CHAPTER 7

THE Z-TRANSFORM

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7.0 Introduction

- *z*-transform is the discrete-time counterpart of the Laplace transform, however, they have some important distinctions that arise from the fundamental differences between continuous-time and discrete-time signals and systems.

- *z*-transform expand the applications in which the discrete-time Fourier transform can or can not be used.

7.1.1 Introduction of The z-Transform

The *z-transform* of *x*[*n*] is defined as $X(z)=\sum_{n=-\infty}^{\infty} x[n]z^{-n}$, where *z* is a complex variable. *X*(*z*) is a Laurent series (洛朗级数).

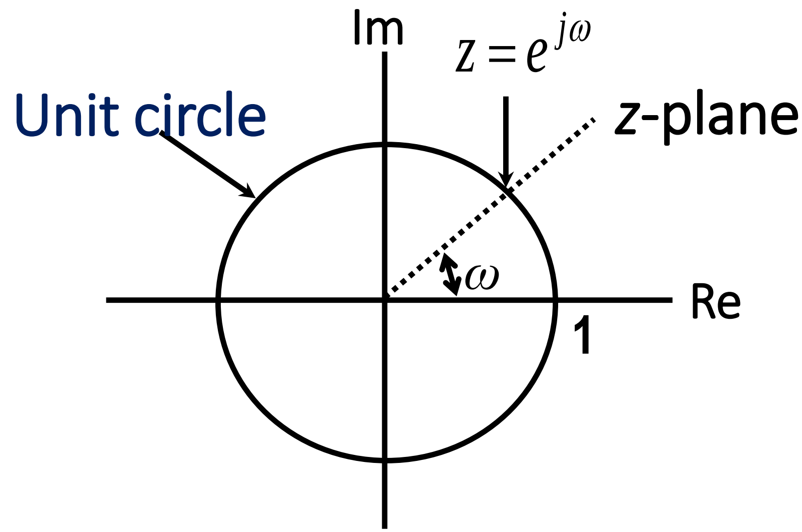
Expressing the complex variable *z* in polar form as $z=re^{j\omega}$,

$$X(re^{j\omega})=\sum_{n=-\infty}^{\infty} x[n](re^{j\omega})^{-n}=\sum_{n=-\infty}^{\infty}\left\{x[n]r^{-n}\right\}e^{-j\omega n}$$

- $X(re^{j\omega})$ is the Fourier transform of *x*[*n*] multiplied by a real exponential r^{-n}.
- For *r* = 1, or equivalently, |*z*| = 1, *z*-transform equation reduces to the Fourier transform. $X(z)\big|_{z=e^{j\omega}}=X(e^{j\omega})=\mathcal{F}\{x[n]\}$
- *Different* from the continuous-time case, the *z*-transform reduces to the Fourier transform *on the contour* in the complex *z*-plane *corresponding to a circle with a radius of unity*.

7.1 The z-Transform

✓The *z*-transform reduces to the Fourier transform for values of *z* on *the unit circle*.



- For convergence of the *z*-transform, we require that the Fourier transform of $x[n]r^{-n}$ converge.

- In general, the *z*-transform of a sequence has associated with it a range of values of *z* for which *X*(*z*) converges, and this range of values is referred to as the *region of convergence* (*ROC*).

- If the *ROC* includes the unit circle, then the Fourier transform also converges.

7.1.2 Examples

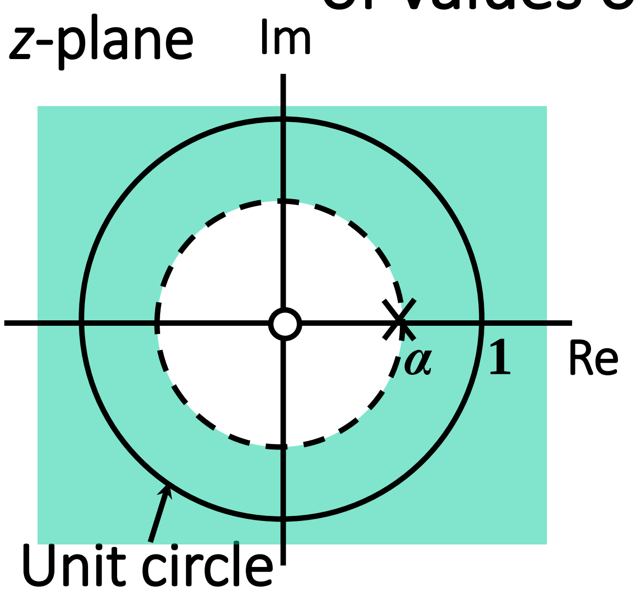
Example 7.1 Consider the signal $x[n]=\alpha^n u[n]$.

Sol :

$$X(z)=\sum_{n=-\infty}^{\infty}\alpha^n u[n] z^{-n}=\sum_{n=0}^{\infty}\left(\alpha z^{-1}\right)^n$$

For convergence of $X(z)$, we require that $\sum_{n=0}^{\infty}\left|\alpha z^{-1}\right|^n<\infty$

Consequently, the region of convergence is the range of values of z for which $\left|\alpha z^{-1}\right|<1$.



Then

$$X(z)=\sum_{n=0}^{\infty}\left(\alpha z^{-1}\right)^n=\frac{1}{1-\alpha z^{-1}}=\frac{z}{z-\alpha},\quad |z|>|\alpha|$$

$$u[n] \overset{ZT}{\longleftrightarrow} \frac{z}{z-1},\quad |z|>1$$

Pole-zero plot and *ROC* for $0<\alpha<1$

Example 7.2 Determine the z-transform of $x[n] = -\alpha^n u[-n-1]$.

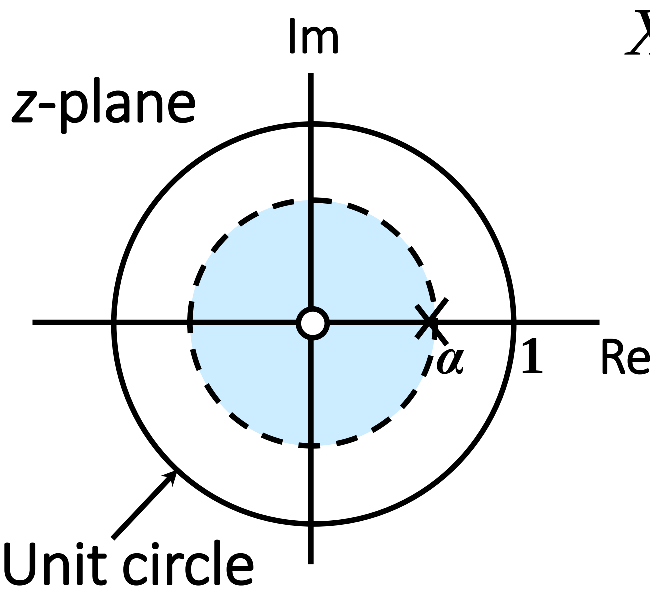
Sol: $$X(z) = -\sum_{n=-\infty}^{\infty} \alpha^n u[-n-1] z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} \alpha^n z^{-n} = -\sum_{n=1}^{\infty} \alpha^{-n} z^n = 1 - \sum_{n=0}^{\infty} (\alpha^{-1} z)^n$$

If $\left|\alpha^{-1} z\right| < 1$, this sum converges and

$$X(z) = 1 - \frac{1}{1-\alpha^{-1} z}$$

$$= \frac{1}{1-\alpha z^{-1}} = \frac{z}{z-\alpha}, \quad |z| < |\alpha|$$



Pole-zero plot and *ROC* for $0 < \alpha < 1$

Example 7.3

Consider a signal that is the sum of two real exponentials:

$$x[n]=7\left(\frac{1}{3}\right)^{n}u[n]-6\left(\frac{1}{2}\right)^{n}u[n]$$

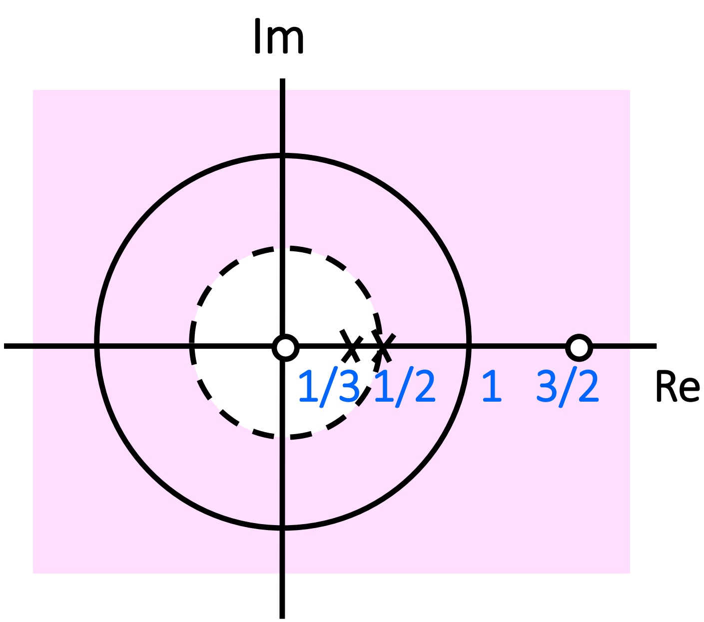
Sol： The *z*-transform is then

$$X(z)=7\sum_{n=-\infty}^{\infty}\left(\frac{1}{3}\right)^{n}u[n]z^{-n}-6\sum_{n=-\infty}^{\infty}\left(\frac{1}{2}\right)^{n}u[n]z^{-n}$$

$$=\frac{7}{1-\frac{1}{3}z^{-1}}-\frac{6}{1-\frac{1}{2}z^{-1}}$$

$$=\frac{1-\frac{3}{2}z^{-1}}{(1-\frac{1}{3}z^{-1})(1-\frac{1}{2}z^{-1})}$$

$$=\frac{z\left(z-\frac{3}{2}\right)}{\left(z-\frac{1}{3}\right)\left(z-\frac{1}{2}\right)}\qquad |z|>\frac{1}{2}$$



