Introduction to linear regression

(Week 05 lecture notes)

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Topics covered in this lecture

- Distinguish between a functional relationship and a statistical relationship.
- Know the Gauss-Markov conditions for simple linear regression.
- Understand the least squares (LS) method.
- Recognize the difference between a population regression line and the estimated regression line.
- Know how to derive and obtain the LS estimates b_1, b_0.
- Interpret the intercept b_0 and slope b_1 of an estimated regression equation.
- Understand the coefficient of determination R^2 and how to intepret it
- Use regression model to perform prediction in R/Python

What is regression?

- Regression means "going back"
- Linear regression/linear models: a procedure to analyze data
- Historically, *Francis Galton* (1822-1911) invented the term and concepts of regression and correlation.
  - He predicted child's height from fathers height
    - Sons of the tallest fathers tended to be taller than average, but shorter than their fathers.
    - Sons of the shortest fathers tended to be shorter than average, but taller than their fathers.
  - He was deeply concerned about "regression to mediocrity".
  - A brief history of Linear Regression and more about Galton, `http://www.amstat.org/publications/jse/v9n3/stanton.html`
- Regression analysis is a statistical method to summarize and study the **relationships** between variables in a data set.

Response and predictor variables

- One variable, denoted Y, is regarded as the response (or outcome, or dependent) variable
  - whose behaviour that we want to study under the impact of the other variables
  - sometimes unavailable, so we would like to be able to **predict**
- The other variable, denoted X, is regarded as the predictor (or explanatory, or independent) variable.
  - variables that gives combinations of conditions under which we want to examine the response variable (e.g. drug treatments, experiemental conditions)
  - variables that provides information (e.g. covariates including demographical information, educational background, etc.)

Types of relationships

Relationship between Y and X

- Functional (or deterministic) Relationships
  - $Y = f(X)$, where f() is some function. eg. Circumference=$\pi \times$ diameter.
  - the relationship is deterministic through the function, for each possible X there is only one Y.
- Statistical Relationships
  - $Y = f(X) + \epsilon$, where ϵ is the random error term. eg. a simple linear regression model.
  - the relationship is probabilistic (contains random noise) due to the random error, for each possible X, there is a unique Y (if ϵ is continuous, then the probability of obtaining two exactly the same value is 0)

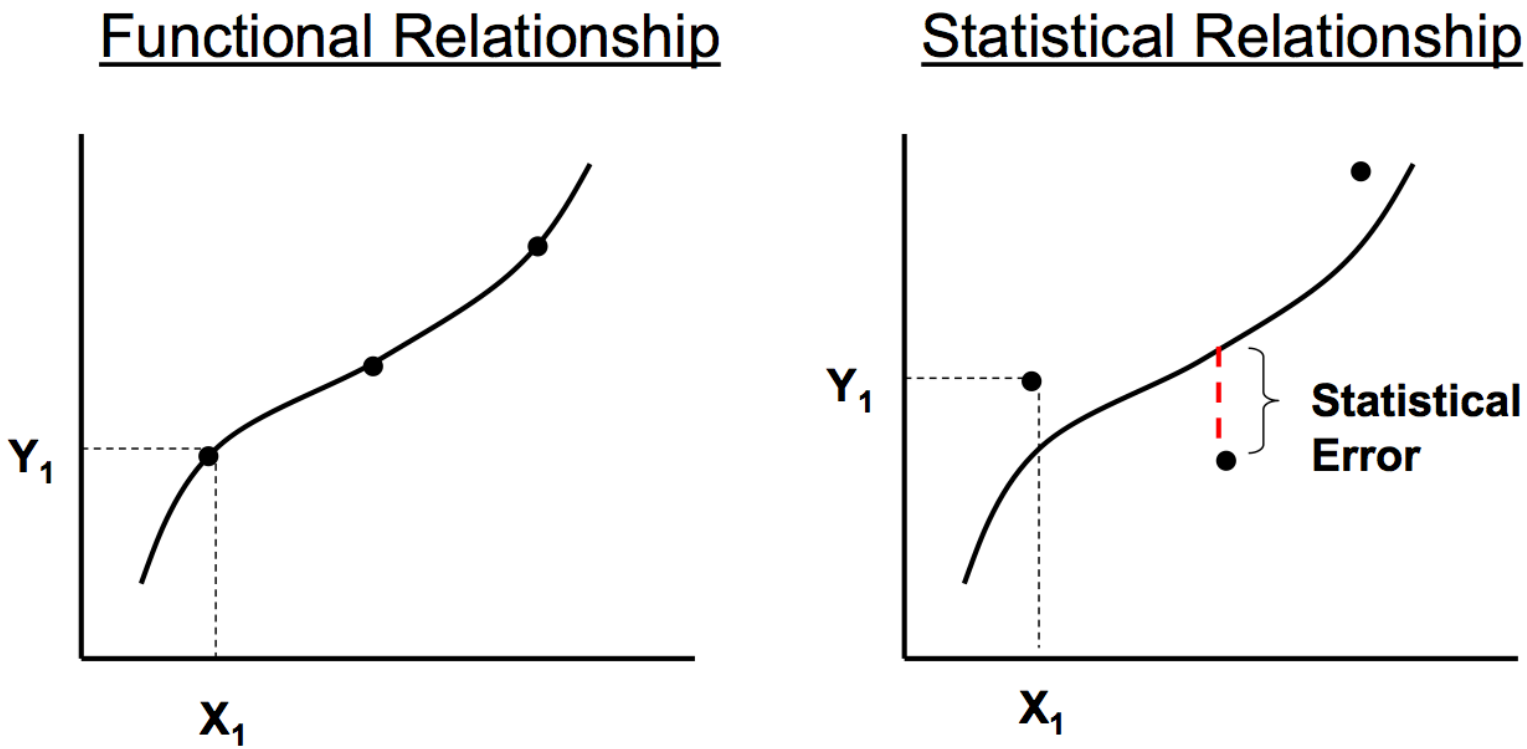
What does the data look like?

i	X	Y
1	0	6.95
2	1	5.22
3	2	6.46
4	3	7.03
5	4	9.71
6	5	9.67
7	6	10.69
8	7	13.85
9	8	13.21
10	9	14.82

There are $n = 10$ observations. The third observation is the pair $ (x_3,y_3)=(2,6.46)$. In a real dataset, usually you don't have the index i column as given in the table.

Types of relationships

- Scatter plots of data pair (Y_i, X_i)



- For each of these relationships, the equation, $Y = f(X)$, describes the relationship between the two variables.
- We are not interested in the functional relationship (deterministic) in this course.
- Instead, we are interested in **statistical relationships**, in which the relationships between the variables is not prefect.

A statistical model for the data

- the functional part of a statistical relationship is described by a **simple** linear model of the form

$$\mathrm{E}(Y|X) = \beta_0 + \beta_1 X$$

- a statistical model on $Y|X$ or the distribution of Y can be described by a **simple** linear regression model

$$Y \sim \beta_0 + \beta_1 X + \epsilon$$

where $\epsilon \sim \mathcal{N}(0, \sigma^2)$.

