Week 6: Feedback Linearization of Nonlinear Systems I

Past and Current Homework

HW 1 Stats – out of 120 pts

Average	97.38461
Median	104.00
Standard Deviation	30.33692

HW 2 Stats – out of 100 pts

Average	78.85714
Median	83.00
Standard Deviation	15.06584

HW 4 – Problem 1 – take Lie brackets k times

HW 4 – Problem 4 – the Jacobi-Lee identity

$$\boxed{[X,[Y,Z]] + [Z,[X,Y]] + [Y,[X,Z]] = 0 \text{ - the Jacobi-Lie identity}}$$

Outline

- Finishing up Controllability:
  - Distributions
  - Rank
  - Accessibility
- State Feedback for Linear Systems
- Feedback Linearization of Nonlinear Systems:
  - Relative Degree
  - Nonlinear Coordinate Transformation
  - Affine Form to Canonical Form

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Recap: Local Accessibility of Affine Nonlinear Systems. What is an accessibility distribution?

$$\dot{x}(t) = f(x) + g(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$

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$$\dot{x}(t) = f(x) + g(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$

<u>Theorem</u>: The Affine Nonlinear System is locally accessible at a point $x_0 \in \mathbb{R}^n$ if the accessibility distribution Q spans $\mathbb{R}^n$, where

$$Q = \ [g_1, .., g_m, \dots, \left[g_i, g_j\right] \dots, \left[ad_{g_i}^k, g_j\right] \dots, [f, g_i], \dots, [ad_f^k, g_i], \dots]$$

The Accessibility Distribution: Local Accessibility of Affine Nonlinear Systems

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<u>Theorem</u>: The Affine Nonlinear System is locally accessible at a point $x_0 \in \mathbb{R}^n$ if the accessibility distribution Q spans $\mathbb{R}^n$, where

$$Q_{m=1} = \left\{ \begin{matrix} f, g, [f, g\,], \\ [f, [f, g\,]], [g, [f, g]], \cdots \end{matrix} \right\}$$

$\rightarrow$ *any Lie bracket combination of f and g you can think of*

The Accessibility Distribution: Local Accessibility of Affine Nonlinear Systems

$$\dot{x}(t) = f(x) + g(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$

<u>Theorem</u>: The Affine Nonlinear System is locally accessible at a point $x_0 \in \mathbb{R}^n$ if the accessibility distribution Q spans $\mathbb{R}^n$, where

$$Q_{m=2} = \left\{ \begin{array}{c} f, g_1, g_2, [g_1, g_2], [f, g_1], [f, g_2], \\ [f, [f, g_1]], [f, [f, g_2]], [[f, g_1], g_2], [[f, g_1], g_1], \\ [[f, g_2], g_2], [[g_1, g_2], g_2], [[g_2, g_1], g_1], \dots \dots \end{array} \right\}$$

$\rightarrow$ *any Lie bracket combination of* f, g_1, g_2 *you can think of*

For the purpose of our class, accessibility at every point is equivalent to controllability

$$\dot{x}(t) = f(x) + g(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$

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$$Q = \ [g_1, \ldots, g_m, \ldots, \left[g_i, g_j\right] \ldots, \left[ad_{g_i}^k, g_j\right] \ldots, [f, g_i], \ldots, [ad_f^k, g_i], \ldots]$$

If you find n independent columns, then you have local accessibility!

If you want to be controllable, you should prove local accessibility everywhere.

Local accessibility example

$$\dot{x}_1 = u \quad \dot{x}_2 = x_1^2$$

Examine accessibility at $(0,0)$.

$$\dot{x}(t) = f(x) + g(x)u = f(x) + \sum_{i=1}^{m} g_i(x)u_i$$

<u>The system is locally accessible at (0, 0) if the accessibility distribution</u> spans $\mathbb{R}^2$

$$Q = \{g_1, \ldots, g_m, \ldots, [g_i, g_j] \ldots, [ad_{g_i}^k, g_j] \ldots, [f, g_i], \ldots, [ad_f^k, g_i], \ldots\}$$

What is m?

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$$Q = \{g_1, .., g_m, \dots, [g_i, g_j] \dots, [ad_{g_i}^k, g_j] \dots, [f, g_i], \dots, [ad_f^k, g_i], \dots\}$$

$$m = number\ of\ columns\ in\ g(x) = 1$$

$$f(x) = (0, x_1)^T, g(x) = (1{,}0)^T, [f, g](x) = (0, -1)^T$$

$$f(0,0) = (0{,}0)^T, g(0,0) = (1{,}0)^T, [f, g](0,0) = (0, -1)^T$$

$$Q = \{g(0\ 0), [f, g](0,0)\} = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$$

Rank 2! No need to compute higher order Lie brackets

Recap: Let's learn about the distributions and their rank

If $g_1, g_2, \dots, g_m$ are a set of vector fields on a manifold M

Their distribution $\Delta_x = span\{g_1(x), g_2(x), \dots, g_m(x)\}$ is a subspace on TM_x

The span consists of all possible linear combinations

The rank of a distribution is the number of independent vector fields

Let's compute the rank of a distribution

$$D = R^3, \Delta = span\{f_1, f_2, f_3\}$$

$$f_1 = \begin{pmatrix} x_1 \\ 1 + x_3 \\ 1 \end{pmatrix}, f_2 = \begin{pmatrix} x_1 x_2 \\ (1 + x_3)x_2 \\ x_2 \end{pmatrix}, f_3 = \begin{pmatrix} x_1 \\ x_1 \\ 0 \end{pmatrix}$$

Be careful because the rank of a distribution may not be the same at all points!!

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When $x_1 = x_2 = 0$, the rank is .....
When $x_1 = 0, x_2 \neq 0$, the rank is .....
When $x_1 \neq 0, x_2 = 0$, the rank is ......
When $x_1 \neq 0, x_2 \neq 0$, the rank is ......

Be careful because the rank of a distribution may not be the same at all points!!

$$D = R^3, \Delta = span\{f_1, f_2, f_3\}$$

$$f_1 = \begin{pmatrix} x_1 \\ 1 + x_3 \\ 1 \end{pmatrix}, f_2 = \begin{pmatrix} x_1 x_2 \\ (1 + x_3)x_2 \\ x_2 \end{pmatrix}, f_3 = \begin{pmatrix} x_1 \\ x_1 \\ 0 \end{pmatrix}$$

When $x_1 = x_2 = 0$, the rank is one.
When $x_1 = 0, x_2 \neq 0$, the rank is one.
When $x_1 \neq 0, x_2 = 0$, the rank is two.
When $x_1 \neq 0, x_2 \neq 0$, the rank is two.
The rank is **two** everywhere <u>except</u> when $x_1 = 0$ when the rank is **one**.

Involutive distributions

A distribution $\Delta_x = span\{g_1(x), g_2(x), \ldots, g_m(x)\}$ is involutive if and only if $[g_i(x), g_j(x)] \in \Delta_x$ for all i and j between 1 and m.

Quick problem: Is this distribution an involution?

$D = R^3, \Delta = span\{f_1, f_2, f_3\}$

$$f_1 = \begin{pmatrix} x_1 \\ 1 + x_3 \\ 1 \end{pmatrix}, f_2 = \begin{pmatrix} x_1 x_2 \\ (1 + x_3)x_2 \\ x_2 \end{pmatrix}, f_3 = \begin{pmatrix} x_1 \\ x_1 \\ 0 \end{pmatrix}$$

To be an involution, we need $[f_i, f_j] \in \Delta$ for all i, j pairs between 1 and 3

Let’s practice distributions more!

Compute the rank of this distribution and decide if it’s involutive:

Example: $D = R^3; \Delta = \text{span}\{f_1, f_2\}$

$$f_1 = \begin{bmatrix} 2x_2 \\ 1 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}$$

$$[f_1, f_2] = \;?$$

Involutive distribution example

Compute the rank of this distribution and decide if it's involutive:

Example: $D = R^3$; $\Delta = \text{span}\{f_1, f_2\}$

$$f_1 = \begin{bmatrix} 2x_2 \\ 1 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}, \quad \dim(\Delta(x)) = 2, \ \forall\, x \in D$$

$$[f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{rank}\ [f_1(x), f_2(x), [f_1, f_2](x)] =$$

$$\text{rank} \begin{bmatrix} 2x_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & x_2 & 1 \end{bmatrix} = 3, \ \forall\, x \in D$$

Δ is not involutive

Involutive distribution example 2

Compute the rank of this distribution and decide if it's involutive:

Example: $D = \{x \in R^3 \mid x_1^2 + x_3^2 \neq 0\}; \Delta = \text{span}\{f_1, f_2\}$

$$f_1 = \begin{bmatrix} 2x_3 \\ -1 \\ 0 \end{bmatrix}, \; f_2 = \begin{bmatrix} -x_1 \\ -2x_2 \\ x_3 \end{bmatrix}$$

$$[f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 = \text{?}$$

Involutive distribution example 2

Compute the rank of this distribution and decide if it's involutive:

Example: $D = \{x \in R^3 \mid x_1^2 + x_3^2 \neq 0\}; \Delta = \text{span}\{f_1, f_2\}$

$$f_1 = \begin{bmatrix} 2x_3 \\ -1 \\ 0 \end{bmatrix}, \ f_2 = \begin{bmatrix} -x_1 \\ -2x_2 \\ x_3 \end{bmatrix}, \ \dim(\Delta(x)) = 2, \ \forall\, x \in D$$

$$[f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 = \begin{bmatrix} -4x_3 \\ 2 \\ 0 \end{bmatrix}$$

$$\text{rank} \begin{bmatrix} 2x_3 & -x_1 & -4x_3 \\ -1 & -2x_2 & 2 \\ 0 & x_3 & 0 \end{bmatrix} = 2, \ \forall\, x \in D$$