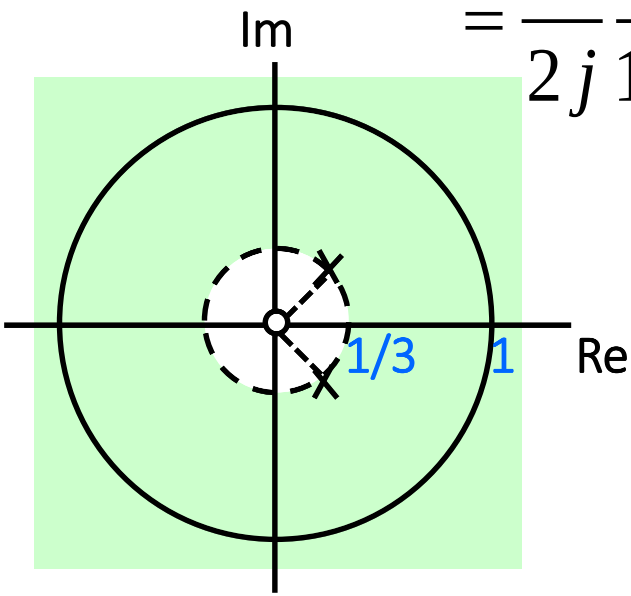
Example 7.4 Consider the signal $x[n]=\left(\frac{1}{3}\right)^n \sin\left(\frac{\pi}{4}n\right)u[n].$

Sol: $x[n]=\frac{1}{2j}\left(\frac{1}{3}e^{j\pi/4}\right)^n u[n]-\frac{1}{2j}\left(\frac{1}{3}e^{-j\pi/4}\right)^n u[n]$

The *z*-transform of this signal is

$$X(z)=\frac{1}{2j}\sum_{n=0}^{\infty}\left(\frac{1}{3}e^{j\pi/4}z^{-1}\right)^n-\frac{1}{2j}\sum_{n=0}^{\infty}\left(\frac{1}{3}e^{-j\pi/4}z^{-1}\right)^n$$

$$=\frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{j\pi/4}z^{-1}}-\frac{1}{2j}\frac{1}{1-\frac{1}{3}e^{-j\pi/4}z^{-1}} \quad |z|>\frac{1}{3}$$



$$X(z)=\frac{\frac{1}{3\sqrt{2}}z}{(z-\frac{1}{3}e^{j\pi/4})(z-\frac{1}{3}e^{-j\pi/4})}$$

7.2 The Region of Convergence For The z-Transform

➤Property 1: The *ROC* of $X(z)$ consists of a ring in the z-plane centered about the origin.

➤Property 2: The *ROC* does not contain any poles.

➤Property 3: If $x[n]$ is of finite duration, then the *ROC* is the entire z-plane, except possibly $z = 0$ and/or $z = \infty$.

Example 7.5

Consider the unit sample signal $\delta[n]$.

$$\mathcal{Z}\{\delta[n]\} = \sum_{n=-\infty}^{\infty} \delta[n] z^{-n} = 1 \qquad \mathbf{0 \le |z| \le \infty}$$

On the other hand,

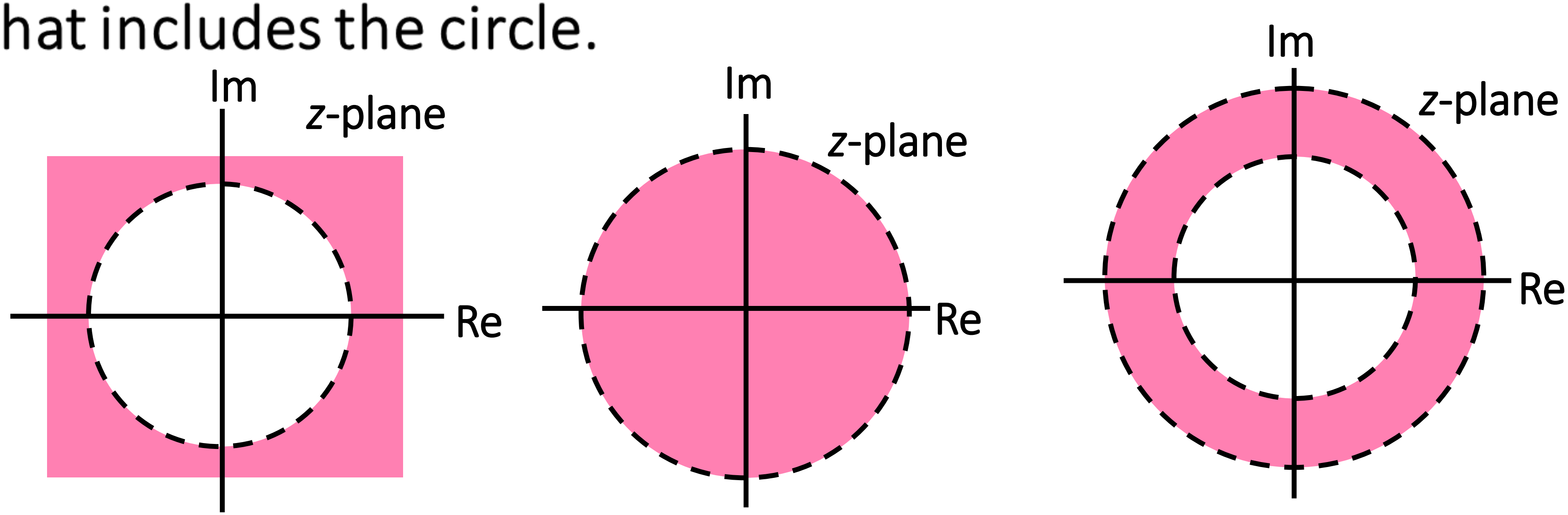
$$\mathcal{Z}\{\delta[n-1]\} = \sum_{n=-\infty}^{\infty} \delta[n-1] z^{-n} = z^{-1}, \quad \mathbf{0 < |z| \le \infty}$$

Similarly,

$$\mathcal{Z}\{\delta[n+1]\} = \sum_{n=-\infty}^{\infty} \delta[n+1] z^{-n} = z, \quad \mathbf{0 \le |z| < \infty}$$

7.2 The Region of Convergence For The z-Transform

- Property 4: If $x[n]$ is a right-sided sequence, and if the circle $|z| = r_0$ is in the *ROC*, then all *finite values* of z for which $|z| > r_0$ will also be in the *ROC*.
- Property 5: If $x[n]$ is a left-sided sequence, and if the circle $|z| = r_0$ is in the *ROC*, then all values of z for which $0 < |z| < r_0$ will also be in the *ROC*.
- Property 6: If $x[n]$ is two sided, and if the circle $|z| = r_0$ is in the *ROC*, then the *ROC* will consist of a ring in the z-plane that includes the circle.



Example 7.6 Consider a two sided sequence $x[n]=b^{|n|}, b>0.$

Sol: $x[n]=b^{n}u[n]+b^{-n}u[-n-1]$

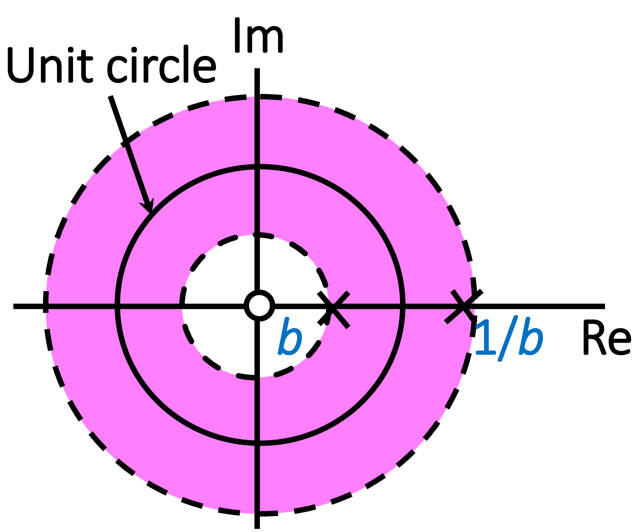
$$b^{n}u[n]\leftrightarrow\frac{1}{1-bz^{-1}},|z|>b,\qquad b^{-n}u[-n-1]\leftrightarrow\frac{-1}{1-b^{-1}z^{-1}},|z|<\frac{1}{b}$$

For $b > 1$, there is no common *ROC*, and thus the sequence will not have a *z*-transform.

For $b < 1$, the *ROC*s overlap, and thus the *z*-transform for the composite sequence is

$$X(z)=\frac{1}{1-bz^{-1}}-\frac{1}{1-b^{-1}z^{-1}}$$

$$=\frac{\frac{b^{2}-1}{b}z}{(z-b)(z-b^{-1})},\quad b<|z|<\frac{1}{b}$$



7.2 The Region of Convergence For The z-Transform

- Property 7: If the z-transform $X(z)$ of $x[n]$ is rational, then its *ROC* is bounded by poles or extends to infinity.

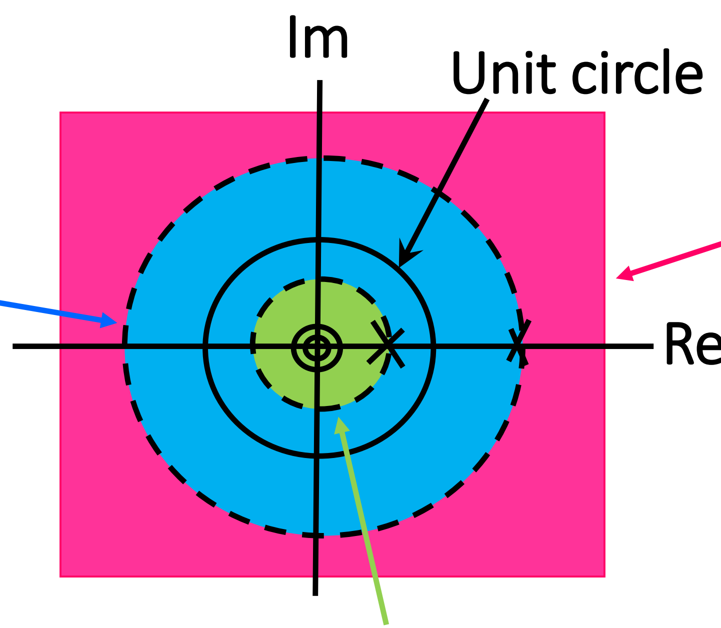
- Property 8: If the z-transform $X(z)$ of $x[n]$ is rational, and if $x[n]$ is right sided, then the *ROC* is the region in the z-plane outside the outermost pole — i.e., outside the circle of radius equal to the largest magnitude of the poles of $X(z)$. Furthermore, if $x[n]$ is causal, then the *ROC* also includes $z = \infty$.

