

Theorem 7.5.1. *The optimal trajectories for the singular optimal control problem 7.5.5, 7.5.6 on a Riemannian symmetric space are given by equations 7.5.34. These equations are Lie Poisson with respect to a singular rigid body Hamiltonian on* $\mathfrak{g}$.

We see therefore that we can obtain the singular optimal trajectories by letting $J\mid_{\mathfrak{k}}\rightarrow\infty$ in the full rigid body Hamiltonian $H(X)=\frac{1}{2}<<X,J^{-1}X>>$, thus obtaining the singular Hamiltonian $H(P)=\frac{1}{2}<<P,J^{-1}P>>$.

This observation also enables us to obtain the singular rigid body equations directly by a limiting process from the full rigid body equations. The key is the correct choice of angular velocity and momentum variables corresponding to the Lie algebra decomposition $\mathfrak{g}=\mathfrak{p}\oplus\mathfrak{k}$.

In the notation of equation 7.5.34 we write an arbitrary element of $\mathfrak{g}$ as $M=JV+Q$, $JV\in\mathfrak{p}$, $Q\in\mathfrak{k}$. Then the generalized rigid body equations 7.5.38 become

$$\begin{aligned} J\dot{V}_t &= [Q_t,V_t]+[JV_t,J^{-1}Q_t] \\ \dot{Q}_t &= [Q_t,J^{-1}Q_t]+[JV_t,V_t]\,. \end{aligned} \tag{7.5.41}$$

Letting $J\mid_{\mathfrak{k}}\rightarrow\infty$ we obtain

$$\begin{aligned} J\dot{V}_t &= [Q_t,V_t] \\ \dot{Q}_t &= [JV_t,V_t]\,. \end{aligned} \tag{7.5.42}$$

We note that this is a mixture between the Lagrangian and Hamiltonian pictures. While the variables in $\mathfrak{k}$ are momenta (and should really be viewed as lying in $\mathfrak{k}^*$) the variables in $\mathfrak{p}$ are velocities. These variables are not only the natural ones in which to take the limit in the full rigid body equations, but are natural from the point of view of the maximum principle, for the variables Q correspond to the constraints and therefore are naturally viewed as costates.

Note also that this reduction may also be viewed as a generalized Routhian reduction as in Marsden and Scheurle [1992]. The group variables are the only ones which are Legendre transformed. This will be discussed further in a future paper.

As mentioned in the previous section, the necessary conditions above may also be derived directly from the maximum principle developed for Lie groups (see e.g. Brockett [1973], Jurdjevic [1991]). The Hamiltonian in the maximum principle of the system on a Lie group as discussed above is precisely $\frac{1}{2}<P_t,J^{-1}P_t>$. This is just the sum of the Hamiltonians corresponding to each of the vector fields X_i.

We note also the following interpretation of the fact that $J\mid_{\mathfrak{k}}\rightarrow\infty$ (due to R. Brockett). Write $M\in\mathfrak{g}$ now as $M=JZ+P$, $Z\in\mathfrak{k}$, $P\in\mathfrak{p}$. Then the Hamiltonian (in the maximum principle) becomes

$$\begin{aligned} H(M)=<M,J^{-1}M> &= <JZ+P,J^{-1}(JZ+P)> \\ &= <JZ,Z>+<P,J^{-1}P> \end{aligned} \tag{7.5.43}$$

Letting $J\mid_{\mathfrak{k}}\rightarrow\infty$ we see that the cost becomes infinite unless $Z=0$, i.e. the constraints are satisfied.

Example: We now consider a simple but non-trivial example: the symmetric space $SO(3)/SO(2)$.

In this case $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ becomes $so(3) = so(2) \oplus \mathbb{R}^2$. Reflecting this decomposition we may represent matrices in $so(3)$ as

$$\begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 0 & \end{bmatrix} \tag{7.5.44}$$

with the lower 2×2 block in $so(2)$.

This example illustrates the importance of writing the optimal equations in the natural variables $M = JV + Q$ in order to understand the limiting process in equations 7.5.41 and 7.5.42.

We write

$$\begin{bmatrix} 0 & -J_3\omega_3 & J_2\omega_2 \\ J_3\omega_3 & 0 & -m_1 \\ -J_2\omega_2 & m_1 & 0 \end{bmatrix} \tag{7.5.45}$$

Here $Q_t \in so(2)$ has "momentum" variable m_1.

Then equations 7.5.42 or

$$(JV_t + Q_t)^{\cdot} = [JV_t + Q_t, V_t] \tag{7.5.46}$$

become for $\mathfrak{g} = so(3)$

$$\begin{aligned} \dot{m}_1 &= (J_2 - J_3)\omega_2\omega_3 \\ J_2\dot{\omega}_2 &= -m_1\omega_3 \\ J_3\dot{\omega}_3 &= m_1\omega_2 \,. \end{aligned} \tag{7.5.47}$$

The full rigid body equations in these variables are

$$(JV_t + Q_t)^{\cdot} = [JV_t + Q_t, V_t + J^{-1}Q_t] \tag{7.5.48}$$

which are for $\mathfrak{g} = so(3)$

$$\begin{aligned} \dot{m}_1 &= (J_2 - J_3)\omega_2\omega_3 \\ J_2\dot{\omega_2} &= \left(\frac{J_3}{J_1} - 1\right)\omega_3 m_1 \\ J_3\dot{\omega}_3 &= \left(1 - \frac{J_2}{J_1}\right) m_1\omega_2 \end{aligned} \tag{7.5.49}$$

which clearly limits to 7.5.47 as $J_1 \to \infty$.

Note that if we write the rigid body equations in the usual form

$$J(V_t + Z_t)^{\cdot} = [J(V_t + Z_t), V_t + Z_t] \,, \tag{7.5.50}$$

yielding for $\mathfrak{g} = so(3)$

$$\begin{aligned} J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 \\ J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_1\omega_3 \\ J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_2\omega_1\,, \end{aligned} \tag{7.5.51}$$

we see that the limiting process $J_1 \to \infty$ makes no sense. The same is true for the rigid body in the momentum representation.

We remark that this set of equations, despite its singular nature is still integrable, for we still have two conserved quantities, the Hamiltonian $H(\omega) = J_2\omega_2^2 + J_3\omega_3^2 (= \frac{1}{2} < P, J^{-1}P >)$ and the Casimir $C(\omega) = m_1^2 + J_2^2\omega_2^2 + J_3^2\omega_2^2$. (Recall that a Casimir function for a Poisson structure is a function that commutes with every other function under the Poisson bracket.)

It is interesting to consider the case $J = I$. Equations 7.5.34 then become

$$\begin{aligned} \dot{P}_t &= [Q_t, P_t] \\ \dot{Q}_t &= 0\,. \end{aligned} \tag{7.5.52}$$

Hence $Q_t = Q$ is constant.

Similarly, considering 7.5.33, we obtain

$$\dot{V}_t = [Z_t, V_t], \quad Z_t = Z \quad \text{constant}\,. \tag{7.5.53}$$

This is of course solvable: $V_t = Ad_{e^{Z_t}} V_0$ and $u_i(t) =<< e_i, Ad_{e^{Z_t}} V_0 >>$ (see the equivalent expression (3.5) of Brockett [1973]). Consider again the case $SO(3)/SO(2)$. Since $V_t \in \mathbb{R}^2$ and $Z_t \in so(2)$ we may set

$$Z = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\phi \\ 0 & \phi & 0 \end{bmatrix} \tag{7.5.54}$$

where ϕ is fixed. Then

$$e^{Zt} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi t & -\sin\phi t \\ 0 & \sin\phi t & \cos\phi t \end{bmatrix}\,. \tag{7.5.55}$$

Hence the optimal evolution of V_t (or equivalently the optimal controls) is given by rotation. This recovers precisely the result of Baillieul who indeed analyzed the case $J = I$ in dimension 3. (See Baillieul [1975], page III-5.)

7.5.4 Optimal control and a particle in a magnetic field

We begin by considering the connection between optimal control of the Heisenberg system and the motion of a particle in a magnetic field. This will be seen be a special case of the more general motion of a particle in a magnetic or Yang Mills field. The control analysis of the Heisenberg model goes back to Brockett [1981] and Bailleul [1975], while a modern treatment of the relationship with a particle in a magnetic field may be found in Montgomery [1993] for example. A nice treatment of the pure mechanical aspects of a particle in a magnetic field may be found in Marsden and Ratiu [1994].

Consider firstly the optimal control of the Heisenberg equations. Recall that the Heisenberg control system is given by:

$$\begin{aligned} \dot{x} &= u \\ \dot{y} &= v \\ \dot{z} &= uy - xv = \dot{x}y - \dot{y}x \,. \end{aligned} \tag{7.5.56}$$

The last equation may be viewed as a nonholonomic constraint on the state space of the system or as a connection on the bundle with base space $\mathbb{R}^2$ coordinatized by x and y and fiber z coordinatized by z.

We now consider the optimal control problem:

$$\min \int (u^2 + v^2) dt \tag{7.5.57}$$

subject to the equations 7.5.56.

This may be reformulated as the constrained optimization problem

$$\min \int (\dot{x}^2 + \dot{y}^2 - \lambda(\dot{z} - \dot{x}y + \dot{y}z)) dt \tag{7.5.58}$$

subject to

$$\dot{z} = \dot{x}y - \dot{y}x \,. \tag{7.5.59}$$

The Euler-Lagrange equation for z is

$$\frac{d}{dt}(\lambda) = 0 \tag{7.5.60}$$

and hence λ is a constant.

Thus the Euler-Lagrange equations for x and y become

$$\begin{aligned} \ddot{x} + \lambda y &= 0 \\ \ddot{y} - \lambda x &= 0 \,. \end{aligned} \tag{7.5.61}$$

Now let $\mathbf{v}$ be the vector $(\dot{x}, \dot{y}, 0)$, let $\tilde{\lambda}$ be the vector $(0, 0, \lambda)$ and let Ω be the matrix

$$\begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then the equations for x and y may be rewritten as

$$\frac{d\mathbf{v}}{dt} + \Omega\mathbf{v} = 0 \tag{7.5.62}$$

or

$$\frac{d\mathbf{v}}{dt} + \mathbf{v} \times \tilde{\lambda} = 0\,. \tag{7.5.63}$$

. (Note also that $\ddot{z} = \ddot{x}y - \ddot{y}x = -\lambda(\dot{x}x + \dot{y}y)$ thus enabling us to solve for $\dot{z}$ in terms of x and y.)

That these equations are a particular case of planar charged particle motion in a magnetic field may be seen by considerting the slightly more general problem below.

We now consider again the optimal control problem:

$$\min \int (u^2 + v^2)dt \tag{7.5.64}$$

but now subject to the equations.

$$\begin{aligned}\dot{x} &= u\\ \dot{y} &= v\\ \dot{z} &= A_1 u + A_2 v\,,\end{aligned} \tag{7.5.65}$$

where A_1 and A_2 are smooth functions of x and y.

Then the Euler Lagrange equations for z again yields $\lambda =$ const while the equations for x and y are

$$\begin{aligned}\ddot{x} &= \frac{\lambda}{2}(A_{2x} - A_{1y})\dot{y}\\ \ddot{y} &= \frac{\lambda}{2}(A_{1y} - A_{2x})\dot{x}\,.\end{aligned} \tag{7.5.66}$$

Now let $\mathbf{A}$ be the vector $(A_1, A_2, 0)$. Then

$$(\nabla \times \mathbf{A})_z = A_{2x} - A_{1y} \equiv B_z.$$

Hence the Euler Lagrange equations may be rewritten

$$\begin{aligned}\ddot{x} &= \frac{\lambda}{2}B_z\dot{y}\\ \ddot{y} &= -\frac{\lambda}{2}B_z\dot{x}\,,\end{aligned} \tag{7.5.67}$$

or, if $\mathbf{v} = (\dot{x}, \dot{y}, 0)$ and $\mathbf{B} = (0, 0, B_z)$, as

$$\frac{d\mathbf{v}}{dt} + \mathbf{v} \times \frac{\lambda}{2}\mathbf{B} = 0\,. \tag{7.5.68}$$

This is indeed the motion of a planar charged particle in a magnetic field (see e.g. Marden and Ratiu [1994]), where the Lagrange multiplier is identified with a multiple of the charge.

This problem may also be solved via the maximum prinicple:

Form the Hamiltonian

$$H = p_1 u + p_2 v + p_3(A_1 u + A_2 v) - \frac{u^2}{2} - \frac{v^2}{2} \,. \tag{7.5.69}$$

The optimality conditions

$$\frac{\partial H}{\partial u} = 0 = \frac{\partial H}{\partial v}$$

yield

$$\begin{aligned} u &= p_1 + p_3 A_1 \\ v &= p_2 + p_3 A_2 \,. \end{aligned} \tag{7.5.70}$$

Hence the optimal Hamiltonian becomes

$$H = \frac{1}{2}\{(p_1 + A_1 p_3)^2 + (p_2 + A_2 p_3)^2\} \,. \tag{7.5.71}$$

This is the Hamiltonian for a particle in a magnetic field. Note that to obtain the sign in, say, Marsden and Ratiu [1994] we replace A_i by $-A_i$ in the $\dot{z}$ equation. Note also that p_3 is a cyclic variable and hence is constant in time. This may be chose to have the value e/c, the charge over the speed of light, as in equations for the particle.

7.5.5 Rigid Extremals

A particularly interesting phenomenon that occurs in sub-Riemannian optimal control problems is the existence of rigid extremals — i.e. isolated extremals which admit no allowable variations. In such a situation, we can obtain an optimal solution to the optimal control problem which does not satisfy the optimal control equations. We can rephrase this, as does Montgomery [ ], by saying that one has a (subRiemannian) geodesic which does not satisfy the geodesic equations.

We will consider here the example of Montgomery [ ]. The problem however has a rather interesting history — for some details on this see the paper of Montgomery and, for example, Strichartz [1983], [1989].

We follow here the treatment in Montgomery [ ], casting things in the language of optimal control theory.

Since we will be considering an optimal trajectory with cylindrical geometry, it is best to phrase the problem in cylindrical coordinates in $\mathbb{R}^3$: (r, θ, z).

We consider then the optimal control problem

$$\min_{u,v} \int (u^2 + r^2 v^2) dt$$

subject to

$$\begin{aligned} \dot{r} &= u \\ \dot{\theta} &= v \\ \dot{z} &= -A(r)v \end{aligned} \tag{7.5.72}$$

where $A(r)$ is a smooth function of r with a single nondegenerate maximum at $r = 1$. Thus we require $\frac{dA}{dr}\big|_{r=1} = 0$ and $\frac{d^2A}{dr^2}\big|_{r=1} < 0$. We can take, for example,

$$A = \frac{1}{2} r^2 - \frac{1}{4} r^4.$$

The control vector fields are

$$\begin{aligned} X_1 &= \frac{\partial}{\partial r} \\ X_2 &= \frac{\partial}{\partial \theta} - A(r) \frac{\partial}{\partial z}. \end{aligned} \tag{7.5.73}$$

The system is controllable since

$$[X_1, X_2] = \frac{-dA}{dr} \frac{\partial}{\partial z} \tag{7.5.74}$$

so X_1, X_2 and $[X_1, X_2]$ span $\mathbb{R}^3$ everywhere except at $r = 1$. But for $r = 1$ $[X_1, [X_1, X_2]] = \frac{-d^2A}{dr^2} \frac{\partial}{\partial z} \neq 0$ so the control vector fields still span $\mathbb{R}^3$.

The examples of Montgomery are the helices with pitch $A(1)$. One can take the curves $(r, \theta, z) = (1, \theta, -A(1)\theta)$. These curves clearly lie in the constraint distribution. Montgomery shows that this helix is minimizing but does not satisfy the optimal control (or subRiemannian geodesic) equations. We will sketch these arguments here.

Consider firstly the geodesic equations. They are given in Cartesian form by equations (5.142) where

$$B_z = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}.$$

To compute B_z here observe that we need to replace $A(r)d\theta$ by $A_1 dx + A_2 dy$. Now

$$A_1 dx + A_2 dy = A(r) d\theta \tag{7.5.75}$$

implies

$$\left(\frac{-\partial A_1}{\partial y} + \frac{\partial A_2}{\partial x} \right) dx \wedge dy = \frac{dA}{dr} dr \wedge d\theta \tag{7.5.76}$$

But $dx \wedge dy = rdr \wedge d\theta$. Hence

$$\frac{1}{r}\frac{dA}{dr} = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} = B_z$$

For our given curve $r = 1$ and hence B_z is zero. But $\ddot{x}$ and $\ddot{y}$ are clearly not zero. Hence this curve cannot satisfy the geodesic equations.

We now show that the helix is an isolated point in the space of all piecewise C^1 curves in the constraint distributins with fixed endpoints. Since it is isolated it is auomatically a local minimum. However, it is not isolated in the C^0 or H^1 topologies. Montgomery however shows that it is still a local minimum, at least for short enough arcs — see Montgomery [ ].

Now consider a curve γ in th constraint distribution connecting the points $(x_0, y_0, 0)$ to (x_0, y_0, z_1). The projected curve onto the $x - y$ plane is thus closed.

Since $dz = -Ad\theta$ along the curve we have, using Stoke's theorem,

$$\begin{aligned} z_1 &= -\int A d\theta \\ &= -\int\int_\Delta \frac{dA}{dr} dr d\theta \\ &= -\int\int_\Delta B(r) r dr d\theta \end{aligned}$$

where Δ is the region in the plane enclosed by the projected curve. Following our earlier magnetic analogy this quantity can be viewed as the flux through the region by the plane enclosed by the curve.

Now recall that in our case

$$B = \frac{1}{r}\frac{dA}{dr} = 1 - r^2 \tag{7.5.77}$$

and thus is positive in the interior of the projected unit disc and negative on the exterior. Hence, if we perturb the helix so as to push part of the projected curve into the interior of the disc we subtract flux since B is positive in the interior. On the other hand if we push the projected curve to the exterior of the disc we add negative flux. In either case z_1 decreases, violating the fixed endpoint conditions. Hence there are no allowable piecewise smooth variations of the helix and it is indeed rigid.

7.6 Dynamic Optimal Control

7.6.1 The Falling Cat Theorem

Falling cat problem. This problem is an abstraction of the problem of how a falling cat should optimally (in some sense) move its body parts so that it achieves a 180° reorientation during its fall.

We begin with a Riemannian manifold Q (the configuration space of the problem) with a free and proper isometric action of a Lie group G on Q (the group $SO(3)$ for the falling cat). Let $\mathcal{A}$ denote the mechanical connection; that is, it is the principal connection whose horizontal space is the metric orthogonal to the group orbits. The quotient space $Q/G = X$, the shape space, inherits a Riemannian metric from that on Q. Given a curve $c(t)$ in Q, we shall denote the corresponding curve in the base space X by $r(t)$.

The optimal control problem under consideration is as follows:

Isoholonomic problem (falling cat problem). *Fixing two points q_1, $q_2 \in Q$, among all curves $q(t) \in Q$, $0 \leq t \leq 1$ such that $q(0) = q_0, q(1) = q_1$ and $\dot{q}(t) \in \mathrm{hor}_{q(t)}$ (horizontal with respect to the mechanical connection $\mathcal{A}$), find the curve or curves $q(t)$ such that the energy of the base space curve, namely,*

$$\frac{1}{2}\int_0^1 \|\dot{r}\|^2 dt,$$

is minimized.

Theorem 7.6.1. (Montgomery [1984, 1990, 1991a]). *If $q(t)$ is a (regular) optimal trajectory for the isoholonomic problem, then there exists a curve $\lambda(t) \in \mathfrak{g}^*$ such that the reduced curve $r(t)$ in $X = Q/G$ together with $\lambda(t)$ satisfies **Wong's equations**:*

$$\begin{aligned}
\dot{p}_\alpha &= -\lambda_a \mathcal{B}^a_{\alpha\beta}\dot{r}^\beta - \frac{1}{2}\frac{\partial g^{\beta\gamma}}{\partial r^\alpha}p_\beta p_\gamma \\
\dot{\lambda}_b &= -\lambda_a C^a_{db}\mathcal{A}^d_\alpha \dot{r}^\alpha
\end{aligned}$$

where $g_{\alpha\beta}$ is the local representation of the metric on the base space X; that is

$$\frac{1}{2}\|\dot{r}\|^2 = \frac{1}{2}g_{\alpha\beta}\dot{r}^\alpha\dot{r}^\beta,$$

$g^{\beta\gamma}$ is the inverse of the matrix $g_{\alpha\beta}$, p_α is defined by

$$p_\alpha = \frac{\partial l}{\partial \dot{r}^\alpha} = g_{\alpha\beta}\dot{r}^\beta,$$

and where we write the components of $\mathcal{A}$ as $\mathcal{A}^b_\alpha$ and similarly for its curvature $\mathcal{B}$.

Proof. As with the Heisenberg system, by general principles in the calculus of variations, given an optimal solution $q(t)$, there is a Lagrange multiplier $\lambda(t)$ such that the new action function defined on the space of curves with fixed endpoints by

$$\mathfrak{S}[q(\,\cdot\,)] = \int_0^1 \left[\frac{1}{2}\|\dot{r}(t)\|^2 + \langle\lambda(t), \mathcal{A}\dot{q}(t)\rangle\right] dt$$

has a critical point at this curve. Using the integrand as a Lagrangian, identifying $\Omega = \mathcal{A}\dot{q}$ and applying the reduced Euler-Lagrange equations from Lecture 1 to the reduced Lagrangian

$$l(r, \dot{r}, \Omega) = \frac{1}{2}\|\dot{r}\|^2 + \langle \lambda, \Omega \rangle$$

then gives Wong's equations by the following simple calculations:

$$\frac{\partial l}{\partial \dot{r}^\alpha} = g_{\alpha\beta}\dot{r}^\beta; \quad \frac{\partial l}{\partial r^\alpha} = \frac{1}{2}\frac{\partial g^{\beta\gamma}}{\partial r^\alpha}\dot{r}^\beta \dot{r}^\gamma; \quad \frac{\partial l}{\partial \Omega^a} = \lambda_a.$$

The constraints are $\Omega = 0$ and so the reduced Euler-Lagrange equations become

$$\begin{aligned} \frac{d}{dt}\frac{\partial l}{\partial \dot{r}^\alpha} - \frac{\partial l}{\partial r^\alpha} &= -\lambda_a(\mathcal{B}^a_{\alpha\beta}\dot{r}^\beta) \\ \frac{d}{dt}\lambda_b &= -\lambda_a(\mathcal{E}^a_{\alpha b}\dot{r}^\alpha) = -\lambda_a C^a_{db}\mathcal{A}^d_\alpha \dot{r}^\alpha. \end{aligned}$$

But

$$\begin{aligned} \frac{d}{dt}\frac{\partial l}{\partial \dot{r}^\alpha} - \frac{\partial l}{\partial r^\alpha} &= \dot{p}_\alpha - \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial r^\alpha}\dot{r}^\beta \dot{r}^\gamma \\ &= \dot{p}_\alpha + \frac{1}{2}\frac{\partial g^{\kappa\sigma}}{\partial r^\alpha} g_{\kappa\beta} g_{\sigma\gamma}\dot{r}^\beta \dot{r}^\gamma \\ &= \dot{p}_\alpha + \frac{1}{2}\frac{\partial g^{\beta\gamma}}{\partial r^\alpha} p_\beta p_\gamma, \end{aligned}$$

and so we have the desired equations. ■

Remark. There is a rich literature on Wong's equations and it was an important ingredient in the development of reduction theory. Some references are Sternberg [1977], Guillemin and Sternberg [1978], Weinstein [1978], Montgomery, Marsden and Ratiu [1984], Montgomery [1984], Koon and Marsden [1997] and Cendra, Holm, Marsden, and Ratiu [1998].

Nonholonomic optimal control. Using a synthesis of the techniques used above for the Heisenberg system and the falling cat problem, Koon and Marsden [1997] generalized these problems to the nonholonomic case. In addition, these methods allow one to treat the falling cat problem even in the case that the angular momentum is not zero.

In this process the momentum equation plays the role of the constraint. It is inserted as a first order differential constraint on the nonholonomic momentum.

7.6.2 Optimal control of the rolling ball

In this section we consider optimal control of the rolling ball (see also Chapter 4). We suppose here that $J = \alpha I_3$, I_3 the 3×3 identity matrix, in which case the equations (27) become, after suitable state feedback, the following control system on $\mathbb{R}^4 \times SO(3)$:

$$\begin{aligned} \ddot{x} &= \widetilde{u}_1 \\ \ddot{y} &= \widetilde{u}_2 \\ \dot{P} &= PS(-\dot{x}\mathbf{e}_2 + \dot{y}\mathbf{e}_1 + \overline{c}\mathbf{e}_3); \overline{c} = c/\alpha\,. \end{aligned} \tag{7.6.1}$$

(28) is evidently controllable (it is accessible, as in the systems analyzed in Bloch et.al. [1992] and the uncontrolled trajectories are periodic).

The obvious minimum energy control problem is

$$\min_u \int_0^T \frac{1}{2}(\widetilde{u}_1^2 + \widetilde{u}_2^2)dt \tag{7.6.2}$$

subject to (28).

This may be viewed as the following constrained variational problem on $SO(3)$:

Set $q = P$ and

$$\begin{aligned} \frac{D\dot{q}}{dt} &= \dot{y}X_1 - \dot{x}X_1 + \overline{c}X_3\,, \\ X_i(q) &= PS(e_i) \end{aligned} \tag{7.6.3}$$

Then the variational problem may be posed as:

$$\min_q \int_0^T \left\langle \frac{D^2q}{dt^2}, \frac{D^2q}{dt^2} \right\rangle dt$$

subject to (30).

In contrast with the minimum energy holonomic control problems of Section 1, this nonholonomic problem introduces constraints into the variational problem. However, this now falls into the class of problems analyzed by Crouch and Leite [1991]. Defining the vector of Lagrange multipliers by $\Lambda = \sum_i \lambda_i X_i$, the resulting extremals satisfy

$$\begin{aligned} x^{(4)} &= \overline{c}y^{(3)} + \lambda_3\dot{y} - \overline{c}\lambda_1 \\ y^{(4)} &= -\overline{c}x^{(3)} - \lambda_3\dot{x} - \overline{c}\lambda_2 \\ \dot{\lambda}_3 &= -\dot{y}x^{(3)} + \dot{x}y^{(3)} - \lambda_2\dot{y} - \lambda_1\dot{x} \\ \dot{\lambda}_1 &= 0 \\ \dot{\lambda}_2 &= 0\,. \end{aligned} \tag{7.6.4}$$

8
Energy Based Methods for Stability

A key notion in dynamics and control is that of stability of a point or set. In this context one normally thinks either of aymptotic or nonlinear stability, the former meaning essentially that all nearby trajectories tend to the set and the latter meaning that all trajectories starting nearby the set remain near the set. Most often one considers stability of an equilibrium point of a given system. Also of interest are relative equilibria – equilibria modulo the action of a symmetry group. Example of such equilibria are the so-called stationary motions of rigid body – rotations about one of the principal axes. In the theory of controlled systems, the problem is often that of achieving nonlinear or asymptotic stability of an initially unstable equilibrium point or set.

8.1 The Energy Momentum Method

In this section we discuss the so called Energy Casimir method and its applications to the analysis of stability and stabilization problems. Since the application to pure stability problems is covered very well in Marsden and Ratiu [1994] we will shall concentrate mainly on the stabilization angle. The key idea is the use of a combination of energy and another conserved quantity, such as momentum, to provide a Lyapunov function for the system.

