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A map $f\colon X\times X\to X\times X$ is called a *Yang–Baxter map* on a set X if it satisfies the relation

$$(f\times \mathrm{id}_X)\circ(\mathrm{id}_X\times f)\circ(f\times \mathrm{id}_X)=(\mathrm{id}_X\times f)\circ(f\times \mathrm{id}_X)\circ(\mathrm{id}_X\times f). \tag{1}$$

The study of these maps was proposed in [1], and we have borrowed the term "Yang–Baxter map" from [2]. Various approaches to the construction of Yang–Baxter maps and several examples can be found in [2]–[4].

Proposition 1. *The map $f\colon \mathbb{R}^4\to\mathbb{R}^4$ given by $f(a,b,c,d)=(a',b',c',d')$, where*

$$\begin{aligned} a'&=\max(a,a+b,b+c), & b'&=d-\max(0,a-b-c+\max(0,b)+\max(0,d)),\\ c'&=a+c+d-\max(a,c,a+d), & d'&=\max(b,a-c+\max(0,b)+\max(0,d)), \end{aligned} \tag{2}$$

is a non-degenerate (*that is*, *bijective*) *Yang–Baxter map on $\mathbb{R}^2$. The inverse map is given by the formula $f^{-1}=\tau\circ f\circ\tau$, where $\tau(a,b,c,d)=(-a,b,-c,d)$.*

Proof. By explicit substitution it is easy to verify that the map

$$f^r(a,b,c,d)=\left(a+ab+bc,\frac{bcd}{bc+a(1+b)(1+d)},\frac{acd}{a+c+ad},\frac{bc+a(1+b)(1+d)}{c}\right) \tag{3}$$

obtained from f by replacing the operators $+$, $-$, max by $\cdot$, $/$, $+$, respectively, is a Yang–Baxter map on $\mathbb{R}^2_+$. In this calculation we use only the commutative and distributive properties of the operators $\cdot$, $/$, $+$, which are also possessed by $+$, $-$, max.

The maps f and f^r define actions ρ and ρ^r of the braid group B_n on $\mathbb{R}^{2n}$ and $\mathbb{R}^{2n}_+$, respectively, in the standard way. Let us clarify the geometric meaning of the actions ρ and ρ^r.

The braid group B_n can be embedded in the group $MCG^{0,n+3}$ of diffeomorphism classes of the sphere S^2 with $n+3$ distinguished points $P_0,P_1,\dots,P_{n+2}$. Here a Dehn semitwist around the points P_i, P_{i+1} corresponds to the generator σ_i. The group $MCG^{0,n+3}$ acts in the standard way on the space $L^{0,n+3}$ of laminations and on the Teichmüller space $T^{0,n+3}$ of the surface $S^2-\{P_0,\dots,P_{n+2}\}$. The action ρ (ρ^r) is just the description in a special system of coordinates of the restriction to the subgroup B_n of the standard action of the group $MCG^{0,n+3}$ on the space $L^{0,n+3}$ (respectively, $T^{0,n+3}$).

We realize the sphere S^2 as $\mathbb{R}^2\cup\{\infty\}$, and put $P_k=(k,0)$ for $k=0,\dots,n+1$, $P_{n+2}=\infty$, $e_0=\{(x,0)\mid x<0\}$, $e_{3k+1}=\{(k+1/2,y)\}$ for $k=0,\dots,n$, $e_{3k-1}=\{(k,y)\mid y>0\}$, $e_{3k}=\{(k,y)\mid y<0\}$ for $k=1,\dots,n$, $e_{3n+2}=\{(x,0)\mid x>n+1\}$. The vertices P_k and the edges e_k define a singular triangulation of S^2.

The action of the group B_n on $T^{0,n+3}$ coincides with ρ^r if as coordinates $(a_1,b_1,\dots,a_n,b_n)$ on $T^{0,n+3}$ we take $a_k=\lambda_{3k-1}/\lambda_{3k}$, $b_k=\lambda_{3k-2}/\lambda_{3k+1}$, where λ_i is the λ-length of the edge e_i defined in [5].

