數值方法 H/W Ch4 E14116401 工科115 張瑋哲

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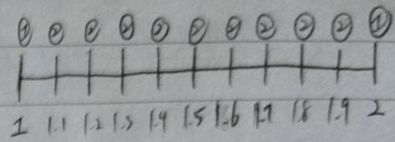
1、 $\int_1^2 e^x \sin(4x)\,dx$, $h=0.1 \Rightarrow n=10$ $f(x)=e^x\sin(4x)$ $\int_1^2 e^x \sin(4x) = 3.85936$

(a) Composite trapezoidal rule : $\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{2}\left[f(x_0)+f(x_n)+2\sum_{i=1}^{n-1} f(x_i)\right]$

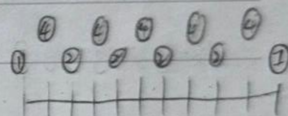


$\int_1^2 e^x \sin(4x)\,dx = \frac{0.1}{2}\left[f(1)+f(2)+2\sum_{i=1}^{9} f(x_i)\right] = 0.05\left[-2.057+2(-2.858+3.307-3.241+-2.559+-1.252\right.$

$\left.+0.5772+2.7047+4.8014+6.4714)+7.3104\right] = \underline{0.39644}$ #，若用 $2\sum_{i=11}^{19}\left(e^{\frac{x}{10}}\sin 4\frac{x}{10}\right)$ 及 $f(1)$ $f(2)$，得 $\underline{0.396156}$ #

完全不四捨五入用計算機按 $0.05\left[f(1)+f(2)+2\sum_{i=11}^{19} e^{\frac{x}{10}}\sin\left(4\frac{x}{10}\right)\right] \doteqdot \underline{0.3961416}$ #

(b) Composite Simpson's method : $\int_{x_0}^{x_n} f(x)\,dx = \frac{h}{3}\left[f(x_0)+f(x_n)+4\sum_{i=1}^{n/2} f(x_{2i-1})+2\sum_{i=1}^{n/2-1} f(x_{2i})\right]$



$\int_1^2 e^x \sin(4x)\,dx = \frac{0.1}{3}\left[f(1)+f(2)+4\left[f(1.1)+f(1.3)\ldots+f(1.9)\right]+2\left[f(1.2)+f(1.4)+\ldots+f(1.8)\right]\right]$

$=\frac{0.1}{3}\left[-2.057+7.3104+4\left[-2.858-3.241-1.252+2.7047+6.4714\right]+2\left[-3.307-2.559+0.5772+4.8014\right]\right] \doteqdot \underline{0.385967}$ #

(c) Composite midpoint rule : $\int_{x_0}^{x_n} f(x)\,dx = h\sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)$, $x_0=1$, $x_1=1.1$, $x_2=1.2$ ---

$\int_1^2 e^x \sin 4x\,dx = 0.1\left(f(1.05)+f(1.15)+f(1.25)+\ldots f(1.95)\right) = 0.1\left(-2.49+-3.138+-3.346-2.98-1.98-0.391\right.$

$\left.+1.6221+3.7806+5.7156+7.0184\right) = \underline{0.38117}$ #

2、 $\int_1^{1.5} x^2 \ln x \, dx$ with $n=3$, $n=4$, $[1,1.5] \to [-1,1]$, $\frac{x-1}{1.5-1} = \frac{\eta-(-1)}{1-(-1)} \Rightarrow x-1 = 0.5 \cdot \frac{\eta+1}{2}$

$x = \frac{\eta+1}{4} + 1 = \frac{\eta+5}{4}$ or $\int_1^{1.5} x^2 \ln x \, dx = \frac{1.5-1}{2} \int_{-1}^{1} f\left(\frac{1+1.5}{2} + \frac{1.5-1}{2}\eta\right) d\eta = \frac{1}{4}\int_{-1}^{1} f(0.25\eta + 1.25) d\eta$

$n=3$ 時、 $\begin{cases} C_1 = 0.5556 \\ C_2 = 0.8889 \\ C_3 = 0.5556 \end{cases}$ $\begin{cases} x_1 = -0.7746 \\ x_2 = 0 \\ x_3 = 0.7746 \end{cases}$ $\Rightarrow \begin{cases} x_1' = 0.25(-0.7746) + 1.25 = 1.05635 \\ x_2' = 0.25(0) + 1.25 = 1.25 \\ x_3' = 0.25(0.7746) + 1.25 = 1.44365 \end{cases}$

