
Mathematical Preliminaries (**Vector & Matrix Algebra**)

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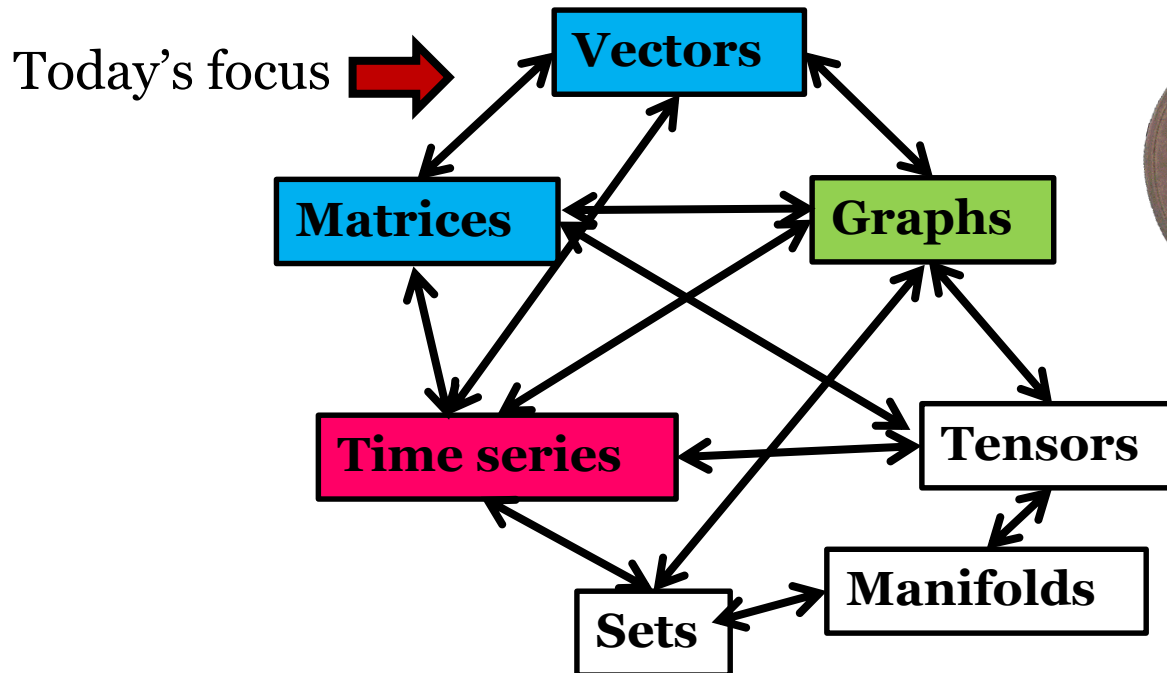
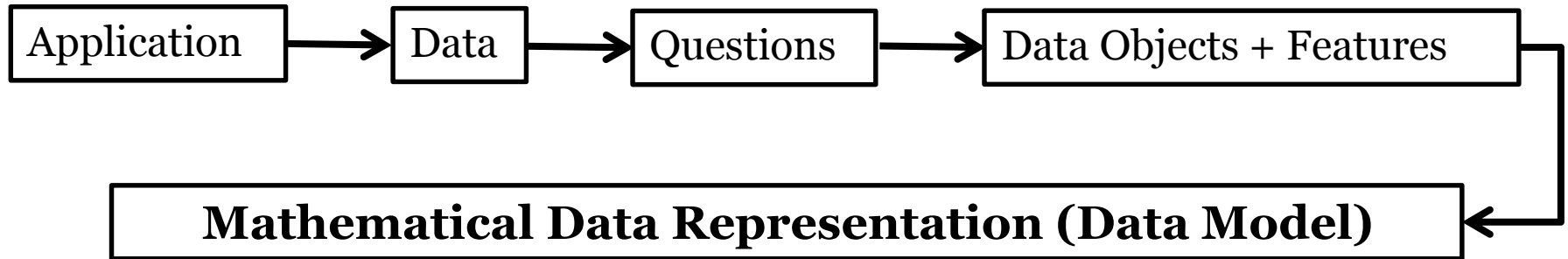
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Mathematical Preliminaries

