# CHAPTER 8

# STATE MODEL REPRESENTATION

# Content

# Content

## 8.0 Introduction

- There are two mathematical models for systems: input-output representation and state-variable representation. The former describes the input/output behavior of systems. The latter describes the internal behavior of systems.

- The objective of this chapter: define the state model and study the basic properties of this model for both continuous- and discrete-time systems.

# 8.1 State Model (状态模型)

For a single-input single-output causal continuous-time system,

input : $v(t)$ output: $y(t)$

Question:

At the time of $t_1$, is it possible to compute the output response $y(t)$ from only the knowledge of the input $v(t)$ for $t \geq t_1$ ?

Obviously it is not. The reason is that the application of the input $v(t)$ for $t < t_1$ may put energy into the system that affects the output response for $t \geq t_1$ .

## 8.1 State Model (状态模型)

For a single-input single-output causal continuous-time system,

input : $v(t)$ output: $y(t)$

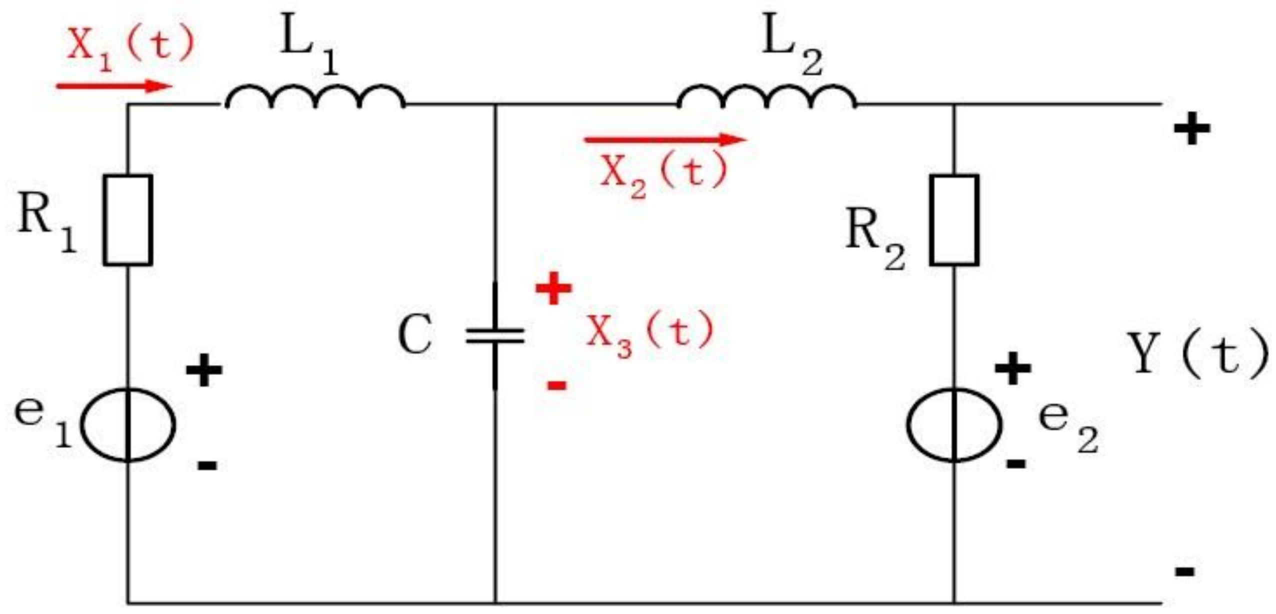
- For any time point $t_1$, the state $x(t)$ of the system at time $t = t_1$ is defined to be that portion of the past history $t \le t_1$ of the system required to determine the output response $y(t)$ for all $t \ge t_1$ given the input $v(t)$ for $t \ge t_1$. A nonzero state $x(t_1)$ at time $t_1$ indicates the presence of energy in the system at time $t_1$.

- If the system is zero at $t_1$, $y(t)$ can be computed from $v(t)$ for $t \ge t_1$.

- If the system is not zero at $t_1$, knowledge of the state is necessary to be able to compute the output $y(t)$.

Example 8.1

Consider the circuit in the right figure. Try to determine the currents in $L_1$ and $L_2$, and the voltage on $C$, besides the output signal $y(t)$.




Sol: Let $x_1$ be the current through $L_1$, $x_2$ be the current through $L_2$, $x_3$ be the voltage on $C$,

$$KVL: \begin{cases} L_1\dot{x}_1 + x_3 + R_1x_1 = e_1 \\ L_2\dot{x}_2 + R_2x_2 - x_3 = -e_2 \end{cases}$$

$$KCL: \quad C\dot{x}_3 = x_1 - x_2$$

$$y = R_2x_2 + e_2$$

# 8.1 State Model

Rewrite the former equations, respectively, as

$$\dot{x}_1 = -\frac{R_1}{L_1}x_1 - \frac{1}{L_1}x_3 + \frac{1}{L_1}e_1$$

$$\dot{x}_2 = -\frac{R_2}{L_2}x_2 + \frac{1}{L_2}x_3 - \frac{1}{L_2}e_2$$

$$\dot{x}_3 = \frac{1}{C}x_1 - \frac{1}{C}x_2$$

$$y = R_2 x_2 + e_2$$

## Matrix form representation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & -\frac{1}{L_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

If we have $x_1$, $x_2$, $x_3$, we can get all the information about the system. So they are necessary and enough.

## 8.1 State Model

From the example, if the given system is *N*-dimensional, the state $\vec{x}(t)$ of the system at time *t* is an *N*-element column vector given by:

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}$$

The components $x_1(t)$, $x_2(t)$, ...., $x_N(t)$ are called the *state variable* (状态变量) of the system.