- Property 9: If the z-transform $X(z)$ of $x[n]$ is rational, and if $x[n]$ is left sided, then the *ROC* is the region in the z-plane inside the innermost nonzero pole — i.e., inside the circle of radius equal to the smallest magnitude of the poles of $X(z)$ other than any at $z = 0$ and extending inward to and possibly including $z = 0$. In particular, if $x[n]$ is anti-causal, then the *ROC* also includes $z = 0$.

Example 7.7

Consider all of the possible ROCs that can be connected with the function $X(z)=\dfrac{1}{(1-\frac{1}{3}z^{-1})(1-2z^{-1})}$

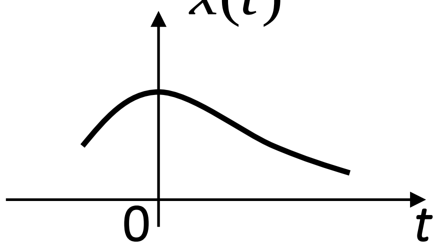
The ROC if $x[n]$ is two sided

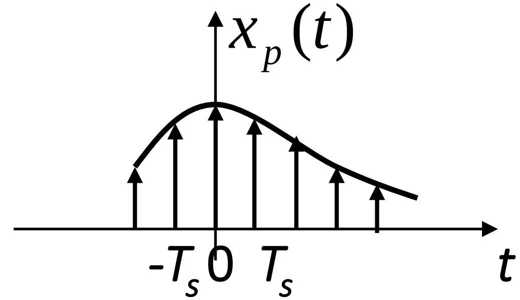


The ROC if $x[n]$ is right sided

The ROC if $x[n]$ is left sided

7.3.1 Relationship Between the Expressions of LT And ZT





$$x_p(t) = x(t) \cdot p(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_p(s) = \int_{-\infty}^{\infty} x_p(t)e^{-st}dt = \int_{-\infty}^{\infty} [\sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)]e^{-st}dt$$

$$= \sum_{n=-\infty}^{\infty} x(nT_s)\int_{-\infty}^{\infty} \delta(t - nT_s)e^{-st}dt = \sum_{n=-\infty}^{\infty} x(nT_s)e^{-snT_s} = \sum_{n=-\infty}^{\infty} x(n)(e^{sT_s})^{-n}$$

Let $e^{sT_s} = z$

$$\boxed{X_p(s)\Big|_{z=e^{sT_s}} = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}}$$

7.3.2 Mapping Between the s-Plane And z-Plane

$$r\,e^{j\theta} = z = e^{sT_s} = e^{(\sigma+j\omega)T_s} = e^{\sigma T_s} e^{j\omega T_s}$$

$$\Downarrow$$

$$r = e^{\sigma T_s} = e^{\frac{2\pi\sigma}{\omega_s}}, \quad \theta = \omega T_s + 2k\pi = 2\pi \frac{\omega}{\omega_s} + 2k\pi$$

✓ The *jω-axis* $\sigma = 0$ on the s-plane is mapped to the unit circle $r = 1$ on the *z*-plane.

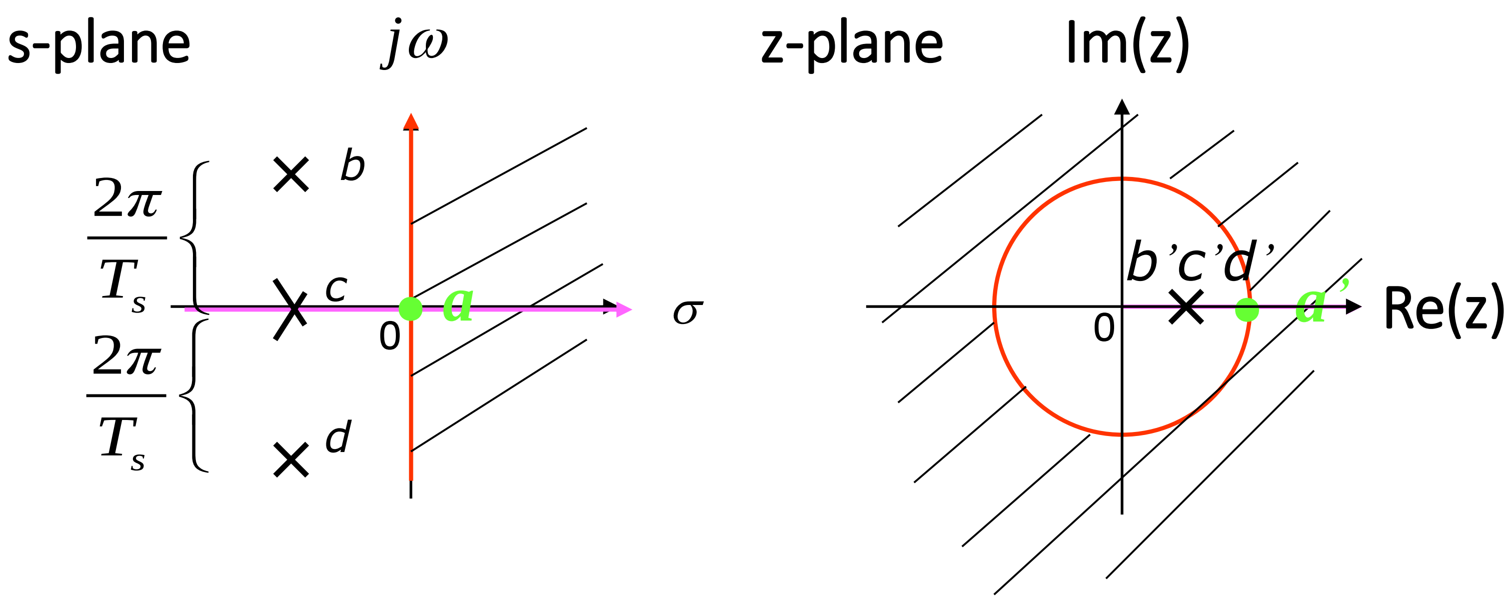
✓The *right half* of the s-plane $\sigma > 0$ is mapped to the region *outside* the unit circle on the *z*-plane, i.e., $r > 1$.

✓The *left half* of the s-plane $\sigma < 0$ is mapped to the region *inside* the unit circle on the *z*-plane, i.e., $r < 1$.

✓The *real axis* on the s-plane $\omega = 0$ is mapped to the *positive real axis* on the *z*-plane, i.e., $\theta = 2k\pi$.

✓The *origin* of the s-plane $\sigma = 0, \omega = 0$ is mapped to the *intersection* of the unit circle and the positive real axis on the *z*-plane, i.e., $z = 1$.

✓ *An arbitrary line parallel to the real axis and pass the points* Of $\frac{j(2k+1)\omega_s}{2}$ on the s-plane is mapped to the *negative real axis* on the *z*-plane, i.e., $\theta = (2k+1)\pi$.



7.4 The Inverse z-Transform

$$x[n]r^{-n} = \mathcal{F}^{-1}\{X(re^{j\omega})\} = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})e^{j\omega n}d\omega$$

Multiplying both sides by r^n yields $x[n] = \frac{1}{2\pi}\int_{2\pi} X(re^{j\omega})(re^{j\omega})^n d\omega$

Changing the variable of integration: $dz = jre^{j\omega}d\omega = jzd\omega$

Thus, the basic inverse z-transform equation is:

$$\boxed{x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz}$$

- The formal evaluation of the integral for a general *X*(*z*) requires the use of contour integration in the complex plane.
- There are two alternative procedures for obtaining a sequence from its *z*-transform: one is partial-fraction expansion, the other is power-series expansion.

X(*z*) can be interpreted as a power series involving both positive and negative powers of *z*. The coefficients in this power series are the sequence values *x*[*n*].

Example 7.8

Consider the *z*-transform $X(z) = \dfrac{3 - \frac{5}{6} z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{3} z^{-1})}, \quad \dfrac{1}{4} < |z| < \dfrac{1}{3}.$

Sol: Performing the partial-fraction expansion, we obtain

$$X(z) = \frac{1}{1 - \frac{1}{4} z^{-1}} + \frac{2}{1 - \frac{1}{3} z^{-1}}$$

$$\left(\frac{1}{4}\right)^n u[n] \xleftrightarrow{ZT} \frac{1}{1 - \frac{1}{4} z^{-1}}, \quad |z| > \tfrac{1}{4}$$

$$-2\left(\frac{1}{3}\right)^n u[-n-1] \xleftrightarrow{ZT} \frac{2}{1 - \frac{1}{3} z^{-1}}, \quad |z| < \tfrac{1}{3}$$

$$x[n] = \left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]$$

If $$X(z)=\frac{3z-\frac{5}{6}}{(z-\frac{1}{4})(z-\frac{1}{3})},\quad \frac{1}{4}<|z|<\frac{1}{3}$$

We have two ways to expand *X*(*z*).

Method 1: Consider $\dfrac{X(z)}{z}=\dfrac{3z-\frac{5}{6}}{z(z-\frac{1}{4})(z-\frac{1}{3})}$

$$\frac{X(z)}{z}=\frac{-10}{z}+\frac{4}{z-\frac{1}{4}}+\frac{6}{z-\frac{1}{3}}\Rightarrow X(z)=-10+\frac{4z}{z-\frac{1}{4}}+\frac{6z}{z-\frac{1}{3}}$$

$$x[n]=-10\delta[n]+4\left(\frac{1}{4}\right)^n u[n]-6\left(\frac{1}{3}\right)^n u[-n-1]$$

Method 2: Directly expanding *X*(*z*) yields $X(z)=\dfrac{1}{z-\frac{1}{4}}+\dfrac{2}{z-\frac{1}{3}}$

$$x[n]=\left(\frac{1}{4}\right)^{n-1}u[n-1]-2\left(\frac{1}{3}\right)^{n-1}u[-n]=4\left(\frac{1}{4}\right)^n u[n-1]-6\left(\frac{1}{3}\right)^n u[-n]$$

Example 7.9

Consider the *z*-transform $X(z) = 4z^2 + 2 + 3z^{-1}, \quad 0 < |z| < \infty.$

Sol: From the power-series definition of the *z*-transform, we can determine the inverse transform of *X*(*z*) by inspection:

$$x[n] = \begin{cases} 4, & n = -2 \\ 2, & n = 0 \\ 3, & n = 1 \\ 0, & otherwise \end{cases}$$

That is,

$$x[n] = 4\delta[n+2] + 2\delta[n] + 3\delta[n-1]$$

Example 7.10 Consider the *z*-transform $X(z) = \dfrac{1}{1-\alpha z^{-1}}, \quad |z| > |\alpha|.$

Sol: From the *ROC*, we can conclude that the corresponding sequence *x*[*n*] is right-sided, so that we arrange the numerator polynomial and the denominator polynomial with a order of the power of *z* decreasing (or a order of the power of z^{-1} increasing).

