CSC165H1 Problem Set 2

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1 AND vs. IMPLIES

(a) WTP: $\forall n \in \mathbb{N}, n > 15 \Rightarrow n^3 - 10n^2 + 3 \geq 165$

Proof: Let $n \in \mathbb{N}$, and assume $n > 15$, WTP:$n^3 - 10n^2 + 3 > 165 \vee n^3 - 10n^2 + 3 = 165$

$$\begin{aligned}
n &> 15 && \text{(by assumption)} \\
n^3 &> 15n^2 && \text{(since } n > 15\text{, then } n^2 > 0\text{)} \\
n^3 - 10n^2 &> 5n^2 \\
n^3 - 10n^2 + 3 &> 5n^2 + 3 \\
n^3 - 10n^2 + 3 &> 5n^2 + 3 > 1128 > 165 && \text{(since } n > 15\text{, then } 5n^2 + 3 > 1128\text{)}
\end{aligned}$$

Therefore, $n^3 - 10n^2 + 3 > 165$
Therefore, $n^3 - 10n^2 + 3 > 165 \vee n^3 - 10n^2 + 3 = 165$ holds
Thus, $\forall n \in \mathbb{N}, n > 15 \Rightarrow n^3 - 10n^2 + 3 \geq 165$

(b) We want to disprove: $\forall n \in \mathbb{N}, n > 15 \wedge n^3 - 10n^2 + 3 \geq 165$, which is equivalent to prove its negation: $\exists n \in \mathbb{N}, n \leq 15 \vee n^3 - 10n^2 + 3 < 165$

Proof: Take $n = 1$ Then $n = 1 \leq 15$
Therefore $\exists n \in \mathbb{N}, n \leq 15 \vee n^3 - 10n^2 + 3 < 165$
Therefore $\forall n \in \mathbb{N}, n > 15 \wedge n^3 - 10n^2 + 3 \geq 165$ is False.

2 Ceiling function

(a) Translate into predicate logic: $\forall n, m \in \mathbb{N}, n < m \Rightarrow \left\lceil \frac{m-1}{m} \cdot n \right\rceil = n$

Proof: Let $n, m \in \mathbb{N}$, and assume $n < m$, WTS: $\left\lceil \frac{m-1}{m} \cdot n \right\rceil = n$

$$\left\lceil \frac{m-1}{m} \cdot n \right\rceil = \left\lceil (1-\frac{1}{m}) \cdot n \right\rceil$$
$$\left\lceil (1-\frac{1}{m}) \cdot n \right\rceil = \left\lceil -\frac{n}{m} + n \right\rceil$$
$$\left\lceil -\frac{n}{m} + n \right\rceil = \left\lceil -\frac{n}{m} \right\rceil + n \qquad \text{(since } n, m \in \mathbb{N}\text{, then } n \in \mathbb{Z} \text{ and } -\tfrac{n}{m} \in \mathbb{R} \text{ and by Fact 2)}$$
$$\left\lceil -\frac{n}{m} \right\rceil + n = n \qquad \text{(since } n < m \text{ and by the definition of ceiling function)}$$

Therefore $\left\lceil \frac{m-1}{m} \cdot n \right\rceil = n$
Therefore $\forall n, m \in \mathbb{N}, n < m \Rightarrow \left\lceil \frac{m-1}{m} \cdot n \right\rceil = n$

(b) Define a predicate $IsMultipleOf50(n)$: $\exists q \in \mathbb{Z}, n = 50q$, where $n \in \mathbb{N}$

Translate into predicate logic: $\forall n \in \mathbb{N}, nextFifty(n) \geq n \wedge IsMultipleOf50(nextFifty(n)) \wedge (\forall x \in \mathbb{N}, x \geq n \wedge IsMultipleOf50(x) \Rightarrow x \geq nextFifty(n))$

Proof: Let $n \in \mathbb{N}$, WTS: $nextFifty(n) \geq n \wedge IsMultipleOf50(nextFifty(n)) \wedge (\forall x \in \mathbb{N}, x \geq n \wedge IsMultipleOf50(x) \Rightarrow x \geq nextFifty(n))$

At first, we want to show: $nextFifty(n) \geq n$

$$\left\lceil \frac{n}{50} \right\rceil \geq \frac{n}{50} \qquad \text{(by the definition of ceiling function)}$$
$$50 \cdot \left\lceil \frac{n}{50} \right\rceil \geq 50 \cdot \frac{n}{50} = n$$
$$nextFifty(n) \geq n$$