VECTOR & MATRIX ALGEBRA

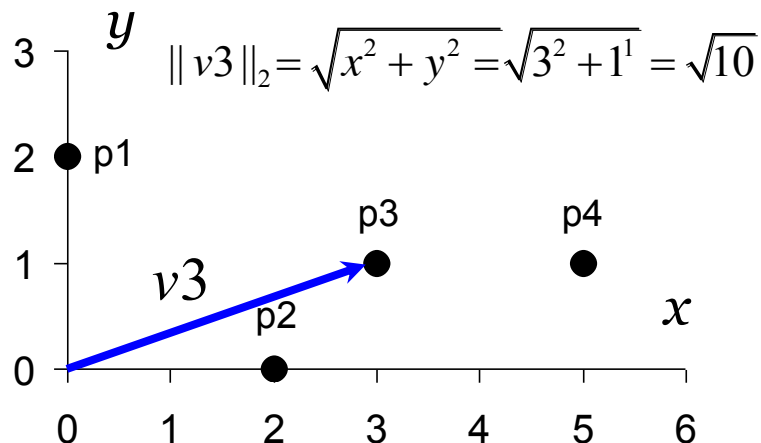
Recap: Data Mining Process



Not one hat fits all
More than one models
are needed
Models are related

Vectors in Low- and High-dimensional Spaces

Points in 2-dimensional space



Data Points in 2-d

point	x	y
p1	0	2
p2	2	0
p3	3	1
p4	5	1

Do not mix with $length(v)$ in R:
 • number of vector components, d

Point \longleftrightarrow Vector

$$p_3 = (x, y) \leftrightarrow v_3 = (x, y)$$

Vector has:

- The origin (0,0)
- The direction
- The length/norm: $\|v\|$

In d -dimensional space:

$$p = (p_1, p_2, \dots, p_d) \in R^d \leftrightarrow$$

$$v = (v_1, v_2, \dots, v_d) \in R^d$$

Vector length/norm (e.g. L_2 -norm):

scalar \rightarrow $\|v\|_2 = \sqrt{\sum_{k=1}^d (v_k)^2} \in R$

Normalized Vector
 (L_2 -norm=1):

$$u = \frac{v}{\|v\|}$$

Ex #1: Vector Norm

Consider the vector in a two-dimensional space:

$$\boldsymbol{v} = (1, -2) \in \mathbb{R}^2$$

- a. What is the length, i.e., the L_2 –norm of this vector? Show calculations by hand.
Validate the result by showing the R code that does the same.

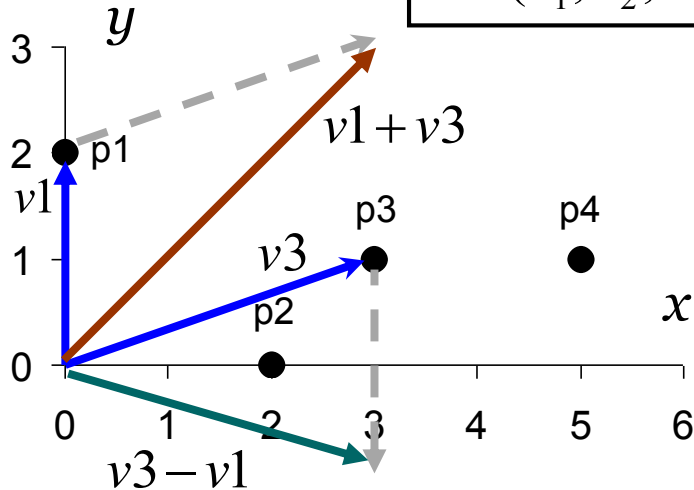
$$||\boldsymbol{v}|| =$$

- b. Normalize this vector to the unit length? Show calculations by hand.
Validate the result by showing the R code that does the same.

$$||\boldsymbol{v}_n|| =$$

Some Vector Operations

$$u = (u_1, u_2, \dots, u_d) \in R^d \text{ and } v = (v_1, v_2, \dots, v_d) \in R^d$$



Scaling:

vector $\rightarrow \alpha \cdot u = (\alpha \cdot u_1, \dots, \alpha \cdot u_d) \in R^d$

