

## Floating Point

### Is Java broken?

It's not only Java. Check this out in Python:

```
>>> x = 0.1
>>> sum = x + x + x
>>> sum == 0.3
False
>>> sum
0.30000000000000004
>>> bigger = 1.0
>>> s = 1.0e-6
>>> sum1 = s + s + s + s + s + s + s + s + s + s + s + s + bigger
>>> sum2 = bigger + s + s + s + s + s + s + s + s + s + s + s
>>> sum1 == sum2
False
>>> sum1
1.00001
>>> sum2
1.0000099999999992
```

## Example 1

- Perform some simple arithmetic, and check that the laws of mathematics hold.
- Code: Adding.java

## Representing numbers

- It all makes sense if you understand how “real” numbers are represented.
- First, consider an `int` like 42. Hardware doesn't directly represent 4s or 2s — everything is binary.
- $42 = 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$
- So 42 can be represented by 101010 (base 2).

## Representing fractions

- Fractions can be handled using the same approach.
- Example:  $0.4375 =$   
 $0 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} + 1 \times 2^{-4}$   
 $= 0/2 + 1/4 + 1/8 + 1/16$   
 $= 0.25 + 0.125 + 0.0625$   
 $= 0.4375$
- So we can represent 0.4375 using 0.0111 (base 2).
- Another example:  $0.1 =$   
 $0.000110011001100110011001100...$
- 0.1 does not have a finite binary representation

## IEEE-754 Floating Point

- Like a binary version of scientific notation
- 32 bits for a float (64 bits for a double) as follows:
  - 1 bit for the **sign**
  - 8 bits for the **exponent** e
  - 23 bits for the **mantissa** (significand) M

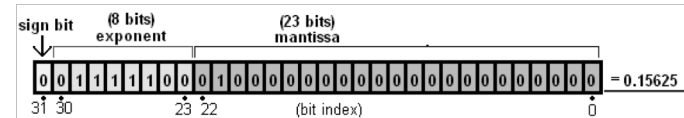
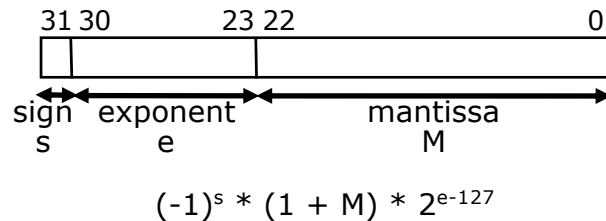
## Some problem numbers

- You already know from math that some numbers do not have a finite representation.
- Even worse, some numbers that have a finite representation in decimal do not in binary!
  - (Thought question: is the reverse true?)
- Computer systems have finite memory. But we need to represent numbers that take an infinite number of bits.
- Solution?

## Allocation of 32 bits

- 1 bit** for the **sign**: 1 for negative and 0 for positive
- 8 bits** for the **exponent** e
  - To allow for negative exponents, 127 is added to that exponent to get the representation. We say that the exponent is "biased" by 127.
  - So the range of possible exponents is not 0 to  $2^8-1 = 0$  to 255, but (0-127) to (255-127) = -127 to 128.
- 23 bits** for the **mantissa** M
  - Since the first bit must be 1, we don't waste space storing it!

## IEEE-754 Floating Point



$$(-1)^{\text{sign}} \left( 1 + \sum_{i=1}^{23} b_{23-i} 2^{-i} \right) \times 2^{(e-127)}$$

- sign = 0
  - $1 + \sum_{i=1}^{23} b_{23-i} 2^{-i} = 1 + 2^{-2} = 1.25$
  - $2^{(e-127)} = 2^{124-127} = 2^{-3}$
- thus:
- value =  $1.25 \times 2^{-3} = 0.15625$

## Rounding

- If we have to lose some digits, we don't just truncate, we round.
- In rounding a decimal to a whole number, an issue arises: If we have a 0.5, do we round up or down?
- If we always round up, we are biasing towards higher values.
- "Proper" rounding: round to the nearest even number.  
E.g., 17.5 is rounded up to 18 but 16.5 is rounded down to 16.
- The IEEE standard uses proper rounding also.

## Historical aside

- 30 years ago, computer manufacturers each had their own standard for floating point.
- Problem? Writing portable software!
- Advantage to manufacturers? Customers got locked in to their particular computers.
- In the late 1980s, the IEEE produced the standard that now virtually all follow.
- William Kahan spearheaded the effort, and won the 1989 Turing Award for it.