Δ is involutive

Special controllability case: no drift

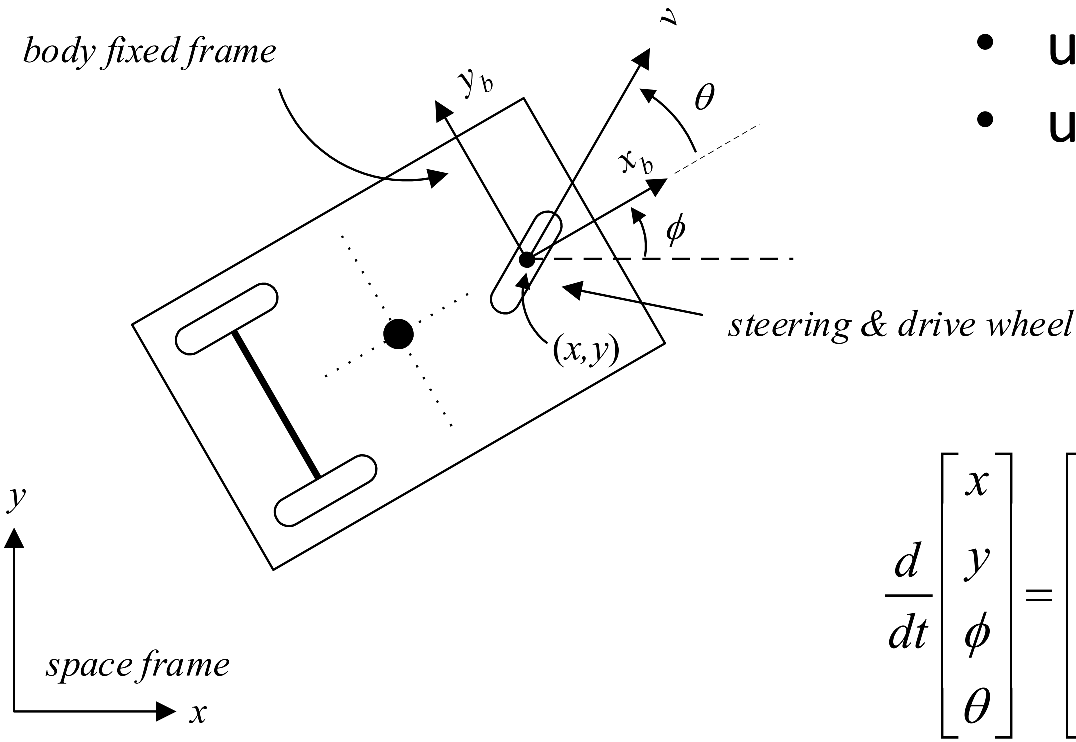
$$\dot{x}(t) = g(x)u = \sum_{i=1}^{m} g_i(x)u_i$$

<u>Theorem</u>: If the drift term $f(x) = 0$, then the system is controllable if the matrix Q has full rank.

$$Q = \{g_1, .., g_m, \dots, [g_i, g_j] \dots, [ad_{g_i}^k, g_j] \ \dots, \}$$

If you find n independent columns, then you have controllability!

Controllability Example: Parking



- u_1 is the driving input
- u_2 is the steering input

$$u_1 = v, u_2 = \omega$$

drive

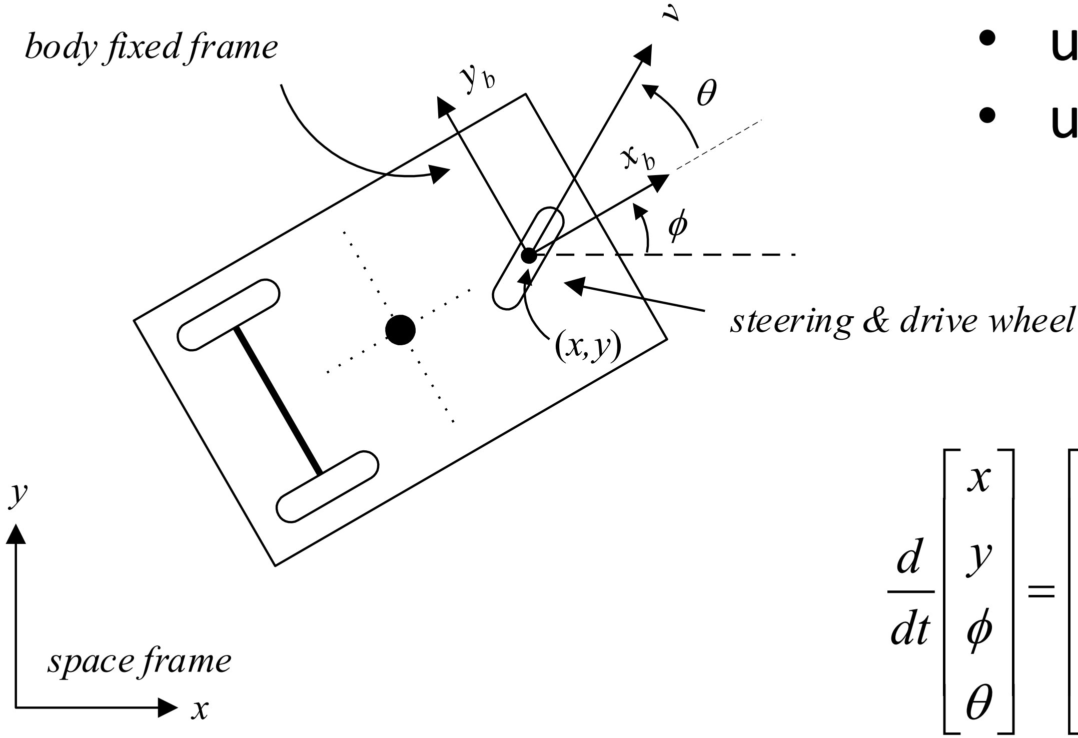
$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \phi \\ \theta \end{bmatrix} = \begin{bmatrix} \cos(\phi+\theta) & 0 \\ \sin(\phi+\theta) & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

steer

Cars are just carts with fixed rear axis

First things to check to determine controllability:

Is the dynamical system in affine form? Are we in the special case f(x) = 0? Who is g(x)?



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steer

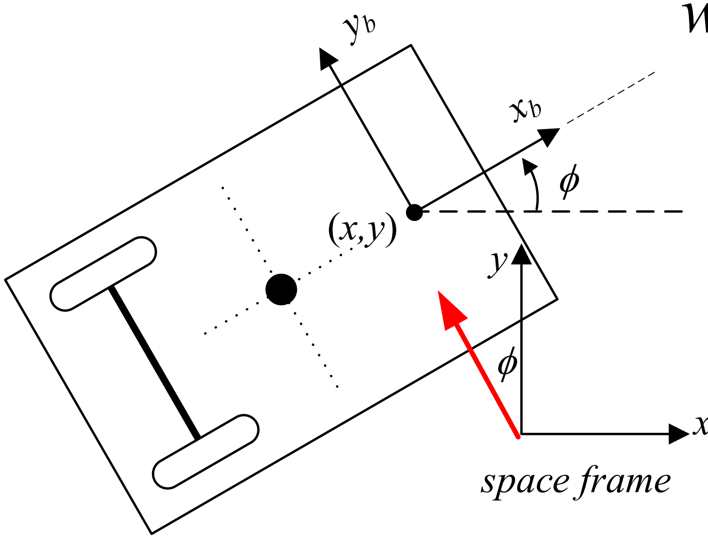
Cars are just carts with fixed rear axis

Parking, Controllability Distribution

Q = $\left\langle f, g_1, \ldots, g_m \middle| \operatorname{span}\left\{ f, g_1, \ldots, g_m \right\} \right\rangle$

Q = $\left\langle \begin{bmatrix} \cos(\phi+\theta) \\ \sin(\phi+\theta) \\ \sin(\theta) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \middle| \operatorname{span}\left\{ \begin{bmatrix} \cos(\phi+\theta) \\ \sin(\phi+\theta) \\ \sin(\theta) \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$

Parking, new directions from Lie bracket



$$wriggle = [steer, drive] = \begin{bmatrix} -\sin(\theta+\phi) \\ \cos(\theta+\phi) \\ \cos\theta \\ 0 \end{bmatrix}$$

$$slide = [wriggle, drive] = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \\ 0 \end{bmatrix}$$

Parking, Span of Controllability Distribution

$$\text{span}\left\{\begin{bmatrix}\cos(\phi+\theta)\\ \sin(\phi+\theta)\\ \sin(\theta)\\ 0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix},\begin{bmatrix}-\sin(\theta+\phi)\\ \cos(\theta+\phi)\\ \cos\theta\\ 0\end{bmatrix},\begin{bmatrix}-\sin\phi\\ \cos\phi\\ 0\\ 0\end{bmatrix}\right\}$$

$$\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$$

This shows that you can manoeuver your car into any parking lot by applying controls corresponding to the 'Slide' direction, i.e., by applying the control sequence {Wriggle, Drive, −Wriggle, −Drive}.

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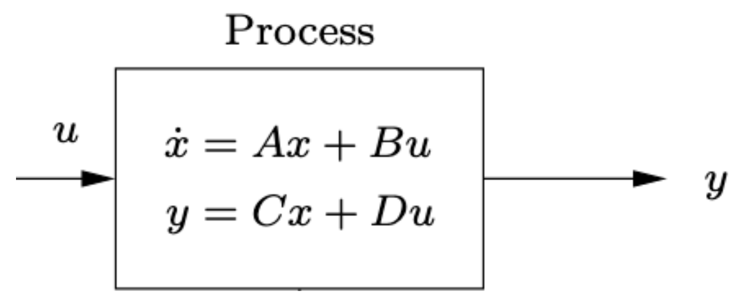
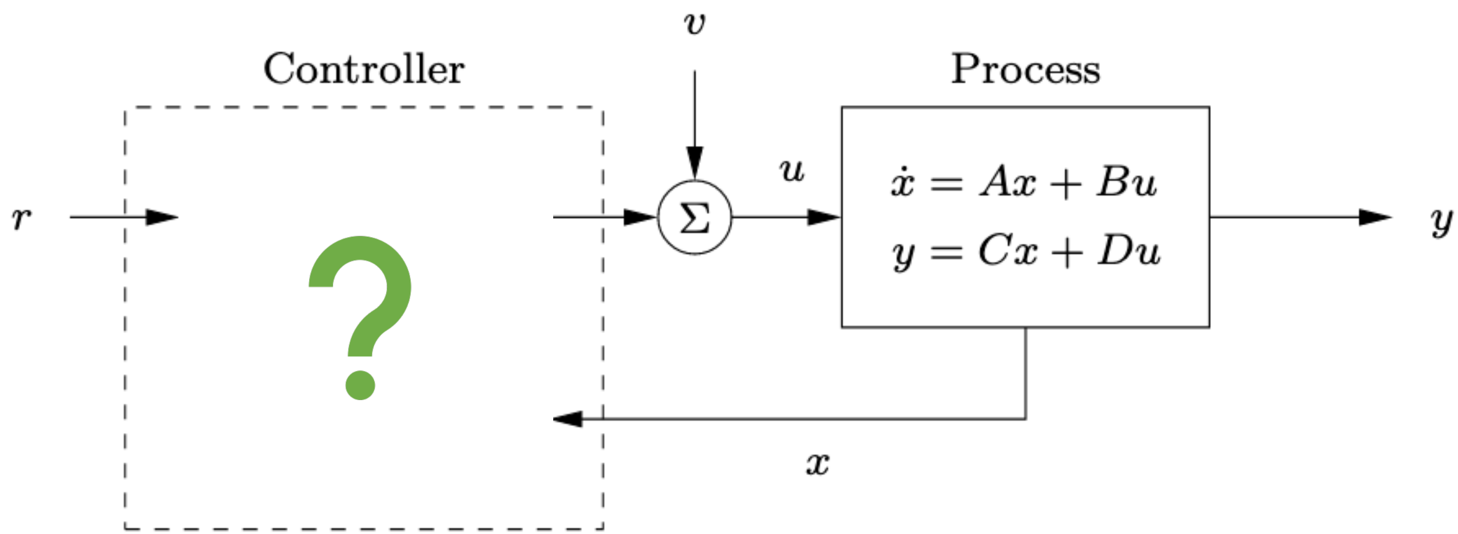
Block Diagram Open Loop: Linear System



