1、

x	4.0	4.2	4.5	4.7	5.1	5.5	5.9	6.3
y	102.6	113.2	130.1	142.1	167.5	195.1	224.9	256.8

(a) $P_2 = ax^2 + bx + c$ ⤵

$$\begin{bmatrix} \sum_1^8 x_i^4 & \sum x_i^3 & \sum x_i^2 \\ \sum x_i^3 & \sum x_i^2 & \sum x_i \\ \sum x_i^2 & \sum x_i & n \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sum x_i^2 y_i \\ \sum x_i y_i \\ \sum y_i \end{bmatrix}$$

計算機

$$= \begin{bmatrix} 5843.815 & 1087.488 & 206.74 \\ 1087.488 & 206.74 & 40.2 \\ 206.74 & 40.2 & 8 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 37691.573 \\ 7011.21 \\ 1332.3 \end{bmatrix}$$

解聯立： $a = 6.69118$, $b = -1.88375$, $c = 3.08639$

$P_2(x) = 6.69118x^2 - 1.88375x + 3.08639$ #

$E_1 = \sum_1^8 [y_i - P_2(x_i)]^2 = 0.00524569$ # by python

(b)

$$y = be^{ax} \Rightarrow \ln y = \ln b + \ln e^{ax} = ax + \ln b$$

$$\ln y = ax + \ln b \Rightarrow Y = AX + B \text{ linear}$$

$$\begin{bmatrix} \Sigma x^2 & \Sigma x \\ \Sigma x & n \end{bmatrix} \begin{bmatrix} a \\ \ln b \end{bmatrix} = \begin{bmatrix} \Sigma x_i \ln y_i \\ \Sigma \ln y_i \end{bmatrix}$$

用(a)&計算機

$$\Rightarrow \begin{bmatrix} 206.74 & 40.2 \\ 40.2 & 8 \end{bmatrix} \begin{bmatrix} a \\ \ln b \end{bmatrix} = \begin{bmatrix} 205.6188 \\ 40.5433 \end{bmatrix}$$

計算機

$$\Rightarrow a = 0.39846199,\ \ln b = 3.06564102,\ b = 21.4482$$

$$\Rightarrow P = 21.4482\, e^{0.398462x} \quad \#$$

但 Python 跑 $b = 21.444544$, $a = 0.3984934$

$$E_w = \sum_1^8 [y_i - P(x_i)]^2 \approx 94.983021 \quad \#$$

(c) $y = bx^n \Rightarrow \ln y = n\ln x + \ln b \Rightarrow Y = AX + B$

$$\begin{bmatrix} \Sigma(\ln x_i)^2 & \Sigma \ln x_i \\ \Sigma \ln x_i & n \end{bmatrix} \begin{bmatrix} n \\ \ln b \end{bmatrix} = \begin{bmatrix} \Sigma \ln x \ln y \\ \Sigma \ln y \end{bmatrix}$$

計算機 $\Rightarrow$

$$\begin{bmatrix} 20.7371 & 12.8225 \\ 12.8225 & 8 \end{bmatrix} \begin{bmatrix} n \\ \ln b \end{bmatrix} = \begin{bmatrix} 65.3571 \\ 40.5433 \end{bmatrix}$$

$n = 2.020096$, $\ln b = 1.830077 \Rightarrow b = 6.234367$

$P = 6.234367\, x^{2.020096}$ #

但 python $\Rightarrow$ $n = 2.019634$, $b = 6.238952$

$P = 6.238952\, x^{2.019634}$

$E_{rr} = 0.011207$ #

2. Legendre $\Rightarrow w(x)=1$, $x \in [-1,1]$

$$L_{n+1}(x)=\frac{2n+1}{n+1}xL_n(x)-\frac{n}{n+1}L_{n-1}(x)$$

$$L_0=1 \quad L_1=x \quad L_2=\frac{3x^2-1}{2}$$

$$f(x)=\frac{1}{2}\cos x+\frac{1}{4}\sin 2x$$

$$a_0=\frac{\int_{-1}^{1}\left(\frac{1}{2}\cos x+\frac{1}{4}\sin 2x\right)1\,dx}{\int_{-1}^{1}1^2\,dx}=0.4207355$$

$$a_1=\frac{\int_{-1}^{1}\left(\frac{1}{2}\cos x+\frac{1}{4}\sin 2x\right)x\,dx}{\int_{-1}^{1}x^2\,dx}=0.3265483$$