$\frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}\eta + \frac{b+a}{2}\right) d\eta = \sum_{i=1}^{3} C_i f(x_i')$

$f(x_1') = 1.05635^2 \ln(1.05635) \doteqdot 0.06117$
$f(x_2') = 1.25^2 \ln(1.25) \doteqdot 0.34866$
$f(x_3') = 1.44365^2 \ln(1.44365) \doteqdot 0.7652$

$\Rightarrow \begin{cases} C_1 f(x_1') \doteqdot 0.03398 \\ C_2 f(x_2') \doteqdot 0.30992 \\ C_3 f(x_3') \doteqdot 0.4251 \end{cases}$

$\frac{1}{4}\int_{-1}^{1} f(0.25\eta + 1.25) d\eta \doteqdot \frac{1}{4}(0.03398 + 0.30992 + 0.4251) = \underline{0.19225}$ # $\varepsilon_r = |0.19225 - 0.19226| = \underline{10^{-5}}$ #

$n=4$ 時 $\begin{cases} C_1 = 0.34785 \\ C_2 = 0.65215 \\ C_3 = 0.65215 \\ C_4 = 0.34785 \end{cases}$ $\begin{cases} x_1 = -0.86114 \\ x_2 = -0.33998 \\ x_3 = 0.33998 \\ x_4 = 0.86114 \end{cases}$ $\Rightarrow \begin{cases} x_1' = 0.25(-0.86114) + 1.25 = 1.03472 \\ x_2' = 0.25(-0.33998) + 1.25 = 1.165 \\ x_3' = 0.25(0.33998) + 1.25 = 1.335 \\ x_4' = 0.25(0.86114) + 1.25 = 1.465 \end{cases}$

$f(x_1') = 1.03472^2 \ln(1.03472) \doteqdot 0.03654$
$f(x_2') = 1.165^2 \ln(1.165) \doteqdot 0.2073$
$f(x_3') = 1.335^2 \ln(1.335) \doteqdot 0.5149$
$f(x_4') = 1.465^2 \ln(1.465) \doteqdot 0.8195$

$\Rightarrow \begin{cases} C_1 f(x_1') \doteqdot 0.01271 \\ C_2 f(x_2') \doteqdot 0.1352 \\ C_2 f(x_3') \doteqdot 0.3358 \\ C_4 f(x_4') \doteqdot 0.28506 \end{cases}$

$\frac{1}{4}\int_{-1}^{1} f(0.25\eta + 1.25) d\eta \doteqdot 0.25(0.01271 + 0.1352 + 0.3358 + 0.28506) \doteqdot \underline{0.19219}$ # $\varepsilon_r = |0.19219 - 0.19226| = \underline{7 \times 10^{-5}}$ #

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3、 $\int_0^{\pi/4}\int_{\sin x}^{\cos x}(2y\sin x+\cos^2 x)\,dy\,dx$, use (a) Simpson, $n=4, m=4$ (b) Gauss, $n=3, m=3$

calc. value: $\int_{\sin x}^{\cos x}(2y\sin x+\cos^2 x)\,dy = \int_{\sin x}^{\cos x} 2y\sin x\,dy + \int_{\sin x}^{\cos x}\cos^2 x\,dy = \sin x(y^2)\Big|_{\sin x}^{\cos x} + \cos^2 x(y)\Big|_{\sin x}^{\cos x}$

$= \sin x(\cos^2 x-\sin^2 x)+\cos^2 x(\cos x-\sin x)$

$u=\cos x$

$\int_0^{\pi/4}\left(\sin x(\cos^2 x-\sin^2 x)+\cos^2 x(\cos x-\sin x)\right)dx = \int_0^{\pi/4}\left(\cancel{\sin x\cos^2 x} - \sin^3 x + \cos^3 x - \cancel{\sin\cos^2 x}\right)dx$

$= \int_0^{\pi/4}(\sin^3 x+\cos^3 x)\,dx \doteqdot 0.5118$

(a)