8.1.1 The Energy-Momentum Method for Holonomic Systems

As mentioned above, we use here an approach to stability which generalizes the energy-momentum method for Hamiltonian systems. Of course the energy-momentum method has a long and distinguished history going back to Routh, Riemann, Poincaré, Lyapunov, Arnold, Smale and many others. The main new feature provided in the more recent work of Simo, Lewis and Marsden [1991] (see Marsden [1992] for an exposition) is to obtain the powerful block diagonalization structure of the second variation of the augmented Hamiltonian as well as the normal form for the symplectic structure. This formulation also allowed for the proof of a *converse* of the energy-momentum method in the context of *dissipation induced instabilities* due to Bloch, Krishnaprasad, Marsden and Ratiu [1994, 1996].

Recall that the key idea for analyzing the stability of relative equilibria in the holonomic setting is to use the energy plus a function of other conserved quantities such as the momentum as a Lyapunov function. In effect, one is analyzing stability subject to the systems lying on a level surface of the momentum.

In a body frame and in the special case of Lie-Poisson systems, the momentum often can be written in terms of a Casimir, a function that commutes with every function under the Poisson bracket, and the method is sometimes called the energy-Casimir method.

While the energy is conserved, it does not provide sufficient information on stability since its second variation will be only semidefinite at a stable equilibrium in general. The algorithm for analyzing stability is thus as follows:

1. write the equations of motion in Hamiltonian form and identify the critical point of interest,

2. identify other conserved quantities such as momentum,

3. choose a function H_C such that the energy plus the function of other conserved quantities has a critical point at the chosen equilibrium and

4. show that H_C is definite at the given equilibrium. This proves nonlinear stability in the sense of Lyapunov.

Of course in special circumstances one has to interpret stability modulo the symmetry or a similar space in order to obtain stability. A good example is the study of two dimensional ABC flows, as in Chern and Marsden [1990].

In the nonholonomic case, while energy is conserved, momentum generally is not. As indicated in the discussion of the three principle cases above, in some cases however the momentum equation is integrable, leading to invariant surfaces which make possible an energy-momentum analysis similar to that of the Hamiltonian case. When the momentum equation is not integrable, one can get asymptotic stability in certain directions and the stability analysis is rather different from the Hamiltonian case. Nonetheless, to show stability we will make use of the conserved energy and the dynamic momentum equation.

8.2 The Energy-Momentum Method for the Stability of Nonholonomic Systems

Here we analyze the stability of relative equilibria for nonholonomic mechanical systems with symmetry using an energy-momentum analysis for nonholonomic systems that is analogous to that for holonomic systems given in Simo, Lewis, and Marsden [1991]. This section is based on the paper Zenkov, Bloch and Marsden [1997] and follows the the spirit of the paper by Bloch, Krishnaprasad, Marsden and Murray [1996], hereafter referred to as [BKMM]. We will illustrate our energy-momentum stability analysis with a low dimensional model example, and then with several mechanical examples of interest including the falling disk, the roller racer, and the rattleback top.

As discussed above symmetries do not always lead to conservation laws as in the classical Noether theorem, but rather to an interesting *momentum equation*.

The momentum equation has the structure of a parallel transport equation for the momentum corrected by additional terms. This parallel transport occurs in a certain vector bundle over shape space. In some instances such as the Routh problem of a sphere rolling inside a surface of revolution (see Zenkov [1995]) this equation is *pure transport*, and in fact is integrable (the curvature of the transporting connection is zero). This leads to nonexplicit conservation laws.

In other important instances, the momentum equation is *partially integrable* in a sense that we shall make precise. Our goal is to make use of, as far as possible, the energy momentum approach to stability for Hamiltonian systems. This method goes back to fundamental work of Routh (and many others in this era), and in more modern works, that of Arnold [1966] and Smale [1970], and Simo, Lewis and Marsden [1991] (see for example, Marsden [1992] for an exposition and additional references). Because of the nature of the momentum equation, the analysis we present is rather different in several important respects. In particular, our energy-momentum analysis varies according to the structure of the momentum equation and, correspondingly, we divide our analysis into several parts.

We consider a number of interesting examples including the rolling disc discussed ealier. Other examples are as follows:

A Mathematical Example

We now consider an instructive, but (so far as we know) nonphysical example. Unlike the rolling disk, it has asymptotically stable relative equilibria, and is a simple example that exhibits the richness of stability in nonholonomic systems. Our general theorems presented later are well illustrated by this example and the reader may find it helpful to return to it again later.

Consider a Lagrangian on $T\mathbb{R}^3$ of the form

$$\begin{aligned} L(r^1, r^2, s, \dot{r}^1, \dot{r}^2, \dot{s}) = \frac{1}{2}\big\{&(1-[a(r^1)]^2)(\dot{r}^1)^2 - 2a(r^1)b(r^1)\dot{r}^1\dot{r}^2 \\ &+ (1-[b(r^1)]^2)(\dot{r}^2)^2 + \dot{s}^2\big\} - V(r^1), \end{aligned} \tag{8.2.1}$$

where a, b, and V are given real valued functions of a single variable. We consider the nonholonomic constraint

$$\dot{s} = a(r^1)\dot{r}^1 + b(r^1)\dot{r}^2. \tag{8.2.2}$$

Using the definitions, straightforward computations show that $B_{12} = \partial_{r^1} b = -B_{12}$. The constrained Lagrangian is $L_c = \frac{1}{2}\left\{(\dot{r}^1)^2 + (\dot{r}^2)^2)\right\} - V(r^1)$ and the equations of motion, namely, $(d/dt)(\partial_{\dot{r}^\alpha} L_c) - \partial_{r^\alpha} L_c = -\dot{s}B_{\alpha\beta}\dot{r}^\beta$ become

$$\frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^1} - \frac{\partial L_c}{\partial r^1} = -\dot{s}B_{12}\dot{r}^2, \quad \frac{d}{dt}\frac{\partial L_c}{\partial \dot{r}^2} = \dot{s}B_{12}\dot{r}^1.$$

The Lagrangian is independent of r^2 and correspondingly, we introduce the nonholonomic momentum defined by

$$p = \frac{\partial L_c}{\partial \dot{r}^2}.$$

We shall review the nonholonomic momentum later on in connection with general symmetries, but for now just regard this as a definition. Taking into account the constraint equation and the equations of motion above, we can rewrite the equations of motion in the form

$$\ddot{r}^1 = -\frac{\partial V}{\partial r^1} - \frac{\partial b}{\partial r^1}\left(a(r^1)\dot{r}^1 + b(r^1)p\right)p, \tag{8.2.3}$$

$$\dot{p} = \frac{\partial b}{\partial r^1}\left(a(r^1)\dot{r}^1 + b(r^1)p\right)\dot{r}^1. \tag{8.2.4}$$

Observe that the momentum equation does not, in any obvious way, imply a conservation law.

A ***relative equilibrium*** is a point (r_0, p_0) that is an equilibrium modulo the variable r^2; thus, from the equations (8.2.3) and (8.2.4), we require $\dot{r}^1 = 0$ and

$$\frac{\partial V}{\partial r^1}(r_0^1) + \frac{\partial b}{\partial r^1}b(r_0^1)p_0^2 = 0.$$

We shall see that relative equilibria are Lyapunov stable and in addition asymptotically stable in certain directions if the following two stability conditions are satisfied:

(i) the energy function $E = \frac{1}{2}(\dot{r}^1)^2 + \frac{1}{2}p^2 + V$, which has a critical point at (r_0, p_0), has a positive definite second derivative at this point.

(ii) the derivative of E along the flow of the auxilliary system

$$\ddot{r}^1 = -\frac{\partial V}{\partial r^1} - \frac{\partial b}{\partial r^1}\left(a(r^1)\dot{r}^1 + b(r^1)p\right)p, \quad \dot{p} = \frac{\partial b}{\partial r^1}(r^1)p\dot{r}^1$$

is strictly negative.

The Roller Racer

We now consider a tricycle-like mechanical system called the ***roller racer***, or the ***Tennessee racer***, that is capable of locomotion by oscillating the front handlebars. This toy was studied using the methods of [BKMM] in Tsakiris [1995]. The methods here may be useful for modeling and studying the stability of other systems, such as aircraft landing gears and train wheels.

The roller racer is modeled as a system of two planar coupled rigid bodies (the main body and the second body) with a pair of wheels attached on each of the bodies at their centers of mass. We assume that the mass and the linear momentum of the second body are negligible, but that the moment of inertia about the vertical axis is not. See Figure 8.2.1.

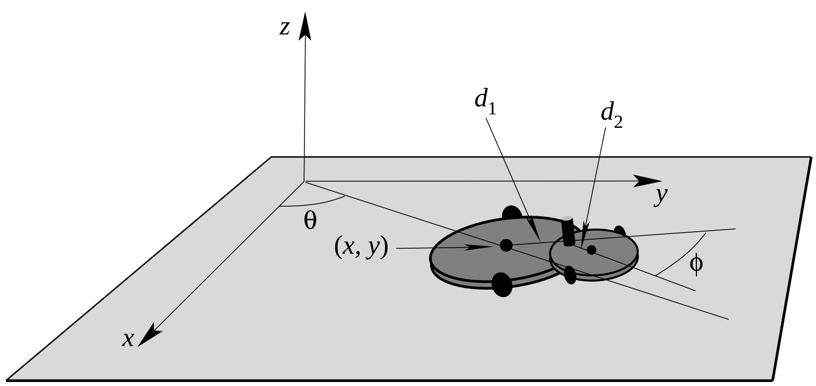


FIGURE 8.2.1. The geometry for the roller racer.

Let (x, y) be the location of the center of mass of the first body and denote the angle in between the inertial reference frame and the line passing through the center of mass of the first body by θ, the angle between the bodies by ϕ, and the distances from the centers of mass to the joint by d_1

and d_2. The mass of body 1 is denoted m and the inertias of the two bodies are written as I_1 and I_2.

The Lagrangian and the constraints are

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2(\dot{\theta} + \dot{\phi})^2$$

and

$$\dot{x} = \cos\theta \left(\frac{d_1 \cos\phi + d_2}{\sin\phi}\dot{\theta} + \frac{d_2}{\sin\phi}\dot{\phi} \right),$$
$$\dot{y} = \sin\theta \left(\frac{d_1 \cos\phi + d_2}{\sin\phi}\dot{\theta} + \frac{d_2}{\sin\phi}\dot{\phi} \right).$$

The configuration space is $SE(2) \times SO(2)$. The Lagrangian and the constraints are invariant under the left action of $SE(2)$ on the first factor of the configuration space.

We shall see later that the roller racer has a two dimensional manifold of equilibria and that under a suitable stability condition some of these equilibria are stable modulo $SE(2)$ and in addition asymptotically stable with respect to $\dot{\phi}$.

The Rattleback

A rattleback is a convex nonsymmetric rigid body rolling without sliding on a horizontal plane. It is known for its ability to spin in one direction and to resist spinning in the opposite direction for some parameter values, and for other values, to exhibit multiple reversals. See Figure 8.2.2.

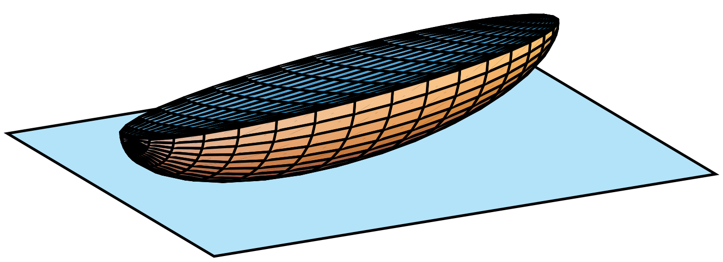


FIGURE 8.2.2. The rattleback.

Basic references on the rattleback are Walker [1896], Karapetyan [1980, 1981], Markeev [1983, 1992], Pascal [1983, 1986], and Bondi [1986]. We adopt the ideal model (with no energy dissipation and no sliding) of these references and within that context, no approximations are made. In particular, the shape need not be ellipsoidal. Walker did some initial stability and instability investigations by computing the spectrum while Bondi extended this analysis and also used what we now recognize as the momentum equation. (See Burdick, Goodwine and Ostrowski [1994]). Karapetyan carried out a stability analysis of the relative equilibria, while Markeev's and Pascal's main contributions were to the study of spin reversals using small parameter and averaging techniques.

Introduce the Euler angles θ, ϕ, ψ using the principal axis body frame relative to an inertial reference frame. These angles together with two horizontal coordinates x, y of the center of mass are coordinates in the configuration space $SO(3) \times \mathbb{R}^2$ of the rattleback.

The Lagrangian of the rattleback is computed to be

$$
\begin{aligned}
L = & \frac{1}{2}\left[A\cos^2\psi + B\sin^2\psi + m(\gamma_1\cos\theta - \zeta\sin\theta)^2\right]\dot\theta^2 \\
& + \frac{1}{2}\left[(A\sin^2\psi + B\cos^2\psi)\sin^2\theta + C\cos^2\theta\right]\dot\phi^2 \\
& + \frac{1}{2}\left(C + m\gamma_2^2\sin^2\theta\right)\dot\psi^2 + \frac{1}{2}m\left(\dot x^2 + \dot y^2\right) \\
& + m(\gamma_1\cos\theta - \zeta\sin\theta)\gamma_2\sin\theta\,\dot\theta\dot\psi + (A - B)\sin\theta\sin\psi\cos\psi\,\dot\theta\dot\phi \\
& + C\cos\theta\,\dot\phi\dot\psi + mg(\gamma_1\sin\theta + \zeta\cos\theta),
\end{aligned}
$$

where

$$
\begin{aligned}
A, B, C &= \text{the principal moments of inertia of the body,} \\
m &= \text{the total mass of the body,} \\
(\xi, \eta, \zeta) &= \text{coordinates of the point of contact relative to the body frame,} \\
\gamma_1 &= \xi\sin\psi + \eta\cos\psi, \\
\gamma_2 &= \xi\cos\psi - \eta\sin\psi.
\end{aligned}
$$

The shape of the body is encoded by the functions ξ, η and ζ. The constraints are

$$
\dot x = \alpha_1\dot\theta + \alpha_2\dot\psi + \alpha_3\dot\phi, \quad \dot y = \beta_1\dot\theta + \beta_2\dot\psi + \beta_3\dot\phi,
$$

where

$$
\begin{aligned}
\alpha_1 &= -(\gamma_1\sin\theta + \zeta\cos\theta)\sin\phi, \\
\alpha_2 &= \gamma_2\cos\theta\sin\phi + \gamma_1\cos\phi, \\
\alpha_3 &= \gamma_2\sin\phi + (\gamma_1\cos\theta - \zeta\sin\theta)\cos\phi, \\
\beta_k &= -\frac{\partial\alpha_k}{\partial\phi}, \quad k = 1, 2, 3.
\end{aligned}
$$

The Lagrangian and the constraints are $SE(2)$-invariant, where the action of an element $(a, b, \alpha) \in SE(2)$ is given by

$$
(x, y, \phi) \mapsto (x\cos\alpha - y\sin\alpha + a, x\sin\alpha + y\cos\alpha + b, \phi + \alpha).
$$

Corresponding to this invariance, ξ, η, and ζ are functions of the variables θ and ψ only.

8.2.1 The Equations of Motion of Nonholonomic Systems with Symmetries

We will need the nonholonomic equations of motion and the momentum equation in the body representation:

The momentum in body representation. Let a local trivialization be chosen on the principle bundle $\pi : Q \to Q/G$, with a local representation having components denoted (r, g). Let r, an element of shape space Q/G have coordinates denoted r^α, and let g be group variables for the fiber, G. In such a representation, the action of G is the left action of G on the second factor.

Put

$$l(r, \dot{r}, \xi) = L(r, g, \dot{r}, \dot{g}).$$

Choose a q-dependent basis $e_a(q)$ for the Lie algebra such that the first m elements span the subspace $\mathfrak{g}^q$ in the following way. First, one chooses, for each r, such a basis at the identity element $g = \mathrm{Id}$, say

$$e_1(r), e_2(r), \ldots, e_m(r), e_{m+1}(r), \ldots, e_k(r).$$

For example, this could be a basis whose generators are orthonormal in the kinetic energy metric. Now define the ***body fixed basis*** by

$$e_a(r, g) = \mathrm{Ad}_g \cdot e_a(r),$$

then the first m elements will indeed span the subspace $\mathfrak{g}^q$ since the distribution is invariant. Define the components of the momentum in body representation to be

$$p_b := \left\langle \frac{\partial l}{\partial \xi}, e_b(r) \right\rangle.$$

Thus, we have

$$J^{\mathrm{nhc}}(r, g, \dot{r}, \dot{g}) = \mathrm{Ad}^*_{g^{-1}} p(r, \dot{r}, \xi).$$

This formula explains why p and J^{nhc} are the body momentum and the spatial momentum respectively.

The Nonholonomic Connection. Assume that the Lagrangian has the form kinetic energy minus potential energy, and that the constraints and the orbit directions span the entire tangent space to the configuration space ([BKMM] call this the "dimension assumption"):

$$\mathcal{D}_q + T_q(\mathrm{Orb}(q)) = T_q Q. \tag{8.2.5}$$

In this case, the momentum equation can be used to augment the constraints and provide a connection on $Q \to Q/G$.

Definition 8.2.1. *Under the dimension assumption in equation (8.2.5), and the assumption that the Lagrangian is of the form kinetic minus potential energies, the* ***nonholonomic connection*** $\mathcal{A}$ *is the connection on the principal bundle* $Q \to Q/G$ *whose horizontal space at the point* $q \in Q$ *is given by the orthogonal complement to the space* $\mathcal{S}_q$ *within the space* $\mathcal{D}_q$.

Let $\mathbb{I}(q) : \mathfrak{g}^{\mathcal{D}} \to \left(\mathfrak{g}^{\mathcal{D}}\right)^*$ be the locked inertia tensor relative to $\mathfrak{g}^{\mathcal{D}}$, defined by

$$\langle \mathbb{I}(q)\xi, \eta \rangle = \langle\langle \xi_Q, \eta_Q \rangle\rangle, \qquad \xi, \eta \in \mathfrak{g}^q,$$

where $\langle\langle \cdot\,, \cdot \rangle\rangle$ is the kinetic energy metric. Define a map $A_q^{\mathrm{sym}} : T_qQ \to \mathcal{S}_q = \mathcal{D}_q \cap T_q(\mathrm{Orb}(q))$ given by

$$A_q^{\mathrm{sym}}(v_q) = (\mathbb{I}^{-1} J^{\mathrm{nhc}}(v_q))_Q.$$

This map is equivariant and is a projection onto $\mathcal{S}_q$. Choose $\mathcal{U}_q \subset T_q(\mathrm{Orb}(q))$ such that $T_q(\mathrm{Orb}(q)) = \mathcal{S}_q \oplus \mathcal{U}_q$. Let $A_q^{\mathrm{kin}} : T_qQ \to \mathcal{U}_q$ be a $\mathcal{U}_q$ valued form that projects $\mathcal{U}_q$ onto itself and maps $\mathcal{D}_q$ to zero; for example, it can be given by orthogonal projection relative to the kinetic energy metric (this will be our default choice).

Proposition 8.2.2. *The nonholonomic connection regarded as an Ehresmann connection is given by*

$$A = A^{\mathrm{kin}} + A^{\mathrm{sym}}. \tag{8.2.6}$$

When the connection is regarded as a principal connection (i.e., takes values in the Lie algebra rather than the vertical space) we will use the symbol $\mathcal{A}$.

Given a velocity vector $\dot{q}$ that satisfies the constraints, we orthogonally decompose it into a piece in $\mathcal{S}_q$ and an orthogonal piece denoted $\dot{r}^h$. We regard $\dot{r}^h$ as the horizontal lift of a velocity vector $\dot{r}$ on the shape space; recall that in a local trivialization, the horizontal lift to the point (r, g) is given by

$$\dot{r}^h = (\dot{r}, -\mathcal{A}_{\mathrm{loc}}\dot{r}) = (\dot{r}^\alpha, -\mathcal{A}_\alpha^a \dot{r}^\alpha),$$

where $\mathcal{A}_\alpha^a$ are the components of the nonholonomic connection which is a principal connection in a local trivialization.

We will denote the decomposition of $\dot{q}$ by

$$\dot{q} = \Omega_Q(q) + \dot{r}^h,$$

so that for each point q, Ω is an element of the Lie algebra and represents the spatial angular velocity of the locked system. In a local trivialization, we can write, at a point (r, g),

$$\Omega = \mathrm{Ad}_g(\Omega_{\mathrm{loc}}),$$

so that $\Omega_{\rm loc}$ represents the body angular velocity. Thus,

$$\Omega_{\rm loc} = \mathcal{A}_{\rm loc}\dot{r} + \xi$$

and, at each point q, the constraints are that Ω belongs to $\mathfrak{g}^q$, *i.e.,*

$$\Omega_{\rm loc} \in \operatorname{span}\{e_1(r), e_2(r), \ldots, e_m(r)\}.$$

The vector $\dot{r}^h$ need not be orthogonal to the whole orbit, just to the piece $\mathcal{S}_q$. Even if $\dot{q}$ does not satisfy the constraints, we can decompose it into three parts and write

$$\dot{q} = \Omega_Q(q) + \dot{r}^h = \Omega_Q^{\rm nh}(q) + \Omega_Q^{\perp}(q) + \dot{r}^h,$$

where $\Omega_Q^{\rm nh}$ and $\Omega_Q^{\perp}$ are orthogonal and where $\Omega_Q^{\rm nh}(q) \in \mathcal{S}_q$. The relation $\Omega_{\rm loc} = \mathcal{A}_{\rm loc}\dot{r} + \xi$ is valid even if the constraints do not hold; also note that this decomposition of Ω corresponds to the decomposition of the nonholonomic connection given by $A = A^{\rm kin} + A^{\rm sym}$ that was given in equation (8.2.6).

To avoid confusion, we will make the following index and summation conventions

1. The first batch of indices range from 1 to m corresponding to the symmetry directions along constraint space. These indices will be denoted $a, b, c, d, \ldots$ and a summation from 1 to m will be understood.

2. The second batch of indices range from $m+1$ to k corresponding to the symmetry directions not aligned with the constraints. Indices for this range or for the whole range 1 to k will be denoted by $a', b', c', \ldots$ and the summations will be given explicitly.

3. The indices $\alpha, \beta, \ldots$ on the shape variables r range from 1 to σ. Thus, σ is the dimension of the shape space Q/G and so $\sigma = n - k$. The summation convention for these indices will be understood.

According to [BKMM], the equations of motion are given by the next theorem

Theorem 8.2.3. *The following **reduced nonholonomic Lagrange-d'Alembert equations** hold for each $1 \leq \alpha \leq \sigma$ and $1 \leq b \leq m$:*

$$\begin{aligned}
\frac{d}{dt}\frac{\partial l_c}{\partial \dot{r}^\alpha} - \frac{\partial l_c}{\partial r^\alpha} &= -\frac{\partial I^{cd}}{\partial r^\alpha} p_c p_d - \mathcal{D}^c_{b\alpha} I^{bd} p_c p_d - \mathcal{B}^c_{\alpha\beta} p_c \dot{r}^\beta \\
&\quad - \mathcal{D}_{\beta\alpha b} I^{bc} p_c \dot{r}^\beta - \mathcal{K}_{\alpha\beta\gamma} \dot{r}^\beta \dot{r}^\gamma, \\
\frac{d}{dt} p_b &= C^c_{ab} I^{ad} p_c p_d + \mathcal{D}^c_{b\alpha} p_c \dot{r}^\alpha + \mathcal{D}_{\alpha\beta b} \dot{r}^\alpha \dot{r}^\beta.
\end{aligned}$$

Here $l_c(r^\alpha, \dot{r}^\alpha, p_a)$ is the constrained Lagrangian; r^α, $1 \leq \alpha \leq \sigma$, are coordinates in the shape space; p_a, $1 \leq a \leq m$, are components of the momentum *map in the body representation,* $p_a = \langle \partial l_c / \partial \Omega_{\rm loc}, e_a(r) \rangle$*;* I^{ad} *are the components of the inverse locked inertia tensor;* $\mathcal{B}^a_{\alpha\beta}$ *are the local coordinates of the curvature* $\mathcal{B}$ *of the nonholonomic connection* $\mathcal{A}$*; and the coefficients* $\mathcal{D}^c_{b\alpha}$*,* $\mathcal{D}_{\alpha\beta b}$*,* $\mathcal{K}_{\alpha\beta\gamma}$ *are given by the formulae*

$$\begin{aligned}
\mathcal{D}^c_{b\alpha} &= \sum_{a'=1}^{k} -C^c_{a'b}\mathcal{A}^{a'}_\alpha + \gamma^c_{b\alpha} + \sum_{a'=m+1}^{k} \lambda_{a'\alpha} C^{a'}_{ab} I^{ac}, \\
\mathcal{D}_{\alpha\beta b} &= \sum_{a'=m+1}^{k} \lambda_{a'\alpha} \left(\gamma^{a'}_{b\beta} - \sum_{b'=1}^{k} C^{a'}_{b'b}\mathcal{A}^{b'}_\beta \right), \\
\mathcal{K}_{\alpha\beta\gamma} &= \sum_{a'=1}^{k} \lambda_{a'\gamma}\mathcal{B}^{a'}_{\alpha\beta},
\end{aligned}$$

where

$$\lambda_{a'\alpha} = l_{a'\alpha} - \sum_{b'=1}^{k} l_{a'b'}\mathcal{A}^{b'}_\alpha := \frac{\partial l}{\partial \xi^{a'} \partial \dot r^\alpha} - \sum_{b'=1}^{k} \frac{\partial l}{\partial \xi^{a'} \partial \xi^{b'}} \mathcal{A}^{b'}_\alpha$$

for $a' = m+1, \ldots, k$*. Here* $C^{b'}_{a'c'}$ *are the structure constants of the Lie algebra defined by* $[e_{a'}, e_{c'}] = C^{b'}_{a'c'} e_{b'}$*,* $a', b', c' = 1, \ldots, k$*; and the coefficients* $\gamma^{c'}_{b\alpha}$ *are defined by*

$$\frac{\partial e_b}{\partial r^\alpha} = \sum_{c'=1}^{k} \gamma^{c'}_{b\alpha} e_{c'}.$$

A ***relative equilibrium*** is an equilibrium of the reduced equations; that is, it is a solution that is given by a one parameter group orbit, just as in the holonomic case (see, e.g., Marsden [1992] for a discussion).

