

x	4.0	4.2	4.5	4.7	5.1	5.5	5.9	6.3
y	102.6	113.2	130.1	142.1	167.5	195.1	224.9	256.8

(a) $P_2 = ax^2 + bx + c$ ↘

$$\begin{bmatrix} \sum_{i=1}^8 x_i^4 & \sum x_i^3 & \sum x_i^2 \\ \sum x_i^3 & \sum x_i^2 & \sum x_i \\ \sum x_i^2 & \sum x_i & n \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum x_i^2 y_i \\ \sum x_i y_i \\ \sum y_i \end{bmatrix}$$

計算機

$$= \begin{bmatrix} 5843.815 & 1087.488 & 206.74 \\ 1087.488 & 206.74 & 40.2 \\ 206.74 & 40.2 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 37691.573 \\ 7011.21 \\ 1332.3 \end{bmatrix}$$

解聯立: $a = 6.69118, b = -1.88375, c = 3.08639$

$$P_2(x) = 6.69118x^2 - 1.88375x + 3.08639 \quad \#$$

$$E_r = \sum_{i=1}^8 [y_i - P_2(x_i)]^2 = 0.00524569 \quad \# \text{ by python}$$

(b) $y = be^{ax} \Rightarrow \ln y = \ln b + \ln e^{ax} = ax + \ln b$

$\ln y = ax + \ln b \Rightarrow Y = Ax + B$ linear

$$\begin{bmatrix} \sum x^2 & \sum x \\ \sum x & n \end{bmatrix} \begin{bmatrix} a \\ \ln b \end{bmatrix} = \begin{bmatrix} \sum x_i \ln y_i \\ \sum \ln y_i \end{bmatrix}$$

同(a) & 計算機

$$\Rightarrow \begin{bmatrix} 206.74 & 40.2 \\ 40.2 & 8 \end{bmatrix} \begin{bmatrix} a \\ \ln b \end{bmatrix} = \begin{bmatrix} 205.6188 \\ 40.5433 \end{bmatrix}$$

計算機

$$\Rightarrow a = 0.39846199, \ln b = 3.06564102, b = 21.4482$$

$$\Rightarrow P = 21.4482 e^{0.398462x} \quad \#$$

但 python 跑 $b = 21.444344, a = 0.3984934$

$$Err = \sum_1^8 [y_i - P(x_i)]^2 \approx 94.983021 \quad \#$$

$$(c) y = b x^n \Rightarrow \ln y = n \ln x + \ln b \Rightarrow Y = AX + B$$

$$\begin{bmatrix} \sum (\ln x_i)^2 & \sum \ln x_i \\ \sum \ln x_i & n \end{bmatrix} \begin{bmatrix} n \\ \ln b \end{bmatrix} = \begin{bmatrix} \sum \ln x \ln y \\ \sum \ln y \end{bmatrix}$$

計算機
⇒

$$\begin{bmatrix} 20.7371 & 12.8225 \\ 12.8225 & 8 \end{bmatrix} \begin{bmatrix} n \\ \ln b \end{bmatrix} = \begin{bmatrix} 65.3571 \\ 40.5433 \end{bmatrix}$$

$$n = 2.020096, \ln b = 1.830077 \Rightarrow b = 6.234367$$

$$p = 6.234367 x^{2.020096} \quad \#$$

但 python $\Rightarrow n = 2.019634, b = 6.238952$

$$p = 6.238952 x^{2.019634}$$

$$Err = 0.0117207 \quad \#$$

2. Legendre $\Rightarrow w(x)=1, x \in [-1, 1]$

$$L_{n+1}(x) = \frac{2n+1}{n+1} x L_n(x) - \frac{n}{n+1} L_{n-1}(x)$$

$$L_0 = 1 \quad L_1 = x \quad L_2 = \frac{3x^2-1}{2}$$

$$f(x) = \frac{1}{2} \cos x + \frac{1}{4} \sin 2x$$

$$a_0 = \frac{\int_{-1}^1 \left(\frac{1}{2} \cos x + \frac{1}{4} \sin 2x \right) 1 dx}{\int_{-1}^1 1^2 dx} = 0.4207355$$

$$a_1 = \frac{\int_{-1}^1 \left(\frac{1}{2} \cos x + \frac{1}{4} \sin 2x \right) x dx}{\int_{-1}^1 x^2 dx} = 0.3265483$$

$$a_2 = \frac{\int_{-1}^1 \left(\frac{1}{2} \cos x + \frac{1}{4} \sin 2x \right) \left(\frac{3x^2-1}{2} \right) dx}{\int_{-1}^1 \left(\frac{3x^2-1}{2} \right)^2 dx} = -0.15508763$$

$$\Rightarrow f(x) \approx P_2(x) = -0.15508763 \left(\frac{3x^2-1}{2} \right) + 0.3265483 x + 0.4207355$$

$$= -0.2326314 x^2 + 0.3265483 x + 0.4982793 \quad \#$$

$$Err = \int_{-1}^1 [f(x) - P(x)]^2 dx$$

$$= \int_{-1}^1 \left[\left(\frac{1}{2} \cos x + \frac{1}{4} \sin 2x \right) + 0.2326314 x^2 - 0.3265483 x - 0.4982793 \right]^2 dx$$

