练习7.

1. 设 $u(x,t) = v(x,t) + \frac{1}{l}x$

则有 $\begin{cases} v_{tt} = a^2 v_{xx} \\ v(0,t) = v(l,t) = 0 \\ v(x,0) = \sin\frac{3\pi x}{l} \quad v_t(x,0) = x(l-x) \end{cases}$

由分离变量法得

$\lambda = \lambda_n = \left(\frac{n\pi}{l}\right)^2 \quad (n=1,2\dots)$

$X_n = B_n \sin\frac{n\pi}{l}x$

$T_n = C_n \cos\frac{an\pi t}{l} + D_n \sin\frac{an\pi t}{l}$

$\therefore v(x,t) = \sum_{n=1}^{\infty}\left(a_n \cos\frac{an\pi t}{l} + b_n \sin\frac{an\pi t}{l}\right)\sin\frac{n\pi}{l}x$

$\therefore \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{l}x = \sin\frac{3\pi x}{l}$

$\sum_{n=1}^{\infty} b_n \frac{an\pi}{l}\sin\frac{n\pi x}{l} = x(l-x)$

$\therefore a_3 = 1 \quad a_n = 0,\ n \neq 3$

$\frac{an\pi}{l}b_n = \frac{2}{l}\int_0^l x(l-x)\sin\frac{n\pi x}{l}\,dx = 4\frac{l^2}{(n\pi)^3}\left[1-(-1)^n\right]$

$\therefore b_n = \frac{4l^3}{an^4\pi^4}\left[1-(-1)^n\right]$

$\therefore u(x,t) = \sin\frac{3\pi x}{l}\cos\frac{3a\pi t}{l} + \sum_{n=1}^{\infty}\frac{4l^3}{an^4\pi^4}\left[1-(-1)^n\right]\sin\frac{an\pi t}{l}\sin\frac{n\pi x}{l}$

练习 8

1. 设 $u(x,t) = v(x,t) + w(x)$

令 $\begin{cases} w''(x) = -e^{-x} - 2 \\ w'(0) = 1 \\ w(3) = -18 - e^{-3} \end{cases}$

$\therefore w(x) = -e^{-x} - x^2 - 9$

则 $\begin{cases} v_t = 2v_{xx} \\ v_x(0,t) = v(3,t) = 0 \\ v(x,0) = -x^2 + 9 \end{cases}$

~~固有函数系为~~ $\cancel{\left\{ \cos \frac{(2n-1)\pi x}{6} \right\} \quad n = 1, 2, \cdots}$

~~设~~ $\cancel{v(x,t) = \sum_{n=1}^{\infty} T(t) \cos \frac{(2n-1)\pi x}{6}}$

~~则~~ $\cancel{\sum_{n=1}^{\infty} \left(T'(t) + 2\left[\frac{(2n-1)\pi}{6}\right]^2 \right) \cos \frac{(2n-1)\pi x}{6} = 0}$

$\cancel{\therefore T'(t) + 2\left(\frac{(2n-1)\pi}{6}\right)^2 = 0}$

设 $v(x,t) = T(t) \cdot X(x)$

分离变量得 $\lambda = \lambda_n = \left(\frac{(2n-1)\pi}{6}\right)^2 \quad X_n = A_n \cos \frac{(2n-1)\pi x}{6}$

由 $T'(t) + 2\lambda T(t) = 0$

得 $T(t) = C_n e^{-2\lambda t} = C_n e^{-\frac{(2n-1)^2\pi^2}{18} t}$

$\therefore v(x,t) = \sum_{n=1}^{\infty} a_n e^{-\frac{(2n-1)^2\pi^2}{18} t} \cdot \cos \frac{(2n-1)\pi x}{6}$

$\because v(x,0) = \sum_{n=1}^{\infty} a_n \cos \frac{(2n-1)\pi x}{6} = -x^2 + 9$

$\therefore a_n = \frac{2}{3} \int_0^3 (-x^2 + 9) \cos \frac{(2n-1)\pi x}{6} = (-1)^{n+1} \frac{288}{(2n-1)^3 \pi^3}$

$\therefore u(x,t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{288}{(2n-1)^3 \pi^3} e^{-\frac{(2n-1)^2\pi^2}{18} t} \cdot \cos \frac{(2n-1)\pi x}{6} - e^{-x} - x^2 - 9.$

3、设 $u(x,y)=v(x,y)-\frac{1}{\pi^2}\sin\pi x+1+x$

则 $\begin{cases} v_{xx}+v_{yy}=0 \\ v(0,y)=v(1,y)=0 \\ v(x,0)=-\frac{1}{\pi^2}\sin\pi x \quad v(x,1)=0 \end{cases}$

