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CHAPTER I

Braid Groups

In this introductory chapter, we briefly explain how the groups B_n arise in several contexts of geometry and algebra, and we mention a few basic results that will be frequently used in the sequel. Our purpose here is not to be exhaustive, and many possible approaches are not mentioned—for instance the connection with configuration spaces. All results in this chapter are classical, and we refer to textbooks for most of the proofs; see for instance [**123**], [**118**], [**14**], or [**176**].

The organization is as follows. In Section 1, we start with the Artin presentation of the group B_n in terms of generators and relations. In Section 2, we describe the connection with the geometric viewpoint of isotopy classes of families of intertwining strands. In Section 3, we address the braid group as the mapping class group of a punctured disk. Finally, in Section 4, we introduce the monoid of positive braids and mention some basic results from Garside's theory.

1. The Artin presentation

Here, we introduce the braid group B_n using the abstract presentation already mentioned in the Introduction, due to E. Artin [**4**].

1.1. Braid relations. Braid groups can be specified using a standard presentation.

Definition 1.1. For $n \geqslant 2$, the *n-strand braid group* B_n is defined to be the presented group

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{lll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for} & |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for} & |i-j| = 1 \end{array} \right\rangle. \tag{1.1}$$

The elements of B_n are called *n-strand braids*. The braid group on infinitely many strands, denoted B_∞, is defined by a presentation with infinitely many generators $\sigma_1, \sigma_2, \dots$ subject to the same relations.

Clearly, the identity mapping on $\{\sigma_1, \dots, \sigma_{n-1}\}$ extends into a homomorphism of B_n to B_{n+1}. It can be proved easily—and it will be clear from the geometric interpretation of Section 2—that this homomorphism is injective, and, therefore, we can identify B_n with the subgroup of B_∞ generated by $\sigma_1, \dots, \sigma_{n-1}$. This is the point of view we shall always adopt in the sequel.

1.2. Braid words. According to Definition 1.1, every braid admits decompositions in terms of the generators σ_i and their inverses. A word on the letters $\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}$ is called an *n-strand braid word*. The *length* of a braid word w is denoted by $\ell(w)$. If the braid β is the equivalence class of the braid word w, we say that *w represents* β, or is an *expression* of β, and we write $\beta = \overline{w}$. We say that two braid words are *equivalent* if they represent the same braid, *i.e.*, if they are equivalent with respect to the least congruence that contains the relations of (1.1). As, for instance, the braid word $\sigma_1^k\sigma_1^{-k}$ represents the unit braid for every k, each braid admits infinitely many representative braid words.

2. Isotopy classes of braid diagrams

We now connect the abstract point of view of Section 1 with the concrete intuition of braids as strands that are intertwined.

2.1. Geometric braids. We denote by D^2 the unit disk with centre 0 in the plane $\mathbb{R}^2$ identified with the complex line $\mathbb{C}$, and by D_n the disk D^2 with n regularly spaced points in the real axis as distinguished points; we call these points the *puncture points* of D^2.

Definition 2.1. We define an *n-strand geometric braid* to be an embedding b of the disjoint union $\coprod_{j=1}^n [0_j, 1_j]$ of n copies of the interval $[0,1]$ into the cylinder $[0,1] \times D^2$ satisfying the following properties:

• for t in $[0_j, 1_j]$, the point $b(t)$ lies in $\{t\} \times D^2$;

• the set $\{b(0_1), \ldots, b(0_n)\}$ is the set of punctures of $\{0\} \times D^2$, and similarly the set $\{b(1_1), \ldots, b(1_n)\}$ is the set of punctures of $\{1\} \times D^2$.

The image of each interval is called a *strand* of the braid. The idea is that, visualizing the unit interval as being horizontal, we have n strands running continuously from left to right, intertwining, but not meeting each other.

Each geometric braid b determines a permutation π in the symmetric group $\mathfrak{S}_n$ as follows. Label the punctures $P_1, \ldots, P_n$. Then we take $\pi(i) = j$ if the strand of b that ends at $P_i \times \{1\}$ begins at $P_j \times \{0\}$.

A geometric braid whose permutation is trivial is called *pure*.

2.2. The group of isotopy classes. To obtain the connection with B_n, and in particular to be able to obtain a group structure, we appeal to isotopies for identifying geometric braids that are topologically equivalent.