$$= 0.00324043 \quad \# \quad \text{by python \& 手機工程計算機}$$

$$3. f(x) = x^2 \sin x, x \in [0, 1], m=16, n=4$$

$$S_4(z) = \frac{1}{2}a_0 + a_4 \cos(4z) + \sum_{k=1}^3 [a_k \cos(kz) + b_k \sin(kz)], z \in [-\pi, \pi]$$

$$x_0 = 0, x_{31} = 1, \Delta x = \frac{1}{31}, x_i = \frac{i}{31}, i \in [0, 31]$$

$$z_i = \pi \left[2 \frac{x_i - 0}{1 - 0} - 1 \right] = \pi [2x_i - 1] \Rightarrow \pi \left[2 \frac{i}{31} - 1 \right]$$

$$\Rightarrow \left(\frac{z_i}{\pi} + 1 \right) / 2 = x_i$$

$$a_0 = \frac{1}{16} \sum_{i=0}^{31} f(z_i)$$

$$a_k = \frac{2}{32} \sum_{i=0}^{31} f(z_i) \cos(kz_i) \quad k=1, 2, 3$$

$$b_k = \frac{2}{32} \sum_{i=0}^{31} f(z_i) \sin(kz_i), k=1, 2, 3$$

$$f(x) = x^2 \sin x = \left[\left(\frac{z_i}{\pi} + 1 \right) / 2 \right]^2 \sin \left[\left(\frac{z_i}{\pi} + 1 \right) / 2 \right] = f(z_i)$$

$$\Rightarrow a_0 = \frac{1}{16} \sum_{i=0}^{31} f(z_i) = \left(\frac{i}{31} \right)^2 \sin \left(\frac{i}{31} \right) = 0.459205$$

$$\Rightarrow a_k = \frac{2}{32} \sum_{i=0}^{31} f(z_i) \cos(kz_i) = \frac{1}{16} \sum_{i=0}^{31} \left(\frac{i}{31} \right)^2 \sin \left(\frac{i}{31} \right) \cos \left[k\pi \left(\frac{2i}{31} - 1 \right) \right]$$

$$a_1 = -0.146756$$

$$a_2 = 0.054608$$

$$a_3 = -0.038929$$

$$a_4 = 0.033542$$

$$\Rightarrow b_k = \frac{2}{32} \sum_{i=0}^{31} f(z_i) \sin(kz_i) = \frac{1}{16} \sum_{i=0}^{31} \left(\frac{i}{31} \right)^2 \sin \left(\frac{i}{31} \right) \sin \left[k\pi \left(\frac{2i}{31} - 1 \right) \right]$$

$$b_1 = 0.232287$$

$$b_2 = -0.124941$$

$$b_3 = 0.082932$$

$$\sum_{x=0}^{31} \left[\left(\frac{x}{31} \right)^2 \sin\left(\frac{x}{31}\right) \right] \div 16$$

$$\approx \frac{\sin\left(\frac{1}{31}\right)}{961} + \frac{4}{961} \sin\left(\frac{2}{31}\right) + \frac{9}{961} \sin\left(\frac{3}{31}\right) + \dots$$

結果格式

SHI 小数

SOLV 0.459 205 305 29

a_0

$$\left(\frac{x}{31} \right)^2 \sin\left(\frac{x}{31}\right) \cos\left(\pi\left(\frac{2x}{31} - 1\right)\right) \div 16$$

$$\approx \frac{-\cos\left(\frac{2}{31}\pi\right) \sin\left(\frac{1}{31}\right) - 4\cos\left(\frac{4}{31}\pi\right) \sin\left(\frac{2}{31}\right) - 9\cos\left(\frac{6}{31}\pi\right) \sin\left(\frac{3}{31}\right) - \dots}{961}$$

結果格式

SHI 小数

SOLV -0.146 756 127 02

a_1

$$\left(\frac{x}{31} \right)^2 \sin\left(\frac{x}{31}\right) \cos\left(2\pi\left(\frac{2x}{31} - 1\right)\right) \div 16$$

$$\approx \frac{\frac{1}{961} \cos\left(\frac{4}{31}\pi\right) \sin\left(\frac{1}{31}\right) + \frac{4}{961} \cos\left(\frac{8}{31}\pi\right) \sin\left(\frac{2}{31}\right) + \frac{9}{961} \cos\left(\frac{12}{31}\pi\right) \sin\left(\frac{3}{31}\right) + \dots}{16}$$

結果格式

SHI 小数

SOLV 0.054 607 988 08

a_2

$$\left(\frac{x}{31} \right)^2 \sin\left(\frac{x}{31}\right) \cos\left(3\pi\left(\frac{2x}{31} - 1\right)\right) \div 16$$

$$\approx \frac{-\cos\left(\frac{6}{31}\pi\right) \sin\left(\frac{1}{31}\right) - 4\cos\left(\frac{12}{31}\pi\right) \sin\left(\frac{2}{31}\right) - 9\cos\left(\frac{18}{31}\pi\right) \sin\left(\frac{3}{31}\right) - \dots}{961}$$