# 8.2 State Equations (状态方程)

For a single-input single-output $N$-dimensional continuous-time system with state $\vec{x}(t)$ given by :

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{bmatrix}$$

It can be modeled by the state equations given by :

derivative of the state vector

$$\dot{\vec{x}}(t) = f\left(\vec{x}(t), v(t), t\right)$$

$$y(t) = g\left(\vec{x}(t), v(t), t\right) \longrightarrow \text{output equation}$$

Here, both $f$ and $g$ are generally vector-valued function of state $\vec{x}(t)$ at time $t$, the input $v(t)$ at time $t$, and time $t$.

## 8.2 State Equations

- The above two equations comprise the *state model* of the system.
- The state equation describes the state response resulting from the application of an input $v(t)$ with initial state.
- The output equation gives the output response as a function of the state and input.

The two parts correspond to a cascade decomposition of the system as illustrated in Figure 1.

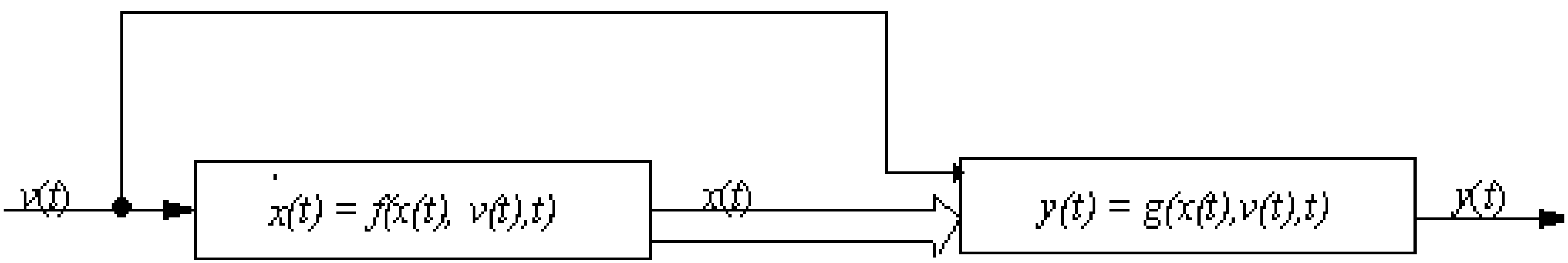



Figure 1  Cascade structure corresponding to state model

## 8.2 State Equations

If *f* and *g* are both linear, the state equations can be written in the form:

$$\dot{\vec{x}}(t)=\vec{A}(t)\vec{x}(t)+\vec{B}(t)v(t)$$

$$y(t)=\vec{C}(t)\vec{x}(t)+D(t)v(t)$$

- $\vec{\boldsymbol{A}}(t)$ is a $N \times N$ matrix whose entries are functions of time $t$;
- $\vec{\boldsymbol{B}}(t)$ is an $N$-element column vector whose components are functions of $t$;
- $\vec{\boldsymbol{C}}(t)$ is an $N$-element row vector with time-varying components;
- $D(t)$ is a real-valued function of time;
- The number N of state variables is called the *dimension* of the state model (or system).

If the system is time invariant, then the state model is given by:

$$\dot{\vec{x}}(t)=\vec{A}\,\vec{x}(t)+\vec{B}v(t) \qquad (1)$$

$$y(t)=\vec{C}\,\vec{x}(t)+Dv(t) \qquad (2)$$

In this case, $\vec{\boldsymbol{A}}(t)$, $\vec{\boldsymbol{B}}(t)$, $\vec{\boldsymbol{C}}(t)$ and $D(t)$ are constant.

## 8.2 State Equations

With $a_{ij}$ equal to the *ij* entry of $\vec{\boldsymbol{A}}$ and $b_i$ equal to the *i*th component of $\vec{\boldsymbol{B}}$, (1) can be written in the expanded form :

$$
\begin{aligned}
\dot{x}_1(t) &= a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1N}x_N(t) + b_1 v(t) \\
\dot{x}_2(t) &= a_{21}x_1(t) + a_{22}x_2(t) + \cdots + a_{2N}x_N(t) + b_2 v(t) \\
&\vdots \\
\dot{x}_N(t) &= a_{N1}x_1(t) + a_{N2}x_2(t) + \cdots + a_{NN}x_N(t) + b_N v(t)
\end{aligned}
$$

With $c = [c_1 \quad c_2 \quad \cdots \ c_N]$, the expanded form of (2) is:

$$y(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_N x_N(t) + dv(t)$$

From the expanded form of the state equations, it is seen that the derivative $\dot{x}_i(t)$ of the *i*th state variable and the output *y*(*t*) are equal to linear combinations of all the state variables and the input.

## 8.3 Construction of State Models

Consider a single-input single-output continuous-time system given by the first-order input/output differential equation:

$$\dot{y}(t) = f\left(y(t), v(t), t\right)$$

Defining the state $x(t)$ of the system to be equal to $y(t)$ results in the state model:

$$\dot{x}(t) = f\left(x(t), v(t), t\right)$$

$$y(t) = x(t)$$

If the given system is LTI so that:

$$\dot{y}(t) = -ay(t) + bv(t)$$

*a* and *b* are constants, then the state model is :

$$\dot{x}(t) = -ax(t) + bv(t)$$

$$y(t) = x(t)$$

## 8.3 Construction of State Models

Suppose that the system has the second-order input/output differential equation:

$$\ddot{y}(t) = f\big(y(t), \dot{y}(t), v(t), t\big)$$

Defining the state variables by :

$$x_1(t) = y(t), \qquad x_2(t) = \dot{y}(t)$$

yields the state model:

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = f\big(x_1(t), x_2(t), v(t), t\big)$$
$$y(t) = x_1(t)$$

## 8.3 Construction of State Models

Example 8.2 Consider a continuous-time second-order LTI system described by the following input-output equation:

$$\ddot{y}(t) = -a_1 \dot{y}(t) - a_0 y(t) + b_0 v(t)$$

Construct its state model.