Vector Sum:

vector $\rightarrow u + v = (u_1 + v_1, \dots, u_d + v_d) \in R^d$

Vector Difference:

vector $\rightarrow u - v = (u_1 - v_1, \dots, u_d - v_d) \in R^d$

Data Points in 2-d

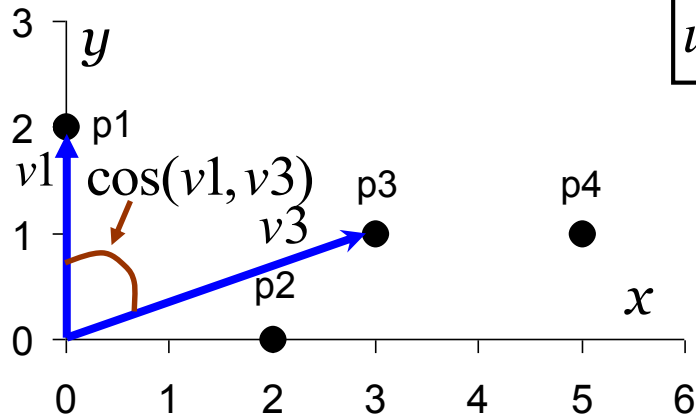
point	x	y
p1	0	2
p2	2	0
p3	3	1
p4	5	1

Scalar Product of Two Vectors:

scalar $\rightarrow (u, v) = u_1 \cdot v_1 + \dots + u_d \cdot v_d \in R$

- $v = c(5, 1, 3); u = c(2, 5, 5)$
- $u+v; u-v;$
- $u \cdot v; 2 \cdot v;$

Cosine between Two Vectors



Data Points in 2-d

point	x	y
p1	0	2
p2	2	0
p3	3	1
p4	5	1

$$\begin{aligned} \|v1\| &= \sqrt{4} = 2 \\ \|v3\| &= \sqrt{10} \\ (v1, v3) &= 3 \cdot 0 + 1 \cdot 2 = 2 \\ \cos(v1, v3) &= \frac{2}{2 \cdot \sqrt{10}} = \frac{1}{\sqrt{10}} \end{aligned}$$

$$u = (u_1, u_2, \dots, u_d) \in R^d \text{ and } v = (v_1, v_2, \dots, v_d) \in R^d$$

Scalar Product of Two Vectors:

scalar $\rightarrow (u, v) = u_1 \cdot v_1 + \dots + u_d \cdot v_d \in R$

Vector length/norm (e.g. L_2 -norm):

scalar $\rightarrow \|v\|_2 = \sqrt{\sum_{k=1}^d (v_k)^2} \in R$

Cosine between Two Vectors:

scalar $\rightarrow \cos(u, v) = \frac{(u, v)}{\|u\| \cdot \|v\|} \in R$

Orthogonal Vectors:

$$\begin{aligned} u \perp v &\Rightarrow \cos(u, v) = 0 \Rightarrow (u, v) = 0 \\ u &= (1, 1), v = (1, -1) \end{aligned}$$

Ex #2: Scalar/Inner Product & Cosine

Consider two vectors in a two-dimensional space:

$$\mathbf{v} = (1, -2), \mathbf{u} = (2, 1) \in \mathbb{R}^2$$

- a. What is the scalar product (aka inner product) of these two vectors?
Show calculations by hand. Validate the result by showing the R code that does the same. Is scalar product symmetric, i.e. $(\mathbf{v}, \mathbf{u}) = (\mathbf{u}, \mathbf{v})$?

$$(\mathbf{v}, \mathbf{u}) =$$

$$(\mathbf{u}, \mathbf{v}) =$$

- b. What is the value of $\cos(\mathbf{u}, \mathbf{v})$? Show calculations by hand.
Validate the result by showing the R code that does the same.
Are these two vectors perpendicular, i.e., angle is 90° ?

$$\cos(\mathbf{u}, \mathbf{v}) =$$

Ex #3: Scalar/Inner Product & Cosine

Consider two vectors in a four-dimensional space:

$$\mathbf{v} = (1, -2, 1, -2), \mathbf{u} = (2, 1, 2, 1) \in \mathbb{R}^4$$

- a. What is the scalar product (aka inner product) of these two vectors?
Show calculations by hand. Validate the result by showing the R code that does the same. Is scalar product symmetric, i.e. $(\mathbf{v}, \mathbf{u}) = (\mathbf{u}, \mathbf{v})$?

