CHAPTER 2

LINEAR TIME-INVARIANT SYSTEMS

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2.0 Introduction

- Convolution property of LTI systems
- Impulse response and LTI systems' properties
- Linear constant-coefficient difference and differential equations (LCCDEs) and their solutions

2.1 Discrete-Time LTI Systems: The Convolution Sum

2.1.1 What is the Convolution Sum?

$$x[n] \longrightarrow \boxed{\begin{array}{c} h[n] \\ \delta[n] \to h[n] \end{array}} \longrightarrow y[n] = x[n] * h[n]$$

Step 1: Decomposing $x[n]$:

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$
$$= \cdots + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + \cdots$$

Step 2: From linearity and time-invariance property, we can write:

$$y[n] = \cdots + x[-2]h[n+2] + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + \cdots$$

convolution sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

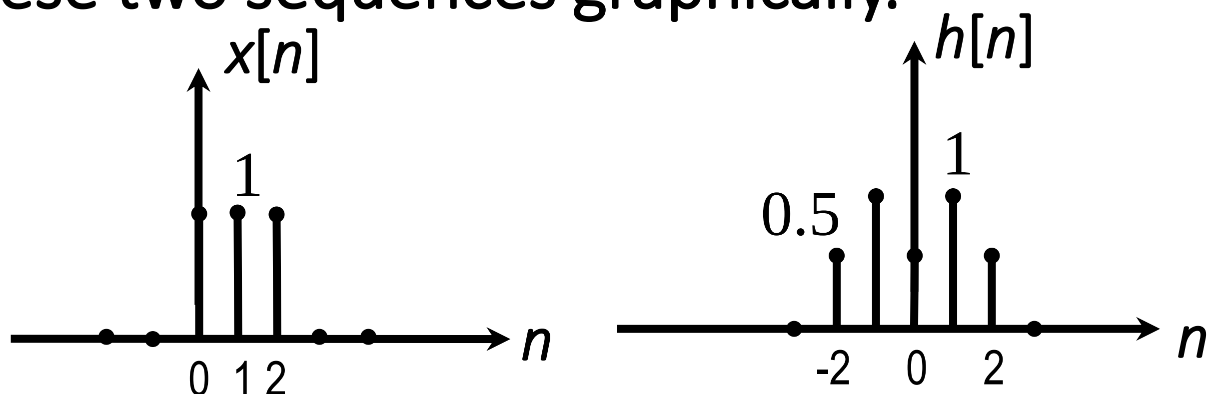
$h[n]$ can completely characterize LTI system !

2.1 Discrete-Time LTI Systems: The Convolution Sum

2.1.2 How to calculate the Convolution Sum?

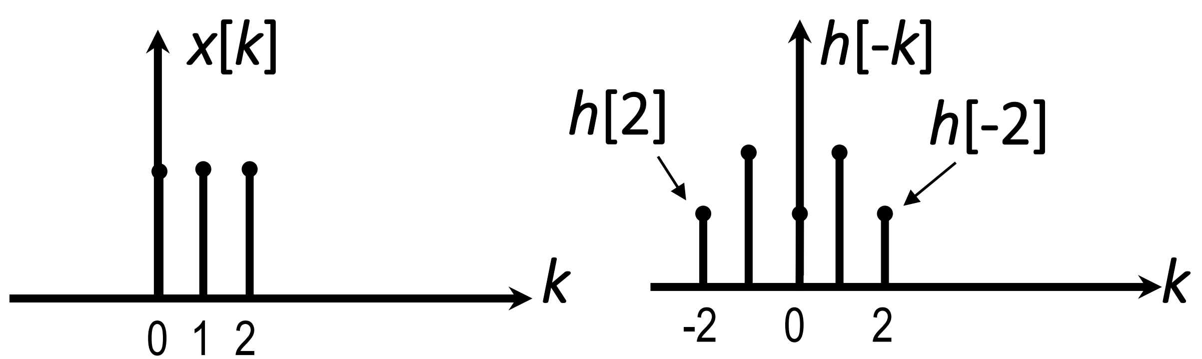
Example 2.1 (Method 1: Graphical Method)

Consider an LTI system with unit sample response $h[n]$ and input $x[n]$, as illustrated in Figure (a). Calculate the convolution sum of these two sequences graphically.

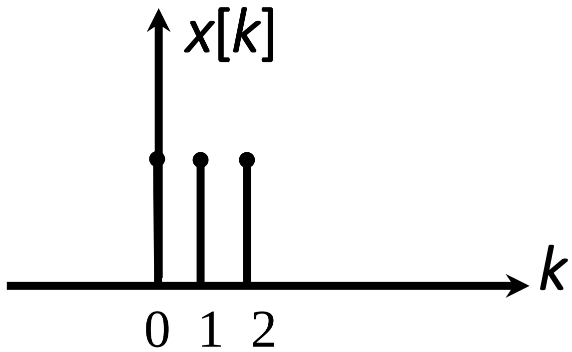


(a)

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

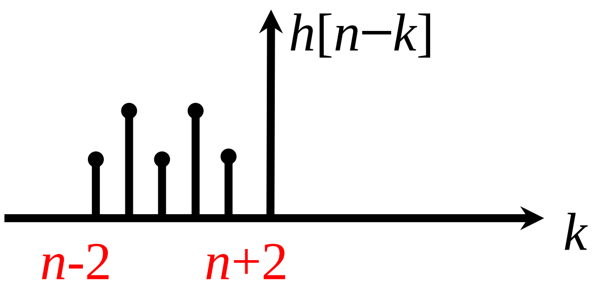


2.1 Discrete-Time LTI Systems: The Convolution Sum



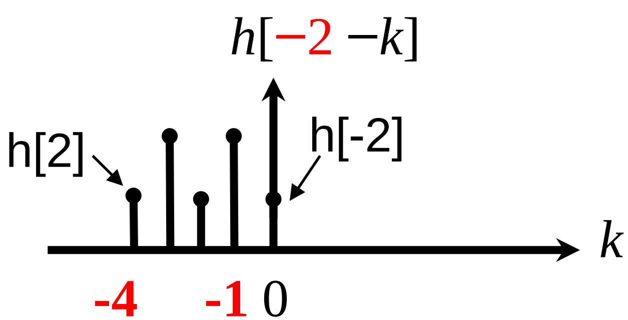
$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

For n<-2



$$\sum_{k=-\infty}^{\infty}x[k]h[n-k]=0$$

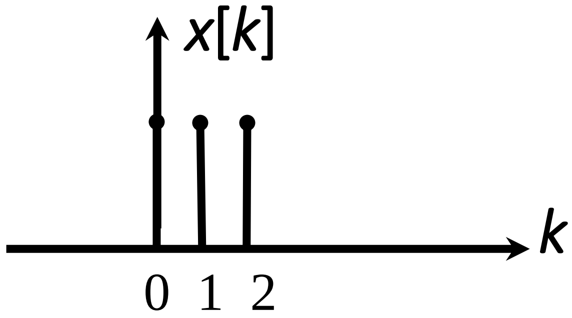
For n= −2,



$$y[-2]=\sum_{k=-\infty}^{\infty}x[k]h[-2-k]$$

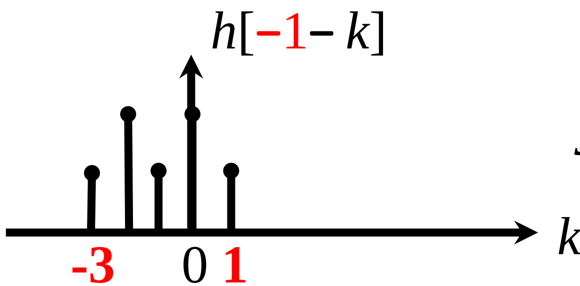
$$=x[0]h[-2]=0.5$$

2.1 Discrete-Time LTI Systems: The Convolution Sum



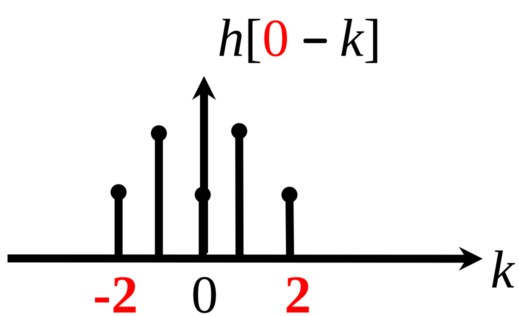
$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