Sol: Let $x_1(t) = y(t), x_2(t) = \dot{y}(t)$ to obtain:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -a_1 x_2(t) - a_0 x_1(t) + b_0 v(t)$$

Thus,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ b_0 \end{bmatrix} v(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

## 8.3 Construction of State Models

The defining of state variables in terms of the output and derivatives of the output extends to any system given by the *N*th-order input/output differential equation:

$$y^{(N)}(t) = f\left(y(t), y^{(1)}(t), \cdots, y^{(N-1)}(t), v(t), t\right)$$

with the state variables defined by

$$x_i(t) = y^{(i-1)}(t), \qquad i = 1, 2, \cdots, N$$

The resulting state equations are:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = x_3(t)$$

$$\vdots$$

$$\dot{x}_{N-1}(t) = x_N(t)$$

$$\dot{x}_N(t) = f\left(x_1(t), x_2(t), \cdots, x_N(t), v(t), t\right)$$

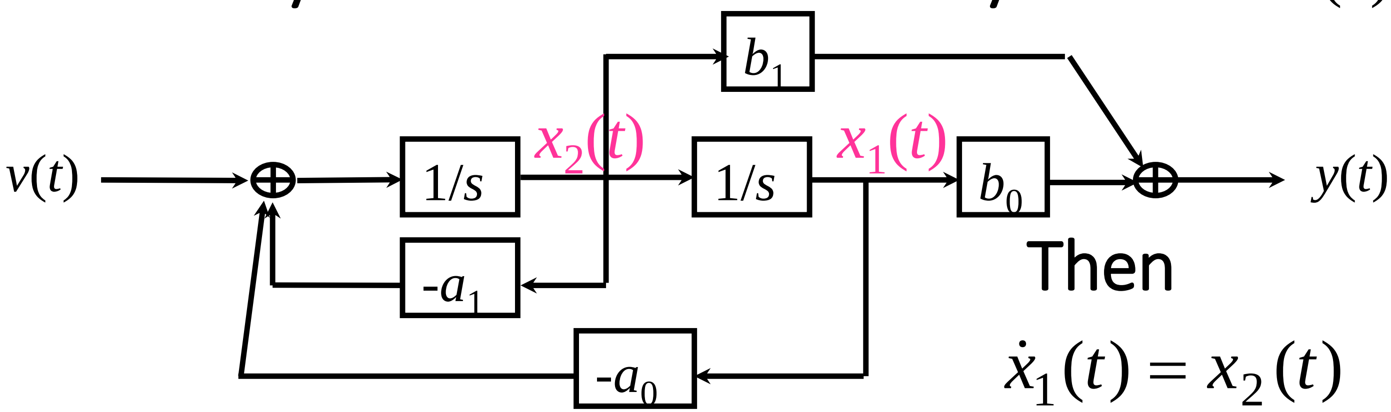
$$y(t) = x_1(t)$$

Example 8.3 If the input-output equation for a system is:

$$\ddot{y}(t)+a_1\dot{y}(t)+a_0 y(t)=b_1\dot{v}(t)+b_0 v(t)$$

Construct its state model.

Sol: The system function for the system is: $H(s)=\dfrac{b_1 s+b_0}{s^2+a_1 s+a_0}$




Then

$$\dot{x}_1(t)=x_2(t)$$

$$\dot{x}_2(t)=-a_1x_2(t)-a_0x_1(t)+v(t)$$

$$y(t)=b_1x_2(t)+b_0x_1(t)$$

The state model is:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix}=\begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}v(t)$$

$$y(t)=\begin{bmatrix}b_0 & b_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

Rewrite the system function as

$$H(s) = H_1(s)H_2(s) = \frac{1}{s^2 + a_1 s + a_0}\left(b_1 s + b_0\right)$$

Let $x_1(t) = z(t), x_2(t) = \dot{z}(t)$, where $z(t)$ is the output of $H_1(s)$. Then

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -a_1 x_2(t) - a_0 x_1(t) + v(t)$$