The Constrained Routhian. This function is defined by analogy with the usual Routhian by

$$R(r^\alpha, \dot r^\alpha, p_a) = l_c(r^\alpha, \dot r^\alpha, I^{ab} p_b) - I^{ab} p_a p_b,$$

and in terms of it, the reduced equations of motion become

$$\begin{aligned}
\frac{d}{dt}\frac{\partial R}{\partial \dot r^\alpha} - \frac{\partial R}{\partial r^\alpha} &= -\mathcal{D}^c_{b\alpha} I^{bd} p_c p_d - \mathcal{B}^c_{\alpha\beta} p_c \dot r^\beta \\
&\quad - \mathcal{D}_{\beta\alpha b} I^{bc} p_c \dot r^\beta - \mathcal{K}_{\alpha\beta\gamma} \dot r^\beta \dot r^\gamma, \qquad (8.2.7) \\
\frac{d}{dt} p_b &= C^c_{ab} I^{ad} p_c p_d + \mathcal{D}^c_{b\alpha} p_c \dot r^\alpha + \mathcal{D}_{\alpha\beta b} \dot r^\alpha \dot r^\beta. \qquad (8.2.8)
\end{aligned}$$

The Reduced Constrained Energy. As in [BKMM], the kinetic energy in the variables r^α, $\dot{r}^\alpha$, Ω^a, $\Omega^{a'}$ equals

$$\begin{aligned}&\frac{1}{2}g_{\alpha\beta}\dot{r}^\alpha\dot{r}^\beta+\frac{1}{2}I_{ac}\Omega^a\Omega^c\\&\qquad+\sum_{a'=m+1}^{k}\left(l_{a'\alpha}-l_{a'c'}\mathcal{A}^{c'}_\alpha\right)\Omega^{a'}\dot{r}^\alpha+\frac{1}{2}\sum_{a',c'=m+1}^{k}l_{a'c'}\Omega^{a'}\Omega^{c'},\end{aligned}\tag{8.2.9}$$

where $g_{\alpha\beta}$ are coefficients of the kinetic energy metric induced on the manifold Q/G. Substituting the relations $\Omega^a = I^{ab}p_b$ and the constraint equations $\Omega^{a'} = 0$ in (8.2.9) and adding the potential energy, we define the function E by

$$E=\frac{1}{2}g_{\alpha\beta}\dot{r}^\alpha\dot{r}^\beta+U(r^\alpha,p_a),\tag{8.2.10}$$

which represents the reduced constrained energy in the coordinates r^α, $\dot{r}^\alpha$, p_a, where $U(r^\alpha,p_a)$ is the ***amended potential*** defined by

$$U(r^\alpha,p_a)=\frac{1}{2}I^{ab}p_ap_b+V(r^\alpha),\tag{8.2.11}$$

and $V(r^\alpha)$ is the potential energy of the system.

Now, we show that the reduced constrained energy is conserved along the solutions of (8.2.7), (8.2.8).

Theorem 8.2.4. *The reduced constrained energy is a constant of motion.*

Proof. One way to prove this is to note that the reduced energy is a constant of motion, because it equals the energy represented in coordinates r, $\dot{r}$, g, ξ and because the energy is conserved, since the Lagrangian and the constraints are time-invariant. Along the trajectories, the constrained energy and the energy are the same. Therefore, the reduced constrained energy is a constant of motion.

One may also prove this fact by a direct computation of the time derivative of the reduced constrained energy (8.2.10) along the vector field defined by the equations of motion. ■

Skew Symmetry Assumption. *We assume that the tensor $C^c_{ab}I^{ad}$ is skew-symmetric in c, d.*

This holds for most physical examples and certainly the systems discussed in this paper. (Exceptions include systems with no shape space such as the homogeneous sphere on the plane and certain cases 'of the 'Suslov" problem of a nonhomogeneous rigid body subject to a linear constraint in the angular velocities.) It is an intrinsic (coordinate independent) condition, since $C^c_{ab}I^{ad}$ represents an intrinsic bilinear map of $\left(\mathfrak{g}^{\mathcal{D}}\right)^*\times\left(\mathfrak{g}^{\mathcal{D}}\right)^*$ to $\left(\mathfrak{g}^{\mathcal{D}}\right)^*$.

Under this assumption, the terms quadratic in p in the momentum equation vanish, and the equations of motion become

$$\frac{d}{dt}\frac{\partial R}{\partial \dot r^\alpha} - \frac{\partial R}{\partial r^\alpha} = -\mathcal{D}^c_{b\alpha} I^{bd} p_c p_d - \mathcal{B}^c_{\alpha\beta} p_c \dot r^\beta \\ - \mathcal{D}_{\beta\alpha b} I^{bc} p_c \dot r^\beta - \mathcal{K}_{\alpha\beta\gamma} \dot r^\beta \dot r^\gamma, \tag{8.2.12}$$

$$\frac{d}{dt} p_b = \mathcal{D}^c_{b\alpha} p_c \dot r^\alpha + \mathcal{D}_{\alpha\beta b} \dot r^\alpha \dot r^\beta. \tag{8.2.13}$$

As a result, *the dimension of the family of the relative equilibria equals the number of components of the (nonholonomic) momentum map.*

In the case when $C^c_{ab} = 0$ (cyclic variables, or internal abelian symmetries) the matrix $C^c_{ab} I^{ad}$ vanishes, and the preceding equations of motion are the same as those obtained by Karapetyan [1983].

Below, three principal cases will be considered:

1. **Pure Transport Case** In this case, terms quadratic in $\dot r$ are not present in the momentum equation, so it is in the form of a transport equation—i.e. the momentum equation is an equation of parallel transport and the equation itself defines the relevant connection.

   Under certain integrability conditions (see below) the transport equation defines invariant surfaces, which allow us to use a type of energy-momentum method for stability analysis in a similar fashion to the manner in which the holonomic case uses the level surfaces defined by the momentum map. The key difference is that in our case, the additional invariant surfaces do not arise from conservation of momentum. In this case, one gets stable, but not asymptotically stable, relative equilibria. Examples include the rolling disk, a body of revolution rolling on a horizontal plane, and the Routh problem.

2. **Integrable Transport Case** In this case, terms quadratic in $\dot r$ are present in the momentum equation and thus it is not a pure transport equation. However, in this case, we assume that the transport part is integrable. As we shall also see, in this case relative equilibria may be asymptotically stable. We are able to find a generalization of the energy-momentum method which gives conditions for asymptotic stability. An example is the roller racer.

3. **Nonintegrable Transport Case** Again, the terms quadratic in $\dot r$ are present in the momentum equation and thus it is not a pure transport equation. However, the transport part is not integrable. Again, we are able to demonstrate asymptotic stability using Lyapunov-Malkin Theorem and to relate it to an energy-momentum type analysis under certain eigenvalue hypotheses, as we will see in 8.2.3. An example is the rattleback top. Another example is a nonhomogeneous

sphere with a center of mass lying off the planes spanned by the principal axis body frame. See Markeev [1992].

In some examples, such as the nonhomogeneous (unbalanced) Kovalevskaya sphere rolling on the plane, these eigenvalue hypotheses do not hold. We intend to investigate this case in a future publication.

In the sections below where these different cases are discussed we will make clear at the beginning of each section what the underlying hypotheses on the systems are by listing the key hypotheses and labeling them by H1, H2, and H3.

8.2.2 The Pure Transport Case

In this section we assume that

H1 $\mathcal{D}_{\alpha\beta b}$ are skew-symmetric in α, β. Under this assumption, the momentum equation can be written as the vanishing of the connection one form defined by $dp_b - \mathcal{D}^c_{b\alpha} p_c dr^\alpha$.

H2 The curvature of the preceding connection form is zero.

A nontrivial example of this case is that of Routh's problem of a sphere rolling in a surface of revolution. See Zenkov [1995].

Under the above two assumptions, the distribution defined by the momentum equation is integrable, and so we get invariant surfaces, which makes further reduction possible. This enables us to use the energy-momentum method in a way that is similar to the holonomic case, as we explained above.

Note that if the number of shape variables is one, the above connection is integrable, because it may be treated as a system of linear ordinary differential equations with coefficients depending on the shape variable r:

$$\frac{dp_b}{dr} = \mathcal{D}^c_b p_c.$$

As a result, we obtain an integrable nonholonomic system, because after solving the momentum equation for p_b and substituting the result in the equation for the shape variable, the latter equation may be viewed as a Lagrangian system with one degree of freedom, which is integrable.

The Nonholonomic Energy-Momentum Method

We now develop the energy-momentum method for the case in which the momentum equation is pure transport. Under the assumptions H1 and H2 made so far, the equations of motion become

$$\frac{d}{dt}\frac{\partial R}{\partial \dot r^\alpha} - \frac{\partial R}{\partial r^\alpha} = -\mathcal{D}^c_{b\alpha} I^{bd} p_c p_d - \mathcal{B}^c_{\alpha\beta} p_c \dot r^\beta - \mathcal{K}_{\alpha\beta\gamma} \dot r^\beta \dot r^\gamma, \tag{8.2.14}$$

$$\frac{d}{dt} p_b = \mathcal{D}^c_{b\alpha} p_c \dot r^\alpha. \tag{8.2.15}$$

A ***relative equilibrium*** is a point $(r, \dot{r}, p) = (r_0, 0, p_0)$ which is a fixed point for the dynamics determined by equations (8.2.14) and (8.2.15). Under assumption H1 the point (r_0, p_0) is seen to be a critical point of the amended potential.

Because of our zero curvature assumption H2, the solutions of the momentum equation lie on surfaces of the form $p_a = P_a(r^\alpha, k_b)$, $a, b = 1, \ldots, m$, where k_b are constants labeling these surfaces.

Using the functions $p_a = P_a(r^\alpha, k_b)$ we introduce the ***reduced amended potential*** $U_k(r^\alpha) = U(r^\alpha, P_a(r^\alpha, k_b))$. We think of the function $U_k(r^\alpha)$ as being the restriction of the function U to the invariant manifold

$$Q_k = \{(r^\alpha, p_a) \mid p_a = P_a(r^\alpha, k_b)\}.$$

Theorem 8.2.5. *Let assumptions H1 and H2 hold and let (r_0, p_0), where $p_0 = P(r_0, k^0)$, be a relative equilibrium. If the reduced amended potential $U_{k^0}(r)$ has a nondegenerate minimum at r_0, then this equilibrium is Lyapunov stable.*

Proof. First, we show that the relative equilibrium

$$r^\alpha = r_0^\alpha, \quad p_a^0 = P_a(r_0^\alpha, k_b^0) \tag{8.2.16}$$

of the system (8.2.14), (8.2.15) is stable modulo perturbations consistent with Q_{k^0}. Consider the phase flow restricted to the invariant manifold Q_{k^0}, where k^0 corresponds to the relative equilibrium. Since $U_{k^0}(r^\alpha)$ has a nondegenerate minimum at r_0^α, the function $E|_{Q_{k^0}}$ is positive definite. By Theorem 8.2.4 its derivative along the flow vanishes. Using $E|_{k^0}$ as a Lyapunov function, we conclude that equations (8.2.14), (8.2.15), restricted to the manifold Q_{k^0}, have a stable equilibrium point r_0^α on Q_{k^0}.

To finish the proof, we need to show that equations (8.2.14), (8.2.15), restricted to nearby invariant manifolds Q_k, have stable equilibria on these manifolds.

If k is sufficiently close to k^0, then by the properties of families of Morse functions (see Milnor [1963]), the function $U_k \colon Q_k \to \mathbb{R}$ has a nondegenerate minimum at the point r^α which is close to r_0^α. This means that for all k sufficiently close to k^0 the system (8.2.14), (8.2.15) restricted to Q_k has a stable equilibrium r^α. Therefore, the equilibrium (8.2.16) of equations (8.2.14), (8.2.15) is stable.

The stability here cannot be asymptotic, since the dynamical systems on Q_k have a positive definite conserved quantity—the reduced energy function. ■

Remark. Even though in general $P_a(r^\alpha, k_b)$ can not be found explicitly, the types of critical points of U_k may be explicitly determined as follows. First of all, note that

$$\frac{\partial p_a}{\partial r^\alpha} = \mathcal{D}_{b\alpha}^c p_c$$

as long as $(r^\alpha, p_a) \in Q_k$. Therefore

$$\frac{\partial U_k}{\partial r^\alpha} = \nabla_\alpha U,$$

where

$$\nabla_\alpha = \frac{\partial}{\partial r^\alpha} + \mathcal{D}^c_{b\alpha} p_c \frac{\partial}{\partial p_b}. \tag{8.2.17}$$

Then the relative equilibria satisfy the condition

$$\nabla_\alpha U = 0,$$

while the condition for stability

$$\frac{\partial^2 U_k}{\partial r^2} \gg 0$$

(*i.e.*, is positive definite) becomes the condition

$$\nabla_\alpha \nabla_\beta U \gg 0.$$

In the commutative case this was shown by Karapetyan [1983].

Now we give the stability condition in a form similar to that of energy-momentum method for holonomic systems given in Simo, Lewis, and Marsden [1991].

Theorem 8.2.6 (The nonholonomic energy-momentum method). *Under assumptions H1 and H2, the point $q_e = (r^\alpha_0, p^0_a)$ is a relative equilibrium if and only if there is a $\xi \in \mathfrak{g}^{q_e}$ such that q_e is a critical point of the **augmented energy** $E_\xi : \mathcal{D}/G \to \mathbb{R}$ (i.e., E_ξ is a function of $(r, \dot{r}, p)$), defined by*

$$E_\xi = E - \langle p - P(r,k), \xi \rangle.$$

This equilibrium is stable if $\delta^2 E_\xi$ restricted to $T_{q_e} Q_k$ is positive definite (here δ denotes differentiation with respect to all variables except ξ).

Proof. A point $q_e \in Q_k$ is a relative equilibrium if $\partial_{r^\alpha} U_k = 0$. This condition is equivalent to $d\left(E|_{Q_k}\right) = 0$. The last equation may be represented as $d\left(E - \langle p - P(r,k), \xi\rangle\right) = 0$ for some $\xi \in \mathfrak{g}^{q_e}$. Similarly, the condition for stability $d^2 U_k \gg 0$ is equivalent to $d^2\left(E|_{Q_k}\right) \gg 0$, which may be represented as $\left(\delta^2 E_\xi\right)|_{T_{q_e} Q_k} \gg 0$. ■

Note that if the momentum map is preserved, then the formula for E_ξ becomes

$$E_\xi = E - \langle p - k, \xi \rangle,$$

which is the same as the formula for the augmented energy E_ξ for holonomic systems.

Examples

There are several examples which illustrate the ideas above. For instance the falling disk, Routh's problem, and a body of revolution rolling on a horizontal plane are systems where the momentum equation defines an integrable distribution and we are left with only one shape variable. Since the stability properties of all these systems are similar, we consider here only the rolling disk. For the body of revolution on the plane see Chaplygin [1897a] and Karapetyan [1983]. For the Routh problem see Zenkov [1995].

The Rolling Disk. Consider again the disk rolling without sliding on the xy-plane. Recall that we have the following: Denote the coordinates of contact of the disk in the xy-plane by x, y. Let θ, ϕ, and ψ denote the angle between the plane of the disk and the vertical axis, the "heading angle" of the disk, and "self-rotation" angle of the disk respectively, as was introduced earlier.

The Lagrangian and the constraints in these coordinates are given by

$$
\begin{aligned}
L &= \frac{m}{2}\left[(\xi - R(\dot\phi\sin\theta + \dot\psi))^2 + \eta^2\sin^2\theta + (\eta\cos\theta + R\dot\theta)^2\right] \\
&\quad + \frac{1}{2}\left[A(\dot\theta^2 + \dot\phi^2\cos^2\theta) + B(\dot\phi\sin\theta + \dot\psi)^2\right] - mgR\cos\theta, \\
\dot x &= -\dot\psi R\cos\phi, \\
\dot y &= -\dot\psi R\sin\phi,
\end{aligned}
$$

where $\xi = \dot x\cos\phi + \dot y\sin\phi + R\dot\psi$, $\eta = -\dot x\sin\phi + \dot y\cos\phi$. Note that the constraints may be written as $\xi = 0$, $\eta = 0$.

This system is invariant under the action of the group $G = SE(2) \times SO(2)$; the action by the group element (a, b, α, β) is given by

$$
(\theta, \phi, \psi, x, y) \mapsto (\theta, \phi + \alpha, \psi + \beta, x\cos\alpha - y\sin\alpha + a, x\sin\alpha + y\cos\alpha + b).
$$

Obviously,

$$
T_q\mathrm{Orb}(q) = \mathrm{span}\left(\frac{\partial}{\partial\phi}, \frac{\partial}{\partial\psi}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right),
$$

and

$$
\mathcal{D}_q = \mathrm{span}\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\phi}, R\cos\phi\frac{\partial}{\partial x} + R\sin\phi\frac{\partial}{\partial y} - \frac{\partial}{\partial\psi}\right),
$$

which imply

$$
\mathcal{S}_q = \mathcal{D}_q \cap T_q\mathrm{Orb}(q) = \mathrm{span}\left(\frac{\partial}{\partial\phi}, -R\cos\phi\frac{\partial}{\partial x} - R\sin\phi\frac{\partial}{\partial y} + \frac{\partial}{\partial\psi}\right).
$$

Choose vectors $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, $(0,0,0,1)$ as a basis of the Lie algebra $\mathfrak{g}$ of the group G. The corresponding generators are ∂_x, ∂_y, $-y\partial_x + x\partial_y + \partial_\phi$, ∂_ψ. Taking into account that the generators ∂_ϕ,

$-R\cos\phi\partial_x - R\sin\phi\partial_y + \partial_\psi$ correspond to the elements $(y, -x, 1, 0)$, $(-R\cos\phi, -R\sin\phi, 0, 1)$ of the Lie algebra $\mathfrak{g}$, we obtain the following momentum equations:

$$\begin{aligned} \dot{p}_1 &= mR^2 \cos\theta\, \dot{\theta}\dot{\psi}, \\ \dot{p}_2 &= -mR^2 \cos\theta\, \dot{\theta}\dot{\phi}, \end{aligned} \tag{8.2.18}$$

where

$$\begin{aligned} p_1 &= A\dot{\phi}\cos^2\theta + (mR^2 + B)(\dot{\phi}\sin\theta + \dot{\psi})\sin\theta, \\ p_2 &= (mR^2 + B)(\dot{\phi}\sin\theta + \dot{\psi}), \end{aligned} \tag{8.2.19}$$

into which the constraints have been substituted. One may notice that

$$p_1 = \frac{\partial l_c}{\partial \dot{\phi}}, \quad p_2 = \frac{\partial l_c}{\partial \dot{\psi}}.$$

Solving (8.2.19) for $\dot{\phi}$, $\dot{\psi}$ and substituting the solutions back in the equations (8.2.18) we obtain another representation of the momentum equations:

$$\begin{aligned} \frac{dp_1}{dt} &= mR^2\cos\theta \left(-\frac{\sin\theta}{A\cos^2\theta} p_1 + \left(\frac{1}{mR^2 + B} + \frac{\sin^2\theta}{A\cos^2\theta} \right) p_2 \right) \dot{\theta}, \\ \frac{dp_2}{dt} &= mR^2\cos\theta \left(-\frac{1}{A\cos^2\theta} p_1 + \frac{\sin\theta}{A\cos^2\theta} p_2 \right) \dot{\theta}. \end{aligned} \tag{8.2.20}$$

The right hand sides of (8.2.20) do not have terms quadratic in the shape variable θ. The distribution, defined by (8.2.20), is integrable and defines two integrals of the form $p_1 = P_1(\theta, k_1, k_2)$, $p_2 = P_2(\theta, k_1, k_2)$. It is known that these integrals may be written down explicitly in terms of the hypergeometric function. See Appel [1900], Chaplygin [1897a], and Korteweg [1899] for details.

To carry out stability analysis, we use the remark following Theorem 8.2.5. Using formulae (8.2.19), we obtain the amended potential

$$U(\theta, p) = \frac{1}{2}\left[\frac{(p_1 - p_2\sin\theta)^2}{A\cos^2\theta} + \frac{p_2^2}{B + mR^2} \right] + mgR\cos\theta.$$

Straightforward computation shows that the condition for stability $\nabla^2 U \gg 0$ of a relative equilibrium $\theta = \theta_0$, $p_1 = p_1^0$, $p_2 = p_2^0$ becomes

$$\begin{aligned} &\frac{B}{A(mR^2 + B)}(p_2^0)^2 + \frac{mR^2\cos^2\theta_0 + 2A\sin^2\theta_0 + A}{A^2}(p_1^0 - p_2^0\sin\theta_0)^2 \\ &\qquad - \frac{(mR^2 + 3B)\sin\theta_0}{A(mR^2 + B)\cos^2\theta_0}(p_1^0 - p_2^0\sin\theta_0)p_2^0 - mgR\cos\theta_0 > 0. \end{aligned}$$

Note that this condition guarantees stability here relative to θ, $\dot{\theta}$, p_1, p_2; in other words we have stability modulo the action of $SE(2) \times SO(2)$.

The falling disk may be considered as a limiting case of the body of revolution which also has an integrable pure transport momentum equation (this example is treated in Chaplygin [1897a] and Karapetyan [1983]). The rolling disc has also been analyzed recently by O'Reilly [1996] and Cushman, Hermans and Kemppainen [1996]. O'Reilly considered bifurcation of relative equilibria, the stability of vertical stationary motions, as well as the possibility of sliding.

8.2.3 The Non-Pure Transport Case

In this section we consider the case in which the coefficients $\mathcal{D}_{\alpha\beta b}$ are not skew symmetric in α, β and the two subcases where the transport part of the momentum equation is integrable or is not integrable, respectively. In either case one may obtain asymptotic stability.

We begin by assembling some preliminary results on center manifold theory and show how they relate to the Lyapunov-Malkin Theorem. The center manifold theorem provides new and useful insight into the existence of integral manifolds. These integral manifolds play a crucial role in our analysis. Lyapunov's original proof of the Lyapunov-Malkin Theorem used a different approach to proving the existence of local integrals, as we shall discuss below. Malkin extended the result to the nonautonomous case.

Center Manifold Theory in Stability Analysis

Here we discuss the center manifold theory and its applications to the stability analysis of nonhyperbolic equilibria.

Consider a system of differential equations

$$\dot{x} = Ax + X(x, y), \tag{8.2.21}$$

$$\dot{y} = By + Y(x, y), \tag{8.2.22}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and A and B are constant matrices. It is supposed that all eigenvalues of A have nonzero real parts, and all eigenvalues of B have zero real parts. The functions X, Y are smooth, and satisfy the conditions $X(0,0) = 0$, $dX(0,0) = 0$, $Y(0,0) = 0$, $dY(0,0) = 0$. We now recall the following definition:

Definition 8.2.7. *A smooth invariant manifold of the form $x = h(y)$ where h satisfies $h(0) = 0$ and $dh(0) = 0$ is called a* ***center manifold****.*

We are going to use the following version of the center manifold theorem following the exposition of Carr [1981] (see also Chow and Hale [1982]).

Theorem 8.2.8 (The center manifold theorem). *If the functions $X(x, y)$, $Y(x, y)$ are C^k, $k \geq 2$, then there exist a (local) center manifold for* (8.2.21), *(8.2.22), $x = h(y)$, $|y| < \delta$, where h is C^k. The flow on the center manifold is governed by the system*

$$\dot{y} = By + Y(h(y), y). \tag{8.2.23}$$

The next theorem explains that the reduced equation (8.2.23) contains information about stability of the zero solution of (8.2.21), (8.2.22).

Theorem 8.2.9. *Suppose that the zero solution of* (8.2.23) *is stable (asymptotically stable) (unstable) and that the eigenvalues of A are in the left half plane. Then the zero solution of* (8.2.21), (8.2.22) *is stable (asymptotically stable) (unstable).*

Let us now look at the special case of (8.2.22) in which the matrix B vanishes. Equations (8.2.21), (8.2.22) become

$$\dot{x} = Ax + X(x, y), \tag{8.2.24}$$

$$\dot{y} = Y(x, y). \tag{8.2.25}$$

Theorem 8.2.10. *Consider the system of equations* (8.2.24), (8.2.25). *If $X(0, y) = 0$, $Y(0, y) = 0$, and the matrix A does not have eigenvalues with zero real parts, then the system* (8.2.24), (8.2.25) *has n local integrals in the neighborhood of $x = 0$, $y = 0$.*

Proof. The center manifold in this case is given by $x = 0$. Each point of the center manifold is an equilibrium of the system (8.2.24), (8.2.25). For each equilibrium point $(0, y_0)$ of our system, consider an m-dimensional smooth invariant manifold

$$S(y_0) = S^s(y_0) \times S^u(y_0),$$

where $S^s(y_0)$ and $S^u(y_0)$ are stable and unstable manifolds at the equilibrium $(0, y_0)$. The center manifold and these manifolds $S(y_0)$ can be used for a (local) substitution $(x, y) \to (\bar{x}, \bar{y})$ such that in the new coordinates the system of differential equations become

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{X}(\bar{x}, \bar{y}), \quad \dot{\bar{y}} = 0.$$

The last system of equations has n integrals $\bar{y} = \text{const}$, so that the original equation has n smooth local integrals. Observe that the tangent spaces to the common level sets of these integrals at the equilibria are the planes $y = y_0$. Therefore, the integrals are of the form $y = f(x, k)$, where $\partial_x f(0, k) = 0$. ■

The following theorem gives stability conditions for equilibria of the system (8.2.24), (8.2.25).

Theorem 8.2.11 (Lyapunov-Malkin). *Consider the system of differential equations* (8.2.24), (8.2.25), *where* $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, A *is an* $m \times m$-*matrix, and* $X(x,y)$, $Y(x,y)$ *represent nonlinear terms. If all eigenvalues of the matrix* A *have negative real parts, and* $X(x,y)$, $Y(x,y)$ *vanish when* $x = 0$, *then the solution* $x = 0$, $y = c$ *of the system* (8.2.24), (8.2.25) *is stable with respect to* x, y, *and asymptotically stable with respect to* x. *If a solution* $x(t)$, $y(t)$ *of* (8.2.24), (8.2.25) *is close enough to the solution* $x = 0$, $y = 0$, *then*

$$\lim_{t \to \infty} x(t) = 0, \quad \lim_{t \to \infty} y(t) = c.$$

The proof of this theorem consists of two steps. The first step is a reduction of the system to the common level set of integrals described in Theorem 8.2.10. The second step is the construction of a Lyapunov function for the reduced system. The details of the proof may be found in Lyapunov [1992] and Malkin [1938].

Historical Note. The proof of the Lyapunov-Malkin Theorem uses the fact that the system of differential equations has local integrals, as discussed in Theorem 4.4. To prove existence of these integrals, Lyapunov uses a theorem of his own about the existence of solutions of PDE's. He does this assuming that the nonlinear terms on the right hand sides are series in x and y with time-dependent bounded coefficients. Malkin generalizes Lyapunov's result for systems for which the matrix A is time-dependent. We consider the nonanalytic case, and to prove existence of these local integrals, we use center manifold theory. This simplifies the arguments to some extent as well as showing how the results are related.

The following Lemma specifies a class of systems of differential equations, that satisfies the conditions of the Lyapunov-Malkin Theorem.

