Involution Example

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Compute the rank of the distribution Δ and and decide if it is involutive. The domain is $D = \mathbb{R}^2$.

$$\Delta(x) = span\{f_1(x), f_2(x)\},\ f_1(x) = \begin{pmatrix} x_1 \\ x_1 + x_2 \end{pmatrix},\ f_2(x) = \begin{pmatrix} x_2 \\ 1 \end{pmatrix}. \tag{1}$$

Step 1: Are the two vector fields linearly dependent or linearly independent?

If they were linearly dependent, then Δ would be a distribution of rank one and involutive by default.

Let's check if f_1 and f_2 are linearly dependent.

This would be equivalent to the existence of a non-zero constant α such that

$$f_1 = \alpha f_2. \tag{2}$$

This equation becomes

$$\begin{pmatrix} x_1 \\ x_1 + x_2 \end{pmatrix} = \alpha \begin{pmatrix} x_2 \\ 1 \end{pmatrix} \tag{3}$$

This is equivalent to the system of equations:

$$x_1 = \alpha x_2, \qquad x_1 + x_2 = \alpha. \tag{4}$$

We can use the first equation to plug in x_1 into the second equation and obtain that

$$(\alpha + 1)x_2 = \alpha \tag{5}$$

So then in equation (5) either $x_2 = \frac{\alpha}{\alpha+1}$ or $\alpha + 1 = 0$.

Case 1: If $\alpha + 1 = 0$, then it must be that $0 \times x_2 = \alpha$ in equation (5). Thus, $\alpha = 0$. But it's not possible for $0 + 1 = 0$. So this case is impossible.

Case 2: In equation (5), it must be that $x_2 = \frac{\alpha}{\alpha+1}$, which means from equation (4) that $x_1 = \frac{\alpha^2}{\alpha+1}$. Alternatively, this can be written as $\alpha = -\frac{x_2}{x_2-1}$ with x_2 different from one and $x_1 = -\frac{x_2^2}{x_2-1}$.

Conclusion: The vector fields f_1 and f_2 can only be linearly dependent if there is a nonzero α such that $x_1 = \frac{\alpha^2}{\alpha+1}$ and $x_2 = \frac{\alpha}{\alpha+1}$. For the rest of x_1 and x_2, they are linearly independent.

Equivalent Conclusion The vector fields f_1 and f_2 can only be linearly dependent if $x_1 = -\frac{x_2^2}{x_2-1}$ with x_2 different from one and zero (else α would be zero). For the rest of x_1 and x_2, they are linearly independent.

Let's Double Check What We Got For Fun Let's say that $x_1 = -\frac{x_2^2}{x_2-1}$ with x_2 different from one and zero. Then

$$f_1(x) = \begin{pmatrix} -\frac{x_2^2}{x_2-1} \\ -\frac{x_2^2}{x_2-1} + x_2 \end{pmatrix} = \begin{pmatrix} -\frac{x_2^2}{x_2-1} \\ -\frac{x_2}{x_2-1} \end{pmatrix},\ f_2(x) = \begin{pmatrix} x_2 \\ 1 \end{pmatrix}. \tag{6}$$

We find that $f_1(x) = f_2(x) \times (-\frac{x_2}{x_2-1})$.

Step 2: Let's assume that the two vector fields are linearly independent. Then in order to know whether Δ is an involution, we need to check whether the Lie bracket $[f_1, f_2]$ is in the span of f_1 and f_2.

Let's compute the Lie bracket. We obtain

$$[f_1, f_2] = \begin{pmatrix} x_1 \\ -x_2 - 1 \end{pmatrix} \tag{7}$$

Step 3: Let's check whether the Lie bracket $[f_1, f_2]$ is in the span of f_1 and f_2. This is equivalent to the existence of constants α and β such that $[f_1, f_2] = \alpha f_1 + \beta f_2$. α and β can't both be zero.

The linear dependence equation $[f_1, f_2] = \alpha f_1 + \beta f_2$ is equivalent to

$$\begin{pmatrix} x_1 \\ -x_2 - 1 \end{pmatrix} = \alpha \begin{pmatrix} x_1 \\ x_1 + x_2 \end{pmatrix} + \beta \begin{pmatrix} x_2 \\ 1 \end{pmatrix}. \tag{8}$$

This turns into a system of equations as follows:

$$x_1 = \alpha x_1 + \beta x_2, \qquad -x_2 - 1 = \alpha(x_1 + x_2) + \beta. \tag{9}$$

The first equation in (9) is equivalent to

$$x_1(1 - \alpha) = \beta x_2. \tag{10}$$

We have two possibilities in equation (10):

Case 1: $\alpha = 1$. This means that $\beta x_2 = 0$. In turn, this creates two possibilities:

Case 1.1: $\alpha = 1$ and $\beta = 0$. In the systems of equations in (9), this case implies that $x_1 = x_1$ (trivially true) and $-x_2 - 1 = x_1 + x_2$. Thus, the system of equations in (9) means that $x_1 = -2x_2 - 1$.