Therefore $nextFifty(n) \geq n$ is true

Next, we want to show that: $IsMultipleOf50(nextFifty(n))$, which is: $\exists q \in \mathbb{Z}, nextFifty(n) = 50q$
Take $q = \left\lceil \frac{n}{50} \right\rceil$ (since by the definition of ceiling function, then $q = \left\lceil \frac{n}{50} \right\rceil \in \mathbb{Z}$)

$$nextFifty(n) = 50 \cdot \left\lceil \frac{n}{50} \right\rceil = 50q$$

Therefore $IsMultipleOf50(nextFifty(n))$ is true

Finally, we want to prove: $\forall x \in \mathbb{N}, x \geq n \wedge IsMultipleOf50(x) \Rightarrow x \geq nextFifty(n)$
Let $x \in \mathbb{N}$
Assume $x \geq n$
Assume $IsMultipleOf50(x)$, which is: $\exists q \in \mathbb{Z}, x = 50q$
Let q be such value, WTS: $x \geq nextFifty(n)$, whcih is: $x = 50q \geq 50 \cdot \left\lceil \frac{n}{50} \right\rceil$

$$
\begin{aligned}
x = 50q &\geq n && \text{(by our assumption)}\\
q \geq \frac{n}{50} &\geq \left\lceil \frac{n}{50} \right\rceil && \text{(since } q \in Z\text{)}\\
50q &\geq 50 \cdot \left\lceil \frac{n}{50} \right\rceil\\
x = 50q &\geq 50 \cdot \left\lceil \frac{n}{50} \right\rceil = nextFifty(n)
\end{aligned}
$$

Therefore $x \geq nextFifty(n)$
Therefore $\forall x \in \mathbb{N}, x \geq n \wedge IsMultipleOf50(x) \Rightarrow x \geq nextFifty(n)$

Thus $\forall n \in \mathbb{N}, nextFifty(n) \geq n \wedge IsMultipleOf50(nextFifty(n)) \wedge (\forall x \in \mathbb{N}, x \geq n \wedge IsMultipleOf50(x) \Rightarrow x \geq nextFifty(n))$.

3 Divisibility

(a) WTP: $\forall n \in \mathbb{N}, n \leq 2300 \Rightarrow (49|n \Leftrightarrow 50 \cdot (nextFifty(n) - n) = nextFifty(n))$

Proof: Let $n \in \mathbb{N}$, and assume $n \leq 2300$, WTS: $49|n \Leftrightarrow 50 \cdot (nextFifty(n) - n) = nextFifty(n)$
At first, we prove " $\Rightarrow$ " direction,
Assume $49|n$, which is: $\exists q \in Z, n = 49q$, WTS: $50 \cdot (nextFifty(n) - n) = nextFifty(n)$

$$
\begin{aligned}
\left\lceil \frac{50-1}{50} \cdot q \right\rceil &= q && \text{(since } n = 49q \leq 2300\text{, then } q < 50 \text{ and by Q2(a))}\\
49 \left\lceil \frac{49d}{50} \right\rceil &= 49d\\
50 \left\lceil \frac{49d}{50} \right\rceil - 49d &= \left\lceil \frac{49d}{50} \right\rceil\\
nextFifty(n) - n &= \frac{nextFifty(n)}{50} && \text{(since } n = 49d\text{)}\\
50 \cdot (nextFifty(n) - n) &= nextFifty(n)
\end{aligned}
$$

Therefore $49|n \Rightarrow 50 \cdot (nextFifty(n) - n) = nextFifty(n)$

Next, we prove " $\Leftarrow$ " direction,
Assume $50 \cdot (nextFifty(n) - n) = nextFifty(n)$, WTP: $49|n$, which is: $\exists q \in Z, n = 49d$
Take $q = \lceil \frac{n}{50} \rceil$ (since by the definition of ceiling function, then $q \in \mathbb{Z}$)

$$
\begin{aligned}
50 \cdot (nextFifty(n) - n) &= nextFifty(n) && \text{(by our assumption)}\\
50 \cdot nextFifty(n) - 50n &= nextFifty(n)\\
49 \cdot nextFifty(n) &= 50n\\
49 \left\lceil \frac{n}{50} \right\rceil &= n\\
n &= 49q
\end{aligned}
$$

Therefore, $49|n \Leftrightarrow 50 \cdot (nextFifty(n) - n) = nextFifty(n)$
Therefore, $\forall n \in \mathbb{N}, n \leq 2300 \Rightarrow (49|n \Leftrightarrow 50 \cdot (nextFifty(n) - n) = nextFifty(n))$

(b) Disprove: $\forall n \in \mathbb{N}, 49|n \Leftrightarrow 50 \cdot (nextFifty(n) - n) = nextFifty(n)$
It is equivalent to prove its negation: $\exists n \in \mathbb{N}, (49|n \wedge 50 \cdot (nextFifty(n) - n) \neq nextFifty(n)) \vee (50 \cdot (nextFifty(n) - n) = nextFifty(n) \wedge 49 \nmid n)$

Proof: Take $n = 2450 \in \mathbb{N}$, since $n = 49 \times 50$, then $49|n$ is true
Next, $nextFifty(n) = 50 \cdot \left\lceil \frac{n}{50} \right\rceil = 50 \left\lceil \frac{2450}{50} \right\rceil = 50 \times 49 = 2450$,
then $50 \cdot (nextFifty(n) - n) = 50 \cdot (2450 - 2450) = 0 \neq 2450 \neq nextFifty(n)$
Therefore, $50 \cdot (nextFifty(n) - n) \neq nextFifty(n))$
Therefore, $49|n \wedge 50 \cdot (nextFifty(n) - n) \neq nextFifty(n))$
Therefore, $(49|n \wedge 50 \cdot (nextFifty(n) - n) \neq nextFifty(n)) \vee (50 \cdot (nextFifty(n) - n) = nextFifty(n) \wedge 49 \nmid n)$ holds

Thus, we have proven its negation.