设 $v(x,y)=Y(y)X(x)$

则 $\lambda=\lambda_n=\left(\frac{n\pi}{1}\right)^2 \ (n=1,2,\cdots)$

$X_n=B_n\sin n\pi x$

$\therefore Y_n=C_n e^{n\pi y}+D_n e^{-n\pi y}$

$\therefore v(x,y)=\sum_{n=1}^{\infty}(a_n e^{n\pi y}+b_n e^{-n\pi y})\sin n\pi x$

$\because \begin{cases} v(x,0)=\sum_{n=1}^{\infty}(a_n+b_n)\sin n\pi x=-\frac{1}{\pi^2}\sin\pi x \\ v(x,1)=\sum_{n=1}^{\infty}(a_n e^{n\pi}+b_n e^{-n\pi})\sin n\pi x=0 \end{cases}$

$\therefore a_1=\frac{1}{(e^{2\pi}-1)\pi^2} \quad b_1=\frac{-e^{2\pi}}{(e^{2\pi}-1)\pi^2}$

$a_n=b_n=0 \ , \ n\neq 1$

$\therefore u(x,y)=\left(\frac{1}{(e^{2\pi}-1)\pi^2}e^{\pi y}+\frac{-e^{2\pi}}{(e^{2\pi}-1)\pi^2}e^{-\pi y}\right)\sin\pi x$

3、设 $u(x,t)=T(t)X(x)$

则 $\begin{cases} T'(t)+\lambda T(t)=0 \\ X''(x)+2X'(x)+\lambda X(x)=0 \\ X(0)=X(1)=0 \end{cases}$

$\lambda=\lambda_n=(n\pi)^2-1 \quad (n=1,2\cdots)$

$X_n(x)=B_n e^{-x}\sin n\pi$

$T_n(t)=C_n e^{-\cancel{\lambda t}}=C_n e^{(1-n^2\pi^2)t}$

$\therefore u(x,t)=\sum_{n=1}^{\infty}a_n e^{(1-n^2\pi^2)t}\cdot e^{-x}\cdot\sin n\pi x$

$\Rightarrow u(x,0)=\sum_{n=1}^{\infty}a_n e^{-x}\sin n\pi=e^{-x}\sin\pi x$

$\Rightarrow a_1=1 \quad a_n=0,\ n\neq 1$

$\therefore u(x,t)=e^{(1-\pi^2)t-x}\cdot\sin\pi x$

2. 设 $u(x,t)=V(x,t)+w(x)$

令 $\begin{cases} w''(x)=6x-6 \\ w(0)=0 \\ w'(2)=1 \end{cases}$

则 $W(x)=x^3-3x^2+x$

∴ $\begin{cases} V_t=V_{xx} \\ V(0,t)=V_x(2,t)=0 \\ V(x,0)=\sin\frac{\pi}{4}x \end{cases}$

$\lambda=\lambda_n=\left[\frac{(2n-1)\pi}{4}\right]^2$

$X_n=A_n\sin\frac{(2n-1)\pi}{4}x \quad (n=1,2\cdots)$

$T_n=C_n e^{-\frac{(2n-1)^2\pi^2}{16}t}$

∴ $v(x,t)=\sum_{n=1}^{\infty}a_n e^{-\frac{(2n-1)^2\pi^2}{16}t}\sin\frac{(2n-1)\pi}{4}x$

∵ $V(x,0)=\sum_{n=1}^{\infty}a_n\sin\frac{(2n-1)\pi}{4}x=\sin\frac{\pi}{4}x$

∴ $a_1=1 \quad a_n=0,\ n\neq 1$

∴ $u(x,t)=e^{-\frac{1}{16}\pi^2 t}\sin\frac{\pi}{4}x+x^3-3x^2+x$

2、设 $u(x,t)=v(x,t)+\sin t$

$$\therefore \begin{cases} v_t = 8u_{xx} + e^{t}\sin\frac{x}{2} \\ v(0,t)=0, \quad v_x(\pi,t)=0 \\ v(x,0)=0 \end{cases}$$

~~设 [illegible]~~

则固有函数系为 $\left\{\sin\frac{2n-1}{2}x\right\}$ $(n=1,2,\cdots)$

设 $v(x,t)=\sum_{n=1}^{\infty} T_n(t)\sin\frac{2n-1}{2}x$

$$\therefore \sum_{n=1}^{\infty}\left[T_n'(t)+2(2n-1)^2T_n(t)\right]\sin\frac{2n-1}{2}x = e^{t}\sin\frac{x}{2}$$

$$\therefore \begin{cases} T_1'(t)+2T(t)=e^{t}, & n=1 \\ T_n'(t)+2T_n(t)=0, & n\neq 1 \\ T(0)=0 \end{cases}$$

$$\therefore T_1(t)=\frac{e^{t}-e^{-2t}}{3} \qquad T_n(t)=0 \quad n\neq 1$$

$$\therefore u(x,t)=\frac{e^{t}-e^{-2t}}{3}\sin\frac{x}{2}+\sin t$$