Definition 2.2. Two geometric braids b, b' are said to be *isotopic* if there is a continuous $[0,1]$-family of geometric braids b_t with $b_0 = b$ and $b_1 = b'$.

The geometric idea is that we can deform one braid into the other while holding their endpoints fixed. Note that isotopic braids induce the same permutation.

There exists a natural way of defining a product of two geometric braids using concatenation: Given two geometric braids b_1 and b_2, we squeeze the image of b_1 into the cylinder $[0, \frac{1}{2}] \times D^2$, the image of b_2 into $[\frac{1}{2}, 1] \times D^2$, and obtain a new, well-defined geometric braid $b_1 \cdot b_2$. Clearly, this product is compatible with isotopy, hence it induces a well-defined operation on the set of isotopy classes of geometric braids.

Lemma 2.3. *For each n, the set of isotopy classes of n-strand geometric braids equipped with the above product is a group.*

Proof (Sketch). The neutral element of the group is the isotopy class of the trivial geometric braid, whose strands are just straight line segments not intertwining each other. The inverse of the class of a geometric braid is the class of its reflection in the disk $\{\frac{1}{2}\} \times D^2$. $\square$

We shall see in a moment that this group is isomorphic to the group B_n of Section 1.

Note that the function that to every braid associates the permutation it induces on the set of punctures yields a homomorphism of the group of isotopy classes of n-strand geometric braids to the symmetric group $\mathfrak{S}_n$. Our definition of the permutation associated with a braid was made precisely to obtain a homomorphism, and not an anti-homomorphism. Here we regard permutations as acting on the left, consistent with the convention for mapping classes, to be discussed in Section 3.

PROPOSITION 2.4. *The group of isotopy classes of n-strand geometric braids is isomorphic to the group B_n.*

PROOF (SKETCH). We define a homomorphism from B_n to our group of isotopy classes by sending the generator σ_i to the class of the clockwise half-twist braid involving the ith and $(i+1)$st strand indicated in Figure 1—in this picture, the cylinder has not been drawn, for simplicity, and our picture represents a side-view of the braid.

The figure also illustrates the fact that this homomorphism is well defined. Indeed, in our group of isotopy classes, we have that crossings which are far apart commute—so that we have $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geqslant 2$—and the Reidemeister III-type relation $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ holds.

We leave it to the reader to verify that this homomorphism is surjective—any geometric braid can be deformed into one in which a side-view offers only finitely many crossings, all of which are transverse—and injective—our two types of relations suffice to relate any two braid diagrams representing isotopic geometric braids. The proofs, which we shall not discuss here, are similar to the proof that isotopy classes of knots are the same as knot diagrams up to Reidemeister-equivalence [**179**]. Details can be found in [**123**]. □

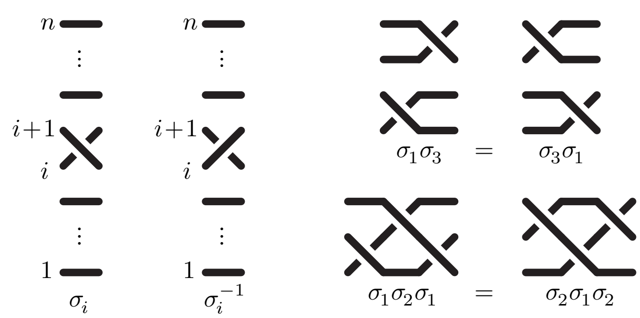

FIGURE 1. The Artin generators σ_i and σ_i^{-1} of the braid group B_n and their relations, when realized as geometric braids.

This completes the definition of B_n in terms of n-strand geometric braids. From now on, we shall no longer distinguish between B_n and the group of isotopy classes of n-strand geometric braids.

We remark that the above definition has a very natural generalization: We can replace the disk D^2 by any compact surface $\mathcal{S}$, possibly with boundary. Choosing n puncture points in $\mathcal{S}$, we can define the *n-strand braid group of the surface $\mathcal{S}$* to be the group of n-strand braids in $\mathcal{S} \times [0,1]$; see Section 3.2 of Chapter XVI.

3. Mapping class groups

We shall now identify the braid group B_n with the group of homotopy classes of self-homeomorphisms of an n-punctured disk. The idea is simply to look at braids from one end rather than from the side.