$$(\mathbf{v}, \mathbf{u}) =$$

$$(\mathbf{u}, \mathbf{v}) =$$

- b. What is the value of $\cos(\mathbf{u}, \mathbf{v})$? Show calculations by hand.
Validate the result by showing the R code that does the same.
Are these two vectors perpendicular, i.e., angle is 90° ?

$$\cos(\mathbf{u}, \mathbf{v}) =$$

Vector Transpose (v^T)

v

v^T

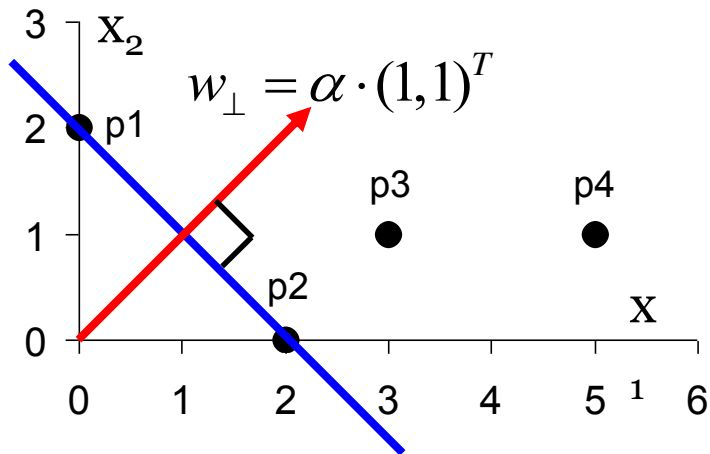
- $v = c(5,1,3);$
- $vt = t(v);$
- $help(t)$

Lines, Planes, Hyperplanes, Normal Vectors

Line in 2-dimensions:

$$y = ax + b, \text{ or equivalently}$$

$$l : a_1x_1 + a_2x_2 + b = 0$$



$$l : x_1 + x_2 - 2 = 0$$

$$w_{\perp} = (1, 1)^T \text{ and } b = -2$$

Line, $l \longleftrightarrow$ **Normal Vector**
(orthogonal to l)

$$l \leftrightarrow w_{\perp} = (a_1, a_2)^T$$

$$x = (x_1, x_2)$$

$$w_{\perp} = (a_1, a_2)^T$$

$$l : x \cdot w_{\perp} + b = 0$$

Plane in 3-dimensions:

$$a_1x_1 + a_2x_2 + a_3x_3 + b = 0$$

Hyper-plane in d -dimensions:

$$a_1x_1 + a_2x_2 + \dots + a_dx_d + b = 0$$

$$l \leftrightarrow w_{\perp} = (a_1, a_2, \dots, a_d)^T$$

Ex #4: Normal Vector to a Line

Consider the line in a 2-dimensional space:

$$l: x_1 - x_2 + 2 = 0 \in \mathbb{R}^2$$

a. What is the normal vector w_\perp for the line, i.e. the perpendicular vector to this line?

$$w_\perp(l) = ?$$

b. What is the value of the intercept b for this line?

$$b = ?$$

c. Choose any point p that lies on this line and give its coordinates:

$$p = (x_1 = \quad, x_2 = \quad) \in l$$

d. Show (by manual calculations) that the following is true:

$$(p, w_\perp) + b = 0$$

Ex #5: Normal Vector to a Plane

Consider the plane in a 3-dim. space:

$$\alpha: x_1 + x_2 - x_3 - 1 = 0 \in \mathbb{R}^3$$

a. What is the normal vector w_\perp for the plane, i.e. perpendicular vector to this plane?

$$w_\perp(\alpha) = ?$$

b. What is the value of the intercept b for this plane?

$$b = ?$$

c. Choose any point p that lies on this plane and give its coordinates:

$$p = (x_1 = \quad, x_2 = \quad, x_3 = \quad) \in \alpha$$

d. Show (by manual calculations) that the following is true:

