# CSC165H1 Problem Set 3

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## 1 Special numbers

**Proof**: we define a predicate $P(n) : F_n - 2 = \prod_{i=0}^{n-1} F_i$, where $n \in \mathbb{N}$
WTP: $\forall n \in \mathbb{N}, P(n)$
**Base Case**: when $n = 0$

$$\begin{aligned}
F_0 - 2 &= 2^{2^0} + 1 - 2 && \text{(by the definition of } F_n\text{)} \\
&= 1 \\
&= \prod_{i=0}^{-1} F_i && \text{(since by the course notes, when } j > k,\ \textstyle\prod_{i=j}^{k} = 1\text{)}
\end{aligned}$$

Therefore $P(0)$ holds

**Inductive Step**: Let $k \in \mathbb{N}$ and assume $P(k)$, which is: $F_k - 2 = \prod_{i=0}^{k-1} F_i$
WTS: $P(k+1)$, which is: $F_{k+1} - 2 = \prod_{i=0}^{k} F_i$

$$\begin{aligned}
\prod_{i=0}^{k} F_i &= \prod_{i=0}^{k-1} F_i \cdot F_k \\
&= (F_k - 2)F_k && \text{(by induction hypothesis)} \\
&= (2^{2^k} + 1 - 2)(2^{2^k} + 1) && \text{(by the definition of } F_n\text{)} \\
&= (2^{2^k} - 1)(2^{2^k} + 1) \\
&= (2^{2^k})^2 - 1 \\
&= 2^{2^{k+1}} - 1 && \text{(by hint, } \forall n \in \mathbb{N},\ 2^{2^{n+1}} = (2^{2^n})^2\text{)} \\
&= 2^{2^{k+1}} + 1 - 2 \\
&= F_{k+1} - 2
\end{aligned}$$

Therefore, $P(k+1)$ is True
Therefore $\forall n \in \mathbb{N}$, $F_n - 2 = \prod_{i=0}^{n-1} F_i$

## 2 Sequences

(a) $a_0 = 1$, $a_1 = \frac{1}{\frac{1}{a_0}+1} = \frac{1}{2}$, $a_2 = \frac{1}{\frac{1}{a_1}+1} = \frac{1}{3}$, $a_3 = \frac{1}{\frac{1}{a_2}+1} = \frac{1}{4}$

(b) **Proof**: we define a predicate $P(n) : a_n = \frac{1}{n+1}$, where $n \in \mathbb{N}$
WTP: $\forall n \in \mathbb{N}$, $P(n)$
**<u>Base Case</u>**: when $n = 0$

$$\begin{aligned} a_0 &= 1 && \text{(by the definition of sequences)} \\ &= \frac{1}{0+1} && \text{(left side equals to right side)} \end{aligned}$$

Therefore, $P(0)$ is True

**<u>Inductive Step</u>**: Let $k \in \mathbb{N}$, and assume $P(k)$ is true, which is: $a_k = \frac{1}{k+1}$
WTS: $P(k+1)$, which is: $a_{k+1} = \frac{1}{(k+1)+1}$

$$\begin{aligned} a_{k+1} &= \frac{1}{\frac{1}{a_k}+1} && \text{(by the definition of sequences)} \\ &= \frac{1}{\frac{1}{\frac{1}{k+1}}+1} && \text{(by induction hypothesis)} \\ &= \frac{1}{(k+1)+1} \end{aligned}$$

Therefore, $P(k+1)$ holds
Therefore $\forall n \in \mathbb{N}$, $a_n = \frac{1}{n+1}$

(c) $a_{2,0} = 2$, $a_{2,1} = \frac{2}{\frac{1}{a_{2,0}}+1} = \frac{4}{3}$, $a_{2,2} = \frac{2}{\frac{1}{a_{2,1}}+1} = \frac{8}{7}$, $a_{2,3} = \frac{2}{\frac{1}{a_{2,2}}+1} = \frac{16}{15}$
$a_{3,0} = 3$, $a_{3,1} = \frac{3}{\frac{1}{a_{3,0}}+1} = \frac{9}{4}$, $a_{3,2} = \frac{3}{\frac{1}{a_{3,1}}+1} = \frac{27}{13}$, $a_{3,3} = \frac{3}{\frac{1}{a_{3,2}}+1} = \frac{81}{40}$