Lemma 8.2.12. *Consider a system of differential equations of the form*

$$\dot{u} = Au + By + \mathcal{U}(u,y), \quad \dot{y} = \mathcal{Y}(u,y), \tag{8.2.26}$$

where $u \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\det A \neq 0$, *and where* $\mathcal{U}$ *and* $\mathcal{Y}$ *represent higher order nonlinear terms. There is a change of variables of the form* $u = x + \phi(y)$ *such that*

(i) *in the new variables* x, y *system* (8.2.26) *becomes*

$$\dot{x} = Ax + X(u,y), \quad \dot{y} = Y(x,y),$$

(ii) *if* $Y(0,y) = 0$, *then* $X(0,y) = 0$ *as well.*

Proof. Put $u = x + \phi(y)$, where $\phi(y)$ is defined by

$$A\phi(y) + By + \mathcal{U}(\phi(y), y) = 0.$$

System (8.2.26) in the variables x, y becomes

$$\dot{x} = Ax + X(x, y), \quad \dot{y} = Y(x, y),$$

where

$$\begin{aligned} X(x, y) &= A\phi(y) + By + \mathcal{U}(x + \phi(y), y) - \frac{\partial \phi}{\partial y} Y(x, y), \\ Y(x, y) &= \mathcal{Y}(x + \phi(y), y). \end{aligned}$$

Note that $Y(0, y) = 0$ implies $X(0, y) = 0$. ■

The Mathematical Example

The Lyapunov-Malkin conditions. Recall from §8.2 that the equations of motion are

$$\begin{aligned} \ddot{r} &= -\frac{\partial V}{\partial r} - \frac{\partial b}{\partial r}\left(a(r)\dot{r} + b(r)p\right)p, \\ \dot{p} &= \frac{\partial b}{\partial r}\left(a(r)\dot{r} + b(r)p\right)\dot{r}, \end{aligned} \tag{8.2.27}$$

here and below we write r instead of r^1.

Recall also that a point $r = r_0$, $p = p_0$ is a relative equilibrium if r_0 and p_0 satisfy the condition

$$\frac{\partial V}{\partial r}(r_0) + \frac{\partial b}{\partial r} b(r_0)\, p_0^2 = 0.$$

Introduce coordinates u_1, u_2, v in the neighborhood of this equilibrium by

$$r = r_0 + u_1, \quad \dot{r} = u_2, \quad p = p_0 + v.$$

The linearized equations of motion are

$$\begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= \mathcal{A}u_2 + \mathcal{B}u_1 + \mathcal{C}v, \\ \dot{v} &= \mathcal{D}u_2, \end{aligned}$$

where

$$\begin{aligned} \mathcal{A} &= -\frac{\partial b}{\partial r} a p_0, \\ \mathcal{B} &= -\frac{\partial^2 V}{\partial r^2} - \left[\frac{\partial^2 b}{\partial r^2} b + \left(\frac{\partial b}{\partial r}\right)^2\right] p_0^2, \\ \mathcal{C} &= -2\frac{\partial b}{\partial r} b p_0, \\ \mathcal{D} &= \frac{\partial b}{\partial r} b p_0, \end{aligned}$$

and where V, a, b, and their derivatives are evaluated at r_0. The characteristic polynomial of these linearized equations is calculated to be

$$\lambda[\lambda^2 - \mathcal{A}\lambda - (\mathcal{B} + \mathcal{C}\mathcal{D})].$$

It obviously has one zero root. The two others have negative real parts if

$$\mathcal{B} + \mathcal{C}\mathcal{D} < 0, \quad \mathcal{A} < 0. \tag{8.2.28}$$

These conditions imply linear stability. We discuss the meaning of these conditions later.

Next, we make the substitution $v = y + \mathcal{D}u_1$, which defines the new variable y. The (nonlinear) equations of motion become

$$\begin{aligned} \dot{u}_1 &= u_2, \\ \dot{u}_2 &= \mathcal{A}u_2 + (\mathcal{B} + \mathcal{C}\mathcal{D})u_1 + \mathcal{C}y + \mathcal{U}(u, y), \\ \dot{y} &= \mathcal{Y}(u, y), \end{aligned}$$

where $\mathcal{U}(u, y)$, $\mathcal{Y}(u, y)$ stand for nonlinear terms, and $\mathcal{Y}(u, y)$ vanishes when $u = 0$. By Lemma 8.2.12 there exists a further substitution $u = x + \phi(y)$ such that the equations of motion in coordinates x, y become

$$\begin{aligned} \dot{x} &= Px + X(x, y), \\ \dot{y} &= Y(x, y), \end{aligned}$$

where $X(x, y)$ and $Y(0, y)$ satisfy the conditions $X(0, y) = 0$, $Y(0, y) = 0$. Here,

$$P = \begin{pmatrix} 0 & 1 \\ \mathcal{B} + \mathcal{C}\mathcal{D} & \mathcal{A} \end{pmatrix}.$$

This form enables us to use the Lyapunov-Malkin Theorem and conclude that the linear stability implies nonlinear stability and in addition that we have asymptotic stability with respect to variables x_1, x_2.

The Energy-Momentum Method. To find a Lyapunov function based approach for analyzing the stability of the mathematical example, we introduce a modified dynamical system and use its energy function and momentum to construct a Lyapunov function for the original system. This modified system is introduced for the purpose of finding the Lyapunov function and is not used in the stability proof. We will generalize this approach below and this example may be viewed as motivation for the general approach.

Consider then the new system obtained from the Lagrangian (8.2.1) and the constraint (8.2.2) by setting $a(r) = 0$. Notice that L_c stays the same and therefore, the equation of motion may be obtained from (8.2.27):

$$\ddot{r} = -\frac{\partial V}{\partial r} - \frac{\partial b}{\partial r} b(r) p^2, \quad \dot{p} = \frac{\partial b}{\partial r} b(r) p \dot{r}.$$

The condition for existence of the relative equilibria also stays the same. However, a crucial observation is that for the new system, the momentum equation is now integrable, in fact explicitly, so that in this example:

$$p = k \exp(b^2(r)/2).$$

Thus, we may proceed and use this invariant surface to perform reduction. The amended potential, defined by $U(r,p) = V(r) + \frac{1}{2}p^2$, becomes

$$U_k(r) = V(r) + \frac{1}{2}\left(k \exp(b^2(r)/2)\right)^2.$$

Consider the function

$$W_k = \frac{1}{2}(\dot{r})^2 + U_k(r) + \epsilon(r - r_0)\dot{r}.$$

If ϵ is small enough and U_k has a nondegenerate minimum, then so does W_k. Suppose that the matrix P has no eigenvalues with zero real parts. Then by Theorem 8.2.10 equations (8.2.27) have a local integral $p = \mathcal{P}(r, \dot{r}, c)$.

Differentiate W_k along the vector field determined by (8.2.27). We obtain

$$\begin{aligned}\dot{W}_k = -\epsilon\left(\frac{\partial^2 V}{\partial r^2}(r_0) + \left(\frac{\partial^2 b}{\partial r^2}b(r_0) + 2\left(\frac{\partial b}{\partial r}b(r_0)\right)^2 + \left(\frac{\partial b}{\partial r}(r_0)\right)^2\right)p_0^2\right) \\ - \frac{\partial b}{\partial r}a(r_0)p_0\dot{r}^2 + \epsilon\dot{r}^2 + \{\text{higher order terms}\}.\end{aligned}$$

Therefore, W_k is a Lyapunov function for the flow restricted to the local invariant manifold $p = \mathcal{P}(r, \dot{r}, c)$ if

$$\frac{\partial^2 V}{\partial r^2}(r_0) + \left(\frac{\partial^2 b}{\partial r^2}b(r_0) + 2\left(\frac{\partial b}{\partial r}b(r_0)\right)^2 + \left(\frac{\partial b}{\partial r}(r_0)\right)^2\right)p_0^2 > 0 \qquad (8.2.29)$$

and

$$\frac{\partial b}{\partial r}a(r_0)p_0\dot{r}^2 > 0. \qquad (8.2.30)$$

Notice that the Lyapunov conditions (8.2.29) and (8.2.30) are the same as conditions (8.2.28).

Introduce the operator

$$\nabla = \frac{\partial}{\partial r} + \frac{\partial b}{\partial r}b(r)p\frac{\partial}{\partial p}$$

(cf. Karapetyan [1983]). Then condition (8.2.29) may be represented as

$$\nabla^2 U > 0,$$

which is the same as the condition for stability of stationary motions of a nonholonomic system with an integrable momentum equation (recall that this means that there are no terms quadratic in $\dot{r}$, only transport terms defining an integrable distribution). The left hand side of formula (8.2.30) may be viewed as a derivative of the energy function

$$E = \frac{1}{2}\dot{r}^2 + \frac{1}{2}p^2 + V$$

along the flow

$$\ddot{r} = -\frac{\partial V}{\partial r} - \frac{\partial b}{\partial r}\left(a(r)\dot{r} + b(r)p\right)p, \quad \dot{p} = \frac{\partial b}{\partial r}b(r)p\dot{r},$$

or as a derivative of the amended potential U along the vector field defined by the nontransport terms of the momentum equations

$$\dot{p} = \frac{\partial b}{\partial r}a(r)\dot{r}^2.$$

The Nonholonomic Energy-Momentum Method

We now generalize the energy-momentum method discussed above for the mathematical example to the general case in which the transport part of the momentum equation is integrable.

Here we assume hypothesis H2 in the present context, namely:

H2 The curvature of the connection form associated with the transport part of the momentum equation, namely $dp_b - \mathcal{D}^c_{b\alpha}p_c dr^\alpha$, is zero.

The momentum equation in this situation is

$$\frac{d}{dt}p_b = \mathcal{D}^c_{b\alpha}p_c\dot{r}^\alpha + \mathcal{D}_{\alpha\beta b}\dot{r}^\alpha\dot{r}^\beta.$$

Hypothesis H2 implies that the form due to the transport part of the momentum equation defines an integrable distribution. Associated to this distribution, there is a family of integral manifolds

$$p_a = P_a(r^\alpha, k_b)$$

with P_a satisfying the equation $dP_b = \mathcal{D}^c_{b\alpha}P_c dr^\alpha$. Note that these manifolds *are not invariant manifolds of the full system* under consideration because the momentum equation has non-transport terms. Substituting the functions $P_a(r^\alpha, k_b)$, $k_b = \text{const}$, into $E(r, \dot{r}, p)$, we obtain a function

$$V_k(r^\alpha, \dot{r}^\alpha) = E(r^\alpha, \dot{r}^\alpha, P_a(r^\alpha, k_b)),$$

that depends only on r^α, $\dot{r}^\alpha$ and parametrically on k. This function will not be our final Lyapunov function but will be used to construct one in the proof to follow.

Pick a relative equilibrium $r^\alpha = r^\alpha_0$, $p_a = p^0_a$. In this context we introduce the following definiteness assumptions:

H3 At the equilibrium $r^\alpha = r_0^\alpha$, $p_a = p_a^0$ the two symmetric matrices $\nabla_\alpha \nabla_\beta U$ and $(\mathcal{D}_{\alpha\beta b} + \mathcal{D}_{\beta\alpha b}) I^{bc} p_c$ are positive definite.

Theorem 8.2.13. *Under assumptions H2 and H3, the equilibrium $r^\alpha = r_0^\alpha$, $p_a = p_a^0$ is Lyapunov stable. Moreover, the system has local invariant manifolds that are tangent to the family of manifolds defined by the integrable transport part of the momentum equation at the relative equilibria. The relative equilibria, that are close enough to r_0, p_0, are asymptotically stable in the directions defined by these invariant manifolds. In addition, for initial conditions close enough to the equilibrium $r^\alpha = r_0^\alpha$, $p_a = p_a^0$, the perturbed solution approaches a nearby equilibrium.*

Proof. The substitution $p_a = p_a^0 + y_a + \mathcal{D}_{a\alpha}^b(r_0) p_b^0 u^\alpha$, where $u^\alpha = r^\alpha - r_0^\alpha$, eliminates the linear terms in the momentum equation. In fact, with this substitution, the equations of motion (8.2.12), (8.2.13) become

$$\begin{aligned}
\frac{d}{dt}\frac{\partial R}{\partial \dot r^\alpha} - \frac{\partial R}{\partial r^\alpha} &= -\mathcal{D}_{b\alpha}^c I^{bd} p_c p_d - \mathcal{B}_{\alpha\beta}^c p_c \dot r^\beta \\
&\quad - \mathcal{D}_{\beta\alpha b} I^{bc} p_c \dot r^\beta - \mathcal{K}_{\alpha\beta\gamma} \dot r^\beta \dot r^\gamma, \\
\frac{d}{dt} y_b &= \mathcal{D}_{b\alpha}^c y_c \dot r^\alpha + (\mathcal{D}_{b\alpha}^c - \mathcal{D}_{b\alpha}^c(r_0)) p_c^0 \dot r^\alpha + \mathcal{D}_{\alpha\beta b} \dot r^\alpha \dot r^\beta.
\end{aligned}$$

We will show in §8.2.3 that H3 implies the hypotheses of Theorem 8.2.10. Thus, the above equations have local integrals $y_a = f_a(r, \dot r, c)$, where the functions f_a are such that $\partial_r f_a = \partial_{\dot r} f_a = 0$ at the equilibria. Therefore, the original equations (8.2.12), (8.2.13) have n local integrals

$$p_a = \mathcal{P}_a(r^\alpha, \dot r^\alpha, c_b), \quad c_b = \text{const}, \tag{8.2.31}$$

where $\mathcal{P}_a$ are such that

$$\frac{\partial \mathcal{P}}{\partial r^\alpha} = \frac{\partial P}{\partial r^\alpha}, \quad \frac{\partial \mathcal{P}}{\partial \dot r^\alpha} = 0$$

at the relative equilibria.

We now use the $V_k(r^\alpha, \dot r^\alpha)$ to construct a Lyapunov function to determine the conditions for asymptotic stability of the relative equilibrium $r^\alpha = r_0^\alpha$, $p_a = p_a^0$. We will do this in a fashion similar to that used by Chetaev [1959] and Bloch, Krishnaprasad, Marsden, and Ratiu [1994]. Without loss of generality, suppose that $g_{\alpha\beta}(r_0) = \delta_{\alpha\beta}$. Introduce the function

$$W_k = V_k + \epsilon \sum_{\alpha=1}^{\sigma} u^\alpha \dot r^\alpha.$$

Consider the following two manifolds at the equilibrium (r_0^α, p_a^0): the integral manifold of the transport equation

$$Q_{k^0} = \left\{ p_a = P_a(r^\alpha, k^0) \right\},$$

and the local invariant manifold

$$\mathcal{Q}_{c^0} = \left\{ p_a = \mathcal{P}_a(r^\alpha, \dot{r}^\alpha, c^0) \right\}.$$

Restrict the flow to the manifold $\mathcal{Q}_{c^0}$. Choose $(r^\alpha, \dot{r}^\alpha)$ as local coordinates on $\mathcal{Q}_{c^0}$, then V_{k^0} and W_{k^0} are functions defined on $\mathcal{Q}_{c^0}$. Since

$$\frac{\partial U_{k^0}}{\partial r^\alpha}(r_0) = \nabla_\alpha U(r_0, p_0) = 0$$

and

$$\frac{\partial^2 U_{k^0}}{\partial r^\alpha \partial r^\beta}(r_0) = \nabla_\alpha \nabla_\beta U(r_0, p_0) \gg 0,$$

the function V_{k^0} is positive definite in some neighborhood of the relative equilibrium $(r_0^\alpha, 0) \in \mathcal{Q}_{c^0}$. The same is valid for the function W_{k^0} if ϵ is small enough.

Now we show that $\dot{W}_{k^0}$ (as a function on $\mathcal{Q}_{c^0}$) is negative definite. Calculate the derivative of W_{k^0} along the flow:

$$\begin{aligned}
\dot{W}_{k^0} = g_{\alpha\beta}\dot{r}^\alpha \ddot{r}^\beta + \frac{1}{2}\dot{g}_{\alpha\beta}\dot{r}^\alpha \dot{r}^\beta + I^{ab}P_a\dot{P}_b \\
+ \frac{1}{2}\dot{I}^{ab}P_aP_b + \dot{V} + \epsilon \sum_{\alpha=1}^{\sigma}\left((\dot{r}^\alpha)^2 + u^\alpha \ddot{r}^\alpha\right). \qquad (8.2.32)
\end{aligned}$$

Using the explicit representation of equation (8.2.12), we obtain

$$\begin{aligned}
g_{\alpha\beta}\ddot{r}^\beta + \dot{g}_{\alpha\beta}\dot{r}^\beta = \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial r^\alpha}\dot{r}^\beta\dot{r}^\gamma - \frac{\partial V}{\partial r^\alpha} - \frac{1}{2}\frac{\partial I^{ab}}{\partial r^\alpha}\mathcal{P}_a\mathcal{P}_b - \mathcal{D}^c_{b\alpha}I^{bd}\mathcal{P}_c\mathcal{P}_d \\
- \mathcal{D}_{\beta\alpha b}I^{bc}\mathcal{P}_c\dot{r}^\beta - \mathcal{B}^c_{\alpha\beta}\mathcal{P}_c\dot{r}^\beta - \mathcal{K}_{\alpha\beta\gamma}\dot{r}^\beta\dot{r}^\gamma. \quad (8.2.33)
\end{aligned}$$

Therefore,

$$\begin{aligned}
& g_{\alpha\beta}\dot{r}^\alpha\ddot{r}^\beta + \frac{1}{2}\dot{g}_{\alpha\beta}\dot{r}^\alpha\dot{r}^\beta + I^{ab}P_a\dot{P}_b + \frac{1}{2}\dot{I}^{ab}P_aP_b + \dot{V} \\
& \quad = -\frac{1}{2}\dot{g}_{\alpha\beta}\dot{r}^\alpha\dot{r}^\beta + \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial r^\alpha}\dot{r}^\alpha\dot{r}^\beta\dot{r}^\gamma - \frac{\partial V}{\partial r^\alpha}\dot{r}^\alpha - \frac{1}{2}\frac{\partial I^{ab}}{\partial r^\alpha}\mathcal{P}_a\mathcal{P}_b\dot{r}^\alpha \\
& \quad - \mathcal{D}^c_{b\alpha}I^{bd}\mathcal{P}_c\mathcal{P}_d\dot{r}^\alpha - \mathcal{D}_{\beta\alpha b}I^{bc}\mathcal{P}_c\dot{r}^\alpha\dot{r}^\beta - \mathcal{B}^c_{\alpha\beta}\mathcal{P}_c\dot{r}^\alpha\dot{r}^\beta \\
& \quad - \mathcal{K}_{\alpha\beta\gamma}\dot{r}^\alpha\dot{r}^\beta\dot{r}^\gamma + I^{ab}\mathcal{D}^c_{b\alpha}P_aP_c\dot{r}^\alpha + \frac{1}{2}\frac{\partial I^{ab}}{\partial r^\alpha}P_aP_b\dot{r}^\alpha + \dot{V}.
\end{aligned}$$

Using skew-symmetry of $\mathcal{B}^c_{\alpha\beta}$ and $\mathcal{K}_{\alpha\beta\gamma}$ with respect to α, β and canceling the terms

$$-\frac{1}{2}\dot{g}_{\alpha\beta}\dot{r}^\alpha\dot{r}^\beta + \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial r^\alpha}\dot{r}^\alpha\dot{r}^\beta\dot{r}^\gamma - \frac{\partial V}{\partial r^\alpha}\dot{r}^\alpha + \dot{V},$$

we obtain

$$
\begin{aligned}
&g_{\alpha\beta}\dot{r}^{\alpha}\ddot{r}^{\beta} + \frac{1}{2}\dot{g}_{\alpha\beta}\dot{r}^{\alpha}\dot{r}^{\beta} + I^{ab}P_a\dot{P}_b + \frac{1}{2}\dot{I}^{ab}P_aP_b + \dot{V} \\
&\quad = -\mathcal{D}_{\beta\alpha b}I^{bc}\mathcal{P}_c\dot{r}^{\alpha}\dot{r}^{\beta} + \left(\frac{1}{2}\frac{\partial I^{ab}}{\partial r^{\alpha}} + I^{ac}\mathcal{D}^b_{c\alpha}\right)\left(P_aP_b - \mathcal{P}_a\mathcal{P}_b\right)\dot{r}^{\alpha}. \qquad (8.2.34)
\end{aligned}
$$

Substituting (8.2.34) in (8.2.32) and determining $\ddot{r}^{\alpha}$ from (8.2.33)

$$
\begin{aligned}
\dot{W}_{k^0} = &-\mathcal{D}_{\beta\alpha b}I^{bc}\mathcal{P}_c\dot{r}^{\alpha}\dot{r}^{\beta} + \epsilon\sum_{\alpha=1}^{\sigma}(\dot{r}^{\alpha})^2 \\
&- \epsilon\sum_{\gamma=1}^{\sigma}g^{\alpha\beta}u^{\gamma}\left(\frac{\partial V}{\partial r^{\alpha}} + \frac{1}{2}\frac{\partial I^{ab}}{\partial r^{\alpha}}\mathcal{P}_a\mathcal{P}_b + \mathcal{D}^c_{b\alpha}I^{bd}\mathcal{P}_c\mathcal{P}_d\right) \\
&+ \epsilon\sum_{\gamma=1}^{\sigma}g^{\alpha\gamma}u^{\gamma}\left(-\dot{g}_{\alpha\beta}\dot{r}^{\beta} + \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial r^{\alpha}}\dot{r}^{\beta}\dot{r}^{\gamma} - \mathcal{B}^c_{\alpha\beta}\mathcal{P}_c\dot{r}^{\beta}\right. \\
&\qquad\qquad\qquad \left. - \mathcal{D}_{\beta\alpha b}I^{bc}\mathcal{P}_c\dot{r}^{\beta} - \mathcal{K}_{\alpha\beta\gamma}\dot{r}^{\beta}\dot{r}^{\gamma}\right) \\
&+ \frac{1}{2}\frac{\partial I^{ab}}{\partial r^{\alpha}}\left(P_aP_b - \mathcal{P}_a\mathcal{P}_b\right)\dot{r}^{\alpha} + I^{ac}\mathcal{D}^b_{c\alpha}\left(P_aP_b - \mathcal{P}_a\mathcal{P}_b\right)\dot{r}^{\alpha}. \qquad (8.2.35)
\end{aligned}
$$

Since

$$
\frac{\partial V}{\partial r^{\alpha}} + \frac{1}{2}\frac{\partial I^{ab}}{\partial r^{\alpha}}\mathcal{P}_a\mathcal{P}_b + \mathcal{D}^c_{b\alpha}I^{bd}\mathcal{P}_c\mathcal{P}_d = 0
$$

at the equilibrium and the linear terms in the Taylor expansions of $\mathcal{P}$ and P are the same,

$$
\frac{\partial V}{\partial r^{\alpha}} + \frac{1}{2}\frac{\partial I^{ab}}{\partial r^{\alpha}}\mathcal{P}_a\mathcal{P}_b + \mathcal{D}^c_{b\alpha}I^{bd}\mathcal{P}_c\mathcal{P}_d = F_{\alpha\beta}u^{\beta} + \{\text{nonlinear terms}\}, \qquad (8.2.36)
$$

where

$$
\begin{aligned}
F_{\alpha\beta} &= \frac{\partial}{\partial r^{\beta}}\left(\frac{\partial V}{\partial r^{\alpha}} + \frac{1}{2}\frac{\partial I^{ab}}{\partial r^{\alpha}}P_aP_b + \mathcal{D}^c_{b\alpha}I^{bd}P_cP_d\right) \\
&= \frac{\partial^2 V}{\partial r^{\alpha}\partial r^{\beta}} + \frac{1}{2}\frac{\partial^2 I^{ab}}{\partial r^{\alpha}\partial r^{\beta}}P_aP_b + \frac{\partial I^{ab}}{\partial r^{\alpha}}P_a\frac{\partial P_b}{\partial r^{\beta}} \\
&\quad + \frac{\partial}{\partial r^{\beta}}\left(\mathcal{D}^c_{b\alpha}I^{bd}\right)P_cP_d + \mathcal{D}^c_{b\alpha}I^{bd}\left(\frac{\partial P_c}{\partial r^{\beta}}P_d + P_c\frac{\partial P_d}{\partial r^{\beta}}\right) \\
&= \frac{\partial^2 V}{\partial r^{\alpha}\partial r^{\beta}} + \frac{1}{2}\frac{\partial^2 I^{ab}}{\partial r^{\alpha}\partial r^{\beta}}P_aP_b + \frac{\partial I^{ab}}{\partial r^{\alpha}}P_a\mathcal{D}^c_{b\beta}P_c \\
&\quad + \frac{\partial}{\partial r^{\beta}}\left(\mathcal{D}^c_{b\alpha}I^{bd}\right)P_cP_d + \mathcal{D}^c_{b\alpha}I^{bd}\left(\mathcal{D}^a_{c\beta}P_d + P_c\mathcal{D}^a_{d\beta}P_a\right) \\
&= \nabla_{\alpha}\nabla_{\beta}U.
\end{aligned}
$$

In the last formula all the terms are evaluated at the equilibrium.

Taking into account that $g_{\alpha\beta} = \delta_{\alpha\beta} + O(u)$, that the Taylor expansion of $P_a P_b - \mathcal{P}_a \mathcal{P}_b$ starts from the terms of the second order, and using (8.2.36), we obtain from (8.2.35)

$$\dot{W}_{k^0} = -\mathcal{D}_{\beta\alpha b} I^{bc}(r_0) p_c^0 \dot{r}^\alpha \dot{r}^\beta - \epsilon F_{\alpha\beta} u^\alpha u^\beta + \epsilon \sum_{\alpha=1}^{\sigma} (\dot{r}^\alpha)^2 \\ - \epsilon \left(\mathcal{D}_{\beta\alpha b} I^{bc}(r_0) p_c^0 + \mathcal{B}^c_{\alpha\beta}(r_0) p_c^0 \right) u^\alpha \dot{r}^\beta + \{\text{cubic terms}\}.$$

Therefore, the condition $(\mathcal{D}_{\alpha\beta b} + \mathcal{D}_{\beta\alpha b}) I^{bc} p_c^0 \gg 0$ implies that $\dot{W}_{k^0}$ is negative definite if ϵ is small enough and positive. Thus, W_{k^0} is a Lyapunov function for the flow on $\mathcal{Q}_{c^0}$, and therefore the equilibrium $(r_0^\alpha, 0)$ for the flow on $\mathcal{Q}_{c^0}$ is asymptotically stable.

Using the same arguments we used in the proof of Theorem 8.2.5, we conclude that the equilibria on the nearby invariant manifolds $\mathcal{Q}_k$ are asymptotically stable as well. ■

There is an alternative way to state the above theorem, which uses the basic intuition we used to find the Lyapunov function.

Theorem 8.2.14 (The nonholonomic energy-momentum method). *Under the assumption that H2 holds, the point $q_e = (r_0^\alpha, p_a^0)$ is a relative equilibrium if and only if there is a $\xi \in \mathfrak{g}^{q_e}$ such that q_e is a critical point of the augmented energy $E_\xi = E - \langle p - P(r,k), \xi \rangle$. Assume that*

(i) *$\delta^2 E_\xi$ restricted to $T_{q_e} Q_k$ is positive definite (here δ denotes differentiation by all variables except ξ);*

(ii) *the quadratic form defined by the flow derivative of the augmented energy is negative definite at q_e.*

Then H3 holds and this equilibrium is Lyapunov stable and asymptotically stable in the directions of due to the invariant manifolds (8.2.31).