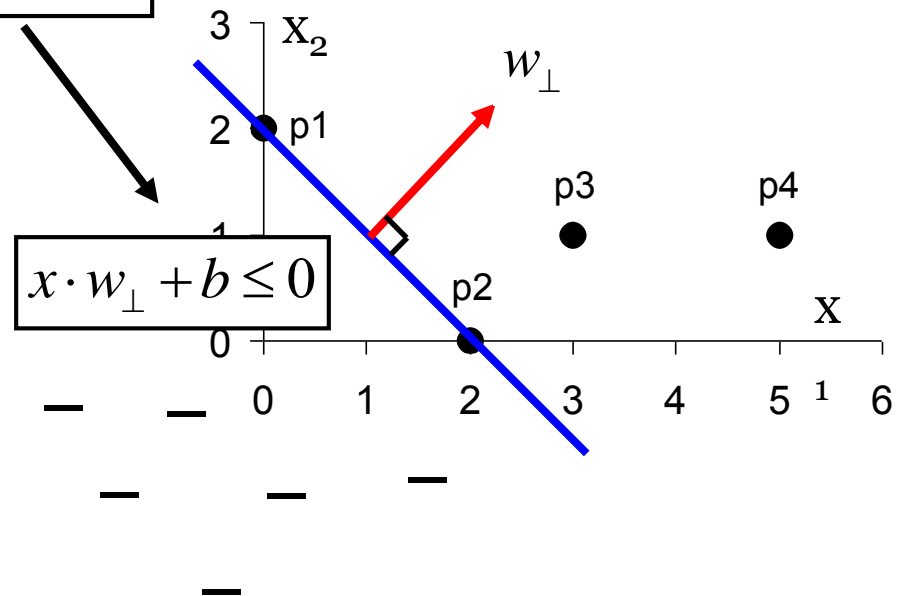
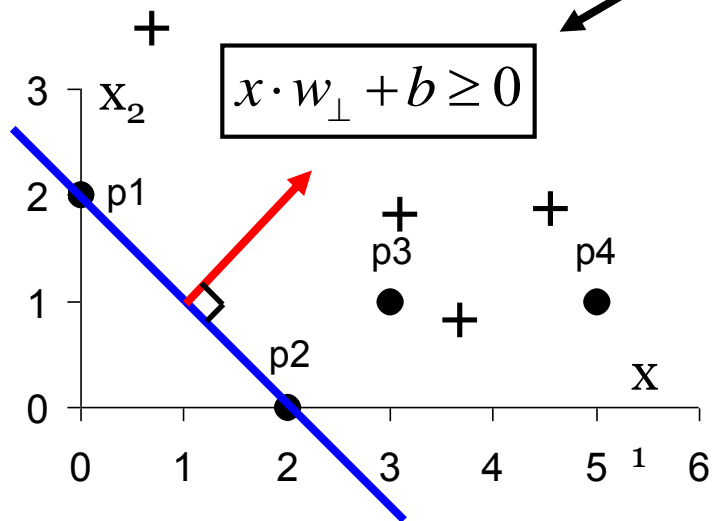
$$(p, w_\perp) + b = 0$$

Half-Planes, Half-Spaces, Half-Hyperspaces

$$x = (x_1, x_2, \dots, x_d)$$

$$w_{\perp} = (a_1, a_2, \dots, a_d)^T$$

$$x \cdot w_{\perp} + b = 0$$



$$x = p3 = (3, 1)$$

$$w_{\perp} = (1, 1)^T \text{ and } b = -2$$

$$p3 \cdot w_{\perp} + b = 3 \cdot 1 + 1 \cdot 1 - 2 = 2 \geq 0$$

Ex #6: Half-Planes

Consider the plane in a 3-dim. space:

$$\alpha: x_1 + x_2 - x_3 - 1 = 0 \in \mathbb{R}^3$$

a. Give coordinates of any point p that lies in the **positive half-plane** of this plane:

$$p = (?, ?, ?) \in \alpha$$

b. Show (by manual calculations) that the following is true:

$$(p, w_{\perp}) + b > 0$$

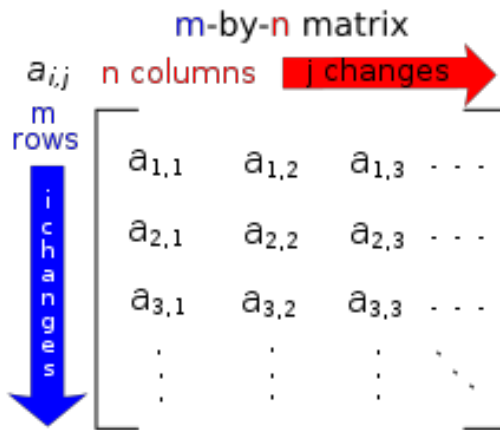
d. Give coordinates of any point q that lies in the **negative half-plane** of this plane:

$$q = (?, ?, ?) \in \alpha$$

e. Show (by manual calculations) that the following is true:

$$(q, w_{\perp}) + b < 0$$

Matrix and its Transpose



$$(A^T)_{i,j} = A_{j,i}$$

$$(A^T)^T = A$$

$$\begin{array}{c} \xrightarrow{\text{blue}} \\ \xrightarrow{\text{brown}} \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -6 & 0 \end{bmatrix}^T = \begin{array}{c} \downarrow \text{blue} \quad \downarrow \text{brown} \\ \begin{bmatrix} 1 & 0 \\ 2 & -6 \\ 3 & 0 \end{bmatrix} \end{array}$$

- $A = \text{matrix}(c(1,0,2,-6,3,0), \text{nrow}=2, \text{ncol}=3);$
- $A;$
- $B = t(A);$
- $B;$
- $t(B);$
- $\text{help}(t)$

Ex #7: Transpose of the Matrix

Consider the following 3-by-2 matrix:

$$\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 0 & 3 \\ 2 & -1 \end{bmatrix}$$

- a. What is the value of $\mathbf{A}[3, 2] =$?
- b. Show its transpose matrix and validate with R code: $\mathbf{A}^T =$
- c. What is the value of $\mathbf{A}^T[1, 2] =$?
- d. If the matrix had m rows and n columns, then how many rows the transpose matrix will have?

Some Matrix Operations: Element-by-Element

$$\dim(A) = \dim(B) = m \times n$$

Diagram illustrating element-by-element operations for matrices A and B of dimension $m \times n$. The operations are:

- $(c \cdot A)_{i,j} = c \cdot A_{i,j}$
- $(A \pm B)_{i,j} = A_{i,j} \pm B_{i,j}$
- $(A \otimes B)_{i,j} = A_{i,j} \cdot B_{i,j}$
- $(A / B)_{i,j} = A_{i,j} / B_{i,j}$

```
A=matrix(c(54,49,49,41,26,43,49,50,58,71),nrow=5,ncol=2))  
B=matrix(c(1:10), nrow=5, ncol=2))
```

To perform element-by-element ops:

```
2*A+3; A+B; A-B; A*B; A/B;
```

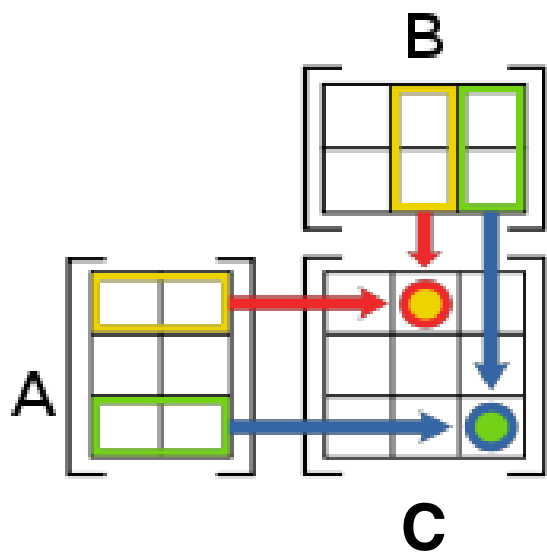
Matrix-Matrix Multiplication

$$\dim(A) = m \times n$$

$$\dim(B) = n \times k$$

$$C = A * B$$

$$\dim(C) = m \times k$$



$$(A * B)_{i,j} = A_{i,1}B_{1,j} + A_{i,2}B_{2,j} + \dots + A_{i,n}B_{n,j}$$

$$= \sum_{r=1}^n A_{i,r}B_{r,j}$$

$$\begin{bmatrix} \underline{1} & \underline{0} & \underline{2} \\ -1 & 3 & 1 \end{bmatrix}_{2 \times 3} \times \begin{bmatrix} 3 & \underline{1} \\ 2 & \underline{1} \\ 1 & \underline{0} \end{bmatrix}_{3 \times 2} = \begin{bmatrix} 5 & \underline{1} \\ 4 & 2 \end{bmatrix}_{2 \times 2}$$