Simpson: $n=4 \Rightarrow h=\frac{\frac{\pi}{4}-0}{4}=\frac{\pi}{16}$, $m=4 \Rightarrow k_i=\frac{\cos x_i-\sin x_i}{4}$,

x_0	x_1	x_2	x_3	x_4
0	$\frac{\pi}{16}$	$\frac{\pi}{8}$	$\frac{3\pi}{16}$	$\frac{\pi}{4}$

$I(x_i)=\frac{k_i}{3}\left[f(x_i,y_{i0})+4f(x_i,y_{i1})+2f(x_i,y_{i2})+4f(x_i,y_{i3})+f(x_i,y_{i4})\right]$

$I(x_0)=I(0)$: $k_0=\frac{1-0}{4}=0.25$,

y_{00}	y_{01}	y_{02}	y_{03}	y_{04}
$\sin x_0$	$\sin x_0+k$	$\sin x_0+2k$	$\sin x_0+3k$	$\cos x_0$
0	0.25	0.5	0.75	1

$f(0,0)=1$ $f(0,0.25)=1$ $f(0,0.5)=1$ $f(0,0.75)=1$ $f(0,1)=1$

$I(x_0)=\frac{0.25}{3}[1+4+2+4+1]=1$

$I(x_1)=I\left(\frac{\pi}{16}\right)$: $k_1=0.1964$

y_{10}	y_{11}	y_{12}	y_{13}	y_{14}
$\sin\left(\frac{\pi}{16}\right)$	$\sin\left(\frac{\pi}{16}\right)+k$	$\sin\left(\frac{\pi}{16}\right)+2k$	$\sin\left(\frac{\pi}{16}\right)+3k$	$\cos\left(\frac{\pi}{16}\right)$
0.195	0.3915	0.5879	0.7843	0.9808

$f\left(\frac{\pi}{16},0.195\right)=1.038$ $f\left(\frac{\pi}{16},0.3915\right)=1.1147$ $f\left(\frac{\pi}{16},0.5879\right)=1.191$ $f\left(\frac{\pi}{16},0.7843\right)=1.268$

$f\left(\frac{\pi}{16},0.9808\right)=1.3446$ $I(x_1)=\frac{0.1964}{3}\left[1.038+4(1.1147+1.268)+2(1.191)+1.3446\right]=0.9359$

	y_{20}	y_{21}	y_{22}	y_{23}	y_{24}
$I(x_2)=I(\frac{\pi}{8})$, $k_2=0.1353$	$\sin\frac{\pi}{8}$	$\sin\frac{\pi}{8}+k$	$\sin\frac{\pi}{8}+2k$	$\sin\frac{\pi}{8}+3k$	$\cos\frac{\pi}{8}$
	0.3827	0.518	0.6533	0.7886	0.9239

$f(\frac{\pi}{8},0.3827)=1.1466$ $\quad f(\frac{\pi}{8},0.518)=1.25$ $\quad f(\frac{\pi}{8},0.6533)=1.354$ $\quad f(\frac{\pi}{8},0.7886)=1.457$

$f(\frac{\pi}{8},0.9239)=1.561$ $\quad I(x_2)=\frac{k_2}{3}(1.1466+1.561+4(1.25+1.457)+2(1.354))=0.7326$

$k_4=0$

	y_{30}	y_{31}	y_{32}	y_{33}	y_{34}
$I(x_3)=I(\frac{3\pi}{16})$, $k_3=0.069$	$\sin\frac{3\pi}{16}$	$\sin\frac{3\pi}{16}+k$	$\sin\frac{3\pi}{16}+2k$	$\sin\frac{3\pi}{16}+3k$	$\cos\frac{3\pi}{16}$
	0.5557	0.6246	0.6936	0.7626	0.8315

$f(\frac{3\pi}{16},0.5557)=1.3088$ $\quad f(\frac{3\pi}{16},0.6246)=1.3854$ $\quad f(\frac{3\pi}{16},0.6936)=1.462$ $\quad f(\frac{3\pi}{16},0.7626)=1.5387$ $\quad f(\frac{3\pi}{16},0.8315)=1.6153$

$I(x_3)=\frac{0.069}{3}(1.3088+1.6153+4(1.3854+1.5387)+2(1.462))=0.4035$

	y_{40}	y_{41}	y_{42}	y_{43}	y_{44}
$I(x_4)=I(\frac{\pi}{4})$, $k=0$	$\sin\frac{\pi}{4}$	$\sin\frac{\pi}{4}$	$\sin\frac{\pi}{4}$	$\sin\frac{\pi}{4}$	$\sin\frac{\pi}{4}$
	0.7071	0.7071	0.7071	0.7071	0.7071