For n= −1,



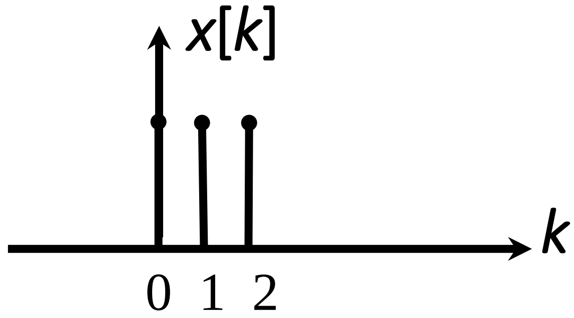
$$y[-1]=x[0]h[-1]+x[1]h[-2]=1.5$$

For n = 0,



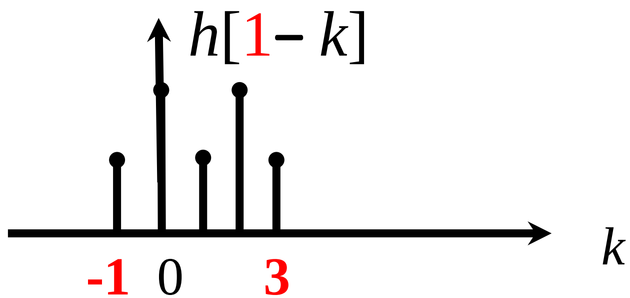
$$y[0]=x[0]h[0]+x[1]h[-1]$$
$$+x[2]h[-2]=2$$

2.1 Discrete-Time LTI Systems: The Convolution Sum



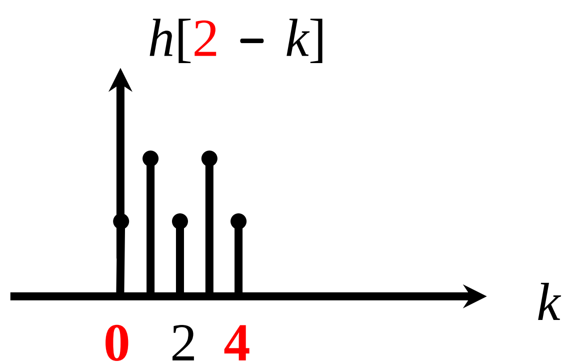
$$y[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]$$

For n = 1,



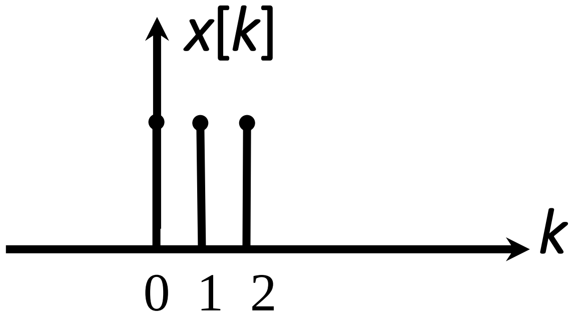
$$y[1]=x[0]h[1]+x[1]h[0]$$
$$+x[2]h[-1]=2.5$$

For n = 2,



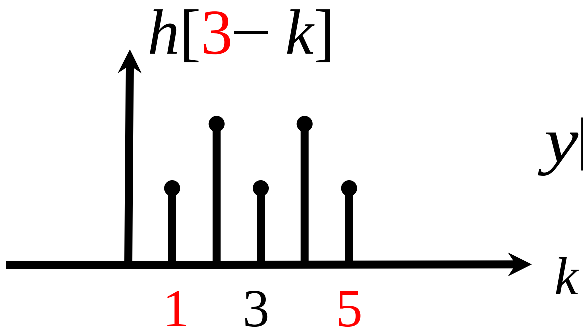
$$y[2]=x[0]h[2]+x[1]h[1]$$
$$+x[2]h[0]=2$$

2.1 Discrete-Time LTI Systems: The Convolution Sum



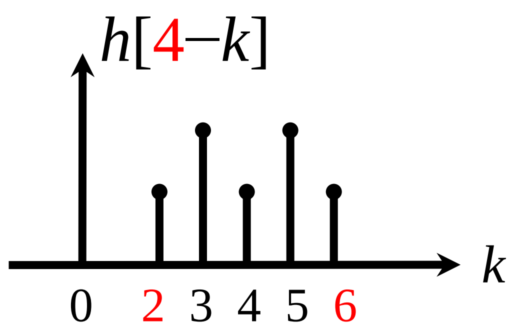
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

For n = 3,



$$y[3] = x[1]h[2] + x[2]h[1] = 1.5$$

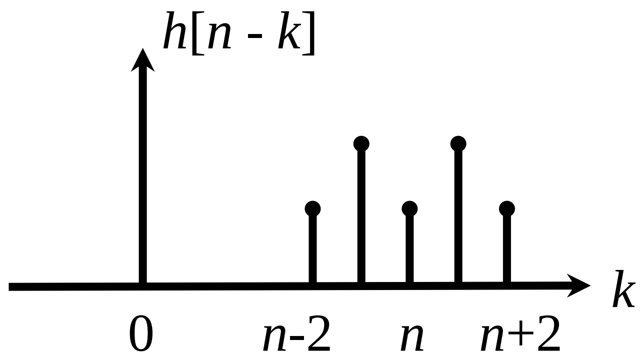
For n = 4,



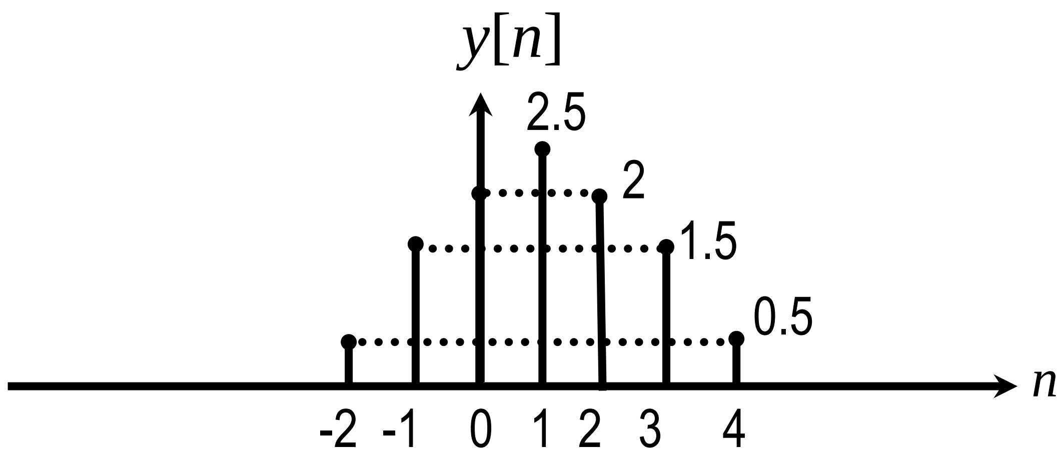
$$y[4] = x[2]h[2] = 0.5$$

2.1 Discrete-Time LTI Systems: The Convolution Sum

For *n*>4,



$$\sum_{k=-\infty}^{\infty} x[k]h[n-k] = 0$$



Method 2: (Table Method)