Astrom, Murray, Feedback Systems, 2012

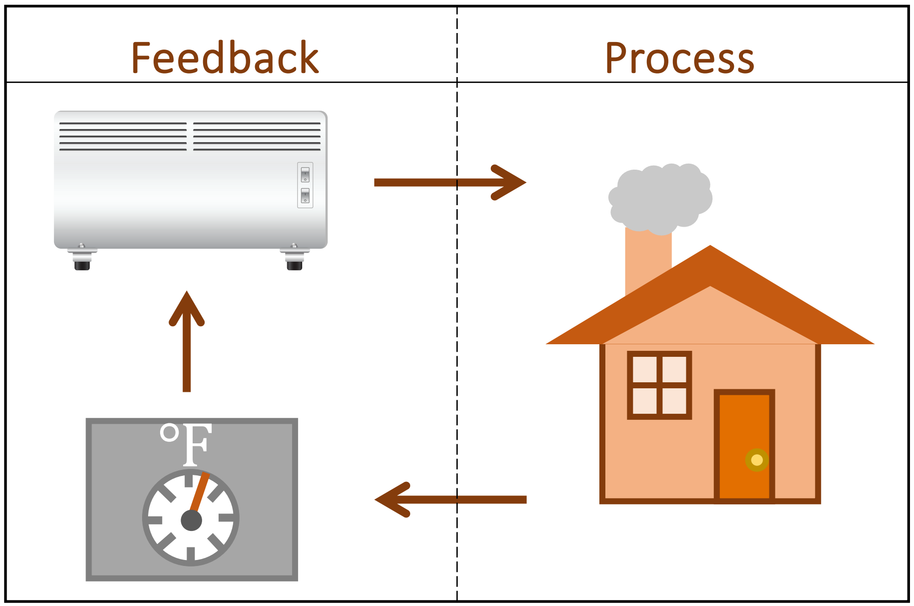
Block Diagram Closed Loop: Controller for Linear Systems



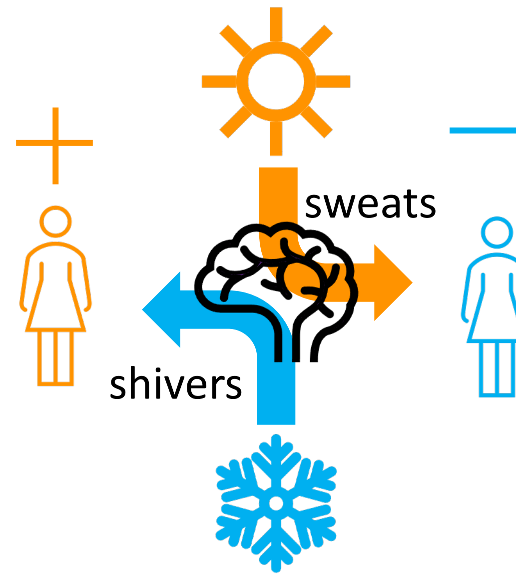
Astrom, Murray, Feedback Systems, 2012

Feedback allows engineered and biological systems to adapt to their changing environment

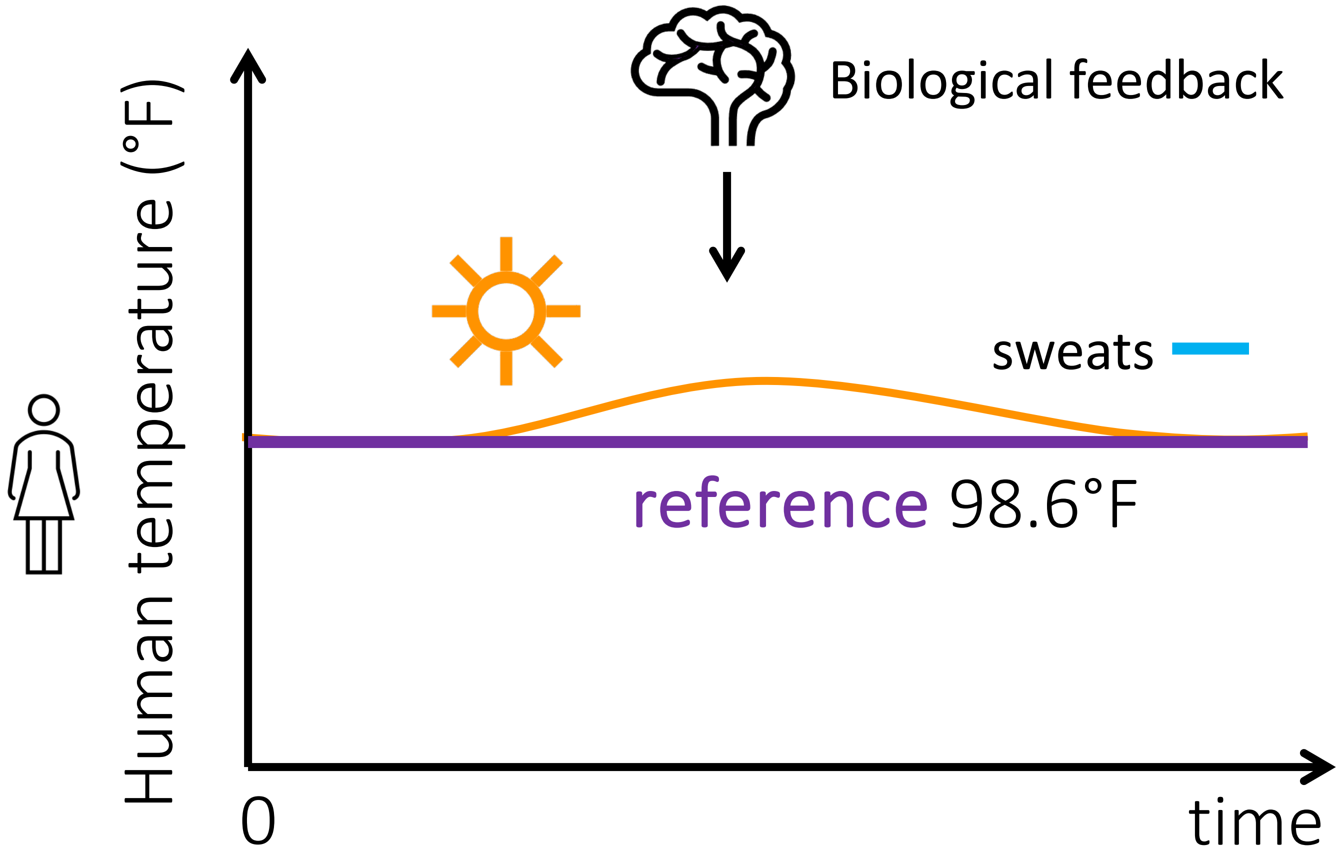
Engineering Feedback



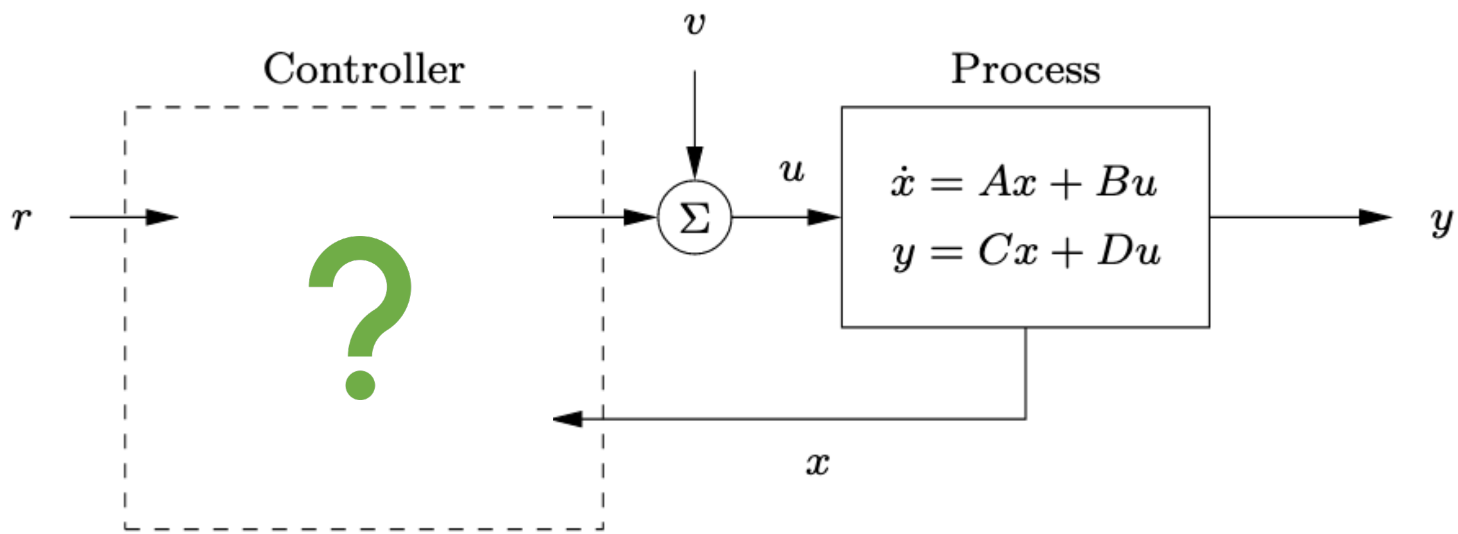
Biological Feedback



One role of engineered and biological feedback is reference tracking



Block Diagram Closed Loop: Controller for Linear Systems



Process

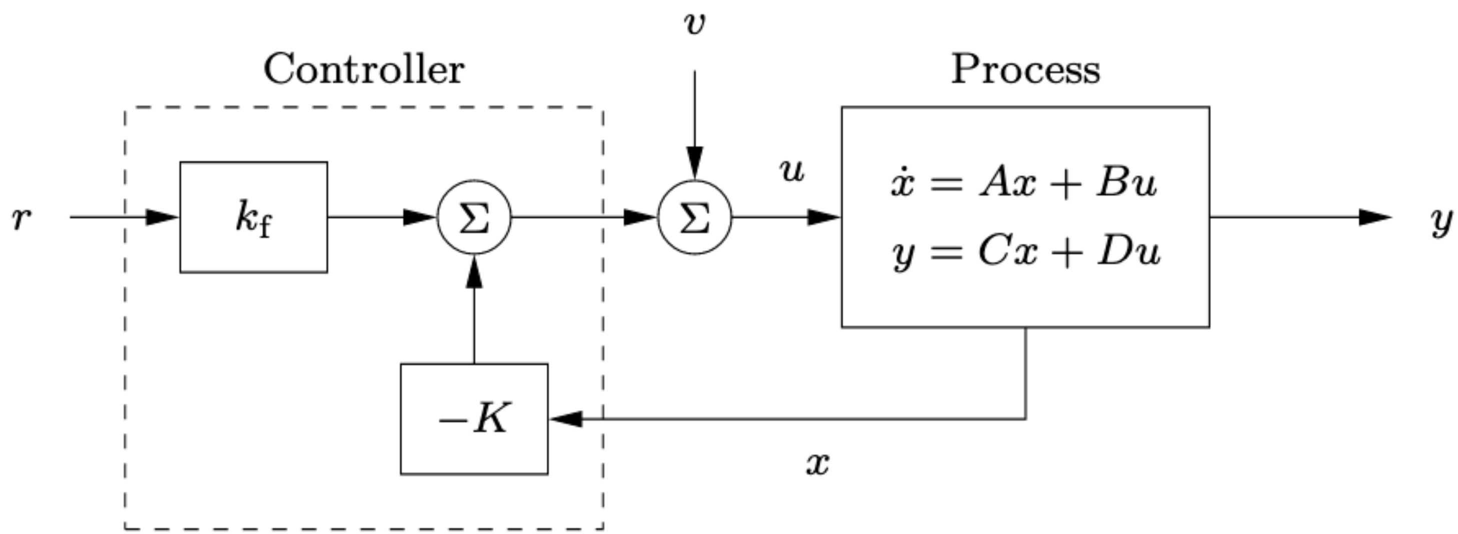
$$\boxed{\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du}$$

Controller?

$$\boxed{u = ?}$$

Astrom, Murray, Feedback Systems, 2012

Block Diagram Closed Loop: A Controller Option: State Feedback for Linear Systems



A feedback control system with state feedback. The controller uses the system state x and the reference input r to command the process through its input u. We model disturbances via the additive input v.