- Or in other words:

$$Y|X \sim \mathcal{N}(\beta_0 + \beta_1 X, \sigma^2)$$

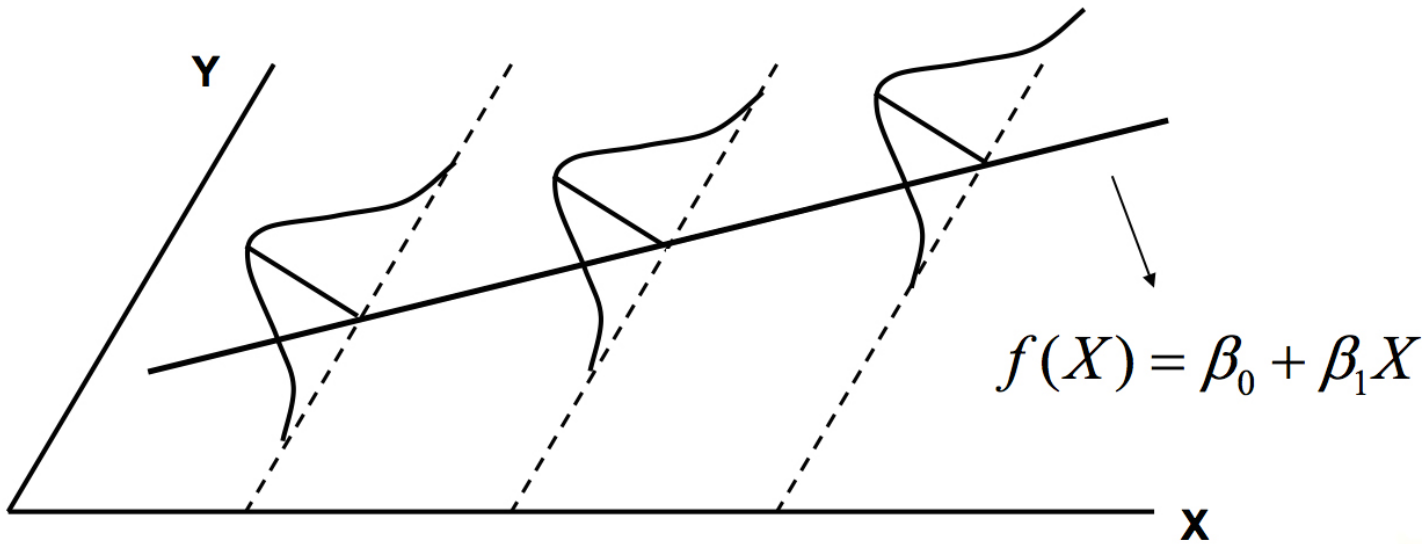
Note $\mathrm{E}(Y|X) = \beta_0 + \beta_1 X$ or $\mathrm{E}(Y|X) = a + bX$ are both acceptable expressions.

Regression Models

- Regression model describes the statistical relationship between the response variabel Y and one or more predictor variable(s)
  - The response variable Y has a tendency to vary with the predictor variable X in a systematic fashion.
  - The data are **scattered** around the regression curve.
- Regression model assumes a distribution for Y at each level of X.
- When the relationship between Y and X is linear, we call it linear regression.
  - In linear regression model, if it concerns study of only one predictor, then we have simple linear regression model.
  - In contrast, we have multiple linear regression.

A simple linear regression

- It concerns about the statistical relationship between Y and a single predictor X.
- The regression curve is a straight line.



The relationship is linear if it is linear in the model parameters (β_0, β_1) and nonlinear, if it is not linear in parameters.

A simple linear regression

- A formal statistical model:

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i \tag{1}$$

  - Y_i is the value of response variable in the i^{th} observation (random but observable).
  - X_i is the predictor in the i^{th} observation (a known constant).
  - β_0 is the intercept of the regression line (model parameter: assume constant but unknown).
  - β_1 is the slope of the regression line (model parameter: assume constant but unknown).
  - ϵ_i is the error term (random and unobservable)

A Simple Linear Regression: summary

R/C	Known	Unknown
Random	Y	ϵ
Constant	X	$\beta_0, \beta_1, \sigma^2$

- The parameters associated with a simple linear regression model are $\beta_0, \beta_1, \sigma^2$.
- We need to find estimators for all three of them in order to make use of the model.

Example 1: hourly wage (Y) and years of education (X)

Variables

- Y: hourly wage(pound)
- X: years of education

Parameter interpretation

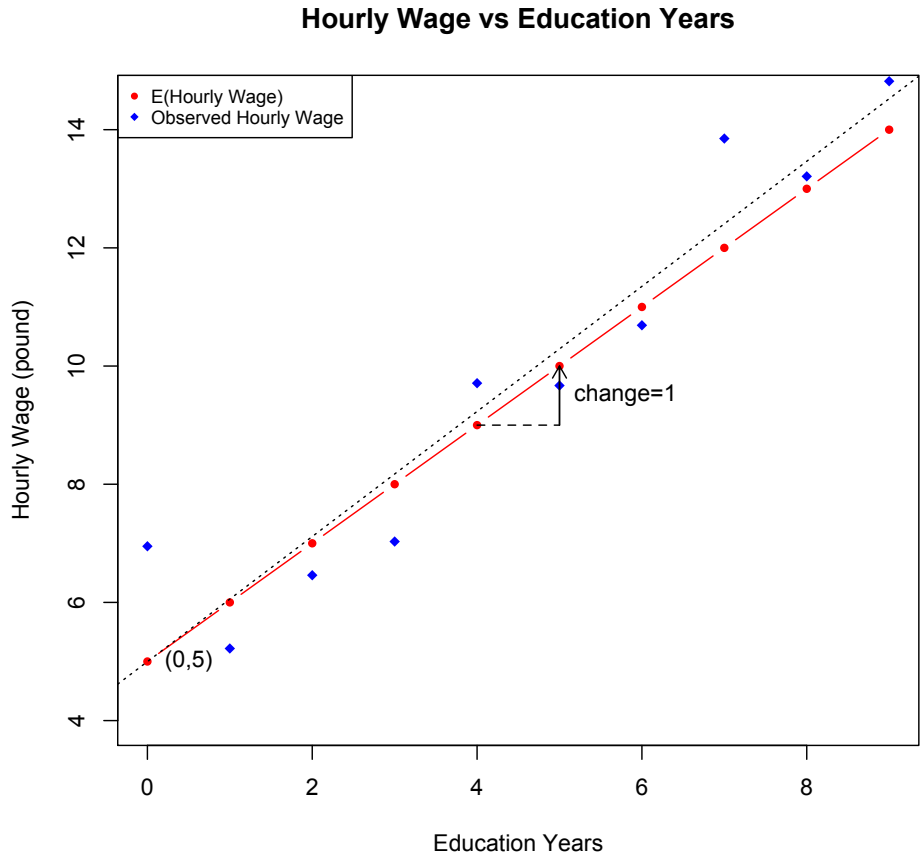
- β_0: Y-intercept, the starting salary or the baseline salary
- β_1: slope, the hourly wage increase per one more year of education

Example 1: hourly wage (Y) and years of education (X)

EducYrs	E(Y)=E(HWage$_T$)	Y=HWage$_O$
0	5	6.95
1	6	5.22
2	7	6.46
3	8	7.03
4	9	9.71
5	10	9.67
6	11	10.69
7	12	13.85
8	13	13.21
9	14	14.82

- EducYrs (X): years of education;
- HWage$_T$ (true E(Y)): the true expected hourly wage (in pounds).
- HWage$_O$ (observed Y): the observed hourly wage (in pounds)

Example 1: hourly wage (Y) and education years (X)



The observed Y goes up and down around the population regression line. In real world, we don't observed the true error term (ϵ), instead we have data (EducYrs, HWage_O). We aim to reveal the true relationship between Y and X using the data we observed. That is, how to use observed data to estimate β_0, β_1?