### Back to the example (Adding.java)

- As we saw, 0.1 cannot be represented exactly in binary, leading to the unexpected result.
- And adding a very small quantity to a very large quantity can mean the smaller quantity falls off the end of the mantissa.
- But if we add small quantities to each other, this doesn't happen.  
And if they accumulate into a larger quantity, they may not be lost when we finally add the big quantity in.

## Lessons

- When adding floating point numbers, add the smallest first.
- More generally, try to avoid adding dissimilar quantities.
- Specific scenario: When adding a list of floating point numbers, sort them first.

## Examples 2 and 3

- This seems contrived, but consider some value that accumulates in a loop.
  - Code: Totalling.java
- Or consider adding up a list of doubles, what should you do?
  - Code: ArrayTotal.java

## Example 4

- Repeat a task for values in a particular range with an increment of 0.1.
  - For example, for values 0.1 to 0.5 with an increment of 0.1.
  - For example, for values 1.1 to 1.5 with an increment of 0.1.
  - Code: FunctionValues.java


## Lessons

- Don't use floating point variables to control what is essentially a counted loop.
- Also: Notice that we wrote  
 $x = 1.0 + i * 0.1;$   
 instead of initializing  $x$  to 1.0 and then repeatedly adding 0.1.  
 Why? Fewer total arithmetic operations means fewer rounding errors are introduced.
- Use fewer arithmetic operations where possible.

## Example 5

- A very simple program that just prints the same variable using different formats.
- Code: `Examine.java`

## What happened?

- We shouldn't be surprised by now to find out that  $4/5$  can't be represented exactly in a float. Lots of things can't.
- But the represented value should be **off by** a tiny bit. What are all these extra digits?? 
- $4/5 = 1.1001100110011001100110011001100110011001100 \dots \times 2^{-1}$
- It gets rounded to  
 $1.10011001100110011001100110011001 \times 2^{-1}$
- When we ask to print it as a decimal number, it gets converted.  
 The exact equivalent is  
 $0.800000011920928955078125000000$
- But only 7 of those digits are significant.

## Lesson

- Don't print more precision in your output than you are holding.

## Why does this matter?

## Ariane 5 Rocket Launch

### Ariane 5 rocket explosion

- In 1996, the European Space Agency's Ariane 5 rocket exploded 40 seconds after launch.
- During conversion of a 64-bit to a 16-bit format, overflow occurred: the number was too big to store in 16 bits.
- This hadn't been expected because the data (acceleration reported by sensors) had never been this large before. But this new rocket was faster than its predecessor.
- \$7 billion of R&D had been invested in this rocket.
- Reference:  
<http://www.around.com/ariane.html>

### Patriot missile accident

- In 1991, an American missile failed to track and destroy an incoming missile. Instead it hit a US Army barracks, killing 28.
- The system tracked time in tenths of seconds. The error in approximating 0.1 with 24 bits was magnified in its calculations.
- At the time of the accident, the error corresponded to 0.34 seconds. A Patriot missile travels about half a km in that time.
- Reference:  
<http://www.ima.umn.edu/~arnold/disasters/patriot.html>

## Sinking of an oil rig

- In 1992, the Sleipner A oil and gas platform sank in the North Sea near Norway.
- Numerical issues in modelling the structure caused shear stresses to be underestimated by 47%.
- As a result, concrete walls were not built thick enough.
- Cost: \$700 million
- Reference:  
<http://www.ima.umn.edu/~arnold/disasters/sleipner.html>

## What should you do?

"95% of folks out there are completely clueless about floating-point."

James Gosling

## Follow the lessons

- Use double instead of float.
- When adding floating point numbers, add the smallest first.
- More generally, try to avoid adding dissimilar quantities.
- Specific scenario: When adding a list of floating point numbers, sort them first.
- Don't use floating point variables to control what is essentially a counted loop.
- Use fewer arithmetic operations where possible.
- Don't print more precision in your output than you are holding.

## CSC336 Numerical Methods

- The study of computational methods for solving problems in linear algebra, non-linear equations, and approximation. The aim is to give students a basic understanding of both floating-point arithmetic and the implementation of algorithms used to solve numerical problems, as well as a familiarity with current numerical computing environments.