UNIVERSITY OF TORONTO

Faculty of Arts and Science

Midterm 1, Version 2
CSC165H1S

Date: Friday February 10, 12:10-1:00pm
Duration: 50 minutes
Instructor(s): David Liu, Toniann Pitassi

No Aids Allowed

Name:

Student Number:

Please read the following guidelines carefully!

- **Please write your name on both the front and back of this exam**.
- This examination has **4** questions. There are a total of **8 pages, DOUBLE-SIDED**.
- Answer questions clearly and completely, with justifications unless explicitly asked not to.
- Unless stated otherwise, your formulas can use *only* the propositional connectives and quantifiers we have seen in class, arithmetic operators (like $+$, $\times$, and exponentiation), comparison operators (like $=$ and $>$), and the divisibility and $Prime$ predicates. You may not define your own sets or predicates unless asked to do so.
- All formulas must have negations applied directly to propositional variables or predicates (e.g., $\neg Prime(n)$). You do *not* need to show your work for computing negations.
- In your proofs, you may always use definitions of predicates. You may *not* use any external facts about rates of growth, divisibility, primes, or greatest common divisor unless you prove them, or they are given to you in the question.
- You may **not** use induction for your proofs on this midterm.

Take a deep breath.

This is your chance to show us

How much you've learned.

We **WANT** to give you the credit

That you've earned.

A number does not define you.

Good luck!

1. **[6 marks] Statements in logic.**

   (a) [**3 marks**] Write the truth table for the following formula. No rough work is required.

$$\big((p \Leftrightarrow q) \land r\big) \Rightarrow \neg r$$

Hint: use vacuous truth to quickly find some rows where the formula is true.

<u>**Solution**</u>

p	q	r	$\big((p \Leftrightarrow q) \land r\big) \Rightarrow \neg r$
F	F	F	T
F	F	T	F
F	T	F	T
F	T	T	T
T	F	F	T
T	F	T	T
T	T	F	T
T	T	T	F

   (b) [**3 marks**] Consider the pair of statements:

(1) $\forall n \in \mathbb{N},\ P(n) \Rightarrow Q(n)$ (2) $\forall n \in \mathbb{N},\ P(n) \Leftrightarrow Q(n)$

Define the predicates P and Q with domain $\mathbb{N}$ so that one of these statements is true and the other one false. Note that you're only defining the predicates *once*: the two statements must use the same definitions for P and Q.

Also, briefly explain which statement is true and which one is false, and why; no formal proofs necessary.

<u>**Solution**</u>

Let $P(n)$ be the predicate "$n > 2$" and $Q(n)$ be the predicate "$n > 1$".

The first statement becomes $\forall n \in \mathbb{N},\ n > 2 \Rightarrow n > 1$, which is true: every number that is greater than 2 is also greater than 1. The second statement becomes $\forall n \in \mathbb{N},\ n > 2 \Leftrightarrow n > 1$, which is false: not every number that is greater than 1 is also greater than 2 ($n = 1$ is a counter-example).

2. **[7 marks] Translating statements**.

A **semiprime** is a natural number that can be written as the product of two (possibly equal) prime numbers. For example, $6 = 2 \cdot 3$ and $49 = 7 \cdot 7$ are semiprimes.

Express each of the following statements using predicate logic. No justification is required.

Note: please review the instructions on the midterm's front page for our expectations in this question. In particular, you may *not* define any helper predicates or sets.

(a) **[4 marks]** There is a semiprime greater than 165.

Solution

$\exists n \in \mathbb{N},\ n > 165 \wedge \big(\exists p, q \in \mathbb{N},\ Prime(p) \wedge Prime(q) \wedge n = p \cdot q\big)$

(b) **[3 marks]** There are no semiprimes.
Hint: it may be easier to first express the negation of this statement.

Solution

$\forall n, p, q \in \mathbb{N}, \neg Prime(p) \vee \neg Prime(q) \vee n \neq p \cdot q.$

3. **[6 marks] Proofs (inequalities).** Consider the following statement: "For every real number x greater than or equal to 3, $x^3 - x^2 - 10 > 2x$."

(a) [**1 mark**] Translate the above statement into predicate logic.

Solution

$\forall x \in \mathbb{R},\ x \geq 3 \Rightarrow x^3 - x^2 - 10 > 2x.$

(b) [**5 marks**] Prove the above statement. In your proof, any chains of calculations must follow a top-down order; don't start with the inequality you're trying to prove!

Hint: try starting with the expression $x^3 - x^2 - 2x$ and factor.

Write any rough work or intuition in the **Discussion** box, and write your formal proof in the **Proof** box. Your rough work/intuition will only be graded if your proof is not completely correct.

Solution

Proof. Let $x \in \mathbb{R}$, and assume that $x \geq 3$. We want to prove that $x^3 - x^2 - 10 > 2x$.
First, we start with the expression $x^3 - x^2 - 2x = x(x+1)(x-2)$.
Since $x \geq 3$, we know that $x+4 \geq 4$ and $x-2 \geq 1$. Multiplying these three inequalities (including the assumption $x \geq 3$), we get:

$$\begin{aligned} x^3 - x^2 - 2x &= x(x+1)(x-2) \\ &\geq 3 \times 4 \times 1 \\ &= 12 \\ &> 10 \end{aligned}$$

So then $x^3 - x^2 - 2x > 10$. Rearranging this inequality gives $x^3 - x^2 - 10 > 2x$. $\square$

4. **[5 marks] Proofs (number theory).** Consider the following statement: "There exists a positive integer x such that for every integer y, if $y \neq 0$ then $x \nmid x + y$."

   (a) [**1 mark**] Translate the above statement into predicate logic. Use $\mathbb{Z}^+$ to represent the set of positive integers, and $\mathbb{Z}$ to represent the set of integers.

Solution

$\exists x \in \mathbb{Z}^+,\ \forall y \in \mathbb{Z},\ y \neq 0 \Rightarrow x \nmid x + y.$

   (b) [**1 mark**] Write the negation of this statement, fully-simplified so that all negation symbols are applied directly to predicates.

Solution

$\forall x \in \mathbb{Z}^+,\ \exists y \in \mathbb{Z},\ y \neq 0 \wedge x \mid x + y.$

   (c) [**3 marks**] Disprove the original statement by proving its negation, using the definition of divisibility:

$$x \mid y : \quad \exists k \in \mathbb{Z},\ y = kx$$

   Do not use any external facts about divisibility.

   Write any rough work or intuition in the **Discussion** box, and write your formal proof in the **Proof** box. Your rough work/intuition will only be looked at if your proof is not completely correct.

Solution

Proof. Let $x \in \mathbb{Z}^+$. Let $y = x$. We want to prove that $y \neq 0$ and that $x \mid x + y$.
First, since $y = x$ and x is positive, $y \neq 0$.
Second, let $k = 2$. Then $x + y = x + x = 2x = kx$, and so $x \mid x + y$. □

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly *at the location of the original question.*

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Name:

Question	Grade	Out of
Q1		6
Q2		7
Q3		6
Q4		5
Total		24