And

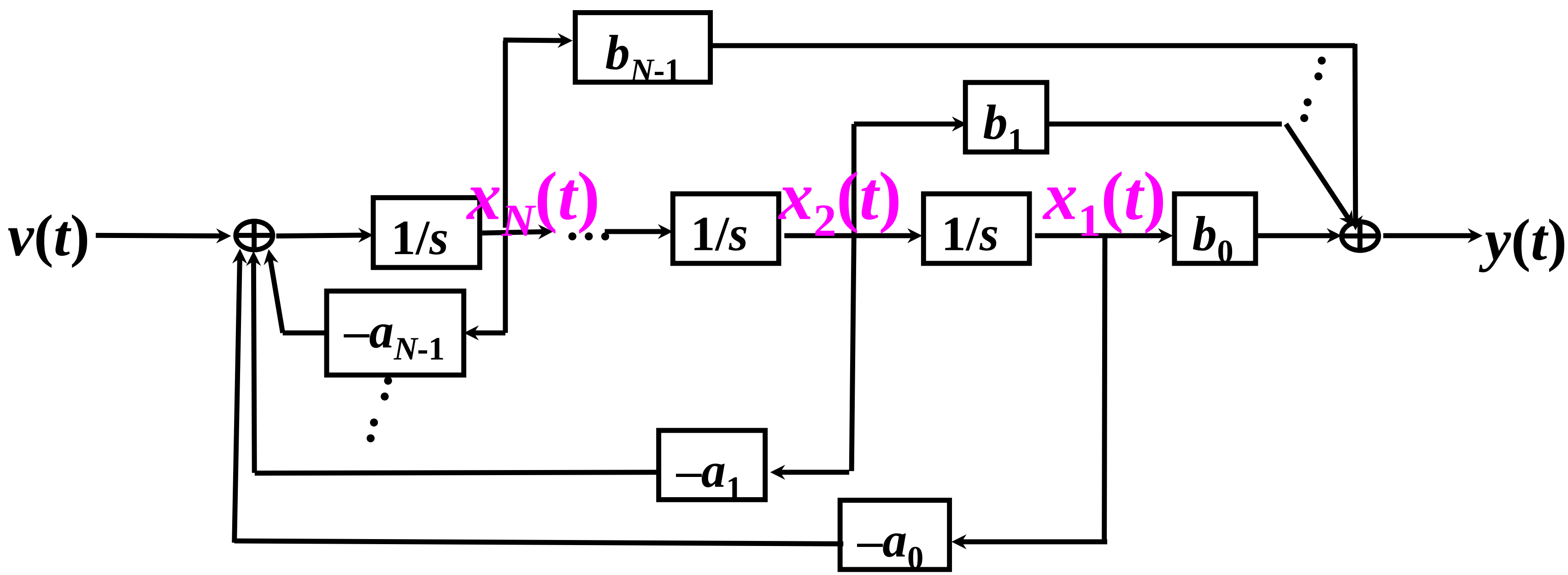
$$\begin{aligned} y(t) &= b_1 \dot{z}(t) + b_0 z(t) \\ &= b_1 x_2(t) + b_0 x_1(t) \end{aligned}$$

## 8.3 Construction of State Models

For a general LTI system given by the *N*th-order input/output differential equation:

$$y^{(N)}(t) + \sum_{i=0}^{N-1} a_i y^{(i)}(t) = \sum_{i=0}^{N-1} b_i v^{(i)}(t)$$

Its block diagram representation is:



This system has the *N*-dimensional state model

$$\dot{\vec{x}}(t) = \vec{A}\vec{x}(t) + \vec{B}v(t), \quad y(t) = \vec{C}\,\vec{x}(t)$$

where :

$$\vec{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{N-1} \end{bmatrix}, \quad \vec{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \vec{C} = \begin{bmatrix} b_0 \; b_1 \cdots b_{N-1} \end{bmatrix}$$

## 8.3 Construction of State Models

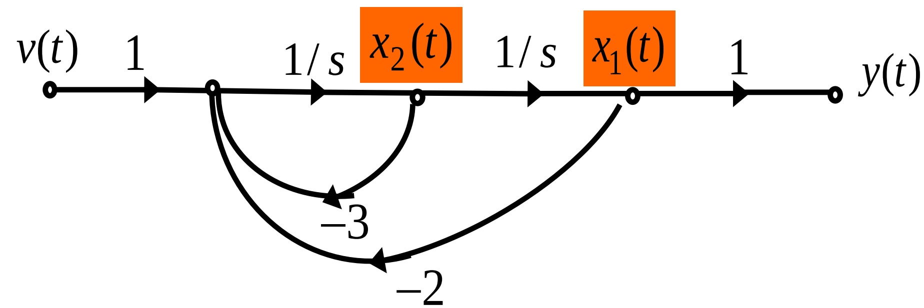
Example 8.4 Consider a continuous-time LTI system with transfer function

$$H(s) = \frac{1}{(s+1)(s+2)}$$

Draw the direct-, cascade- and parallel form signal flow graph of the system, respectively. And construct the state models of the system based on the signal flow graph, respectively.

**Direct-form:**

$$H(s) = \frac{1}{s^2 + 3s + 2}$$




State model:

$$\dot{x}_1(t) = x_2(t)$$

$$\dot{x}_2(t) = -2x_1(t) - 3x_2(t) + v(t)$$

$$y(t) = x_1(t)$$

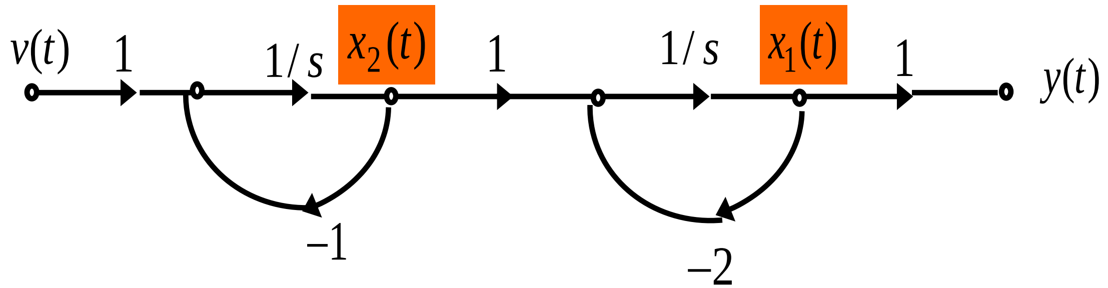
Matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

