UNIVERSITY OF TORONTO
Faculty of Arts and Science

Midterm 1, Version 3
CSC165H1S

Date: Friday February 10, 2:10-3:00pm
Duration: 50 minutes
Instructor(s): David Liu, Toniann Pitassi

No Aids Allowed

Name:

Student Number:

Please read the following guidelines carefully!

- **Please write your name on both the front and back of this exam**.
- This examination has **4** questions. There are a total of **8 pages, DOUBLE-SIDED**.
- Answer questions clearly and completely, with justifications unless explicitly asked not to.
- Unless stated otherwise, your formulas can use *only* the propositional connectives and quantifiers we have seen in class, arithmetic operators (like $+$, $\times$, and exponentiation), comparison operators (like $=$ and $>$), and the divisibility and *Prime* predicates. You may not define your own sets or predicates unless asked to do so.
- All formulas must have negations applied directly to propositional variables or predicates (e.g., $\neg Prime(n)$). You do *not* need to show your work for computing negations.
- In your proofs, you may always use definitions of predicates. You may *not* use any external facts about rates of growth, divisibility, primes, or greatest common divisor unless you prove them, or they are given to you in the question.
- You may **not** use induction for your proofs on this midterm.

Take a deep breath.

This is your chance to show us
How much you've learned.
We **WANT** to give you the credit
That you've earned.
A number does not define you.

Good luck!

1. **[6 marks] Statements in logic.**

(a) [**3 marks**] Write the truth table for the following formula. No rough work is required.

$$\big((p \Leftrightarrow q) \wedge \neg r\big) \Rightarrow r$$

Hint: use vacuous truth to quickly find some rows where the formula is true.

<u>Solution</u>

p	q	r	$\big((p \Leftrightarrow q) \wedge \neg r\big) \Rightarrow r$
F	F	F	F
F	F	T	T
F	T	F	T
F	T	T	T
T	F	F	T
T	F	T	T
T	T	F	F
T	T	T	T

(b) [**3 marks**] Consider the pair of statements:

(1) $\big(\exists n \in \mathbb{N},\ P(n)\big) \wedge \big(\exists m \in \mathbb{N}, Q(m)\big)$ (2) $\exists n \in \mathbb{N},\ P(n) \wedge Q(n)$

Define the predicates P and Q with domain $\mathbb{N}$ so that one of these statements is true and the other one false. Note that you're only defining the predicates *once*: the two statements must use the same definitions for P and Q.

Also, briefly explain which statement is true and which one is false, and why; no formal proofs necessary.

<u>Solution</u>

Let $P(n)$ be the predicate "$n = 0$" and $Q(n)$ be the predicate "$n = 1$".

The first statement becomes $\big(\exists n \in \mathbb{N},\ n = 0\big) \wedge \big(\exists m \in \mathbb{N},\ m = 1\big)$, which is true: there is a natural number that is equal to 0, and a natural number that is equal to 1.

The second statement becomes $\exists n \in \mathbb{N},\ n = 0 \wedge n = 1$, which is false: there is no number that is equal to both 0 and 1 at the same time!

2. **[7 marks] Translating statements**.

Let $x \in \mathbb{N}$. We say that x is a **twin prime** if and only if both x and $x+2$ are prime. For example, 5 is a twin prime, because both 5 and 7 are prime. Note that 7 is *not* a twin prime, since 9 is not prime.

Express each of the following statements using predicate logic. No justification is required.

Note: please review the instructions on the midterm's front page for our expectations in this question. In particular, you may *not* define any helper predicates or sets.

(a) **[3 marks]** For every twin prime p, $2p+1$ is also a twin prime.

Solution

$\forall p \in \mathbb{N},\ Prime(p) \land Prime(p+2) \Rightarrow Prime(2p+1) \land Prime(2p+3).$

(b) **[4 marks]** There are infinitely many numbers that are *not* a twin prime.
Hint: it may be easier to first express "n is not a twin prime."

Solution

$\forall n_0 \in \mathbb{N},\ \exists n \in \mathbb{N},\ n > n_0 \land \big(\neg Prime(n) \lor \neg Prime(n+2)\big).$

3. **[6 marks] Proofs (inequalities).** Consider the following statement: "There exists a real number x such that x is less than 3 and for every real number y, $x + y^2 > 25$."

   (a) [**1 mark**] Translate the above statement into predicate logic.

   Solution

   $\exists x \in \mathbb{R},\ x < 3 \land \big(\forall y \in \mathbb{R},\ x + y^2 > 25\big)$

   (b) [**1 mark**] Write the negation of this statement, fully-simplified so that all negation symbols are applied directly to predicates. You can simplify $\neg(a > b)$ to $a \leq b$ and $\neg(a < b)$ to $a \geq b$.

   Solution

   $\forall x \in \mathbb{R},\ x \geq 3 \lor \big(\exists y \in \mathbb{R},\ x + y^2 \leq 25\big)$

   (c) [**4 marks**] Disprove the original statement by proving its negation. In your proof, any chains of calculations must follow a top-down order; don't start with the inequality you're trying to prove!

   Hint: use the fact that $\neg p \lor q$ is equivalent to $p \Rightarrow q$ to rewrite the negation into an implication.

   Write any rough work or intuition in the **Discussion** box, and write your formal proof in the **Proof** box. Your rough work/intuition will only be graded if your proof is not completely correct.

   Solution

   Using the hint, we rewrite the negation from part (b) as

   $$\forall x \in \mathbb{R},\ x < 3 \Rightarrow \big(\exists y \in \mathbb{R},\ x + y^2 \leq 25\big)$$

   We'll prove this statement.

   Proof. Let $x \in \mathbb{R}$, and assume that $x < 3$. Let $y = 0$. We want to prove that $x + y^2 \leq 25$. Since $y = 0$, we know that $x + y^2 = x$. So then $x + y^2 < 3$ (by our assumption), and so $x + y^2 \leq 25$. $\square$

4. **[5 marks] Proofs (number theory).** Consider the following statement: "For all two integers x and y, if x and y are both divisible by 3, then $x^2 + 2y^2$ is divisible by 3."

(a) **[1 mark]** Translate the above statement into predicate logic.

Solution

$\forall x, y \in \mathbb{Z},\ 3 \mid x \wedge 3 \mid y \Rightarrow 3 \mid x^2 + 2y^2$

(b) **[4 marks]** Prove the above statement using the definition of divisibility:

$$x \mid y : \quad \exists k \in \mathbb{Z},\ y = kx$$

Do not use any external facts about divisibility.

Write any rough work or intuition in the **Discussion** box, and write your formal proof in the **Proof** box. Your rough work/intuition will only be looked at if your proof is not completely correct.

Solution

Proof. Let $x, y \in \mathbb{Z}$. Assume that x and y are divisible by 3, i.e., that there exist $k_1, k_2 \in \mathbb{Z}$ such that $x = 3k_1$ and $y = 3k_2$. We want to prove that there exists $k_3 \in \mathbb{Z}$ such that $x^2 + 2y^2 = 3k_3$. Let $k_3 = 3k_1^2 + 6k_2^2$.

Then we can calculate, starting with $x^2 + 2y^2$:

$$\begin{aligned} x^2 + 2y^2 &= (3k_1)^2 + 2(3k_2)^2 \\ &= 9k_1^2 + 18k_2^2 \\ &= 3(3k_1^2 + 6k_2^2) \\ &= 3k_3 \end{aligned}$$

□

Use this page for rough work. If you want work on this page to be marked, please indicate this clearly *at the location of the original question.*

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Name:

Question	Grade	Out of
Q1		6
Q2		7
Q3		6
Q4		5
Total		24