In R:

$$C = A \%*\% B;$$

Ex #8: Matrix-Matrix Multiplication

1. Generate in R a 3-by-3 matrix A filled with one's everywhere but the diagonal; and with 3's on the diagonal.
2. Generate in R a 3-by-2 matrix B filled with two's.

a. Multiply in R matrix A by matrix B and print the resulting matrix C:

$$\mathbf{C} = \mathbf{AB} =$$

b. What is the size of this new matrix **C**, i.e. number of rows and cols?

c. Can you multiply matrix B by A to get matrix D (why or why not):

$$\mathbf{D} = \mathbf{BA} =$$

d. Show with manual calculations how you will get the value of:

$$\mathbf{C}[2, 2] =$$

Ex #9: Projection Matrix

1. Generate in R a 4-by-2 matrix A and plot its 4 rows as points in 2-dim.
2. Generate a 2-by-2 matrix with diagonal elements as one's and off-diagonal matrix as zero's (aka *identity* matrix, I).

a. Multiply in R matrix A by matrix I and print the resulting matrix C:

$$C = A I =$$

b. Set the $I[2, 2]$ element to zero and assign the new matrix to P.

c. Multiple matrix A by the modified matrix I to get matrix D:

$$D = A P =$$

d. Add the rows of matrix D as four points in 2-dim. to your original plot of A.

Do you observe that these new points are projections of the original points? What axis are the points of A projected to in D?

e. How will you modify the identity matrix I to project the points of A on the other axis?

Advanced Topics: Optional

Inverse of a Square Matrix, $\text{nrow}(A)=\text{ncol}(A)$

Identity Matrix, I_n

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determinant, $\det(A)$, $|A|$

scalar: $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \textcircled{ad} - \textcircled{bc}$

```
A = matrix(rnorm(16), nrow=4, ncol=4);  
IA = solve(A);  
help(solve);
```

Matrix Inverse, A^{-1}

$$A * A^{-1} = A^{-1} * A = I$$

$$A^{-1} \text{ exists} \Leftrightarrow \det(A) \neq 0$$

2-by-2 Matrix Inverse

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

***n*-by-*n* Matrix Inverse**

- LU-factorization
- Gaussian elimination
- Gauss-Jordan elimination

Mathematical Preliminaries

MATRIX DECOMPOSITION & TRANSFORMATIONS

Linear Transformation Matrix

Matrix-based Transformation of Vectors

$$V_{old} \xrightarrow{\text{matrix, } A} V_{new}$$

- **Scaling Matrix**
- **Reflection Matrix**
- **Rotation Matrix**
- **Projection Matrix**

Notation

$\mathbf{u} = [u_1, u_2, \dots, u_p]$ – row vector

$\mathbf{v} = \begin{bmatrix} v_1 \\ \dots \\ v_m \end{bmatrix}$ – column vector

$\mathbf{A} = \mathbf{A}_{m \times p} = \begin{bmatrix} a_{11} & \dots & a_{1p} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mp} \end{bmatrix}$ – $m \times p$ matrix

$\mathbf{I} = \mathbf{I}_{m \times m} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 1 \end{bmatrix}$ – identity matrix

Scaling Matrix

Row vectors:

$$v_{old} \in R^p \xrightarrow[\text{matrix, } A]{\text{scalar, } \lambda \in R^+} v_{new} = \lambda \cdot v_{old} \in R^p$$

$$A_\lambda = \lambda \cdot I_{p \times p} = \begin{bmatrix} \lambda & 0 & \dots & 0 \\ 0 & \lambda & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \lambda \end{bmatrix} \text{—scaling matrix}$$