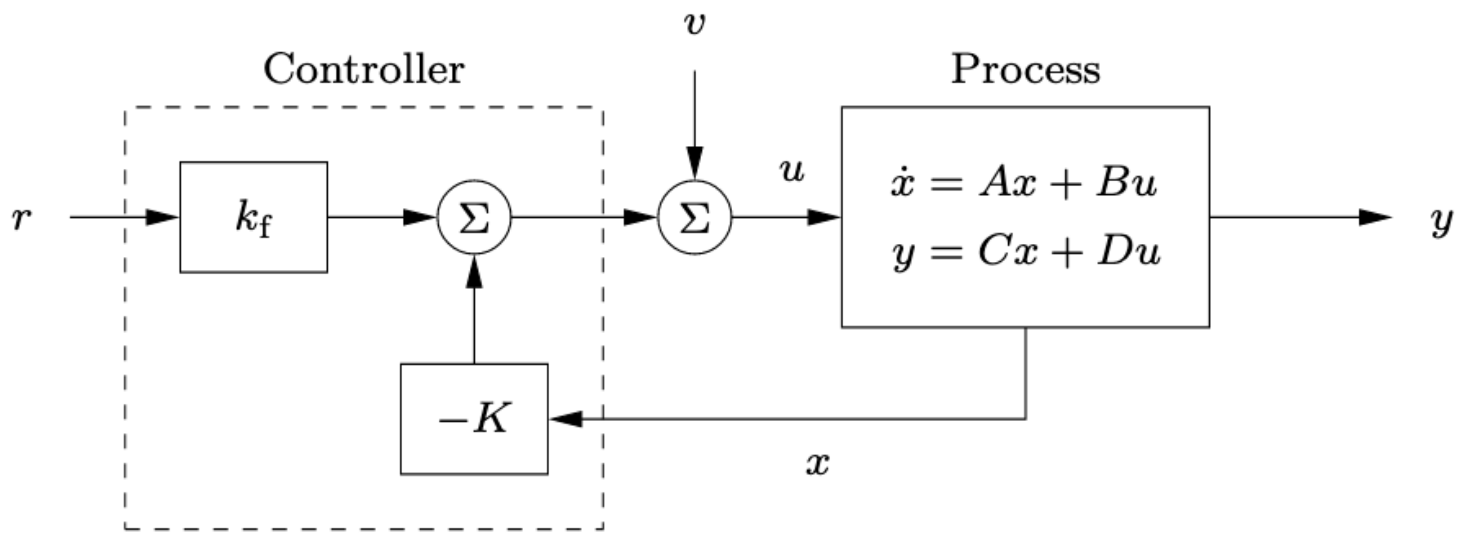
$$\boxed{\frac{dx}{dt} = Ax + Bu, \qquad y = Cx + Du}$$

$$\boxed{u = -Kx + k_f r}$$

State Feedback Controller

Astrom and Murray, 2020

Block Diagram Closed Loop: A Controller Option: State Feedback for Linear Systems



A feedback control system with state feedback. The controller uses the system state x and the reference input r to command the process through its input u. We model disturbances via the additive input v.

$$\boxed{\frac{dx}{dt} = (A - BK)x + Bk_f r.}$$

$$\boxed{u = -Kx + k_f r}$$

State Feedback Controller

Astrom and Murray, 2020

A Controller Option: State Feedback for Linear Systems

$$\boxed{\frac{dx}{dt} = (A - BK)x + Bk_\text{f}r,}$$

$$\boxed{y = Cx + Du,}$$

$$\boxed{u = -Kx + k_\text{f}r}$$

State Feedback Controller

Astrom and Murray, 2020

Who should K the gain of state feedback be?

$$\boxed{\frac{dx}{dt} = (A - BK)x + Bk_f r.}$$

$$\boxed{y = Cx + Du}$$

$$\boxed{u = -Kx + k_f r}$$

State Feedback Controller

Astrom and Murray, 2020

Gain K should at least be chosen to achieve stability; other considerations may be important too

$$\boxed{\frac{dx}{dt} = (A - BK)x + Bk_f r,}$$

$$\boxed{y = Cx + Du}$$

$$\boxed{u = -Kx + k_f r}$$

State Feedback Controller

Pole placement: We choose K such that the roots of the characteristic polynomial λ have real part < 0

$$\lambda(s) = \det(sI - A + BK)$$

Astrom and Murray, 2020

Lastly, to successfully track the reference, we should choose gain k_f

$$\frac{dx}{dt} = (A - BK)x + Bk_f r,$$

$$y = Cx + Du$$

$$\boxed{u = -Kx + k_f r}$$

State Feedback Controller

When D = 0, we choose k_f as follows to obtain tracking:

$$k_f = -1/\left(C(A - BK)^{-1}B\right)$$

Astrom and Murray, 2020

Example 1: Controlling the number of lynxes to equal 30 through the food supply for hares in the predator-prey model

$$\frac{dH}{dt} = (r+u)H\left(1 - \frac{H}{k}\right) - \frac{aHL}{c+H}, \qquad H \geq 0,$$
$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL, \qquad L \geq 0.$$

u = ? such that L_{eq} = 30 lynxes

$a = 3.2, \quad b = 0.6, \quad c = 50,$
$d = 0.56, \quad k = 125 \quad r = 1.6.$

Astrom and Murray, 2020

Example 1: Recall that the open loop predator-prey model (u = 0) produces limit cycle oscillations of hares and lynxes

$$\frac{dH}{dt} = (r + \overset{0}{u})H\left(1 - \frac{H}{k}\right) - \frac{aHL}{c+H}, \qquad H \geq 0,$$

$$\frac{dL}{dt} = b\frac{aHL}{c+H} - dL, \qquad L \geq 0.$$

u = ? such that L_{eq} = 30 lynxes

$$a = 3.2, \qquad b = 0.6, \qquad c = 50,$$
$$d = 0.56, \qquad k = 125 \qquad r = 1.6.$$

Astrom and Murray, 2020

Example 1: Step 1) linearize and find the equilibrium of the dynamical system

At equilibrium: set $\frac{dH}{dt} = \frac{dL}{dt} = 0.$

We get H_{eq} = 20.6 hares, L_{eq} = 29.5 lynxes

Hmmm… not (0 hares, 0 lynxes). What to do?

Astrom and Murray, 2020

Example 1: Step 2) use linear transformation to shift the equilibrium to (0 hares, 0 lynxes)

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0.13 & -0.93 \\ 0.57 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 17.2 \\ 0 \end{pmatrix} v,$$

Define new variables $z_1 = H - H_{eq}$, $z_2 = L - L_{eq}$.

- $H_{eq} = 20.6$, $L_{eq} = 29.5$

Astrom and Murray, 20207677

Example 1: Step 2) use linear transformation to shift the equilibrium to (0 hares, 0 lynxes)

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0.13 & -0.93 \\ 0.57 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 17.2 \\ 0 \end{pmatrix} v$$

Define new variables $z_1 = H - H_{eq}$, $z_2 = L - L_{eq.}$

- $H_{eq} = 20.6$, $L_{eq} = 29.5$

$$\frac{dz_1}{dt} = \frac{dH}{dt}$$

$$\frac{dz_2}{dt} = \frac{dL}{dt}$$

$$v = u$$

Astrom and Murray, 20207677

Example 1: Step 3) check that the linearly shifted system is controllable to (0 hares, 0 lynxes). How can we do that?

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0.13 & -0.93 \\ 0.57 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 17.2 \\ 0 \end{pmatrix} v,$$

$$Q = \begin{bmatrix} 17.2 & 2.23 \\ 0 & 9.8 \end{bmatrix} \longrightarrow \text{Controllable}$$

Astrom and Murray, 2020

Example 1: Step 4) assign the eigenvalues of the characteristic polynomial of the system using state feedback

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0.13 & -0.93 \\ 0.57 & 0 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 17.2 \\ 0 \end{pmatrix} v,$$

- For simplicity, we choose them to be $\lambda_1 = -0.1, \lambda_2 = -0.2$ and we solve for the feedback gain.