# 8.3 Construction of State Models

**Cascade-form:**

$$H(s) = \frac{1}{s+1} \cdot \frac{1}{s+2}$$




**State model:**

$$\dot{x}_1(t) = -2x_1(t) + x_2(t)$$

$$\dot{x}_2(t) = -x_2(t) + v(t)$$

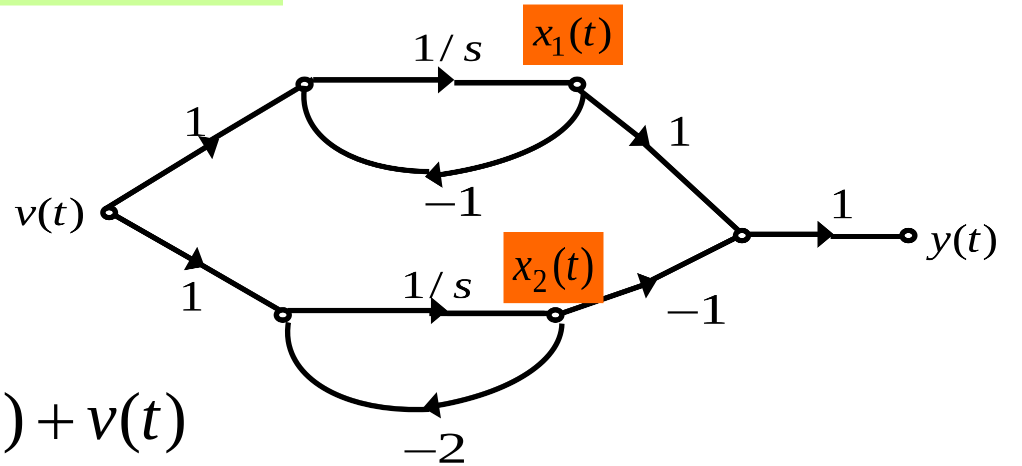
$$y(t) = x_1(t)$$

**Matrix form:**

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

# 8.3 Construction of State Models

**Parallel-form:** $H(s)=\dfrac{1}{s+1}+\dfrac{-1}{s+2}$




State model:

$$\dot{x}_1(t)=-x_1(t)+v(t)$$

$$\dot{x}_2(t)=-2x_2(t)+v(t)$$

$$y(t)=x_1(t)-x_2(t)$$

Matrix form:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}v \qquad y=\begin{bmatrix}1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

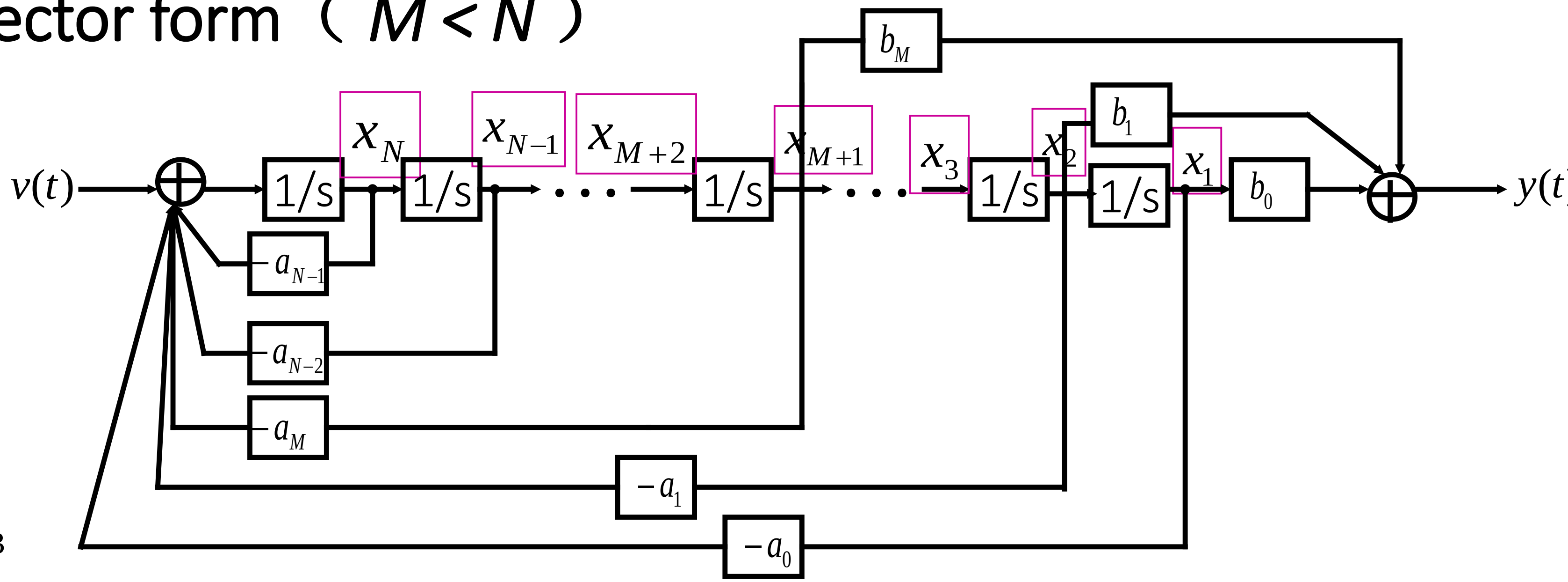
You may obtain different state equations depending on the different choice of state variables!

Summary on the general form of the state model:

***Nth-order differential equation (Scalar)*** $\rightarrow$ ***First-order differential equations in N-dimensional space (Vector)***

$$H(s)=\frac{b_M s^M+b_{M-1}s^{M-1}+\cdots+b_1 s+b_0}{s^N+a_{N-1}s^{N-1}+\cdots+a_1 s+a_0}$$

① Director form（ $M<N$ ）




$$\begin{cases}\dot{x}_1=x_2\\ \dot{x}_2=x_3\\ \cdots\\ \dot{x}_{N-1}=x_N\\ \dot{x}_N=-a_{N-1}x_n-a_{N-2}x_{n-1}-\cdots-a_1x_2-a_0x_1+v\end{cases}$$

$$y=b_0x_1+b_1x_2+\cdots+b_Mx_{M+1}$$

# 8.3 Construction of State Models

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{N-1} \\ \dot{x}_N \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{N-2} & -a_{N-1} \end{bmatrix}}_{\vec{A}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\vec{B}} v$$

$$y = \underbrace{\begin{bmatrix} b_0 & b_1 & \cdots & b_M & 0 & \cdots & 0 \end{bmatrix}}_{\vec{C}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix}$$