Then performing *long division*:

$$\begin{array}{r|l} & 1+\alpha z^{-1}+\alpha^2 z^{-2}+\cdots \\ \hline 1-\alpha z^{-1} & 1 \\ & \underline{1-\alpha z^{-1}} \\ & \alpha z^{-1} \\ & \underline{\alpha z^{-1}-\alpha^2 z^{-2}} \\ & \alpha^2 z^{-2} \\ & \vdots \end{array}$$

$$\frac{1}{1-\alpha z^{-1}} = 1+\alpha z^{-1}+\alpha^2 z^{-2}+\cdots$$

$$x[n] = \alpha^n u[n]$$

7.4 The Inverse z-Transform

If the *ROC* is $|z|<|\alpha|$, before performing long division, we arrange the numerator polynomial and the denominator polynomial with a order of the power of *z* increasing (or a order of the power of z^{-1} decreasing).

$$
\begin{array}{r|l}
 & -\alpha^{-1}z-\alpha^{-2}z^{2}-\cdots \\
\hline
-\alpha z^{-1}+1 & 1 \\
 & \underline{1-\alpha^{-1}z} \\
 & \alpha^{-1}z
\end{array}
$$

$$\frac{1}{1-\alpha z^{-1}}=-\alpha^{-1}z-\alpha^{-2}z^{2}-\cdots$$

$$x[n]=-\alpha^{n}u[-n-1]$$

Example 7.11

Consider the *z*-transform $X(z) = e^{\frac{a}{z}} \quad |z| > 0$, determine *x*[*n*].

Sol: Besides long-division method which is used in the case of *X*(*z*) rational, any methods can be used to obtain the power series expression of *X*(*z*). Thus, in this case

$$X(z) = e^{\frac{a}{z}} = \sum_{n=0}^{\infty} \frac{\left(\frac{a}{z}\right)^n}{n!} = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$$

$$x[n] = \frac{a^n}{n!} u[n]$$

Example 7.12 Consider the *z*-transform $X(z)=\dfrac{z^3+2z^2+1}{z(z-1)(z-0.5)}, |z|>1.$

Sol: From the definition of the inverse z-transform,

$$x[n]=\frac{1}{2\pi j}\oint X(z)z^{n-1}dz \underset{\text{Residue Theorem}}{=} \sum_k \operatorname{Res}\left[X(z)z^{n-1}\right]_{z=p_k}$$

$$=\sum_k \operatorname{Res}\left[\frac{z^3+2z^2+1}{(z-1)(z-0.5)}z^{n-2}\right]_{z=p_k}$$

x[*n*]: right sided !

For $n\geq 2$, $X(z)z^{n-1}$ has only two first-order poles: $z_1=1,\ z_2=0.5$

Thus

$$x[n]=\left[\left(\frac{z^3+2z^2+1}{z-0.5}z^{n-2}\right)\right]_{z=1}+\left[\left(\frac{z^3+2z^2+1}{z-1}z^{n-2}\right)\right]_{z=0.5}=8-13(0.5)^n$$

For n=1, $X(z)z^{n-1}$ has three first-order poles: $z_1=1,\ z_2=0.5,\ z_3=0$

Then $$x[n]=8-6.5+2=3.5$$

For $\boldsymbol{n}$=0, $\boldsymbol{X(z)z^{n-1}}$ has two first-order poles: $z_1 = 1$, $z_2 = 0.5$, and a second-order pole: $z_3 = 0$.

$$\mathrm{Res}\left[X(z)z^{n-1}\right]_{z=1} = \left[\left(\frac{z^3+2z^2+1}{z-0.5}z^{-2}\right)\right]_{z=1} = 8,$$

$$\mathrm{Res}\left[X(z)z^{n-1}\right]_{z=0.5} = \left[\left(\frac{z^3+2z^2+1}{z-1}z^{-2}\right)\right]_{z=0.5} = -13$$

$$\mathrm{Res}\left[X(z)z^{n-1}\right]_{z=0} = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left(\frac{z^3+2z^2+1}{z^2(z-1)(z-0.5)}z^2\right)\right]_{z=0} = 6$$

Then

$$x[n] = 8 - 13 + 6 = 1$$

Consequently, $x[n] = \delta[n] + 3.5\delta[n-1] + \left[8 - 13(0.5)^n\right]u[n-2]$

Question: from the *ROC*, we can only know that $x[n]$ is right sided rather than causal, why we finish the discussion at n=0?

In fact, even if we calculate we will find that $x[n]$=0 for n=-1,-2, …

Because

$$x[n]=\sum_k \mathrm{Re}\,s\left[X(z)z^{n-1}\right]_{z=p_k}=-\mathrm{Re}\,s\left[X(z)z^{n-1}\right]_{z=\infty}=\mathrm{Re}\,s\left[X\left(\frac{1}{z}\right)z^{-n+1}\frac{1}{z^2},0\right]$$

$$x[n]=\mathrm{Re}\,s\left[\frac{\frac{1}{z^3}+2\frac{1}{z^2}+1}{\left(\frac{1}{z}-1\right)\left(\frac{1}{z}-0.5\right)}\left(\frac{1}{z}\right)^{n-2}\frac{1}{z^2},0\right]=\mathrm{Re}\,s\left[\frac{z^3+2z+1}{z^{n+1}(z-1)(0.5z-1)},0\right]$$

For $n>-1$, $z=0$ is $(n+1)th$-order pole; for $n\le -1$, $z=0$ is not the singularity of $X(z)z^{n-1}$ any more, thus we have

$$x[n]=-\mathrm{Re}\,s\left[X(z)z^{n-1},\infty\right]=0 \quad for \quad n\le -1$$

In fact, by checking whether or not $z=\infty$ is included in the *ROC*, we can find that $x[n]$ is right-sided, furthermore it is causal, so we just need to consider non-negative values of n.

In the discrete-time case, the Fourier transform can again be evaluated geometrically by considering the pole and zero vectors in the z-plane. However, since in this case the rational function is to be evaluated on the contour $|z| = 1$, we consider the vectors from the poles and zeros to the unit circle rather than to the imaginary axis.

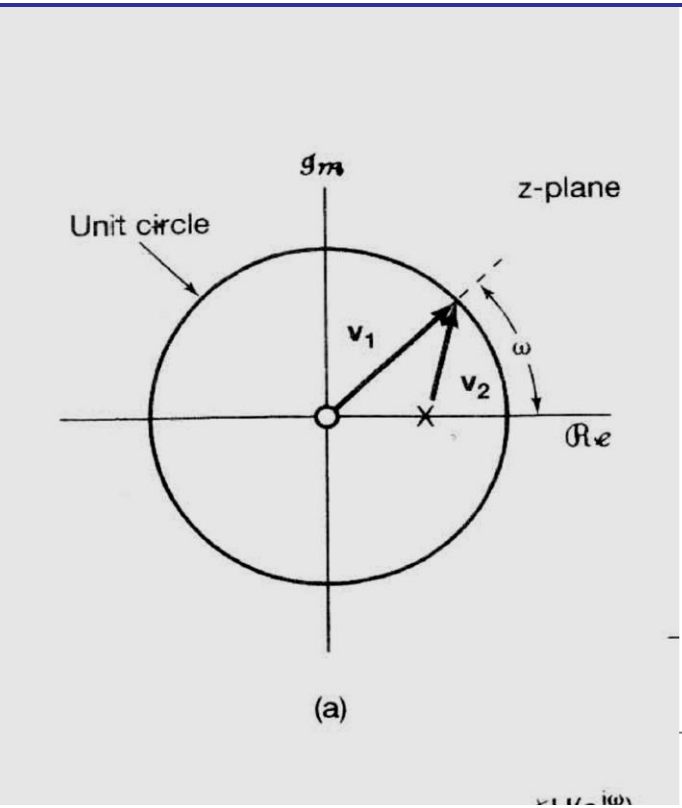
Consider a *first-order* causal discrete-time system with a impulse response: $h[n] = a^n u[n]$

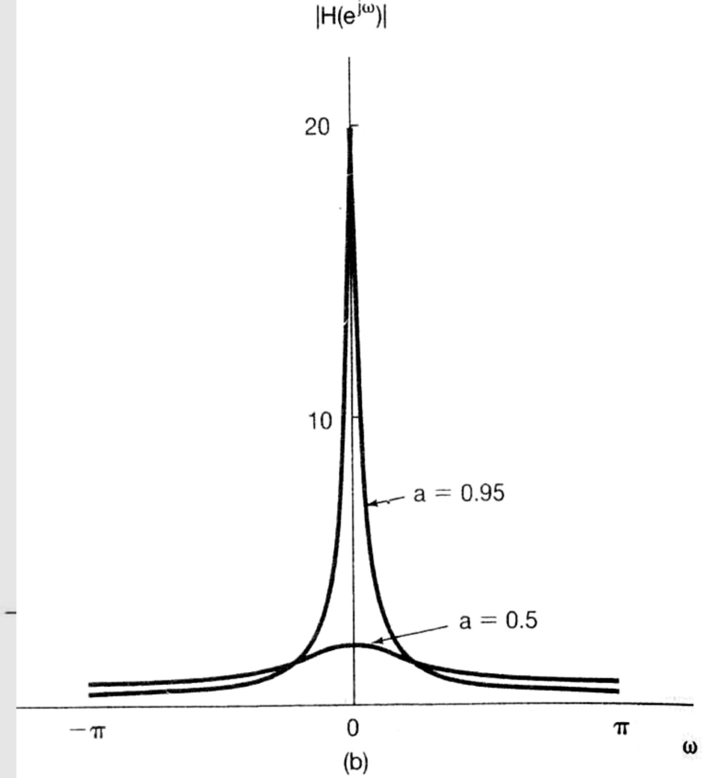
Its *z*-transform is

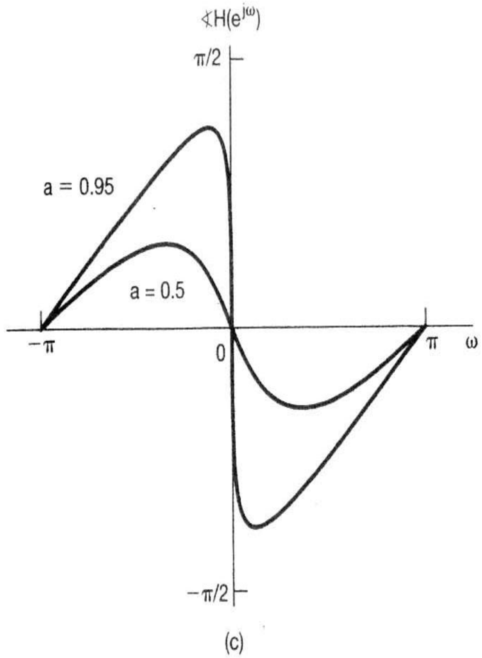
$$H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, \quad |z| > |a|$$

For $|a| < 1$, the *ROC* includes the unit circle, and consequently, the Fourier transform of *h*[*n*] converges and is equal to *H*(*z*) for $z = e^{j\omega}$.