- We obtain $K = \begin{pmatrix} 0.025 & -0.052 \end{pmatrix}$

Astrom and Murray, 2020

Example 1: Step 5) solve for the gain k_f to obtain tracking

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} 0.13 & -0.93 \\ 0.57 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 17.2 \\ 0 \end{pmatrix} v$$

$$k_\mathrm{f} = -1/\big(C(A - BK)^{-1}B\big) \quad \Rightarrow \quad k_f = 0.002$$

$$\begin{aligned} u &= u_\mathrm{e} - K(x - x_\mathrm{e}) + k_\mathrm{f}(L_\mathrm{d} - y_\mathrm{e}) \\ &= -\begin{pmatrix} 0.025 & -0.052 \end{pmatrix} \begin{pmatrix} H - 20.6 \\ L - 29.5 \end{pmatrix} + 0.002\,(L_\mathrm{d} - 29.5 \end{aligned}$$

Astrom and Murray, 2020

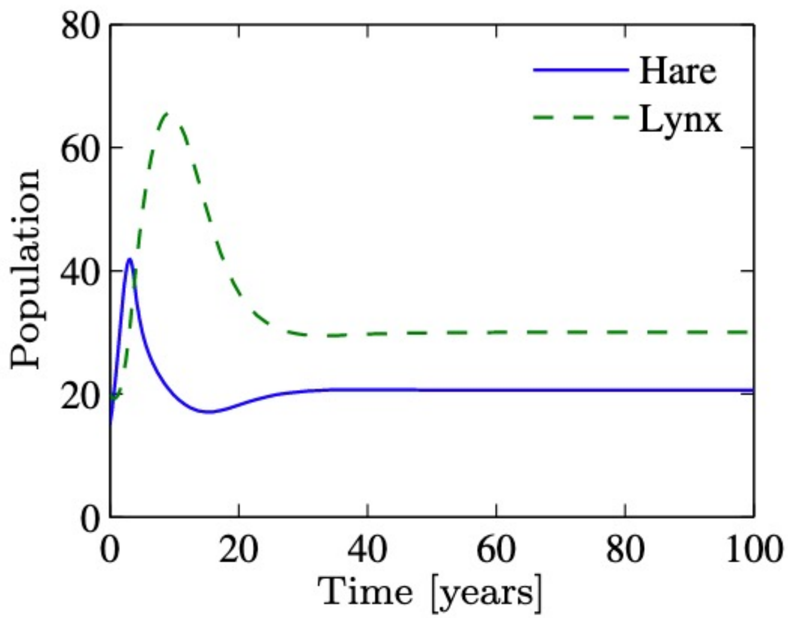
Example 1: Step 6) write down the state feedback

$$
\begin{aligned}
u = \quad & - K(x - x_e) + k_f(L_d - y_e) \\
= - & \begin{pmatrix} 0.025 & -0.052 \end{pmatrix} \begin{pmatrix} H - 20.6 \\ L - 29.5 \end{pmatrix} + 0.002\,(L_d - 29.5)
\end{aligned}
$$

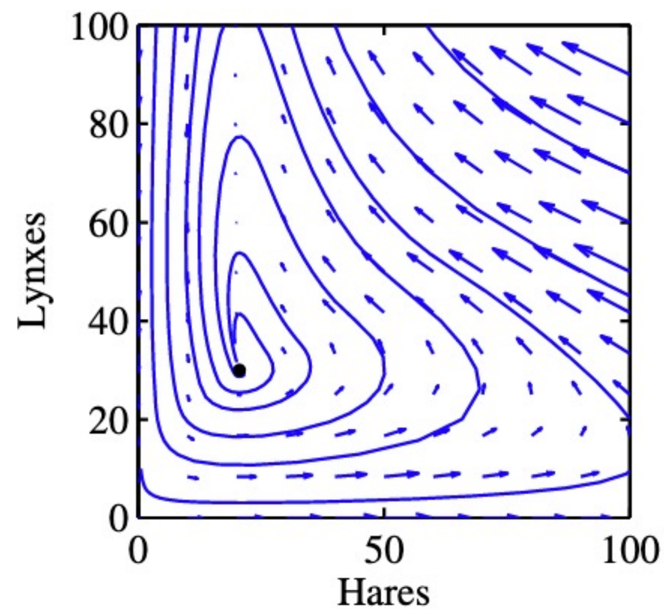
$$L_d = 30$$

Astrom and Murray, 2020

Example 1: Did we control the number of lynxes to 30 using state feedback?



(a) Initial condition response

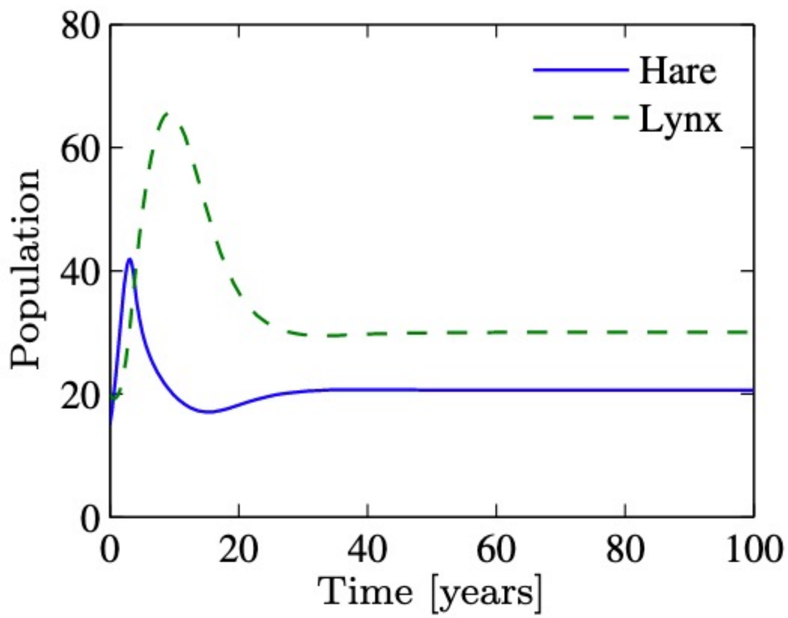


(b) Phase portrait

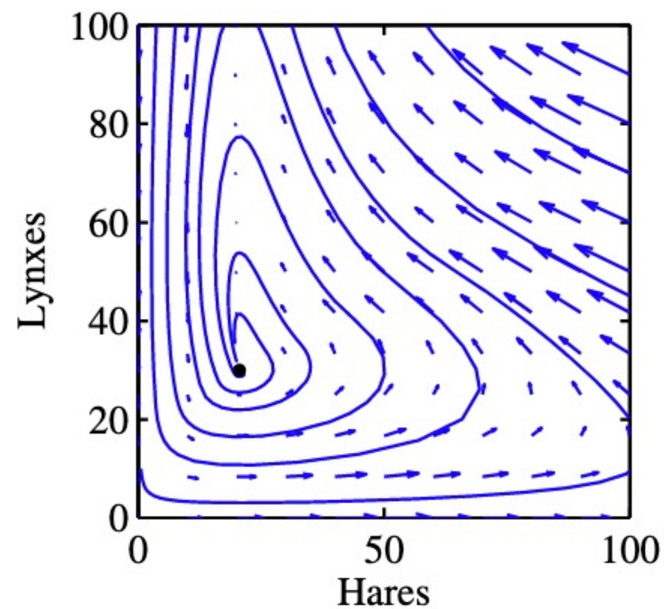
Simulation results for the controlled predator–prey system. The population of lynxes and hares as a function of time is shown in (a), and a phase portrait for the controlled system is shown in (b). Feedback is used to make the population stable at $H_e = 20.6$ and $L_e = 30$.

Astrom and Murray, 2020

Example 1: Is this state feedback controller unique?



(a) Initial condition response



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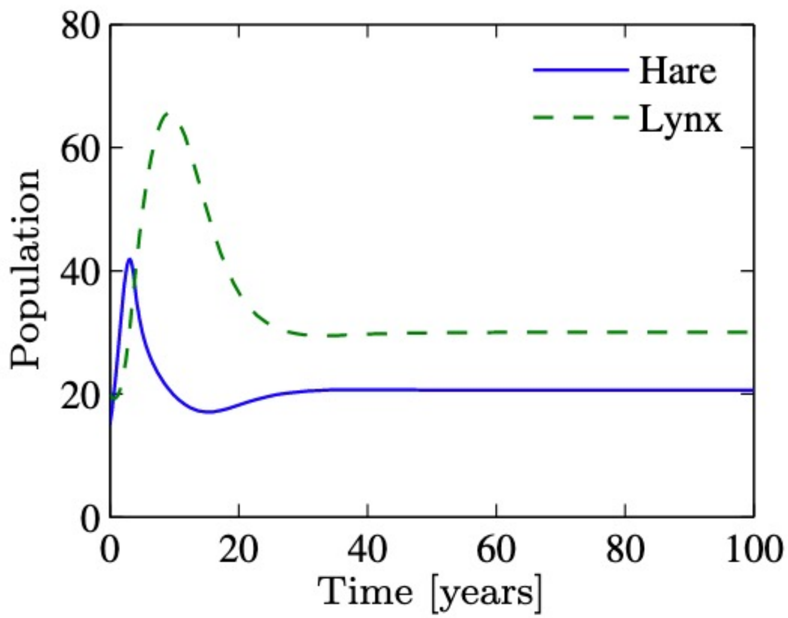
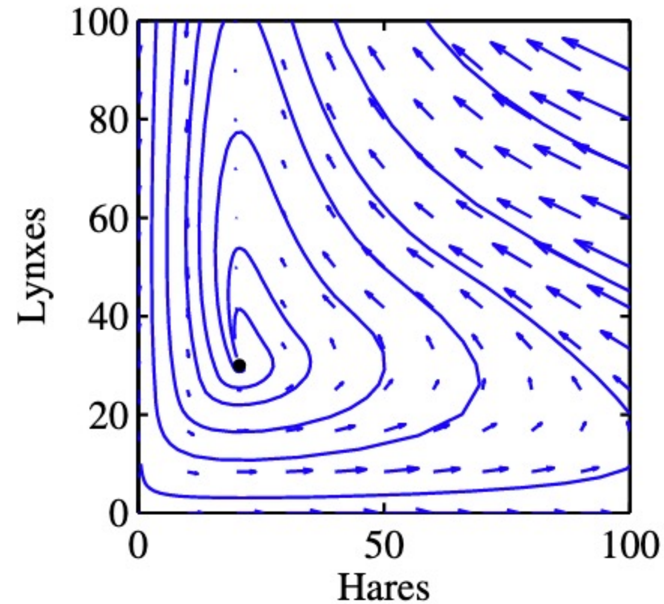
Astrom and Murray, 2020

Example 1: Is this state feedback controller unique? NO!!



(a) Initial condition response



(b) Phase portrait

Simulation results for the controlled predator–prey system. The population of lynxes and hares as a function of time is shown in (a), and a phase portrait for the controlled system is shown in (b). Feedback is used to make the population stable at $H_e = 20.6$ and $L_e = 30$.

Astrom and Murray, 2020

Summary Slide

- A useful controller for linear dynamical system is state feedback
- There are many types of controllers for linear systems that achieve different objectives
- State feedback can be used in the tracking problem i.e.

