Exercise sheet: Review of Probability

The following exercises have different levels of difficulty indicated by (*), (**), (***). An exercise with (*) is a simple exercise requiring less time to solve compared to an exercise with (***), which is a more complex exercise.

1. (*) The table below gives details of symptoms that patients presented and whether they were suffering from meningitis Using this dataset, calculate the following probabilities

ID	Headache	Fever	Vomiting	Meningitis
1	true	true	false	false
2	false	true	false	false
3	true	false	true	false
4	true	false	true	false
5	false	true	false	true
6	true	false	true	false
7	true	false	true	false
8	true	false	true	true
9	false	true	false	false
10	true	false	true	true

   (a) $P(\text{Vomiting} = \text{true})$.
   (b) $P(\text{Headache} = \text{false})$.
   (c) $P(\text{Headache} = \text{true}, \text{Vomiting} = \text{false})$.
   (d) $P(\text{Vomiting} = \text{false}|\text{Headache} = \text{true})$.
   (e) $P(\text{Meningitis} = \text{true}|\text{Fever} = \text{true}, \text{Vomiting} = \text{false})$.

2. (*) Consider the experiment of tossing a coin three times. Let X be the RV giving the number of heads obtained. We assume that the tosses are independent and the probability of a head is p. Find the probabilities $P(X = 0), P(X = 1), P(X = 2)$, and $P(X = 3)$.

3. (***) Suppose that the two RVs X and Z are statistically independent. Show that the mean and variance of their sum satisfies

$$E\{X + Z\} = E\{X\} + E\{Z\}$$
$$\text{var}\{X + Z\} = \text{var}\{X\} + \text{var}\{Z\}.$$

4. (*) Consider a discrete RV X whose pmf is given as

$$P(X) = \begin{cases} \frac{1}{3}, & \text{if } x = -1, 0, 1, \\ 0 & \text{otherwise} \end{cases}$$

Find the mean and variance of X.

5. (**) The RV X can take values $x_1 = 1$ and $x_2 = 2$. Likewise, the RV Y can take values $y_1 = 1$ and $y_2 = 2$. The joint pmf of the RVs X and Y is given as

$$P(X,Y) = \begin{cases} k(2x_i + y_j), & \text{for } i = 1,2\,; j = 1,2 \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant.

(a) Find the value of k.

(b) Find the marginal pmf for X and Y.

(c) Are X and Y independent?

6. (**) The joint pdf of the RVs X and Y is given by

$$p(x,y) = \begin{cases} k(x + y), & \text{for } 0 < x < 2, 0 < y < 2 \\ 0 & \text{otherwise,} \end{cases}$$

where k is a constant.

(a) Find the value of k.

(b) Find the marginal pdf for X and Y.

(c) Are X and Y independent?

7. (**) Suppose that we have three coloured boxes r (red), b (blue), and g (green). Box r contains 3 apples, 4 oranges, and 3 limes, box b contains 1 apple, 1 orange, and 0 limes, and box g contains 3 apples, 3 oranges, and 4 limes. If a box is chosen at random with probabilities $P(r) = 0.2, P(b) = 0.2, P(g) = 0.6$, and a piece of fruit is removed from the box (with equal probability of selecting any of the items in the box), then what is the probability of selecting an apple? If we observe that the selected fruit is in fact an orange, what is the probability that it came from the green box?