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CHAPTER II

A Linear Ordering of Braids

In this chapter, we introduce the linear ordering of braids, sometimes called the *Dehornoy ordering*, that is the main subject of this book, and we list its main properties known so far. The construction starts with the notion of a σ-positive braid, and it relies on three basic properties, called **A**, **C**, and **S**, from which the σ-ordering can easily be constructed and investigated. In this chapter, we take Properties **A**, **C**, and **S** for granted, and explore their consequences. The many different proofs of these statements will be found in subsequent chapters.

The chapter is organized as follows. In Section 1, we introduce the σ-ordering and its variant the σ^{Φ}-ordering starting from Properties **A** and **C**. In Section 2, we give many examples of the sometimes surprising behaviour of the σ-ordering, and we introduce Property **S**. In Section 3, we develop global properties of the σ-ordering, involving Archimedian property, discreteness, density, and convex subgroups. Finally, in Section 4, we investigate the restriction of the σ-ordering to the monoid B_n^+ of positive braids, showing that this restriction is a well-ordering, and we give an inductive construction of the σ-ordering of B_n^+ from the σ-ordering of B_{n-1}^+.

Convention. In this chapter and everywhere in this book, when we speak of positive braids, we always mean those braids that lie in the monoid B_∞^+, *i.e.*, those braids that admit at least one expression by a word containing no letter σ_i^{-1}. Such braids are sometimes called Garside positive braids, but we shall not use that name here. So the word "positive" never refers to any of the specific linear orderings we shall investigate hereafter. For the latter case, we shall introduce specific names for the braids that are larger than 1, typically σ-positive and σ^{Φ}-positive in the case of the σ-ordering and of the σ^{Φ}-ordering.

1. The σ-ordering of B_n

In this section we give a first definition of the σ-ordering of braids, based on the notion of a σ-positive braid word—many alternative definitions will be given in subsequent chapters. We explain how to construct the σ-ordering from two specific properties of braids called **A** and **C**. We also introduce a useful variant of the σ-ordering, called the σ^{Φ}-ordering, which is its image under the flip automorphism. Finally, we briefly discuss the algorithmic issues involving the σ–ordering.

1.1. Ordering a group. We start with premilinary remarks about what can be expected here. First, we recall that a *strict ordering* of a set Ω is a binary relation $\prec$ that is antireflexive ($x \prec x$ never holds) and transitive (the conjunction of $x \prec y$ and $y \prec z$ implies $x \prec z$). A strict ordering of Ω is called *linear* (or *total*) if, for all x, x' in Ω, one of $x = x'$, $x \prec x'$, $x' \prec x$ holds. Then, we recall the notion of an orderable group.

Definition 1.1. (i) A *left-invariant ordering*, or *left-ordering*, of a group G is a strict linear ordering $\prec$ of G such that $g \prec h$ implies $fg \prec fh$ for all f, g, h in G. A group G is said to be *left-orderable* if there exists at least one left-invariant ordering of G.

(ii) A *bi-invariant ordering*, or *bi-ordering*, of a group G is a left-ordering of G that is also right-invariant, *i.e.*, $g \prec h$ implies $gf \prec hf$ for all f, g, h in G. A group G is said to be *bi-orderable* if there exists at least one bi-invariant ordering of G.

Proposition 1.2. *For $n \geqslant 3$, the group B_n is not bi-orderable.*

Proof. If $\prec$ is a bi-invariant ordering of a group G, then $g \prec h$ implies $\varphi(g) \prec \varphi(h)$ for each inner automorphism φ of G. Now, in the case of B_n, the inner automorphism Φ_n associated with Garside's fundamental braid Δ_n of (I.4.1) exchanges σ_i and σ_{n-i} for each i. Hence it is impossible to have $\sigma_1 \prec \sigma_{n-1}$ and $\Phi_n(\sigma_1) \prec \Phi_n(\sigma_{n-1})$ simultaneously. □

Therefore, in the best case, we shall be interested in orders that are invariant under multiplication on one side. Then, both sides play symmetric roles, as an immediate verification gives

Lemma 1.3. *Assume that G is a group and $\prec$ is a left-invariant ordering of G. Define $g \mathrel{\tilde{\prec}} h$ to mean $g^{-1} \prec h^{-1}$. Then $\tilde{\prec}$ is a right-invariant ordering of G.*

We shall concentrate hereafter on left-invariant orderings. Specifying such an ordering is actually equivalent to specifying a subsemigroup of a certain type, called a positive cone.

Definition 1.4. A subset P of a group G is called a *positive cone* on G if P is closed under multiplication and $G \setminus \{1\}$ is the disjoint union of P and P^{-1}.

Lemma 1.5. (i) *Assume that $\prec$ is a left-invariant ordering of a group G. Then the set P of all elements in G that are larger than 1 is a positive cone on G, and $g \prec h$ is equivalent to $g^{-1}h \in P$.*

(ii) *Assume that P is a positive cone on a group G. Then the relation $g^{-1}h \in P$ is a left-invariant ordering of G, and P is then the set of all elements of G that are larger than 1.*

The verification is easy. Note that the formula $hg^{-1} \in P$ would define a right-invariant ordering.

1.2. The σ-ordering of braids. We now introduce on B_n a certain binary relation that will turn out to be a left-invariant ordering. The construction involves particular braid words defined in terms of the letters they contain.

Definition 1.6. A braid word w is said to be *σ-positive* (*resp.* *σ-negative*) if, among the letters $\sigma_i^{\pm 1}$ that occur in w, the one with lowest index occurs positively only, *i.e.*, σ_i occurs but σ_i^{-1} does not (*resp.* negatively only, *i.e.*, σ_i^{-1} occurs but σ_i does not).

For instance, $\sigma_3\sigma_2\sigma_3^{-1}$ is a σ-positive braid word: the letter with lowest index is σ_2 (there is no $\sigma_1^{\pm 1}$), and there is one σ_2 but no σ_2^{-1}. By contrast, the word $\sigma_2^{-1}\sigma_3\sigma_2$, which is equivalent to $\sigma_3\sigma_2\sigma_3^{-1}$, is neither σ-positive nor σ-negative: the letter with lowest index is σ_2 again, but, here, both σ_2 and σ_2^{-1} appear.

Definition 1.7. For β, β' in B_n, we say that $\beta <_n \beta'$ is true if $\beta^{-1}\beta'$ admits an n-strand representative word that is σ-positive.

Example 1.8. Let $\beta = \sigma_2$ and $\beta' = \sigma_3\sigma_2$. Among the 4-strand braid words that represent the quotient $(\sigma_2)^{-1}(\sigma_3\sigma_2)$, there is the word $\sigma_2^{-1}\sigma_3\sigma_2$, which is neither σ-positive nor σ-negative, but there is also the word $\sigma_3\sigma_2\sigma_3^{-1}$—and many others. As the latter word is a 4-strand braid word that is σ-positive, $\beta <_4 \beta'$ is true.

Similarly, we have

$$\sigma_1 >_\infty \sigma_2 >_\infty \sigma_3 >_\infty \cdots \tag{1.1}$$

since, for each i, the braid word $\sigma_{i+1}^{-1}\sigma_i$ is σ-positive.

The central property is the following result of [**47**] (see Remark 1.16) which implies the first part of the theorem mentioned in the Introduction:

Proposition 1.9. (i) *For $2 \leqslant n \leqslant \infty$, the relation $<_n$ is a left-invariant ordering of B_n.*

(ii) *For each n, the relation $<_n$ is the restriction of $<_\infty$ to B_n.*

Owing to (ii) above, we shall drop the subscripts and simply write $<$ for $<_n$. The order $<$ will be called the *σ-ordering* of braids, which is coherent with its definition in terms of the generators σ_i.

By definition, the relation $\beta >_n 1$ is true if and only if β admits at least one σ-positive n-strand representative word. According to Lemma 1.5, proving Proposition 1.9(i) amounts to proving that the set of all such braids is a positive cone. The latter result is a consequence of the following two statements:

Property A (Acyclicity). *A σ-positive braid word is not trivial.*

Property C (Comparison). *Every nontrivial braid of B_n admits an n-strand representative word that is σ-positive or σ-negative.*

Proof of Proposition 1.9 from Properties **A** and **C**. (i) Let P_n be the set of all n-strand braids that admit a σ-positive n-strand representative word. We shall prove that P_n is a positive cone in B_n. First, the concatenation of two σ-positive n-strand braid words is a σ-positive n-strand braid word; hence P_n is closed under multiplication.

Then, we claim that $B_n \setminus \{1\}$ is the disjoint union of P_n and P_n^{-1}. Indeed, Property **A** implies $1 \notin P_n$, and therefore $1 \notin P_n^{-1}$ as $1^{-1} = 1$ holds. So $P_n \cup P_n^{-1}$ is included in $B_n \setminus \{1\}$. Now assume $\beta \in P_n \cap P_n^{-1}$. We deduce $\beta^{-1} \in P_n$, whence

$$1 = \beta\beta^{-1} \in P_n \cdot P_n \subseteq P_n,$$

which contradicts $1 \notin P_n$. So P_n and P_n^{-1} must be disjoint. Finally, Property **C** (for B_n) means that $P_n \cup P_n^{-1}$ covers $B_n \setminus \{1\}$.

(ii) Assume $\beta, \beta' \in B_n$. Any σ-positive n-strand braid word representing $\beta^{-1}\beta'$ *a fortiori* witnesses the relation $\beta <_\infty \beta'$, so $\beta <_n \beta'$ implies $\beta <_\infty \beta'$. Conversely, assume $\beta <_\infty \beta'$. As $<_n$ is a linear ordering of B_n, one of $\beta <_n \beta'$ or $\beta \geqslant_n \beta'$ holds. In the latter case, we would deduce $\beta \geqslant_\infty \beta'$, which contradicts the hypothesis $\beta <_\infty \beta'$. So $\beta <_n \beta'$ is the only possibility. ☐

Property **A** has four different proofs in this text: they can be found on pages 73, 175, 190, and 224. As for Property **C**, no fewer than eight proofs are given, on pages 60, 89, 116, 148, 164, 190, 201, and 205.

In addition to being invariant under left multiplication, the σ-ordering of braids is invariant under the shift endomorphism, defined as follows.

DEFINITION 1.10. For w a braid word, the *shifting* of w is the braid word $\mathrm{sh}(w)$ obtained by replacing each letter σ_i with σ_{i+1}, and each letter σ_i^{-1} with σ_{i+1}^{-1}.

The explicit form of the braid relations implies that the shift mapping induces an endomorphism of B_∞, still denoted sh and called the *shift endomorphism*. The same argument guaranteeing that the canonical morphism of B_{n-1} into B_n is an embedding shows that the shift endomorphism of B_∞ is injective.

PROPOSITION 1.11. *For all braids β, β', the relation $\beta < \beta'$ is equivalent to* $\mathrm{sh}(\beta) < \mathrm{sh}(\beta')$.