3.1. Homeomorphisms of a surface. Let $\mathcal{S}$ be an oriented compact surface, possibly with boundary, and let $\mathcal{P}$ be a finite set of distinguished interior points of $\mathcal{S}$. The most important example in this section will be the n-punctured disk D_n.

DEFINITION 3.1. The *mapping class group* $\mathcal{MCG}(\mathcal{S}, \mathcal{P})$ of the surface $\mathcal{S}$ relative to $\mathcal{P}$ is the group of all isotopy classes of orientation-preserving self-homeomorphisms of $\mathcal{S}$ that fix $\partial\mathcal{S}$ pointwise and preserve $\mathcal{P}$ globally.

This means that any homeomorphism φ from $\mathcal{S}$ to itself and taking punctures to punctures represents an element of the mapping class group, provided it acts as the identity on the boundary of $\mathcal{S}$. Note that the punctures may be permuted by φ. Two homeomorphisms φ, ψ represent the same element if and only if they are isotopic through a family of boundary-fixing homeomorphisms which also fix $\mathcal{P}$. They will then induce the same permutation of the punctures.

CONVENTION 3.2. In the sequel, the product we consider on a mapping class group $\mathcal{MCG}(\mathcal{S}, \mathcal{P})$ is composition: $\varphi\psi$ simply means “first apply ψ, then φ”.

We remark that, in the previous paragraph, the word “isotopic” could have been replaced by the word “homotopic”: by a theorem of Epstein [**76**], two homeomorphisms of a compact surface are homotopic if and only if they are isotopic.

Mapping class groups, also known as modular groups, play a prominent role in the study of the topology and geometry of surfaces, as well as in 3-dimensional topology. To illustrate the difficulty of understanding them, we note that simply proving that they admit finite—and in fact quite elegant—presentations already requires deep arguments [**105, 193**].

3.2. Connection with geometric braids. Our aim now is to sketch a proof of the following result; for details see, for instance, [**14**] or [**123**]:

PROPOSITION 3.3. *There is an isomorphism of B_n with $\mathcal{MCG}(D_n)$.*

PROOF (SKETCH). We outline a proof that $\mathcal{MCG}(D_n)$ is naturally isomorphic to the group of isotopy classes of geometric braids defined above. Let b be an n-strand geometric braid, sitting in the cylinder $[0, 1] \times D^2$, whose n strands are starting at the puncture points of $\{0\} \times D_n$ and ending at the puncture points of $\{1\} \times D_n$. Then b may be considered to be the graph of the motion, as time goes from 1 to 0, of n points moving in the disk, starting and ending at the puncture points—according to Convention 3.2, letting time go from 0 to 1 would lead to an anti-isomorphism. It can be proved that this motion extends to a continuous family of homeomorphisms of the disk, starting with the identity and fixed on the boundary at all times. The end map of this isotopy is the corresponding homeomorphism $\varphi\colon D_n \to D_n$, which is well defined up to isotopy fixed on the punctures and the boundary.

Conversely, given a homeomorphism $\varphi\colon D_n \to D_n$, representing some element of the mapping class group, we want to get an n-strand geometric braid. By a well-known trick of Alexander, every homeomorphism of a disk that fixes the boundary

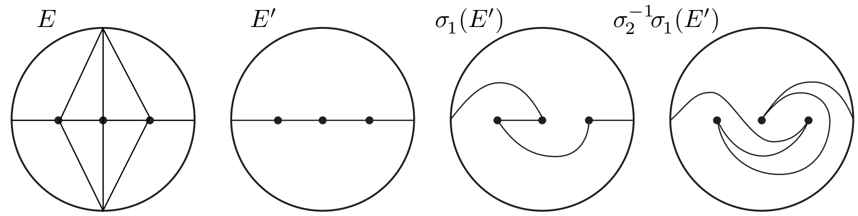

FIGURE 2. Two possible curve diagrams on D_n, and the image of one of them under the homeomorphisms σ_1 and $\sigma_2^{-1}\sigma_1$.

is isotopic to the identity, through homeomorphisms fixing the boundary. The corresponding braid is then the graph of the restriction of such an isotopy to the puncture points. Again, we must regard the isotopy parameter as going from 1 to 0 here. □