$f(\frac{\pi}{4},0.7071)=1.7071$ $\quad I(x_4)=0$ $\quad$ 所求：$\frac{\frac{\pi}{16}}{3}\left[I(x_0)+4(I(x_1)+I(x_3))+2(I(x_2))+I(x_4)\right]$

$=\frac{\pi}{48}\left(1+4(0.9359+0.4035)+2(0.7326)\right)=\underline{0.512}$ # $\quad \varepsilon_{rr}=|0.512-0.5118|=\underline{0.0002}$ #

(b) $\int_0^{\frac{\pi}{4}}\int_{\sin x}^{\cos x}(2y\sin x+\cos^2 x)\,dy\,dx = \frac{\frac{\pi}{4}-0}{2}\sum_{i=1}^{3} C_i I(x_i) = \frac{\pi}{8}\sum_{i=1}^{3} C_i I(x_i)$, $x_i = \frac{\frac{\pi}{4}-0}{2}\eta + \frac{\frac{\pi}{4}+0}{2} = \frac{\pi}{8}\eta + \frac{\pi}{8}$

$I(x_i) = \int_{\sin x_i}^{\cos x_i} 2y\sin x_i + \cos^2 x_i \, dy = \frac{\cos x_i - \sin x_i}{2}\sum_{j=1}^{3} C_j f(x_i, y_{ij})$, $y_{ij}\ \frac{\cos x_i - \sin x_i}{2}\eta_j + \frac{\cos x_i + \sin x_i}{2}$

$\eta_1 = -\sqrt{\frac{3}{5}} = -0.7746$ $C_1 = \frac{5}{9}$ $x_1 = 0.0885$ $y_{11} = 0.1907$, $y_{12} = 0.5422$, $y_{13} = 0.8938$ $\frac{\cos x_1 + \sin x_1}{2} = 0.5422$

$\eta_2 = 0$ $C_2 = \frac{8}{9} \Rightarrow x_2 = 0.3927 \Rightarrow y_{21} = 0.4437$, $y_{22} = 0.6533$, $y_{23} = 0.8629$ $\frac{\cos x_2 + \sin x_2}{2} = 0.6533$

$\eta_3 = \sqrt{\frac{3}{5}} = 0.7746$ $C_3 = \frac{5}{9}$ $x_3 = 0.6969$ $y_{31} = 0.6559$, $y_{32} = 0.7043$, $y_{33} = 0.7527$ $\frac{\cos x_3 + \sin x_3}{2} = 0.7043$

$f(x_1, y_{11}) = 1.0259$ $f(x_1, y_{12}) = 1.088$ $f(x_1, y_{13}) = 1.1502$ $\frac{\cos(x_1) - \sin(x_1)}{2} = 0.4538$

$f(x_2, y_{21}) = 1.1931$ $f(x_2, y_{22}) = 1.3536$ $f(x_2, y_{23}) = 1.514$ $\frac{\cos(x_2) - \sin(x_2)}{2} = 0.2706$

$f(x_3, y_{31}) = 1.43$ $f(x_3, y_{32}) = 1.4921$ $f(x_3, y_{33}) = 1.5543.$ $\frac{\cos(x_3) - \sin(x_3)}{2} = 0.0625$

$\sum_{j=1}^{3} C_1 f(x_1, y_j) = \frac{5}{9}\times 1.0259 + \frac{8}{9}\times 1.088 + \frac{5}{9}\times 1.1502 = 2.176$ $I(x_1) = 2.176 \times 0.4538 = 0.9875$

$\sum_{j=1}^{3} C_2 f(x_2, y_j) = \frac{5}{9}\times 1.1931 + \frac{8}{9}\times 1.3536 + \frac{5}{9}\times 1.514 = 2.7071$ $I(x_2) = 2.7071 \times 0.2706 = 0.7325$

$\sum_{j=1}^{3} C_3 f(x_3, y_j) = \frac{5}{9}\times 1.43 + \frac{8}{9}\times 1.4921 + \frac{5}{9}\times 1.5543 = 2.984$ $I(x_3) = 2.984 \times 0.0625 = 0.1865$