PROOF. The shifting of a σ-positive braid word is a σ-positive braid word, so $\beta < \beta'$ implies $\mathrm{sh}(\beta) < \mathrm{sh}(\beta')$. Conversely, as $<$ is a linear ordering, the only possibility when $\mathrm{sh}(\beta) < \mathrm{sh}(\beta')$ is true is that $\beta < \beta'$ is true as well, as $\beta \geqslant \beta'$ would imply $\mathrm{sh}(\beta) \geqslant \mathrm{sh}(\beta')$. $\square$

It is straightforward to check that, conversely, the σ-ordering is the only partial ordering on B_∞ that is invariant under multiplication on the left and under the shift endomorphism, and satisfies for all braids β, β' the inequality

$$1 < \mathrm{sh}(\beta)\,\sigma_1\,\mathrm{sh}(\beta').$$

1.3. Equivalent formulations. Before proceeding, we introduce derived notions in order to restate Properties **A** and **C** in slightly different forms. First, we can refine the notion of a σ-positive braid word by taking into account the specific index i that is involved.

DEFINITION 1.12. A braid word is said to be *σ_i-positive* if it contains at least one letter σ_i, but no σ_i^{-1} and no $\sigma_j^{\pm 1}$ with $j < i$. Similarly, it is said to be *σ_i-negative* if it contains at least one σ_i^{-1}, but no σ_i and no $\sigma_j^{\pm 1}$ with $j < i$. It is said to be *σ_i-free* if it contains no $\sigma_j^{\pm 1}$ with $j \leqslant i$.

So a braid word is σ-positive if and only if it is σ_i-positive for some i. Note that, for $i \geqslant 2$, a word w is σ_i-positive if and only if it is $\mathrm{sh}^{i-1}(w_1)$ for some σ_1-positive word w_1—we recall that sh is the shift mapping of Definition 1.10. Similarly, a braid word w is σ_i-free if and only if it is $\mathrm{sh}^i(w_1)$ for some w_1.

Then Properties **A** and **C** can be expressed in terms of σ_1-positive, σ_1-negative, and σ_1-free words.

PROPOSITION 1.13. *Property* **A** *is equivalent to:*
Property A (second form). *A σ_1-positive braid word is not trivial.*

PROOF. Every σ_1-positive braid word is σ-positive, so the first form of Property **A** implies the second form.

Conversely, assume the second form of Property **A**. Let w be a σ-positive word. Then w is σ_i-positive for some i. As observed above, this means that we have $w = \mathrm{sh}^{i-1}(w_1)$ for some σ_1-positive word w_1. By the second form of Property **A**, the word w_1 is not trivial, *i.e.*, it does not represent the unit braid. As the shift endomorphism of B_∞ is injective, this implies that w is not trivial either. So, the first form of Property **A** is satisfied. $\square$

PROPOSITION 1.14. *Property* **C** *is equivalent to:*
Property C (second form). *Every braid of B_n admits an n-strand representative word that is σ_1-positive, σ_1-negative, or σ_1-free.*

PROOF. A σ-positive braid word is either σ_1-positive or σ_1-free, so the first form of Property **C** implies the second form.

Conversely, assume the second form of Property **C**. We prove the first form using induction on $n \geqslant 2$. For $n = 2$, the two forms coincide. Assume $n \geqslant 3$. Let β be a nontrivial n-strand braid. By the second form of Property **C**, we find an n-strand braid word w representing β that is σ_1-positive, σ_1-negative, or σ_1-free. In the first two cases, we are done. Otherwise, let $w_1 = \mathrm{sh}^{-1}(w)$, which makes sense as, by hypothesis, w contains no letter $\sigma_1^{\pm 1}$. As the shift endomorphism of B_∞ is injective, the word w_1 does not represent 1, so the induction hypothesis implies that w_1 is equivalent to some $(n-1)$-strand braid word w_1' that is σ-positive or σ-negative. By construction, the word $\mathrm{sh}(w_1')$ represents β and it is σ-positive or σ-negative. $\square$

On the other hand, it will be often convenient in the sequel to have a name for the braids that admit a σ-positive word representative. So, we introduce the following natural terminology.

DEFINITION 1.15. A braid β is said to be *σ-positive inside B_n* (*resp. σ-negative, σ_i-positive, σ_i-negative, σ_i-free*) if, among all word representatives of β, there is at least one n-strand braid word that is σ-positive (*resp.* σ-negative, σ_i-positive, σ_i-negative, σ_i-free).

We insist that, in Definition 1.15, we only demand that there exists *at least one* word representative with the considered property. So, for instance, the braid $\sigma_2^{-1}\sigma_3\sigma_2$ is σ_2-positive since, among its many word representatives, there is one, namely $\sigma_3\sigma_2\sigma_3^{-1}$, that is σ_2-positive—there are many more: $\sigma_3\sigma_2\sigma_3^{-1}\sigma_3\sigma_3^{-1}$ is another σ_2-positive 4-strand braid word that represents the braid $\sigma_2^{-1}\sigma_3\sigma_2$.

With this terminology, $\beta <_n \beta'$ is equivalent to $\beta^{-1}\beta'$ being σ-positive inside B_n. Similarly, Property **A** means that a σ-positive braid is not trivial, and Property **C** that every nontrivial braid of B_n is σ-positive or σ-negative inside B_n.

REMARK 1.16. By Proposition 1.9(ii), a braid β of B_n satisfies $\beta >_n 1$ if and only if it satisfies $\beta >_\infty 1$, hence β is σ-positive inside B_n if and only if it is σ-positive inside B_∞. In other words, if an n-strand braid admits a word representative that is σ-positive, then it admits a word representative that is σ-positive and is an n-strand braid word, an *a priori* stronger property. Building on this result, we shall often drop the mention "inside B_n", exactly as when we write $<$ for $<_n$. However, a careful distinction has to be made when proving Property **C**. It can be mentioned that the original argument of [**47**] only leads to a proof of Property **C** in B_∞: this is enough to order every braid group B_n, but not to deduce Property **C** in B_n; see Chapter IV.

1.4. The σ^Φ-ordering of braids. If $\prec$ is an ordering of a group G and φ is an automorphism of G, then the relation $\varphi(g) \prec \varphi(h)$ defines a new ordering of G with the same invariance properties as $\prec$. In the case of B_n, the flip automorphism, *i.e.*, the inner automorphism Φ_n associated with the braid Δ_n, plays an important role, and it is natural to introduce the image of the σ-ordering under Φ_n, *i.e.*, the flipped version of the σ-ordering. As will be seen in Section 4, the new ordering so obtained has some nice properties not shared by the original version, particularly in terms of avoiding the infinite descending sequence of (1.1).

We recall from Lemma I.4.4 that Φ_n exchanges σ_i and σ_{n-i} for $1 \leqslant i < n$, thus corresponding to a symmetry in the associated braid diagrams.

DEFINITION 1.17. For $2 \leqslant n < \infty$ and β, β' in B_n, we declare that $\beta <^{\Phi}_n \beta'$ is true if we have $\Phi_n(\beta) < \Phi_n(\beta')$.

PROPOSITION 1.18. *The relation $<^{\Phi}_n$ is a left-invariant ordering of B_n. Moreover, for all β, β' in B_n, the relations $\beta <^{\Phi}_n \beta'$ and $\beta <^{\Phi}_{n+1} \beta'$ are equivalent.*

PROOF. The first part is clear as Φ_n is an automorphism of B_n.

Assume $\beta, \beta \in B_n$ and $\beta <^{\Phi}_n \beta'$. By definition, we have $\Phi_n(\beta) < \Phi_n(\beta')$, hence $\mathrm{sh}(\Phi_n(\beta)) < \mathrm{sh}(\Phi_n(\beta'))$ by Proposition 1.11. By construction, we have

$$\Phi_{n+1}(\beta) = \mathrm{sh}(\Phi_n(\beta)) \quad \text{and} \quad \Phi_{n+1}(\beta') = \mathrm{sh}(\Phi_n(\beta')),$$

so $\beta <^{\Phi}_{n+1} \beta'$ follows. As $<^{\Phi}_n$ is a linear ordering, this is enough to conclude that $<^{\Phi}_n$ coincides with the restriction of $<^{\Phi}_{n+1}$ to B_n. □

Owing to Proposition 1.18, we shall drop the subscripts and simply write $<^{\Phi}$ for the ordering of B_∞ whose restriction to B_n is $<^{\Phi}_n$. For instance, we have

$$1 <^{\Phi} \sigma_1 <^{\Phi} \sigma_2 <^{\Phi} \dots .$$

The flipped order $<^{\Phi}$ is easily described in terms of word representatives.

DEFINITION 1.19. (i) A braid word w is said to be σ^{Φ}*-positive* (*resp.* σ^{Φ}*-negative*) if, among the letters $\sigma_i^{\pm 1}$ that occur in w, the one with *highest* index occurs positively only (*resp.* negatively only).

(ii) A braid β is said to be σ^{Φ}*-positive* (*resp.* σ^{Φ}*-negative*) if it admits at least one braid word representative that is σ^{Φ}-positive (*resp.* σ^{Φ}-negative).

The only difference between a σ-positive and a σ^{Φ}-positive braid word is that, in the former case, we consider the letter σ_i with lowest index, while, in the latter case, we consider the letter σ_i with highest index.

PROPOSITION 1.20. *For all braids β, β', the relation $\beta <^{\Phi} \beta'$ holds if and only if $\beta^{-1}\beta'$ is σ^{Φ}-positive.*

PROOF. By construction, an n-strand braid word w is σ^{Φ}-positive if and only if the n-strand braid word $\Phi_n(w)$ is σ-positive. □

Thus the flipped order $<^{\Phi}$ is the counterpart of the σ-order $<$ in which the highest index replaces the lowest index, and σ^{Φ}-positive words replace σ-positive words. It is therefore natural to call it the σ^{Φ}*-ordering* of braids.

As the flip Φ_n is an automorphism of the group B_n, the properties of $<$ and $<^{\Phi}$ are similar. However, there are at least two reasons for considering both $<$ and $<^{\Phi}$. First, there is no flip on B_∞, and the two orderings differ radically on B_∞: (1.1) shows that $(B^+_\infty, <)$ has infinite descending sequences, while we shall see in Section 4.1 below that $(B^+_\infty, <^{\Phi})$ is a well-ordering, and, therefore, it has no infinite descending chain. The second reason is that, in subsequent chapters, certain approaches demand that one specific version be used: the original version $<$ in Chapter IV, the flipped version $<^{\Phi}$ in Chapters VII and VIII.

1.5. Algorithmic aspects. The σ-ordering of braids is a complicated object. However, it is completely effective in that there exist efficient comparison algorithms. In this section (and everywhere in the sequel) we denote by $\overline{w}$ the braid represented by a braid word w—but, as usual, we use σ_i both for the letter and for the braid it represents.

PROPOSITION 1.21. *For each n, the σ-ordering of B_n has at most a quadratic complexity: there exists an algorithm that, starting with two n-strand braid words w, w' of length ℓ, runs in time $O(\ell^2)$ and decides whether $\overline{w} < \overline{w'}$ holds.*

At this early stage, we cannot yet describe the algorithms witnessing to the above upper complexity bound. It turns out that most of the proofs of Property **C** alluded to in Section 1.2 provide an effective comparison algorithm. Some of them are quite inefficient—typically the one of Chapter IV—but several lead to a quadratic complexity. This is particularly the case with those based on the Φ-normal form of Chapter VII and on the ϕ-normal form of Chapter VIII: in both cases, the normal form can be computed in quadratic time, and, then, the comparison itself can be made in linear time. This is also the case with the lamination method of Chapter XII: in this case, the coordinates of a braid can be computed in quadratic time, and the comparison (with the unit braid) can then be made in (sub)linear time. Similar results are conjectured in the case of the handle reduction method of Chapter V and the Tetris algorithm of Chapter XI—see Chapter XVI for further discussion.