結果格式

SHI 小数

SOLV -0.038 928 891 42

a_3

$$\left(\frac{x}{31} \right)^2 \sin\left(\frac{x}{31}\right) \cos\left(4\pi\left(\frac{2x}{31} - 1\right)\right) \div 16$$

$$\approx \frac{\frac{1}{961} \cos\left(\frac{8}{31}\pi\right) \sin\left(\frac{1}{31}\right) - \frac{4}{961} \cos\left(\frac{16}{31}\pi\right) \sin\left(\frac{2}{31}\right) + \frac{9}{961} \cos\left(\frac{24}{31}\pi\right) \sin\left(\frac{3}{31}\right) - \dots}{16}$$

結果格式

SHI 小数

SOLV 0.033 542 251 17

a_4

$$\left(\frac{x}{31} \right)^2 \sin\left(\frac{x}{31}\right) \sin\left(\pi\left(\frac{2x}{31} - 1\right)\right) \div 16$$

$$\approx \frac{-225 \sin\left(\frac{15}{31}\pi\right) \sin\left(\frac{\pi}{31}\right) + 256 \sin\left(\frac{16}{31}\pi\right) \sin\left(\frac{2\pi}{31}\right) - 225 \sin\left(\frac{17}{31}\pi\right) \sin\left(\frac{3\pi}{31}\right) + \dots}{961}$$

結果格式

SHI 小数

SOLV 0.232 287 224 87

b_1

$$\left(\frac{x}{31} \right)^2 \sin\left(\frac{x}{31}\right) \sin\left(2\pi\left(\frac{2x}{31} - 1\right)\right) \div 16$$

$$\approx \frac{-64 \sin\left(\frac{8}{31}\pi\right) \sin\left(\frac{\pi}{31}\right) + 529 \sin\left(\frac{16}{31}\pi\right) \sin\left(\frac{2\pi}{31}\right) - 225 \sin\left(\frac{24}{31}\pi\right) \sin\left(\frac{3\pi}{31}\right) + \dots}{961}$$

結果格式

SHI 小数

SOLV -0.124 941 108 04

b_2

$$\left(\frac{x}{31} \right)^2 \sin\left(\frac{x}{31}\right) \sin\left(3\pi\left(\frac{2x}{31} - 1\right)\right) \div 16$$

$$\approx \frac{-25 \sin\left(\frac{5}{31}\pi\right) \sin\left(\frac{\pi}{31}\right) + 676 \sin\left(\frac{10}{31}\pi\right) \sin\left(\frac{2\pi}{31}\right) - 225 \sin\left(\frac{15}{31}\pi\right) \sin\left(\frac{3\pi}{31}\right) + \dots}{961}$$

結果格式

SHI 小数

SOLV 0.082 932 037 89

b_3

$$f_4(z_j) = \frac{0.459205}{2} + 0.033542 \cos 4z$$

$$- 0.146756 \cos z + 0.232287 \sin z$$

$$+ 0.034608 \cos 2z - 0.124941 \sin 2z$$

$$- 0.038929 \cos 3z + 0.082932 \sin 3z$$

#

$$(b) \int_0^1 \zeta_4(x) dx \Rightarrow \begin{cases} z = \pi[2x_i - 1] \\ dz = 2\pi dx_i \end{cases} \Rightarrow dx_i = \frac{dz}{2\pi}$$

$$\Rightarrow \int_{-\pi}^{\pi} \zeta_4(z) \frac{dz}{2\pi} = 0.2296025 \quad \#$$

$$(c) \int_0^1 x^2 \sin x dx = 0.2232443$$

$$E_{\text{err}} = |0.2232443 - 0.2296025| = 6.3582 \times 10^{-3}$$

$$\text{rel. } E_{\text{err}} \approx 0.02848 \approx 2.85\% \quad \#$$

$$(d) E(\zeta_4) = \sum_{i=0}^{31} [f(x_i) - \zeta_4(x_i)]^2$$

$$= 2.4342069 \quad \# \text{ python}$$

```

1 import math
2 import numpy as np
3 PI = np.pi
4
5 def f(x):
6     return x**2 * math.sin(x)
7
8 def S4(x):
9     a0 = 0.459205
10    a = [-0.146756, 0.054608, -0.038929, 0.033542]
11    b = [0.232287, -0.124941, 0.082932, 0]
12
13    total = a0
14    for k in range(1, 5): # k=1~4
15        angle = k * PI * (2*x - 1)
16        total += a[k-1] * math.cos(angle)
17        total += b[k-1] * math.sin(angle)
18    return total
19
20 def calculate_error():
21    m = 32
22    error = 0.0
23
24    for j in range(m):
25        xj = j / m
26        diff = f(xj) - S4(xj)
27        error += diff**2
28
29    return error
30
31 if __name__ == "__main__":
32     error = calculate_error()
33     print(f"E(S4) = {error}")
34

```

PS D:\Library\WQ\0524alpha_running> & C:/ProgramData/anaconda3/python.exe "d:/ForClass/1132/1132Numerical/HW8/3(d).py"
E(S4) = 2.43420692849035