True vs Estimated model

Assume we have a data set of size $n : (Y_i, X_i), i = 1, \ldots, n$.

True regression model (or population regression model)

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i = f(X) + \epsilon_i, \quad f(X) = \beta_0 + \beta_1 X_i$$

Estimated regression model (or sample regression model)

$$\hat{Y}_i = b_0 + b_1 X_i = \hat{f}(X), \quad \hat{f}(X) = b_0 + b_1 X_i$$

- Point estimators of β_0, β_1 are denoted by b_0, b_1 respectively.
- An estimate of Y_i (for a given X_i) is denoted by $\hat{Y}_i$.
- An estimate of ϵ_i (for given X_i) is denoted by e_i

$$e_i = Y_i - \hat{Y}_i = Y_i - (b_0 + b_1 X_i)$$

This implies that

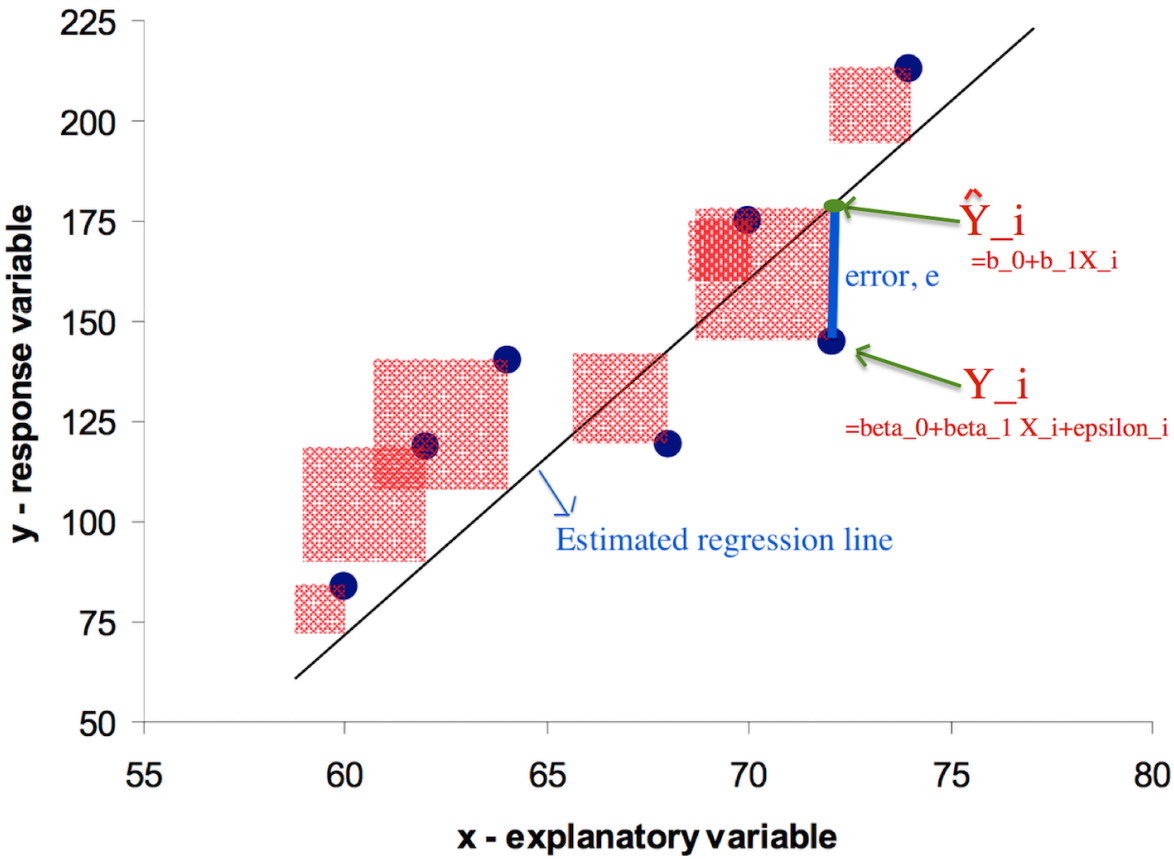
$$Y_i = \hat{Y}_i + e_i = (b_0 + b_1 X_i) + e_i$$

True vs Estimated model

- Difference between $\hat{Y}_i = b_0 + b_1 X_i$ and $Y_i = \beta_0 + \beta_1 X_i + \epsilon_i$.
- Note that we never observed ϵ_i, but we could estimate it by e_i.

$$Y_i = \hat{Y}_i + e_i = \hat{f}(X) + \text{estimated error}_i,$$

where $e_i = Y_i - \hat{Y}_i$.