Let us mention that, for a convenient definition for the RAM complexity of the input braids, the algorithm of Chapter XII even leads to a complexity upper bound which is quadratic independently of the braid index n, *i.e.*, there exists an absolute constant C so that the running time for complexity ℓ input braids in B_∞ is bounded above by $C \cdot \ell^2$.

We also point out that every comparison algorithm for the σ-ordering of braids automatically gives a solution to the braid word problem, *i.e.*, to the braid isotopy problem: indeed, we have $\overline{w} = \overline{w'}$ if and only if we have neither $\overline{w} < \overline{w'}$ nor $\overline{w} > \overline{w'}$. It also leads to a comparison for the flipped version $<^{\Phi}$ of the σ-ordering, as, if w, w' are n-strand braid words, $\overline{w} <^{\Phi} \overline{w'}$ is equivalent to $\overline{\Phi_n(w)} < \overline{\Phi_n(w')}$, and the flip automorphism Φ_n can be computed in linear time.

Another related question is that of effectively finding σ-positive representative words, *i.e.*, starting with a braid word w, finding an equivalent braid word w' that is σ-positive, σ-negative, or empty. Property **C** asserts that this is always possible. Every algorithmic solution to that problem gives a comparison algorithm as, by Property **A**, w' being σ-positive implies $\overline{w} = \overline{w'} > 1$, but, conversely, deciding $\overline{w} > 1$ does not require that we exhibit a σ-positive witness.

PROPOSITION 1.22. *The σ-positive representative problem has at most an exponential complexity: there exist a polynomial $P(n, \ell)$ and an algorithm that, starting with an n-strand braid word w of length ℓ, runs in time $2^{P(n,\ell)}$ and returns a braid word of length bounded by $2^{P(n,\ell)}$ that is equivalent to w and is σ-positive, σ-negative, or empty.*

The handle reduction approach of Chapter V gives the precise form of such a polynomial: $P(n, \ell) = n^4\ell$. From the transmission-relaxation approach of Chapter XI, an asymptotically better estimate can be extracted: $P(n, \ell) = \mathrm{const} \cdot n\ell$. However, the algorithm outlined in Chapter XI is just polynomial, but the output of the algorithm is not a braid word in the standard sense but a zipped word, this meaning that, sometimes, instead of writing one and the same subword many times, the algorithm outputs the subword once and specifies the number of repetitions. This allows us to make the size of the output bounded above by a polynomial in n and ℓ though the length of the word after unzipping is not known to be of polynomial size so far.

It is likely that the approach of Chapter VIII leads to much better results: very recently, J. Fromentin announced a new algorithm that solves the σ-positive representative problem with a quadratic time complexity and a linear space complexity, without zipping the output. We refer to Chapter XVI for further discussion.

2. Local properties of the σ-ordering

We shall now list—with or without proof—some properties of the σ-ordering of braids. In this section, we consider properties that can be called local in that they involve finitely many braids at a time.

2.1. Curious examples. We start with a series of examples, including some rather surprising ones, that illustrate the complexity of the σ-ordering. The reader should note that all examples below live in B_3. This shows that, despite its simple definition, even the σ-ordering of 3-strand braids is a quite complicated object.

The first example shows that the σ-ordering is not invariant under multiplication on the right, as was already known from Proposition 1.2.

Example 2.1. Let $\beta = \sigma_1\sigma_2^{-1}$, and $\gamma = \sigma_1\sigma_2\sigma_1$, *i.e.*, $\gamma = \Delta_3$. The word $\sigma_1\sigma_2^{-1}$ contains one occurrence of σ_1 and no occurrence of σ_1^{-1}, so the braid β is σ-positive, and $\beta > 1$ is true. On the other hand, the braid $\gamma^{-1}\beta\gamma$ is represented by the word $\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_1\sigma_2^{-1}\sigma_1\sigma_2\sigma_1$, hence also by the equivalent word $\sigma_2\sigma_1^{-1}$, as, by Lemma I.4.4, we have $\Delta_3^{-1}\sigma_i\Delta_3 = \sigma_{3-i}$ for $i = 1, 2$. The word $\sigma_2\sigma_1^{-1}$ contains one letter σ_1^{-1} and no letter σ_1. So, by definition, we have $\gamma^{-1}\beta\gamma < 1$, and, therefore, $\beta\gamma < \gamma$. So $1 < \beta$ does not imply $\gamma < \beta\gamma$.

A phenomenon connected with the noninvariance under right multiplication is that a conjugate of a braid that is larger than 1 may be smaller than 1. Example 2.1 actually gives us an illustration of this situation: in fact, in this case, the conjugate is the inverse.

Example 2.2. Let $\beta = \sigma_1\sigma_2^{-1}$ again. Then β is σ_1-positive, hence larger than 1. By Lemma I.4.4, conjugating by Δ_3 amounts to exchanging σ_1 and σ_2. So we have $\Delta_3\beta\Delta_3^{-1} = \sigma_2\sigma_1^{-1}$, a σ_1-negative braid, hence smaller than 1, *i.e.*, we have $\beta > 1$ and $\Delta_3\beta\Delta_3^{-1} < 1$; however, we shall see in Corollary 3.7 below that the conjugates of a braid β cannot be too far from β.

An easy exercise is that every left-invariant ordering such that $g \prec h$ implies $g^{-1} \succ h^{-1}$ is also right-invariant. As the braid ordering is not right-invariant, there must exist counterexamples, *i.e.*, braids β, γ satisfying $\beta < \gamma$ and $\beta^{-1} < \gamma^{-1}$. Here are examples of this situation.

Example 2.3. Let $\beta = \Delta_3$ and $\gamma = \sigma_2^2\sigma_1$. Then we find $\beta^{-1}\gamma = \sigma_1\sigma_2^{-1}$, a σ_1-positive word, and $\beta\gamma^{-1} = \sigma_1\sigma_2^{-1}$, again a σ_1-positive word, So, in this case, we have $1 < \beta < \gamma$ and $\beta^{-1} < \gamma^{-1}$.

EXAMPLE 2.4. Here is a stronger example. Let $\beta = \sigma_2^{-1}\sigma_1^2\sigma_2$ and $\gamma = \Delta_3$. We now find $\beta^{-1}\gamma = \sigma_1\sigma_2^{-1}\sigma_1$ (see below), a σ_1-positive word, and $\beta\gamma^{-1} = \sigma_2^{-1}\sigma_1\sigma_2^{-1}$, a σ_1-positive word. So we obtain again $1 < \beta < \gamma$ and $\beta^{-1} < \gamma^{-1}$. But there is more. We claim that $\beta^{-p}\gamma = \sigma_1\sigma_2^{-2p+1}\sigma_1$ holds for $p \geqslant 1$. Indeed, for $p = 1$, we have

$$\beta^{-1}\gamma = \sigma_2^{-1}\sigma_1^{-1} \cdot \sigma_1^{-1}\sigma_2\sigma_1\sigma_2 \cdot \sigma_1 = \sigma_2^{-1}\sigma_1^{-1} \cdot \sigma_2\sigma_1 \cdot \sigma_1 = \sigma_1\sigma_2^{-1}\sigma_1.$$

For $p \geqslant 2$, applying the induction hypothesis, we find

$$\begin{aligned}\beta^{-p}\gamma &= \sigma_2^{-1}\sigma_1^{-2}\sigma_2 \cdot \beta^{-p+1}\gamma \\ &= \sigma_2^{-1}\sigma_1^{-2}\sigma_2 \cdot \sigma_1\sigma_2^{-2p+3}\sigma_1 = \sigma_1\sigma_2^{-2} \cdot \sigma_2^{-2p+3}\sigma_1 = \sigma_1\sigma_2^{-2p+1}\sigma_1.\end{aligned}$$

As $\sigma_1\sigma_2^{-2p+1}\sigma_1$ is a σ_1-positive word for each p, we have in this case $1 < \beta^p < \gamma$ for each positive p, and $\beta^{-1} < \gamma^{-1}$.

Even more curious situations occur. Assume that β is a σ_1-positive braid. Then the sequence $1, \beta, \beta^2, \ldots$ is strictly increasing, and its entries admit expressions in which more and more letters σ_1 occur. One might therefore expect that, eventually, the braid β^p dominates σ_1, which only contains one letter σ_1. The next example shows this is not the case.

EXAMPLE 2.5. Consider $\beta = \sigma_2^{-1}\sigma_1$. Then $\beta^p < \sigma_1$ holds for each p. The inequality clearly holds for $p \leqslant 0$. For positive p, we will show that $\sigma_1^{-1}\beta^p$ is σ_1-negative. To this end, we prove the equality

$$\sigma_1^{-1}\beta^p = \sigma_2(\sigma_2\sigma_1^{-1})^{p-1}\sigma_1^{-1}\sigma_2^{-1} \tag{2.1}$$

using induction on $p \geqslant 1$. For $p = 1$, (2.1) reduces to $\sigma_1^{-1}\sigma_2^{-1}\sigma_1 = \sigma_2\sigma_1^{-1}\sigma_2^{-1}$, which directly follows from the braid relation. For $p \geqslant 2$, we find

$$\begin{aligned}\sigma_1^{-1}\beta^p &= (\sigma_1^{-1}\beta^{p-1}) \cdot \sigma_2^{-1}\sigma_1 \\ &= \sigma_2(\sigma_2\sigma_1^{-1})^{p-2}\sigma_1^{-1}\sigma_2^{-1} \cdot \sigma_2^{-1}\sigma_1 \\ &= \sigma_2(\sigma_2\sigma_1^{-1})^{p-2}\sigma_2\sigma_1^{-2}\sigma_2^{-1} = \sigma_2(\sigma_2\sigma_1^{-1})^{p-1}\sigma_1^{-1}\sigma_2^{-1},\end{aligned}$$

using the induction hypothesis and the equality $\sigma_1^{-1}\sigma_2^{-2}\sigma_1 = \sigma_2\sigma_1^{-2}\sigma_2^{-1}$.

It can be observed that, more generally, $\beta^p < \sigma_2^{-q}\sigma_1$ holds for all nonnegative p and q. So the ascending sequence β^p does not even approach σ_1, as it remains below each entry in the descending sequence $\sigma_2^{-q}\sigma_1$.

Our last example will demonstrate that the σ-ordering of B_n is not Conradian.

DEFINITION 2.6. A left-invariant ordering $\prec$ of a group G is *Conradian* if for all g, h in G that are greater than 1, there exists a positive integer p satisfying $h \prec gh^p$.

Conrad used this property in [**38**] to show that such left-ordered groups share many of the properties of bi-orderable groups; see Section XV.5 for more details.