7.5 Geometric Evaluation of The Fourier Transform From The Pole-Zero Plot







The pole-zero plot for $H(z)$, including the vectors from the pole (at $z = a$) and zero (at $z = 0$) to the unit circle.

Magnitude of the frequency response for $a = 0.95$ and $a = 0.5$

Phase of the frequency response for $a = 0.95$ and $a = 0.5$

7.6 Properties of The z-Transform

7.6.1 Linearity

If $\quad x_1[n] \leftrightarrow X_1(z), \quad ROC = R_1$ and $\quad x_2[n] \leftrightarrow X_2(z), \quad ROC = R_2$

then $\quad ax_1[n] + bx_2[n] \leftrightarrow aX_1(z) + bX_2(z), \quad ROC\ containing\ R_1 \cap R_2$

Note: *ROC* is at least the intersection of R_1 and R_2, which could be empty, also can be larger than the intersection.

7.6.2 Time Shifting

If $\quad x[n] \leftrightarrow X(z), \quad ROC = R$

then $\quad x[n-n_0] \leftrightarrow z^{-n_0} X(z), \quad ROC = R$

Except for the possible addition/deletion of the origin or infinity.

7.6.3 Scaling in the *z*-Domain

If $\quad x[n] \leftrightarrow X(z), \quad ROC = R$

then $\quad z_0{}^n x[n] \leftrightarrow X\left(\dfrac{z}{z_0}\right), \quad ROC = |z_0| R$

Special case: when $z_0 = e^{j\omega_0}$, $e^{j\omega_0 n} x[n] \leftrightarrow X(e^{-j\omega_0} z), \quad ROC = R$

Example 7.13

Find the z-transform of the signal $x[n]=\alpha^n u[n]-\alpha^n u[n-1]$.

Sol: From Example 7.1, we know $\alpha^n u[n] \xleftrightarrow{ZT} \dfrac{z}{z-\alpha}, \quad |z|>|\alpha|$

Then from the time shifting property,

$$\alpha^n u[n-1]=\alpha\alpha^{n-1}u[n-1] \xleftrightarrow{ZT} \alpha\frac{z}{z-\alpha}z^{-1}=\frac{\alpha}{z-\alpha}, \quad |z|>|\alpha| \quad (\mathbf{I})$$

So that, $X(z)=\dfrac{z}{z-\alpha}-\dfrac{\alpha}{z-\alpha}=\dfrac{z-\alpha}{z-\alpha}=1, \quad ROC=entire\ z-plane$

In fact, $x[n]=\delta[n] \xleftrightarrow{ZT} 1, \quad entire\ z-plane$

In step ($\mathbf{I}$), one pole at z = 0 was introduced, and it cancelled a zero at the same location.

Be careful when you use the time-shifting property because there is maybe pole-zero cancellation !

7.6.4 Time Reversal

$$x[-n] \leftrightarrow X\left(\frac{1}{z}\right), \quad ROC = \frac{1}{R}$$

Consequence: If $\boldsymbol{z_0}$ is in the *ROC* for $x[n]$, then $\frac{\mathbf{1}}{\boldsymbol{z_0}}$ is in the *ROC* for $x[-n]$.

7.6.5 Time Expansion

$$x_{(k)}[n] \leftrightarrow X(z^k), \quad ROC = R^{1/k}$$

7.6.6 Conjugation

$$x^*[n] \leftrightarrow X^*(z^*), \quad ROC = R$$

Consequence: If $x[n]$ is real, $X(z) = X^*(z^*)$

Thus, if $X(z)$ has a pole (or zero) at $z = z_0$, it must also have a pole (or zero) at the complex conjugate point $z = z_0$*****.

7.6.7 The Convolution Property

If $x_1[n] \leftrightarrow X_1(z), \quad ROC = R_1$ and $x_2[n] \leftrightarrow X_2(z), \quad ROC = R_2$

then $x_1[n] * x_2[n] \leftrightarrow X_1(z) X_2(z), \quad ROC\ containing\ R_1 \cap R_2$

7.6.8 Differentiation in the *z*-Domain

$$nx[n] \leftrightarrow -z\frac{d\,X(z)}{d\,z}, \quad ROC = R$$

7.6.9 The Initial- and Final-Value Theorems

If *x*[*n*] is a causal sequence, i.e., *x*[*n*] = 0, for *n* < 0, then

Initial-value theorem : $x[0] = \lim_{z \to \infty} X(z)$

Final -value theorem : $\lim_{n \to \infty} x[n] = \lim_{z \to 1} (z-1) X(z)$

Example 7.14

let *w*[*n*] be the running sum of *x*[*n*]: $w[n]=\sum_{k=-\infty}^{n} x[k]$, find the Z-transform in terms of *X*(*z*), the z-transform of *x*[*n*].

Sol: Since $$w[n]=u[n]*x[n]$$

From the convolution property,

$$W(z)=\mathcal{ZT}\{u[n]\}\cdot X(z)=\frac{z}{z-1}X(z)$$

If the *ROC* of *X*(*z*) is *R*, the *ROC* of *W*(*z*) must includes at least the interconnection of *R* with |*z*| > 1.

Example 7.15

Determine the inverse *z*-transform for $X(z)=\dfrac{a}{\left(1-az^{-1}\right)^2}, \quad |z|>|a|$

Sol: From Example 7.1,

$$a^n u[n] \xleftrightarrow{ZT} \frac{1}{1-az^{-1}}, \quad |z|>|a|$$

Hence

$$na^n u[n] \xleftrightarrow{ZT} -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right)=\frac{az^{-1}}{\left(1-az^{-1}\right)^2}, \quad |z|>|a|$$

Finally,

$$(n+1)a^{n+1}u[n+1]=(n+1)a^{n+1}u[n] \xleftrightarrow{ZT} \frac{a}{\left(1-az^{-1}\right)^2}, \quad |z|>|a|$$

7.7 Analysis And Characterization of LTI Systems Using z-Transform

- The z-transforms of the input and the output of an LTI system are related through multiplication by the z-transform of the impulse response of the system:

$$Y(z) = X(z)\, H(z) \longleftarrow \textit{system function}$$

- The ROC associated with the system function for a causal system is the exterior of a circle in the z-plane.

✓A discrete-time LTI system is causal if and only if the ROC of its system function is the exterior of a circle, *including infinity*.

✓A discrete-time LTI system with rational system function $H(z)$ is causal if and only if: (a) the ROC is the exterior of a circle outside the outermost pole; and (b) with $H(z)$ expressed as a ratio of polynomials in z, the order of the numerator *cannot be greater* than the order of the denominator.

7.7 Analysis And Characterization of LTI Systems Using z-Transform

- An LTI system is stable if and only if the ROC of its system function $H(z)$ *includes the unit circle*, $|z| = 1$.
- A causal LTI system with rational system function $H(z)$ is stable if and only if *all of the poles of H(z) lie inside the unit circle* — i.e., they must all have magnitude smaller than 1.
- For an LTI system which is described by a linear constant-coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Its system function (transfer function) takes the form of

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Example 7.16 Consider a system with system function whose algebraic expression is

$$H(z) = \frac{z^3 - 2z^2 + z}{z^2 + \frac{1}{4}z + \frac{1}{8}}.$$

Sol: We can conclude that the system is not causal, because the numerator of $H(z)$ is of higher order than the denominator.

In fact, since $$H(z) = z - \frac{\frac{9}{4}z^2 - \frac{7}{8}z}{z^2 + \frac{1}{4}z + \frac{1}{8}}$$

Even we don't know the *ROC* for this system function. However, from the point that the inverse transform of z is $\delta[n+1]$, we can conclude this system is non-causal.

Example 7.17 Consider a system with system function

$$H(z)=\frac{1}{1-\frac{1}{2}z^{-1}}+\frac{1}{1-2z^{-1}},\quad |z|>2$$

Sol: From

$$H(z)=\frac{2-\frac{5}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}=\frac{2z^{2}-\frac{5}{2}z}{z^{2}-\frac{5}{2}z+1}$$

Argument 1: We see that *H*(*z*) is rational and the numerator and denominator of *H*(*z*) are both of degree two. So it is causal !

Argument 2: The *ROC* for this system function is the exterior of a circle outside the outermost pole. Furthermore, the *ROC* contains the point $z=\infty$. So it is causal !