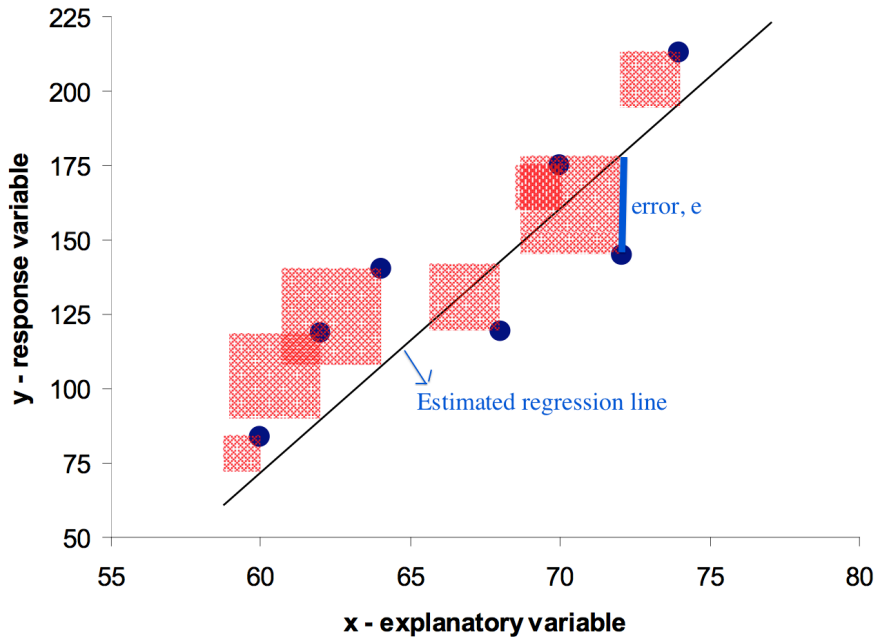
Estimation by Least Squares method

Gauss-Markov Assumptions

- **Gauss-Markov Assumptions**:
  1. Dependent variable is linear in parameter and can be written as : $Y = \beta_0 + \beta_1 X + \epsilon$
  2. $E(\epsilon_i) = 0$. ϵ_i is R.V. with mean 0.
  3. $V(\epsilon_i) = \sigma^2$, this homoskedasticity implies that the model uncertainty is identical across observations.
  4. $Cov(\epsilon_i, \epsilon_j) = 0$ for $i \neq j$. ϵ_i and ϵ_j are uncorrelated:
- X is assumed to be constant, ie, X is uncorrelated with the error term ($Cov(X_i, \epsilon_i) = 0$).
- $cov(\epsilon_i, \epsilon_j)$=0 does not guarantee ϵ_i and ϵ_j are independent. But if they are independent, their covariance must be 0.
- **Above assumptions imply**:
  - $E(Y_i|X_i) = \mu_i = \beta_0 + \beta_1 X_i$, that is $f(X) = \beta_0 + \beta_1 X$
  - $V(Y_i|X_i) = V(\mu_i + \epsilon_i) = V(\epsilon_i) = \sigma^2$
  - $Cov(Y_i, Y_j|X_i) = E\{(Y_i - \mu_i)(Y_j - \mu_j)\} = E(\epsilon_i \epsilon_j) = Cov(\epsilon_i, \epsilon_j) = 0$

We often drop $|X$ notation in above because X is non-random.

Least Square Method



- The equation of the estimated model (or best fitting line) is: $\hat{Y}_i = b_0 + b_1 X_i$
- We need to find the values b_0, b_1 that make the sum of the squared prediction error the smallest it can be. That is, find b_0 and b_1 that miniminze the objective function Q.

$$Q = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{n} (Y_i - b_0 - b_1 X_i)^2$$

Least Square Estimates b_0, b_1

$$Q = \sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{n}(Y_i - b_0 - b_1 X_i)^2$$

Minimizing Q gives

$$b_0 = \hat{\beta}_0 = \bar{y} - b_1 \bar{x} \tag{2}$$

$$b_1 = \hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} = \frac{S_{xy}}{S_{xx}} \tag{3}$$

where

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i, \quad S_{xy} = \sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}), S_{xx} = \sum_{i=1}^{n}(x_i - \bar{x})^2$$

Substituting b_0 in the estimated model, it can be rewritten as

$$\hat{Y}_i = b_0 + b_1 X_i = \bar{Y} + b_1(X_i - \bar{X}),$$

this also implies

$$Y_i = \bar{Y} + b_1(X_i - \bar{X}) + e_i$$

i.e. The estimated regression line always goes through the point data point $(\bar{X}, \bar{Y})$.

Proof

$$\frac{\partial Q}{\partial b_0} = -2\sum_{i=1}^{n}(Y_i - b_0 - b_1 X_i) = 0 \tag{4}$$

$$\frac{\partial Q}{\partial b_1} = -2\sum_{i=1}^{n}(Y_i - b_0 - b_1 X_i)X_i = 0 \tag{5}$$

These lead to the **Normal equations:**

$$\sum_{i=1}^{n} Y_i = nb_o + b_1 \sum_{i=1}^{n} X_i$$

$$\sum_{i=1}^{n} X_i Y_i = b_0 \sum_{i=1}^{n} X_i + b_1 \sum_{i=1}^{n} X_i^2$$

The normal equations can be solved simultaneously for b_0 and b_1 given in equation (2) and (3) respectively.

proof (not tested)

The Hessian matrix which is the matrix of second order partial derivatives in this case is given as

$$H = \begin{pmatrix} \frac{\partial Q}{\partial \beta_0^2} & \frac{\partial Q}{\partial \beta_0 \beta_1} \\ \frac{\partial Q}{\partial \beta_0 \beta_1} & \frac{\partial Q}{\partial \beta_1^2} \end{pmatrix} = 2 \begin{pmatrix} n & n\bar{x} \\ n\bar{x} & \sum x_i^2 \end{pmatrix}$$

- The 2 by 2 matrix H is positive definite if its determinant and the element in the first row and column of H are positive.
- The determinant of H is given by $|H| = 4n \sum (x_i - \bar{x})^2 > 0$ given $x \neq c$(some constant).
- So H is positive definite for any (β_0, β_1), therefore Q has a global minimum at (b_0, b_1).

Equivalent formula for b_1

$$b_1 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(X_i - \bar{X})^2} = \frac{S_{xy}}{S_{xx}} \tag{6}$$

$$= \frac{\sum_{i=1}^{n}(X_i - \bar{X})Y_i}{S_{xx}} \tag{7}$$

$$= \sum_{i=1}^{n} \frac{X_i - \bar{X}}{S_{xx}} y_i = \sum_{i=1}^{n} k_i Y_i \tag{8}$$

$$= \frac{\sum_{i=1}^{n} X_i Y_i - n\bar{X}\bar{Y}}{S_{xx}} \tag{9}$$

$$= \sum_{i=1}^{n} k_i Y_i \tag{10}$$

where

$$k_i = \frac{X_i - \bar{X}}{S_{xx}} = \frac{X_i - \bar{X}}{\sum_{i=1}^{n}(X_i - \bar{X})^2},$$

suggesting that b_1 is a linear combination of Y_i (assume constant X) and hence is a linear estimator.

Equivalent formula for b_0

$$
\begin{aligned}
b_0 = \bar{Y} - b_1 \bar{X} &= \sum_{i=1}^{n} \frac{1}{n} Y_i - \bar{X} \sum_{i=1}^{n} k_i Y_i \\
&= \sum_{i=1}^{n} (\frac{1}{n} - k_i \bar{X}) Y_i \\
&= \sum_{i=1}^{n} w_i Y_i
\end{aligned}
$$

where

$$
w_i = \frac{1}{n} - k_i \bar{X},
$$

suggesting that b_0 is also a linear combination of Y_i and hence is a linear estimator.

Estimation of error terms variance σ^2

- Error sum of squares (SSE) or residual sum of square (RSS)

$$SSE = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{n} (Y_i - b_0 - b_1 X_i)^2$$

- SSE has n-2 degrees of freedom associated with it. Two degrees of freedom are lost because both β_0 and β_1 had to be estimated in obtaining estimated means $\hat{Y}_i$
- In LS method, the error term variance $\sigma^2 = V(\epsilon_i)$ for all i, is estimated by the error mean square (MSE)

$$s^2 = MSE = \frac{SSE}{n-2} = \frac{\sum_{i=1}^{n} e_i^2}{n-2} = \frac{(Y_i - \hat{Y}_i)^2}{n-2}$$

Example 2: Estimation (by hand)

- Annual salary (Y) and years of service (X)

	x_i	y_i	$x_i - \bar{x}$	$y_i - \bar{y}$	$(x_i - \bar{x})^2$	$(y_i - \bar{y})^2$	$(x_i - \bar{x})(y_i - \bar{y})$
i=1	3	34	-5	-4	25	16	20
i=2	6	34	-2	-4	4	16	8
i=3	10	38	2	0	4	0	0
i=4	8	37	9	-1	0	1	0
i=5	13	47	5	9	25	81	45
Sum	40	190	0	0	58	114	73

Above calculation gives $\bar{X} = 40/5 = 8$ and $\bar{Y} = 190/5 = 38$.