PROPOSITION 2.7. *For $n \geqslant 3$, the σ-ordering of the braid group B_n is not Conradian.*

PROOF. Let $\beta = \sigma_2^{-1}\sigma_1$ and $\gamma = \sigma_2^{-2}\sigma_1$. Clearly, β and γ are σ_1-positive, so $\beta > 1$ and $\gamma > 1$ hold. We claim that $\gamma\beta^p < \beta$ holds for each $p \geqslant 0$. To see that, using induction on $p \geqslant 0$, we prove the equality

$$\beta^{-1}\gamma\beta^p = \sigma_2^2(\sigma_1^{-1}\sigma_2)^{p-1}\sigma_1^{-2}\sigma_2^{-1}. \tag{2.2}$$

For $p = 0$, using the braid relations, we find

$$\beta^{-1}\gamma = \sigma_1^{-1}\sigma_2\sigma_2^{-2}\sigma_1 = \sigma_1^{-1}\sigma_2^{-1}\sigma_1 = \sigma_2\sigma_1^{-1}\sigma_2^{-1} = \sigma_2^2(\sigma_1^{-1}\sigma_2)^{-1}\sigma_1^{-2}\sigma_2^{-1}.$$

For $p = 1$, we have

$$\beta^{-1}\gamma\beta = \sigma_1^{-1}\sigma_2\sigma_2^{-2}\sigma_1\sigma_2^{-1}\sigma_1 = \sigma_1^{-1}\sigma_2^{-1}\sigma_1\sigma_2^{-1}\sigma_1 = \sigma_2\sigma_1^{-1}\sigma_2^{-2}\sigma_1 = \sigma_2^2\sigma_1^{-2}\sigma_2^{-1}.$$

For $p \geqslant 1$, again using the equality $\sigma_1^{-1}\sigma_2^{-2}\sigma_1 = \sigma_2\sigma_1^{-2}\sigma_2^{-1}$ of Example 2.5, we find

$$\begin{aligned}\beta^{-1}\gamma\beta^p &= (\sigma_2^2(\sigma_1^{-1}\sigma_2)^{p-2}\sigma_1^{-2}\sigma_2^{-1})(\sigma_2^{-1}\sigma_1)\\ &= \sigma_2^2(\sigma_1^{-1}\sigma_2)^{p-2}\sigma_1^{-1}\sigma_2\sigma_1^{-2}\sigma_2^{-1} = \sigma_2^2(\sigma_1^{-1}\sigma_2)^{p-1}\sigma_1^{-2}\sigma_2^{-1}.\end{aligned}$$

For $p \geqslant 1$, the right-hand side of (2.2) is σ_1-negative, and, for $p = 0$, it is equivalent to the σ_1-negative word $\sigma_2\sigma_1^{-1}\sigma_2$, so, in each case, we obtain $\beta < \gamma\beta^p$. □

2.2. Property S. After the many counterexamples of Section 2.1, we turn to positive results.

We have seen in Example 2.1 that the σ-ordering of braids is not invariant under multiplication on the right, and, therefore, that a conjugate of a braid larger than 1 need not be larger than 1. This phenomenon cannot, however, occur with conjugates of positive braids, *i.e.*, of braids that can be expressed using the generators σ_i only, and not their inverses. The core of the question is the last of the three fundamental properties of braids we shall develop here:

Property S (Subword). *Every braid of the form $\beta^{-1}\sigma_i\beta$ is σ-positive.*

Property **S** was first proved by Richard Laver in [**137**]. In this text, proofs of Property **S** appear on pages 83, 152, 193, and 263.

Using the compatibility of $<$ with multiplication on the left and a straighforward induction, we deduce the following result, which explains our terminology:

PROPOSITION 2.8. *Assume that β, β' are braids and some braid word representing β' is obtained by inserting positive letters σ_i in a braid word representing β. Then we have $\beta' > \beta$.*

We recall that B_∞^+ denotes the submonoid of B_∞ generated by the braids σ_i. Another consequence of Property **S** is:

PROPOSITION 2.9. *If β belongs to B_∞^+ and is not 1, then $\beta' > 1$ is true for every conjugate β' of β. More generally, $\beta > 1$ is true for every quasi-positive braid β, the latter being defined as a braid that can be expressed as a product of conjugates of positive braids.*

PROOF. Assume $\beta' = \gamma^{-1}\beta\gamma$ with $\beta \in B_\infty^+$. By definition, β is a product of finitely many braids σ_i, so, in order to prove $\beta' > 1$, it suffices to establish that $\gamma^{-1}\sigma_i\gamma > 1$ holds for each i, and this is Property **S**. □

As was noted by Stepan Orevkov [**166**], the converse implication is not true: the braid $\sigma_2^{-5}\sigma_1\sigma_2^2\sigma_1$ is a non-quasi-positive braid but every conjugate of it is σ-positive.

By applying the flip automorphism Φ_n, we immediately deduce from Property **S** that every braid of the form $\beta^{-1}\sigma_i\beta$ is also σ^Φ-positive, and that the counterpart of Proposition 2.8 involving the ordering $<^\Phi$ is true. A direct application is the following result, which is important for analysing the restriction of $<^\Phi$ to B_∞^+:

PROPOSITION 2.10. *For each n, the set B_n^+ is the initial segment of $(B_\infty^+, <^\Phi)$ determined by σ_n, i.e., we have $B_n^+ = \{\beta \in B_\infty^+ \mid \beta <^\Phi \sigma_n\}$.*

Proof. By definition, $\beta <^{\Phi} \sigma_n$ holds for every β in B_n^+. Indeed, if w is any n-strand braid word representing β, then $w^{-1}\sigma_n$ is a σ_n^{Φ}-positive word representing $\beta^{-1}\sigma_n$.

Conversely, assume that β is a positive braid satisfying $\beta <^{\Phi} \sigma_n$. Let w be a positive braid word representing β, and let σ_i be the generator with highest index occurring in w. By the counterpart of Proposition 2.8, we have $\beta \geqslant^{\Phi} \sigma_i$, and, therefore, $i \geqslant n$ would contradict the hypothesis $\beta <^{\Phi} \sigma_n$. $\square$

Another application of Property **S** is the following property from [**51**]. We recall that sh denotes the shift endomorphism of B_∞ that maps σ_i to σ_{i+1} for every i.

Proposition 2.11. *For each braid β, we have $\beta < \mathrm{sh}(\beta)\,\sigma_1$.*

Proof. Let β be an arbitrary braid in B_n. We claim that the braid $\beta^{-1}\mathrm{sh}(\beta)\sigma_1$ is σ_1-positive. To see that, we write, inside B_{n+1},

$$\beta^{-1}\mathrm{sh}(\beta)\sigma_1 = (\underline{\beta^{-1}\sigma_2 \dots \sigma_n\beta}) \cdot (\underline{\sigma_n^{-1} \dots \sigma_2^{-1}}) \cdot (\underline{\sigma_2 \dots \sigma_n\beta^{-1}\sigma_n^{-1} \dots \sigma_2^{-1}}) \cdot \mathrm{sh}(\beta)\sigma_1.$$

The first underlined fragment is a conjugate of the positive braid $\sigma_2 \dots \sigma_n$, so, by Property **S**, it is σ-positive, hence either σ_1-positive or σ_1-free. The second underlined fragment is σ_1-free. Next, it is easy to check with a picture that the third underlined fragment is equal to $\sigma_1^{-1}\mathrm{sh}(\beta^{-1})\sigma_1$. Putting things together, we obtain

$$\beta^{-1}\mathrm{sh}(\beta)\sigma_1 = \beta' \cdot \underline{\sigma_1^{-1}\mathrm{sh}(\beta^{-1}) \cdot \sigma_1 \cdot \mathrm{sh}(\beta)\sigma_1},$$

where β' is a braid that is either σ_1-positive or σ_1-free. But, now, we see that the underlined expression is a conjugate of σ_1, so, by Property **S**, it is σ-positive, hence σ_1-positive or σ_1-free. We deduce that $\beta^{-1}\mathrm{sh}(\beta)\sigma_1$ itself is σ_1-positive or σ_1-free.

Finally, it is impossible that $\beta^{-1}\mathrm{sh}(\beta)\sigma_1$ be σ_1-free. Indeed, let π be the permutation of $\{1, \dots, n\}$ induced by β. Then the initial position of the strand that finishes at position 1 in any diagram representing $\beta^{-1}\mathrm{sh}(\beta)\sigma_1$ is $\pi^{-1}(\pi(1) + 1)$, which cannot be 1.

So the only possibility is that $\beta^{-1}\mathrm{sh}(\beta)\sigma_1$ is σ_1-positive, hence σ-positive. $\square$

3. Global properties of the σ-ordering

We turn to more global properties, involving infinitely many braids at a time. Here we successively consider the Archimedean property, the question of density and the associated topology, and convex subgroups.

3.1. The Archimedean property. We shall show that the σ-ordering and, more generally, any left-invariant ordering of B_n fails to be Archimedean for $n \geqslant 3$. However, certain partial Archimedean properties involving the central elements Δ_n^2 are satisfied.

Definition 3.1. A left-ordered group $(G, \prec)$ is said to be *Archimedean* if, for all g, h larger than 1 in G, there exists a positive integer p for which $g \prec h^p$ holds.

In other words, the powers of any nontrivial element are cofinal in the ordering. For example, an infinite cyclic group, with either of the two possible orderings, is Archimedean. On the other hand, $\mathbb{Z} \times \mathbb{Z}$ with lexicographic ordering is not Archimedean, whereas Archimedean orderings for the same group do exist, by embedding $\mathbb{Z} \times \mathbb{Z}$ in the additive real numbers, sending the generators to rationally independent numbers, and taking the induced ordering.

PROPOSITION 3.2. *The σ-ordering of B_n is not Archimedean for $n \geqslant 3$.*

PROOF. For every positive integer p, we have $1 < \sigma_2^p < \sigma_1$. □

One can say more.

PROPOSITION 3.3. *For $n \geqslant 3$, every left-invariant ordering of B_n fails to be Archimedean.*

This follows from the the fact that B_n is not Abelian for $n \geqslant 3$ and from a result of P. Conrad [**38**] generalizing the classical theorem of Hölder [**111**]: any left-invariant Archimedean ordering of a group must also be right-invariant, and the group embeds, simultaneously in the algebraic and order senses, in the additive real numbers. In particular, such a group is Abelian.

By contrast to the previous negative result, there is a partial Archimedean property involving the central element Δ_n^2, namely that every braid is dominated by some power of the braid Δ_n^2.

The results we shall establish turn out to be true not only for the σ-ordering, but also for any left-invariant ordering of B_n. So, for the rest of this section, we consider this extended framework. When $\prec$ denotes a strict ordering, $\preccurlyeq$ denotes the corresponding nonstrict ordering, *i.e.*, $x \preccurlyeq y$ stands for "$x \prec y$ or $x = y$".

LEMMA 3.4. *Assume that $\prec$ is a left-invariant ordering of B_n. Then $\Delta_n^{2p} \prec \beta$ implies $\beta^{-1} \prec \Delta_n^{-2p}$, and the conjunction of $\Delta_n^{2p} \prec \beta$ and $\Delta_n^{2q} \prec \gamma$ implies $\Delta_n^{2p+2q} \prec \beta\gamma$. The same implications hold for $\preccurlyeq$.*

PROOF. Assume $\Delta_n^{2p} \prec \beta$. Multiplying by β^{-1} on the left, we get $\beta^{-1}\Delta_n^{2p} \prec 1$, which is also $\Delta_n^{2p}\beta^{-1} \prec 1$. Multiplying by Δ_n^{-2p} on the left, we deduce $\beta^{-1} \prec \Delta_n^{-2p}$.