Proof. We have already shown in Theorem 8.2.6 that positive definiteness of $\delta^2 E_\xi|_{T_{q_e} Q_k}$ is equivalent to the condition $\nabla_\alpha \nabla_\beta U \gg 0$. To complete the proof, we need to show that the requirement (ii) of the theorem is equivalent to the condition $(\mathcal{D}_{\alpha\beta b} + \mathcal{D}_{\beta\alpha b}) I^{bc}(r_0) p_c^0 \gg 0$. Compute the flow derivative of E_ξ:

$$\dot{E}_\xi = \dot{E} - \langle \dot{p} - \dot{P}, \xi \rangle = -\langle \dot{p} - \dot{P}, \xi \rangle \\ = (\mathcal{D}^b_{a\alpha} p_b \dot{r}^\alpha + \mathcal{D}_{\alpha\beta a} \dot{r}^\alpha \dot{r}^\beta - \mathcal{D}^b_{a\alpha} P_b \dot{r}^\alpha) \xi^a.$$

Since at the equilibrium $p = P$, $\xi^a = I^{ab} p_b$, and $\dot{E} = 0$ (Theorem 8.2.4), we obtain

$$\dot{E}_\xi = -\mathcal{D}_{\alpha\beta a} I^{ab}(r_0) p_b^0 \dot{r}^\alpha \dot{r}^\beta.$$

The condition $\dot{E}_\xi \gg 0$ is thus equivalent to $(\mathcal{D}_{\alpha\beta b} + \mathcal{D}_{\beta\alpha b}) I^{bc}(r_0) p_c^0 \gg 0$. ■

For some examples, such as the roller racer, we need to consider a degenerate case of the above analysis. Namely, we consider a nongeneric case, when $U = \frac{1}{2} I^{ab}(r) p_a p_b$ (the original system has no potential energy), and the components of the locked inertia tensor I^{ab} satisfy the condition

$$\frac{1}{2} \frac{\partial I^{ab}}{\partial r^\alpha} + I^{ac} \mathcal{D}^b_{c\alpha} = 0. \tag{8.2.37}$$

Consequently, the covariant derivatives of the amended potential are equal to zero, and the equations of motion (8.2.12), (8.2.13) become

$$\frac{d}{dt} \left(g_{\alpha\beta} \dot{r}^\beta \right) - \frac{1}{2} \frac{\partial g_{\beta\gamma}}{\partial r^\alpha} \dot{r}^\beta \dot{r}^\gamma = -\mathcal{D}_{\beta\alpha b} I^{bc} p_c \dot{r}^\beta - \mathcal{B}^c_{\alpha\beta} p_c \dot{r}^\beta - \mathcal{K}_{\alpha\beta\gamma} \dot{r}^\beta \dot{r}^\gamma, \tag{8.2.38}$$

$$\frac{d}{dt} p_b = \mathcal{D}^c_{b\alpha} p_c \dot{r}^\alpha + \mathcal{D}_{\alpha\beta b} \dot{r}^\alpha \dot{r}^\beta. \tag{8.2.39}$$

Thus, we obtain an $m+\sigma$-dimensional *manifold* of equilibria $r = r_0$, $p = p_0$ of these equations. Further, we cannot apply Theorem 8.2.13 because the condition $\nabla^2 U \gg 0$ fails. However, we can do a similar type of stability analysis as follows.

As before, set

$$V_k = E(r, \dot{r}, P(r,k)) = \frac{1}{2} g_{\alpha\beta} \dot{r}^\alpha \dot{r}^\beta + \frac{1}{2} I^{ab}(r) P_a(r,k) P_b(r,k).$$

Note that P satisfies the equation

$$\frac{\partial P_b}{\partial r^\alpha} = \mathcal{D}^c_{b\alpha} P_c,$$

which implies that

$$\begin{aligned} \frac{\partial}{\partial r^\alpha} \left(\frac{1}{2} I^{ab}(r) P_a P_b \right) &= \frac{1}{2} \frac{\partial I^{ab}}{\partial r^\alpha} P_a P_b + I^{ab} P_a \frac{\partial P_b}{\partial r^\alpha} \\ &= \left(\frac{1}{2} \frac{\partial I^{ab}}{\partial r^\alpha} + I^{ab} \mathcal{D}^c_{b\alpha} \right) P_a P_b = 0. \end{aligned}$$

Therefore

$$\frac{1}{2} I^{ab} P_a P_b = \text{const}$$

and

$$V_k = \frac{1}{2} g_{\alpha\beta} \dot{r}^\alpha \dot{r}^\beta$$

(up to an additive constant). Thus, V_k is a positive definite function with respect to $\dot{r}$. Compute $\dot{V}_k$:

$$\dot{V}_k = g_{\alpha\beta} \dot{r}^\alpha \ddot{r}^\beta + \dot{g}_{\alpha\beta} \dot{r}^\alpha \dot{r}^\beta = -\mathcal{D}_{\beta\alpha b} I^{bc} p_c \dot{r}^\alpha \dot{r}^\beta + O(\dot{r}^3).$$

Suppose that $(\mathcal{D}_{\alpha\beta b} + \mathcal{D}_{\beta\alpha b})(r_0)I^{bc}(r_0)p_c^0 \gg 0$. Now the linearization of equations (8.2.38) and (8.2.39) about the relative equilibria given by setting $\dot{r} = 0$ has $m + \sigma$ zero eigenvalues corresponding to the r and p directions. Since the matrix corresponding to $\dot{r}$-directions of the linearized system is of the form $D + G$, where D is positive definite and symmetric (in fact, $D = \frac{1}{2}(\mathcal{D}_{\alpha\beta b} + \mathcal{D}_{\beta\alpha b})(r_0)I^{bc}(r_0)p_c^0$) and G is skew-symmetric, the determinant of $D+G$ is not equal to zero. This follows from the observation that $x^t(D+G)x = x^t Dx > 0$ for D positive-definite and G skew-symmetric. Thus using Theorem 8.2.10, we find that the equations of motion have local integrals

$$r = \mathcal{R}(\dot{r}, k), \quad p = \mathcal{P}(\dot{r}, k).$$

Therefore V_k restricted to a common level set of these integrals is a Lyapunov function for the restricted system. Thus, an equilibrium $r = r_0$, $p = p_0$ is stable with respect to r, $\dot{r}$, p and asymptotically stable with respect to $\dot{r}$ if

$$(\mathcal{D}_{\alpha\beta b} + \mathcal{D}_{\beta\alpha b})I^{bc}(r_0)p_c^0 \gg 0. \tag{8.2.40}$$

Summarizing, we have:

Theorem 8.2.15. *Under assumptions H2 if $U = 0$ and the conditions* (8.2.37) *and* (8.2.40) *hold, the nonholonomic equations of motion have an $m+\sigma$-dimensional manifold of equilibria parametrized by r and p. An equilibrium $r = r_0$, $p = p_0$ is stable with respect to r, $\dot{r}$, p and asymptotically stable with respect to $\dot{r}$.*

The Roller Racer

The roller racer provides an illustration of Theorem 8.2.15. Recall that the Lagrangian and the constraints are

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_1\dot{\theta}^2 + \frac{1}{2}I_2(\dot{\theta} + \dot{\phi})^2$$

and

$$\begin{aligned} \dot{x} &= \cos\theta\left(\frac{d_1\cos\phi + d_2}{\sin\phi}\dot{\theta} + \frac{d_2}{\sin\phi}\dot{\phi}\right), \\ \dot{y} &= \sin\theta\left(\frac{d_1\cos\phi + d_2}{\sin\phi}\dot{\theta} + \frac{d_2}{\sin\phi}\dot{\phi}\right). \end{aligned}$$

The configuration space is $SE(2) \times SO(2)$ and, as observed earlier, the Lagrangian and the constraints are invariant under the left action of $SE(2)$ on the first factor of the configuration space.

The nonholonomic momentum is

$$p = m(d_1\cos\phi + d_2)(\dot{x}\cos\theta + \dot{y}\sin\theta) + [(I_1 + I_2)\dot{\theta} + I_2\dot{\phi}]\sin\phi.$$

See Tsakiris [1995] for details of this calculation. The momentum equation is

$$\dot{p} = \frac{((I_1 + I_2)\cos\phi - md_1(d_1\cos\phi + d_2))}{m(d_1\cos\phi + d_2)^2 + (I_1 + I_2)\sin^2\phi}\, p\dot{\phi} + \frac{(d_1 + d_2\cos\phi)(I_2 d_1\cos\phi - I_1 d_2)}{m(d_1\cos\phi + d_2)^2 + (I_1 + I_2)\sin^2\phi}\,\dot{\phi}^2.$$

Rewriting the Lagrangian using p instead of $\dot{\theta}$, we obtain the energy function for the roller racer:

$$E = \frac{1}{2}g(\phi)\dot{\phi}^2 + \frac{1}{2}I(\phi)p^2,$$

where

$$g(\phi) = I_2 + \frac{md_2^2}{\sin^2\phi} - \frac{\left[m(d_1\cos\phi + d_2)d_2 + I_2\sin^2\phi\right]^2}{\sin^2\phi\left[m(d_1\cos\phi + d_2)^2 + (I_1 + I_2)\sin^2\phi\right]}$$

and

$$I(\phi) = \frac{1}{(d_1\cos\phi + d_2)^2 + (I_1 + I_2)\sin^2\phi}. \qquad (8.2.41)$$

The amended potential is given by

$$U = \frac{p^2}{2[(d_1\cos\phi + d_2)^2 + (I_1 + I_2)\sin^2\phi]},$$

which follows directly from (8.2.11) and (8.2.41).

Straightforward computations show that the locked inertia tensor $I(\phi)$ satisfies the condition (8.2.37), and thus the roller racer has a two dimensional manifold of relative equilibria parametrized by ϕ and p. These relative equilibria are motions of the roller racer in circles about the point of intersection of lines through the axles. For such motions, p is the system momentum about this point and ϕ is the relative angle between the two bodies.

Therefore, we may apply the energy-momentum stability conditions (8.2.40) obtained above for the degenerate case. Multiplying the coefficient of the nontransport term of the momentum equation, evaluated at ϕ_0, by $I(\phi)p_0$ and omitting a positive denominator, we obtain the condition for stability of a relative equilibrium $\phi = \phi_0$, $p = p_0$ of the roller racer:

$$(d_1 + d_2\cos\phi_0)(I_2 d_1\cos\phi_0 - I_1 d_2)p_0 > 0.$$

Note that this equilibrium is stable modulo $SE(2)$ and in addition asymptotically stable with respect to $\dot{\phi}$.

Nonlinear Stability by the Lyapunov-Malkin Method

Here we study stability using the Lyapunov-Malkin approach; correspondingly, we do not a priori assume the hypotheses H1 (skewness of $\mathcal{D}_{\alpha\beta b}$ in α, β), H2 (a curvature is zero) or H3 (definiteness of second variations). Rather, at the end of this section we will make eigenvalue hypotheses.

We consider the most general case, when the connection due to the transport part of the momentum equation is not necessary flat and when the nontransport terms of the momentum equation are not equal to zero. In the case when $\mathfrak{g}^{q_e}$ is commutative, this analysis was done by Karapetyan [1980]. Our main goal here is to show that this method extends to the noncommutative case as well.

We start by computing the linearization of equations (8.2.12) and (8.2.13). Introduce coordinates u^α, v^α, and w_a in the neighborhood of the equilibrium $r = r_0$, $p = p_0$ by the formulae

$$r^\alpha = r_0^\alpha + u^\alpha, \quad \dot{r}^\alpha = v^\alpha, \quad p_a = p_a^0 + w_a.$$

The linearized momentum equation is

$$\dot{w}_b = \mathcal{D}_{b\alpha}^c(r_0)p_c^0 v^\alpha.$$

To find the linearization of (8.2.12), we start by rewriting its right hand side explicitly. Since $R = \frac{1}{2} g_{\alpha\beta}\dot{r}^\alpha \dot{r}^\beta - \frac{1}{2} I^{ab} p_a p_b - V$, equation (8.2.12) becomes

$$\begin{aligned} & g_{\alpha\beta}\ddot{r}^\beta + \dot{g}_{\alpha\beta}\dot{r}^\alpha\dot{r}^\beta - \frac{1}{2}\frac{\partial g_{\beta\gamma}}{\partial r^\alpha}\dot{r}^\beta\dot{r}^\gamma + \frac{1}{2}\frac{\partial I^{ab}}{\partial r^\alpha} p_a p_b + \frac{\partial V}{\partial r^\alpha} \\ & \quad = -\mathcal{D}_{c\alpha}^a I^{cd} p_a p_d - \mathcal{D}_{\beta\alpha c} I^{ca} p_a \dot{r}^\beta - \mathcal{B}_{\alpha\beta}^a p_a \dot{r}^\beta - \mathcal{K}_{\alpha\beta\gamma}\dot{r}^\beta\dot{r}^\gamma. \end{aligned}$$

Keeping only the linear terms, we obtain

$$\begin{aligned} & g_{\alpha\beta}(r_0)\ddot{r}^\beta + \frac{\partial^2 V}{\partial r^\alpha \partial r^\beta}(r_0)\, u^\beta + \frac{1}{2}\frac{\partial^2 I^{ab}}{\partial r^\alpha \partial r^\beta}(r_0) p_a^0 p_b^0\, u^\beta + \frac{\partial I^{ab}}{\partial r^\alpha}(r_0) p_a^0\, w_b \\ & \quad = -\mathcal{D}_{c\alpha}^a I^{cd}(r_0) p_a^0\, w_d - \mathcal{D}_{c\alpha}^a I^{cd}(r_0) p_d^0\, w_a - \frac{\partial \mathcal{D}_{c\alpha}^a I^{cd}}{\partial r^\beta}(r_0) p_a^0 p_d^0\, u^\beta \\ & \qquad\qquad - \mathcal{D}_{\beta\alpha c} I^{ca}(r_0) p_a^0\, v^\beta - \mathcal{B}_{\alpha\beta}^a(r_0) p_a^0\, v^\beta. \end{aligned}$$

Next, introduce matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ by

$$\begin{aligned} \mathcal{A}_{\alpha\beta} &= -\left(\mathcal{D}_{\beta\alpha c} I^{ca}(r_0) p_a^0 + \mathcal{B}_{\alpha\beta}^a(r_0) p_a^0\right), \\ \mathcal{B}_{\alpha\beta} &= -\left(\frac{\partial^2 V}{\partial r^\alpha \partial r^\beta}(r_0) + \frac{1}{2}\frac{\partial^2 I^{ab}}{\partial r^\alpha \partial r^\beta}(r_0) p_a^0 p_b^0 + \frac{\partial \mathcal{D}_{c\alpha}^a I^{cb}}{\partial r^\beta}(r_0) p_a^0 p_b^0\right), \end{aligned} \tag{8.2.42}$$

$$\mathcal{C}_\alpha^a = -\left(\frac{\partial I^{ab}}{\partial r^\alpha}(r_0) p_b^0 + \mathcal{D}_{c\alpha}^b I^{ca}(r_0) p_b^0 + \mathcal{D}_{c\alpha}^a I^{cb}(r_0) p_b^0\right), \tag{8.2.43}$$

$$\mathcal{D}_{a\alpha} = \mathcal{D}_{a\alpha}^c(r_0) p_c^0. \tag{8.2.44}$$

Using these notations and making a choice of r^α such that $g_{\alpha\beta}(r_0) = \delta_{\alpha\beta}$, we can represent the equation of motion in the form

$$\dot{u}^\alpha = v^\alpha, \tag{8.2.45}$$

$$\dot{v}^\alpha = \mathcal{A}^\alpha_\beta v^\beta + \mathcal{B}^\alpha_\beta u^\beta + \mathcal{C}^{\alpha a} w_a + \mathcal{V}^\alpha(u, v, w), \tag{8.2.46}$$

$$\dot{w}_a = \mathcal{D}_{a\alpha} v^\alpha + \mathcal{W}_a(u, v, w), \tag{8.2.47}$$

where $\mathcal{V}$ and $\mathcal{W}$ stand for nonlinear terms, and where

$$\mathcal{A}^\alpha_\beta = \delta^{\alpha\gamma}\mathcal{A}_{\gamma\beta},$$
$$\mathcal{B}^\alpha_\beta = \delta^{\alpha\gamma}\mathcal{B}_{\gamma\beta},$$
$$\mathcal{C}^{\alpha a} = \delta^{\alpha\gamma}\mathcal{C}^a_\gamma.$$

(Or $\mathcal{A}^\alpha_\beta = g^{\alpha\gamma}\mathcal{A}_{\gamma\beta}$, $\mathcal{B}^\alpha_\beta = g^{\alpha\gamma}\mathcal{B}_{\gamma\beta}$, $\mathcal{C}^{\alpha a} = g^{\alpha\gamma}\mathcal{C}^a_\gamma$ if $g_{\alpha\beta}(r_0) \neq \delta_{\alpha\beta}$.) Note that

$$\mathcal{W}_a = \left(\mathcal{D}^c_{a\alpha}(p^0_c + w_c) - \mathcal{D}_{a\alpha}\right) v^\alpha + \mathcal{D}_{\alpha\beta a} v^\alpha v^\beta. \tag{8.2.48}$$

The next step is to eliminate the linear terms from (8.2.47). Putting

$$w_a = \mathcal{D}_{a\alpha} u^\alpha + z_a,$$

(8.2.47) becomes

$$\dot{z}_a = \mathcal{Z}_a(u, v, z),$$

where $\mathcal{Z}_a(u, v, z)$ represents nonlinear terms. Formula (8.2.48) leads to

$$\mathcal{Z}_a(u, v, z) = \mathcal{Z}_{a\alpha}(u, v, z) v^\alpha.$$

In particular, $\mathcal{Z}_a(u, 0, z) = 0$. The equations (8.2.45), (8.2.46), (8.2.47) in the variables u, v, z become

$$\begin{aligned}
\dot{u}^\alpha &= v^\alpha,\\
\dot{v}^\alpha &= \mathcal{A}^\alpha_\beta v^\beta + (\mathcal{B}^\alpha_\beta + \mathcal{C}^{\alpha a}\mathcal{D}_{a\beta}) u^\beta + \mathcal{C}^{\alpha a} z_a + \mathcal{V}^\alpha(u, v, z_a + \mathcal{D}_{a\alpha} u^\alpha),\\
\dot{z}_a &= \mathcal{Z}_a(u, v, w).
\end{aligned}$$

Using Lemma 8.2.12, we find a substitution $x^\alpha = u^\alpha + \phi^\alpha(z)$, $y^\alpha = v^\alpha$ such that in the variables x, y, z we obtain

$$\begin{aligned}
\dot{x}^\alpha &= y^\alpha + X^\alpha(x, y, z),\\
\dot{y}^\alpha &= \mathcal{A}^\alpha_\beta y^\beta + (\mathcal{B}^\alpha_\beta + \mathcal{C}^{\alpha a}\mathcal{D}_{a\beta}) x^\beta + Y^\alpha(x, y, z),\\
\dot{z}_a &= Z_a(x, y, z),
\end{aligned} \tag{8.2.49}$$

where the nonlinear terms $X(x, y, z)$, $Y(x, y, z,)$, $Z(x, y, z)$ vanish if $x = 0$ and $y = 0$. Therefore, we can apply the Lyapunov-Malkin Theorem and conclude:

Theorem 8.2.16. *The equilibrium $x = 0$, $y = 0$, $z = 0$ of the system* (8.2.49) *is stable with respect to x, y, z and asymptotically stable with respect to x, y, if all eigenvalues of the matrix*

$$\begin{pmatrix} 0 & I \\ \mathcal{B} + \mathcal{C}\mathcal{D} & \mathcal{A} \end{pmatrix} \tag{8.2.50}$$

have negative real parts.

The Lyapunov-Malkin and the Energy-Momentum Methods

Here we introduce a forced linear Lagrangian system associated with our nonholonomic system. The linear system will have the matrix (8.2.50). Then we compare the Lyapunov-Malkin approach and the energy-momentum approach for systems satisfying hypothesis H2.

Thus, we consider the system with matrix (8.2.50)

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= \mathcal{A}y + (\mathcal{B} + \mathcal{C}\mathcal{D})x. \end{aligned} \tag{8.2.51}$$

According to Theorem 8.2.16, the equilibrium $x = 0$, $y = 0$, $z = 0$ of (8.2.49) is stable with respect to x, y, z and asymptotically stable with respect to x, y if and only if the equilibrium $x = 0$, $y = 0$ of (8.2.51) is asymptotically stable. System (8.2.51) may be viewed as a linear unconstrained Lagrangian system with additional forces imposed on it. Put

$$\begin{aligned} C &= -\frac{1}{2}\left((\mathcal{B} + \mathcal{C}\mathcal{D}) + (\mathcal{B} + \mathcal{C}\mathcal{D})^t\right), \\ F &= \frac{1}{2}\left((\mathcal{B} + \mathcal{C}\mathcal{D}) - (\mathcal{B} + \mathcal{C}\mathcal{D})^t\right), \\ D &= -\frac{1}{2}\left(\mathcal{A} + \mathcal{A}^t\right), \\ G &= \frac{1}{2}\left(\mathcal{A} - \mathcal{A}^t\right). \end{aligned}$$

The equations become

$$\ddot{x}^\alpha = -C^\alpha_\beta x^\beta + F^\alpha_\beta x^\beta - D^\alpha_\beta \dot{x}^\beta + G^\alpha_\beta \dot{x}^\beta. \tag{8.2.52}$$

These equations are the Euler-Lagrange equations with dissipation and forcing for the Lagrangian

$$L = \frac{1}{2}\sum_{\alpha=1}^{\sigma}(\dot{x}^\alpha - G^\alpha_\beta x^\beta)^2 - \frac{1}{2}C_{\alpha\beta}x^\alpha x^\beta,$$

with the Rayleigh dissipation function

$$\frac{1}{2}D_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta,$$

and the nonconservative forces

$$F^\alpha_\beta x^\beta.$$

Note that $D_{\alpha\beta} = (\mathcal{D}_{\alpha\beta b} + \mathcal{D}_{\beta\alpha b}) I^{ab}(r_0) p^0_a$.

The next theorem explains how to compute the matrices C and F using the amended potential of our nonholonomic system.

Theorem 8.2.17. *The entries of the matrices C and F in the dissipative forced system* (8.2.52)*, which is equivalent to the linear system* (8.2.51)*, are*

$$C_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha\nabla_\beta + \nabla_\beta\nabla_\alpha)U(r_0, p_0), \quad F_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)U(r_0, p_0).$$

Proof. Recall that the operators of covariant differentiation due to the transport equation are (see (8.2.17))

$$\nabla_\alpha = \frac{\partial}{\partial r^\alpha} + \mathcal{D}^c_{a\alpha} p_c \frac{\partial}{\partial p_a}.$$

Consequently,

$$\begin{aligned}
\nabla_\beta\nabla_\alpha &= \nabla_\beta\left(\frac{\partial}{\partial r^\alpha} + \mathcal{D}^c_{a\alpha} p_c \frac{\partial}{\partial p_a}\right) \\
&= \left(\frac{\partial}{\partial r^\beta} + \mathcal{D}^d_{b\beta} p_d \frac{\partial}{\partial p_b}\right)\left(\frac{\partial}{\partial r^\alpha} + \mathcal{D}^c_{a\alpha} p_c \frac{\partial}{\partial p_a}\right) \\
&= \frac{\partial^2}{\partial r^\beta \partial r^\alpha} + \frac{\partial}{\partial r^\beta}\left(\mathcal{D}^c_{a\alpha}\frac{\partial}{\partial p_a}\right) p_c + \mathcal{D}^d_{b\beta} p_d \frac{\partial^2}{\partial p_b \partial r^\alpha} \\
&\qquad + \mathcal{D}^d_{b\beta} p_d \mathcal{D}^c_{a\alpha} \frac{\partial}{\partial p_b}\left(p_c \frac{\partial}{\partial p_a}\right).
\end{aligned}$$

Therefore, for the amended potential $U = V + \frac{1}{2} I^{ab} p_a p_b$ we obtain

$$\begin{aligned}
\nabla_\beta\nabla_\alpha U &= \frac{\partial^2 V}{\partial r^\beta \partial r^\alpha} + \frac{1}{2}\frac{\partial^2 I^{ab}}{\partial r^\beta \partial r^\alpha} p_a p_b + \frac{\partial}{\partial r^\beta}\left(\mathcal{D}^c_{a\alpha} I^{ab}\right) p_c p_d \\
&\quad + \mathcal{D}^d_{b\beta}\frac{\partial I^{ab}}{\partial r^\alpha} p_a p_d + \mathcal{D}^d_{b\beta}\mathcal{D}^c_{a\alpha} I^{ab} p_c p_d + \mathcal{D}^d_{b\beta}\mathcal{D}^b_{a\alpha} I^{ac} p_c p_d.
\end{aligned}$$

Formulae (8.2.42), (8.2.43), and (8.2.44) imply that

$$\begin{aligned}
(\mathcal{BC} + \mathcal{D})_{\alpha\beta} &= \frac{\partial^2 V}{\partial r^\beta \partial r^\alpha}(r_0) + \frac{1}{2}\frac{\partial^2 I^{ab}}{\partial r^\beta \partial r^\alpha}(r_0) p^0_a p^0_b \\
&\quad + \frac{\partial}{\partial r^\beta}\left(\mathcal{D}^c_{a\alpha} I^{ab}\right)(r_0) p^0_c p^0_d + \mathcal{D}^d_{b\beta}\frac{\partial I^{ab}}{\partial r^\alpha}(r_0) p^0_a p^0_d \\
&\quad + \mathcal{D}^d_{b\beta}\mathcal{D}^c_{a\alpha} I^{ab}(r_0) p^0_c p^0_d + \mathcal{D}^d_{b\beta}\mathcal{D}^b_{a\alpha} I^{ac}(r_0) p^0_c p^0_d \\
&= \nabla_\beta\nabla_\alpha U(r_0, p_0).
\end{aligned}$$

Therefore

$$C_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha \nabla_\beta + \nabla_\beta \nabla_\alpha) U(r_0, p_0), \quad F_{\alpha\beta} = \frac{1}{2}(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) U(r_0, p_0).$$

■

Observe that the equilibrium $x = 0$, $y = 0$, $z = 0$ of (8.2.49) is stable with respect to x, y, z and asymptotically stable with respect to x, y if and only if the equilibrium $x = 0$, $y = 0$ of the above linear Lagrangian system is asymptotically stable. The condition for stability of the equilibrium $r = r_0$, $p = p_0$ of our nonholonomic system becomes: all eigenvalues of the matrix

$$\begin{pmatrix} 0 & I \\ \nabla_\beta \nabla_\alpha U(r_0, p_0) & -(\mathcal{D}_{\beta\alpha b} I^{ab}(r_0) + \mathcal{B}^a_{\alpha\beta}(r_0)) p^0_a \end{pmatrix}$$

have negative real parts.

If the transport equation is integrable (hypothesis H2), then the operators ∇_α and ∇_β commute, and the corresponding linear Lagrangian system (8.2.52) has no nonconservative forces imposed on it. In this case the sufficient conditions for stability are given by the Thompson Theorem (Thompson and Tait [1987], Chetaev[1959]): the equilibrium $x = 0$ of (8.2.52) is asymptotically stable if the matrices C and D are positive definite. These conditions are identical to the energy-momentum conditions for stability obtained in Theorem 8.2.13. Notice that if C and D are positive definite, then the matrix (8.2.50) is positive definite. This implies that the matrix A in Theorem 8.2.10 has spectrum in the left half plane. Further, our coordinate transformations here give the required form for the nonlinear terms of Theorem 8.2.10. *Therefore, the above analysis shows that hypothesis H3 implies the hypotheses of Theorem 8.2.10.*

Remark. On the other hand (cf. Chetaev [1959]), if the matrix C is not positive definite (and thus the equilibrium of the system $\ddot{x} = -Cx$ is unstable), and the matrix D is degenerate, then in certain cases the equilibrium of the equations $\ddot{x} = -Cx - D\dot{x} + G\dot{x}$ may be stable. Therefore, the conditions of Theorem 8.2.13 are sufficient, but not necessary.