$$b_1 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^{n}(X_i - \bar{X})^2} = \frac{73}{58} = 1.258621$$

$$b_0 = \bar{Y} - b_1\bar{X} = 38 - 1.258621 \times 8 = 27.931$$

Example 2: Estimation (by hand)

- Find $\hat{Y}_i = 27.931 + 1.25862 X_i$
  - $\hat{Y}_i = c(31.70686, 35.48272, 40.51720, 37.99996, 44.29306)$
- Find $e_i = Y_i - \hat{Y}_i$
  - $e_i = c(2.29314, -1.48272, -2.51720, -0.99996, 2.70694)$
- Estimate σ^2 by MSE: $s^2 = \hat{\sigma}^2 = \sum e_i^2/(n-2) = 7.373563$
  - $\hat{\sigma} = \sqrt{7.373563} = 2.715431$

Extended to multiple predictors (multiple input variables)

- the output is an **outcome** variable Y
- the inputs are the **predictor** variables $X_1, \ldots, X_p$
- the relationship is described by a linear regression model of the form

$$\mathrm{E}(Y|X_1, \ldots, X_p) = \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p$$

and a model on $Y|X_1, \ldots, X_p$ or the distribution of Y can be described by

$$Y \sim \beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p + \epsilon$$

where $\epsilon \sim \mathcal{N}(0, \sigma^2)$.
Or in other words:

$$Y|X_1, \ldots, X_p \sim \mathcal{N}(\beta_0 + \beta_1 X_1 + \cdots + \beta_p X_p, \sigma^2)$$

Inference on model parameters β_1 and β_0

Motivations

- Recall the assumptions:
  1. Dependent variable is linear in parameter and can be written as : $Y = \beta_0 + \beta_1 X + \epsilon$
  2. $E(\epsilon_i) = 0$. ϵ_i is R.V. with mean 0.
  3. $V(\epsilon_i) = \sigma^2$, this homoskedasticity implies that the model uncertainty is identical across observations.
  4. $Cov(\epsilon_i, \epsilon_j) = 0$ for $i \neq j$. ϵ_i and ϵ_j are uncorrelated:
- In other words, the least squares estimators hold no matter what the distribution of *epsilon*.
- However, if we want to make inference on β_0 or β_1, we need to make distributional assumptions.

Normal error regression models

Similar to the previous model, the true regression model (or population regression model)

$$Y_i = \beta_0 + \beta_1 X_i + \epsilon_i = f(X) + \epsilon_i, \quad f(X) = \beta_0 + \beta_1 X_i$$

The only difference is that for $i = 1, \ldots, n$:

$$\epsilon_i \sim \mathcal{N}(0, \sigma^2)$$

In other words,

- The errors are IID normal with mean 0 and **unknown** variance σ^2
- $Y_i \sim \mathcal{N}(\beta_0 + \beta_1 X_i, \sigma^2)$ are independent.

Normal error regression models

This formulation enables

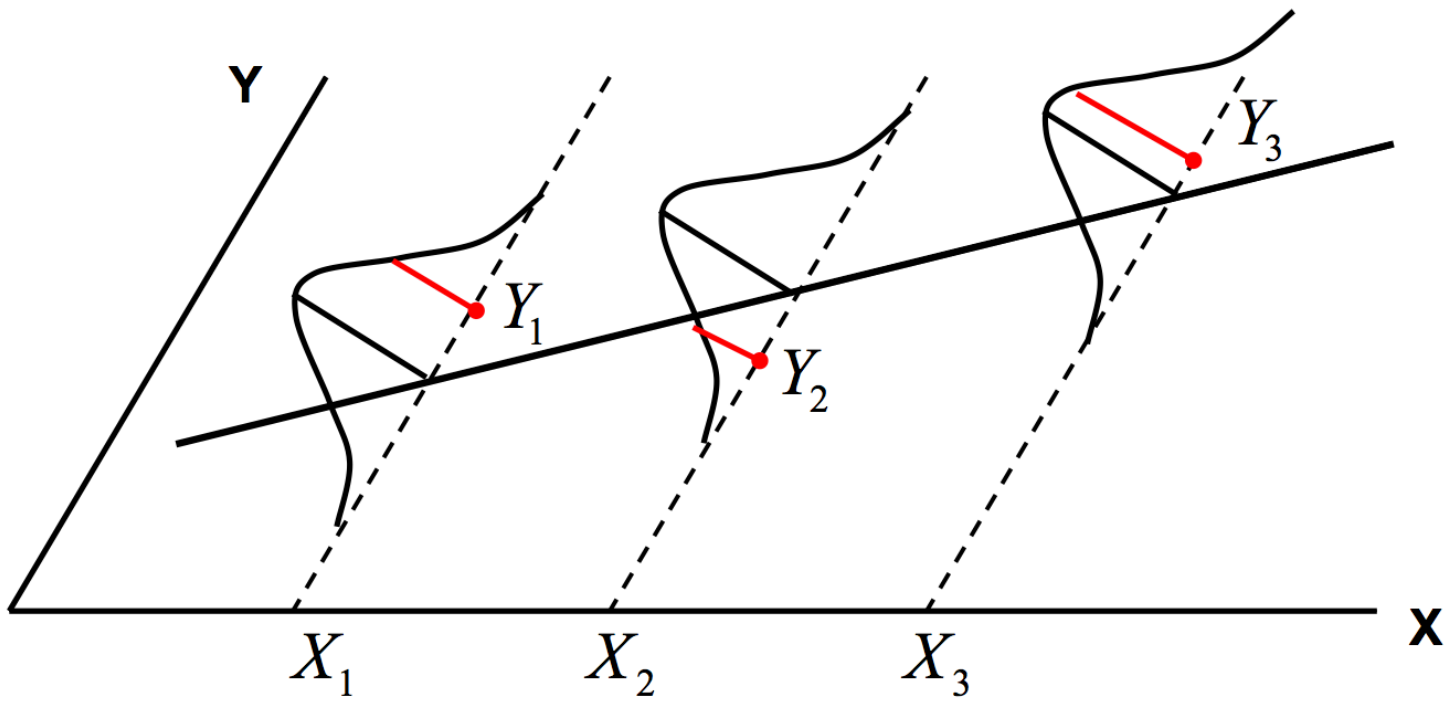
- estimation of β_0, β_1 and σ^2 through maximum likelihood method

and

- inference on these parameters via a known sampling distribution

Maximum Likelihood Estimation for Regression

- Similarly, we find the values of parameters $\theta = (\beta_0, \beta_1, \sigma^2)$ in the normal error model that maximize the relevant likelihood function.