$\frac{\pi}{8}\sum_{i=1}^{3} C_i I(x_i) = \frac{\pi}{8}\left(\frac{5}{9}\times 0.9875 + \frac{8}{9}\times 0.7325 + \frac{5}{9}\times 0.1865\right) \doteq 0.5118$ # $\varepsilon_r = |0.5118 - 0.5118| = 0$ #

(a) 0.512 , (b) 0.5118 , (c) ε_r of Simpson = 0.0002 , ε_r of Gauss = 0 #

4. (a) $\int_0^1 x^{-\frac{1}{4}} \sin x \, dx \Rightarrow$ left-endpoint singularity

0° $f(x) = \frac{\sin x}{x^{\frac{1}{4}}}$

1° $P_4(x) = \sin(0) + \cos(0)(x) + \frac{-\sin(0)}{2!}x^2 + \frac{-\cos(0)}{3!}x^3 + \frac{\sin(0)}{4!}x^4 = x - \frac{x^3}{6}$

2° $\int_0^1 \frac{P_4(x)}{x^{\frac{1}{4}}} dx = \int_0^1 (x - \frac{x^3}{6})(x^{-\frac{1}{4}}) \, dx = \int_0^1 x^{\frac{3}{4}} - \frac{x^{\frac{11}{4}}}{6} \, dx = \left(\frac{4}{7}x^{\frac{7}{4}} - \frac{4^2}{90}x^{\frac{15}{4}}\right)\Big|_{0^+}^1 = \frac{166}{315}$

3° $G(x) = \begin{cases} \frac{1}{x^{\frac{1}{4}}}(\sin x - P_4(x)) & \text{if } 0 \le x \le 1 \\ 0, & \text{if } x = 0 \end{cases}$

4° $n = 4,\ h = \frac{1-0}{4} = 0.25 \quad \int_0^1 G(x)\,dx = \frac{0.25}{3}\left[G(0) + G(1) + 4\left(G(0.25) + G(0.75)\right) + 2G(0.5)\right]$

$= \frac{1}{12}\left[0 + 8.137\times10^{-3} + 4(1.149\times10^{-5} + 2.0968\times10^{-3}) + 2(3.0785\times10^{-4})\right] = 1.432\times10^{-3}$

5° $\int_0^1 f(x)\,dx = \int_0^1 \frac{P(x)}{x^{\frac{1}{4}}}dx + \int_0^1 G(x)\,dx = \underline{0.528416}$ #

$\boxed{\text{計算機 } \int_{0.0001}^1 x^{-\frac{1}{4}} \sin x \, dx = 0.5284}$

4. (b) $\int_1^\infty x^{-4} \sin x \, dx$, $t = x^{-1} \Rightarrow x = t^{-1}$ $dx = -t^{-2}dt$ $\begin{cases} x = 1, & t = 1 \\ x = \infty, & t = 0 \end{cases}$

$\int_1^\infty f(x)\,dx = \int_1^0 t^4 \sin(\frac{1}{t})(-t^{-2})dt = \int_0^1 t^2 \sin(\frac{1}{t})\,dt$

$n = 4,\ h = \frac{1-0}{4} = \frac{1}{4} \quad \int_0^1 t^2 \sin(\frac{1}{t})\,dt = \frac{1}{12}\left[f(0) + 4\left(f(0.25) + f(0.75)\right) + 2\left(f(0.5)\right) + f(1)\right]$

$= \frac{1}{12}\left[0 + 4(-0.0473 + 0.5462) + 2(0.2273) + 0.8415\right] = \underline{0.2743}$ #

$n = 6 \quad h = \frac{1}{6} \quad \int_0^1 t^2 \sin(\frac{1}{t})\,dt = \frac{1}{18}\left[f(0) + 4\left(f(\frac{1}{6}) + f(\frac{3}{6}) + f(\frac{5}{6})\right) + 2\left(f(\frac{2}{6}) + f(\frac{4}{6})\right) + f(1)\right]$

$= \frac{1}{18}\left(0 + 4(-7.76\times10^{-3} + 0.2273 + 0.6472) + 2(0.0157 + 0.4433) + 0.8415\right)$

$= 0.2904$

$\boxed{\text{計算機} \Rightarrow \int_{0.0001}^1 t^2 (\sin\frac{1}{t})\,dt \approx 0.2865}$