② Parallel form

$$H(s) = \frac{k_1}{s-\lambda_1} + \frac{k_2}{s-\lambda_2} + \cdots + \frac{k_N}{s-\lambda_N}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{N-1} \\ \dot{x}_N \end{bmatrix} = \underbrace{\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_{N-1} & 0 \\ 0 & 0 & \cdots & 0 & \lambda_N \end{bmatrix}}_{\vec{A}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}}_{\vec{B}} v$$

$$y = \underbrace{\begin{bmatrix} k_1 & k_2 & \cdots & k_{N-1} & k_N \end{bmatrix}}_{\vec{C}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{N-1} \\ x_N \end{bmatrix}$$

Example 8.5 *Integrator Realization*

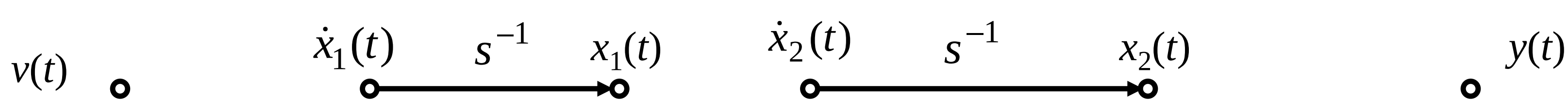
Consider a two-dimensional state model with arbitrary coefficients; that is,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} v(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

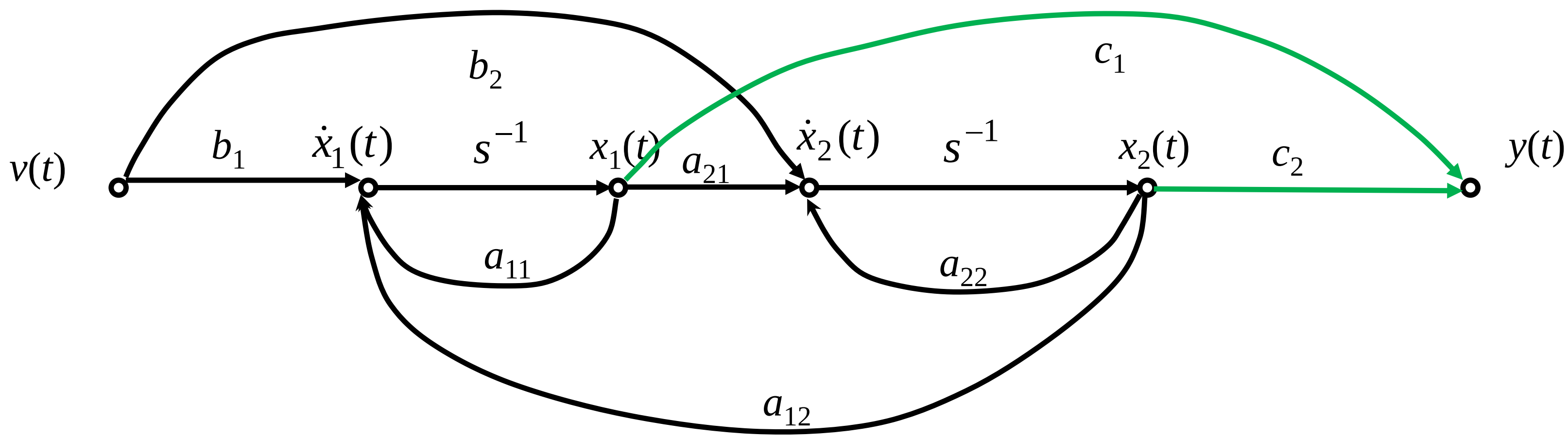
Draw the signal flow graph of the system.

Sol: Step 1: Define the output of each integrator in the interconnection to be a state variable. Then if the output of the *i*th integrator is $\dot{x}_i(t)$, the input to this integrator is $x_i(t)$ .



## Step 2: Realize the state equation

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} v(t)$$




## Step 3: Realize the output equation

$$y(t) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

# 8.4 Multi-Input Multi-Output Systems

The state model of a *p*-input *r*-output LTI *N*th-order continuous-time system is given by:

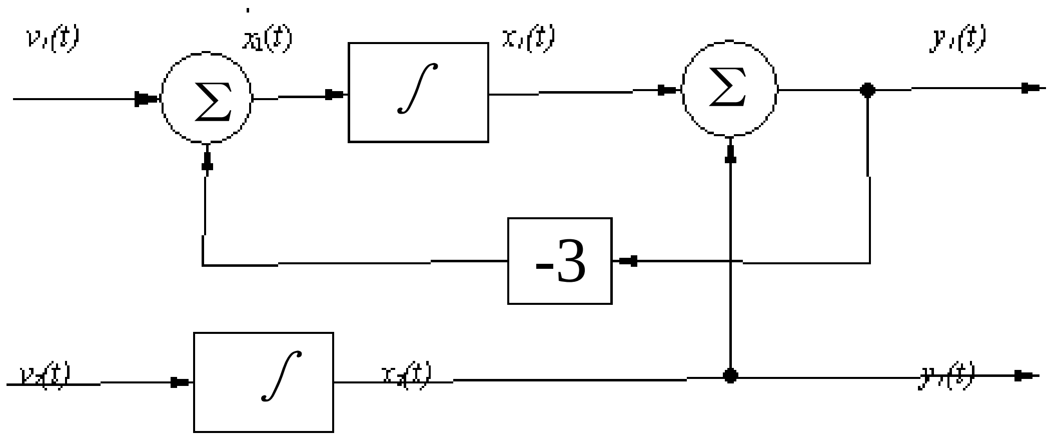
$$\dot{\vec{x}}(t) = \vec{A}\vec{x}(t) + \vec{B}\vec{v}(t)$$

$$\vec{y}(t) = \vec{C}\vec{x}(t) + \vec{D}\vec{v}(t)$$

Where now $\overrightarrow{\boldsymbol{B}}$ is a $N \times p$ matrix of real numbers, $\overrightarrow{\boldsymbol{C}}$ is a $r \times N$ matrix of real numbers, and $\overrightarrow{\boldsymbol{D}}$ is a $r \times p$ matrix.