Now assume $\Delta_n^{2p} \prec \beta$ and $\Delta_n^{2q} \prec \gamma$. By multiplying the first inequality by Δ_n^{2q} on the left, we obtain $\Delta_n^{2p+2q} \prec \Delta_n^{2q}\beta = \beta\Delta_n^{2q}$. By multiplying the second inequality by β on the left, we obtain $\beta\Delta_n^{2q} \prec \beta\gamma$. We deduce $\Delta_n^{2p+2q} \prec \beta\gamma$. □

LEMMA 3.5. *Assume that $\prec$ is a left-invariant ordering of B_n satisfying $1 \prec \Delta_n$. Then, for each i in $\{1, \ldots, n-1\}$, we have $\Delta_n^{-2} \prec \sigma_i \prec \Delta_n^2$.*

PROOF. By Lemma I.4.4, we have $\delta_n^n = \Delta_n^2$, so the hypothesis $1 \prec \Delta_n$ implies $1 \prec \Delta_n^2 = \delta_n^n$, hence $1 \prec \delta_n$, and, therefore, $1 \prec \delta_n \prec \delta_n^2 \prec \ldots \prec \delta_n^n = \Delta_n^2$.

Assume that $\Delta_n^2 \preccurlyeq \sigma_i$ holds for some i. Let j be any element of $\{1, \ldots, n-1\}$. By formulas (I.4.3) and (I.4.4), we can find p with $0 \leqslant p \leqslant n-1$ satisfying $\sigma_j = \delta_n^{-p}\sigma_i\delta_n^p$. Then we obtain

$$1 \prec \delta_n^{n-p} = \delta_n^{-p}\Delta_n^2 \preccurlyeq \delta_n^{-p}\sigma_i \preccurlyeq \delta_n^{-p}\sigma_i\delta_n^p = \sigma_j.$$

So $1 \prec \sigma_j$ holds for each generator σ_j. Applying Lemma 3.4, we deduce that, if a braid β can be represented by a positive braid word that contains at least one letter σ_i, then $\Delta_n^2 \preccurlyeq \beta$ holds. This applies in particular to Δ_n, and we deduce $\Delta_n^2 \preccurlyeq \Delta_n$, which contradicts the assumption $1 \prec \Delta_n$.

Similarly, assume that $\sigma_i \preccurlyeq \Delta_n^{-2}$ holds. Consider again any σ_j. If p is as above, we also have $\sigma_j = \delta_n^{n-p}\sigma_i\delta_n^{p-n}$, since δ_n^n lies in the center of B_n. Then we find

$$\sigma_j = \delta_n^{n-p}\sigma_i\delta_n^{p-n} \prec \delta_n^{n-p}\sigma_i \preccurlyeq \delta_n^{n-p}\Delta_n^{-2} = \delta_n^{-p} \preccurlyeq 1.$$

This time, $\sigma_j \prec 1$ holds for each j. As Δ_n is a positive braid, this implies $\Delta_n \prec 1$, which contradicts the assumption $1 \prec \Delta_n$. □

Gathering the results, we immediately deduce:

PROPOSITION 3.6. *Assume $\prec$ is a left-invariant ordering of B_n and $1 \prec \Delta_n$ holds. Then, for each braid β in B_n, there exists a unique integer p for which $\Delta_n^{2p} \preccurlyeq \beta \prec \Delta_n^{2p+2}$ is true. Moreover, if β can be represented by a braid word of length ℓ, we have $|p| \leqslant \ell$.*

PROOF. Lemma 3.5 implies that each generator σ_i lies in the interval $(\Delta_n^{-2}, \Delta_n^2)$. Then Lemma 3.4 implies that every braid that can be represented by a word of length ℓ lies in the interval $[\Delta_n^{-2\ell}, \Delta_n^{2\ell})$. As this interval is the disjoint union of the intervals $[\Delta_n^{2p}, \Delta_n^{2p+2})$ for $-\ell \leqslant p < \ell$, the result of the proposition follows. ☐

In this way, we obtain a decomposition of $(B_n, \prec)$ into a sequence of disjoint intervals of size Δ_n^2, as suggested in Figure 1.

As noted by A. Malyutin and N.Yu. Netsvetaev in [**150**], the previous result implies that the action of conjugacy cannot move a braid too far.

COROLLARY 3.7 (Figure 1). *Assume that $\prec$ is a left-invariant ordering of B_n satisfying $1 \prec \Delta_n$. Then, if β and β' are conjugate,*

$$\Delta_n^{2p} \preccurlyeq \beta \prec \Delta_n^{2p+2} \quad \textit{implies} \quad \Delta_n^{2p-2} \preccurlyeq \beta' \prec \Delta_n^{2p+4}. \tag{3.1}$$

So, in particular, $\beta\Delta_n^{-4} \prec \beta' \prec \beta\Delta_n^4$ is always true.

PROOF. Assume $\Delta_n^{2p} \preccurlyeq \beta \prec \Delta_n^{2p+2}$ and $\beta' = \gamma\beta\gamma^{-1}$. By Proposition 3.6, we have $\Delta_n^{2q} \preccurlyeq \gamma \prec \Delta_n^{2q+2}$ for some q. Lemma 3.4 first implies $\Delta_n^{-2q-2} \prec \gamma^{-1} \preccurlyeq \Delta_n^{-2q}$, and then

$$\Delta_n^{2q+2p-2q-2} \prec \gamma\beta\gamma^{-1} \prec \Delta_n^{2q+2+2p+2-2q},$$

which gives $\Delta_n^{2p-2} \prec \beta' \prec \Delta_n^{2p+4}$. ☐

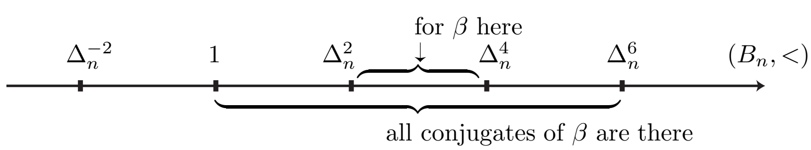

FIGURE 1. Powers of Δ_n^2 and the action of conjugacy on $(B_n, <)$.

All the previous results apply to the σ-ordering, as it is a left-invariant ordering of B_n and $1 < \Delta_n$ is satisfied. Note that, in this case, Corollary 3.7 is optimal in the sense that we cannot replace intervals of length Δ_n^2 with intervals of length Δ_n in Lemma 3.4: for instance, we have $1 < \sigma_1^2\sigma_2 < \Delta_3$ and $\Delta_3^2 < \Delta_3\sigma_1^2\sigma_2 < \Delta_3^3$.

3.2. Discreteness and density. Left-invariant orderings of a group have a sort of homogeneity—the ordering near any two group elements has similar order properties, because of invariance under left translation. In particular, there is a basic dichotomy between discrete and dense orders.

DEFINITION 3.8. A left-invariant ordering of a group is said to be *discrete* if its positive cone has a least element; it is said to be *dense* if the positive cone does not have a least element.

Equivalently, a left-invariant ordering of a group is discrete if every group element has an immediate successor and predecessor, and it is dense if between any two group elements one can find another element of the group. One verifies easily that, in a discretely left-ordered group, with least element ε larger than 1, the immediate successor of a group element g is $g\varepsilon$ and its immediate predecessor is $g\varepsilon^{-1}$.

The braid orderings display both types.

PROPOSITION 3.9. *The σ-ordering of B_n is discrete, with least σ-positive element σ_{n-1}.*

PROOF. Clearly σ_{n-1} is σ-positive. Conversely, assume that β belongs to B_n and is σ-positive. If β is σ_i-positive for some i with $i \leqslant n-2$, then $\sigma_{n-1}^{-1}\beta$ is σ_i-positive as well, so $\sigma_{n-1} < \beta$ holds. On the other hand, if β is σ_{n-1}-positive, it must be σ_{n-1}^p for some $p \geqslant 1$, and we find $\sigma_{n-1}^{-1}\beta = \sigma_{n-1}^{p-1}$, hence $\sigma_{n-1} \leqslant \beta$. □

As the flip automorphism Φ_n is an isomorphism of $(B_n, <)$ to $(B_n, <^{\Phi})$, the flipped version $<^{\Phi}$ of the σ-ordering is also discrete on B_n, and σ_1 is the least σ^{Φ}-positive element. In the inclusions $B_n \subseteq B_{n+1}$, the σ^{Φ}-ordering has the pleasant property that the same element σ_1 is least σ-positive in each braid group. For this reason, we see a difference in the two orderings in the limit. The reader may easily verify the following.

PROPOSITION 3.10. *The σ-ordering of B_∞ is dense, whereas the σ^{Φ}-ordering of B_∞ is discrete, with σ_1 being the least element larger than* 1.

COROLLARY 3.11. *The ordered set $(B_\infty, <)$ is order-isomorphic to $(\mathbb{Q}, <)$.*

PROOF. A well-known result of Cantor says that any two countable linearly ordered sets that are dense—there always exists an element between any two elements—and unbounded—there is no minimal or maximal element—are isomorphic: assuming that the sets are $\{a_n \mid n \in \mathbb{N}\}$ and $\{b_n \mid n \in \mathbb{N}\}$, one alternatively defines $f(a_0)$, $f^{-1}(b_0)$, $f(a_1)$, $f^{-1}(b_1)$, etc. so as to keep f order-preserving.

Here the rationals are eligible, and the set B_∞ is countable. So, in order to apply Cantor's criterion, it suffices to prove that $(B_\infty, <)$ is dense and unbounded. The former result is Proposition 3.10. The latter is clear: for every braid β, we have $\beta\sigma_1^{-1} < \beta < \beta\sigma_1$. □

Of course, the order-isomorphism of Corollary 3.11 could not be an isomorphism in the algebraic sense, as B_∞ is non-Abelian.

Every linearly ordered set has an order topology, with open intervals forming a basis for the topology. If the ordering is discrete, as is the case for the σ-ordering of B_n for $n < \infty$, then the topology is also discrete. Since B_∞, with the σ-ordering, is order isomorphic with the rational numbers, its order topology is metrizable. In fact, it has a natural metric, as follows.

PROPOSITION 3.12. *For $\beta \neq \beta'$ in B_∞, define $d(\beta, \beta')$ to be 2^{-p} where p is the greatest integer satisfying $\beta^{-1}\beta' \in \mathrm{sh}^p(B_\infty)$, completed with $d(\beta, \beta) = 0$. Then d is a distance on B_∞, and the topology of B_∞ associated with the linear order $<$ is the topology associated with d.*

PROOF. It is routine to verify that d is a distance. The open disk of radius 2^{-p} centered at β is the left coset $\beta\,\mathrm{sh}^p(B_\infty)$, *i.e.*, the set of all braids of the form $\beta\,\mathrm{sh}^p(\gamma)$.