$$a_2=\frac{\int_{-1}^{1}\left(\frac{1}{2}\cos x+\frac{1}{4}\sin 2x\right)\left(\frac{3x^2-1}{2}\right)dx}{\int_{-1}^{1}\left(\frac{3x^2-1}{2}\right)^2dx}=-0.15508763$$

$$\Rightarrow f(x)\simeq P_2(x)=-0.15508763\left(\frac{3x^2-1}{2}\right)+0.3265483x+0.4207355$$

$$=-0.232631 4x^2+0.3265483x+0.4982793 \ \#$$

$$Err=\int_{-1}^{1}[f(x)-P(x)]^2dx$$

$$=\int_{-1}^{1}\left[\left(\frac{1}{2}\cos x+\frac{1}{4}\sin 2x\right)+0.232631 4x^2-0.3265483x-0.4982793\right]^2dx$$

$$=0.00324043 \ \#$$ by python & 手機工程計算機

3. $f(x) = x^2 \sin x$, $x \in [0,1]$, $m = 16$, $n = 4$

$$S_4(z) = \frac{1}{2}a_0 + a_4 \cos(4z) + \sum_{k=1}^{3}\left[a_k \cos(kz) + b_k \sin(kz)\right], \quad z \in [-\pi, \pi)$$

$$x_0 = 0, \quad x_{31} = 1, \quad \Delta x = \frac{1}{31}, \quad x_i = \frac{i}{31}, \quad i \in [0, 31]$$

$$z_i = \pi\left[2\frac{x_i - 0}{1-0} - 1\right] = \pi\left[2x_i - 1\right] \Rightarrow \pi\left[2\frac{i}{31} - 1\right]$$

$$\Rightarrow \left(\frac{z_i}{\pi} + 1\right)/2 = x_i$$

$$a_0 = \frac{1}{16}\sum_{i=0}^{31} f(z_i)$$

$$a_k = \frac{2}{32}\sum_{i=0}^{31} f(z_i)\cos(kz) \quad k = 1, 2, 3$$

$$b_k = \frac{2}{32}\sum_{i=0}^{31} f(z_i)\sin(kz), \quad k = 1, 2, 3$$

$$f(x) = x^2 \sin x = \left[\left(\frac{z_i}{\pi} + 1\right)/2\right]^2 \sin\left[\left(\frac{z_i}{\pi} + 1\right)/2\right] = f(z_i)$$

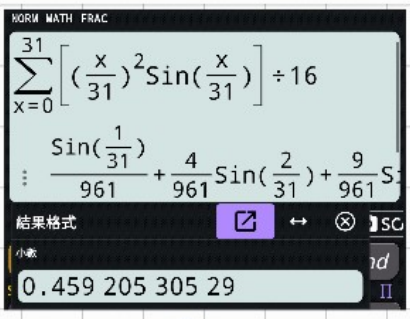
$$\Rightarrow a_0 = \frac{1}{16}\sum_{i=0}^{31} f(z_i) = \left(\frac{i}{31}\right)^2 \sin\left(\frac{i}{31}\right) = 0.459205$$

$$\Rightarrow a_k = \frac{2}{32}\sum_{i=0}^{31} f(z_i)\cos(kz_i) = \frac{1}{16}\sum_{i=0}^{31}\left(\frac{i}{31}\right)^2 \sin\left(\frac{i}{31}\right)\cos\left[k\pi\left(\frac{2i}{31} - 1\right)\right]$$

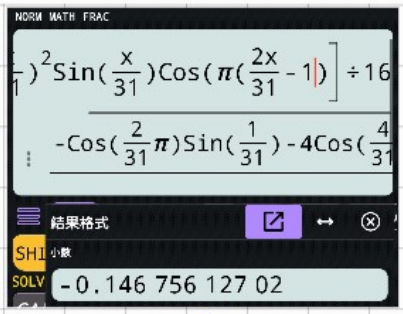
$a_1 = -0.146756$ $\quad a_2 = 0.054608$ $\quad a_3 = -0.038929$ $\quad a_4 = 0.033542$

$$\Rightarrow b_k = \frac{2}{32}\sum_{i=0}^{31} f(z_i)\sin(kz_i) = \frac{1}{16}\sum_{i=0}^{31}\left(\frac{i}{31}\right)^2 \sin\left(\frac{i}{31}\right)\sin\left[k\pi\left(\frac{2i}{31} - 1\right)\right]$$

