

UNIVERSITY OF TORONTO  
Faculty of Arts and Science

Midterm 1, Version 3  
CSC165H1S

Date: Friday February 10, 2:10-3:00pm

Duration: 50 minutes

Instructor(s): David Liu, Toniann Pitassi

No Aids Allowed

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Name:

Student Number:

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Please read the following guidelines carefully!

- Please write your name on both the front and back of this exam.
  - This examination has 4 questions. There are a total of 8 pages, **DOUBLE-SIDED**.
  - Answer questions clearly and completely, with justifications unless explicitly asked not to.
  - Unless stated otherwise, your formulas can use *only* the propositional connectives and quantifiers we have seen in class, arithmetic operators (like  $+$ ,  $\times$ , and exponentiation), comparison operators (like  $=$  and  $>$ ), and the divisibility and *Prime* predicates. You may not define your own sets or predicates unless asked to do so.
  - All formulas must have negations applied directly to propositional variables or predicates (e.g.,  $\neg \text{Prime}(n)$ ). You do *not* need to show your work for computing negations.
  - In your proofs, you may always use definitions of predicates. You may *not* use any external facts about rates of growth, divisibility, primes, or greatest common divisor unless you prove them, or they are given to you in the question.
  - You may **not** use induction for your proofs on this midterm.
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Take a deep breath.

This is your chance to show us

How much you've learned.

We **WANT** to give you the credit

That you've earned.

A number does not define you.

Good luck!

1. [6 marks] **Statements in logic.**

- (a) [3 marks] Write the truth table for the following formula. No rough work is required.

$$((p \Leftrightarrow q) \wedge \neg r) \Rightarrow r$$

**Hint:** use vacuous truth to quickly find some rows where the formula is true.

- (b) [3 marks] Consider the pair of statements:

$$(1) \quad (\exists n \in \mathbb{N}, P(n)) \wedge (\exists m \in \mathbb{N}, Q(m))$$

$$(2) \quad \exists n \in \mathbb{N}, P(n) \wedge Q(n)$$

Define the predicates  $P$  and  $Q$  with domain  $\mathbb{N}$  so that one of these statements is true and the other one false. Note that you're only defining the predicates *once*: the two statements must use the same definitions for  $P$  and  $Q$ .

Also, briefly explain which statement is true and which one is false, and why; no formal proofs necessary.

2. [7 marks] **Translating statements.**

Let  $x \in \mathbb{N}$ . We say that  $x$  is a **twin prime** if and only if both  $x$  and  $x + 2$  are prime. For example, 5 is a twin prime, because both 5 and 7 are prime. Note that 7 is *not* a twin prime, since 9 is not prime.

Express each of the following statements using predicate logic. No justification is required.

Note: please review the instructions on the midterm's front page for our expectations in this question. In particular, you may *not* define any helper predicates or sets.

(a) For every twin prime  $p$ ,  $2p + 1$  is also a twin prime.

(b) There are infinitely many numbers that are *not* a twin prime.

**Hint:** it may be easier to first express “ $n$  is not a twin prime.”

3. [6 marks] **Proofs (inequalities).** Consider the following statement: “There exists a real number  $x$  such that  $x$  is less than 3 and for every real number  $y$ ,  $x + y^2 > 25$ .”

(a) [1 mark] Translate the above statement into predicate logic.

(b) [1 mark] Write the negation of this statement, fully-simplified so that all negation symbols are applied directly to predicates. You can simplify  $\neg(a > b)$  to  $a \leq b$  and  $\neg(a < b)$  to  $a \geq b$ .

(c) [4 marks] Disprove the original statement by proving its negation. In your proof, any chains of calculations must follow a top-down order; don't start with the inequality you're trying to prove!

**Hint:** use the fact that  $\neg p \vee q$  is equivalent to  $p \Rightarrow q$  to rewrite the negation into an implication.

Write any rough work or intuition in the **Discussion** box, and write your formal proof in the **Proof** box. Your rough work/intuition will only be graded if your proof is not completely correct.

*Discussion.*

*Proof.*

4. [5 marks] **Proofs (number theory)**. Consider the following statement: “For all two integers  $x$  and  $y$ , if  $x$  and  $y$  are both divisible by 3, then  $x^2 + 2y^2$  is divisible by 3.”

(a) [1 mark] Translate the above statement into predicate logic.

(b) [4 marks] Prove the above statement using the definition of divisibility:

$$x \mid y : \exists k \in \mathbb{Z}, y = kx$$

Do not use any external facts about divisibility.

Write any rough work or intuition in the **Discussion** box, and write your formal proof in the **Proof** box. Your rough work/intuition will only be looked at if your proof is not completely correct.

*Discussion.*

*Proof.*

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly *at the location of the original question*.

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Name:

Question	Grade	Out of
Q1		6
Q2		7
Q3		6
Q4		5
<b>Total</b>		24