Example 7.18

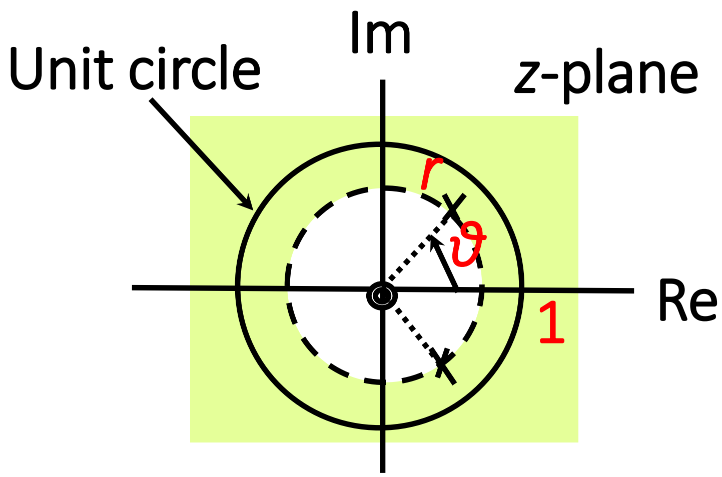
Consider a second-order causal system with system function

$$H(z)=\frac{1}{1-\left(2r\cos\theta\right)z^{-1}+r^2z^{-2}}$$

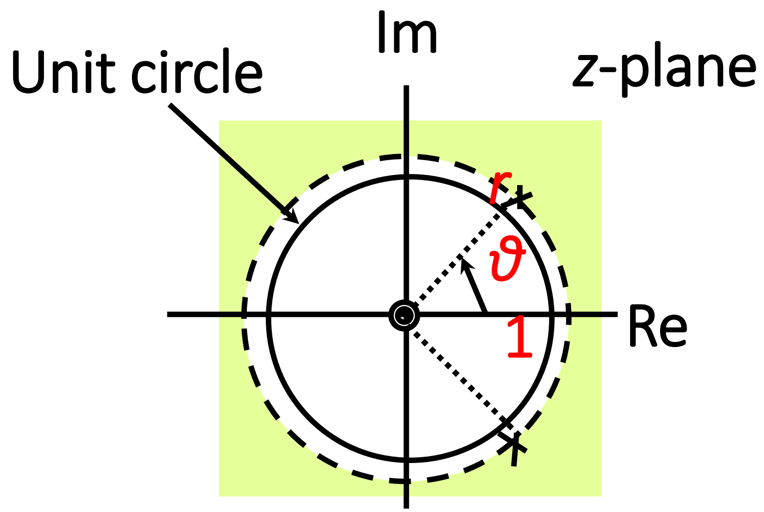
Sol: Since $1-\left(2r\cos\theta\right)z^{-1}+r^2z^{-2}=1-(re^{j\theta}+re^{-j\theta})z^{-1}+r^2z^{-2}$

The poles located at $\boldsymbol{z_1}=\boldsymbol{re^{j\theta}}$ and $\boldsymbol{z_2}=\boldsymbol{re^{-j\theta}}$.

$\boldsymbol{r<1}$ ***Stable***



$\boldsymbol{r>1}$ ***Unstable***



Example 7.19

Consider a stable and causal system with impulse response $h[n]$ and rational system function $H(z)$. Suppose it is known that $H(z)$ contains a pole at $z = 1/2$ and a zero somewhere on the unit circle. The precise number and location of all of the other poles and zeros are unknown. For each of the following statements, let us determine whether we can definitely say that it is true, whether we can definitely say that it is false, or whether there is insufficient information given to determine if it is true or not:

(a) $\mathcal{F}\{(1/2)^n h[n]\}$ converges. ✓

(b) $H(e^{j\omega}) = 0$ for some ω. ✓

(c) $h[n]$ has finite duration. ×

(d) $h[n]$ is real. Insufficient!

(e) $g[n]=n\{h[n]*h[n]\}$ is the impulse response of a stable system. ✓

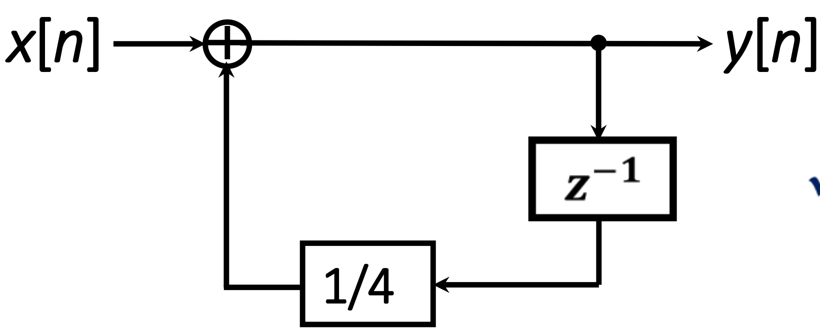
7.8 System Function Algebra And Block Diagram Representations

The use of the *z*-transform to convert system descriptions to algebraic equations is also helpful in analyzing interconnections of LTI systems and in representing and synthesizing systems as interconnections of basic system building blocks.

Example 7.20

Consider the causal LTI system with system function $H(z) = \dfrac{1}{1 - \frac{1}{4} z^{-1}}$

Sol: This system can also be described by the difference equation $y[n] - \frac{1}{4} y[n-1] = x[n]$



✓ z^{-1} is the system function of a unit delayer.

Block diagram representation of the causal LTI system

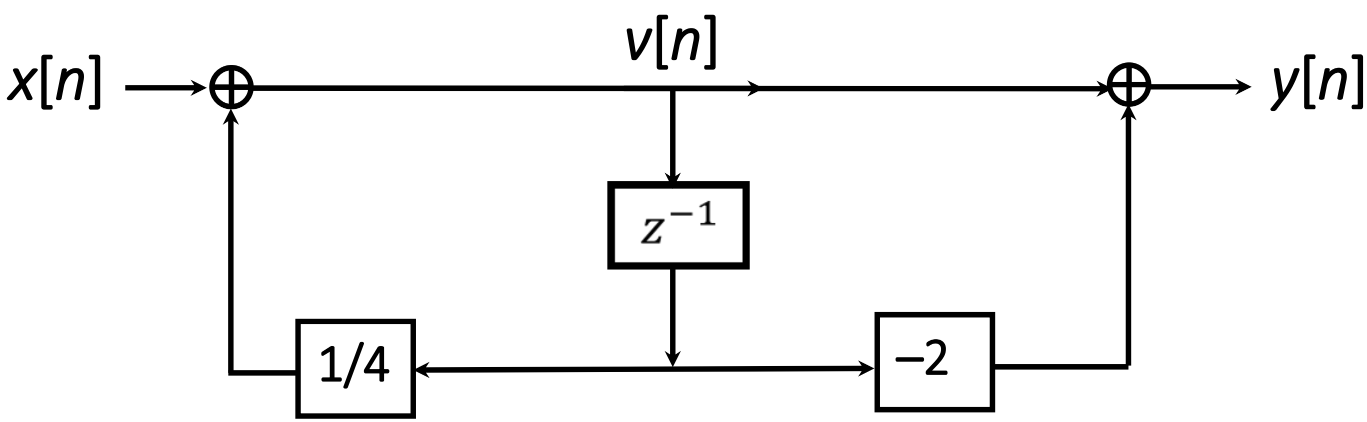
Example 7.21

Consider a causal LTI system with system function $H(z)=\dfrac{1-2z^{-1}}{1-\frac{1}{4}z^{-1}}$.

Sol: $$H(z)=\left(\frac{1}{1-\frac{1}{4}z^{-1}}\right)\left(1-2z^{-1}\right)$$

Let *v*[*n*] be the output of the first subsystem, *y*[*n*] the output of the overall system, the relationship between *v*[*n*] and *y*[*n*] is

$$y[n]=v[n]-2v[n-1]$$



Direct-Form block diagram representation for the system

Example 7.22

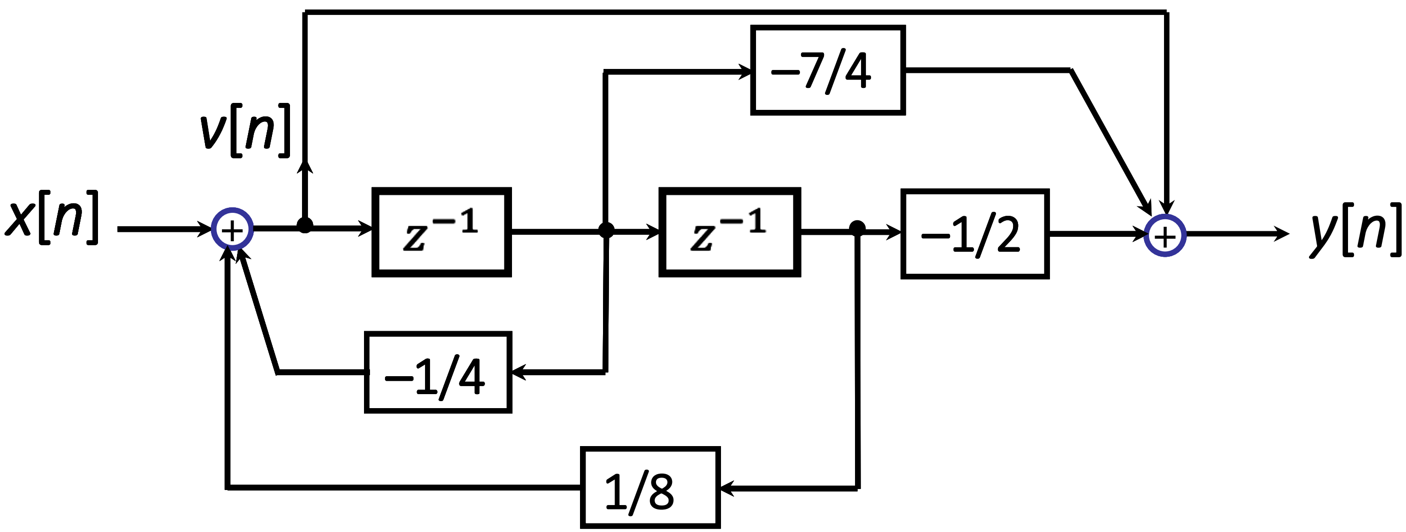
Consider a second-order LTI system with system function

$$H(z)=\frac{1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

Sol:

$$H(z)=\underbrace{\left(\frac{1}{1+\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}\right)}_{H_1(z)}\underbrace{\left(1-\frac{7}{4}z^{-1}-\frac{1}{2}z^{-2}\right)}_{H_2(z)}$$

$H_1(z):\ v[n]+\frac{1}{4}v[n-1]-\frac{1}{8}v[n-2]=x[n]$ $\quad H_2(z):\ y[n]=v[n]-\frac{7}{4}v[n-1]-\frac{1}{2}v[n-2]$