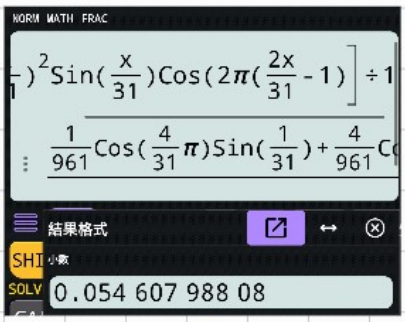
$b_1 = 0.232287$ $\quad b_2 = -0.124941$ $\quad b_3 = 0.082932$



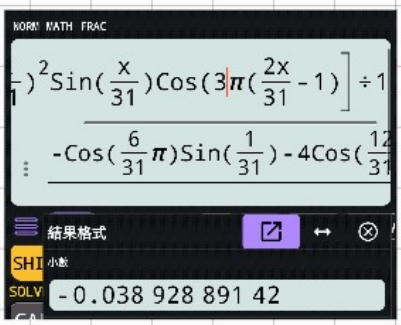
a_0



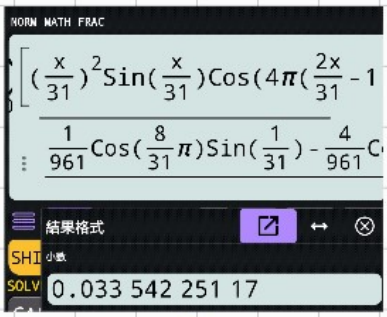
a_1



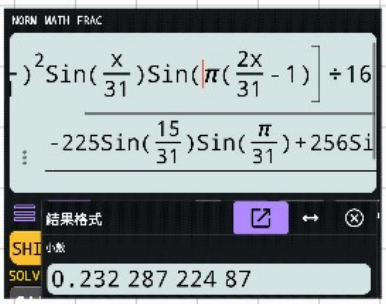
a_2



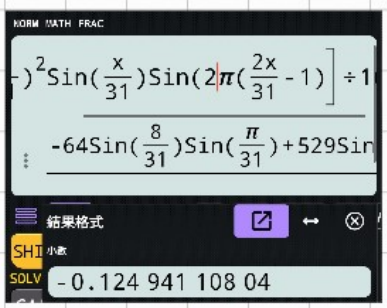
a_3



a_4



b_1



b_2

b_3

$$S_4(z) = \frac{0.459205}{2} + 0.033542\cos 4z$$

$$- 0.146756\cos z + 0.232287\sin z$$

$$+ 0.034608\cos 2z - 0.124941\sin 2z$$

$$- 0.038929\cos 3z + 0.082932\sin 3z \quad \#$$

(b)

$$\int_0^1 S_4(x)\,dx \Rightarrow \begin{cases} z = \pi[2x_i - 1] \\ dz = 2\pi\,dx_i \end{cases} \Rightarrow dx_i = \frac{dz}{2\pi}$$

$$\Rightarrow \int_{-\pi}^{\pi} S_4(z)\frac{dz}{2\pi} = 0.229625 \quad \#$$

(c) $\int_0^1 x^2 \sin x\,dx = 0.232443$

$$E_{rr} = |0.232443 - 0.229625| = 6.3582 \times 10^{-3}$$

$$rel.E_{rr} \approx 0.0248 \approx 2.85\% \quad \#$$

(d)

$$E(S_4) = \sum_{j=0}^{31}\left[f(x_j) - S_4(x_j)\right]^2$$

$$= 2.4342089 \quad \# \text{ 用 python}$$

```python
import math
import numpy as np
PI = np.pi

def f(x):
    return x**2 * math.sin(x)

def S4(x):
    a0 = 0.459205
    a = [-0.146756, 0.054608, -0.038929, 0.033542]
    b = [0.232287, -0.124941, 0.082932, 0]

    total = a0
    for k in range(1, 5):  # k=1~4
        angle = k * PI * (2*x - 1)
        total += a[k-1] * math.cos(angle)
        total += b[k-1] * math.sin(angle)
    return total

def calculate_error():
    m = 32
    error = 0.0

    for j in range(m):
        xj = j / m
        diff = f(xj) - S4(xj)
        error += diff**2

    return error

if __name__ == "__main__":
    error = calculate_error()
    print(f"E(S4) = {error}")

```

```
PS D:\Library\WQ\0524alpha_running> & C:/ProgramData/anaconda3/python.exe "d:/ForClass/1132/1132Numerical/HW8/3(d).py"
E(S4) = 2.43420692849035
```