3.3. Curve diagrams. In order to visualize a mapping of a surface, it is useful to consider images of certain subsets of the surface. Let E be a diagram on the surface $\mathcal{S}$, consisting of a finite number of disjoint, properly embedded arcs, meaning the arcs terminate either on $\partial\mathcal{S}$ or in a puncture of $\mathcal{S}$. Suppose in addition that E fills $\mathcal{S}$, in the sense that the interiors of all components of the surface obtained by cutting $\mathcal{S}$ along the arcs of E are homeomorphic to open disks. Typical examples in the case $\mathcal{S} = D_n$ are the standard triangulation, as well as the collection of $n+1$ horizontal line segments indicated in Figure 2. Then the isotopy class of a homeomorphism $\varphi\colon \mathcal{S} \to \mathcal{S}$ is uniquely determined by the isotopy class of the diagram $\varphi(E)$. This fact is also illustrated in Figure 2.

It is also well known that a homeomorphism of D_n can be recovered up to homotopy from the induced isomorphism of the fundamental group $\pi_1(D_n, *)$, where $*$ is a fixed point of the boundary ∂D_n. The group $\pi_1(D_n, *)$ is a free group on n generators, say F_n. So we obtain an embedding $B_n \cong \mathcal{MCG}(D_n) \to \mathrm{Aut}(F_n)$, which can be written explicitly if we choose a base point $*$ and generators of $\pi_1(D_n, *)$. Two choices will play an important role in this text, namely when the base point $*$ is the leftmost point of the disk, and the generators are the loops $x_1, \dots, x_n$ in the first case, and $y_1, \dots, y_n$ in the second case, as shown in Figure 3.

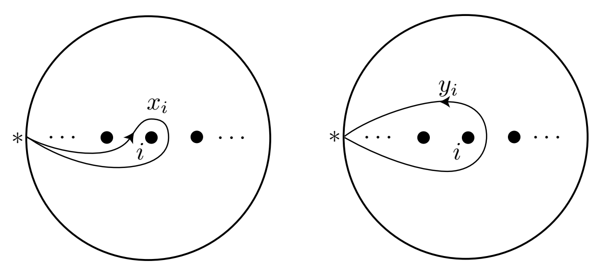

FIGURE 3. Two choices for the generators for the fundamental group of D_n.

4. Positive braids

A useful perspective is to restrict attention to a special class of braids, namely *positive* braids. This approach turns out to be extremely fruitful because the braid monoids that appear in this way turn out to have a very rich theory, based on Garside's seminal work of [**95**].

4.1. Braid monoids. As the relations of (1.1) involve no negative letter σ_i^{-1}, they also define a monoid. So, as we did in Section 1, we can introduce the following abstract definition:

DEFINITION 4.1. The *positive braid monoid* B_n^+ is defined to be the monoid that admits, as a monoid, the presentation (1.1). The elements of B_n^+ are called *positive braids*.

So the elements of B_n^+ are represented by words in the letters σ_i, but not σ_i^{-1}. Such words are called *positive*. As the relations of (1.1) preserve the word length, all positive braid words representing a given positive braid β have the same length, denoted $\ell(\beta)$—of course, the same property does not hold in the full braid group, as, there, the length two word $\sigma_1\sigma_1^{-1}$ is equivalent to the length zero empty word. In particular because of the previous seemingly trivial observation, the positive braid monoid is often easier to handle than the full braid group. Note that the length function is a morphism of the *monoid* B_n^+ to the monoid $(\mathbb{N}, +)$.

Geometrically, we may think of B_n^+ as the monoid of geometric braids with only positive crossings in their diagram, up to positive isotopy, where the isotopies are deformations through a family of braids that are again positive in the same sense.

The reason why this point of view is so fruitful is the following fundamental result due to Garside; for a proof, see, for instance, [**52**] or [**123**], or see [**95**] itself:

PROPOSITION 4.2. *The canonical mapping of B_n^+ to B_n is injective.*

Equivalently, if two positive geometric braids are isotopic, then they are isotopic through a family of positive braids. So the braid monoid embeds in the braid group, and it identifies with the subset of B_n consisting of braids which are representable by words in the letters σ_i, not using any σ_i^{-1}.

