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COMS W4721

Problem 1.

(a) because we already known that the sequence is iid

$$\rightarrow L(\pi, r) = \prod_{i=1}^{N} P(x_i \mid \pi, r) = \prod_{i=1}^{N} \binom{x_i+r-1}{x_i} \pi^{\sum_{i=1}^{N} x_i} (1-\pi)^{N\cdot r}$$

(b) $$\ell(\pi, r) = \ln L(\pi, r) = \sum_{i=1}^{N} \ln \binom{x_i+r-1}{x_i} + \sum_{i=1}^{N} x_i \ln \pi + N\cdot r \ln(1-\pi)$$

$$\frac{\partial \ell(\pi, r)}{\partial \pi} = \frac{\sum_{i=1}^{N} x_i}{\pi} + (-1)\frac{N\cdot r}{1-\pi} = 0$$

$$\sum_{i=1}^{N} x_i (1-\pi) = N r \cdot \pi$$

$$(N\cdot r + \sum_{i=1}^{N} x_i)\pi = \sum_{i=1}^{N} x_i$$

$$\rightarrow \pi = \frac{\sum_{i=1}^{N} x_i}{\sum_{i=1}^{N} x_i + N\cdot r} = \frac{\bar{x}}{\bar{x}+r} \quad (\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i)$$

$$\rightarrow \hat{\pi}_{ML} = \frac{\bar{x}}{\bar{x}+r}$$

(c). $$\hat{\pi}_{MAP} = \arg\max_{\pi} P(\pi \mid x) = \arg\max_{\pi} \frac{P(x\mid\pi)P(\pi)}{P(x)} = \arg\max_{\pi} P(x\mid\pi)P(\pi) \quad (P(x) \text{ don't have } \pi)$$

$$\rightarrow \hat{\pi}_{map} = \arg\max_{\pi} \prod_{i=1}^{N} P(x_i \mid \pi) P(\pi)$$

$$= \arg\max_{\pi} \sum_{i=1}^{N} \left(\ln P(x_i\mid\pi)\right) + \ln P(\pi)$$

$$\rightarrow \frac{\partial}{\partial \pi} \sum_{i=1}^{N} \left(\ln P(x_i \mid \pi)\right) + \ln P(\pi) = 0$$

$$\frac{\partial}{\partial \pi}\left(\sum_{i=1}^{N} \ln \binom{x_i+r-1}{x_i} + \sum_{i=1}^{N} x_i \cdot \ln\pi + N\cdot r \ln(1-\pi) + \ln \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \pi^{a-1}(1-\pi)^{b-1} \right) = 0$$

$$\frac{\sum_{i=1}^{N} x_i}{\pi} - N\cdot r \frac{1}{1-\pi} + (a-1)\frac{1}{\pi} - (b-1)\frac{1}{1-\pi} = 0$$

$$\frac{1}{\pi}\left(\sum_{i=1}^{N} x_i + a - 1\right) = \frac{1}{1-\pi}(Nr + b - 1)$$

$$\rightarrow \hat{\pi}_{MAP} = \frac{\sum_{i=1}^{N} x_i + a - 1}{\sum_{i=1}^{N} x_i + Nr + a + b - 2} = \frac{\bar{x} + \frac{a-1}{N}}{\bar{x} + r + \frac{a+b-2}{N}}$$

(d) $$P(\pi \mid x) = \frac{\prod_{i=1}^{N} P(x_i \mid \pi)\, P(\pi)}{\int_0^1 \prod_{i=1}^{N} P(x_i \mid \pi)\, P(\pi)\, d\pi}$$

$$P(\pi \mid x) \propto \prod_{i=1}^{N} \left[\binom{x_i + r - 1}{x_i} \pi^{x_i} (1-\pi)^r \right] \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \pi^{a-1} (1-\pi)^{b-1}$$

$$\propto \pi^{\Sigma x_i + a - 1} (1-\pi)^{Nr + b - 1}$$

$$\therefore \rightarrow P(\pi \mid x) \sim \text{Beta}\left(\sum_{i=1}^{N} x_i + a,\ Nr + b\right)$$

(e). $$E(\pi \mid x) = \frac{\sum_{i=1}^{N} x_i + a}{\sum_{i=1}^{N} x_i + Nr + a + b}$$

$$Var(\pi \mid x) = \frac{\left(\sum_{i=1}^{N} x_i + a\right)(Nr + b)}{\left(\sum_{i=1}^{N} x_i + Nr + a + b\right)^2 \left(\sum_{i=1}^{N} x_i + Nr + a + b + 1\right)}$$

$\triangle$(if $x \sim \text{Beta}(\alpha, \beta)$, $E(x) = \frac{\alpha}{\alpha+\beta}$, $Var(x) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$)

$\therefore$ $\hat{\pi}_{ML}$ is the mean of $\text{Beta}\left(\sum_{i=1}^{N} x_i,\ Nr\right)$ $(a' = 0,\ b' = 0)$

$\hat{\pi}_{MAP}$ is the mean of $\text{Beta}\left(\sum_{i=1}^{N} x_i + a - 1,\ Nr + b - 1\right)$ $(a' = a-1,\ b' = b-1)$