4 Functions.

(a) "*f is bounded*" in predicate logic: $\exists k \in \mathbb{R}, \forall x \in \mathbb{N}, f(x) \leq k$

(b) Define a predicate $Bounded(f) : \exists k \in \mathbb{R}, \forall x \in \mathbb{N}, f(x) \leq k$, where $f : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$.
Translate into predicate logic: $\forall f_1, f_2 : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}, Bounded(f_1) \wedge Bounded(f_2) \Rightarrow Bounded(f_1 + f_2)$
Proof: Let $f_1, f_2 : \mathbb{N} \rightarrow \mathbb{R}^{\geq 0}$, and assume $Bounded(f_1) \wedge Bounded(f_2)$,which is:
$\exists k_1 \in \mathbb{R}, \forall x \in \mathbb{N}, f_1(x) \leq k_1$
$\exists k_2 \in \mathbb{R}, \forall x \in \mathbb{N}, f_2(x) \leq k_2$
Let k_1, k_2 be such values, WTP: $Bounded(f_1 + f_2)$, which is: $\exists k_3 \in \mathbb{R}, \forall x \in \mathbb{N}, f_1(x) + f_2(x) \leq k_3$
Let $x \in \mathbb{N}$. Take $k_3 = k_1 + k_2$ (since $k_1, k_2 \in \mathbb{R}$, then $k_3 \in \mathbb{R}$)

$$\begin{aligned} f_1(x) &\leq k_1 && \text{(by our assumption)} \\ f_2(x) &\leq k_2 && \text{(by our assumption)} \\ f_1(x) + f_2(x) &\leq k_1 + k_2 \\ f_1(x) + f_2(x) &\leq k_3 \end{aligned}$$

Therefore, $\exists k_3 \in \mathbb{R}, \forall x \in \mathbb{N}, f_1(x) + f_2(x) \leq k_3$
Therefore, if f_1 and f_2 is bounded, then $f_1 + f_2$ is bounded.

(c) Prove: $\forall f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}, Bounded(f_1 + f_2) \Rightarrow Bounded(f_1) \wedge Bounded(f_2)$

Proof: Let $f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}$, and assume $Bounded(f_1 + f_2)$, which is: $\exists k_3 \in \mathbb{R}, \forall x \in \mathbb{N}, f_1(x) + f_2(x) \leq k_3$, let k_3 be such value, WTP:$Bounded(f_1) \wedge Bounded(f_2)$,which is:
$\exists k_1 \in \mathbb{R}, \forall x \in \mathbb{N}, f_1(x) \leq k_1$
$\exists k_2 \in \mathbb{R}, \forall x \in \mathbb{N}, f_2(x) \leq k_2$
Let $x \in \mathbb{N}$, and take $k_1 = k_2 = k_3$ (since $k_3 \in \mathbb{R}$, then $k_1, k_2 \in \mathbb{R}$)

$$\begin{aligned}
f_1(x) + f_2(x) &\leq k_3 && \text{(by our assumption)} \\
f_1(x) &\leq k_3 - f_2(x) \\
f_1(x) &\leq k_3 && \text{(since } f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}\text{, then } \forall x \in \mathbb{N}, f_2(x) \geq 0) \\
f_1(x) &\leq k_1 && \text{(since } k_3 = k_1)
\end{aligned}$$

Therefore, $\exists k_1 \in \mathbb{R}, \forall x \in \mathbb{N}, f_1(x) \leq k_1$
Similarly,

$$\begin{aligned}
f_1(x) + f_2(x) &\leq k_3 && \text{(by our assumption)} \\
f_2(x) &\leq k_3 - f_1(x) \\
f_2(x) &\leq k_3 && \text{(since } f_1 : \mathbb{N} \to \mathbb{R}^{\geq 0}\text{, then } \forall x \in \mathbb{N}, f_1(x) \geq 0) \\
f_2(x) &\leq k_2 && \text{(since } k_3 = k_2)
\end{aligned}$$

Therefore, $\exists k_2 \in \mathbb{R}, \forall x \in \mathbb{N}, f_2(x) \leq k_2$
Therefore, $Bounded(f_1) \wedge Bounded(f_2)$ holds

Thus, $\forall f_1, f_2 : \mathbb{N} \to \mathbb{R}^{\geq 0}, Bounded(f_1 + f_2) \Rightarrow Bounded(f_1) \wedge Bounded(f_2)$