Assume now that β_1, β, β_2 lie in B_n and $\beta_1 < \beta < \beta_2$ holds. We will show that the open d-disk around β of radius 2^{-n+1} is included in the interval (β_1, β_2). Indeed, if $d(\beta, \gamma) < 2^{-n+1}$, then $\beta^{-1}\gamma$ belongs to $\mathrm{sh}^n(B_\infty)$. The hypothesis $\beta_1 < \beta$ implies that $\beta_1^{-1}\beta$ is σ_i-positive for some $i \leqslant n-1$. Writing $\beta_1^{-1}\gamma = (\beta_1^{-1}\beta)(\beta^{-1}\gamma)$, we see that $\beta_1^{-1}\gamma$ is also σ_i-positive and, therefore, $\beta_1 < \gamma$ is true. A similar argument gives $\gamma < \beta_2$.

Conversely, let us start with an arbitrary open d-disk $\beta\,\mathrm{sh}^p(B_\infty)$. Let β' be a braid in this disk; we have to find an open $<$-interval containing β' which lies entirely in the disk. By hypothesis, we have $\beta' = \beta\,\mathrm{sh}^p(\gamma)$ for some γ of B_∞. Let γ_1 and γ_2 be any braids satisfying $\gamma_1 < \gamma < \gamma_2$. Then the interval $(\beta\,\mathrm{sh}^p(\gamma_1), \beta\,\mathrm{sh}^p(\gamma_2))$ contains $\beta\,\mathrm{sh}^p(\gamma)$ and is included in the disk, because $\mathrm{sh}^p(B_\infty)$ is convex—see Proposition 3.17 below. This completes the proof that the topologies associated with $<$ and with d coincide. $\square$

3.3. Dense subgroups. It is clear that densely ordered groups can have subgroups which are discretely ordered (by the same ordering)—witness $\mathbb{Z}$ in $\mathbb{Q}$. But the reverse can happen, too. For example, the lexicographic ordering on $\mathbb{Q} \times \mathbb{Z}$ is discrete—with least positive element $(0, 1)$—whereas the subgroup $\mathbb{Q} \times \{0\}$ is densely ordered. This latter phenomenon happens quite naturally also for the braid groups.

Note that, if one allows the generators σ_i to commute, the braid relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ implies that σ_i and σ_{i+1} become equal. From this one sees that the Abelianization of B_n is infinite cyclic, and the Abelianization map $B_n \to \mathbb{Z}$ can be identified with the sum of the exponents of a word in the σ_i generators. The commutator subgroup $[B_n, B_n]$ consists exactly of braids expressed in the generators σ_i with exponent sum zero.

Proposition 3.13 ([**37**]). *For $n \geqslant 3$, the commutator subgroup $[B_n, B_n]$ is densely ordered under the σ-ordering.*

Proof. For simplicity, we will prove this just for $n = 3$, referring the reader to [**37**] for the general case, whose proof is similar.

For contradiction, suppose $[B_3, B_3]$ has a least σ-positive element β. We consider the braid $\beta\sigma_2\beta^{-1}$. There are three possibilities:

Case 1: $\beta\sigma_2\beta^{-1}$ is σ_1-positive. Then β must be σ_1-positive. So is $\beta\sigma_2\beta^{-1}\sigma_2^{-1}$ and we have $1 < \beta\sigma_2\beta^{-1}\sigma_2^{-1}$. On the other hand, as β is σ_1-positive, $\sigma_2\beta^{-1}\sigma_2^{-1}$ is σ_1-negative, and we have $\sigma_2\beta^{-1}\sigma_2^{-1} < 1$ and $\beta\sigma_2\beta^{-1}\sigma_2^{-1} < \beta$. So the commutator $\beta\sigma_2\beta^{-1}\sigma_2^{-1}$ is a smaller σ-positive element of $[B_3, B_3]$ than β, contradicting the hypothesis on β.

Case 2: $\beta\sigma_2\beta^{-1}$ is σ_1-negative. A similar argument gives $1 < \beta\sigma_2^{-1}\beta^{-1}\sigma_2 < \beta$, again a contradiction.

Case 3: $\beta\sigma_2\beta^{-1}$ is σ_2^p for some p. Counting the exponents, we see that the only possibility is $p = 1$, *i.e.*, β commutes with σ_2. It is shown in [**84**] that the centralizer of the subgroup of B_3 generated by σ_2 is the subgroup (isomorphic to $\mathbb{Z} \times \mathbb{Z}$) generated by σ_2 and Δ_3^2, so we must have $\beta = (\sigma_1\sigma_2\sigma_1)^{2q}\sigma_2^r$ for some integers q, r. But, since β is σ_1-positive and a commutator, we have $q > 0$ and $6q + r = 0$. Now, consider $\beta' = \sigma_1\sigma_2^{-1}$. We have $\beta' > 1$ and $\beta' \in [B_3, B_3]$, and an easy calculation gives $\beta' < \beta$, again contradicting the hypothesis on β. $\square$

Other subgroups of B_n with $n \geqslant 3$ which are shown to be densely ordered by the σ-ordering in [**37**] include the following:

• $[PB_n, PB_n]$, the commutator subgroup of the pure braid group; but PB_n itself is discretely ordered, with least positive element σ_{n-1}^2;

• the subgroup of Brunnian braids—defined as braids such that, for every strand, its removal results in a trivial braid;

• the subgroup of homotopically trivial braids, as considered in [**99**];

• kernels of the Burau representation for those n for which this representation is unfaithful—it is known to be unfaithful for $n \geqslant 5$ and faithful for $n \leqslant 3$.

The method of proof is to identify explicitly which braids can possibly be the least σ-positive elements of a given normal subgroup of B_n.

3.4. Convex subgroups. Convex subgroups play an important role in the theory of orderable groups.

Definition 3.14. If $(G, \prec)$ is a left-ordered group, a subgroup H of G is said to be *convex* if, for all h, h' in H and g in G satisfying $h \prec g \prec h'$, one has $g \in H$.

An equivalent criterion for convexity of H is the conjunction of $1 \prec g \prec h$, $g \in G$, and $h \in H$ implies $g \in H$. It is easy to verify that the collection of convex subgroups of a given group is linearly ordered by inclusion. Moreover, if N is a normal convex subgroup of the left-ordered group G, then the quotient group G/N is left-orderable by ordering cosets according to their representatives.

If the ordering of G is discrete, and H is a convex subgroup distinct from $\{1\}$, then the ordering on H is also discrete, and H contains the minimal positive element of G, which is also minimal positive in H.

We shall see that there are rather few convex subgroups in the braid groups under the σ-ordering.

Proposition 3.15. *The group B_n has no proper normal convex subgroup.*

Proof. Suppose H is a normal and convex subgroup of B_n distinct of $\{1\}$. As remarked above, the minimal positive element σ_{n-1} of B_n belongs to H by convexity. Since H is normal, σ_1 also belongs to H, as the Garside braid Δ_n conjugates it to σ_{n-1}. All the other σ_i generators are positive and less than σ_1, so they must also be in H, and therefore we have $H = B_n$, alternatively, we can observe that all generators σ_i are conjugated to σ_{n-1} in B_n, as seen in Lemma I.4.4. □

Proposition 3.16. *For i in $\{1, \dots, n-1\}$, let H_i be the subgroup of B_n generated by $\sigma_i, \dots, \sigma_{n-1}$. Then each subgroup H_i is convex in B_n and these are the only nontrivial convex subgroups.*

Proof. First, we verify that H_i is convex. Suppose $1 < \gamma < \beta$ with $\beta \in H_i$ and $\gamma \in B_n$. Note that the σ-positive elements of H_i are exactly the σ_j-positive braids in B_n with $j \geqslant i$. So β is σ_j-positive for some $j \geqslant i$. By hypothesis, γ is σ_k-positive for some k in $\{1, \dots, n-1\}$. If we had $k < j$, then $\beta^{-1}\gamma$ would be σ_j-positive, implying $\beta < \gamma$ and contradicting the hypothesis. Therefore, we have $k \geqslant j \geqslant i$ and γ lies in H_i.

It remains to show that there are no other nontrivial convex subgroups. Assume that C is a convex subgroup of B_n distinct of $\{1\}$. Let i be the least positive integer such that C contains a σ_i-positive braid, say β. We will show that $C = H_i$. First note that C contains each σ_j with $j > i$, because $\sigma_j^{-1}\beta$ is σ_i-positive and we have $1 < \sigma_j < \beta \in C$.

Now we may write $\beta = \beta_0\sigma_i\beta_1\sigma_i \dots \sigma_i\beta_m$ for some $m \geqslant 1$ and some β_i belonging to H_{i+1}, hence to C. Since C is a subgroup and β_0 belongs to C, the braid β' defined by $\beta' = \sigma_i\beta_1\sigma_i \dots \sigma_i\beta_m$ also belongs to C. In case $m > 1$, we conclude $\sigma_i^{-1}\beta'$ is also σ_i-positive and therefore we have $1 < \sigma_i < \beta'$. On the other hand, if $m = 1$ holds, we have $\beta' = \sigma_i\beta_1$. In either case, we conclude that σ_i belongs to C. We have shown that C is included in H_i. If the inclusion were proper, then C would contain a braid which is σ_j-positive for some $j < i$, contradicting our choice of i. □

Almost exactly the same argument shows the following.

PROPOSITION 3.17. *The nontrivial convex subgroups of B_∞ are exactly those of the form* $\mathrm{sh}^i(B_\infty)$*. None of these is normal.*

Finally, using the flip automorphism Φ_n, we see that, when the σ^Φ-ordering $<^\Phi$ replaces the σ-ordering, then the convex subgroups of B_n are the groups B_i with $i \leqslant n$. The same holds for B_∞.

4. The σ-ordering of positive braids

In this section, we review some results about the restriction of the orderings $<$ and $<^\Phi$ to the braid monoids B_n^+, most of which will be further developed in Chapters VII and VIII. As the many examples of Section 2.1 showed, the σ-ordering is a quite complicated ordering. By contrast, its restriction to the monoid B_n^+ is a simple ordering, namely a well-ordering. In particular, every nonempty set of positive braids has a least element, and, if it is bounded, it has a least upper bound.

We give two proofs of the well-order property for the σ-ordering of B_n^+. Due to Laver [**137**] and based on Property **S**, the first one uses Higman's subword lemma, and it is not constructive. Then, we give another argument, which is constructive and much more precise. It is based on Serge Burckel's approach in [**27**]. Here we follow the new description of [**62**], which relies on an operation called the Φ_n-splitting of a braid. It shows that the ordering of B_n^+ is a sort of lexicographical extension of the ordering of B_{n-1}^+.

Most of the properties described in this section for the monoids B_n^+ extend to the case of the so-called dual braid monoids B_n^{+*}. Introduced by Birman, Ko, and Lee in [**15**], the dual monoid B_n^{+*} is a submonoid of B_n that properly includes B_n^+. Interestingly, the proofs turn out to be easier in the case of B_n^{+*} than in the case of B_n^+. We refer to Chapter VIII for details.

4.1. The well-order property. Restricting a linear ordering to a proper subset always gives a linear ordering, but the properties of the initial ordering and of its restriction may be very different—we already saw examples in Section 3.3. This is what happens with the σ-ordering of B_n and its restriction to B_n^+. For instance, we saw in Proposition 3.9 that $(B_n, <)$ is discrete, and that every braid β has an immediate predecessor, namely $\beta\sigma_{n-1}^{-1}$. The situation is radically different with B_n^+. In particular, $(B_n^+, <)$ has limit points: for instance, in $(B_3^+, <)$, the braid σ_1 is the least upper bound of the increasing sequence $(\sigma_2^p)_{p \geqslant 0}$; see Figure 2.