Let's double check that linear dependence occurs in this relationship between x_1 and x_2. First, the Lie bracket is

$$[f_1, f_2] = \begin{pmatrix} -2x_2 - 1 \\ -x_2 - 1 \end{pmatrix}. \tag{11}$$

Furthermore,

$$f_1 = \begin{pmatrix} -2x_2 - 1 \\ -2x_2 - 1 + x_2 \end{pmatrix}. \tag{12}$$

We can observe that $[f_1, f_2] = f_1$. This means that Δ is an involution in the case $x_1 = -2x_2 - 1$.

Case 1.2: $\alpha = 1$, $x_2 = 0$, β is nonzero. In the systems of equations in (9), this means that $-1 = x_1 + \beta$. Thus, $\beta = -1 - x_1$.

Let's double check that linear dependence occurs in this case. First, the Lie bracket is

$$[f_1, f_2] = \begin{pmatrix} x_1 \\ -1 \end{pmatrix}. \tag{13}$$

Furthermore,

$$f_1 = \begin{pmatrix} x_1 \\ x_1 \end{pmatrix} \tag{14}$$

and

$$f_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{15}$$

We can see that $[f_1, f_2] = f_1 + (-1 - x_1) f_2$. Therefore, we got an involution when $x_2 = 0$, x_1 is not -1 (because β was nonzero in this case).

Case 2: α is not one. Therefore, from equation (10), it must be that

$$x_1 = \frac{\beta x_2}{1 - \alpha}. \tag{16}$$

We can substitute in the second equation in the system (9) to obtain

$$-x_2 - 1 = \alpha\left(\frac{\beta x_2}{1-\alpha} + x_2\right) + \beta \tag{17}$$

We can manipulate this equation to obtain

$$x_2 = \frac{(\beta+1)(1-\alpha)}{-\alpha^2 + \alpha\beta + 1} \tag{18}$$

or that $-\alpha^2 + \alpha\beta + 1 = 0$.

Case 2.1 α is not one and $-\alpha^2 + \alpha\beta + 1 = 0$. Therefore, $\beta = \frac{\alpha^2-1}{\alpha}$ or $\alpha = 0$.

Case 2.1.1 $\alpha = 0$ and $-\alpha^2 + \alpha\beta + 1 = 0$. This implies that $-1 = 0$. It is impossible.

Case 2.1.2 α is not one and $-\alpha^2 + \alpha\beta + 1 = 0$.

If we plug β back into equation (17), then it must be that $\alpha^2 + \alpha - 1 = 0$. We can solve this equation to get the roots $\alpha_{1,2} = \frac{-1\pm\sqrt{5}}{2}$. Then $\beta = -1$.

From the first equation in (9), we obtain that $x_1 = \frac{-2}{3\pm\sqrt{5}} x_2$.

In this case when x_1 and x_2 are proportional by value $\frac{-2}{3\pm\sqrt{5}}$, Δ is an involution.

Case 2.2 α is not one and

$$x_2 = \frac{(\beta+1)(1-\alpha)}{\alpha^2 - \alpha\beta - 1} \tag{19}$$

This means that

$$x_1 = \frac{\beta}{1-\alpha} \times \frac{(\beta+1)(1-\alpha)}{\alpha^2 - \alpha\beta - 1}. \tag{20}$$

Equivalently, this means that

$$x_1 = \frac{(\beta+1)\beta}{\alpha^2 - \alpha\beta - 1}. \tag{21}$$

In the case where α is not one and $\alpha^2 - \alpha\beta - 1$ is not zero and we can find these constants such that

$$x_1 = \frac{(\beta+1)\beta}{\alpha^2 - \alpha\beta - 1}, \qquad x_2 = \frac{(\beta+1)(1-\alpha)}{-\alpha^2 + \alpha\beta + 1}, \tag{22}$$

the distribution Δ is an involution.

In all other cases, it is not.