(d) **Proof**: Let $k \in \mathbb{N}$, and assume $k > 1$
we define a predicate $P(n) : a_{k,n} = \frac{k^{n+1}-k^{n+2}}{1-k^{n+1}}$, where $n \in \mathbb{N}$
WTP: $\forall n \in \mathbb{N}$, $P(n)$
**<u>Base Case</u>**: when $n = 0$

left side:

$$a_{k,0} = k \qquad \text{(by the definition of sequences)}$$

right side:

$$\begin{aligned} \frac{k^{0+1}-k^{0+2}}{1-k^{0+1}} &= \frac{k-k^2}{1-k} \\ &= \frac{k(1-k)}{1-k} \\ &= k && \text{(since } k > 1 \text{, then } 1-k \neq 0) \end{aligned}$$

Therefore, left side equals to right side
Therefore, $P(0)$ holds

<u>**Inductive Step**</u>: Let $j \in \mathbb{N}$, and assume $P(j)$ is True, which is: $a_{k,j} = \frac{k^{j+1}-k^{j+2}}{1-k^{j+1}}$
WTS: $P(j+1)$ holds, which is: $a_{k,j+1} = \frac{k^{j+2}-k^{j+3}}{1-k^{j+2}}$

$$\begin{aligned} a_{k,j+1} &= \frac{k}{\frac{1}{a_{k,j}}+1} && \text{(by the definition of sequences)} \\ &= \frac{k}{\frac{1}{\frac{k^{j+1}-k^{j+2}}{1-k^{j+1}}}+1} && \text{(by induction hypothesis)} \\ &= \frac{k}{\frac{1-k^{j+1}+k^{j+1}-k^{j+2}}{k^{j+1}-k^{j+2}}} \\ &= \frac{k^{j+2}-k^{j+3}}{1-k^{j+2}} \end{aligned}$$

Therefore, $P(j+1)$ is True
This completes the proof of the inductive step and thus the proof

# 3 Properties of Asymptotic notation

(a) **Proof:** Translate into predicate logic: $\forall f : \mathbb{N} \to \mathbb{R}^{\geq 0},\ f \in \mathcal{O}(n) \Rightarrow Sum_f \in \mathcal{O}(n^2)$
Let $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$, and assume $f \in \mathcal{O}(n)$, which is: $\exists n_0, c \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow f \leq cn$
WTS: $Sum_f \in \mathcal{O}(n^2)$, which is: $\exists n_1, c_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow Sum_f \leq c_1 n^2$

Let $n_0, c$ be such values, and take $n_1 = \lceil n_0 \rceil$ (since $n_0 \in \mathbb{R}^+$, then $n_1 \in \mathbb{R}^+$), take $c_1 = \sum_{i=0}^{n_1-1} f(i) + (\frac{n_1 c}{2} + c)$ (since $f : \mathbb{N} \to \mathbb{R}^{\geq 0}$ and $c \in \mathbb{R}^+$, then $c_1 \in \mathbb{R}^+$)
Let $n \in \mathbb{N}$, and assume $n \geq n_1$, WTP: $Sum_f \leq c_1 n^2$