- $Y_i \sim N(\beta_0 + \beta_1 X_i, \sigma^2)$, the density function at y_i is

$$f(y_i; \beta_0, \beta_1, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}\}$$

MLE for Regression

- Likelihood function

$$L(\beta_0, \beta_1, \sigma^2) = \prod_{i=1}^{n} f(y_i; \theta) = \{\frac{1}{2\pi\sigma^2}\}^{n/2} \exp\{-\frac{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}\}$$

- Log likelihood function

$$\ell(\beta_0, \beta_1, \sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2}{2\sigma^2}$$

Partial differentiation of $\ell(\beta_0, \beta_1, \sigma^2)$ yields

$$\frac{\partial \ell(\beta_0, \beta_1, \sigma^2)}{\partial \beta_0} = \frac{1}{\sigma^2}\sum(y_i - \beta_0 - \beta_1 x_i) \tag{11}$$

$$\frac{\partial \ell(\beta_0, \beta_1, \sigma^2)}{\partial \beta_1} = \frac{1}{\sigma^2}\sum x_i(y_i - \beta_0 - \beta_1 x_i) \tag{12}$$

$$\frac{\partial \ell(\beta_0, \beta_1, \sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum(y_i - \beta_0 - \beta_1 x_i)^2 \tag{13}$$

MLE vs LSE for Regression

Set partial derivatives equal to zero and solve them to obtain the MLE of $\beta_0, \beta_1, \sigma^2$.

Parameters	MLE	Same as LSE?
β_0	$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$	Same. $\hat{\beta}_0 = b_0$
β_1	$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$	Same. $\hat{\beta}_1 = b_1$
σ^2	$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2}{n}$	Different. $\hat{\sigma}^2 = \frac{n-2}{n} MSE$

Note that the MLE of σ^2 is a biased estimator of the error term variance.

Inference on β_0 and β_1

- We knew how to point estimate of β_0, β_1 in SLR from data, Using Ordinary Least Squares (OLS) or MLE.
- Now we take up the inferences concerning the regresssion parameters β_0, β_1
  - How accurate are their estimates?
  - How to obtain an interval estimate for each of them?
  - How to test a specific parameter value of interest?

Sampling distribution of b_1

- From earlier, we know (no need to show)
  - $b_1 = \frac{S_{XY}}{S_{XX}} = \sum_{i=1}^{n} k_i Y_i$, so the b_1 is normally distributed.
  - b_1 is unbiased $E(b_1) = \beta_1, \sigma^2(b_1) = V(b_1) = \frac{\sigma^2}{S_{XX}}$
  - $\hat{\sigma}^2$=MSE=SSE/(n-2), so $s^2(b_1) = MSE/S_{XX}$, $\sqrt{s^2(b_1)} = s(b_1)$
- (Common) Sampling distribution of b_1 Normal error model assumption when σ^2 is unknown:

$$\frac{b_1 - \beta_1}{s(b_1)} \sim t_{n-2}$$

- (Not common) Sampling distribution of b_1 Normal error model assumption when σ^2 is known: $\frac{b_1 - \beta_1}{\sigma(b_1)} \sim N(0, 1)$

Inferences concerning β_1

- $1-\alpha$ confidence limits for β_1 are

$$b_1 \pm t_{1-\alpha/2,n-2} s(b_1) = b_1 \pm t_{1-\alpha/2,n-2} s(b_1)$$

- Testing concerning β_1

$$H_0 : \beta_1 = 0 \quad H_a : \beta_1 \neq 0$$

Note that when $\beta_1 = 0$, there is no linear association between Y and X.

  - Test statistics

$$t^* = \frac{b_1 - 0}{s(b_1)} |_{H_0} \sim t_{n-2}$$

  - If $|t^*| \leq t_{1-\alpha/2,n-2}$, fail to reject H_0.
  - If $|t^*| > t_{1-\alpha/2,n-2}$, reject H_0.
  - Or find relevant p-value, and we reject H_0 if p-value is less than α.
  - If we change $H_a : \beta > 0$ then we reject H_0 if $t^* > t_{1-\alpha,n-2}$ otherwise we fail to reject null hypothesis.

Sampling distribution of b_0

- From earlier, we know (no need to show)
  - $b_0 = \bar{Y} - b_1\bar{X} = \sum_{i=1}^{n} w_i Y_i$, so the b_0 is normally distributed.
  - b_0 is unbiased. $E(b_0) = \beta_0, \sigma^2(b_0) = V(b_0) = \sigma^2\{\frac{1}{n} + \frac{\bar{X}^2}{S_{XX}}\}$
  - $s^2(b_0) = MSE\{\frac{1}{n} + \frac{\bar{X}^2}{S_{XX}}\}$
- (Common) Sampling distribution of b_0 Normal error model assumption when σ^2 is unknown:

$$\frac{b_0 - \beta_0}{s(b_0)} \sim t_{n-2}$$

- (Not Common) Sampling distribution of b_0 Normal error model assumption when σ^2 is known:

$$\frac{b_0 - \beta_0}{\sigma(b_0)} \sim N(0, 1)$$

Inferences concerning β_0

There are only infrequent occasions when we wish to make inferences concerning β_0. We do inferences cencerning β_0 only when the scope of the model includes $X = 0$.

- $1 - \alpha$ confidence limits for β_0 are

$$b_0 \pm t_{1-\alpha/2,n-2} s(b_0) = b_0 \pm t_{1-\alpha/2,n-2} s(b_0)$$

- Testing concerning β_0 is less interest.

Summarizing and intepreting a linear regression model

Key information from a simple linear regression model

- Test whether the slope is zero (establishing the presence linear relationship)
  - significance is the presence or absence of a linear relationship
  - intepretation is the amount of per unit increase/decrease in Y due to a per unit increase in X
- Calculate the coefficient of determination (assess the strength of the linear relationship)
  - intepretation is the amount of variability in Y that is explained by X
  - for a simple linear regression the pearson's correlation coefficient is exactly the same as the coefficient of determination
- The intercept provides a baseline for discussion.
  - the intepretation is the baseline value of Y when $X = 0$ (this might not make sense in certain applications when range of X does not contain 0).

Example 2. Annual Salary (Y) vs years of service (X)

```r
X=c(3,6,10,8,13)     # assign predictor observations to object X
Y=c(34,34,38,37,47) # assign response observations to object Y
lmfit = lm(Y~X)      # fitting data with a simple linear regression
summary(lmfit)     # summary of the fitted model
```

```
##
## Call:
## lm(formula = Y ~ X)
##
## Residuals:
##      1      2      3      4      5
##  2.293 -1.483 -2.517 -1.000  2.707
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  27.9310     3.1002    9.01  0.00289 **
## X             1.2586     0.3566    3.53  0.03864 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.715 on 3 degrees of freedom
## Multiple R-squared:  0.806,  Adjusted R-squared:  0.7413
## F-statistic: 12.46 on 1 and 3 DF,  p-value: 0.03864
```