4.2. The braids δ_n and Δ_n. For each n, two particular positive n-strand braids play a fundamental role in the study of B_n^+ and, more generally, of B_n, namely the so-called *fundamental* braids δ_n and Δ_n. We also introduce the associated conjugacy automorphisms.

DEFINITION 4.3 (Figure 4). We define $\delta_1 = \Delta_1 = 1$ and, for $n \geqslant 2$,

$$\delta_n = \sigma_1\sigma_2 \ldots \sigma_{n-1} \quad \text{and} \quad \Delta_n = \delta_n\delta_{n-1} \ldots \delta_2. \tag{4.1}$$

We define the *cycling* automorphism ϕ_n to be the conjugation by δ_n, and the *flip* automorphism Φ_n to be the conjugation by Δ_n, *i.e.*, for β in B_n,

$$\phi_n : \beta \mapsto \delta_n \, \beta \, \delta_n^{-1} \quad \text{and} \quad \Phi_n : \beta \mapsto \Delta_n \, \beta \, \Delta_n^{-1}. \tag{4.2}$$

The following formulas can be read from Figure 4.

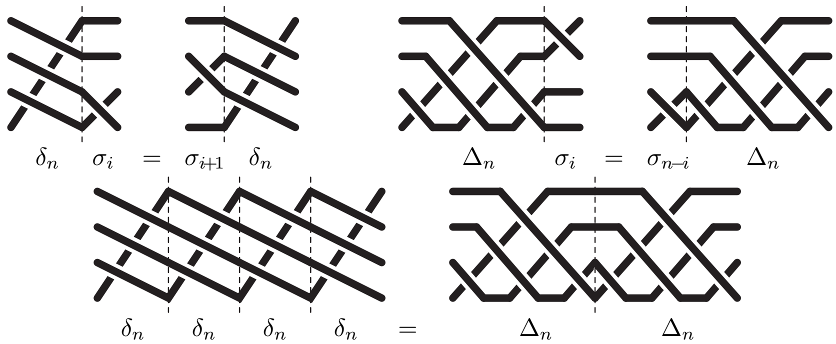

FIGURE 4. The braids δ_n, Δ_n, and their relations: δ_n corresponds to shifting the strands, while Δ_n corresponds to a global half-twist of the n strands.

LEMMA 4.4. *For each n, we have*

$$\delta_n^n = \Delta_n^2 \quad \textit{and} \quad \phi_n^n = \Phi_n^2 = \mathrm{id}_{B_n}, \tag{4.3}$$

$$\phi_n(\sigma_i) = \sigma_{i+1} \quad \textit{for } 1 \leqslant i \leqslant n-2, \tag{4.4}$$

$$\Phi_n(\sigma_i) = \sigma_{n-i} \quad \textit{for } 1 \leqslant i \leqslant n-1. \tag{4.5}$$

Formula (4.4) implies that the generators σ_i of B_n all are pairwise conjugate, whereas (4.5) explains the name “flip automorphism” for Φ_n: in a braid diagram, Φ_n corresponds to a horizontal symmetry. Its being involutive means that the braid Δ_n^2 belongs to the centre of B_n. Actually, it is known [**36**] that, for $n \geqslant 3$, the centre of B_n consists of the powers of Δ_n^2 only. As for ϕ_n, formula (4.3) shows that it has order n, and it should rather be seen as a rotation by $2\pi/n$; see Chapter VIII for details.

4.3. Fractionary decompositions (I). The study of B_n^+ is fundamental for the understanding of B_n, because B_n is a group of fractions for B_n^+, *i.e.*, every braid is a quotient of two positive braids. Here we consider particular decompositions in which the denominator is a power of the braid Δ_n.

First, using an induction on n and playing with the braid relations, one easily obtains:

LEMMA 4.5. *For each i between 1 and $n-1$, the braid Δ_n can be expressed by a positive word that begins with σ_i.*

Then we deduce:

PROPOSITION 4.6. *Every braid β of B_n admits a decomposition $\beta = \Delta_n^{-2p}\beta'$ with $p \geqslant 0$ and $\beta' \in B_n^+$.*

PROOF. By Lemma 4.5, for each i in $\{1, \dots, n-1\}$, we can choose a positive braid word w_i such that $\sigma_i w_i$ represents Δ_n^2, *i.e.*, w_i represents $\sigma_i^{-1}\Delta_n^2$.