The Rattleback

Here we outline the stability theory of the rattleback to illustrate the results discussed above. The details may be found in Karapetyan [1980, 1981] and Markeev [1992].

Recall that the Lagrangian and the constraints are

$$\begin{aligned}
L = {} & \frac{1}{2}\left[A\cos^2\psi + B\sin^2\psi + m(\gamma_1\cos\theta - \zeta\sin\theta)^2\right]\dot{\theta}^2 \\
& + \frac{1}{2}\left[(A\sin^2\psi + B\cos^2\psi)\sin^2\theta + C\cos^2\theta\right]\dot{\phi}^2 \\
& + \frac{1}{2}\left(C + m\gamma_2^2\sin^2\theta\right)\dot{\psi}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \\
& + m(\gamma_1\cos\theta - \zeta\sin\theta)\gamma_2\sin\theta\,\dot{\theta}\dot{\psi} + (A-B)\sin\theta\sin\psi\cos\psi\,\dot{\theta}\dot{\phi} \\
& + C\cos\theta\,\dot{\phi}\dot{\psi} + mg(\gamma_1\sin\theta + \zeta\cos\theta)
\end{aligned}$$

and

$$\dot{x} = \alpha_1\dot{\theta} + \alpha_2\dot{\psi} + \alpha_3\dot{\phi}, \quad \dot{y} = \beta_1\dot{\theta} + \beta_2\dot{\psi} + \beta_3\dot{\phi},$$

where the terms were defined in §8.2.

Using the Lie algebra element corresponding to the generator $\xi_Q = \alpha_3\partial_x + \beta_3\partial_y + \partial_\phi$ we find the nonholonomic momentum to be

$$\begin{aligned}
p = {} & I(\theta,\psi)\dot{\phi} + [(A-B)\sin\theta\sin\psi\cos\psi - m(\gamma_1\sin\theta + \zeta\cos\theta)\gamma_2]\,\dot{\theta} \\
& + \left[C\cos\theta + m(\gamma_2^2\cos\theta + \gamma_1(\gamma_1\cos\theta - \zeta\sin\theta))\right]\dot{\psi},
\end{aligned}$$

where

$$\begin{aligned}
I(\theta,\psi) = {} & (A\sin^2\psi + B\cos^2\psi)\sin^2\theta + C\cos^2\theta \\
& + m(\gamma_2^2 + (\gamma_1\cos\theta - \zeta\sin\theta)^2)..
\end{aligned}$$

The amended potential becomes

$$U = \frac{p^2}{2I(\theta,\psi)} - mg(\gamma_1\sin\theta + \zeta\cos\theta).$$

The relative equilibria of the rattleback are

$$\theta = \theta_0, \quad \psi = \psi_0, \quad p = p_0$$

where θ_0, ψ_0, p_0 satisfy the conditions

$$\begin{aligned}
& mg(\gamma_1\cos\theta_0 - \zeta\sin\theta_0)I^2(\theta_0,\psi_0) \\
& \quad + \left[(A\sin^2\psi_0 + B\cos^2\psi_0 - C)\sin\theta_0\cos\theta_0\right. \\
& \quad \left. - m(\gamma_1\cos\theta_0 - \zeta\sin\theta_0)(\gamma_1\sin\theta_0 + \zeta\cos\theta_0)\right]p_0^2 = 0, \\
& mg\gamma_2 I^2(\theta_0,\psi_0) + [(A-B)\sin\theta_0\sin\psi_0\cos\psi_0 \\
& \quad - m\gamma_2(\gamma_1\sin\theta_0 + \zeta\cos\theta_0)]\,p_0^2 = 0,
\end{aligned}$$

which are derived from $\nabla_\theta U = 0$, $\nabla_\psi U = 0$.

In particular, consider the relative equilibria

$$\theta = \frac{\pi}{2}, \quad \psi = 0, \quad p = p_0,$$

that represent the rotations of the rattleback about the vertical axis of inertia. For such relative equilibria $\xi = \zeta = 0$, and therefore the conditions for existence of relative equilibria are trivially satisfied with an arbitrary value of p_0. Omitting the computations of the linearized equations for the rattleback, which have the form discussed in Section 8.2.3 (see Karapetyan [1980] for details), and the corresponding characteristic polynomial, we just state here the Routh-Hurwitz conditions for all eigenvalues to have negative real parts:

$$\left(R - P\frac{p_0^2}{B^2}\right)\frac{p_0^2}{B^2} - S > 0, \qquad S > 0, \tag{8.2.53}$$

$$(A - C)(r_2 - r_1)p_0 \sin\alpha \cos\alpha > 0. \tag{8.2.54}$$

If these conditions are satisfied, then the relative equilibrium is stable, and it is asymptotically stable with respect to θ, $\dot{\theta}$, ψ, $\dot{\psi}$.

In the above formulae r_1, r_2 stand for the radii of curvature of the body at the contact point, α is the angle between horizontal inertia axis ξ and the r_1-curvature direction, and

$$\begin{aligned}
P = {} & (A + ma^2)(C + ma^2), \\
R = {} & \big[(A + C - B + 2ma^2)^2 \\
& \qquad - (A + C - B + 2ma^2)ma(r_1 + r_2) + m^2a^2r_1r_2\big]\frac{p_0^2}{B^2} \\
& - \left[(A - B)\frac{p_0^2}{B^2} + m(a - r_1\sin^2\alpha - r_2\cos^2\alpha)\left(g + a\frac{p_0^2}{B^2}\right)\right](A + ma^2) \\
& - \left[(C - B)\frac{p_0^2}{B^2} + m(a - r_2\sin^2\alpha - r_1\cos^2\alpha)\left(g + a\frac{p_0^2}{B^2}\right)\right](C + ma^2), \\
S = {} & (A - B)(C - B)\frac{p_0^4}{B^4} + m^2(a - r_1)(a - r_2)\left(g + m\frac{p_0^2}{B^2}\right)^2 \\
& + m\frac{p_0^2}{B^2}\left(g + a\frac{p^2}{B^2}\right)\big[A(a - r_1\cos^2\alpha - r_2\sin^2\alpha) \\
& + C\,(a - r_1\sin^2\alpha - r^2\cos^2\alpha) - B(2a - r_1 - r_2)\big].
\end{aligned}$$

Conditions (8.2.53) impose restrictions on the mass distribution, the magnitude of the angular velocity, and the shape of the rattleback only. Condition (8.2.54) distinguishes the direction of rotation corresponding to the stable relative equilibrium. The rotation will be stable if the largest (smallest) principal inertia axis precedes the largest (smallest) direction of curvature at the point of contact.

The rattleback is also capable of performing stationary rotations with its center of mass moving at a constant rate along a circle. A similar argument gives the stability conditions in this case. The details may be found in Karapetyan [1981] and Markeev [1992].

9
Energy Based Methods for Stabilization

9.1 Controlled Lagrangian Methods

9.1.1 Rigid Body with Rotors

In this sections we apply the Energy Casimir method to the stabilization of a rigid body with rotors (see Bloch, Krishnaprasad, Marsden and Sanchez [1992]).

Recall that this system consists of a rigid body with three rotors aligned along the principal axes of the rigid body. The rotors spin under the influence of a vector **u** of torques acting on the rotors.

The equations of motion are:

$$\begin{aligned}\dot{m} &= m \times \omega \\ \dot{l} &= u\end{aligned}$$

Here, u is the vector of torques exerted on the rotors, so that $u = 0$ corresponds to the uncontrolled case.

$I_1 > I_2 > I_3$ are the rigid body moments of inertia, $J_1 = J_2$ and J_3 are the rotor moments of inertia, $\omega = (\omega_1, \omega_2, \omega_3)$ is the angular velocity of the body, and α is the relative angle of the rotor. The momenta are

$$\begin{aligned}m_i &= (J_1 + I_1)\omega_i \quad i = 1, 2 \\ m_3 &= (J_3 + I_3)\omega_3 + J_3\dot{\alpha} \\ l_3 &= J_3(\omega_3 + \dot{\alpha}).\end{aligned}$$

Writing the equations out, we get

$$\begin{aligned}
\dot{m}_1 &= \left(\frac{1}{I_3} - \frac{1}{\lambda_2}\right) m_2 m_3 - \frac{l_3 m_2}{I_3} \\
\dot{m}_2 &= \left(\frac{1}{\lambda_1} - \frac{1}{I_3}\right) m_1 m_3 + \frac{l_3 m_1}{I_3} \\
\dot{m}_3 &= \left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) m_1 m_2 \doteq a_3 m_1 m_2 \\
\dot{l}_3 &= u.
\end{aligned}$$

Here, $\lambda_i = I_i + J_i$.

If $u = 0$, then l_3 is a constant of motion and the reduced system is Hamiltonian where

$$H = \frac{1}{2}\left(\frac{m_1^2}{\lambda_1} + \frac{m_2^2}{\lambda^2} + \frac{(m_3 - l_3)^2}{I_3}\right) + \frac{1}{2} l_3^2.$$

Choose the feedback control law

$$u = k a_3 m_1 m_2,$$

where k is a gain parameter, then the system retains the S^1 symmetry and $P_k = l_3 - k m_2$ is a new conserved quantity. The closed loop equations are

$$\begin{aligned}
\dot{m}_1 &= m_2\left(\frac{(1-k)m_3 - P_k}{I_3}\right) - \frac{m_3 m_2}{\lambda_2} \\
\dot{m}_2 &= -m_1\left(\frac{(1-k)m_3 - P_k}{I_3}\right) + \frac{m_1 m_3}{\lambda_1} \\
\dot{m}_3 &= a_3 m_1 m_2.
\end{aligned}$$

These equations are Hamiltonian with

$$H = \frac{1}{2}\left(\frac{m_1^2}{\lambda_1} + \frac{m_2^2}{\lambda_2} + \frac{((1-k)m_3 - P_k)^2}{(1-k)I_3}\right) + \frac{1}{2}\frac{P_k^2}{J_3(1-k)},$$

using the Lie-Poisson (rigid body) Poisson structure on $so(3)^*$.

Two special cases are of interest: $k = 0$, the uncontrolled case, and $k = J_3/\lambda_3$, the *driven case* where $\dot{\alpha} =$ constant.

Now consider the case $P = 0$ and the special equilibrium $(0, M, 0)$. For $k > 1 - J_3/\lambda_2$, the equilibrium $(0, M, 0)$ is stable. This is proved by the Energy-Casimir method. We look at $H + C$ where $C = \varphi(||m||^2)$. Pick φ so that

$$\delta(H + C)|_{(0,M,0)} = 0,$$

then one computes that $\delta^2(H+C)$ is negative definite if $k > 1 - J_3/\lambda_2$ and $\varphi''(M^2) < 0$.

9.1.2 The pendulum on a cart

9.2 Controlled Lagrangian methods for Nonholonomic Systems

9.3 Averaging and Energy Methods for the Dynamics of Mechanical Systems

9.3.1 Energy and Curvature

The chapter so far has given us an indication of the role energy-like functions play in the design and analysis of control laws for physical systems. Continuing the discussion, we consider a simple (in the sense of Abraham and Marsden, [1978]) mechanical system characterized by a kinetic energy, $\frac{1}{2}\dot{q}^T M(q)\dot{q}$, and a potential energy,$V(q)$. We have seen in Section that the equations of motion for this system are written:

Fix after Tony's reorganization.

$$\sum_{j=1}^{n} m_{ij}(q)\ddot{q}_j + \sum_{j,k=1}^{n} \Gamma_{kji}\dot{q}_k\dot{q}_j + \frac{\partial V}{\partial q_i} = 0, \quad (i = 1, \ldots, n), \tag{9.3.1}$$

where

$$\Gamma_{kji} = \frac{1}{2}(\frac{\partial m_{ij}}{\partial q_k} + \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{kj}}{\partial q_i}).$$

The Γ_{kji} are called *Christoffel symbols of the first kind*. The terms they define in equation (9.3.1) describe the inertial forces (Coriolis and centrifugal) which affect the system. It is clear from this equation that if M does not depend on q, $\Gamma_{kji} = 0$ for all k, j, i, and inertial forces will not play a role in the system's dynamics. An interesting question is "When can we find a change of coordinates such that in the new coordinate system the inertia matrix is constant?" Note that the inertia matrix M is constant (i.e. does not depend on the configuration variable q) if and only if there is a system of generalized coordinates $(y_1, \ldots, y_n)$ in terms of which the kinetic energy is expressed as $\frac{1}{2}\sum_{i=1}^{n} \dot{y}_i^2$. (*Proof:* In the case that M is constant, we let Y denote the constant, symmetric, positive definite matrix square root of M and let $y = Yq$. The $\dot{y} = Y\dot{q}$, and the kinetic energy is $\frac{1}{2}\sum_{i,j} m_{ij}\dot{q}_i\dot{q}_j = \frac{1}{2}\sum_i \dot{y}_i^2$. The reverse implication requires no proof.)

Hence, whether we can find the desired change of coordinates amounts to whether there exists a diffeomorphism $y = F(q)$ such that $\frac{1}{2}\dot{q}^T M\dot{q} = \frac{1}{2}\|\dot{y}\|^2$. Suppose such an F exists. Then $\dot{y} = \frac{\partial F}{\partial q}\dot{q}$, and $\frac{\partial F}{\partial q}^T \frac{\partial F}{\partial q} = M$. For such an M we compute

$$\frac{\partial m_{ij}}{\partial q_k} = \sum_{\ell=1}^{n} (\frac{\partial^2 F_\ell}{\partial q_i \partial q_j}\frac{\partial F_\ell}{\partial q_j} + \frac{\partial F_\ell}{\partial q_i}\frac{\partial^2 F_\ell}{\partial q_j \partial q_k}).$$

From this it follows from an easy calculation that

$$\begin{aligned}\Gamma_{kji} &= \frac{1}{2}\Big(\frac{\partial m_{ij}}{\partial q_k} + \frac{\partial m_{ik}}{\partial q_j} - \frac{\partial m_{kj}}{\partial q_i}\Big) \\ &= \sum_{\ell=1}^{n} \frac{\partial^2 F_\ell}{\partial q_j \partial q_k}\frac{\partial F_\ell}{\partial q_i}.\end{aligned} \tag{9.3.2}$$

As in Section xxx, the Christoffel symbols (of the first kind) Γ_{kji} are associated with a *connection* (the so-called Levi-Civita connection defined by the inertia tensor M) and a *Riemannian curvature tensor*. Being able to find this F which transforms the given generalized coordinates into a new system in terms of which the inertia matrix is the identity implies the Riemannian curvature tensor is zero. To see this, we first write down the corresponding *Christoffel symbols of the second kind*:

$$\Gamma^{\ell}_{kj} = \sum_{i=1}^{n} m^{\ell i}\Gamma_{kji}.$$

A curvature tensor is prescribed by the corresponding Riemann sysmbol of the first kind:

$$R_{ijk\ell} = \frac{\partial \Gamma_{j\ell i}}{\partial q_k} - \frac{\partial \Gamma_{jki}}{\partial q_\ell} + \sum_{\sigma=1}^{n}\big(\Gamma^{\sigma}_{jk}\Gamma_{i\ell\sigma} - \Gamma^{\sigma}_{j\ell}\Gamma_{ik\sigma}\big).$$

Working this out for the case at hand,

$$\frac{\partial \Gamma_{j\ell i}}{\partial q_k} = \sum_{\sigma=1}^{n}\big(\frac{\partial^3 F_\sigma}{\partial q_k \partial q_j \partial q_\ell}\frac{\partial F_\sigma}{\partial q_i} + \frac{\partial^2 F_\sigma}{\partial q_j \partial q_\ell}\frac{\partial^2 F_\sigma}{\partial q_i \partial q_k}\big),$$

and a permutation of indices gives a similar expression for $\frac{\partial \Gamma_{jki}}{\partial q_k \ell}$. It is straightforward, although a little tedious, to show that

$$\sum_{\sigma=1}^{n}\big(\Gamma^{\sigma}_{jk}\Gamma_{i\ell\sigma} - \Gamma^{\sigma}_{j\ell}\Gamma_{ik\sigma}\big) = \sum_{\beta=1}^{n}\big(\frac{\partial^2 F_\beta}{\partial q_j \partial q_k}\frac{\partial^2 F_\beta}{\partial q_i \partial q_\ell} - \frac{\partial^2 F_\beta}{\partial q_j \partial q_\ell}\frac{\partial^2 F_\beta}{\partial q_i \partial q_k}\big).$$

Hence $R_{ijkl} = 0$, proving the claim. It is convenient to think of the inertia tensor M as a Riemannian metric on the configuration manifold. The Christoffel symbolds Γ_{kji} (or equivalently Γ^{i}_{kj}) define the corresponding *Riemannian connection* (also called the *Levi-Civita connection*). The Reimann symbols of the first kind are components of the Riemannian curvature tensor. Their vanishing identically is essentially a necessary and sufficient condition for the exisitence of a coordinate transformation $F(q) = y$ such that in $y-$coordinates the inertia tensor (=Riemannian metric) is the identity matrix. The reader is referred to Kobayashi and Nomizu [1963] or Wolf [1972] for a broader discussion of curvature and its vanishing.

Definition 9.3.1. *A simple mechanical system of the form (9.3.1) will be called* flat *if the corresponding Riemannian curvature tensor (i.e. the set of Riemann symbols of the first kind) is zero.*

It is of interest to see how this type of flatness simplifies control designs. We begin by considering the case of fully actuated Lagrangian and Hamiltonian systems; in the following section we turn our attention to the more complex (and more intersting) case of super-articulated (or underactuated) systems. Consider a mechanical control system specified by a Lagrangian $L(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} - V(q)$ and equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = u_i \quad (i = 1, \ldots, n),$$

where each $u_i(\cdot)$ is a piecewise continuous control input (function of time defined on a suitable interval). The conjugate momentum vector for this system is $p = M(q)\dot{q}$ and in terms of the Hamiltonian defined by the Legendre transformation

$$\begin{aligned} H(q,p) &= \dot{q}^T p - L(q,\dot{q}) \\ &= \frac{1}{2} p^T M(q)^{-1} p + V(q), \end{aligned} \tag{9.3.3}$$

we define the corresponding Hamiltonian control system

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} + u. \end{aligned} \tag{9.3.4}$$

It has been noted (See, e.g. Bedrosian [1992] and Spong [1998].) that if a system is *flat* there is a canonical transformation together with a control and feedback transformation such that in terms of the new phase space and control variables, the system (9.3.4) takes the form of a system of double integrators.

To understand this, suppose the mechanical system is flat. Let $N(q)$ be the $n \times n$ positive definite symmetric square root of the inertia tensor $M(q)$. If there exists a function $F(q)$ such that $\frac{\partial F}{\partial q} = N(q)$ as above, then the coordinate transformation $Q = F(q)$, $P = N(q)\dot{q}$ is canonical—in the sense that it preserves the symplectic form. We refer the reader to Marsden and Ratiu [1994] or Arnold [1989] for a discussion of canonical transformations. That this transformation is canonical is easily seem if we write

$$\begin{aligned} P \cdot dQ &= \dot{q}^T N(q) \cdot \frac{\partial F}{\partial q} dq \\ &= \dot{q}^T M(q) dq \\ &= p \cdot dq. \end{aligned}$$

We rewrite the Hamiltonian in terms of $Q, P-$coordinates:

$$\begin{aligned} h(q,p) &= \frac{1}{2}p^T M(q)^{-1}p + V(q) \\ &= \frac{1}{2}\dot{q}^T N(q)N(q)\dot{q} + V(q) \\ &= \frac{1}{2}\|P\|^2 + \mathcal{V}(Q), \\ &= \mathcal{H}(Q,P), \end{aligned}$$

where $\mathcal{V}$ is defined by $\mathcal{V}(Q) = V(q)$.

Proposition 9.3.2. *In terms of the variables Q, P, the equations (9.3.4) may be written as*

$$\begin{aligned} \dot{Q} &= P \\ \dot{P} &= -\frac{\partial \mathcal{V}}{\partial Q} + N^{-1}u. \end{aligned} \tag{9.3.5}$$

Under the feedback $u = Nv + N\frac{\partial \mathcal{V}}{\partial Q}$, which is independent of the conjugate momentum variables P, this Hamiltonian control system is transformed into the system of double integrators

$$\begin{aligned} \dot{Q} &= P \\ \dot{P} &= v \end{aligned} \tag{9.3.6}$$

Proof. The coordinate change associated with the Legendre transformation may be written explicitly as

$$Q = F(q) \tag{9.3.7}$$

$$P = \left(\frac{\partial F}{\partial q}\right)^{-1} p \tag{9.3.8}$$

(since $\frac{\partial F}{\partial q}\dot{q} = (\frac{\partial F}{\partial q})^{-1}p$). We wish to find the equations of motion in terms of $Q, P-$coordinates. Differentiating (9.3.7), it is clear that $\dot{Q} = P$. Differentiating (9.3.8), we find that

$$\begin{aligned} \dot{P} &= [\frac{d}{dt}\left(\frac{\partial F}{\partial q}\right)^{-1}]p + \left(\frac{\partial F}{\partial q}\right)^{-1}\dot{p} \\ &= [\frac{d}{dt}\left(\frac{\partial F}{\partial q}\right)^{-1}]p - \left(\frac{\partial F}{\partial q}\right)^{-1}\frac{\partial H}{\partial q} + \left(\frac{\partial F}{\partial q}\right)^{-1}u. \end{aligned}$$

Using the definition of the coordinate transformation (9.3.7)-(9.3.8) together with the symmetry properties of $\frac{\partial F}{\partial q} = N(q)$, it can be shown that

$$\begin{pmatrix} \frac{\partial Q}{\partial q} & 0 \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial Q}{\partial q}^T & \frac{\partial P}{\partial q}^T \\ 0 & \frac{\partial P}{\partial p}^T \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

From this it follows that

$$\Big[\frac{d}{dt}\Big(\frac{\partial F}{\partial q}\Big)^{-1}\Big]p - \Big(\frac{\partial F}{\partial q}\Big)^{-1}\frac{\partial H}{\partial q} = -\frac{\partial \mathcal{H}}{\partial Q} = -\frac{\partial \mathcal{V}}{\partial Q},$$

proving the first part of the proposition. The second part is clear. ■

9.3.2 Second Order *Generalized* Control Systems

Curvature plays an important role in characterizing second order nonlinear control systems. We shall be especially interested in velocity-controlled super-articulated mechanical systems (as introduced and studied in Baillieul [1990,1993,1995,1997a,1997b]). These are systems having two characteristic features: (*i*) fewer control inputs than configuration space dimensions, and (*ii*) certain generalized coordinates together with the corresponding generalized velocities and accelerations are viewed as control inputs. There are several routes leading to a theory of velocity-controlled super-articulated mechanical systems. The reduction process outlined in Baillieul [1997b] will be discussed, but first we shall recall the global definition of a smooth, finite dimensional nonlinear control system.

In 1976, Brockett [1976] noted that local descriptions of nonlinear control system dynamics in the form

$$\dot{q} = f(q, x), \tag{9.3.9}$$

where $f : M \times U \to TM$, were not adequate descriptions of systems where the inputs depend on the states, and even on the time histories of the states. In the natural global extension of (**??**), we view f as a bundle map.

Definition 9.3.3. *A* ***smooth nonlinear control system*** *is a quadruple* (B, M, π, f) *such that*

(i) (B, M, π) *is a fiber bundle with* total space B, base space M, *and canonical projection* $\pi : B \to M$, *and*

(ii) $f : B \to TM$ *is a bundle morphism such that for each* $q \in M$ *and* $x \in U_q = \pi^{-1}(q)$, $f(q, x) \in T_qM$.

For a complete introduction to the theory of fiber bundles, the reader is referred to Husemoller [1994] or the classical treatise Steenrod [1951]. Recall that for each $q \in M$, there is a neighborhood V of q and a diffeomorphism ϕ mapping $\pi^{-1}(V)$ onto $V \times \pi^{-1}(q)$. Thus, as Brockett points out (Brockett, [1976], p. 16), by restricting our attention to such neighborhoods, it is always possible to find a *local* representation of a smooth nonlinear control system of the form (9.3.9).