$x[n]$ \ $h[n]$	$h[-2]$	$h[-1]$	$h[0]$	$h[1]$	$h[2]$	...	
	$y[-2]$	**$y[-1]$**	**$y[0]$**	**$y[1]$**	**$y[2]$**	**$y[3]$**	**$y[4]$**
$x[0]$	$x[0]h[-2]$	$x[0]h[-1]$	$x[0]h[0]$	$x[0]h[1]$	$x[0]h[2]$	0	0
$x[1]$	$x[1]h[-2]$	$x[1]h[-1]$	$x[1]h[0]$	$x[1]h[1]$	$x[1]h[2]$	0	0
$x[2]$	$x[2]h[-2]$	$x[2]h[-1]$	$x[2]h[0]$	$x[2]h[1]$	$x[2]h[2]$	0	0
:	0	0	0	0 0	0	0	

Method 3: Multiplying if two sequences are short.

			0.5	**1**	**0.5**	**1**	**0.5**
×					**1**	**1**	**1**
			0.5	**1**	**0.5**	**1**	**0.5**
		0.5	**1**	**0.5**	**1**	**0.5**	
+	**0.5**	**1**	**0.5**	**1**	**0.5**		
	0.5	**1.5**	**2**	**2.5**	**2**	**1.5**	**0.5**

$$x[n] = \{1,1,1\}_0$$

$$h[n] = \{0.5, 1, 0.5, 1, 0.5\}_{-2}$$

$$x[n]*h[n] = \{0.5,1.5,2,2.5,2,1.5,0.5\}_{-2}$$

Method 4: (Definition)

Example 2.2

Determine and plot the output $y[n] = x[n] * h[n]$ where input $x[n]$ and unit sample response $h[n]$ given by

$$x[n] = \left(\frac{1}{2}\right)^{n-2} u[n-2], \qquad h[n] = u[n+2]$$

Sol: By definition

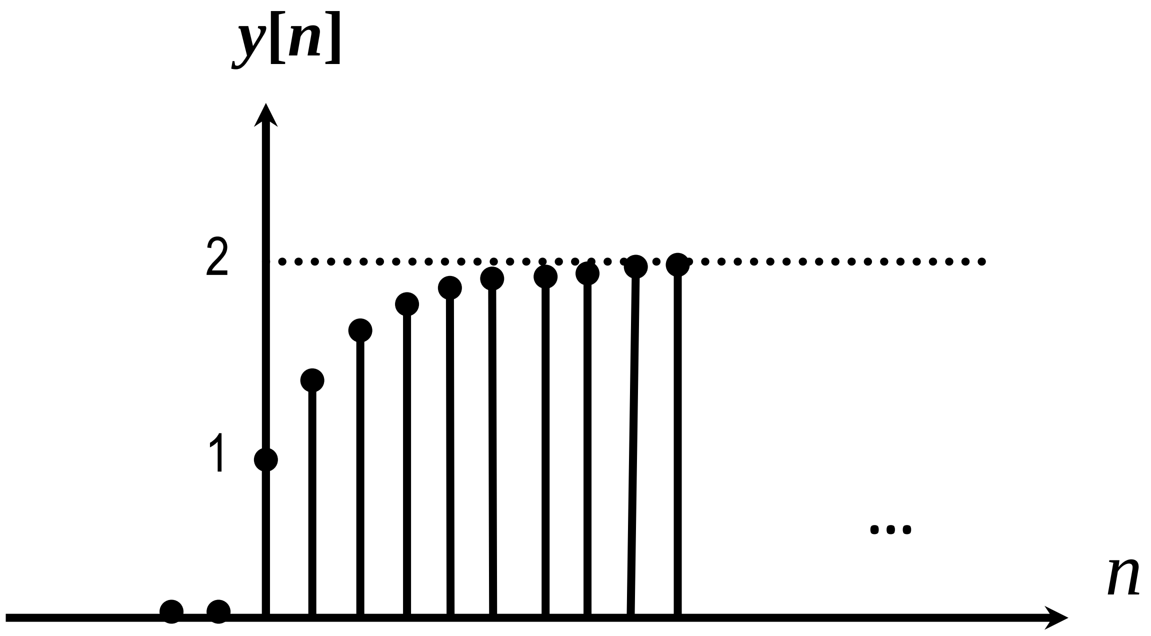
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{k-2} u[k-2] \cdot u[n-k+2]$$

$$y[n] = \sum_{k=2}^{n+2} \left(\frac{1}{2}\right)^{k-2} u[n] = \left(\frac{1}{2}\right)^{-2} \sum_{k=2}^{n+2} \left(\frac{1}{2}\right)^{k} u[n]$$

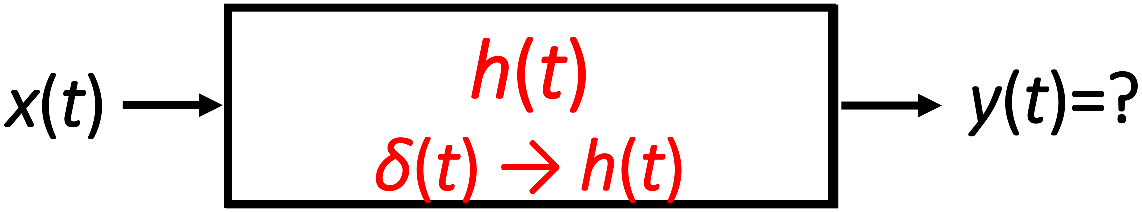
From the geometric sum formula,

$$y[n] = 4 \cdot \frac{\frac{1}{4}\left[1-0.5^{n+1}\right]}{1-0.5} u[n] = 2\left[1-\left(\frac{1}{2}\right)^{n+1}\right] u[n]$$

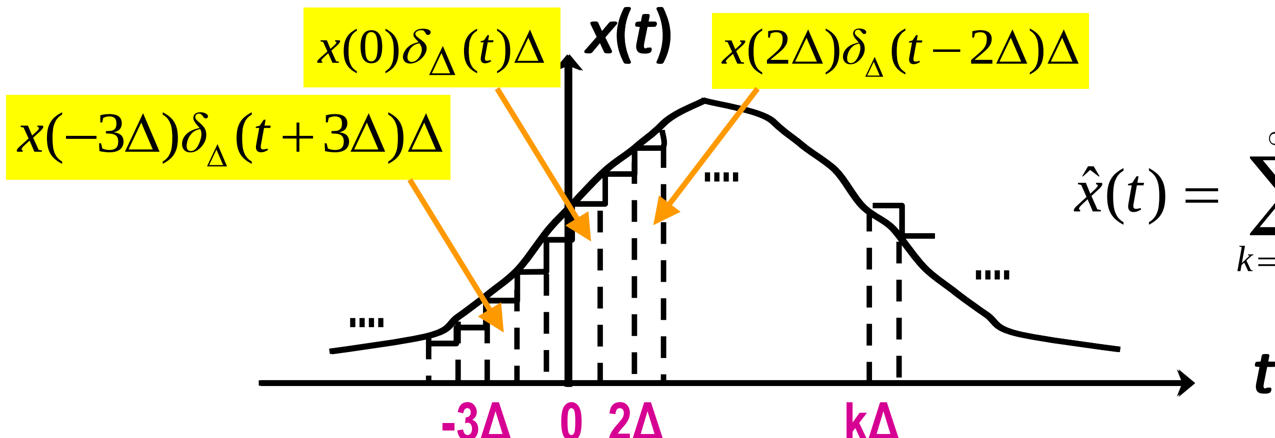
2.1 Discrete-Time LTI Systems: The Convolution Sum