We recall that a linear ordering is called a *well-ordering* if every nonempty subset has a least element, or, equivalently, provided some very weak form of the Axiom of Choice is assumed, if it admits no infinite descending sequence. A direct consequence of Property **S** is the following important result.

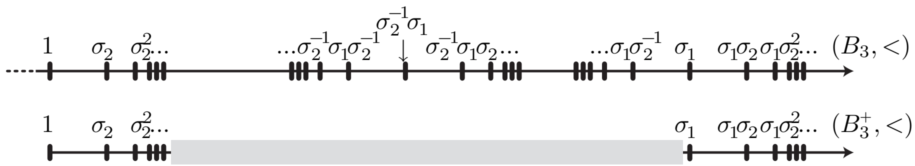

FIGURE 2. Restricting to positive braids completely changes the ordering: for instance, in $(B_3^+, <)$, the braid σ_1 is the limit of σ_2^p, whereas, in $(B_3, <)$, it is an isolated point with immediate predecessor $\sigma_2^{-1}\sigma_1$; the grey part in B_3 includes infinitely many braids, such as $\sigma_2^{-1}\sigma_1$ and its neighbours—and much more—but none of them lies in B_3^+.

PROPOSITION 4.1. *For every n, the restriction of $<$ to B_n^+ is a well-ordering.*

PROOF. A theorem of Higman [**108**], known as Higman's subword lemma, says: An infinite set of words over a finite alphabet necessarily contains two elements w, w' such that w' can be obtained from w by inserting intermediate letters (in not necessarily adjacent positions). Let $\beta_1, \beta_2, ...$ be an infinite sequence of braids in B_n^+. Our aim is to prove that this sequence is not strictly decreasing. For each p, choose a positive braid word w_p representing β_p. There are only finitely many n-strand braid words of a given length, so, for each p, there exists $p' > p$ such that $w_{p'}$ is at least as long as w_p. So, inductively, we can extract a subsequence $w_{p_1}, w_{p_2}, ...$ in which the lengths are nondecreasing. If the set $\{w_{p_1}, w_{p_2}, ...\}$ is finite, there exist k, k' such that w_{p_k} and $w_{p_{k'}}$ are equal, and then we have $\beta_{p_k} = \beta_{p_{k'}}$. Otherwise, by Higman's theorem, there exist k, k' such that w_{p_k} is a subword of $w_{p_{k'}}$, and, by construction, we must have $p_k < p_{k'}$. By Property **S**, this implies $\beta_{p_k} \leqslant \beta_{p_{k'}}$ in B_n^+. So, in any case, the sequence $\beta_1, \beta_2, ...$ is not strictly decreasing. □

The previous proof actually shows more.

PROPOSITION 4.2. *Assume that M is a submonoid of B_∞ generated by finitely many braids, each of which is a conjugate of some σ_i—hence of σ_1. Then the restriction of $<$ to M is a well-ordering.*

PROOF. In the proof of Proposition 4.1, Property **S** is used to ensure that, if a word w in the generators σ_i of B_n is a subword of another word w', then we have $\overline{w} \leqslant \overline{w'}$, where $\overline{w}$ denotes the braid represented by w. Now the same property holds for the generators of M, as each of them is a conjugate of some σ_i. Indeed, inserting a pattern of the form $v\sigma_i v^{-1}$ after w_1 in a braid word $w_1 w_2$ amounts to inserting σ_i in the equivalent braid word $w_1 v v^{-1} w_2$, and, therefore, the braid represented by $w_1 \cdot v\sigma_i v^{-1} \cdot w_2$ is larger than the braid represented by $w_1 w_2$. □

Typically, the dual braid monoids investigated in Chapter VIII are eligible for Proposition 4.2.

REMARK 4.3. The hypothesis that the monoid M is finitely generated is crucial in Proposition 4.2. For instance, we already observed that the submonoid B_∞^+ of B_∞ is not well-ordered by the σ-ordering, as we have an infinite descending sequence $\sigma_1 > \sigma_2 > ...$. Such phenomena already occur inside B_3: for instance, the submonoid of B_3 generated by all conjugates $\sigma_2^{-p}\sigma_1\sigma_2^p$ of σ_1—and, more generally, the submonoid of all quasi-positive n-strand braids, defined to be the submonoid of B_n generated by all conjugates of $\sigma_1, \ldots, \sigma_{n-1}$—contains the infinite descending sequence $\sigma_1 > \sigma_2^{-1}\sigma_1\sigma_2 > \sigma_2^{-2}\sigma_1\sigma_2^2 > \ldots$.

Being a well-ordering has strong consequences. In particular, in contrast to what the examples of Section 2.1 showed, the well-order property implies the most general form of the phenomenon observed in Figure 2:

COROLLARY 4.4. *Every nonempty subset of B_n^+ is either cofinal or it has a least upper bound inside $(B_n^+, <)$.*

Indeed, for X included in B_n^+, unless X is unbounded in B_n^+, the set of all upper bounds of X is nonempty, hence it admits a least element.

4.2. The recursive construction of the ordering on B_n^+. We gave above a quick proof for Proposition 4.1, but the latter is not constructive, and it gives no direct description of the well-ordering $(B_n^+, <)$. We shall now give such a description, based on a recursive construction that connects $(B_{n-1}^+, <)$ and $(B_n^+, <)$. This approach leads in particular to considering the ordering of B_n^+ as an iterated extension of the ordering of B_2^+, *i.e.*, of the standard ordering of natural numbers.

To explain the results, it is crucial to use the flipped version of the σ-ordering, *i.e.*, the ordering $<^{\Phi}$ defined from σ^{Φ}-positive braids. The reason is that, although $(B_n^+, <)$ and $(B_n^+, <^{\Phi})$ are isomorphic, the pairs $(B_n^+, B_{n-1}^+, <)$ and $(B_n^+, B_{n-1}^+, <^{\Phi})$ are not, and the connection between B_n^+ and B_{n-1}^+ is more easily described in the case of $<^{\Phi}$.

The starting point of the approach is the following result from [**62**]. We recall that Φ_n denotes the flip automorphism (both of B_n and of B_n^+) that exchanges σ_i and σ_{n-i} for $1 \leqslant i \leqslant n-1$.

PROPOSITION 4.5. *Assume $n \geqslant 3$. Then, for each braid β in B_n^+, there exists a unique sequence $(\beta_p, \ldots, \beta_1)$ in B_{n-1}^+ such that β admits the decomposition*

$$\beta = \Phi_n^{p-1}(\beta_p) \cdot \ldots \cdot \Phi_n(\beta_2) \cdot \beta_1, \tag{4.1}$$

and for each r the only generator σ_i that right divides $\Phi_n^{p-r}(\beta_p) \cdot \ldots \cdot \beta_r$ is σ_1. The sequence $(\beta_p, \ldots, \beta_1)$ is called the Φ_n-splitting of β.

The result easily follows from the fact that every positive braid β of B_n^+ admits a unique maximal right divisor that lies in B_{n-1}^+. The unusual enumeration of the sequence from the right emphasizes that the construction starts from the right and involves right divisors.

Now, the main result says that, through the Φ_n-splitting, the ordering of B_n^+ is just a lexicographical extension of the ordering of B_{n-1}^+, more exactly a `ShortLex`-extension in the sense of [**77**], *i.e.*, the variant of the lexicographical extension in which the length is first taken into account.

PROPOSITION 4.6. *Assume $n \geqslant 3$. Let β, β' belong to B_n^+, and let $(\beta_p, \ldots, \beta_1)$ and $(\beta'_{p'}, \ldots, \beta'_1)$ be their Φ_n-splittings. Then $\beta <^{\Phi} \beta'$ holds if and only if $(\beta_p, \ldots, \beta_1)$ is smaller than $(\beta'_{p'}, \ldots, \beta'_1)$ for the `ShortLex`-extension of $(B_{n-1}^+, <^{\Phi})$, i.e., we have either $p < p'$, or $p = p'$ and there exists $q \leqslant p$ satisfying $\beta_r = \beta'_r$ for $r > q$ and $\beta_q <^{\Phi} \beta'_q$.*

The result appears as Corollary VII.4.6, and it is also a consequence of Corollary VIII.3.3, with a disjoint argument.

The Φ_n-splitting of a positive braid can be computed easily, and a direct outcome of Proposition 4.6 is the existence, already mentioned in Section 1.5, of a quadratic upper bound for the complexity of the σ- and σ^Φ-orderings.

Corollary 4.7. *For each n, the orderings $<^\Phi$ and $<$ of B_n can be recognized in quadratic time.*

Proof. We use induction on $n \geqslant 2$. Let w be an n-strand braid word of length ℓ. By Proposition I.4.6, we can obtain in time $O(\ell)$ two positive n-strand braid words w_1, w_2 such that w is equivalent to $w_1^{-1}w_2$. Then $\overline{w} >^\Phi 1$ is equivalent to $\overline{w_2} >^\Phi \overline{w_1}$. The Φ_n-splittings of the braids $\overline{w_1}$ and $\overline{w_2}$ can be computed in time $O(\ell^2)$; see Chapter VII. The induction hypothesis implies that the comparison of the sequences so obtained can be done in time $O(\ell^2)$ as well. The argument is similar for the σ-ordering as the shift automorphism Φ_n is computable in linear time. $\square$

4.3. The length of $(B_n^+, <^\Phi)$. Contrary to an arbitrary linear ordering, a well-ordering is completely determined up to isomorphism by a unique parameter, namely its length, usually specified by an ordinal number. In the case of the braid ordering on B_n^+, the length easily follows from the recursive characterization of Proposition 4.6.

We recall that ordinals are a transfinite continuation of the sequence of natural numbers: after the natural numbers comes ω, the first infinite ordinal, then $\omega + 1$, $\omega + 2$, etc. For our purposes, it is enough to know that ordinals come equipped with a well-ordering and with arithmetic operations (addition, multiplication, exponentiation) that extend those of $\mathbb{N}$. For more background information about ordinals, we refer to any textbook in set theory, for instance [**138**].

Proposition 4.8. *For each n, the ordered set $(B_n^+, <^\Phi)$ has ordinal type $\omega^{\omega^{n-2}}$.*

In other words, the length of $(B_n^+, <^\Phi)$ is the ordinal $\omega^{\omega^{n-2}}$. The proof is an easy induction on n.

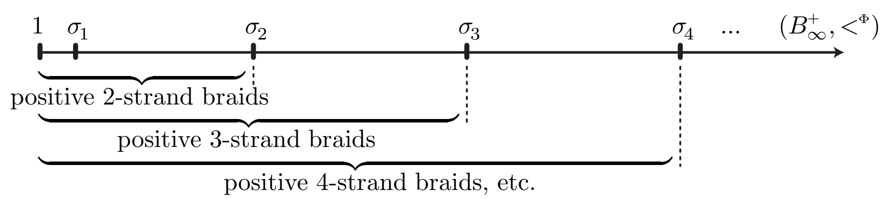

Figure 3. The well-ordered set $(B_\infty^+, <^\Phi)$: an increasing union of end-extensions; for each n, the subset B_n^+ is the initial interval determined by σ_n.

