MEM 636

Nonlinear Control Systems I

Homework IV

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Nonlinear Controls - Accessibility and Controllability
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1 Problem 1 - Analyzing Local Accessibility and Controllability of the System

1.1 Problem Statement Recap

The system is defined as:

$$\dot{x}_1 = u, \quad \dot{x}_2 = x_1^k, \quad k \geq 2$$

where $x = [x_1, x_2]^T$ and $u \in R$ is the control input.

We need to determine:

1. Whether the system is locally accessible and, if so, at what points in R^2.
2. Whether the system is locally controllable.

1.2 My Step-by-Step Solution

1.2.1 Step 1: Define the Vector Fields

From the system dynamics, we identify the following vector fields:

- **Control vector field** $g(x)$: This is associated with the control input u. Since $\dot{x}_1 = u$, we define:

$$g(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

- **Drift vector field** $f(x)$: This represents the part of the dynamics independent of the control input u. From $\dot{x}_2 = x_1^k$, we define:

$$f(x) = \begin{bmatrix} 0 \\ x_1^k \end{bmatrix}$$

1.2.2 Step 2: Compute the First Lie Bracket $[f, g]$

To determine accessibility, we need to compute the Lie bracket $[f, g](x)$, which measures the interaction between the drift and control vector fields.

The Lie bracket $[f, g](x)$ is defined as:

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$$

Calculating each term:

1. **Partial derivative of $g(x)$ with respect to x**: Since $g(x) = [1, 0]^T$, $\frac{\partial g}{\partial x} = 0$, so:

$$\frac{\partial g}{\partial x} f(x) = 0$$

2. **Partial derivative of $f(x)$ with respect to x**: Since $f(x) = [0, x_1^k]^T$, we have:

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 0 \\ kx_1^{k-1} & 0 \end{bmatrix}$$

   Therefore:

$$\frac{\partial f}{\partial x} g(x) = \begin{bmatrix} 0 \\ kx_1^{k-1} \end{bmatrix}$$

Combining these results:

$$[f, g](x) = 0 - \begin{bmatrix} 0 \\ kx_1^{k-1} \end{bmatrix} = \begin{bmatrix} 0 \\ -kx_1^{k-1} \end{bmatrix}$$

1.2.3 Step 3: Construct the Controllability Matrix

The controllability matrix Q is formed by placing the vectors $g(x)$ and $[f, g](x)$ as columns:

$$Q = [g(x)\ [f, g](x)] = \begin{bmatrix} 1 & 0 \\ 0 & -kx_1^{k-1} \end{bmatrix}$$

1.2.4 Step 4: Determine Accessibility and Controllability

To check local accessibility and controllability, we examine the rank of Q:

- **Rank of Q at the origin $x = (0, 0)$**:

$$Q|_{x=(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

  The rank of Q at the origin is 1, which is less than 2 (the dimension of the state space), meaning the system is **not locally accessible** at the origin.

- **Rank of Q when $x_1 \neq 0$**: When $x_1 \neq 0$, the entry $-kx_1^{k-1}$ in the second column of Q is nonzero. Thus:

$$\text{rank}(Q) = 2$$

  This means the system is locally accessible (and therefore locally controllable) at any point where $x_1 \neq 0$.

1.3 Conclusion

1. The system is **locally accessible and controllable** at points where $x_1 \neq 0$.
2. The system is **not locally accessible or controllable** at the origin $(0, 0)$.

2 Problem 2 - Rigid Body Controllability

2.1 Problem Summary

We are given a system representing an actuated rotating rigid body, described by the dynamics:

$$\begin{aligned} \dot{\omega}_1 &= a_1 \omega_2 \omega_3 + u_1, \\ \dot{\omega}_2 &= a_2 \omega_1 \omega_3 + u_2, \\ \dot{\omega}_3 &= a_3 \omega_1 \omega_2, \end{aligned}$$

where:

- $\omega = (\omega_1, \omega_2, \omega_3)$ are the angular velocities,

- a_1, a_2, and a_3 are constants (related to moments of inertia),
- u_1 and u_2 are control inputs, with no control on ω_3 (i.e., $u_3 = 0$).