Example 8.6 *Two-Input Two-Output System*

A two-input two-output system is shown in the following figure




Sol: From the figure,

$$\dot{x}_1(t) = -3y_1(t) + v_1(t) \qquad y_1(t) = x_1(t) + x_2(t)$$

$$\dot{x}_2(t) = v_2(t) \qquad y_2(t) = x_2(t)$$

Inserting the expression for $y_1(t)$ into the expression for $\dot{x}_1(t)$ gives

$$\dot{x}_1(t) = -3\left[x_1(t) + x_2(t)\right] + v_1(t)$$

Putting these equations in matrix from results in the state model:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

## 8.5 Solution of State Equations

*Matrix Exponential* $e^{\vec{A}t}$ (矩阵指数函数):

For each real value of *t*, $e^{\vec{A}t}$ is defined by the matrix power series:

$$e^{\vec{A}t} = \vec{I} + \vec{A}t + \frac{\vec{A}^2 t^2}{2!} + \frac{\vec{A}^3 t^3}{3!} + \frac{\vec{A}^4 t^4}{4!} + \cdots$$

Where $\vec{I}$ is the $N \times N$ identity matrix.

Properties of $e^{\vec{A}t}$ :

- For any real numbers *t* and $\lambda$ , $e^{\vec{A}(t+\lambda)} = e^{\vec{A}t} \bullet e^{\vec{A}\lambda}$
- $e^{\vec{A}t}$ always has an inverse, which is equal to the matrix $e^{-\vec{A}t}$

$$e^{\vec{A}t} \bullet e^{-\vec{A}t} = e^{\vec{A}(t-t)} = \vec{I}_N$$

- The derivative of the matrix exponential is

$$\frac{d}{dt} e^{\vec{A}t} = \vec{A} + \vec{A}^2 t + \frac{\vec{A}^3 t^2}{2!} + \frac{\vec{A}^4 t^3}{3!} + \cdots = \vec{A}\left(\vec{I} + \vec{A}t + \frac{\vec{A}^2 t^2}{2!} + \frac{\vec{A}^3 t^3}{3!} + \cdots\right)$$

$$= \vec{A} \bullet e^{\vec{A}t} = e^{\vec{A}t} \bullet \vec{A}$$

From the derivative property of $e^{\vec{A}t}$, we have that the solution of $\dot{\vec{x}}(t) = \vec{A}\vec{x}(t), t > 0$ is:

$$\vec{x}(t) = e^{\vec{A}t} \bullet \vec{x}(0), \qquad t \geq 0$$

It is seen that the state $\vec{\boldsymbol{x}}(t)$ at time $t$ resulting from state $\vec{\boldsymbol{x}}(0)$ at time $t = 0$ with no input applied for $t \geq 0$ can be computed by multiplying $\vec{\boldsymbol{x}}(0)$ by the matrix $e^{\vec{A}t}$.

As a result of this property, The matrix $e^{\vec{A}t}$ is called the *state-transition matrix* (状态转移矩阵, 状态过渡矩阵) of the system.

## 8.5 Solution of State Equations

For the state eqution $\boxed{\dot{\vec{x}}(t) = \vec{A}\vec{x}(t) + \vec{B}\vec{v}(t)}$ ,

Multiplying both sides on the left by $e^{-\vec{A}t}$ and rearranging terms yields:

$$e^{-\vec{A}t}\left[\dot{\vec{x}}(t) - \vec{A}\vec{x}(t)\right] = e^{-\vec{A}t}\vec{B}\vec{v}(t)$$

From the derivative property we can get

$$\frac{d}{dt}\left[e^{-\vec{A}t}\vec{x}(t)\right] = e^{-\vec{A}t}\vec{B}\vec{v}(t)$$

$$e^{-\vec{A}t}\vec{x}(t) = \vec{x}(0) + \int_0^t e^{-\vec{A}\lambda}\vec{B}\vec{v}(\lambda)d\lambda$$

$$\vec{x}(t) = e^{\vec{A}t}\vec{x}(0) + \int_0^t e^{\vec{A}(t-\lambda)}\vec{B}\vec{v}(\lambda)d\lambda, \qquad t \geq 0$$

$$\vec{x}(t) = e^{\vec{A}t}\vec{x}(0) + e^{\vec{A}t} * \vec{B}\vec{v}(t), \qquad t \geq 0$$

This is the complete solution of the state equation resulting from initial state $\vec{\boldsymbol{x}}(0)$ and input $\vec{\boldsymbol{v}}(t)$ applied for $t \geq 0$.