2.2.1 What is the Convolution Integral?



Step 1: Representing $x(t)$ in terms of impulses (Decomposing):



$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

Staircase approximation to $x(t)$

$$x(t) = \lim_{\Delta\to 0} \hat{x}(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Comparing with the *Sampling property* of $\delta(t)$:

$$x(t_0) = \int_{-\infty}^{\infty} x(t)\delta(t-t_0)dt$$

2.2 Continuous-Time LTI Systems: The Convolution Integral

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta \longrightarrow \boxed{h(t)} \longrightarrow \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_{\Delta}(t-k\Delta)\Delta$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \qquad\qquad y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- *convolution integral*

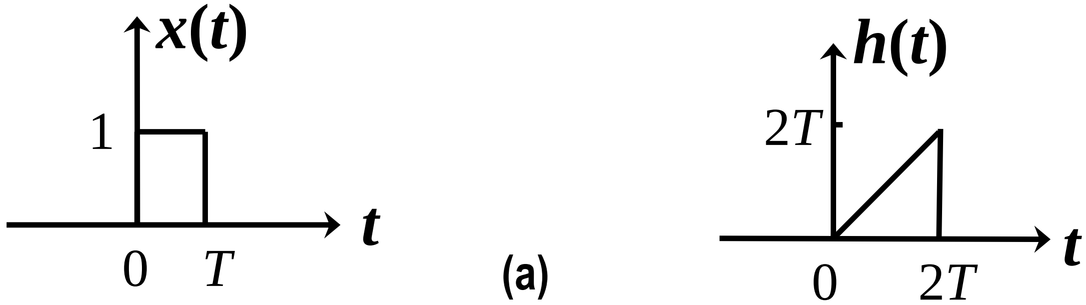
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

- *unit impulse response h*(t) can completely characterize continuous-time LTI systems.

2.2.2 How to calculate the Convolution Integral?

Example 2.3 (Method 1: Graphical Method)

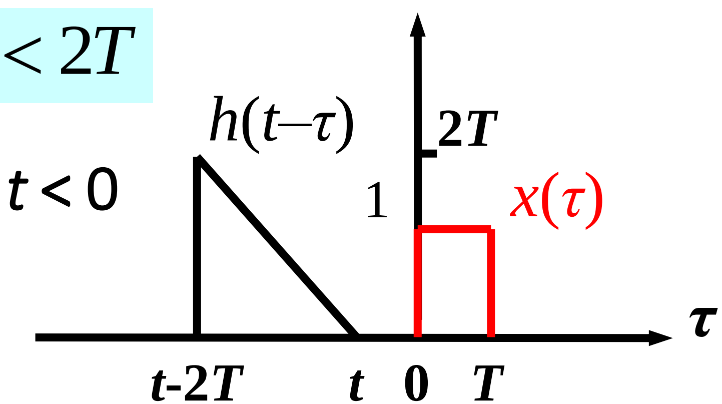
Compute the convolution of the following two signals in Figure (a).



$$x(t) = 1, 0 < t < T \quad h(t) = t, 0 < t < 2T$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

For $t < 0$

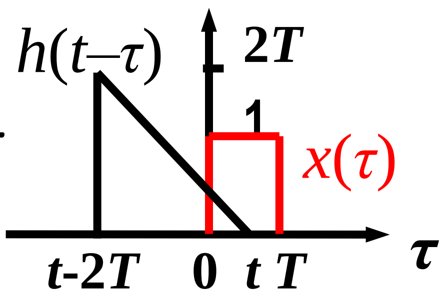


Interval 1. For $t < 0$, there is no overlap between the nonzero portions of $x(\tau)$ and $h(t-\tau)$, and consequently, $y(t) = 0$.

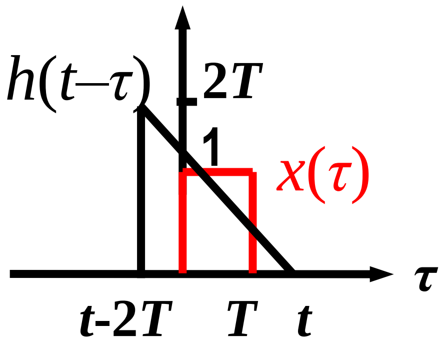
2.2 Continuous-Time LTI Systems: The Convolution Integral

$$x(\tau)h(t-\tau) = \begin{cases} t-\tau, & 0 < \tau < t \\ 0, & otherwise \end{cases}$$

For $0 < t < T$



For $T < t < 2T$



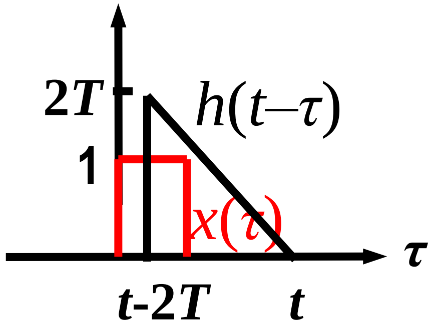
Interval 2. For $0 < t < T$,

$$y(t) = \int_0^t (t-\tau)d\tau = \frac{1}{2}t^2$$

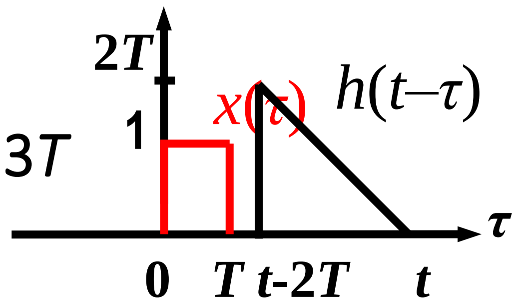
Interval 3. For $T < t < 2T$,

$$y(t) = \int_0^T (t-\tau)d\tau = Tt - \frac{1}{2}T^2$$

For $2T < t < 3T$



For $t > 3T$



Interval 4. For $2T < t < 3T$

$$y(t) = \int_{t-2T}^{T} (t-\tau)d\tau = -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2$$

Interval 5. For $t > 3T$, there is no overlap between the nonzero portions of $x(\tau)$ and $h(t-\tau)$, hence, $y(t) = 0$.