I want to control the food supply of the hares to keep a population of 30 lynxes

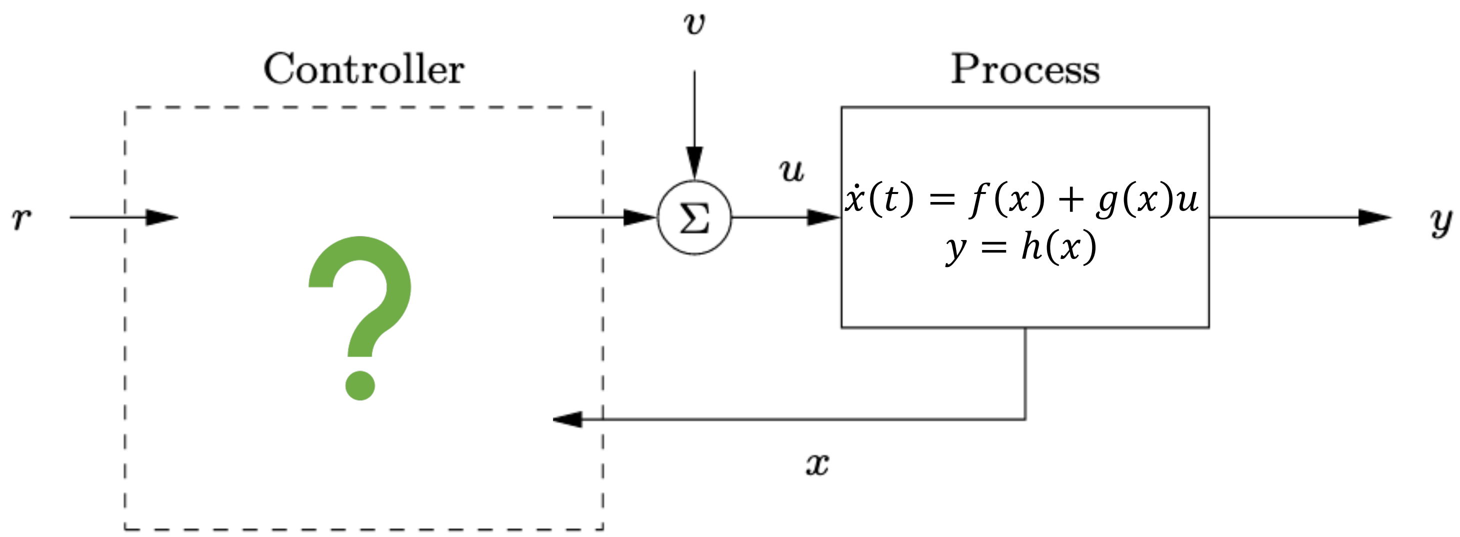
Summary State Feedback Steps:

1) Find the equilibrium of the dynamical system
2) Use linear transformation to shift the equilibrium to (0, 0, .., 0)
3) Check that the linearly shifted system is controllable to the desired reference
4) Assign the eigenvalues of the characteristic polynomial of the system with state feedback
5) Solve for the gain k_f to ensure tracking
6) Write down the state feedback

Outline

- Finishing up Controllability:
  - Distributions
  - Rank
- State Feedback for Linear Systems
- Feedback Linearization of Nonlinear Systems:
  - Relative Degree
  - Nonlinear Coordinate Transformation
  - Affine Form to Canonical Form

What are some Controllers for Nonlinear Affine Systems?



Astrom, Murray, Feedback Systems, 2012

Recap: We study Affine Nonlinear Systems because they have clear criteria for controllability and we can formulate Lie Derivatives and brackets associated to their vector fields

General Form of a Nonlinear System

$$\dot{x}(t) = f(x, u)$$
$$y = h(x, u)$$

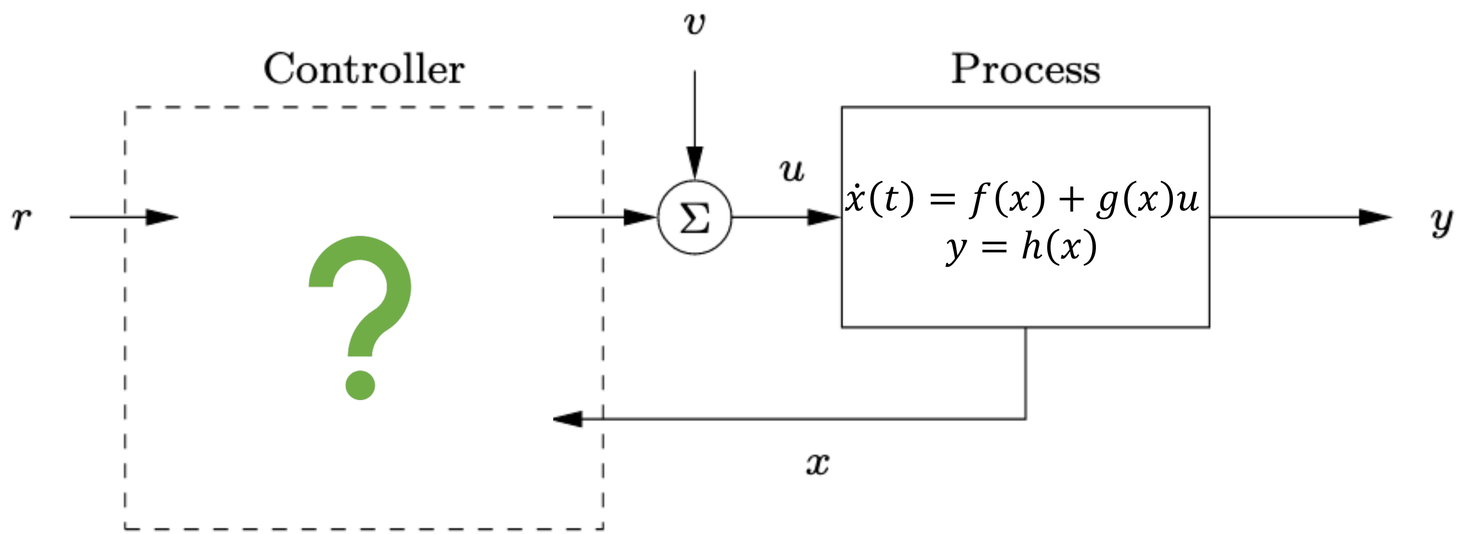
Affine Nonlinear System

$$\dot{x}(t) = f(x) + g(x)u$$
$$y = h(x)$$

Lie Derivative of Affine Nonlinear System

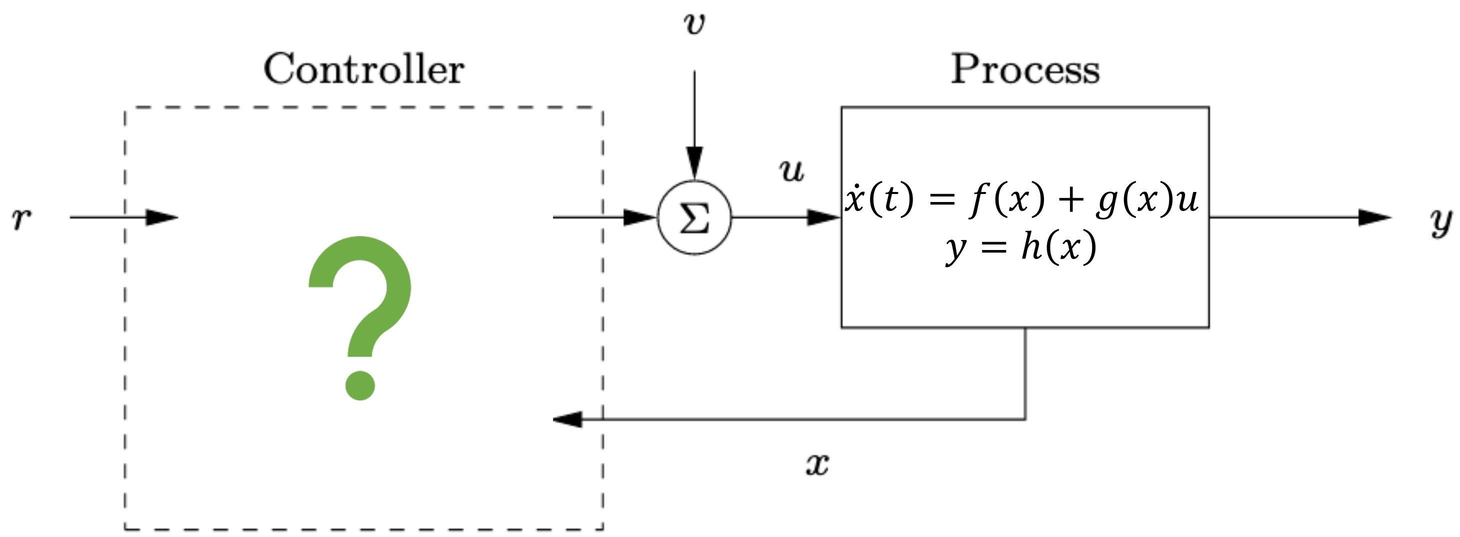
$$L_f h(x) = \frac{dh}{dx} f(x)$$

A useful controller for Nonlinear Affine Systems performs Feedback Linearization



Astrom, Murray, Feedback Systems, 2012

The main idea is that the controller gets rid of the nonlinearity in the system

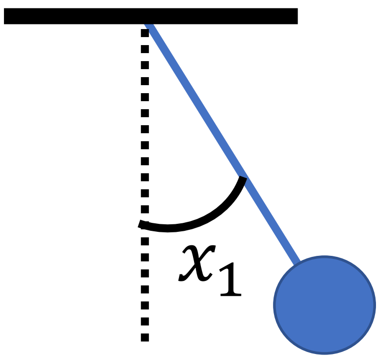


Astrom, Murray, Feedback Systems, 2012

Example 2: Feedback Linearization of a Pendulum

Mathematical Model

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a\sin x_1 - bx_2 + c\boldsymbol{u}$$



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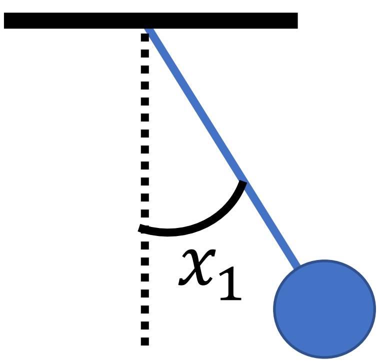

Choose the Controller:

$$\boldsymbol{u} = \frac{a}{c}\sin x_1 + \frac{1}{c}\boldsymbol{v}$$

Example 2: Feedback Linearization of a Pendulum

Mathematical Model

$$\dot{x_1} = x_2$$
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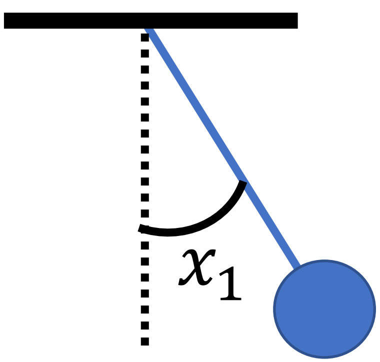

Choose the Controller:

$$\boldsymbol{u} = \frac{a}{c} \sin x_1 + \frac{1}{c} \boldsymbol{v}$$

Linear Dynamical System

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -b x_2 + \boldsymbol{v}$$

Example 2: Feedback Linearization of a Pendulum

Mathematical Model

$$\dot{x}_1 = x_2$$
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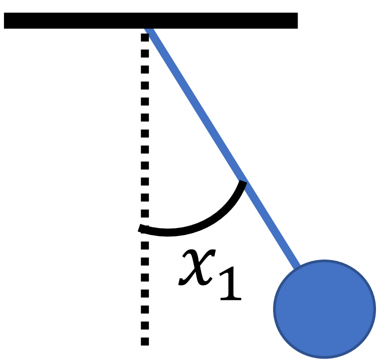


Choose the Controller:

$$\boldsymbol{u} = \frac{a}{c}\sin x_1 + \frac{1}{c}\boldsymbol{v}$$

$$\Downarrow$$

Linear Dynamical System

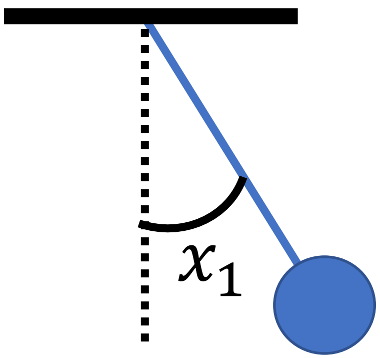
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -bx_2 + \boldsymbol{v}$$

- For the linear dynamical system, we can choose $\boldsymbol{v}$ to be one of many controllers. In this example, we can choose $\boldsymbol{v}$ to be a state feedback controller from before.