Let β be an arbitrary braid in B_n, and let w be an n-strand braid word representing β. Let p be the number of negative letters (letters σ_i^{-1}) in w. Let w' be the positive braid word obtained from w by replacing each letter σ_i^{-1} with w_i.

As w_i represents $\sigma_i^{-1}\Delta_n^2$ and Δ_n^2 commutes with every braid, the word w' represents $\Delta_n^{2p}\beta$, and we deduce $\beta = \Delta_n^{-2p}\beta'$, where β' is the positive braid represented by w'. □

A priori, the fractionary decomposition provided by Proposition 4.6 is not unique. Actually, we can obtain uniqueness by demanding that the exponent of Δ_n is minimal: every braid β in B_n admits a *unique* decomposition $\beta = \Delta_n^{-d}\beta'$ with $d \in \mathbb{Z}$, $\beta' \in B_n^+$ and d minimal such that such a decomposition exists.

4.4. The lattice of divisibility. In order to go further, in particular both to establish Proposition 4.2 and to improve Proposition 4.6, one needs to know more about the monoid B_n^+ and its divisibility relations, as described by Garside's theory.

DEFINITION 4.7. For β, β' in B_n, we say that β' is a *left divisor* of β, denoted $\beta' \preccurlyeq \beta$, if $\beta = \beta'\gamma$ holds for some γ in B_n^+.

Symmetrically, we say that β' is a *right* divisor of β if we have $\beta = \gamma\beta'$ for some γ in B_n^+, but we shall not introduce specific notation.

It can be noted that, for β, β' in B_n, every positive braid γ possibly satisfying $\beta = \beta'\gamma$ has to lie in B_n^+ and, therefore, the divisibility relation in B_n is the restriction of that of B_∞: there is no need to worry about the index n.

In the monoid B_n^+, no element except 1 is invertible, so the divisibility relation $\preccurlyeq$ is a partial ordering. The main result of Garside's theory [**95**] is that, for each n, the poset $(B_n^+, \preccurlyeq)$ is a lattice:

PROPOSITION 4.8 ([**77**, Chapter 9]). *Any two positive braids admit a greatest common left divisor (gcd) and a least common right multiple (lcm).*

Using Proposition 4.6, one can deduce that the poset $(B_n, \preccurlyeq)$ is also a lattice.

4.5. Fractionary decompositions (II). Proposition 4.6 gives a specific role to the powers of the braid Δ_n, and the fractionary decomposition it gives is, in general, not irreducible, in that the numerator and the denominator may have non-trivial common divisors. Garside's results enable us to improve the result. The next refinement says that, among all fractionary decompositions of a braid, there is a distinguished one through which every fractionary decomposition factorizes.

PROPOSITION 4.9. *For each braid β in B_n, there exists a unique pair of positive braids (β_1, β_2) such that $\beta = \beta_1^{-1}\beta_2$ holds and 1 is the only common left divisor of β_1 and β_2 in B_n^+. Then, for each decomposition $\beta = \gamma_1^{-1}\gamma_2$ with γ_1, γ_2 in B_n^+, we have $\gamma_1 = \gamma\beta_1$ and $\gamma_2 = \gamma\beta_2$ for some γ in B_n^+.*

PROOF. Let β belong to B_n, and let $\beta = \beta_1^{-1}\beta_2$ be a fractionary decomposition of β, *i.e.*, we assume $\beta_1, \beta_2 \in B_n^+$, such that $\ell(\beta_1) + \ell(\beta_2)$ is minimal. As the length function takes its values in $\mathbb{N}$, such a pair must exist. First, we observe that 1 is the only common left divisor of β_1 and β_2 since, if we have $\beta_1 = \gamma\beta_1'$ and $\beta_2 = \gamma\beta_2'$, then ${\beta_1'}^{-1}\beta_2'$ is another decomposition of β satisfying $\ell(\beta_1')+\ell(\beta_2') = \ell(\beta_1)+\ell(\beta_2)-2\ell(\gamma)$, hence $\gamma = 1$ by the choice of β_1 and β_2.