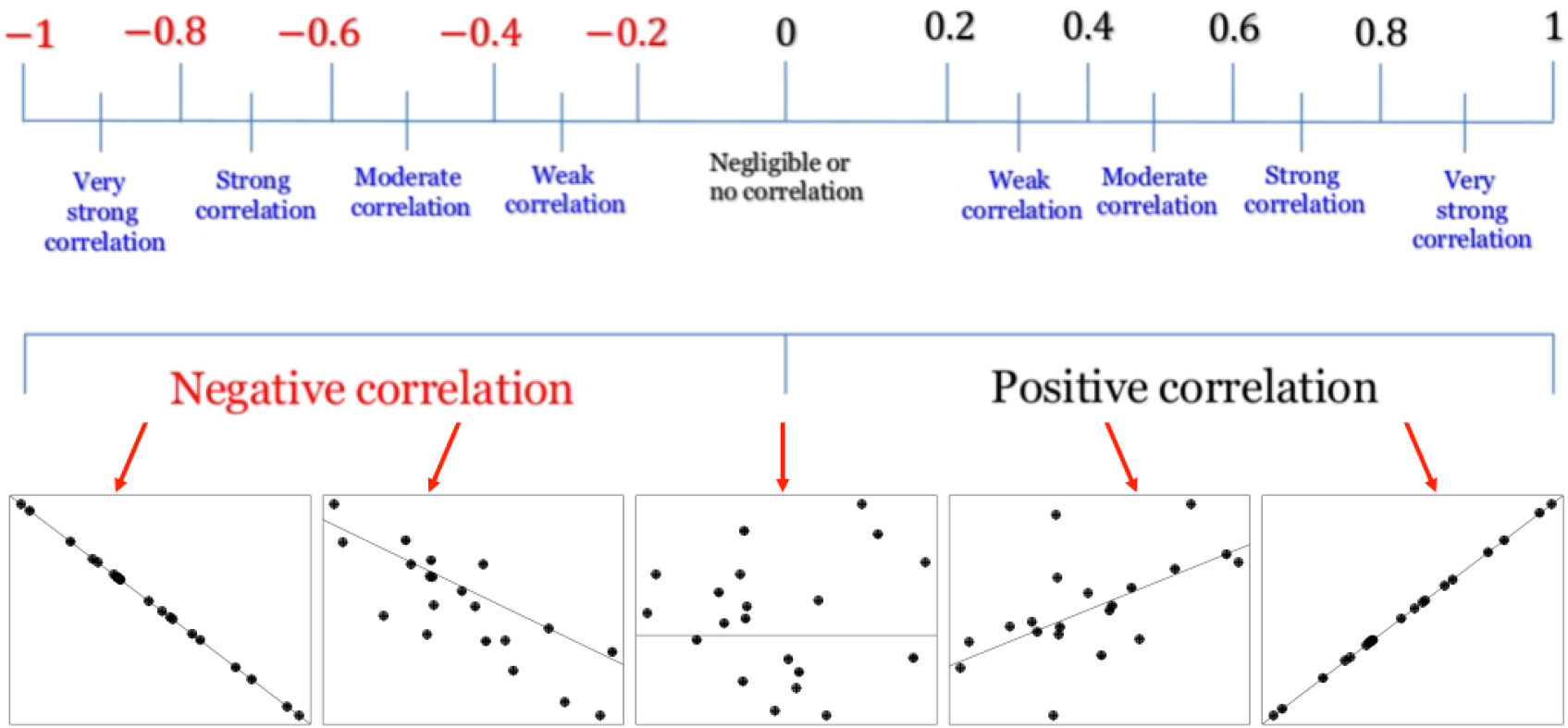
(Pearson) Coefficient of Correlation

- r is the basic quantity in correlation analysis

$$r = \frac{\sum(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{[\sum(X_i - \bar{X})^2][\sum(Y_i - \bar{Y})^2]}} = \frac{S_{XY}}{\sqrt{S_{XX}S_{YY}}}$$

- r takes values in [-1,1], the $\pm$ signs are used for positive and negative linear correlations, respectively.
- r measures strength & direction of linear relationship
  - Value of -1 or +1 indicates a perfect positive or negative correlation, all data points all lie exactly on a straight line
  - Value of 0.0 indicates no linear correlation
  - Positive values indicate a direct relationship, X and Y move in the same directions.
  - Negative values indicate an inverse relationship, X and Y move in opposite directions.
- Note that r is a dimensionless quantity; that is, it does not depend on the units employed.

Interprestation of Correlation coefficient



- $0.2 \leq |r| \leq 0.4$: weak correlation.
- $0.4 < |r| \leq 0.6$: moderate correlation.
- $0.6 < |r| \leq 0.8$: strong correlation.
- $|r| > 0.8$: very strong correlation.

Coefficient of Determination

$$R^2 = \frac{SSR}{SSTO} = 1 - \frac{SSE}{SSTO}$$

- The measure R^2 is called the coefficient of determination.
- R^2 is a proportion. Since $0 \leq SSE \leq SSTO$, it follows that $0 \leq R^2 \leq 1$
- R^2 measures strength of linear relationship.
  - $R^2 = 1$, all data points fall perfectly on the regression line and X accounts for all of the variation in Y.
  - $R^2 = 0$, the estimated regression line is pefectly horizontal. X accounts none of the variation in Y.
- Interpretation of R^2:
  - "$R^2 \times$ 100 percent of the variation in Y is reduced by taking into account predictor X"
  - "$R^2 \times$ 100 percent of the variation in Y is 'explained by' the variation in predictor X"

Exercise: show $R^2 = r^2$ under a simple linear regression model.

Hints:

$$R^2 = \frac{\sum_{i=1}^{n}(b_0 + b_1 X_i - \bar{Y})^2}{\sum_{i=1}^{n}(Y_i - \bar{Y})^2}$$

$$r_{X,Y} = \frac{\Sigma(X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2 \sum_{i=1}^{n}(Y_i - \bar{Y})^2}}$$

where

$$S_{YY} = SSTO = \sum_{i=1}^{n}(Y_i - \bar{Y})^2$$

$$S_{XX} = \sum_{i=1}^{n}(X_i - \bar{X})^2$$

From R^2 to get r

$$r = \pm\sqrt{R^2} = \text{sign}(b_1)\sqrt{R^2}$$

- If b_1 is negative, then r takes a negative sign.
- If b_1 is positive, then r takes a positive sign.

$$r = b_1\sqrt{\frac{S_{XX}}{S_{YY}}}$$

- The estimated slope b_1 of the regression line and the correlation coefficient r always share the same sign.
- If the estimated slope b_1 of the regression line is 0, then the correlation coefficient r must also be 0.