Example 2: Feedback Linearization of a Pendulum

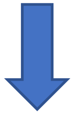
Mathematical Model

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin x_1 - b x_2 + c\boldsymbol{u}$$



Choose the Controller:

$$\boldsymbol{u} = \frac{a}{c} \sin x_1 + \frac{1}{c}\boldsymbol{v}$$



Linear Dynamical System

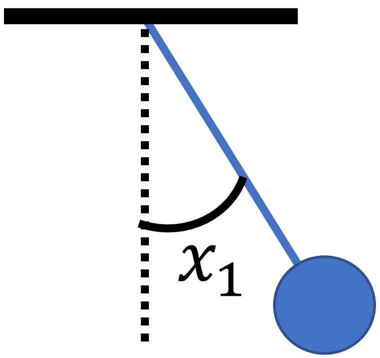
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -b x_2 + \boldsymbol{v}$$

$$\boldsymbol{v} = -Kx = -k_1 x_1 - k_2 x_2$$

Example 2: Feedback Linearization of a Pendulum

Mathematical Model

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin x_1 - b x_2 + c\boldsymbol{u}$$



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Linear Dynamical System

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -b x_2 + \boldsymbol{v}$$

$$\boldsymbol{v} = -Kx = -k_1 x_1 - k_2 x_2$$

- We can choose the gain K as before to guarantee stability

Example 2: Feedback Linearization of a Pendulum

Mathematical Model

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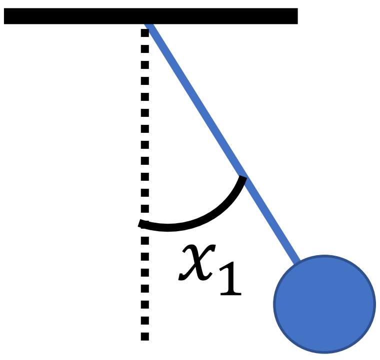
$$\dot{x}_2 = -a\sin x_1 - bx_2 + c\boldsymbol{u}$$



Choose the Controller:

$$\boldsymbol{u} = \frac{a}{c}\sin x_1 + \frac{1}{c}\boldsymbol{v}$$

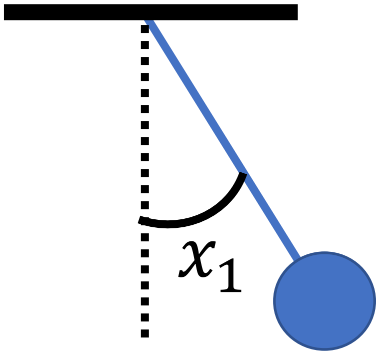
Feedback Linearization

$$\boldsymbol{u} = \frac{a}{c}\sin x_1 + \frac{1}{c}(-k_1x_1 - k_2x_2)$$

We found a controller for a nonlinear system. Can we always do feedback linearization?

Mathematical Model

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a\sin x_1 - bx_2 + c\boldsymbol{u}$$



Choose the Controller:

$$\boldsymbol{u} = \frac{a}{c}\sin x_1 + \frac{1}{c}\boldsymbol{v}$$

$$\Downarrow$$

Feedback Linearization

$$\boldsymbol{u} = \frac{a}{c}\sin x_1 + \frac{1}{c}(-k_1 x_1 - k_2 x_2)$$

Can we always do feedback linearization for nonlinear affine systems?

- Yes if the relative degree is n

Affine Nonlinear System

$$\dot{x}(t) = f(x) + g(x)u$$
$$y = h(x)$$

Let's learn about the relative degree of an affine nonlinear system

Affine Nonlinear System

$$\dot{x}(t) = f(x) + g(x)u$$
$$y = h(x)$$

The Affine Nonlinear System has relative degree ρ if u does not appear in the equation of $y, \dot{y}, \ddot{y}, y^{\rho\,-1}$, but it appears in the equation of y^{ρ}.

Let's learn about the relative degree of an affine nonlinear system

Affine Nonlinear System

$$\dot{x}(t) = f(x) + g(x)u$$
$$y = h(x)$$

The Affine Nonlinear System has relative degree ρ if u does not appear in the equation of $y, \dot{y}, \ddot{y}, y^{\rho\,-1}$, but it appears in the equation of y^{ρ}.

If u does not appear in any of the equations $y, \dot{y}, \ddot{y}, \dots, y^{n}$, then the system has infinite relative degree.

Quick problem 1: Compute the relative degree ρ of the pendulum

Mathematical Model

$$\dot{x_1} = x_2$$
$$\dot{x_2} = -a\sin x_1 - bx_2 + c\boldsymbol{u}$$
$$y = x_1$$

relative degree ρ if u does not appear in the equation of $y, \dot{y}, \ddot{y}, y^{\rho-1}$, but it appears in the equation of y^{ρ}.

Quick problem 2: Compute the relative degree ρ

Mathematical Model

$$\dot{x_1} = x_1$$
$$\dot{x_2} = 2\,x_2 + \boldsymbol{u}$$
$$y = x_1$$

relative degree ρ if u does not appear in the equation of $y, \dot{y}, \ddot{y}, y^{\rho-1}$, but it appears in the equation of y^{ρ}.

The relative degree can be expressed using Lie derivatives

Affine Nonlinear System

$$\dot{x}(t) = f(x) + g(x)u$$
$$y = h(x)$$

The relative degree is ρ on domain D if and only if

- $L_g L_f^{i-1} h(x) = 0$ for i between 0 and $\rho - 1$ and
- $L_g L_f^{\rho-1} h(x) \neq 0$

for all $x \in D$.

To construct feedback linearization for an affine nonlinear system of relative degree n, we need to first perform a Nonlinear Coordinate Transformation

Affine Nonlinear System

$$\dot{x}(t) = f(x) + g(x)u$$
$$y = h(x)$$

We performed a Linear Coordinate Transformation for the hares and the lynxes earlier to push the equilibrium point to (0, 0)

At equilibrium: set $\frac{dH}{dt} = \frac{dL}{dt} = 0$.

We get H_{eq} = 20.6 hares, L_{eq} = 29.5 lynxes

Hmmm… not (0 hares, 0 lynxes). What to do?

Define new variables $z_1 = H - H_{eq}$, $z_2 = L - L_{eq}$.

How can we perform a Nonlinear Coordinate Transformation?

Affine Nonlinear System

$$\dot{x}(t) = f(x) + g(x)u$$
$$y = h(x)$$

Nonlinear Coordinate Transformation

$$z = T(x)$$

How can we perform a Nonlinear Coordinate Transformation?

Affine Nonlinear System

$$\dot{x}(t) = f(x) + g(x)u$$
$$y = h(x)$$

Nonlinear Coordinate Transformation

$$z = T(x)$$

- T(x) is invertible if there exists a function $T^{-1}(z)$ such that $T^{-1}\big(T(x)\big) = x$ for all $x \in$ domain D

How can we perform a Nonlinear Coordinate Transformation?

Affine Nonlinear System

$$\dot{x}(t) = f(x) + g(x)u$$
$$y = h(x)$$

Nonlinear Coordinate Transformation

$$z = T(x)$$

- This definition is equivalent to T being a diffeomorphism

Quick examples of nonlinear coordinate transformations

$T_1(x) = [\begin{matrix} x_1 + x_2 \\ x_2 \end{matrix}]$ is a diffeomorphism everywhere

$T_2(x) = [\begin{matrix} x_1 \\ \sin x_2 \end{matrix}]$ is a local diffeomorphism on the set

$D = \left\{x \in \mathbb{R}^2, |x_2 < \frac{\pi}{2}|\right\}$

How can we perform a Nonlinear Coordinate Transformation?

Affine Nonlinear System with relative degree $\rho = n$

$$\dot{x}(t) = f(x) + g(x)u$$
$$y = h(x)$$

Nonlinear Coordinate Transformation for Feedback Linearization

$$z = T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}$$

What is the result of the Nonlinear Coordinate Transformation?

Affine Nonlinear System with relative degree $\rho = n$

$$\dot{x}(t) = f(x) + g(x)u$$
$$y = h(x)$$

Normal Canonical Form

$$\dot{z} = Az + b\gamma(x)[u - \alpha(x)]$$
$$y = cz$$

$$A = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & 0 & \\ \vdots & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

Chain of Integrators

When can we perform the nonlinear transformation to the Normal Canonical Form?

When the Affine Nonlinear System has relative degree $\rho = n$:

- $L_g L_f^{i-1} h(x) = 0$ for i between 0 and n−1 and $L_g L_f^{n-1} h(x) \neq 0$

for all $x \in D_0$.

When can we perform the nonlinear transformation to the Normal Canonical Form?

When the Affine Nonlinear System has relative degree $\rho = n$:

- $L_g L_f^{i-1} h(x) = 0$ for i between 0 and n−1 and $L_g L_f^{n-1} h(x) \neq 0$ for all $x \in D_0$.