$$v_{new} = v_{old} \cdot A = v_{old} \cdot \lambda \cdot I_{p \times p}$$

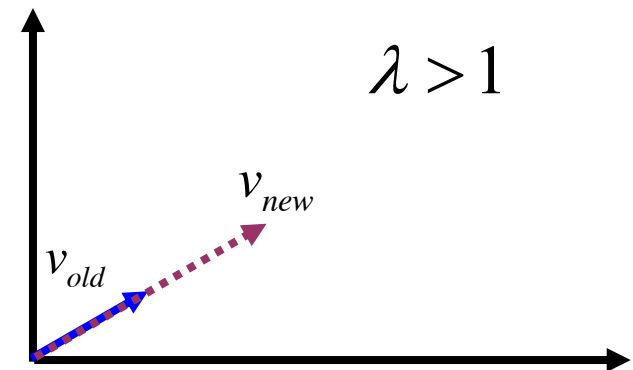
• **Unchanged:**

• **Direction** of a vector

• **Changed:**

• **Vector norm/length**

$$\|v_{new}\| = \lambda \cdot \|v_{old}\|$$



Reflection Matrix

Row vectors:

$$v_{old} \in R^p \xrightarrow{\text{matrix, } A} v_{new} = v_{old} \cdot A \in R^p$$

$$A = I'_{p \times p} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{—reflection matrix; some elements are -1}$$

$$v_{new} = v_{old} \cdot A$$

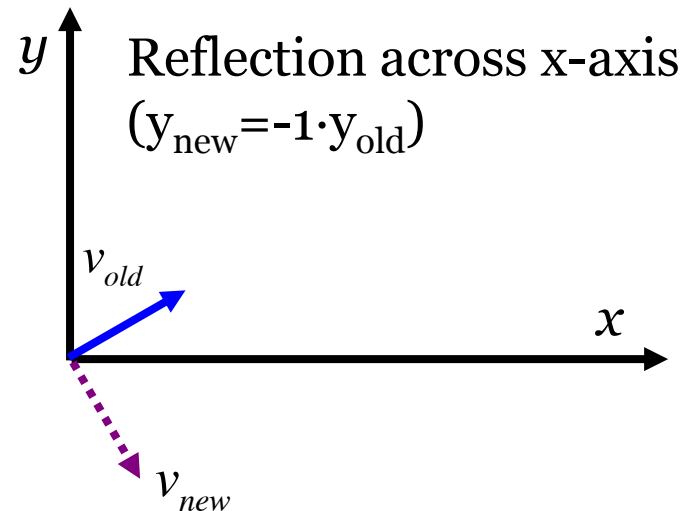
Equivalent to: multiplying one or more vector components by -1

•**Changed:**

- **Direction** of a vector
- Reflects across ≥ 1 axis

•**Unchanged:**

- Vector **norm/length**



Rotation Matrix

Row vectors:

$$v_{old} \in R^p \xrightarrow[\substack{\text{matrix, } A \\ \det(A)=1 \\ A^T=A^{-1}}]{ } v_{new} = v_{old} \cdot A \in R^p$$

•Changed:

•**Direction** of a vector

•Unchanged:

•**Vector norm/length**

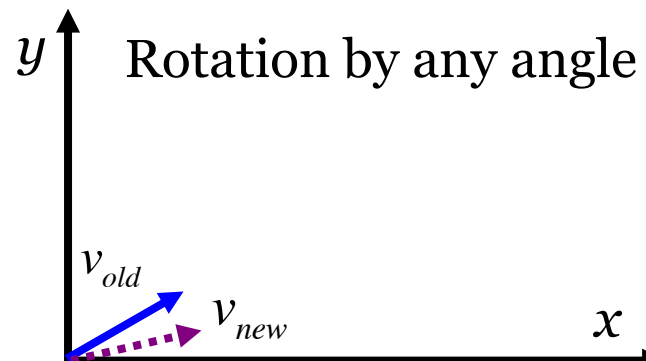
$$A = A_{p \times p} = \begin{bmatrix} a_{11} & \dots & a_{1p} \\ \dots & \dots & \dots \\ a_{p1} & \dots & a_{pp} \end{bmatrix} - p \times p \text{ orthogonal matrix}$$

$$A^T = A^{-1} - \text{orthogonal matrix}$$

$$\det(A)=1$$

Example: $p=2$

$$A(\theta) = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} - \text{clockwise rotation by } \theta$$



Projection Matrix

Row vectors:

$$v_{old} \in R^p \xrightarrow[\text{matrix, } A]{} v_{new} = v_{old} \cdot A \in R^d, d < p$$

•Changed:

- Dimensionality** of a vector
- Direction** of a vector
- Vector norm/length**