The goal is to determine:

1. **Local Accessibility**: Determine at which points in R^3 the system is accessible.
2. **Local Controllability**: Assess whether the system is fully controllable at any points.

2.2 Step-by-Step Solution and Interpretation

2.2.1 Step 1: Define the Vector Fields

Based on the system equations, we identify:

- **Drift vector field** $f(x)$ (represents the uncontrolled dynamics):

$$f = \begin{bmatrix} a_1\omega_2\omega_3 \\ a_2\omega_1\omega_3 \\ a_3\omega_1\omega_2 \end{bmatrix}$$

- **Control vector fields**:
  - $g_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, associated with u_1,
  - $g_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, associated with u_2.

2.2.2 Step 2: Compute the Jacobian of f

To find the Lie brackets, we need the Jacobian of f with respect to ω:

$$J_f = \frac{\partial f}{\partial \omega} = \begin{bmatrix} 0 & a_1\omega_3 & a_1\omega_2 \\ a_2\omega_3 & 0 & a_2\omega_1 \\ a_3\omega_2 & a_3\omega_1 & 0 \end{bmatrix}$$

This Jacobian represents how the drift vector field f changes with the state variables ω_1, ω_2, and ω_3.

2.2.3 Step 3: Compute the First-Order Lie Brackets

Lie Bracket $[f, g_1]$ Using the formula $[f, g_1] = -J_f \cdot g_1$, we get:

$$[f, g_1] = \begin{bmatrix} 0 \\ -a_2\omega_3 \\ -a_3\omega_2 \end{bmatrix}$$

This Lie bracket indicates an additional direction in the state space affected by f and g_1.

Lie Bracket $[f, g_2]$ Similarly, for $[f, g_2] = -J_f \cdot g_2$, we have:

$$[f, g_2] = \begin{bmatrix} -a_1\omega_3 \\ 0 \\ -a_3\omega_1 \end{bmatrix}$$

This provides another direction influenced by the combined effects of f and g_2.

2.2.4 Step 4: Construct the Controllability Matrix Q

The controllability matrix Q is constructed from the vectors g_1, g_2, $[f, g_1]$, and $[f, g_2]$:

$$Q = \begin{bmatrix} g_1 & g_2 & [f, g_1] & [f, g_2] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -a_1\omega_3 \\ 0 & 1 & -a_2\omega_3 & 0 \\ 0 & 0 & -a_3\omega_2 & -a_3\omega_1 \end{bmatrix}$$

2.2.5 Step 5: Determine the Rank of Q

- **At the Origin** $(\omega_1, \omega_2, \omega_3) = (0, 0, 0)$:

$$Q|_{\omega=0} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

  The rank of Q at the origin is 2, which is less than 3 (the dimension of the state space). This means the system is **not locally accessible** at the origin, as we cannot reach every direction in the state space from the origin.

- **At a Non-Origin Point**, such as $\omega_1 = 1$, $\omega_2 = 0$, $\omega_3 = 1$:

$$Q|_{\omega_1=1,\omega_2=0,\omega_3=1} = \begin{bmatrix} 1 & 0 & 0 & -a_1 \\ 0 & 1 & -a_2 & 0 \\ 0 & 0 & 0 & -a_3 \end{bmatrix}$$

  The rank of Q at this point is 3, which is full rank. This indicates that the system is **locally controllable** at points where ω_1 and ω_3 are non-zero.

2.3 Interpretation and Conclusion

- **Local Accessibility**: The system is not accessible at the origin, as the rank of Q there is 2. This is due to the fact that we cannot reach all directions in the state space starting from the origin.
- **Local Controllability**: The system is fully controllable at non-origin points where $\omega_1 \neq 0$ and $\omega_3 \neq 0$. At these points, the controllability matrix Q has rank 3, allowing us to span the entire state space.

Thus, **the system is locally controllable at points where ω_1 and ω_3 are non-zero, but it is not controllable at the origin**. This conclusion aligns with the provided results and verifies the controllability properties of the system.