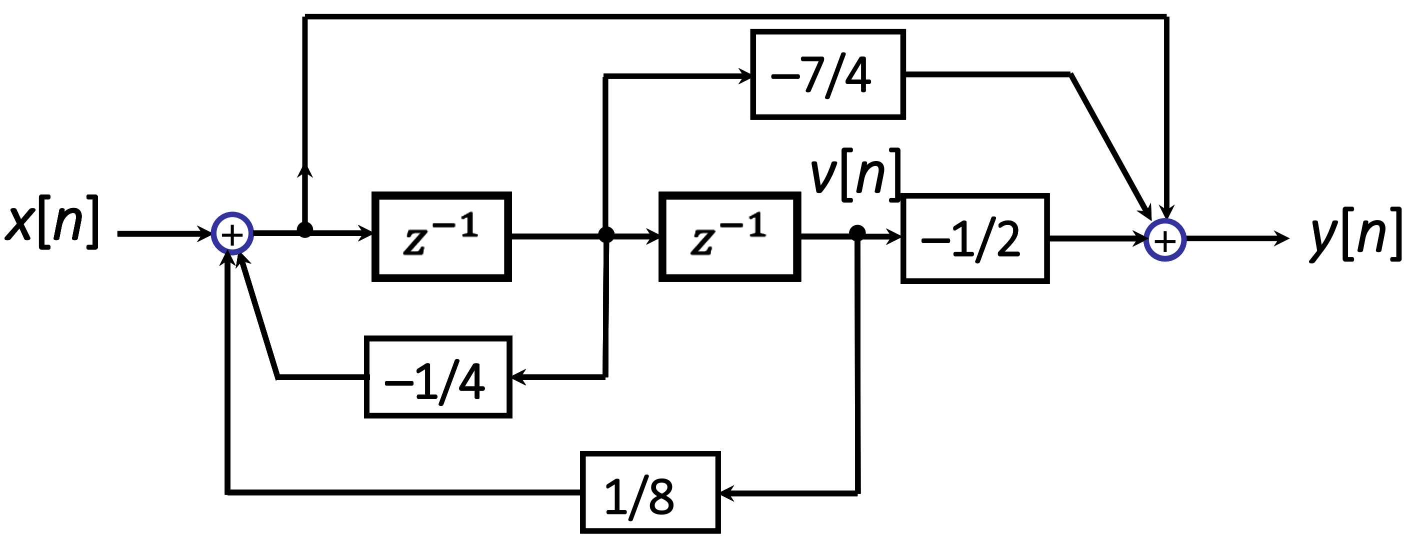
Direct-form representation for the system

7.8 System Function Algebra And Block Diagram Representations

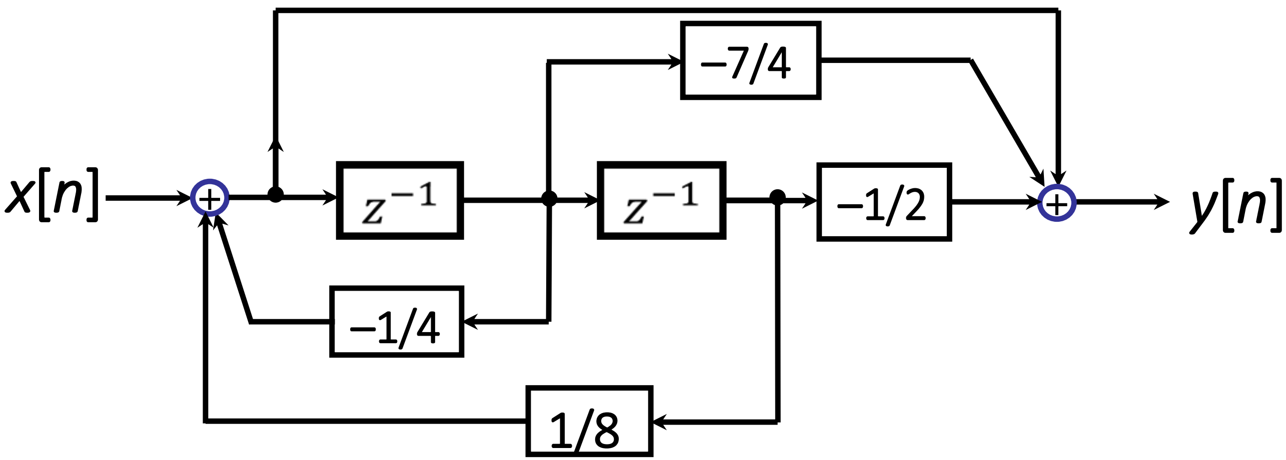
If $$H(z)=\frac{z^2-\frac{7}{4}z-\frac{1}{2}}{z^2+\frac{1}{4}z-\frac{1}{8}}=\frac{1}{z^2+\frac{1}{4}z-\frac{1}{8}}\left(z^2-\frac{7}{4}z-\frac{1}{2}\right)$$

$$H_1(z): \qquad v[n+2]+\frac{1}{4}v[n+1]-\frac{1}{8}v[n]=x[n]$$

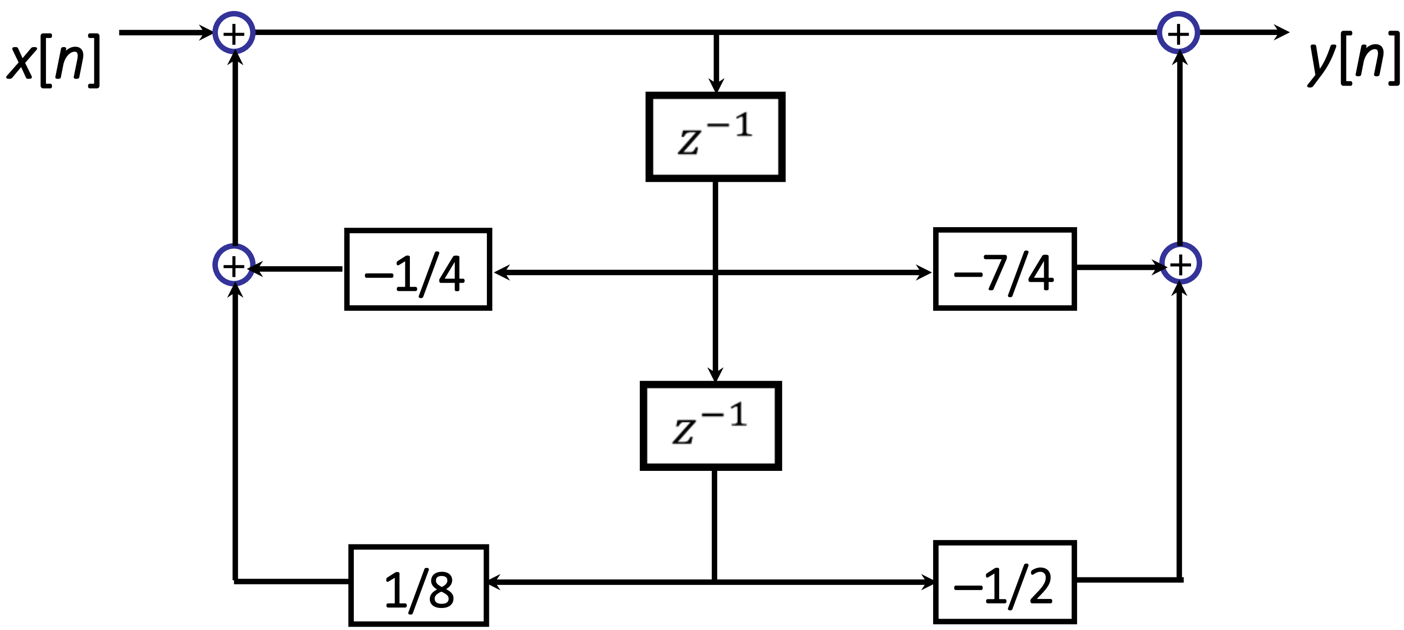
$$H_2(z): \qquad y[n]=v[n+2]-\tfrac{7}{4}v[n+1]-\tfrac{1}{2}v[n]$$



7.8 System Function Algebra And Block Diagram Representations

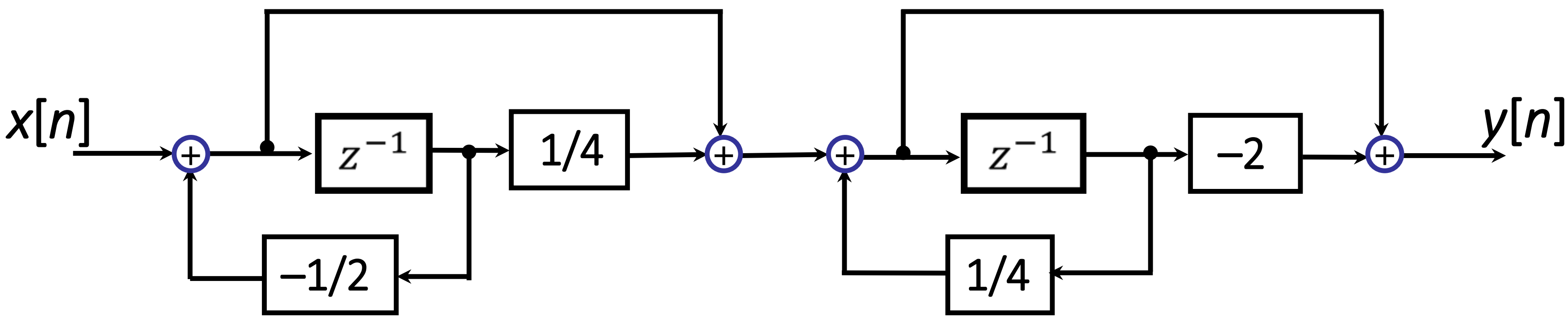


Rotate clockwise it by 90^0 leading to the following diagram:



7.8 System Function Algebra And Block Diagram Representations

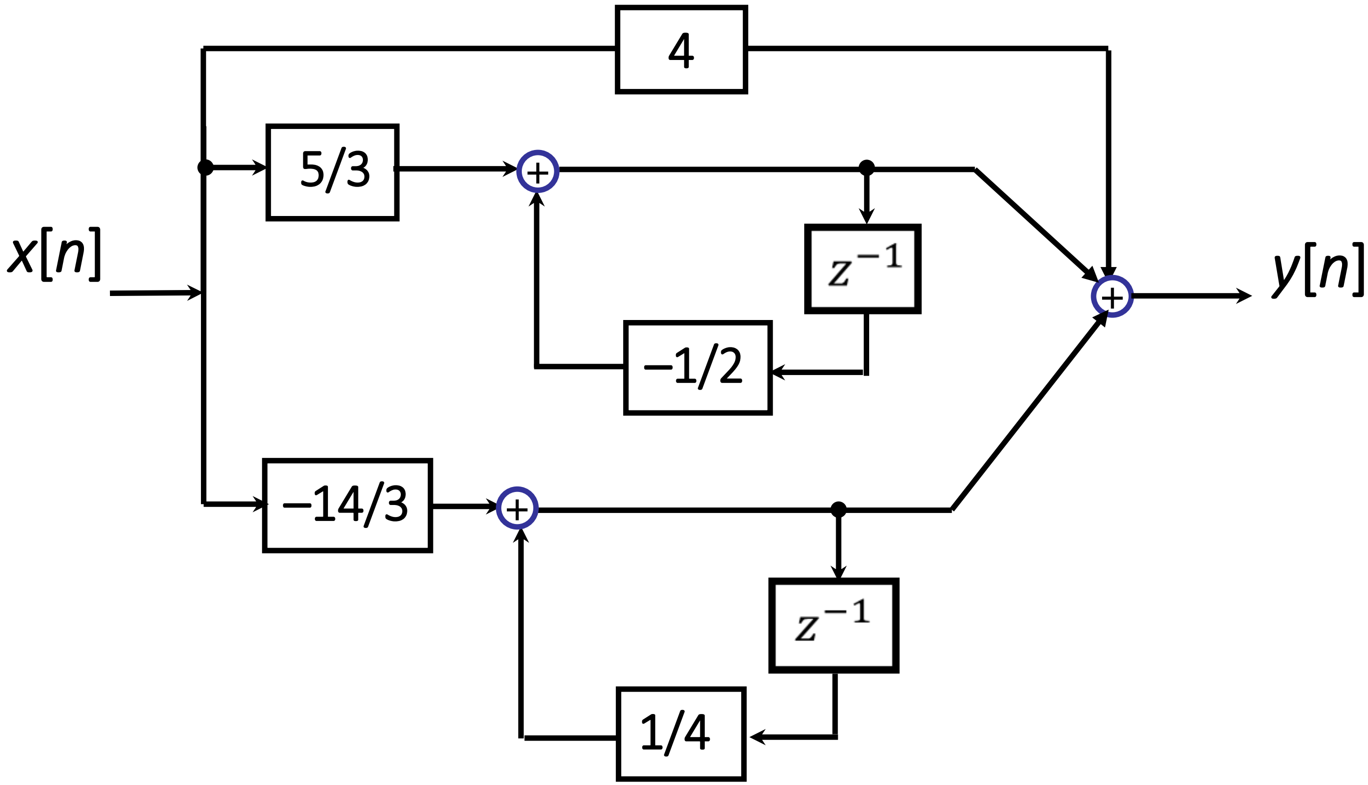
$$H(z) = \left(\frac{1 + \frac{1}{4} z^{-1}}{1 + \frac{1}{2} z^{-1}} \right) \left(\frac{1 - 2z^{-1}}{1 - \frac{1}{4} z^{-1}} \right)$$



Cascade-form representation for the system in Example 7.22

7.8 System Function Algebra And Block Diagram Representations

$$H(z) = 4 + \frac{5/3}{1+\frac{1}{2}z^{-1}} - \frac{14/3}{1-\frac{1}{4}z^{-1}}$$



Parallel-form representation for the system in Example 7.22

7.9 The Unilateral z-Transform

7.9.1 Introduction of the Unilateral *z*-Transform

Bilateral *z*-transform:

$$X(z)=\sum_{n=-\infty}^{\infty}x[n]z^{-n}$$

Unilateral *z*-transform:

$$\mathcal{X}(z)=\sum_{n=0}^{\infty}x[n]z^{-n}$$

- The unilateral ZT differs from the bilateral ZT in that the summation is carried out only over nonnegative values of *n*.
- The unilateral *z*-transform of *x*[*n*] can be thought of as the bilateral transform of *x*[*n*]*u*[*n*].
- The bilateral transform and the unilateral transform of a causal signal are identical.
- The *ROC* for the unilateral ZT is always the exterior of a circle.
- The calculation of the inverse unilateral ZT is basically the same as for bilateral ZT, with the constraint that the *ROC* for a unilateral transform must always be the exterior of a circle.

7.9 The Unilateral z-Transform

Example 7.23 Consider the signal $x[n] = a^{n+1}u[n+1]$.

Sol: The bilateral transform $X(z)$ for this example can be obtained from Example 7.1 and the time-shifting property:

$$X(z) = \frac{z}{1 - az^{-1}}, \quad |z| > |a|$$

By contrast, the unilateral transform is

$$\mathcal{X}(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} a^{n+1}z^{-n} = \frac{a}{1 - az^{-1}}, \quad |z| > |a|$$

We could recognize $\mathcal{X}(z)$ as the bilateral transform of $x[n]u[n]$.

Since $\quad x[n]u[n] = a^{n+1}u[n] = aa^{n}u[n]$

Thus, $\quad \mathcal{X}(z) = \dfrac{za}{z - a}, \quad |z| > |a|$

Example 7.24 Consider the unilateral z-transform

$$X(z)=\frac{3-\frac{5}{6}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{3}z^{-1})}$$

Determine $x[n]$.

Sol: In this case, the *ROC* must be the region outside the outermost pole of $X(z)$, that is $|\boldsymbol{z}| > \frac{\mathbf{1}}{\mathbf{3}}$.

$$X(z)=\frac{1}{1-\frac{1}{4}z^{-1}}+\frac{2}{1-\frac{1}{3}z^{-1}}$$

Thus, $$x[n]=\left(\frac{1}{4}\right)^n u[n]+2\left(\frac{1}{3}\right)^n u[n]$$

✓unilateral z-transforms provide us with information about signals only for $n \geq 0$.

7.9.2 Properties of the Unilateral *z*-Transform

✓Convolution: assuming that $x_1[n]$ and $x_2[n]$ are identically zero for $n < 0$.

$$x_1[n] * x_2[n] \xleftrightarrow{UZ} X_1(z)X_2(z)$$

✓Time delay :

$$x[n-1] \xleftrightarrow{UZ} z^{-1}X(z) + x[-1]$$

$$\boxed{x[n-m] \xleftrightarrow{UZ} z^{-m}\left(X(z) + \sum_{n=-m}^{-1} x[n]z^{-n} \right)}$$

✓Time advance:

$$x[n+1] \xleftrightarrow{UZ} zX(z) - zx[0]$$

$$\boxed{x[n+m] \xleftrightarrow{UZ} z^{m}\left(X(z) - \sum_{n=0}^{m-1} x[n]z^{-n} \right)}$$

Proof of the time-delay property :

$$\mathcal{UZ}\{x[n-m]\} = \sum_{n=0}^{\infty} x[n-m]z^{-n}$$

$$\overset{k=n-m}{=} \sum_{k=-m}^{\infty} x[k]z^{-k-m}$$

$$= z^{-m}\left(\sum_{k=0}^{\infty} x[k]z^{-k} + \sum_{k=-m}^{-1} x[k]z^{-k}\right)$$

$$= z^{-m}\left(\mathcal{X}(z) + \sum_{k=-m}^{-1} x[k]z^{-k}\right)$$

7.9.3 Solving Difference Equations Using the Unilateral z-Transform

Example 7.25

Consider the causal system characterized by the difference equation $$y[n]-\tfrac{3}{8}y[n-1]+\tfrac{1}{32}y[n-2]=x[n],$$

with initial conditions $y[-1]=\beta, y[-2]=\gamma$ and input signal $x[n]=\alpha u[n]$. Determine the output $y[n]$.

Sol: Applying the unilateral transform to both sides of the difference equation to obtain

$$\mathcal{Y}(z)-\tfrac{3}{8}z^{-1}\mathcal{Y}(z)-\tfrac{3}{8}y[-1]+\tfrac{1}{32}z^{-2}\mathcal{Y}(z)+\tfrac{1}{32}y[-2]+\tfrac{1}{32}y[-1]z^{-1}=\mathcal{X}(z)$$

or equivalently,

$$\left(1-\tfrac{3}{8}z^{-1}+\tfrac{1}{32}z^{-2}\right)\mathcal{Y}(z)-\tfrac{3}{8}\beta+\tfrac{1}{32}\gamma+\tfrac{1}{32}\beta z^{-1}=\frac{\alpha}{1-z^{-1}}$$

Thus, we obtain

$$\mathcal{Y}(z)=\frac{\tfrac{3}{8}\beta-\tfrac{1}{32}\gamma-\tfrac{1}{32}z^{-1}}{(1-\tfrac{1}{4}z^{-1})(1-\tfrac{1}{8}z^{-1})}+\frac{\alpha}{(1-z^{-1})(1-\tfrac{1}{4}z^{-1})(1-\tfrac{1}{8}z^{-1})}$$

$$Y(z)=\underbrace{\frac{\frac{3}{8}\beta-\frac{1}{32}\gamma-\frac{1}{32}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{8}z^{-1})}}_{zero-input\,response}+\underbrace{\frac{\alpha}{(1-z^{-1})(1-\frac{1}{4}z^{-1})(1-\frac{1}{8}z^{-1})}}_{zero-state\,response}$$

If the given difference equation is

$$y[n+2]-\tfrac{3}{8}y[n+1]+\tfrac{1}{32}y[n]=x[n+2]$$

We should use the time advance property to obtain

$$z^2\mathcal{Y}(z)-z^2y[0]-zy[1]-\tfrac{3}{8}z\mathcal{Y}(z)+\tfrac{3}{8}y[0]+\tfrac{1}{32}\mathcal{Y}(z)$$
$$=z^2\mathcal{X}(z)-z^2x[0]-zx[1]$$

From the given conditions, we can obtain $x[0] = x[1] = a$, and *recursively* obtain

$$y[0]=\tfrac{3}{8}\beta+\tfrac{1}{32}\gamma+\alpha$$

$$y[1]=\tfrac{11}{64}\beta+\tfrac{3}{256}\gamma+\tfrac{11}{8}\alpha$$

Taking these values into the above equation, we get the $Y(z)$, then calculating the inverse transform to obtain $y[n]$.

7.10 SUMMARY

- The bilateral and unilateral *z*-transform;
- The properties of the *ROC* of the *z*-transform and the relationship between the *ROC* and the poles;
- Methods to calculate the inverse *z*-transform;
- The properties of the bilateral and unilateral *z*-transforms (note the similarities and the differences);
- Significance of the poles and zeros of ZT in characterizing discrete-time signals and systems;
- The computations of the zero-state response and the zero-input response by z-transform;
- The block diagram and signal flow graph representations of discrete-time LTI systems.

Homework

10.21 (b) (d) (f) (g) 10.24

10.27 10.29 (b) (d) (e) 10.31

10.36 10.38 10.42 (a) (c)