Summarizing,
$$y(t) = \begin{cases} 0, & t < 0, \quad t > 3T \\ 0.5t^2, & 0 < t < T \\ Tt - 0.5T^2, & T < t < 2T \\ -0.5t^2 + Tt + 1.5T^2, & 2T < t < 3T \end{cases}$$

2.3.1 Properties of Convolution and Systems' Construction

- The Commutative Property (交换律)

$$x[n]*h[n]=\sum_{k=-\infty}^{\infty}x[k]h[n-k]=h[n]*x[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]$$

$$x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau=h(t)*x(t)=\int_{-\infty}^{\infty}h(\tau)x(t-\tau)d\tau$$

- The Distributive Property (分配律)

$$x[n]*(h_1[n]+h_2[n])=x[n]*h_1[n]+x[n]*h_2[n]$$

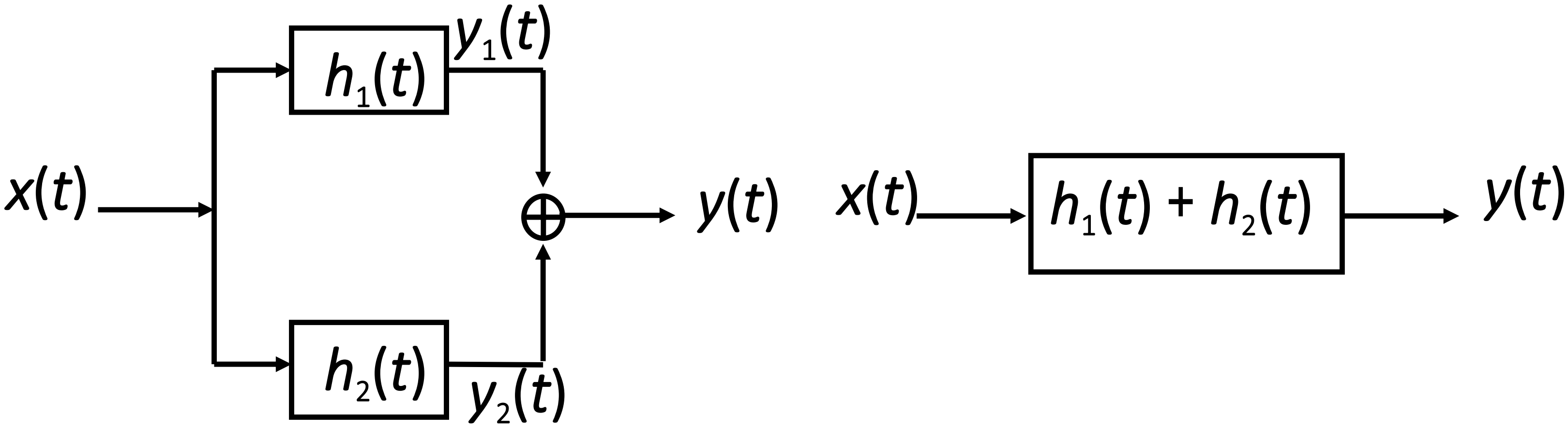
$$x(t)*[h_1(t)+h_2(t)]=x(t)*h_1(t)+x(t)*h_2(t)$$

- The Associative Property (结合律)

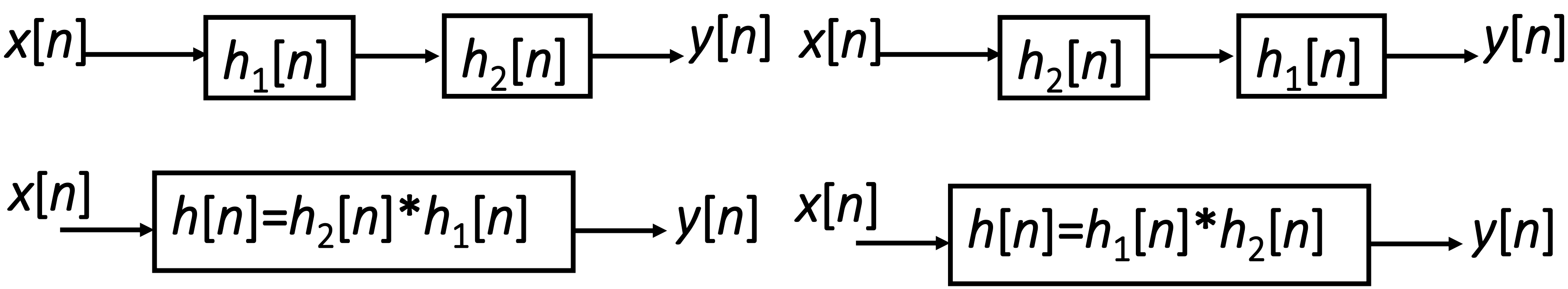
$$x[n]*(h_1[n]*h_2[n])=(x[n]*h_1[n])*h_2[n]$$

$$x(t)*[h_1(t)*h_2(t)]=[x(t)*h_1(t)]*h_2(t)$$

2.3 Properties of Linear Time-Invariant Systems



Two equivalent systems: they having same impulse responses



Four equivalent systems

2.3 Properties of Linear Time-Invariant Systems

- Convolving with Impulses

$$x(t) * \delta(t) = x(t)$$
$$x(t) * \delta(t - t_0) = x(t - t_0)$$

$$x[n] * \delta[n] = x[n]$$
$$x[n] * \delta[n - n_0] = x[n - n_0]$$

- Differentiation and Integration of Convolution Integral

$$y'(t) = x(t) * h'(t) = x'(t) * h(t)$$

$$\int_{-\infty}^{t} y(\tau)d\tau = [\int_{-\infty}^{t} x(\tau)d\tau] * h(t) = x(t) * [\int_{-\infty}^{t} h(\tau)d\tau]$$

Combining the two properties leading to

$$y(t) = [\int_{-\infty}^{t} x(\tau)d\tau] * h'(t) = x'(t) * [\int_{-\infty}^{t} h(\tau)d\tau] = \int_{-\infty}^{t} [x'(\tau) * h(\tau)]d\tau$$

- First Difference and Accumulation of Convolution Sum

$$\nabla y[n] = \{\nabla x[n]\} * h[n] = x[n] * \{\nabla h[n]\}$$

$$\sum_{k=-\infty}^{n} y[k] = \left\{ \sum_{k=-\infty}^{n} x[k] \right\} * h[n] = x[n] * \left\{ \sum_{k=-\infty}^{n} h[k] \right\}$$

$$\boxed{y[n] = \left\{ \sum_{k=-\infty}^{n} x[k] \right\} * \{\nabla h[n]\} = \{\nabla x[n]\} * \left\{ \sum_{k=-\infty}^{n} h[k] \right\}}$$

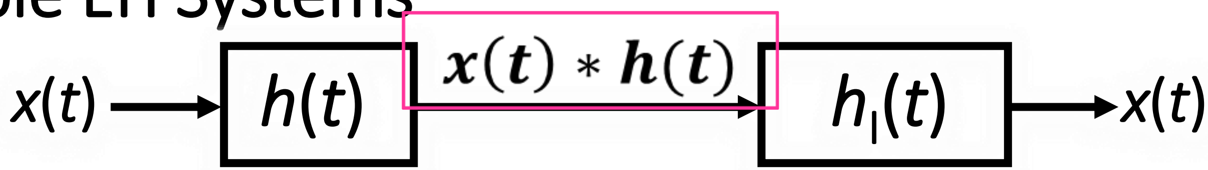
2.3.2 Relations between *h*(*t*)/*h*[*n*] and Properties of LTI Systems

- **LTI Systems with and without Memory**

LTI systems without memory must have its impulse response

$$h[n] = K\delta[n] \qquad h(t) = K\delta(t)$$

- **Invertible LTI Systems**



$$x(t) * h(t) * h_I(t) = x(t)$$

$$h(t) * h_I(t) = \delta(t) \qquad h[n] * h_I[n] = \delta[n]$$

- **Causal LTI Systems**

Causal LTI systems must have its impulse response satisfying

$$h[n] = 0 \quad for\ n < 0 \qquad h(t) = 0 \quad for\ t < 0$$

2.3 Properties of Linear Time-Invariant Systems

➢ Stable LTI Systems

Stable LTI systems must have its impulse response satisfying

absolutely summable $\leftarrow$ $\displaystyle\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

$\displaystyle\int_{-\infty}^{\infty} |h(t)| dt < \infty$ $\rightarrow$ *absolutely integrable*

Proof: (Sufficient condition)

$$|y[n]| = |x[n] * h[n]| = \left| \sum_{k=-\infty}^{\infty} x[n-k] h[k] \right| < \sum_{k=-\infty}^{\infty} |x[n-k]| |h[k]|$$

Let $|x[n]| < B < \infty$ Then $|y[n]| < \displaystyle\sum_{k=-\infty}^{\infty} B |h[k]|$

If $\displaystyle\sum_{k=-\infty}^{\infty} |h[n]| < \infty$ Then $|y[n]| < \infty$

Therefore, the absolutely summable is a sufficient condition to guarantee the stability of a discrete-time LTI system.