$$
\begin{aligned}
Sum_f = \sum_{i=0}^{n} f(i) &= \sum_{i=0}^{n_1-1} f(i) + \sum_{i=n_1}^{n} f(i) \\
&\leq \sum_{i=0}^{n_1-1} f(i) + \sum_{i=n_1}^{n} ci \qquad \text{(since } n_1 = \lceil n_0 \rceil \geq n_0\text{, and by our assumption)} \\
&= \sum_{i=0}^{n_1-1} f(i) + c\sum_{i=n_1}^{n} i \\
&= \sum_{i=0}^{n_1-1} f(i) + c\frac{(n_1+n)(n-n_1+1)}{2} \\
&= \sum_{i=0}^{n_1-1} f(i) + c\frac{n_1 n - n_1^2 + n_1 + n^2 - n_1 n + n}{2} \\
&= \sum_{i=0}^{n_1-1} f(i) + c\frac{n - n_1^2 + n_1 + n^2}{2} \\
&\leq \sum_{i=0}^{n_1-1} f(i) + c\frac{n + n_1 + n^2}{2} \qquad \text{(since } n_1^2 \geq 0\text{)} \\
&= \sum_{i=0}^{n_1-1} f(i) + \frac{c}{2}n_1 + \frac{c}{2}n + \frac{c}{2}n^2 \\
&\leq \left[\sum_{i=0}^{n_1-1} f(i)\right] n^2 + \frac{c}{2}n_1 n^2 + \frac{c}{2}n^2 + \frac{c}{2}n^2 \\
&\qquad \text{(since } n \geq n_1 = \lceil n_0 \rceil\text{, then } n \geq 1\text{)} \\
&= \left[\sum_{i=0}^{n_1-1} f(i) + (\frac{n_1 c}{2} + c)\right] n^2 \\
&= c_1 n^2
\end{aligned}
$$

Therefore, $Sum_f \leq c_1 n^2$
Therefore, $\forall f : \mathbb{N} \to \mathbb{R}^{\geq 0},\ f \in \mathcal{O}(n) \Rightarrow Sum_f \in \mathcal{O}(n^2)$

(b) **Proof:** we define a predicate $P(n) : \sum_{i=1}^{2^n} \frac{1}{i} \geq \frac{n}{2}$, where $n \in \mathbb{N}$
WTP: $\forall n \in \mathbb{N}, P(n)$
**Base Case**: when $n = 0$

$$\sum_{i=1}^{2^0} \frac{1}{i} = 1 \geq 0 = \frac{0}{2}$$

Therefore, $P(0)$ holds

**Inductive Step**: Let $k \in \mathbb{N}$, and assume $P(k)$ is True, which is: $\sum_{i=1}^{2^k} \frac{1}{i} \geq \frac{k}{2}$
WTS: $P(k+1)$, which is: $\sum_{i=1}^{2^{k+1}} \frac{1}{i} \geq \frac{k+1}{2}$

$$\begin{aligned}
\sum_{i=1}^{2^{k+1}} \frac{1}{i} &= \sum_{i=1}^{2^k} \frac{1}{i} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} \\
&\geq \frac{k}{2} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{i} && \text{(by induction hypothesis)} \\
&\geq \frac{k}{2} + \sum_{i=2^k+1}^{2^{k+1}} \frac{1}{2^{k+1}} && \text{(since } i \leq 2^{k+1}\text{, then } \tfrac{1}{i} \geq \tfrac{1}{2^{k+1}}\text{)} \\
&= \frac{k}{2} + \frac{1}{2^{k+1}} \sum_{i=2^k+1}^{2^{k+1}} 1 \\
&= \frac{k}{2} + \frac{1}{2^{k+1}}(2^{k+1} - 2^k) \\
&= \frac{k}{2} + 1 - \frac{1}{2} \\
&= \frac{k+1}{2}
\end{aligned}$$

Therefore, $P(k+1)$ is True
Therefore, $\forall n \in \mathbb{N}, \sum_{i=1}^{2^n} \frac{1}{i} \geq \frac{n}{2}$

(c) **Proof:** we want to disprove the claim, then it is equivalent to prove its negation
Translate into predicate logic: $\exists f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}, f(n) \in \mathcal{O}(g(n)) \wedge Sum_f(n) \notin \mathcal{O}(n \cdot g(n))$
Let $f(n) = \frac{1}{n+1}$ and $g(n) = \frac{1}{n+1}$, First we will prove that $f(n) \in \mathcal{O}(g(n))$