$\Updownarrow$ EQUIVALENT

<u>Theorem:</u>

$$\dot{x} = f(x) + g(x)u$$

is transformable into the canonical form if and only if there is a domain D_0 such that

$$\text{rank}[g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)] = n, \ \ \forall\, x \in D_0$$

and

$$\text{span}\,\{g, ad_f g, \ldots, ad_f^{n-2} g\}$$ is involutive in D_0

Proof of the Nonlinear Transformation to the Canonical Form

The relative degree is $\rho = n$ if and only if

- $L_g L_f^{i-1} h(x) = 0$ for i between 0 and

n−1 and

- $L_g L_f^{n-1} h(x) \neq 0$

The transformed dynamical system is:

$$z_1 = h(x)$$

$$\dot{z}_1 = \frac{dh(x)}{dx}\dot{x} = \frac{dh(x)}{dx}(f(x) + g(x)u) = L_f h(x) + \cancelto{0}{L_g} h(x)u = \boldsymbol{L_f h(x)} = \boldsymbol{z_2}$$

Proof of the Nonlinear Transformation to the Canonical Form

The relative degree is $\rho = n$ if and only if

- $L_g L_f^{i-1} h(x) = 0$ for i between 0 and n−1 and
- $L_g L_f^{n-1} h(x) \neq 0$

$$z = T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}$$

The transformed dynamical system is:

$$z_1 = h(x)$$

$$z_2 = L_f h(x)$$

$$\dot{z}_1 = \frac{dL_f h(x)}{dx}\dot{x} = \frac{dL_f h(x)}{dx}(f(x) + g(x)u) = \frac{dL_f h(x)}{dx}f(x) + \frac{dL_f h(x)}{dx}g(x)u =$$

$$= L_f^2 h(x) + \cancelto{0}{L_g L_f h(x)u} = \boldsymbol{L_f^2 h(x) = z_3}$$

Proof of the Nonlinear Transformation to the Canonical Form

The relative degree is $\rho = n$ if and only if

- $L_g L_f^{i-1} h(x) = 0$ for i between 0 and n−1 and
- $L_g L_f^{n-1} h(x) \neq 0$

$$z = T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}$$

The transformed dynamical system is:

$$\begin{aligned} z_1 &= h(x) \\ z_2 &= L_f h(x) \\ &\vdots \\ z_1^{(n-1)} &= L_f^{n-1} h(x) + \overbrace{L_g L_f^{n-2} h(x)}^{0} u = \boldsymbol{L_f^{n-1} h(x) = z_n} \end{aligned}$$

Proof of the Nonlinear Transformation to the Canonical Form

The relative degree is $\rho = n$ if and only if

- $L_g L_f^{i-1} h(x) = 0$ for i between 0 and n−1 and
- $L_g L_f^{n-1} h(x) \neq 0$

$$z = T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}$$

The transformed dynamical system is:

$$\begin{aligned} z_1 &= h(x) \\ z_2 &= L_f h(x) \\ &\vdots \\ z_n &= L_f^{n-1} h(x) \end{aligned}$$

$$\boldsymbol{z_1^{(n)} = L_f^n h(x) + L_g L_f^{n-1} h(x) u}$$

Proof of the Nonlinear Transformation to the Canonical Form

The transformed dynamical system is:

$$z_1 = h(x)$$
$$z_2 = L_f h(x)$$
$$\vdots$$
$$\mathrm{z_n = L_f^{n-1} h(x)}$$
$$\boldsymbol{z_1^{(n)} = L_f^n h(x) + L_g L_f^{n-1} h(x) u}$$

An alternative form of the same dynamical system is:

$$\dot{z_1} = z_2$$
$$\vdots$$
$$\dot{z_{n-1}} = z_n$$
$$\dot{z_n} = L_f^n h(x) + L_g L_f^{n-1} h(x) u$$

Proof of the Nonlinear Transformation to the Canonical Form

The transformed dynamical system is:

$$z_1 = h(x)$$
$$z_2 = L_f h(x)$$
$$\vdots$$
$$z_n = L_f^{n-1} h(x)$$
$$z_1^{(n)} = L_f^n h(x) + L_g L_f^{n-1} h(x) u$$

An alternative form of the same system is the Canonical Form:

$$\dot{\boldsymbol{z}} = \boldsymbol{Az} + \boldsymbol{b}\boldsymbol{\gamma}(\boldsymbol{x})[\boldsymbol{u} - \boldsymbol{\alpha}(\boldsymbol{x})]$$
$$\boldsymbol{y} = \boldsymbol{cz}$$
$$\gamma(x) = L_g L_f^{n-1} h(x),$$
$$\alpha(x) = \frac{L_f^n h(x)}{L_g L_f^{n-1} h(x)}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & & \\ 0 & 0 & 1 & 0 & \\ \vdots & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

For the Canonical Form of an affine nonlinear system, we can find feedback linearization as follows

Canonical Form:

$$\dot{z} = Az + b\gamma(x)[u - \alpha(x)]$$
$$y = cz$$

Feedback Linearization:

$$\boldsymbol{u = \alpha(x) + \gamma^{-1}(x)v}$$

For the Canonical Form of an affine nonlinear system, we can find feedback linearization as follows

Canonical Form:

$$\dot{z} = Az + b\gamma(x)[u - \alpha(x)]$$
$$y = cz$$

Feedback Linearization:

$$\boldsymbol{u = \alpha(x) + \gamma^{-1}(x)v}$$

Linearized Form:

$$\dot{z} = Az + b\boldsymbol{v}$$
$$y = cz$$

For the Canonical Form of an affine nonlinear system, we can find feedback linearization as follows

Canonical Form:

$$\dot{z} = Az + b\gamma(x)[u - \alpha(x)]$$
$$y = cz$$

Feedback Linearization:

$$\boldsymbol{u = \alpha(x) + \gamma^{-1}(x)v}$$

Linearized Form:

$$\dot{z} = Az + b\boldsymbol{v}$$
$$y = cz$$

We can choose state fb $\boldsymbol{v = -Kz}$

$$\dot{z} = (A - bK)z$$
$$y = cz$$
$$\boldsymbol{u = \alpha(x) - \gamma^{-1}(x)Kz}$$

Summary for Feedback Linearization Control of Nonlinear Affine Systems

1) Start with an Affine Nonlinear System
2) Check whether the relative degree is ρ by showing $L_g L_f^{i-1} h(x) = 0$ for i between 1 and n−1 and $L_g L_f^{n-1} h(x) \neq 0$
3) Check whether your system is already in Canonical Form
4) If not, Perform Nonlinear Coordinate Feedback Linearization

$$z = T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1} h(x) \end{bmatrix}$$

5) Put the system in Canonical Form

$$\dot{z} = Az + b\gamma(x)[u - \alpha(x)]$$
$$y = cz$$

6) Choose feedback linearization to linearize the affine system in Canonical Form
7) Choose state feedback $\boldsymbol{v = -Kz}$ to achieve stability of the dynamical system in Canonical Form
8) Compute the feedback linearization $\boldsymbol{u = \alpha(x) - \gamma^{-1}(x) K\, T(x)}$

What happens if the Affine Nonlinear System has relative degree $\rho < n$?

We'll find out next class

Example 3: Feedback Linearization Example. Step 1) Affine Nonlinear System

Affine Nonlinear System

$$\dot{x_1} = a\sin x_2$$
$$\dot{x_2} = -x_1^2 + \boldsymbol{u}$$
$$y = x_1$$

Example 3: Feedback Linearization Example. Step 2) Is the relative degree equal to two?

Affine Nonlinear System

$$\dot{x_1} = a\sin x_2$$
$$\dot{x_2} = -x_1^2 + \boldsymbol{u}$$
$$y = x_1$$

$$f(x) = \begin{bmatrix} a\sin x_2 \\ -x_1^2 \end{bmatrix}, g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$y = h(x) = x_1$$

1) Is $L_g h(x) = 0$?

2) Is $L_g L_f h(x) \neq 0$

$$L_g h(x) = \frac{dh}{dx} g(x) = [1\ 0]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$
$$L_f h(x) = a\sin x_2$$
$$L_g L_f h(x) = \frac{dL_f h}{dx} g(x) = [0\ a\cos x_2]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = a\cos x_2 \neq 0 \text{ for } |x_2| < \frac{\pi}{2}$$

Example 3: Feedback Linearization Example. Step 2) Is the relative degree equal to two? Yes in the appropriate domain

Affine Nonlinear System

$$\dot{x_1} = a\sin x_2$$
$$\dot{x_2} = -x_1^2 + \boldsymbol{u}$$
$$y = x_1$$

$$f(x) = \begin{bmatrix} a\sin x_2 \\ -x_1^2 \end{bmatrix}, g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$y = h(x) = x_1$$

1) Is $L_g h(x) = 0$?

2) Is $L_g L_f h(x) \neq 0$

$$L_g h(x) = \frac{dh}{dx} g(x) = [1\ 0] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$
$$L_f h(x) = a\sin x_2$$

$$L_g L_f h(x) = \frac{dL_f h}{dx} g(x) = [0\ a\cos x_2] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = a\cos x_2 \neq 0 \text{ for } |x_2| < \frac{\pi}{2}$$

Example 3: Feedback Linearization Example. Steps 3), 4), 5) Checked that the system was not in canonical form and did nonlinear transformation

Affine Nonlinear System

$$\dot{x}_1 = a\sin x_2$$
$$\dot{x}_2 = -x_1^2 + \boldsymbol{u}$$
$$y = x_1$$

Nonlinear Coordinate Transformation

$$z_1 = x_1$$
$$z_2 = a\sin x_2$$

Example 3: Feedback Linearization Example. Steps 3), 4), 5) Checked that the system was not in canonical form and did nonlinear transformation

Affine Nonlinear System

$$\dot{x}_1 = a \sin x_2$$
$$\dot{x}_2 = -x_1^2 + \boldsymbol{u}$$
$$y = x_1$$

Nonlinear Coordinate Transformation

$$z_1 = x_1$$
$$z_2 = a \sin x_2$$

System in Canonical Form

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = a \cos x_2 (u - x_1^2)$$

Example 3: Feedback Linearization Example. Step 6) Choose the feedback linearization controller

Affine Nonlinear System

$$\dot{x}_1 = a \sin x_2$$
$$\dot{x}_2 = -x_1^2 + \boldsymbol{u}$$
$$y = x_1$$

Nonlinear Coordinate Transformation

$$z_1 = x_1$$
$$z_2 = a \sin x_2$$

System in Canonical Form

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = a \cos x_2 (u - x_1^2)$$

Choose the Controller:

$$\boldsymbol{u} = x_1^2 + \frac{1}{\cos x_2} \boldsymbol{v}$$

Example 3: Feedback Linearization Example. Step 7) Choose state feedback

Affine Nonlinear System

$$\dot{x}_1 = a \sin x_2$$
$$\dot{x}_2 = -x_1^2 + \boldsymbol{u}$$
$$y = x_1$$

Nonlinear Coordinate Transformation

$$z_1 = x_1$$
$$z_2 = a \sin x_2$$

System in Canonical Form

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = a \cos x_2 (u - x_1^2)$$

Choose the Controller:

$$\boldsymbol{u} = x_1^2 + \frac{1}{\cos x_2} \boldsymbol{v}$$

$$\Downarrow$$

Linear Dynamical System

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = \boldsymbol{v}$$

$$\boldsymbol{v} = -Kx = -k_1 z_1 - k_2 z_2$$

Example 3: Feedback Linearization Example. Step 8) Compute feedback linearization

Affine Nonlinear System

$$\dot{x}_1 = a \sin x_2$$
$$\dot{x}_2 = -x_1^2 + \boldsymbol{u}$$
$$y = x_1$$

Nonlinear Coordinate Transformation

$$z_1 = x_1$$
$$z_2 = a \sin x_2$$

System in Canonical Form

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = a \cos x_2 (u - x_1^2)$$

Choose the Controller:

$$\boldsymbol{u} = x_1^2 + \frac{1}{\cos x_2} \boldsymbol{v}$$

$\Downarrow$

Linear Dynamical System

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = \boldsymbol{v}$$

$$\boldsymbol{v} = -Kx = -k_1 z_1 - k_2 z_2$$

$$\mathbf{u} = x_1^2 + \frac{1}{\cos x_2} \boldsymbol{v}(-k_1 x_1 - k_2 a \sin x_2)$$

Summary

- Learned about State Feedback
- Learned about Feedback Linearization of Nonlinear Affine Systems