3 Problem 3 - Video Summary

Didn't really understand this so note to self - read further into this.

3.1 (a) Why Can You Move a Car Sideways Using Controllability Concepts?

In nonholonomic systems like a car, sideways movement isn't directly possible because the wheels don't allow lateral sliding. However, by using controllability concepts and combining movements forward and backward with rotations (steering), the car can reach positions off its forward path, effectively achieving sideways displacement over time. This maneuverability is enabled by carefully planning movements to achieve a combination of heading and position adjustments, exploiting the control vector fields' combined effect on the configuration space.

Did not really understand from the video but a further check

3.2 Understanding Lateral Movement Through Lie Brackets

To understand how a car can achieve lateral (sideways) movement through a series of maneuvers, we delve into the mathematical framework of nonlinear controllability, particularly focusing on **Lie brackets**.

3.2.1 Car's Kinematic Model

Consider a simplified kinematic model of a car moving in a plane. The state of the car is defined by its position (x, y) and orientation θ. The control inputs are the forward velocity v and the steering angle ϕ. The equations governing the car's motion are:

$$\begin{aligned}\dot{x} &= v\cos(\theta),\\ \dot{y} &= v\sin(\theta),\\ \dot{\theta} &= \frac{v}{L}\tan(\phi),\end{aligned}$$

where L is the wheelbase of the car.

3.2.2 Nonholonomic Constraint

The car cannot move directly sideways due to its wheel configuration. This constraint is expressed as:

$$\dot{y}\cos(\theta) - \dot{x}\sin(\theta) = 0,$$

indicating that the car's velocity vector is aligned with its orientation.

3.2.3 Control Vector Fields

Define two control vector fields corresponding to the car's inputs:

$$g_1 = \begin{bmatrix}\cos(\theta)\\ \sin(\theta)\\ 0\end{bmatrix}, \quad g_2 = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}.$$

Here, g_1 represents motion in the direction of the car's orientation (forward/backward), and g_2 represents changes in orientation (steering).

3.2.4 Lie Bracket Calculation

The Lie bracket $[g_1, g_2]$ captures the effect of combining the two control inputs in a specific sequence. It is computed as:

$$[g_1, g_2] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2.$$

Calculating the partial derivatives and performing the operations, we obtain:

$$[g_1, g_2] = \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\\ 0\end{bmatrix}.$$

This new vector field corresponds to lateral movement perpendicular to the car's orientation.

3.2.5 Controllability Lie Algebra

The controllability of the system is determined by the Lie algebra generated by the control vector fields and their Lie brackets. In this case, the set $\{g_1, g_2, [g_1, g_2]\}$ spans the state space. This means that, through appropriate sequences of the available controls, the car can achieve movement in any direction, including sideways.

3.3 (b) Describe the LARC Condition and Its Use in Determining Controllability

The **Lie Algebra Rank Condition (LARC)** is a criterion used to determine whether a nonholonomic system is controllable. For the system to be controllable, the Lie algebra generated by the vector fields (control vector fields) must span the full configuration space. In the context of wheeled mobile robots, we express the rate of change of configuration as a matrix $G(q)$ times control inputs. LARC ensures that by calculating Lie brackets of the control vector fields (such as g_1 and g_2), the span of these fields and their Lie brackets can cover all accessible directions in the configuration space. This coverage implies that the system can reach a neighborhood around any point in the configuration space, even if some directions (like lateral movement) aren't directly achievable through simple controls.

4 Problem 4 - Proof of the Jacobi-Lie Identity

Prove the identity.

4.1 Jacobi Identity for Lie Brackets

4.1.1 Definitions and Setup

A **Lie algebra** g is a vector space equipped with a binary operation called the **Lie bracket**, denoted $[x, y]$ for $x, y \in g$, and satisfying the following properties:

1. **Bilinearity**: The bracket is bilinear over the field F.
2. **Skew-Symmetry**:
$$[x, y] = -[y, x]$$
for all $x, y \in g$.
3. **Jacobi Identity**:
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$
for all $x, y, z \in g$.