By Proposition 2.10, the ordered set $(B_\infty^+, <^\Phi)$ is the increasing union of the sets $(B_n^+, <^\Phi)$, each set B_n^+ being an initial segment of the next one; see Figure 3. It is easy to deduce

Proposition 4.9. *The ordered set $(B_\infty^+, <^\Phi)$ is a well-ordering with ordinal type ω^{ω^ω}.*

As the flip automorphism Φ_n preserves B_n^+ globally, the results about $(B_n^+, <^\Phi)$ translate into similar results about $(B_n^+, <)$. In particular, Proposition 4.8 implies

COROLLARY 4.10. *For each n, the well-ordering $(B_n^+, <)$ has ordinal type $\omega^{\omega^{n-2}}$.*

However, we have no counterpart of Proposition 4.9 for $<$: the set B_n^+ is not an initial segment of $(B_\infty^+, <)$, and the latter is not a well-ordered set since it contains the infinite descending sequence of (1.1).

4.4. The rank of a positive braid. One of the nice features when an ordering $\prec$ of a set Ω is a well-ordering is that, for $x \in \Omega$, the position of x in $(\Omega, \prec)$ is unambiguously specified by an ordinal number, called the *rank* of x, namely the order type of the initial segment $\{y \in \Omega \mid y \prec x\}$. The rank function establishes an isomorphism between $(\Omega, \prec)$ and an initial segment of the sequence of ordinals: by construction, $x \prec x'$ is true if and only if the rank of x is smaller than the rank of x'.

So, in our current case, every positive braid β in B_n^+ is associated with a well-defined ordinal number, the rank of β, that specifies its position in $(B_n^+, <^\Phi)$. Moreover, Proposition 2.10 (or simply Figure 3) shows that the rank of β in $(B_n^+, <^\Phi)$ coincides with its rank in $(B_\infty^+, <^\Phi)$, and we can forget about the braid index.

Some values of the rank function are easily computed. For instance, the rank of the braid σ_i is the ordinal $\omega^{\omega^{i-2}}$ for $i \geqslant 2$: indeed, it is the ordinal type of the initial interval determined by σ_i. By Proposition 2.10, the latter is B_i, which, by Proposition 4.8, has ordinal type $\omega^{\omega^{i-2}}$. More values can be read in Figure 4.

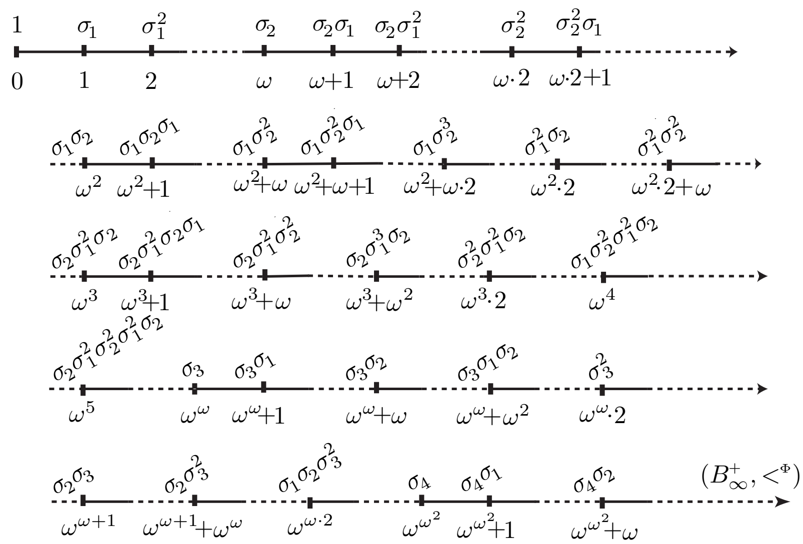

FIGURE 4. Ranks in the well-ordering $(B_\infty^+, <^\Phi)$: the position of each braid is unambiguously specified by an ordinal number that measures the length of the initial interval it determines.

REMARK 4.11. By construction, the rank mapping provides an order-isomorphism between positive braids and ordinals. Except for 2-strand braids, this mapping is *not* an algebraic homomorphism with respect to the ordinal sum: in general, the rank of $\beta_1\beta_2$ is not the sum of the ranks of β_1 and β_2. This happens to be true for $\beta_2 = \sigma_1$, which has rank 1, but, for instance, we can read in Figure 4 that the rank of σ_2 is ω, while that of $\sigma_1\sigma_2$ is ω^2, which is not $1 + \omega$.

Arguably, an optimal description of $(B_\infty^+, <^\Phi)$ would consist of a closed formula explicitly computing, for each positive braid β, the rank of β, *i.e.*, determining the absolute position of β in $(B_\infty^+, <^\Phi)$. An algorithmic method has been described in [**28**], but, so far, it leads to no closed formula in the general case. However, in the case of 3-strand braids, such a formula exists. It relies on identifying distinguished word representatives called Φ-*normal*, from which the rank can be directly read.

Definition 4.12. A nonempty positive 3-strand braid word $\sigma_{[p]}^{e_p} ... \sigma_2^{e_2}\sigma_1^{e_1}$ is said to be Φ-*normal* if the inequalities $e_p \geqslant 1$ and $e_r \geqslant e_r^{\min}$ for $r < p$ are satisfied, where we set $e_1^{\min} = 0$, $e_2^{\min} = 1$, and $e_r^{\min} = 2$ for $r \geqslant 3$, and use $[p]$ to denote 1 for odd p, and 2 for even p.

So the criterion is that a positive 3-strand braid word w is Φ-normal if the successive blocks of letters σ_1 and σ_2 in w, enumerated from the right, and insisting that the rightmost block is a (possibly empty) block of σ_1, have a minimal legal size prescribed by the absolute numbers $e_r^{\min}$. It is easy to check that every nontrivial braid β of B_3^+ is represented by a unique Φ-normal word, naturally called its Φ-*normal form*. Then we have the following explicit formula for the rank.

Proposition 4.13. *For each braid β in B_3^+, the rank of β in $(B_\infty^+, <^\Phi)$ is*

$$\omega^{p-1} \cdot e_p + \sum_{p>r\geqslant 1} \omega^{r-1} \cdot (e_r - e_r^{\min}), \tag{4.2}$$

where $\sigma_{[p]}^{e_p} ... \sigma_2^{e_2}\sigma_1^{e_1}$ is the Φ-normal form of β.

This makes the description of the ordered set $(B_3^+, <^\Phi)$ complete.

Example 4.14. The Φ-normal form of Δ_3 is $\sigma_1\sigma_2\sigma_1$, as the latter word satisfies the defining inequalities, contrary to $\sigma_2\sigma_1\sigma_2$, *i.e.*, $\sigma_2^1\sigma_1^1\sigma_2^1\sigma_1^0$, in which the third exponent from the right, namely 1, is smaller than the minimal legal value $e_3^{\min} = 2$. So, in this case, the sequence $(e_p, ..., e_1)$ is $(1, 1, 1)$, and, applying (4.2), we deduce that the rank of Δ_3 in $(B_3^+, <^\Phi)$ is $\omega^2 \cdot 1 + \omega \cdot (1-1) + 1 \cdot (1-0)$, *i.e.*, $\omega^2 + 1$. The reader can check that, more generally, the flip normal form of Δ_3^d corresponds to the length $d+2$ exponent sequence $(1, 2, ..., 2, 1, d)$, implying that the rank of Δ_3^d is the ordinal $\omega^{d+1} + d$. More values can be read in Figure 4.

4.5. Connection between positive and arbitrary braids. By Proposition I.4.6, every braid is a quotient of two positive braids. It follows that, in theory, the ordering of arbitrary braids is determined by its restriction to positive braids.

Proposition 4.15. *Let $\beta_1, ..., \beta_p$ be a finite family of braids in B_n. Then, for d large enough, $\Delta_n^d\beta_1$, ..., $\Delta_n^d\beta_p$ lie in B_n^+, and the mutual positions of $\beta_1, ..., \beta_p$ in $(B_n, <)$ are the same as the mutual positions of the positive braids $\Delta_n^d\beta_1, ..., \Delta_n^d\beta_p$ in $(B_n^+, <)$.*

The result is clear, as the braid ordering $<$ is left-invariant. A similar result holds for $<^\Phi$.

However, it turns out that this result is of little help in establishing global properties of the braid ordering, and so far there is not much to say about the connection. We just mention two easy remarks involving the left numerators and denominators introduced in Proposition I.4.9 and their right counterpart.

PROPOSITION 4.16. *For each braid β, the right denominator $D_R(\beta)$ (resp. the left denominator $D_L(\beta)$) is the $<$-minimal positive braid β_1 such that $\beta\beta_1$ (resp. $\beta_1\beta$) is positive.*

PROOF. By construction, we have $\beta \cdot D_R(\beta) = N_R(\beta)$ and $D_L(\beta) \cdot \beta = N_L(\beta)$, and both $N_R(\beta)$ and $N_L(\beta)$ are positive braids.

Conversely, assume that β_1 and $\beta\beta_1$ lie in B_∞^+. Then we have $\beta = (\beta\beta_1)\beta_1^{-1}$. By the right counterpart of Proposition I.4.9, we have $\beta_1 = D_R(\beta)\gamma$ for some γ in B_∞^+. Necessarily γ is trivial or σ-positive, and, therefore, we have both $\beta_1 \geqslant D_R(\beta)$ and $\beta_1 \geqslant^\Phi D_R(\beta)$.

Symmetrically, assume that β_1 and $\beta_1\beta$ lie in B_∞^+. Then we have $\beta = \beta_1^{-1}(\beta_1\beta)$. By Proposition I.4.9, there exists γ in B_∞^+ satisfying $\beta_1 = \gamma D_L(\beta)$. As γ belongs to B_∞^+, Property **S** implies both $\beta_1 \geqslant D_L(\beta)$ and $\beta_1 \geqslant^\Phi D_L(\beta)$. □

PROPOSITION 4.17. *For each braid β, the relations $\beta > 1$ and $N_L(\beta) > D_L(\beta)$ are equivalent. Similarly, $\beta >^\Phi 1$ and $N_L(\beta) >^\Phi D_L(\beta)$ are equivalent.*

The verification is straightforward as $<$ and $<^\Phi$ are left-invariant. Note that no such relation exists with the right numerators and denominators: for instance, for $\beta = \sigma_2^{-1}\sigma_1$, we have $\beta > 1$, but $N_R(\beta) = \sigma_1\sigma_2 < D_R(\beta) = \sigma_2\sigma_1$.

The previous observations are rather trivial and do not shed much light on the structure of $(B_n, <)$. The point is that the fractionary decompositions defines two injections ι_L and ι_R of B_n into a subset of $B_n^+ \times B_n^+$, but neither of them preserves the ordered structure. On the other hand, we can easily define a well-ordering on $B_n^+ \times B_n^+$ by using a lexicographical extension of the ordering of B_n^+, and, appealing to ι_L or ι_R, deduce a well-ordering of B_n, but the latter will not be invariant under left (or right) multiplication.