To continue our study of controlled mechanical systems, it will be useful to refine Brockett's global approach to include systems which are second order (in the configuration variables q) and in which the controlling effects of the input variables are primarily due to their accelerations, $\ddot{x}(\cdot)$. It will be useful to develop the notion of parallel displacements of vectorfields in TM along curves in B. More specifically, given a vectorfield $X \in TB$ and any curve $\tau(t) \in B$, the lifting of τ to a curve $X(\tau(t)) \in TB$ is projected by the tangent mapping π_* onto a curve $X(\tau(t)) \in TM$. This association leads to the desired notion of covariant differentiation of vectorfields in TM along curves in B, and we shall prescribe this in terms of local coordinates as follows. Let $(q_1, \ldots, q_n, x_1, \ldots, x_m)$ denote local coordinates defined in a neighborhood of a point $p \in B$. Let

$$X_i = \frac{\partial}{\partial q_i}, \quad (i = 1, \ldots, n),$$

$$\tilde{X}_i = \frac{\partial}{\partial x_i}, \quad (i = 1, \ldots, m)$$

be the associated vectorfields in TB. We may also view $(q_1, \ldots, q_n)$ as defining local coordinates in a neighborhood of $\pi(p) \in M$. Then we similarly let

$$Y_i = \frac{\partial}{\partial q_i}, \quad (i = 1, \ldots, n)$$

be the associated vectorfields in TM. In terms of this local coordinate description, for each X_i and each $(q, x) \in B$, the tangent mapping π_* associates a vector $Y_i^* \in T_qM$ by means of the formula

$$Y_i^*(q, x) = \pi_* X_i(q, x).$$

In general, $Y_i^*(q, x) \neq Y_i(q)$, but we may write

$$Y_i^*(q, x) = \sum_{j=1}^{n} \alpha_{ij}(x) Y_j(q), \quad (i = 1, \ldots, n).$$

By restricting to a smaller neighborhood if necessary, there is no loss of generality in assuming that the $n \times n$ matrix $A(x)$, whose ij-th entry is $\alpha_{ij}(x)$ is nonsingular for each x. For any point $q \in M$, $\pi^{-1}(q) \in B$, each vectorfield X_i $(i = 1, \ldots, n)$ and $\tilde{X}_j$ $(j = 1 \ldots, m)$, defines an integral curve passing through $\pi^{-1}(q)$. For each such integral curve, expressed in our coordinate neighborhood as $\tau(t) = (q(t), x(t)$, there is a corresponding set of curves $Y_i^*(q(t), x(t))$ $(i = 1 \ldots, n)$ in the tangent bundle TM. Tangents to these curves define a *covariant derivative operator* $\nabla : TB \times TM \to TM$ which is prescribed in terms of the given local coordinate by means of $(n+m)n^2$ functions $\{\Gamma_{ij}^k\}$, $(i, j, k = 1, \ldots, n)$, and $\{\tilde{\Gamma}_{ij}^k\}$, $(i = 1, \ldots, m;$ $j, k = 1, \ldots, n)$ and the formulas

$$\nabla_{X_i} Y_j^* = \textstyle\sum_{k=1}^{n} \Gamma_{ij}^k Y_k^*, \quad i, j = 1, \ldots, n; \tag{9.3.10}$$

$$\nabla_{\tilde{X}_i} Y_j^* = \textstyle\sum_{k=1}^{n} \tilde{\Gamma}_{ij}^k Y_k^*, \quad i = 1, \ldots, m; \; j = 1, \ldots, n. \tag{9.3.11}$$

Since the vectorfields X_i (and $\tilde{X}_i$) lie in the coordinate directions, it is clear what is meant if we write

$$\nabla_{q_i} Y_j^* = \sum_{k=1}^{n} \Gamma_{ij}^k Y_k^*; \quad \nabla_{x_i} Y_j^* = \sum_{k=1}^{n} \tilde{\Gamma}_{ij}^k Y_k^*. \tag{9.3.12}$$

The covariant derivative operator allows us to study displacement of vectorfields in TM along curves $(q(t), x(t)) \in B$. We refer to Kobayashi and Nomizu, [**?**], for a clssical treatment of this topic. In the present development, we shall view points q in the base manifold as states of a nonlinear control system (and eventually as generalized coordinates of a mechanical control system) and the points x in the fiber as control inputs. We shall be interested in lifting curves in the total space B to curves in TB and projecting these liftings onto TM. It is useful to keep in mind the following commutative diagram

$$\begin{tikzcd} TB \arrow[r, "\pi_*"] \arrow[d] & TM \arrow[d] \\ B \arrow[r, "\pi"] & M \end{tikzcd}$$

where the vertical arrows are the canonical tangent bundle projections. A curve $(q(t), x(t))$ in B lifts to a curve $X_i(q(t), x(t))$ in TB for each $i = 1, \ldots, n$. The tangent mapping of the canonical projection $\pi : B \to M$ allows us to define a curve $Y_i^*(q(t), x(t)) = \pi_* X_i(q(t), x(t))$ in TM. Expressed in terms of the given local coordinates, let $(q(t), x(t))$ $(0 \le t < t_0)$ be any smooth curve defined in our coordinate neighborhood in B. Let

$$\sum_{i=1}^{n} \alpha_i(t) X_i(q(t), x(t)) \, + \, \sum_{i=1}^{m} \beta_i(t) \tilde{X}_i(q(t), x(t))$$

be the corresponding tangent vector for each t in $0 \le t < t_0$. (This is a curve in TB.) The vectorfields $Y_i^*(q, x) = \pi_* X_i(q, x)$, appearing in the formulas (9.3.10)-(9.3.11) depend on points in the total space B. In terms of the vectorfields Y_i^*, the image curve in TM may be expressed as

$$Y(t) = \sum_{i=1}^{n} \alpha_i(t) Y_i^*(q(t), x(t)). \tag{9.3.13}$$

This projection defines a vectorfield (in TM) *along the curve* $\tau(t) = (q(t), x(t))$ in B. Differentiating Y with respect to t and using our covariant derivative operator to project the result back onto TM, we obtain the *covariant derivative in the direction* $\dot{\tau}$, $\nabla_{\dot{\tau}} Y$, for each t in $0 \le t < t_0$:

$$\sum_{i=1}^{n} \bigg(\dot{\alpha}_i(t) Y_i^* \, + \, \alpha_i(t) (\sum_{j=1}^{n} \nabla_{q_j} Y_i^* \dot{q}_j \, + \, \sum_{j=1}^{m} \nabla_{x_j} Y_i^* \dot{x}_j) \bigg).$$

(Cf. Kobayashi and Nomizu, [**?**] p. 114.) Using the relationships (9.3.12), this expression may be rendered

$$\sum_{k=1}^{n} \bigg(\dot{\alpha}_k(t) + \sum_{i,j=1}^{n} \Gamma_{ji}^k \alpha_i \dot{q}_j + \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{\Gamma}_{ji}^k \alpha_i \dot{x}_j \bigg) Y_k^*. \tag{9.3.14}$$

Curves (9.3.13) which arise as models of velocity or accelration controlled mechanical systems are *parallel* along $\tau(t)$ in the sense of the following definition.

Definition 9.3.4. *We say that a curve*

$$V(t) = \sum_{i=1}^{n} v_i(t) Y_i^*(q(t), x(t))$$

is parallel along $\tau(t) = (q(t), x(t))$ *if the covariant derivative in the direction* $\dot{\tau}$, $\nabla_{\dot{\tau}} V$ *is zero. Equivalently, in terms of local coordinates,* $V(t)$ *is parallel along* $\tau(t)$ *if*

$$\sum_{i=1}^{n}\Big(\dot{v}_i(t) Y_i^* + v_i(t)(\sum_{j=1}^{n}\big[\nabla_{q_j} Y_i^*\big]\dot{q}_j + \sum_{j=1}^{m}\big[\nabla_{x_j} Y_i^*\big]\dot{x}_j)\Big) = 0.$$

An even more explicit rendering of the condition for Y in (9.3.13) to be parallel along τ is that $\alpha(\cdot)$ is the unique solution to the system of differential equations

$$\dot{\alpha}_k(t) + \sum_{i,j=1}^{n} \Gamma_{ji}^{k} \alpha_i \dot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{m} \tilde{\Gamma}_{ji}^{k} \alpha_i \dot{x}_j = 0 \quad (k = 1, \ldots, n).$$

Now we say that (9.3.13) defines a *smooth acceleration-controlled second order system* if *(i)* $\alpha_i(t) = \dot{q}_i(t) + \sum_{j=1}^{m} \gamma_{ij}(q)\dot{x}_j(t)$ and *(ii)* the vector field (9.3.13) is parallel along the curve $(q(t), x(t))$. In particular, if (9.3.13) is a trajectory of smooth acceleration-controlled second order system, the coordinate functions $q_i(t)$ and $x_i(t)$ satisfy

$$\ddot{q}_k + \sum_{j=1}^{m} \gamma_{ij}(q)\ddot{x}_j \sum_{i,j=1}^{n} \Gamma_{ji}^{k} \dot{q}_i \dot{q}_j + \sum_{i=1}^{n}\sum_{j=1}^{m} \hat{\Gamma}_{ji}^{k} \dot{q}_i \dot{x}_j = 0, \quad \text{for } i = 1, \ldots, n, \tag{9.3.15}$$

where $\hat{\Gamma}_{ji}^{k} = \tilde{\Gamma}_{ji}^{k} + \frac{\partial \gamma_{ij}}{\partial q_i}$. This is a direct extension of the geometric characterization of second-order differential equations found in Abraham and Marsden, 1988, [?]. In general, all three components of the triple $(x, \dot{x}, \ddot{x})$ enter the equation (9.3.15), but because we shall be principally concerned with high-frequency periodic inputs, the most significant terms will be those involving $\ddot{x}$. (Suppose $x(\cdot)$ is a periodic vector valued function whose fundamental frequency is ω and whose ℓ_∞ norm is $\mathcal{O}(1)$. Then the norm of $\dot{x}(\cdot)$ is $\mathcal{O}(\omega)$ and the norm of $\ddot{x}(\cdot)$ is $\mathcal{O}(\omega^2)$.) It is precisely because (9.3.15) depends formally on $\ddot{x}$ that we have called these systems *acceleration controlled*. In the next section, we shall study systems in which all $\gamma_{ij}(q) \equiv 0$. We shall also show that the vanishing of a curvature-like quantity is a necessary condition for there to be a trnaformation to coordinates in terms of which on x and $\dot{x}$ (but not $\ddot{x}$ enter the equations of motion.

We have called systems of the form (9.3.15) *acceleration controlled* because of the explicit dependence on the second derivative of the input function $x(\cdot)$. Second order acceleration-controlled systems arise in the study of *super-articulated* mechanical systems (Seto [1994]). (These have also been called *under-actuated* systems in the literature.) The general framework assumes there is given a Lagrangian $L(y, \dot{y})$ defined on the tangent bundle TQ of the configuration space of a mechanical system. Suppose that the generalized coordinates can be partitioned as $y = (r, q)$ and that exogenous generalized forces (control inputs) can be applied to only the coordinates r, while the coordinate variables comprising q evolve freely, subject only to dynamic interactions with the r-variables. We assume $dim\ r = m$ and $dim\ q = n$. The equations of motion for the system take the form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = u, \tag{9.3.16}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = 0, \tag{9.3.17}$$

where u is an m-vector of controls, and we suppose the mapping $\dot{q} \mapsto \frac{\partial \mathcal{L}}{\partial \dot{q}}$ is invertible. I.e. if the Lagrangian takes the form "kinetic minus potential energy," i.e. $\mathcal{L} = \frac{1}{2}\dot{y}^T M \dot{y} - V(y)$, and the inertia matrix is partitioned conformably with (r, q), $M = \begin{pmatrix} \mathcal{N} & \mathcal{A} \\ \mathcal{A}^T & \mathcal{M} \end{pmatrix}$, then $\mathcal{M} = \mathcal{M}(r, q)$ is an $n \times n$ invertible matrix. For the purposes of the model, we assume that the components $u_i(\cdot)$ are each piecewise analytic functions on $[0, \infty)$, although more general inputs (e.g. impulse trains) are also of interest and amenable to study.

A simple reduction process eliminating the explicit dependence on the input $u(\cdot)$ leads to a system of equations of the desired form. We assume that the dynamics of (9.3.16) allow $u(\cdot)$ to be chosen such that r can be specified exactly. We may thus view $r, \dot{r}, \ddot{r}$ as control inputs in equation (9.3.17). To emphasize this viewpoint, we write $(x, v, a) = (r, \dot{r}, \ddot{r})$. Equation (9.3.17) then relates the state variables q and $\dot{q}$ to the inputs (x, v, a). This dynamical relationship may be obtained by applying the Euler-Lagrange operator $\frac{d}{dt}\frac{\partial}{\partial \dot{q}} - \frac{\partial}{\partial q}$ to the *reduced Lagarangian*

$$\mathcal{L}(q, \dot{q}; x, v) = \frac{1}{2}\dot{q}^T \mathcal{M}(q, x)\dot{q} + v^T \mathcal{A}(q, x)\dot{q} - \mathcal{V}(q; x, v), \tag{9.3.18}$$

where $\mathcal{V}(q : x, v) = V(x, q) - \frac{1}{2}v^T \mathcal{N} v$. Although the reduction process described above will play no further role in our theory, it is the main route by which we are lead to velocity controlled Lagrangian systems of the form

(9.3.18). The Lagrangian (9.3.18) gives rise to the Lagrangian dynamics

$$\sum_{j=1}^{n} m_{kj}\ddot{q}_j + \sum_{\ell=1}^{m} a_{\ell k}\dot{v}_\ell + \sum_{i,j=1}^{n} \Gamma_{ijk}\dot{q}_i\dot{q}_j + \sum_{j=1}^{n}\sum_{\ell=1}^{m} \hat{\Gamma}_{\ell jk} v_\ell \dot{q}_j = F(t), (k = 1, \dots, n), \tag{9.3.19}$$

where

$$\Gamma_{ijk} = \frac{1}{2}\Big(\frac{\partial m_{ki}}{\partial q_j} + \frac{\partial m_{kj}}{\partial q_i} - \frac{\partial m_{ij}}{\partial q_k}\Big),$$

$$\hat{\Gamma}_{\ell jk} = \frac{\partial m_{kj}}{\partial x_\ell} + \frac{\partial a_{\ell k}}{\partial q_j} - \frac{\partial a_{\ell j}}{\partial q_k},$$

and a_{ij} and m_{ij} are the ij-th entries in the $m \times n$ and $n \times n$ matrices $\mathcal{A}(q, x)$ and $\mathcal{M}(q, x)$ respectively. $F(t)$ is a vector of *generalized forces* $F_i(t) = \frac{\partial \mathcal{V}}{\partial q_i} - \sum_{k,\ell=1}^{m} \frac{\partial a_{\ell i}}{\partial x_k} v_\ell v_k$. These may be thought of as coming from a velocity-dependent potential if

$$\frac{\partial^2 a_{\ell i}}{\partial q_j \partial x_k} = \frac{\partial^2 a_{\ell j}}{\partial q_i \partial x_k}$$

for all $k, \ell = 1, \dots, m$ and $i, j = 1 \dots, n$.

Just as m_{ij} denotes the $ij-$th element of $\mathcal{M}$, let m^{ij} denote the $ij-$th element of $\mathcal{M}^{-1}$. Multiplying both sides of (9.3.19) by $m^{\sigma k}$ and summing over the index values $k = 1, \dots, n$, we obtain

$$\ddot{q}_\sigma + \sum_{\ell=1}^{m} \gamma_{\sigma\ell}\dot{v}_\ell + \sum_{i,j=1}^{n} \Gamma^\sigma_{ij}\dot{q}_i\dot{q}_j + \sum_{j=1}^{n}\sum_{\ell=1}^{m} \hat{\Gamma}^\sigma_{\ell j} v_\ell \dot{q}_j = \tilde{F}_\sigma(t), \quad (\sigma = 1, \dots, n), \tag{9.3.20}$$

where

$$\gamma_{\sigma\ell} = \sum_{k=1}^{n} m^{\sigma k} a_{k\ell},$$

$$\Gamma^\sigma_{ij} = \sum_{k=1}^{n} m^{\sigma k}\Gamma_{ijk},$$

$$\hat{\Gamma}^\sigma_{\ell j} = \sum_{k=1}^{n} m^{\sigma k}\hat{\Gamma}_{\ell jk}, \text{ and}$$

$$\tilde{F}_\sigma(t) = \sum_{k=1}^{n} m^{\sigma k} F_k(t).$$

Modulo a minor change of notation ($\dot{x}_j = v_j$), when the generalize potential forces $F_k(t) = 0$, equations (9.3.15) and (9.3.20) are the same.

It is of interest to develop a theory of normal forms for systems of the form (9.3.15) and (9.3.20). In this pursuit, we are looking for the appropriate analog of the double integrator form characterized in Proposition 9.3.2. The particular way in which our input variables $(x, \dot{x}, \ddot{x})$ enter the equations together with the fact that the number m of input variables is generally less than the number n of configuration variables rules out the double integrator as a general normal form for acceleration controlled super-articulated systems. Nevertheless, we shall show that when certain curvature-like quantities vanish, we can find a coordinate system in which there is a high degree of decoupling between inputs and configuration variables. Indeed, we shall show that under certain "vanishing" hypotheses, there is a choice of generalized coordinates with respect to which the dynamic effects of the inputs enter through velocity-dependent potential terms.

9.4 Flat Systems and Systems with Flat Inputs

We begin by considering the structure of systems which are *flat* in the sense of Definition 9.3.1. In this setting, a system of the form (9.3.15) or (9.3.20) is said to be flat if the Riemannian curvature tensor prescribed by the Christoffel symbols Γ_{ij}^k vanishes. In terms of the Lagrangian system (9.3.18), if the inertia tensor $\mathcal{M}$ does not depend on the input x, then the vanishing of the curvature tensor is necessary and sufficient for there to be a change of generalized coordinates $\overline{q} = F(q)$ such that the inertia tensor expressed in terms of the $\bar{q}-$coordinates has entries

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

Fot the case in which $\mathcal{M} = \mathcal{M}(q, x)$, explicitly depends on the input x, it is still possible to have the Riemannian curvature tensor vanish. (Of course, in this case, the Riemann symbols will have no dependence on the variable x.) It is then possible to find an $x-$dependent change of coordinates, $\bar{q} = F(q, x)$ such that in the $\bar{q}-$coordinate system, the Lagrangian (9.3.18) takes the form

$$\bar{\mathcal{L}}(\bar{q}, \dot{\bar{q}}; x, \dot{x}) = \frac{1}{2}\|\dot{\bar{q}}\|^2 + \dot{\bar{q}}^T \bar{A}^T(\bar{q}, x)\dot{x} - \bar{\mathcal{V}}(\bar{q}; x, \dot{x}), \tag{9.4.1}$$

where

$$\bar{A}^T(\bar{q}, x) = \frac{\partial G}{\partial \bar{q}}^T A^T(G(\bar{q}, x), x) - \frac{\partial F}{\partial x}(G(\bar{q}, x), x),$$

and

$$\bar{\mathcal{V}}(\bar{q}; x, \dot{x}) = \mathcal{V}(G(\bar{q}, x); x, \dot{x}) + \frac{1}{2}\dot{x}^T \frac{\partial F}{\partial x}^T \frac{\partial F}{\partial x}\dot{x},$$

with $q = G(\bar{q}, x)$ being the $x-$dependent inverse of the diffeomorphism $\bar{q} = F(q, x)$. An interesitng question is "When do the coupling terms $\bar{A}(\bar{q}, x)$ vanish?" The following definitions are useful in formulating the answer.

Definition 9.4.1. (i) *For the second-order generalized control system (9.3.20), we say that the hatted symbols $\hat{\Gamma}^k_{ij}$ define the **input connection**. The input connection is said to be* flat *if the associated Riemann symbols of the second kind*

$$\hat{R}^{\gamma_{\ell i}}_{j} = \frac{\partial \hat{\Gamma}^{\gamma}_{\ell j}}{\partial x_i} - \frac{\partial \hat{\Gamma}^{\gamma}_{ij}}{\partial x_\ell} + \sum_{k=1}^{n} (\hat{\Gamma}^k_{\ell j} \Gamma^{\gamma}_{ik} - \hat{\Gamma}^k_{ij} \Gamma^{\gamma}_{\ell k})$$

vanish for all $i\ell = 1, \ldots, m$; $j, \gamma = 1, \ldots, n$. (ii) *Given the second order generalized control system (9.3.20), we say that the pair $(x, \dot{x})$ constitutes a set of* flat inputs *if the input connection is flat.*■

The desired result on the structure of the Lagrangian control system (9.3.20) is the following.

Theorem 9.4.2. *Consider the Lagrangian control system (9.3.18). Let $U \times V \subset \mathbb{R}^n \times \mathbb{R}^m$, and suppose that $F : U \times V \to \mathbb{R}^n$ is an input-dependent change of coordinates $\bar{q} = F(q, x)$ such that for each x the metric tensor $\mathcal{M}$ expressed in $\bar{q}-$coordinates has δ_{ij} as its $ij-$th entry. Suppose, moreover, that in the $\bar{q}-$coordinates all cross coupling terms $\bar{A}(\bar{q}, x)$ vanish. Then the system is flat and has flat inputs, which is to say both the hatted and unhatted Riemann symbols of the second kind,*

$$R_j{}^{\gamma}{}_{\alpha\beta} = \frac{\partial \Gamma^{\gamma}_{\alpha j}}{\partial q_\beta} - \frac{\partial \Gamma^{\gamma}_{\beta j}}{\partial q_\alpha} + \sum_{k=1}^{n} (\Gamma^k_{\alpha j} \Gamma^{\gamma}_{\beta k} - \Gamma^k_{\beta j} \Gamma^{\gamma}_{\alpha k})$$
$$\alpha, \beta, j, \gamma = 1, \ldots, n,$$

and

$$\hat{R}^{\gamma_{\ell i}}_{j} = \frac{\partial \hat{\Gamma}^{\gamma}_{\ell j}}{\partial x_i} - \frac{\partial \hat{\Gamma}^{\gamma}_{ij}}{\partial x_\ell} + \sum_{k=1}^{n} (\hat{\Gamma}^k_{\ell j} \hat{\Gamma}^{\gamma}_{ik} - \hat{\Gamma}^k_{ij} \hat{\Gamma}^{\gamma}_{\ell k})$$
$$i, \ell = 1, \ldots, m; \quad j, \gamma = 1, \ldots, n,$$

vanish.

The proof of this theorem uses the following lemma.

Lemma 9.4.3. *Let $U \times V \subset \mathbb{R}^m \times \mathbb{R}^n$ be some neighborhood of* $(0, 0,$ *and let $f_j : U \times V \to \mathbb{R}^n$ be a smooth mapping for $j = 1, \ldots, m$. Given $q \in V$, there is defined a neighborhood W of 0 in $\mathbb{R}^m$ and a smooth function $g : W \to \mathbb{R}^n$ such that*

(i) $g(0) = q$, and

(ii) $\frac{\partial g}{\partial r_j}(r) = f_j(r, g(r))$ for all $r \in W$, $j = 1, \ldots, m$

if and only if there is a neighborhood of $(0, q)$ *on which*

$$\frac{\partial f_j}{\partial r_i} - \frac{\partial f_i}{\partial r_j} + \sum_{k=1}^{n}(\frac{\partial f_j}{\partial q_k} f_i^k - \frac{\partial f_i}{\partial q_k} f_j^k) = 0.$$

This lemma is proved in Spivak, 1970. We can now prove Theorem 9.4.2.

Proof of Theorem 9.4.2. Let $\bar{q} = F(q, x)$ be as in the hypothesis of the theorem. Then

$$\mathcal{M} = \frac{\partial F}{\partial q}^T \frac{\partial F}{\partial q},$$

and we may write the kinetic energy in terms of $(\bar{q}, x)-$coordinates as

$$\frac{1}{2}\dot{q}^T \mathcal{M}(q, x)\dot{q} = \frac{1}{2}\|\dot{\bar{q}}\|^2 + \dot{\bar{q}}^T \frac{\partial G}{\partial \bar{q}}^T \mathcal{A}^T \dot{x} - \dot{\bar{q}}^T \frac{\partial F}{\partial x}\dot{x} - \frac{1}{2}\dot{x}^T \frac{\partial F}{\partial x}^T \frac{\partial F}{\partial x}\dot{x},$$

where $q = G(\bar{q}, x)$ is the inverse of the $x-$dependent diffeomorphism $\bar{q} = F(q, x)$, and all terms are written in terms of the variables $\bar{q}$ and x. If the cross coupling terms vanish as in the hypothesis of the theorem, then

$$\mathcal{A}(G(\bar{q}, x), x)\frac{\partial G}{\partial \bar{q}}(G(\bar{q}, x), x) \equiv \frac{\partial F}{\partial x}^T (G(\bar{q}, x), x),$$

which is equivalent to writing

$$\mathcal{A}(q, x) = \frac{\partial F}{\partial x}^T \frac{\partial F}{\partial q} \tag{9.4.2}$$

in terms of the original coordinates. The componentwise rendering of this equation is

$$a_{\ell i} = \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_\ell}\frac{\partial F_k}{\partial q_i} \qquad \begin{array}{l} \ell = 1, \ldots, m \\ i = 1, \ldots, n. \end{array}$$

Similarly, under the hypothesis of the theorem,

$$m_{ij} = \sum_{k=1}^{n} \frac{\partial F_k}{\partial q_i}\frac{\partial F_k}{\partial q_j} \quad i, j = 1 \ldots, n.$$

It is a straightforward calculation to show that

$$\frac{\partial m_{kj}}{\partial x_\ell} + \frac{\partial a_{\ell k}}{\partial q_j} - \frac{\partial a_{\ell j}}{\partial q_k} = 2\sum_{\sigma} \frac{\partial^2 F_\sigma}{\partial q_j \partial x_\ell}\frac{\partial F_\sigma}{\partial q_k}.$$

. Recall that we have called this quantity $\hat{\Gamma}_{\ell jk}$ and in terms of this defined

$$\hat{\Gamma}^{\mu}_{\ell j} = \sum_{k=1}^{n} M^{\mu k}\hat{\Gamma}_{\ell jk}$$

where $m^{\mu k}$ is the $\mu k-$entry in $\mathcal{M}^{-1}$. Using these definitions, one can show that

$$\frac{\partial}{\partial x_\ell}\begin{pmatrix} \frac{\partial F_1}{\partial q_j} \\ \vdots \\ \frac{\partial F_n}{\partial q_j} \end{pmatrix} = \sum_{k=1}^{n} \hat{\Gamma}^k_{\ell j} \begin{pmatrix} \frac{\partial F_1}{\partial q_k} \\ \vdots \\ \frac{\partial F_n}{\partial q_k} \end{pmatrix}.$$

This is a partial differential equation, condidtions for the solution of which are covered by Lemma 9.4.3. Specifically, we are here interested in conditions under which there exisits a fucntion $g : \mathbb{R}^m \to \mathbb{R}^n$ satisfying

$$\frac{\partial g}{\partial x_\ell} = f_\ell(x, g(x)) \tag{9.4.3}$$

where

$$f^j_\ell(x,z) = \sum_{\gamma=1}^{n} \hat{\Gamma}^\gamma_{\ell j} z_\gamma.$$

According to Lemma 9.4.3, a necessary condition for the solution of (9.4.3) is that

$$\frac{\partial f_\ell}{\partial x_i} - \frac{\partial f_i}{\partial x_\ell} + \sum_{k=1}^{n}(\frac{\partial f_\ell}{\partial z_k} f^k_i - \frac{\partial f_i}{\partial z_k} f^k_\ell) = 0.$$

This is rendered (componentwise)

$$\sum_{\gamma=1}^{n}\Big(\frac{\partial \hat{\Gamma}^\gamma_{\ell j}}{\partial x_i} - \frac{\partial \hat{\Gamma}^\gamma_{ij}}{\partial x_\ell}\Big) z_\gamma + \sum_{k\gamma=1}^{n}\Big(\hat{\Gamma}^k_{\ell j}\hat{\Gamma}^\gamma_{ik} z_\gamma - \hat{\Gamma}^k_{ij}\hat{\Gamma}^\gamma_{\ell k} z_\gamma\Big) = 0.$$

Since this must hold identically in $z = (z_1, \ldots, z_n)$, we have the desired conclusion that

$$\frac{\partial \hat{\Gamma}^\gamma_{\ell j}}{\partial x_i} - \frac{\partial \hat{\Gamma}^\gamma_{ij}}{\partial x_\ell} + \sum_{k=1}^{n}\Big(\hat{\Gamma}^k_{\ell j}\hat{\Gamma}^\gamma_{ik} - \hat{\Gamma}^k_{ij}\hat{\Gamma}^\gamma_{\ell k}\Big) = 0.$$

■

The theorem shows that in order for there to be a choice of coordinates, q such that the coefficients $a_{\ell k}$ of $\dot{v}_\ell$ in (9.3.19) all vanish for $k = 1, \ldots, n; \ell = 1, \ldots, m$, the geometric condition that the Riemann symbols $R^\gamma_{j\ \alpha\beta}$ $\hat{R}^\gamma_{j\ \ell i}$ vanish must be satisfied. To understand the physical meaning of these vanishings, we consider some prototypical, beginning with rotating heavy chains. For the present discussion, the salient features are illustrated by the two rotating pendulum systems in Figure 9.4.1. Both pendula undergo controlled rotations about the vertical axis. In Figure 9.4.1(a), the joint by which the pendulum is suspended is a single degree-of-freedom

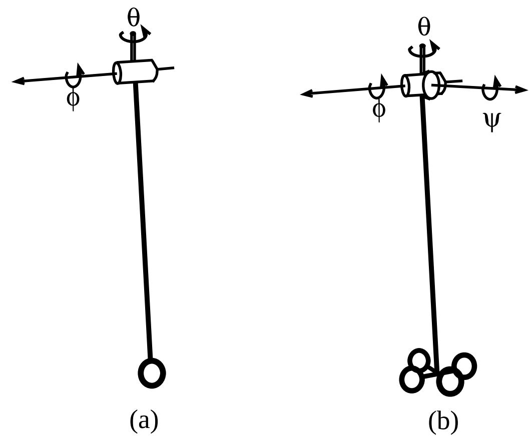


FIGURE 9.4.1. (a) Rotating planar pendulum; (b) Rotating universal joint pendulum.