(Necessary condition)

Let $x[n]=\begin{cases}\dfrac{h^*[-n]}{|h[-n]|}, & h[n]\neq 0\\ 0, & h[n]=0\end{cases}$, $h^*[-n]$ is conjugate complex of *h*[-*n*].

Obviously *x*[*n*] is bounded by 1, i.e. $|x[n]|\le 1$.

However,

$$y[0]=\sum_{k=-\infty}^{\infty}x[-k]h[k]=\sum_{k=-\infty}^{\infty}\frac{|h[k]|^2}{|h[k]|}=\sum_{k=-\infty}^{\infty}|h[k]|$$

If $\sum_{k=-\infty}^{\infty}|h[k]|\to\infty$ Then $y[0]\to\infty$

This showing that the absolutely summable is also a necessary condition.

2.3.3 The Unit Step Response *s*(*t*)/*s*[*n*] of an LTI System

（单位阶跃响应）

$$u(t)/u[n] \longrightarrow \boxed{h(t)\,/\,h[n]} \longrightarrow s(t)/s[n]$$

$$s(t)=u(t)*h(t)=\int_{-\infty}^{\infty}h(\tau)u(t-\tau)d\tau=\int_{-\infty}^{t}h(\tau)d\tau$$

$$h(t)=\frac{ds(t)}{dt}=s'(t)$$

- For a continuous-time LTI system, *s*(*t*) is the running integral of *h*(*t*). *h*(*t*) is first derivative of *s*(*t*).

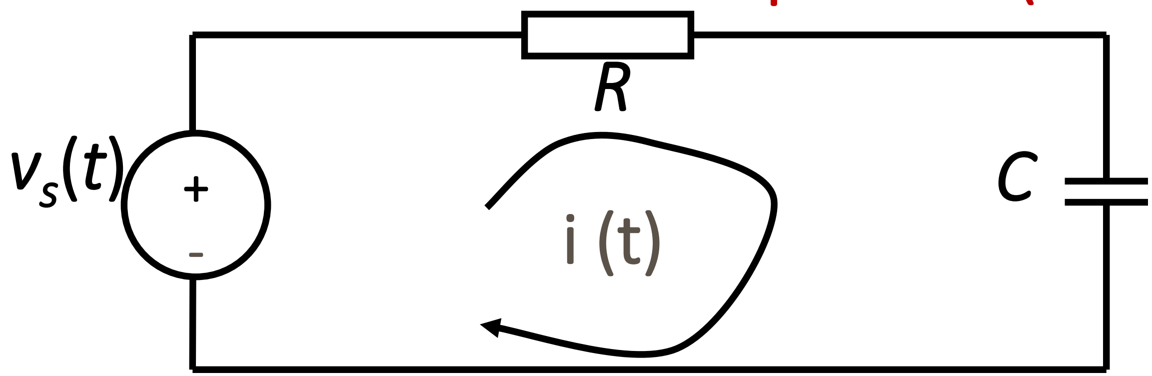
$$s[n]=u[n]*h[n]=\sum_{k=-\infty}^{\infty}h[k]u\left[n-k\right]=\sum_{k=-\infty}^{n}h[k]$$

$$h[n]=s[n]-s[n-1]=\nabla s[n]$$

- For a discrete-time LTI system, *s*[*n*] is the running sum of *h*[*n*]. *h*[*n*] is the first difference of *s*[*n*].

2.4.1 Linear Constant-Coefficient Differential Equations (LCCDE)

Example 2.4



$v_s(t)$: input signal;

$v_c(t)$: output signal.

$$\frac{dv_c(t)}{dt}+\frac{1}{RC}v_c(t)=\frac{1}{RC}v_s(t)$$

- *Linear constant-coefficient differential equation* is the mathematical representation of a continuous-time LTI system.
- General *N*th-order LCCDE: $$\sum_{i=0}^{N}a_i\frac{d^i y(t)}{dt^i}=\sum_{j=0}^{M}b_j\frac{d^j x(t)}{dt^j}$$
- Note: If $v_s(t)$ is a causal signal and $v_c(0_-)\neq 0$, the circuit can also be represented by above equation. In this case it is an Incrementally Linear system.

2.4.2 Solutions to LCCDEs

One or more auxiliary conditions must be specified to solve a differential equation. For a causal LTI system, we will use the condition of initial rest (初始松弛), that is if $x(t)=0$ for $t \le t_0$, $y(t)=0$ for $t \le t_0$ and therefore for a Nth-order equation, the N initial conditions are

$$y(t_0)=\frac{dy(t_0)}{dt}=\frac{d^2y(t_0)}{dt^2}=\cdots=\frac{d^{N-1}y(t_0)}{dt^{N-1}}=0$$

- Classic solution:

$$y(t)=y_p(t)+y_h(t)$$

Particular solution (特解) *homogeneous solution* (齐次解)

‖ ‖

Forced response (受迫响应) *Natural response* (自然响应)

- The second solution: Decomposing response by the reason leading to the part of outputs, i.e., $y(t)=y_{zi}(t)+y_{zs}(t)$

Example 2.4 (Continued) (Reviewing Classic Method)

Sol: Let *R*=4, *C*=1/2, and $v_s(t) = 3e^{-t}u(t)$

Then $\dfrac{dy(t)}{dt} + 0.5y(t) = 0.5x(t)$ ① Obviously, $y_h(t) = Be^{-0.5t}$

Since $x(t) = 3e^{-t}$ for *t*>0, so let $y_p(t) = Ae^{-t} \quad t > 0$

Taking *x*(*t*) and $y_p(t)$ for *t* >0 into equation ① yields

$$-Ae^{-t} + 0.5Ae^{-t} = 1.5e^{-t}$$

Thus $$A = -3$$

So for *t*>0, $$y(t) = Be^{-0.5t} - 3e^{-t}, \quad t > 0$$

Taking use of the condition of initial rest, we get $B = 3$

Consequently, $y(t) = 3e^{-0.5t} - 3e^{-t}$ for *t*>0

or $$y(t) = 3(e^{-0.5t} - e^{-t})u(t)$$

2.4.3 Linear Constant-Coefficient Difference Equations (LCCDE)

➢ *Linear constant-coefficient difference equation* is the mathematical representation of a discrete-time LTI system.

Example 2.5

Jack saves money every month and the interest rate of bank is α per month. He saves into the bank $x[n]$ yuan at the beginning of the n^{th} month and $y[n]$ is the deposits in his account at the end of the n^{th} month (before the bank calculates the interest). Try to write a difference equation relating $x[n]$ and $y[n]$. (For simplicity suppose he wouldn't withdraw his money in bank.)