Limitations of R^2

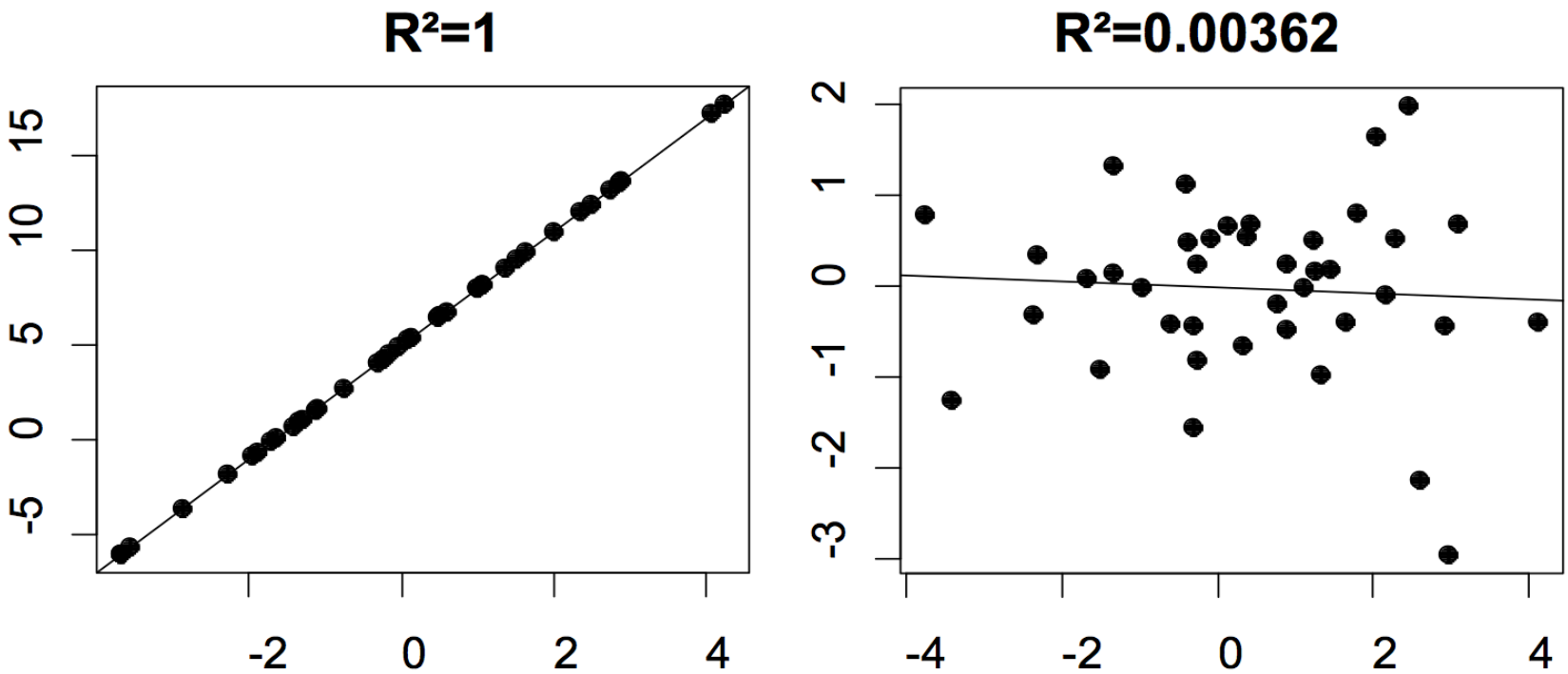
- R^2 is frequently used for assessing and comparing model fits, but
  - A high R^2 does not necessarily imply that you can make useful predictions.
  - A high R^2 does not necessarily imply that the estimated line is a good fit.
  - A low R^2 does not necessarily imply that X and Y are not related or independent.
- R^2 measures degree of *linear* association but does not measure the evidence of a linear relationship between X and Y.
- R^2 usually can be made larger by including a larger number of predictors.(More predictors, MSE goes down,SSTO remains unchanged, so R^2 goes up)
- Adjusted R^2, R_a^2(p=number of predictors in the model+1, +1 for β_0)

$$R_a^2 = 1 - \frac{SSE/(n-p)}{SSTO/(n-1)} = 1 - \frac{n-1}{n-p}\frac{SSE}{SSTO} = 1 - (n-1)\frac{MSE}{SSTO}$$

$$R_a^2 = 1 - \frac{(1-R^2)(n-1)}{n-p}$$

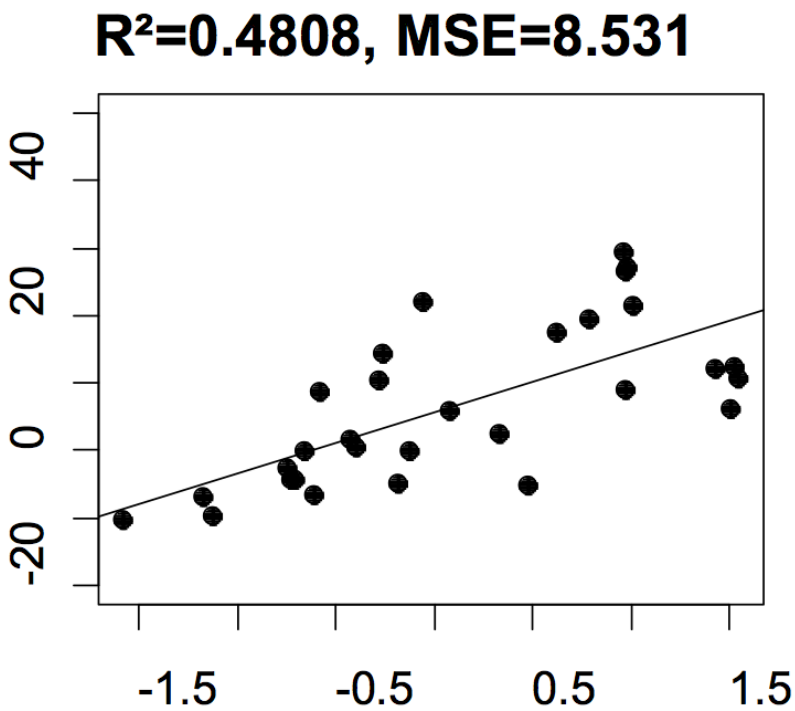
Interpretation of R^2

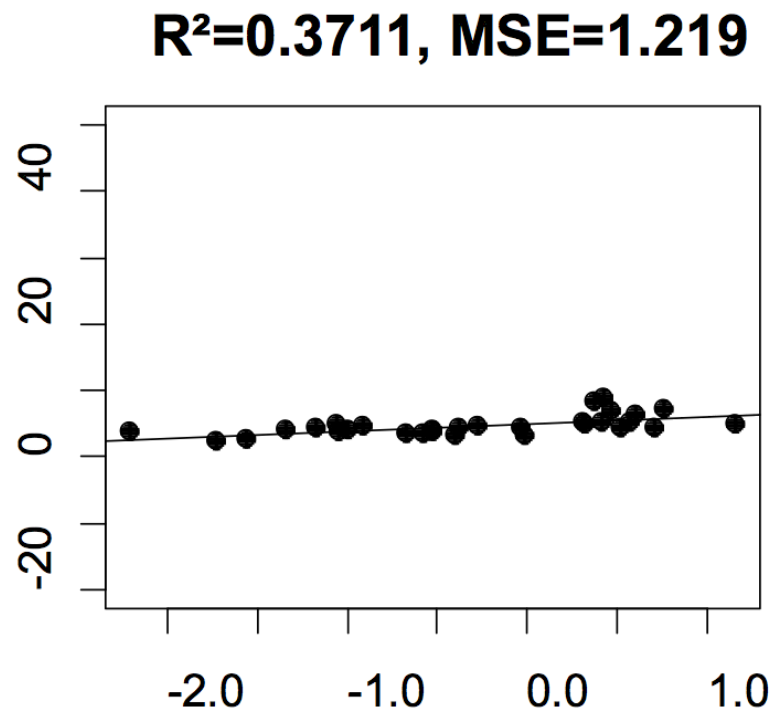
- R^2 measures degree of *linear* association between X and Y.



Interpretation of R^2

- R^2 measures only a relative reduction from SSTO.
- R^2 might be large but MSE may still be too large for inference to be useful in prediction.
- R^2 might be small but MSE may still be small which is useful in prediction





Interpretation of R^2

- R^2 measures degree of *linear* association.
- R^2 might be large or small if the true regression association between X and Y is curvilinear.