revolute joint, while in Figure 9.4.1(b), the link is suspended by a (two degree-of-freedom) universal joint. The respective Lagrangians are

$$L_a(\phi, \dot{\phi}; \dot{\theta}) = \frac{m}{2}(a^2 \cos^2 \phi + \ell^2 \sin^2 \phi)\dot{\theta}^2 + (\ell^2 + \frac{a^2}{2})\dot{\phi}^2 + mg\ell \cos \phi,$$

and

$$\begin{array}{rcl} L_b(\phi, \psi, \dot{\phi}, \dot{\psi}; \dot{\theta}) & = & (l^2 + \frac{a^2}{2}(1 + \sin^2 \psi))\, \dot{\phi}^2 + (\frac{a^2}{2} + l^2)\, \dot{\psi}^2 \\ & + & (\frac{a^2}{2} - l^2) \cos \phi \, \sin(2\psi)\, \dot{\phi}\, \dot{\theta} + (a^2 + 2\, l^2) \sin \phi\, \dot{\psi}\, \dot{\theta} \\ & + & \left(a^2 \cos^2 \phi \cos^2 \psi + (\frac{a^2}{2} + l^2)\, (\sin^2 \phi + \cos \phi \sin^2 \psi) \right) \dot{\theta}^2 \\ & + & mg\ell \, \cos \phi \, \cos \psi, \end{array}$$

and the dynamics of the controlled equations are

$$\begin{array}{rcl} \frac{d}{dt}\left[\frac{m}{2} (a^2 \, \cos^2 \phi + \ell^2 \, \sin^2 \phi)\dot{\theta} \right] & = & u \\ (\ell^2 + \frac{a^2}{2})\ddot{\phi} + m\, (a^2 - \ell^2) \, \sin \phi \, \cos \phi \, \dot{\theta}^2 + mg\ell \, \sin \phi & = & 0, \end{array} \qquad (9.4.4)$$

and (in abbreviated form)

$$\frac{d}{dt} \frac{\partial L_b}{\partial \dot{\theta}} = u,$$

$$\frac{d}{dt} \frac{\partial L_b}{\partial \dot{\phi}} - \frac{\partial L_b}{\partial \phi} = 0, \qquad \frac{d}{dt} \frac{\partial L_b}{\partial \dot{\psi}} - \frac{\partial L_b}{\partial \psi} = 0. \qquad (9.4.5)$$

In both cases a torque u is applied to control the angular velocity $\dot{\theta}$ of rotation about the vertical axis. (See Baillieul [1987] and Marsden and Scheurle [1993] for more information about rotating heavy chains.) Applying our formal reduction procedure, (9.3.18) is rendered respectively

$$\mathcal{L}_a(\phi, \dot{\phi}; \dot{\theta}) = \frac{m}{2}(\ell^2 + \frac{a^2}{2})\dot{\phi}^2 - \mathcal{V}_a(\phi; \dot{\theta}) \tag{9.4.6}$$

where

$$\mathcal{V}_a(\phi; \dot{\theta}) = -(a^2 \cos^2 \phi + \ell^2 \sin^2 \phi)\dot{\theta}^2 - mg\ell \cos \phi,$$

and

$$\mathcal{L}_b(\phi, \psi, \dot{\phi}, \dot{\psi}; \dot{\theta}) \;=\; \frac{1}{2}(\dot{\phi}, \dot{\psi})\, \mathcal{M}_b \begin{pmatrix} \dot{\phi} \\ \dot{\psi} \end{pmatrix} + \dot{\theta}\, \mathcal{A}_b \begin{pmatrix} \dot{\phi} \\ \dot{\psi} \end{pmatrix} - \mathcal{V}_b(\phi, \psi; \dot{\theta}) \tag{9.4.7}$$

where

$$\mathcal{M}_b(\phi, \psi) = \begin{pmatrix} l^2 + \frac{a^2\left(1+\sin\psi^2\right)}{2} & 0 \\ 0 & \frac{a^2}{2} + l^2 \end{pmatrix},$$

$$\mathcal{A}_b(\phi, \psi) = \begin{pmatrix} \left(\frac{a^2}{2} - l^2\right) \cos\phi\, \cos\psi\, \sin\psi \\ \left(\frac{a^2}{2} + l^2\right) \sin\phi \end{pmatrix},$$

and

$$\begin{aligned} \mathcal{V}_b(\phi, \psi; \dot{\theta}) \quad &= \quad -\tfrac{1}{2}(a^2 \cos\phi^2 \cos\psi^2 + \left(\tfrac{a^2}{2} + l^2\right)\, \left(\sin\phi^2 + \cos\phi \sin\psi^2\right))\, \dot{\theta}^2 \\ &\quad -mg\ell\, \cos\phi\, \cos\psi. \end{aligned}$$

The significant difference between the two rotating pendulum systems is the absence or presence of the coupling terms $\mathcal{A}$ in (9.4.6) and (9.4.7) respectively. The dynamics of the rotating planar pendulum (9.4.4) feel the influence of rotation *only* through velocity terms involving $\dot{\theta}^2$. The dynamics of the universal joint pendulum (9.4.5), however, depend on both the angular velocity $\dot{\theta}$ and angular acceleration $\ddot{\theta}$. A natural question which arises is whether there is a change of coordinates which makes the dynamics (9.4.5) dependent only on $\dot{\theta}$ and not on the acceleration $\ddot{\theta}$. The answer to this is negative—meaning that (9.4.5) is truly an acceleration controlled system. This may be seen in terms of the system curvatures discussed above. First, we examine the inertia tensor for the unreduced rotating universal joint system:

$$\begin{pmatrix} \begin{matrix} a^2 \cos^2\phi \cos^2\psi + \\ \left(\frac{a^2}{2} + l^2\right) \left(\sin^2\phi + \cos\phi \sin^2\psi\right) \end{matrix} & \left(\frac{a^2}{2} - l^2\right) \cos\phi\, \cos\psi\, \sin\psi & \left(\frac{a^2}{2} + l^2\right) \sin\phi \\ \left(\frac{a^2}{2} - l^2\right) \cos\phi\, \cos\psi\, \sin\psi & l^2 + \frac{a^2\left(1+\sin^2\psi\right)}{2} & 0 \\ \left(\frac{a^2}{2} + l^2\right) \sin\phi & 0 & \frac{a^2}{2} + l^2 \end{pmatrix}.$$

It is a straightforward (and tedious, if you don't have computer algebra software) calculation to show that not all the Riemann symbols which define the curvature tensor are zero. Indeed...

Interestingly, we obtain . . .`DETAILS have to be added regarding the differences between the planar and nonplanar pendulum systems and how these are related to CURVATURE.`

quite a bit more stuff needed here. jb

9.5 Averaging Lagrangian and Hamiltonian Systems with Oscillatory Inputs

For flat systems with flat inputs, we may express the Lagrangian as the sum of a simple kinetic energy term plus a time-dependent potential. It is natural to conjecture that given small-amplitude high-frequency periodic inputs $x(\cdot)$, their influence is approximately equal to the effect of a potential force determined by the time-averaged potential. The theory we develop next shows this to be the case. Even when the system is not flat and depends unavoidably on second derivatives of the inputs, however, it remains possible to write down an energy-like quantity called the *averaged potential* around whose critical points the motions of the system with oscillatory forcing is organized.

As a general starting point, we consider Lagrangian control system (9.3.18) which we rewrite here:

$$\mathcal{L}(q,\dot{q};x,v) = \frac{1}{2}\dot{q}^T\mathcal{M}(q,x)\dot{q} + v^T\mathcal{A}(q,x)\dot{q} - \mathcal{V}(q;x,v).$$

As above, we assume $x(\cdot)$ and $v(\cdot)$ are inputs with $v = \dot{x}$. The system dynamics are obtained from the corresponding Euler-Lagrange equation

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{\partial\mathcal{L}}{\partial q} = 0.$$

The associated Hamiltonian is

$$\mathcal{H}(q,p;x,v) = \frac{1}{2}(p - \mathcal{A}^Tv)^T\mathcal{M}^{-1}(p - \mathcal{M}^Tv) + \mathcal{V} \qquad (9.5.1)$$

We expand equation (9.5.1) and apply simple averaging to yield the *averaged Hamiltonian*:

$$\overline{\mathcal{H}}(q,p) = \frac{1}{2}p^T\overline{\mathcal{M}^{-1}}p - \overline{v^T\mathcal{A}\mathcal{M}^{-1}}p + \frac{1}{2}\overline{v^T\mathcal{A}\mathcal{M}^{-1}\mathcal{A}^Tv} + \overline{\mathcal{V}}. \qquad (9.5.2)$$

Here the overbars indicate simple averages over one period (T) have been taken: given any piecewise continuous function $F(q,p,t)$, the simple average is given by $\bar{F} = \int_0^T F(q,p,t)\,dt$, where for the purpose of evaluating this integral q and p are regarded as constants.

For this averaged Hamiltonian, there is an obvious decomposition into kinematic and potential energy terms in the case that $\overline{v^T \mathcal{A}\mathcal{M}^{-1}} = 0$:

$$\overline{\mathcal{H}}(q,p) = \underbrace{\frac{1}{2} p^T \overline{\mathcal{M}^{-1}} p}_{\text{avg. kin. energy}} + \underbrace{\frac{1}{2}\,\overline{v^T \mathcal{A}\mathcal{M}^{-1}\mathcal{A}v} + \overline{\mathcal{V}}}_{\text{averaged potential}}. \tag{9.5.3}$$

In the case that $\overline{v^T \mathcal{A}\mathcal{M}^{-1}} \neq 0$, it remains possible to decompose the averaged Hamiltonian (9.5.2) into the sum of averaged kinetic and potential energies, although the description becomes more involved. If $\bar{v} \neq 0$, then the corresponding input variable $x(t)$ will not be not periodic. There will be a "drift" in the value of $x(t)$ which changes by an amount $\bar{v} \cdot T$ every T-units of time. We may nevertheless study averaged Hamiltonian systems in this context. We rewrite the averaged Hamiltonian (9.5.2) as

$$\begin{aligned}
\overline{\mathcal{H}}(q,p) &= \frac{1}{2} p^T \overline{\mathcal{M}^{-1}} p - \overline{v^T \mathcal{A}\mathcal{M}^{-1}} p + \frac{1}{2}\overline{v^T \mathcal{A}\mathcal{M}^{-1}} \left(\overline{\mathcal{M}^{-1}}\right)^{-1} \overline{\mathcal{M}^{-1}\mathcal{A}^T v} \\
&\quad + \frac{1}{2}\,\overline{v^T \mathcal{A}\mathcal{M}^{-1}\mathcal{A}v} - \frac{1}{2}\overline{v^T \mathcal{A}\mathcal{M}^{-1}} \left(\overline{\mathcal{M}^{-1}}\right)^{-1} \overline{\mathcal{M}^{-1}\mathcal{A}^T v} + \overline{\mathcal{V}} \\
&= \underbrace{\frac{1}{2}(\overline{\mathcal{M}^{-1}} p - \overline{\mathcal{M}^{-1}\mathcal{A}^T v})^T \left(\overline{\mathcal{M}^{-1}}\right)^{-1} (\overline{\mathcal{M}^{-1}} p - \overline{\mathcal{M}^{-1}\mathcal{A}^T v})}_{\text{averaged kinetic energy}} \qquad (9.5.4) \\
&\quad \underbrace{+ \frac{1}{2}\overline{v^T \mathcal{A}\mathcal{M}^{-1}\mathcal{A}^T v} - \frac{1}{2}\overline{v^T \mathcal{A}\mathcal{M}^{-1}} \left(\overline{\mathcal{M}^{-1}}\right)^{-1} \overline{\mathcal{M}^{-1}\mathcal{A}^T v} + \overline{\mathcal{V}}}_{\text{averaged potential}}.
\end{aligned}$$

The formal distinction which appears between the zero-mean and non-zero-mean cases ((9.5.4) and (9.5.5) respectively) is that in the latter case both the averaged kinetic and potential energy terms are adjusted to reflect the net (average) motions of the input variables $(x(\cdot), v(\cdot))$. There are important relationships which can be established between the dynamics associated with the averaged Hamltonian (9.5.5) and the dynamics of the periodically forced system. The reader is referred to Weibel, 1997, Weibel *et al.*, 1997, and Weibel & Baillieul, 1998 for details. It is important to mention that for mechanical systems in which $\bar{v} \neq 0$ and $\mathcal{M}$ depends explicitly on x in (9.5.1), the averaging analysis of this chapter may not provide an adequate description of the dynamics. Indeed, in this case, $\|x(t)\|$ will not remain bounded as $t \to \infty$, and if $\mathcal{M}(x(t), q_2)$ also fails to remain bounded, the averaged potential will inherit a dependence on time which will make it difficult to apply the critical point analysis proposed below. Despite this cautionary remark, we shall indicate how our methods may be applied in many instances where $\bar{v} \neq 0$.

9.6 Stability of Mechanical Systems with Oscillatory Inputs

The Lagrangian (9.3.18) gives rise to the Lagrangian dynamics

$$\sum_{j=1}^{n} m_{kj}\ddot{q}_j + \sum_{\ell=1}^{m} a_{\ell k}\dot{v}_\ell + \sum_{i,j=1}^{n} \Gamma_{ijk}\dot{q}_i\dot{q}_j + \sum_{j=1}^{n}\sum_{\ell=1}^{m} \hat{\Gamma}_{\ell jk} v_\ell \dot{q}_j = F(t), (k = 1, \dots, n),$$

where Γ_{ijk} and $\hat{\Gamma}_{\ell jk}$ are defined as in Section 6.4.2 in terms of the entries a_{ij} and m_{ij} in the $m \times n$ and $n \times n$ matrices $\mathcal{A}(q,x)$ and $\mathcal{M}(q,x)$ respectively. To retain our general perspective, we continue to assume that the terms in these equations may depend on x. The explicit form of this dependence will play no role in the present section, however, and hence we simplify our notation by omitting any further mention of the variable x. We seek to understand the stability of (9.3.19) in terms of the corresponding *averaged potential*

$$\mathcal{V}_A(q) = \frac{1}{2}\overline{v^T\mathcal{A}\mathcal{M}^{-1}\mathcal{A}^T v} - \frac{1}{2}\overline{v^T\mathcal{A}\mathcal{M}^{-1}}\left(\overline{\mathcal{M}^{-1}}\right)^{-1}\overline{\mathcal{M}^{-1}\mathcal{A}^T v} + \overline{\mathcal{V}}. \quad (9.6.1)$$

The *averaging principle* of Section 6.x.x states that the effect of forcing (9.3.19) with an oscillatory input $v(\cdot)$ will be to produce stable motions confined to neighborhoods of relative minima of $\mathcal{V}_A(\cdot)$. While this principle appears to govern the dynamics encountered in both simulation and experiments, there is as yet no complete theory describing the observed behavior. Results reported in Baillieul, 1995, [**?**] showed that for a certain class of systems (9.3.19) within a larger class of so-called *linear Lagrangian systems*, strict local minima of the averaged potential are Lyapunov stable rest points of (9.3.19) for all periodic forcing of a given amplitude and sufficiently high frequency. In the present section, we shall review this result, and show that the extension of our analysis to arbitrary systems (9.3.19) is complicated by the fact that in general, a linearization of (9.3.19) fails to capture the stabilizing effects implied by an analysis of the averaged potential. Indeed, we shall show that the averaged potential depends on second order jets of the coefficient functions $\mathcal{A}(q)$ and $\mathcal{M}(q)$. To simplify the presentation, we shall restrict our attention to the case of zero-mean oscillatory forcing in which $\overline{\mathcal{M}^{-1}\mathcal{A}^T v} = 0$.

(Need to write stuff about the averaging principle.)

Suppose q_0 is a strict local minimum of (9.6.1). Applying a high-frequency oscillatory input $v(\cdot)$, we shall look for stable motions of (9.3.19) in neighborhoods of $(q,\dot{q}) = (q_0, 0)$. Of course, even when there are such stable motions, $(q_0, 0)$ need not be a rest point of (9.3.19) for any choice of forcing function $v(\cdot)$. (Cf. the "cart-pendulum" dynamics of Example `XX (See how much of Steve's stuff to include.)` in the case $\alpha \neq \pm\pi/2$.)

To analyze this relationship between (9.3.19) and (9.6.1) in more detail, let us assume $q_0 = 0$. This assumption is made without loss of generality, since we may always change coordinates to make it true. Write

$$\mathcal{A}(q) = \mathcal{A}_0 + \mathcal{A}_1(q) + \mathcal{A}_2(q) + \text{h.o.t.};$$

$$\mathcal{M}(q) = \mathcal{M}_0 + \mathcal{M}_1(q) + \mathcal{M}_2(q) + \text{h.o.t.},$$

where the entries in the $n \times n$ matrix $\mathcal{M}_k(q)$ are homogeneous polynomials of degree k in the components of the vector q, and similarly for the $m \times n$ matrix $\mathcal{A}_k(q)$. It is easy to show that

$$\begin{aligned}\mathcal{M}^{-1}(q) \quad = \quad & \mathcal{M}_0^{-1} - \mathcal{M}_0^{-1}\mathcal{M}_1(q)\mathcal{M}_0^{-1} + \mathcal{M}_0^{-1}\mathcal{M}_1(q)\mathcal{M}_0^{-1}\mathcal{M}_1(q)\mathcal{M}_0^{-1} \\ & -\mathcal{M}_0^{-1}\mathcal{M}_2(q)\mathcal{M}_0^{-1} + \text{h.o.t.}\end{aligned}$$

Using this, we write an expansion of $\mathcal{V}_A$ up through terms of order 2:

$$\mathcal{V}_A(q) = \overline{\mathcal{V}}_0 + \overline{\mathcal{V}}_1(q) + \overline{\mathcal{V}}_2(q) + \text{h.o.t.},$$

where as above, $\overline{\mathcal{V}}_k(q)$ denotes the sum of terms which are homogeneous polynomials of degree k in the components of the vector q, and "h.o.t." refers to a quantity which is of order $o(\|q\|^2)$. Explicitly, under our assumption that $\overline{\mathcal{M}^{-1}\mathcal{A}^T v} = 0$:

$$\overline{\mathcal{V}}_0 = \frac{1}{2}\overline{v^T\mathcal{A}_0\mathcal{M}_0^{-1}\mathcal{A}_0^T v} + \mathcal{V}_0,$$

$$\overline{\mathcal{V}}_1(q) = \frac{1}{2}\overline{v^T(\mathcal{A}_1(q)\mathcal{M}_0^{-1}\mathcal{A}_0^T - \mathcal{A}_0\mathcal{M}_0^{-1}\mathcal{M}_1(q)\mathcal{M}_0^{-1}\mathcal{A}_0^T + \mathcal{A}_0\mathcal{M}_0^{-1}\mathcal{A}_1^T(q))v} + \mathcal{V}_1 \cdot q,$$

and

$$\begin{aligned}\overline{\mathcal{V}}_2(q) = & \frac{1}{2}\overline{v^T\mathcal{A}_1(q)\mathcal{M}_0^{-1}\mathcal{A}_1^T(q)v} - \frac{1}{2}\overline{v^T\mathcal{A}_1(q)\mathcal{M}_0^{-1}\mathcal{M}_1(q)\mathcal{M}_0^{-1}\mathcal{A}_0^T v} \\ & -\frac{1}{2}\overline{v^T\mathcal{A}_0\mathcal{M}_0^{-1}\mathcal{M}_1(q)\mathcal{M}_0^{-1}\mathcal{A}_1^T(q)v} \\ & +\frac{1}{2}\overline{v^T\mathcal{A}_2(q)\mathcal{M}_0^{-1}\mathcal{A}_0^T v} + \frac{1}{2}\overline{v^T\mathcal{A}_0\mathcal{M}_0^{-1}\mathcal{M}_1(q)\mathcal{M}_0^{-1}\mathcal{M}_1(q)\mathcal{M}_0^{-1}\mathcal{A}_0^T v} \\ & -\frac{1}{2}\overline{v^T\mathcal{A}_0\mathcal{M}_0^{-1}\mathcal{M}_2(q)\mathcal{M}_0^{-1}\mathcal{A}_0^T v} + \frac{1}{2}\overline{v^T\mathcal{A}_0\mathcal{M}_0^{-1}\mathcal{A}_2^T(q)v} + q^T\mathcal{V}_2 q.\end{aligned}$$

where $\mathcal{V}_0$, $\mathcal{V}_1$, and $\mathcal{V}_2$ define the jets of the potential $\mathcal{V}(q)$ of orders 0,1, and 2 respectively. Writing $\mathcal{V}_A(\cdot)$ in this way shows its dependence on jets of coefficient functions of (9.3.19) of order up to 2. This dependence implies that the observed stabilizing effects produced by high-frequency forcing cannot be understood in terms of a linearization of the dynamics (9.3.19). We shall examine this remark in a bit greater detail.

Having assumed that $q_0 = 0$ is a strict local minimum of $V_A(\cdot)$, it follows that $\frac{\partial \mathcal{V}_A}{\partial q}(0) = \frac{\partial \overline{\mathcal{V}}_1}{\partial q} = 0$. There are two cases to consider here: *i)* $\frac{\partial \overline{\mathcal{V}}_1}{\partial q} = 0$ for a particular choice of oscillatory input $v(\cdot)$, and *ii)* $\frac{\partial \overline{\mathcal{V}}_1}{\partial q} \equiv 0$ independent of the choice of zero-mean oscillatory (periodic) forcing $v(\cdot)$.

Case i): In this case, the location of the local minimum of $\mathcal{V}_A(\cdot)$ depends on $v(\cdot)$, and it will not generally coincide with a rest point of (9.3.19). While the *averaging principle* suggests that there will be stable motions of (9.3.19) in neighborhoods of local minima of $\mathcal{V}_A$, the analysis of this case has involved either the introduction of dissipation into the model (as was done in Baillieul [1993]) or the use of machinery from the theory of dynamical systems (as was done in the thesis of S. Weibel [1997]). The significant point to note here is that the dependence of $\mathcal{V}_A$ on $v(\cdot)$ implies that $\mathcal{A}_0 \neq 0$, which in turn implies both that $q_0 = 0$ will not be a rest point of (9.3.19) for any oscillatory input $v(\cdot)$, and also that the stabilizing effect of $v(\cdot)$ on motions of (9.3.19) will depend on jets of order up to 2 in the coefficient functions. We shall defer further discussion of this case and present some results on the qualitative theory of such systems in the next section. For further details the reader is referred to Weibel [1997].

Case ii): If $\frac{\partial \overline{\mathcal{V}}_1}{\partial q} \equiv 0$, by which we mean that the first partial derivatives of $\mathcal{V}_A$ evaluated at $q_0 = 0$ are zero independent of coefficients due to $v(\cdot)$, then we also find that $\mathcal{V}_1 = 0$. It then follows that $2\mathcal{A}_0\mathcal{M}_0^{-1}\mathcal{A}_1^T(q) - \mathcal{A}_0\mathcal{M}_0^{-1}\mathcal{M}_1(q)\mathcal{M}_0^{-1}\mathcal{A}_0^T \equiv 0$, and either of two sub-cases can occur: *a*) $\mathcal{A}_0 = 0$, or else *b*) there is a polynomial relationship among the coefficients in the 0-th and 1-st order jets of $\mathcal{A}(q)$ and $\mathcal{M}(q)$. In case *b*), $q_0 = 0$ will correspond to a rest point of (9.3.19) in the absence of forcing, but it will not generally define a rest point when $v(t) \neq 0$. Stabilizing effects of the oscillatory input $v(\cdot)$ appear to again depend on jets of order up to two in the coefficients of the Lagrangian vector field (9.3.19). This case will not be treated further in this section. We shall consider case *a*), $\mathcal{A}_0 = 0$. In this case, the averaged potential $\mathcal{V}_A$ only depends on terms up through first order in the coefficients of (9.3.19).

Slightly refining our notation, let $\mathcal{A}^\ell(q)$ denote the ℓ-th column of the $n \times m$ matrix $\mathcal{A}^T(q)$. Then we have

$$\mathcal{A}^\ell(q) = \mathcal{A}_1^\ell \cdot q + (\text{terms of order } \geq 2), \text{ and}$$

$$\mathcal{M}(q) = \mathcal{M}_0 + (\text{terms of order } \geq 1),$$

where we interpret $\mathcal{M}_0, \mathcal{A}_1^1, \ldots, \mathcal{A}_1^m$ as $n \times n$ coefficient matrices. The following result is now clear.

Proposition 9.6.1. *Suppose $v(\cdot)$ is an $\mathbb{R}^m$-valued piecewise continuous periodic function of period $T > 0$ such that $\bar{v} = \frac{1}{T}\int_0^T v(s)\,ds = 0$. Suppose, moreover, that $\mathcal{A}_0 = 0$. Then the averaged potential of the Lagrangian system (9.3.19) agrees up to terms of order 2 with the averaged potential associated with the linear Lagrangian system*

$$\mathcal{M}_0\ddot{q} + \sum_{\ell=1}^m \Big(\dot{v}_\ell \mathcal{A}_1^\ell q + v_\ell(\mathcal{A}_1^\ell - {\mathcal{A}_1^\ell}^T)\dot{q}\Big) + \mathcal{V}_2 \cdot q = 0. \tag{9.6.2}$$

Proof. The proof follows immediately from examining the above expansion of $\mathcal{V}_A$. ■

A deeper connection with stability is now expressed in terms of the following theorem.

Theorem 9.6.2. *Suppose $w(\cdot)$ is an $\mathbb{R}^m$-valued piecewise continuous periodic function of period $T > 0$ such that $\bar{w} = \frac{1}{T}\int_0^T w(s)\,ds = 0$. Consider the linear Lagrangian system (9.6.2) with input $v(t) = w(\omega t)$, and suppose ${\mathcal{A}_1^\ell}^T = \mathcal{A}_1^\ell$ for $\ell = 1, \ldots, m$. The averaged potential for this system is given by*

$$\mathcal{V}_A(q) = \frac{1}{2} q^T \Big(\mathcal{V}_2 + \sum_{i,j=1}^{m} \sigma_{ij} \mathcal{A}_1^i \mathcal{M}_0^{-1} {\mathcal{A}_1^j}^T \Big) q, \tag{9.6.3}$$

where $\sigma_{ij} = (1/T)\int_0^T w_i(s) w_j(s)\,ds$. If the matrix $\frac{\partial^2 \mathcal{V}_A}{\partial q^2}$ is positive definite, the origin $(q, \dot{q}) = (0, 0)$ of the phase space is stable in the sense of Lyapunov provided ω is sufficiently large.

This theorem has been proved in Baillieul [1995]. In Baillieul [1993], it was shown in the presence of dissipation, the positive definiteness of the Hessian matrix $\frac{\partial^2 \mathcal{V}_A}{\partial q^2}(0)$ is a sufficient condition for (9.3.19) to execute stable motions in a neighborhood of $(q_0, 0)$. Theorem 3 shows that in Case *iib*), it is precisely the conditions of the averaging principle from which we may infer the Lyapunov stability of (9.6.2) based on the positive definiteness of the Hessian of the averaged potential. Clearly this result is special and related to the property that the averaged potential depends only on first order jets of the coefficients of (9.3.19) when $\mathcal{A}(q_0) = 0$. In Weibel and Baillieul [1998] it has been shown that the condition $\mathcal{A}(q_0) = 0$ is also necessary and sufficient for the local minimum q_0 of the averaged potential to define a corresponding fixed point (rather than a periodic orbit) of the *forced* (nonautonomous) Hamiltonian system associated with (9.3.18).

9.7 Stabilization of Rotating Systems

9.7.1 Heavy chains

9.7.2 Body-beam systems

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