Sol: $y[n]$ is consists of the sum of the following three parts:

(1) $x[n]$ — saved at the beginning of the n^{th} month

(2) $y[n-1]$ — deposit of the $(n-1)^{th}$ month

(3) $\alpha y[n-1]$ — interest at the end of the $(n-1)^{th}$ month

So
$$y[n]=x[n]+y[n-1]+\alpha y[n-1]$$

or
$$y[n]-(1+\alpha)y[n-1]=x[n]$$

Additionally,

For sequence $x[n]$, its *First forward difference* (一阶前向差分) is defined as $\Delta x[n] = x[n+1] - x[n]$

First backward difference (一阶后向差分) is defined as

$$\nabla x[n] = x[n] - x[n-1]$$

Analogously, *Second forward difference* can be constructed as

$$\Delta^2 x[n] = \Delta\{\Delta x[n]\}$$

$$= \Delta x[n+1] - \Delta x[n] = x[n+2]-2x[n+1]+x[n]$$

Second backward difference as

$$\nabla^2 x[n] = \nabla x[n] - \nabla x[n-1] = x[n]-2x[n-1]+x[n-2]$$

2.4.4 Solutions to LCCDEs

- General *N*th-order linear constant-coefficient difference equation: $$\sum_{i=0}^{N} a_i y[n-i] = \sum_{j=0}^{M} b_j x[n-j]$$
- For causal LTI systems, initial rest condition is that if $x[n]=0$ for $n<n_0$, $y[n]=0$ for $n<n_0$ and therefore for a *N*th-order equation, the *N* initial conditions are
$$y[n_0-1] = y[n_0-2] = \cdots = y[n_0-N] = 0$$
- Classic solution: $$y[n] = y_p[n] + y_h[n]$$

Forced response *Natural response*

- Recursive method: (递归法) $$y[n] = \frac{1}{a_0}\left\{\sum_{j=0}^{M} b_j x[n-j] - \sum_{i=1}^{N} a_i y[n-i]\right\}$$
- $y[n] = y_{zi}[n] + y_{zs}[n]$

Example 2.6 (Classic Method)

Solve the difference equation $y[n]+2y[n-1]=n-2$ with the initial condition *y*[0]=1.

Sol: The characteristic equation is $a+2=0$, the root is $a = -2$.

So $y_h[n]=C(-2)^n$, Let $y_p[n]=D_1 n+D_2$

Taking $y_p[n]$ into the original equation yields

$$D_1 n+D_2+2D_1(n-1)+2D_2=n-2$$

$$D_1=\frac{1}{3},\quad D_2=-\frac{4}{9}$$

$$y[n]=y_h[n]+y_p[n]=C(-2)^n+\frac{1}{3}n-\frac{4}{9}$$

From the initial condition of *y*[0]=1, we have $C=\frac{13}{9}$

Consequently, $y[n]=\frac{1}{9}\left[13(-2)^n+3n-4\right]$

Example 2.7 (Recursive Method)

A first-order LTI system is represented by equation

$$y[n]-0.5y[n-1]=3x[n]$$

Determine the output recursively with the condition of initial rest and $x[n]=\delta[n-1]$.

Sol: Rewrite the given difference equation as

$$y[n]=3x[n]+0.5y[n-1]$$

Starting from initial condition, we can solve for successive values of *y*[*n*] for *n*≥1:

$$y[1]=3x[1]+0.5y[0]=3$$
$$y[2]=3x[2]+0.5y[1]=3\cdot 0.5$$
$$y[3]=3x[3]+0.5y[2]=3\cdot(0.5)^2$$
$$y[4]=3x[4]+0.5y[3]=3\cdot(0.5)^3$$
$$\vdots$$
$$y[n]=3x[n]+0.5y[n-1]=3\cdot(0.5)^{n-1}$$

Considering *y*[*n*] =0 for *n*≤0, the solution is

$$y[n]=3\cdot(0.5)^{n-1}u[n-1]$$

2.4.5 Relationships between y_p, y_h, y_{zi} and y_{zs}

$$x(t)/x[n] \longrightarrow \boxed{h(t)/h[n]} \longrightarrow$$

$$y(t)/y[n] = y_p(t)/y_p[n] + y_h(t)/y_h[n]$$

$$y(t)/y[n] = y_{zi}(t)/y_{zi}[n] + y_{zs}(t)/y_{zs}[n]$$

Example 2.8 ($y(t) = y_{zi}(t) + y_{zs}(t)$)

A second-order causal LTI system is described by differential equation $y''(t)+5y'(t)+6y(t)=x'(t)+x(t)$. Determine the response with initial conditions $y(0_-)=2, y'(0_-)=-3$ and input $x(t)=3e^{-4t}u(t)$.

Sol: The roots of characteristic equation are a=−2 and a=−3, so

$$y_{zi}(t) = C_1e^{-2t} + C_2e^{-3t}, \quad t>0$$

$$y_{zi}(0_+) = y_{zi}(0_-) = y(0_-) = 2,$$
$$y'_{zi}(0_+) = y'_{zi}(0_-) = y'(0_-) = -3$$

$$\begin{cases} y(0_-) = 2 = C_1 + C_2 \\ y'(0_-) = -3 = -2C_1 - 3C_2 \end{cases}$$

Solve for $C_1 = 3, \quad C_2 = -1$

$$\boxed{y_{zi}(t) = (3e^{-2t} - e^{-3t})u(t)}$$

2.4 Causal LTI Systems Described By Differential and Difference Equations

The impulse response of this system is

$$h(t) = (2e^{-3t} - e^{-2t})u(t)$$

Thus

Relation?

$$y_{zs}(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$$

$$= \int_0^t 3e^{-4(t-\tau)}(2e^{-3\tau} - e^{-2\tau})d\tau$$

$$= 6e^{-4t}\int_0^t e^{\tau}d\tau - 3e^{-4t}\int_0^t e^{2\tau}d\tau$$

$$= \left(-\frac{3}{2}e^{-2t} + 6e^{-3t} - \frac{9}{2}e^{-4t}\right)u(t)$$

y_{zi} y_h y_{zs} y_p

$$y(t) = y_{zi}(t) + y_{zs}(t) = \left(\underbrace{\frac{3}{2}e^{-2t} + 5e^{-3t}}_{\text{Natural response}} \underbrace{- \frac{9}{2}e^{-4t}}_{\text{forced response}}\right)u(t)$$

If applying $y(t) = y_h(t) + y_p(t)$, then from the characteristic roots

$$y_h(t) = A_1 e^{-2t} + A_2 e^{-3t}, \quad t > 0$$

and
$$y_p(t) = Be^{-4t}, \quad t > 0$$

Taking $x(t)$ and $y_p(t)$ for $t > 0$ into input-output equation yields

$$16Be^{-4t} - 20Be^{-4t} + 6Be^{-4t} = -12Be^{-4t} + 3Be^{-4t}$$

$$B = -\frac{9}{2}$$

Then

$$y(t) = y_h(t) + y_p(t) = A_1 e^{-2t} + A_2 e^{-3t} - \frac{9}{2} e^{-4t}, \quad t > 0$$

Finally plug in $y'(0_+) = y'(0_-) + 3 = 0, y(0_+) = y(0_-) = 2$ to find

$$A_1 = \frac{3}{2}, \quad A_2 = 5$$

Not given! Found by impulse balance

Example 2.9 (Is there anything between $h[n]/h(t)$ and y_h?)