Assume that $\beta = \gamma_1^{-1}\gamma_2$ is any fractionary decomposition of β with $\gamma_1, \gamma_2 \in B_n^+$. Using the existence of gcd's and lcm's in B_n^+ (Proposition 4.8), we shall prove that $\gamma_1^{-1}\gamma_2$ factors through $\beta_1^{-1}\beta_2$, *i.e.*, that $\gamma_1 = \gamma\beta_1$ and $\gamma_2 = \gamma\beta_2$ holds for some γ in B_n^+. The argument is illustrated on Figure 5. First, by Proposition 4.6, we can find β_0, γ_0 in B_n^+ satisfying $\gamma_0^{-1}\beta_0 = \beta_1\gamma_1^{-1}$. We deduce $\gamma_0\beta_1 = \beta_0\gamma_1$

and $\gamma_0\beta_2 = \beta_0\gamma_2$ in B_n, hence in B_n^+ by Proposition 4.2. Let γ_0' be the least common right multiple of β_0 and γ_0. By construction, we have $\beta_0 \preccurlyeq \gamma_0\beta_1$ and $\gamma_0 \preccurlyeq \gamma_0\beta_1 = \beta_0\gamma_1$, hence $\gamma_0' \preccurlyeq \gamma_0\beta_1$. Similarly, we have $\beta_0 \preccurlyeq \gamma_0\beta_2 = \beta_0\gamma_2$ and $\gamma_0 \preccurlyeq \gamma_0\beta_2$, hence $\gamma_0' \preccurlyeq \gamma_0\beta_2$. We deduce that γ_0' left-divides the left gcd of $\gamma_0\beta_1$ and $\gamma_0\beta_2$. By hypothesis, β_1 and β_2 have no nontrivial common left divisor, so the left gcd of $\gamma_0\beta_1$ and $\gamma_0\beta_2$ is γ_0, and we must have $\gamma_0' \preccurlyeq \gamma_0$, hence $\gamma_0' = \gamma_0$ since, by definition, γ_0' is a right multiple of γ_0. Hence, we have $\gamma_0 = \beta_0\gamma$ for some γ in B_n^+. As we have $\gamma_0\beta_1 = \beta_0\gamma_1$ and $\gamma_0\beta_2 = \beta_0\gamma_2$, the equality $\gamma_0 = \beta_0\gamma$ implies $\gamma_1 = \gamma\beta_1$ and, similarly, $\gamma_2 = \gamma\beta_2$. We obtained the expected factorization result.

Finally, the pair (β_1, β_2) is unique. Indeed, if $\gamma_1^{-1}\gamma_2$ is any fractionary decomposition of β satisfying $\ell(\gamma_1) + \ell(\gamma_2) = \ell(\beta_1) + \ell(\beta_2)$, the above result gives gives a braid γ satisfying $\gamma_1 = \gamma\beta_1$ and $\gamma_2 = \gamma\beta_2$ and the hypothesis on the length implies $\gamma = 1$, hence $\gamma_1 = \beta_1$ and $\gamma_2 = \beta_2$. □

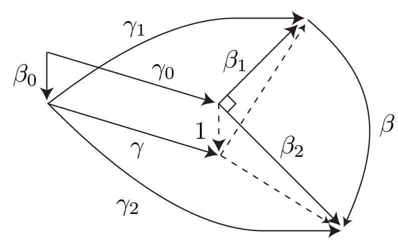

FIGURE 5. Decomposition of a braid into an irreducible fraction: the hypothesis that β_1 and β_2 have no common left divisor implies that every other fractionary decomposition $\gamma_1^{-1}\gamma_2$ factors through $\beta_1^{-1}\beta_2$.

Thus, every braid has a distinguished minimal expression as an irreducible fraction. The situation with B_n and B_n^+ is therefore similar to that of positive rational numbers and their unique expression as an irreducible fraction.

DEFINITION 4.10. For each braid β, the unique braids β_1 and β_2 associated with β by Proposition 4.9 are called the *left denominator* and the *left numerator* of β, and respectively denoted $D_L(\beta)$ and $N_L(\beta)$.

Of course, as B_n is not commutative for $n \geqslant 3$, the position of the denominator matters. Owing to the symmetry of the braid relations, each braid admits a similar irreducible decomposition $N_R(\beta)D_R(\beta)^{-1}$ in which the denominator lies on the right.