The Jacobi identity essentially states that the sum of cyclic permutations of the nested bracket $[x, [y, z]]$ should equal zero.

4.1.2 Proof of the Jacobi Identity

To prove the Jacobi identity, we will expand each term in the identity and demonstrate that they sum to zero.

1. **Expand** $[x, [y, z]]$

   Using the properties of the Lie bracket, we write:
$$[x, [y, z]] = -[y, [x, z]] - [z, [x, y]]$$
   based on the skew-symmetry and linearity of the bracket.

2. **Expand** $[y, [z, x]]$

   Similarly, we have:
$$[y, [z, x]] = -[z, [y, x]] - [x, [y, z]]$$

3. **Expand** $[z, [x, y]]$

   Again, applying skew-symmetry:
$$[z, [x, y]] = -[x, [z, y]] - [y, [z, x]]$$

4. **Combine the Terms**

   Substitute these expanded terms back into the Jacobi identity expression:

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]]$$

   Expanding this, we obtain:

$$= -([y, [x, z]] + [z, [x, y]]) - ([z, [y, x]] + [x, [y, z]]) - ([x, [z, y]] + [y, [z, x]])$$

   Rearranging terms and observing symmetry, each term cancels out, resulting in:

$$= 0$$

4.2 Jacobi Identity Verification for Given Vector Fields

Since I wasn't sure this was enough, I decided to do an example test

Problem Setup

We are given three vector fields:
1. $f(x) = \begin{pmatrix} x_2 \\ -\sin(x_1) - x_2 \end{pmatrix}$ 2. $g(x) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}$ 3. $h(x) = \begin{pmatrix} \cos(x_1) \\ x_2^2 \end{pmatrix}$

   We aim to verify the **Jacobi identity**:

$$[f, [g, h]](x) + [g, [h, f]](x) + [h, [f, g]](x) = 0$$

Step 1: Compute Jacobians of f, g, and h

1. **Jacobian of $f(x)$:**

$$\frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 \\ -\cos(x_1) & -1 \end{pmatrix}$$

2. **Jacobian of $g(x)$:**

$$\frac{\partial g}{\partial x} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

3. **Jacobian of $h(x)$:**

$$\frac{\partial h}{\partial x} = \begin{pmatrix} -\sin(x_1) & 0 \\ 0 & 2x_2 \end{pmatrix}$$

Step 2: Compute Lie Brackets $[f, g](x)$, $[g, h](x)$, and $[h, f](x)$

The Lie bracket $[a, b](x)$ of two vector fields $a(x)$ and $b(x)$ is given by:

$$[a, b](x) = \frac{\partial b}{\partial x} a(x) - \frac{\partial a}{\partial x} b(x)$$

1. Compute $[f, g](x)$:

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$$

   After calculations, we find:

$$[f, g](x) = \begin{pmatrix} -x_1 \\ x_1 + x_2 \end{pmatrix}$$

2. **Compute $[g,h](x)$:**

$$[g,h](x) = \frac{\partial h}{\partial x} g(x) - \frac{\partial g}{\partial x} h(x)$$

This results in:

$$[g,h](x) = \begin{pmatrix} 0 \\ 2x_1x_2 - \cos(x_1) \end{pmatrix}$$

3. **Compute $[h,f](x)$:**

$$[h,f](x) = \frac{\partial f}{\partial x} h(x) - \frac{\partial h}{\partial x} f(x)$$

This results in:

$$[h,f](x) = \begin{pmatrix} x_2(\sin(x_1) + 1) \\ \sin(x_1)^2 + x_2^2 - 1 \end{pmatrix}$$

Step 3: Compute the Nested Lie Brackets

Using these, we calculate the nested Lie brackets $[f,[g,h]]$, $[g,[h,f]]$, and $[h,[f,g]]$, and check if their sum is zero.

After performing these calculations, we arrive at:

$$[f,[g,h]](x) + [g,[h,f]](x) + [h,[f,g]](x) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

I tested this in the MATLAB script "Problem4JacobiLie.mlx" and it matched