UNIVERSITY OF TORONTO
Faculty of Arts and Science

Midterm 1, Version 1
CSC165H1S

Date: Thursday February 9, 6:10-7:00pm
Duration: 50 minutes
Instructor(s): David Liu, Toniann Pitassi

No Aids Allowed

Name:

Student Number:

Please read the following guidelines carefully!

- **Please write your name on both the front and back of this exam**.
- This examination has **4** questions. There are a total of **8 pages, DOUBLE-SIDED**.
- Answer questions clearly and completely, with justifications unless explicitly asked not to.
- Unless stated otherwise, your formulas can use *only* the propositional connectives and quantifiers we have seen in class, arithmetic operators (like $+$, $\times$, and exponentiation), comparison operators (like $=$ and $>$), and the divisibility and $Prime$ predicates. You may not define your own sets or predicates unless asked to do so.
- All formulas must have negations applied directly to propositional variables or predicates (e.g., $\neg Prime(n)$). You do *not* need to show your work for computing negations.
- In your proofs, you may always use definitions of predicates. You may *not* use any external facts about rates of growth, divisibility, primes, or greatest common divisor unless you prove them, or they are given to you in the question.
- You may **not** use induction for your proofs on this midterm.

Take a deep breath.

This is your chance to show us
How much you've learned.
We **WANT** to give you the credit
That you've earned.
A number does not define you.

Good luck!

1. **[6 marks] Statements in logic.**

   (a) **[3 marks]** Write the truth table for the following formula. No rough work is required.

$$\big((p \Leftrightarrow q) \wedge r\big) \Rightarrow \neg p$$

Hint: use vacuous truth to quickly find some rows where the formula is true.

Solution

p	q	r	$\big((p \Leftrightarrow q) \wedge r\big) \Rightarrow \neg p$
F	F	F	T
F	F	T	T
F	T	F	T
F	T	T	T
T	F	F	T
T	F	T	T
T	T	F	T
T	T	T	F

   (b) **[3 marks]** Consider the pair of statements:

(1) $\forall n \in \mathbb{N},\ P(n) \wedge Q(n)$ (2) $\forall n \in \mathbb{N},\ P(n) \Rightarrow Q(n)$

Define the predicates P and Q with domain $\mathbb{N}$ so that one of these statements is true and the other one false. Note that you're only defining the predicates *once*: the two statements must use the same definitions for P and Q.

Also, briefly explain which statement is true and which one is false, and why; no formal proofs necessary.

Solution

Let $P(n)$ be the predicate "$n > 2$" and $Q(n)$ be the predicate "$n > 1$".

The first statement becomes $\forall n \in \mathbb{N},\ n > 2 \wedge n > 1$, which is false (0 is a counter-example). The second statement becomes $\forall n \in \mathbb{N},\ n > 2 \Rightarrow n > 1$, which is true: every number that is greater than 2 is also greater than 1.

2. **[7 marks] Translating statements**.

A **power of two** is a natural number that can be written in the form 2^n, where n is some natural number. A **Mersenne prime** is a natural number that is prime and one less than some power of two. For example, $2^2 - 1 = 3$ and $2^5 - 1 = 31$ are Mersenne primes.

Express each of the following statements using predicate logic. No justification is required.

Note: please review the instructions on the midterm's front page for our expectations in this question. In particular, you may *not* define any helper predicates or sets.

(a) **[3 marks]** 165165 is not a Mersenne prime.
Hint: it may be easier to think about how to express "165165 *is* a Mersenne prime" first.

Solution

$\neg Prime(165165) \lor \big(\forall n \in \mathbb{N},\ 165165 \neq 2^n - 1\big).$

(b) **[4 marks]** There are infinitely many Mersenne primes.

Solution

$\forall n_0 \in \mathbb{N},\ \exists n \in \mathbb{N},\ n > n_0 \land Prime(n) \land \big(\exists k \in \mathbb{N},\ n = 2^k - 1\big)\Big)$

3. **[6 marks] Proofs (inequalities).** Consider the following statement: "There exists a positive real number y such that for every positive real number x, $x^2 + 165 < 2y$."

   (a) [**1 mark**] Translate the above statement into predicate logic. Use the symbol $\mathbb{R}^+$ to denote the set of positive real numbers.

Solution

$\exists y \in \mathbb{R}^+,\ \forall x \in \mathbb{R}^+,\ x^2 + 165 < 2y.$

   (b) [**1 mark**] Write the negation of this statement, fully-simplified so that all negation symbols are applied directly to predicates. You can simplify $\neg(a < b)$ to $a \geq b$.

Solution

$\forall y \in \mathbb{R}^+,\ \exists x \in \mathbb{R}^+,\ x^2 + 165 \geq 2y.$

   (c) [**4 marks**] Disprove the original statement by proving its negation. In your proof, any chains of calculations must follow a top-down order; don't start with the inequality you're trying to prove!
   Write any rough work or intuition in the **Discussion** box, and write your formal proof in the **Proof** box. Your rough work/intuition will only be looked at if your proof is not completely correct.

Solution

Proof. Let $y \in \mathbb{R}^+$. Let $x = \sqrt{2y}$. We want to prove that $x^2 + 165 \geq 2y$.
We start with our definition of x:

$$x = \sqrt{2y}$$
$$x^2 = 2y$$
$$x^2 + 165 \geq 2y \qquad \text{(made the left side bigger)}$$

$\square$

4. **[5 marks] Proofs (number theory).** Consider the following statement: "Every integer that is divisible by 100 is divisible by 2 and by 5."

(a) **[1 mark]** Translate the above statement into predicate logic.

Solution

$\forall x \in \mathbb{Z},\ 100 \mid x \Rightarrow 2 \mid x \wedge 5 \mid x$

(b) **[4 marks]** Prove the above statement using the definition of divisibility:

$$x \mid y : \quad \exists k \in \mathbb{Z},\ y = kx$$

Do not use any external facts about divisibility.

Write any rough work or intuition in the **Discussion** box, and write your formal proof in the **Proof** box. Your rough work/intuition will only be looked at if your proof is not completely correct.

Solution

Proof. Let $x \in \mathbb{Z}$. Assume that $100 \mid x$; that is, assume there exists $k_1 \in \mathbb{Z}$ such that $x = 100k_1$. We want to prove that $2 \mid x$ and $5 \mid x$.

Part 1: proving that $2 \mid x$, i.e., $\exists k_2 \in \mathbb{Z},\ x = 2k_2$.
Let $k_2 = 50k_1$. Then by our assumption, $x = 100k_1 = 2(50k_1) = 2k_2$.

Part 2: proving that $2 \mid x$, i.e., $\exists k_3 \in \mathbb{Z},\ x = 5k_3$.
Let $k_3 = 20k_1$. Then by our assumption, $x = 100k_1 = 5(20k_1) = 5k_3$. $\square$

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly *at the location of the original question.*

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Name:

Question	Grade	Out of
Q1		6
Q2		7
Q3		6
Q4		5
Total		24