To explain the geometric meaning of the action ρ more simply, we shall restrict ourselves to the integer points $\mathbb{Z}^{2n}\subset\mathbb{R}^{2n}$. By an *integer lamination* on $S^2-\{P_0,\dots,P_{n+2}\}$ we mean any isotopy class of a union of non-intersecting simple closed curves $S^2-\{P_0,\dots,P_{n+2}\}$, none of which is homotopic to zero nor bounds a disc containing exactly one of the point P_k. For an integer lamination l we shall denote by $\mu_k(l)$ the smallest possible number of points of intersection of a

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representative of l with the edge e_k. On the set $L_{\mathbb{Z}}^{0,n+3}$ of integer laminations we introduce the following collection of functions: $a_i(l) = (\mu_{3i-1}(l) - \mu_{3i}(l))/2$, $b_i(l) = (\mu_{3i-2}(l) - \mu_{3i+1}(l))/2$.

Lemma 1. *The collection of functions* $(a_1, b_1, \dots, a_n, b_n)$ *defines a one-to-one map* $L_{\mathbb{Z}}^{0,n+3} \to \mathbb{Z}^{2n}$. *Under this identification the action of the group* B_n *on* $L_{\mathbb{Z}}^{0,n+3}$ *coincides with the action* ρ.

The proof differs only in details from the proof of the analogous theorem in [6].

Recall that a braid $b \in B_n$ is called σ-positive (see [7]) if for some k, $0 < k < n$, it can be represented by a word in the standard generators σ_i that does not contain $\sigma_1, \dots, \sigma_{k-1}$ and contains σ_k only to a positive power. A braid that is the inverse of a σ-positive braid is called σ-negative. It was proved in [7] that every non-trivial braid is either σ-positive or σ-negative. Thus in the group B_n we may define a right-invariant ordering by putting $b_1 > b_2$ if $b_1 b_2^{-1}$ is σ-positive. An algorithm was proposed in [7] for determining whether a given braid was σ-positive, σ-negative, or trivial. Although this algorithm works fast in practice, a polynomial bound for its speed has not yet been proved.

We consider a partial order $<_{\mathrm{oddlex}}$ in $\mathbb{Z}^{2n}$ defined as follows: $x <_{\mathrm{oddlex}} y$ if for some $j \leqslant n$ we have $x_{2j-1} < y_{2j-1}$ and $x_{2i-1} = y_{2i-1}$ for all $0 < i < j$. Let L_0 denote the vector $(0, 1, 0, 1, \dots, 0, 1) \in \mathbb{Z}^{2n}$.

Proposition 2. *A braid* $b \in B_n$ *is* σ*-positive* (σ*-negative*) *if and only if* $L_0 <_{\mathrm{oddlex}} b \cdot L_0$ (*respectively*, $b \cdot L_0 <_{\mathrm{oddlex}} L_0$).

This assertion gives a simple algorithm for determining which of the inequalities $b > 1$ or $b < 1$ holds for a given braid b.[1] To estimate its speed we introduce the following definition.

Suppose the braid b is written in the form $b = \Delta_{i_1 j_1}^{p_1} \dots \Delta_{i_m j_m}^{p_m}$, where Δ_{ij}^{p} is an element of Garside type: $\Delta_{ij}^{p} = (\sigma_i \sigma_{i+1} \dots \sigma_{j-1})(\sigma_i \sigma_{i+1} \dots \sigma_{j-2}) \dots \sigma_i$. We define the Δ-*length* of the word b as the quantity $|b|_\Delta = \sum_{s=1}^{m} (1 + \ln(j_s - i_s) + \ln|p_s|)$.

We note that for any word b defining a braid we have the bound $|b|_\Delta \leqslant |b|$. We shall use the following norm in the space $\mathbb{Z}^{2n}$: $\|x\| = \max_{i=1}^{n} |x_i|$.

Proposition 3. *Any braid* $b \in B_n$ *satisfies the bound* $\ln \|b \cdot L_0\| \leqslant 2|b|_\Delta$. *Hence the element* $b \cdot L_0$ *can be computed in* $\mathrm{const} \cdot |b|_\Delta \cdot |b|$ *operations.*

We note that a bound for the time needed to compare b with 1 in the σ-ordering depends only on the length of the word b and not on the number of strings. Until recently the best time bound among algorithms for the solution of the word problem in B_n was that for an algorithm in [9], where the bound is linear in n. An algorithm similar to ours recently appeared independently in [10], where, however, it was not noted that the action on laminations is related to a certain Yang–Baxter map.

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[1] Originally the author had only noticed that the action ρ gave a fast algorithm for recognising a trivial braid. The question of the connection between this action and σ-ordering was posed by Dehornoy, and the answer conjectured by S. Orevkov, who referred the author to [8], where similar geometric ideas are used.

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