Determine the $h[n]$ of a causal LTI system described by

$$y[n]-0.5y[n-1]=3x[n]$$

Sol: $h[n]$ satisfies $h[n]-0.5h[n-1]=0 \qquad n>0$

with initial condition $h[-1]=0$

It's obvious that $h[n]=C\left(0.5\right)^{n} \quad n>0 \quad ①$

From $h[0] - 0.5h[-1] = 3\delta[0]$ we have $h[0]=3$

Continuingly from $h[1] - 0.5h[0] = 3\delta[1]$, $h[1]=1.5$

Taking $h[1]$ into equation ① yields $C=3$

Thus $h[n]=3(0.5)^{n} \quad for \quad n>0 \quad ②$

In fact, $h[0]$ also satisfies equation ②, so we can write

$$h[n]=3\left(0.5\right)^{n}u[n]$$

Trying one more time by recursive method!

2.5 Block Diagram Representations of First-Order Systems Described By Differential and Difference Equations

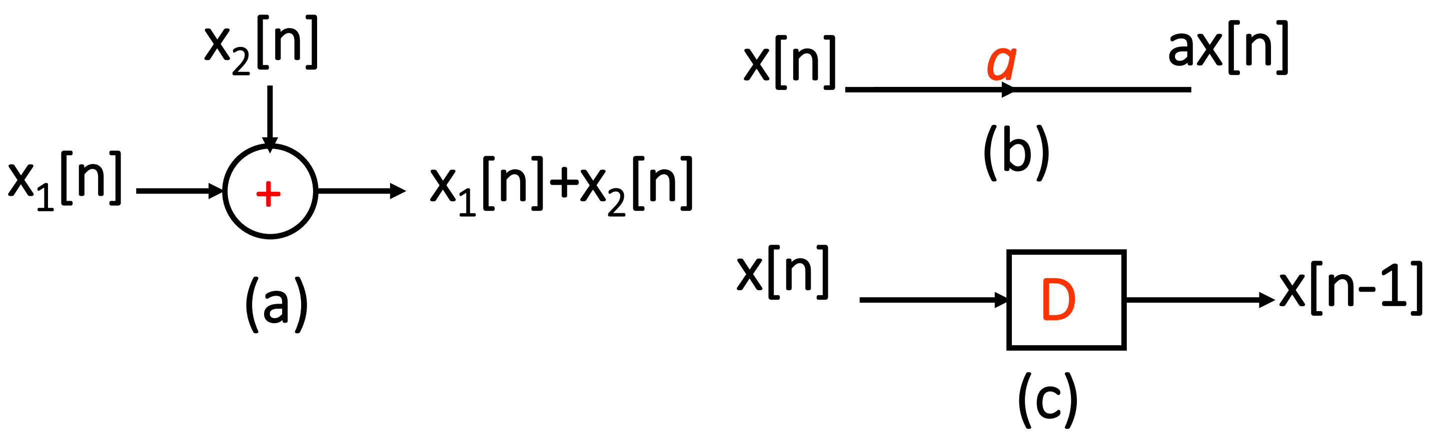
2.5.1 Block Diagram Representations of Discrete-Time Systems

- First-order difference equation :

$$y[n]+ay[n-1]=bx[n]$$

addition delay multiplication

- Three basic elements in block diagram (方框图) : adder (加法器), multiplier (乘法器)and delayer (延时器).

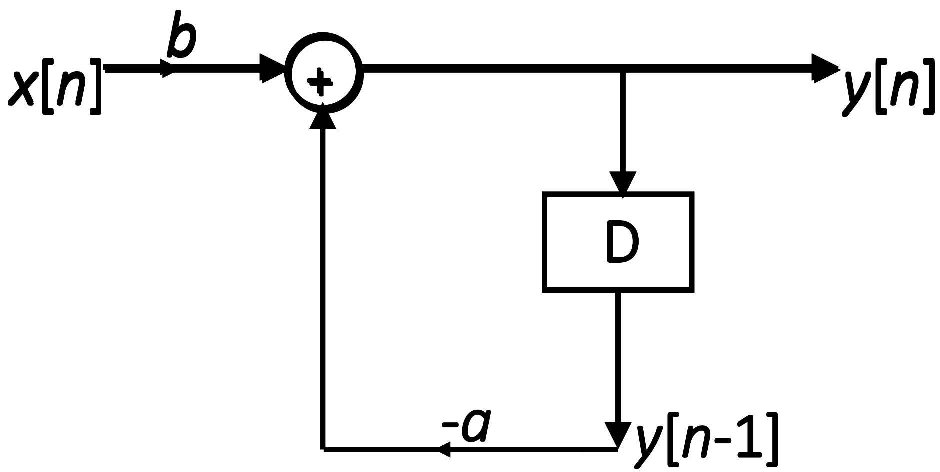


Basic elements for the block diagram representation of causal discrete-time systems: (a) an adder;(b) a multiplier;(c) a delayer.

- Steps to draw the block diagram of causal system represented by first-order difference equation

$$y[n] + ay[n-1] = bx[n]$$

$$y[n] = bx[n] - ay[n-1]$$



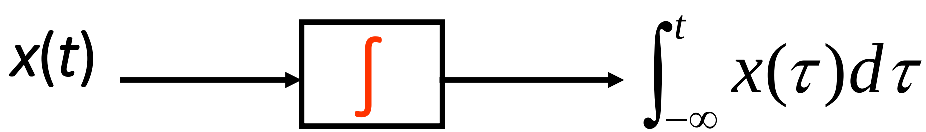
2.5.2 Block Diagram Representations of Continuous-Time Systems

- First-order Differential equation :

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

differentiation addition multiplication

- Three basic elements in block diagram are adder, multiplier and integrator (积分器).

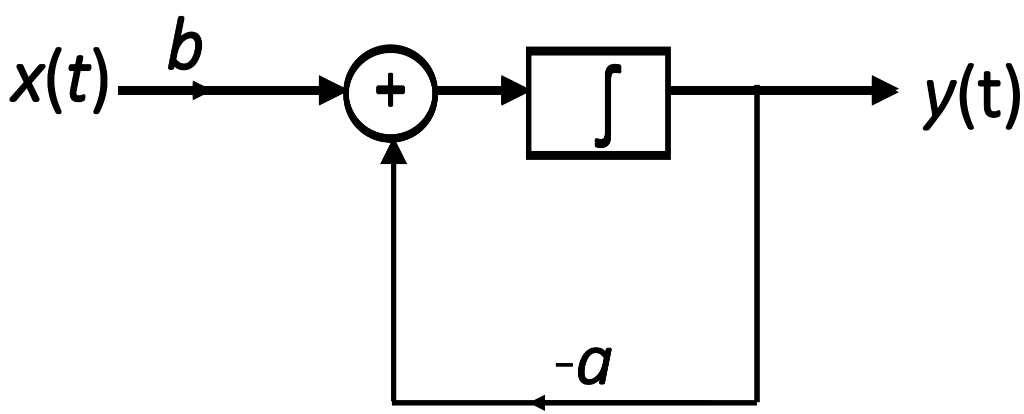


Block diagram representation of an integrator

- Steps to draw the block diagram of causal system represented by first-order differential equation

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

$$\boxed{\frac{dy(t)}{dt} = bx(t) - ay(t)}$$



2.6 SUMMARY

- A representation of an arbitrary signal as weighted sum/integral of shifted unit impulses;
- Convolution sum / convolution integral representation for the response of a LTI system;
- Properties including causality and stability of LTI system ;
- Solutions to LCCDEs ;
- Relationships between y_h, y_p, y_{zi} and y_{zs} ;
- Understanding of initial conditions used to solve LCCDEs, like $y(0_-)$, $y(0_+)$ and initial rest.

Homework

2.21 (a) (c)   2.22 (a) (c)   2.23

2.28 (b) (e) (g)   2.29 (b) (e) (f)

2.31