WTP: $f(n) \in \mathcal{O}(g(n))$, which is: $\exists n_0, c \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow \frac{1}{n+1} \leq c\frac{1}{n+1}$
Let $c = 1$, and $n_0 = 1$, let $n \in \mathbb{N}$, and assume $n \geq n_0$, WTS: $\frac{1}{n+1} \leq c\frac{1}{n+1}$

$$\begin{aligned}
1 &\leq 1 \\
\frac{1}{n+1} &\leq \frac{1}{n+1} && \text{(since by our assumption, } n \geq n_0 = 1\text{, then } n+1 > 0\text{)} \\
&= 1 \cdot \frac{1}{n+1} \\
&= c\frac{1}{n+1}
\end{aligned}$$

Therefore, $\frac{1}{n+1} \leq c\frac{1}{n+1}$
Therefore, $f(n) \in \mathcal{O}(g(n))$

Next, we will prove $Sum_f(n) \notin \mathcal{O}(n \cdot g(n))$ by contradiction
Assume $Sum_f(n) \in \mathcal{O}(n \cdot g(n))$, which is: $\exists n_1, c_1 \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n_1 \Rightarrow \sum_{i=0}^{n} \frac{1}{i+1} \leq c_1 n \cdot \frac{1}{n+1}$

Let $n_1, c_1$ be such values. Let $k \in \mathbb{N}$, and let $n = 2^k - 1$
Take $k = \lceil 4c_1 + \log_2(n_1+1) \rceil$, since $n = 2^k - 1 \geq n_1$, then by our assumption, we have $\sum_{i=0}^{n} \frac{1}{i+1} \leq c_1 n \cdot \frac{1}{n+1}$

$$\begin{aligned}
\sum_{i=0}^{2^k-1} \frac{1}{i+1} &\leq c_1 \cdot \frac{2^k-1}{2^k} && \text{(since } n = 2^k - 1\text{, and } n \geq n_1\text{)} \\
\sum_{i=1}^{2^k} \frac{1}{i} &\leq c_1 \cdot \frac{2^k-1}{2^k} \\
\frac{k}{2} \leq \sum_{i=1}^{2^k} \frac{1}{i} &\leq c_1 \cdot \frac{2^k-1}{2^k} && \text{(by part b)} \\
\frac{k}{2} &\leq c_1 \cdot \frac{2^k-1}{2^k} \\
c_1 &\geq \frac{k}{2} \cdot \frac{2^k}{2^k-1} && \text{(since } k = \lceil 4c_1 + \log_2(n_1+1) \rceil\text{, then } 2^k - 1 \geq 0\text{)} \\
c_1 &\geq \frac{k}{2} && \text{(since } \tfrac{2^k}{2^k-1} > 1\text{)} \\
c_1 &\geq \frac{\lceil 4c_1 + \log_2(n_1+1) \rceil}{2} && \text{(since } k = \lceil 4c_1 + \log_2(n_1+1) \rceil\text{)} \\
c_1 &\geq \frac{4c_1 + \log_2(n_1+1)}{2} \\
c_1 &\geq 2c_1 + \frac{\log_2(n_1+1)}{2} \\
c_1 &\geq 2c_1 && \text{(since } n_1 \in \mathbb{R}^+\text{, then } \tfrac{\log_2(n_1+1)}{2} > 0\text{)}
\end{aligned}$$

Since $c_1 \in \mathbb{R}^+$, then $c_1 \neq 0$, therefore we have a contradiction
Therefore, $Sum_f(n) \notin \mathcal{O}(n \cdot g(n))$
Therefore, $\exists f, g : \mathbb{N} \to \mathbb{R}^{\geq 0}, f(n) \in \mathcal{O}(g(n)) \wedge Sum_f(n) \notin \mathcal{O}(n \cdot g(n))$
Thus, the original claim is False.