## 8.6 Output Response

From $\vec{y}(t)=\vec{C}\vec{x}(t)+\vec{D}\vec{v}(t)$ and the solution for the state equations, we can get:

$$\vec{y}(t)=\vec{C}e^{\vec{A}t}\vec{x}(0)+\int_0^t \vec{C}e^{\vec{A}(t-\lambda)}\vec{B}\vec{v}(\lambda)d\lambda+\vec{D}\vec{v}(t), \qquad t\geq 0$$

From the definition of the unit impulse,we can rewrite the former equation as:

$$\vec{y}(t)=\vec{C}e^{\vec{A}t}\vec{x}(0)+\int_0^t \left\{\vec{C}e^{\vec{A}(t-\lambda)}\vec{B}\vec{v}(\lambda)+\vec{D}\vec{\delta}\left(t-\lambda\right)\vec{v}(\lambda)\right\}d\lambda, \qquad t\geq 0$$

Where the *zero-input response* and the *zero-state response* are:

$$\vec{y}_{zi}(t)=\vec{C}e^{\vec{A}t}\vec{x}(0)$$

$$\vec{y}_{zs}(t)=\int_0^t \left\{\vec{C}e^{\vec{A}(t-\lambda)}\vec{B}\vec{v}(\lambda)+\vec{D}\vec{\delta}\left(t-\lambda\right)\vec{v}(\lambda)\right\}d\lambda=\left[\vec{C}e^{\vec{A}t}\vec{B}+\vec{D}\vec{\delta}(t)\right]*\vec{v}(t)$$

The *impulse response matrix* is : $\vec{h}(t)=\vec{C}e^{\vec{A}t}\vec{B}+\vec{D}\vec{\delta}\left(t\right), \quad t\geq 0$

## 8.7 Solution via The Laplace Transform

Taking the Laplace transform of the equation $\dot{\vec{x}}(t) = \vec{A}\vec{x}(t) + \vec{B}\vec{v}(t)$ gives:

$$s\vec{X}(s) - \vec{x}(0) = \vec{A}\vec{X}(s) + \vec{B}\vec{V}(s)$$

$$\vec{X}(s) = (s\vec{I} - \vec{A})^{-1}\vec{x}(0) + (s\vec{I} - \vec{A})^{-1}\vec{B}\vec{V}(s)$$

From this we can get:

$e^{\vec{A}t} = inverse$ Laplace transform of $(s\vec{I} - \vec{A})^{-1}$

Where $(s\vec{I} - \vec{A})^{-1}$ is the Laplace transform of the state-

transition matrix $e^{\vec{A}t}$ .

## 8.7 Solution via The Laplace Transform

Taking the Laplace transform of the output equation

$\vec{y}(t) = \vec{C}\,\vec{x}(t) + \vec{D}\vec{v}(t)$ yields:

$$\vec{Y}(s) = \vec{C}\,\vec{X}(s) + \vec{D}\vec{V}(s)$$

From the Laplace transform solution for state variable $\vec{\boldsymbol{x}}(t)$, we can get:

$$\vec{Y}(s) = \vec{C}(s\vec{I} - \vec{A})^{-1}\vec{x}(0) + [\vec{C}(s\vec{I} - \vec{A})^{-1}\vec{B} + \vec{D}]\vec{V}(s)$$

If $\vec{x}(0)=0$, then $\vec{Y}(s) = \vec{Y}_{zs}(s) = \vec{H}(s)\vec{V}(s)$

where $\vec{\boldsymbol{H}}(s)$ is the *transfer function matrix* of the system given by

$$\vec{H}(s) = \vec{C}(s\vec{I} - \vec{A})^{-1}\vec{B} + \vec{D}$$

Example 8.7: Consider the two-input three-output two-dimensional system with state model $\dot{\vec{x}}(t)=\vec{A}\vec{x}(t)+\vec{B}\vec{v}(t), \vec{y}(t)=\vec{C}\vec{x}(t)$, where

$$\vec{A}=\begin{bmatrix}-3 & 1\\ -2 & -1\end{bmatrix},\quad \vec{B}=\begin{bmatrix}3 & 2\\ 2 & 1\end{bmatrix},\quad \vec{C}=\begin{bmatrix}1 & 2\\ -2 & 2\\ 1 & -1\end{bmatrix}$$

if the initial state $\vec{x}(0)=\begin{bmatrix}1\\ 1\end{bmatrix}$ and input $\vec{v}(t)=\begin{bmatrix}u(t)\\ e^{-t}u(t)\end{bmatrix}$, compute the output $\vec{y}(t)$.

Sol: First compute the state-transition matrix. Since

$$(s\vec{I}-\vec{A})^{-1}=\begin{bmatrix}s+3 & -1\\ 2 & s+1\end{bmatrix}^{-1}=\frac{1}{s^2+4s+5}\begin{bmatrix}s+1 & 1\\ -2 & s+3\end{bmatrix}=\frac{1}{(s+2)^2+1}\begin{bmatrix}s+1 & 1\\ -2 & s+3\end{bmatrix}$$

The state-transition matrix

$$e^{\vec{A}t}=e^{-2t}\begin{bmatrix}\cos t-\sin t & \sin t\\ -2\sin t & \cos t+\sin t\end{bmatrix}u(t)$$

# 8.7 Solution via The Laplace Transform

The state response $\vec{x}(t)$ resulting from the initial state $\vec{x}(0)$ with zero input is given by $\vec{x}(t) = e^{\vec{A}t}\vec{x}(0), t \geq 0$ , so

$$\vec{x}_{zi}(t) = e^{-2t}\begin{bmatrix} \cos t - \sin t & \sin t \\ -2\sin t & \cos t + \sin t \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^{-2t}\begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix}, \quad t \geq 0$$

The state response $\vec{x}(t)$ resulting from the input $\vec{v}(t) = \begin{bmatrix} u(t) \\ e^{-t}u(t) \end{bmatrix}$ is to be computed.

Since $\vec{V}(s) = \begin{bmatrix} \frac{1}{s} \\ \frac{1}{s+1} \end{bmatrix}$, From $\vec{X}_{zs}(s) = (s\vec{I} - \vec{A})^{-1}\vec{B}\vec{V}(s)$ , we have

$$\vec{X}_{zs}(s) = \begin{bmatrix} s+3 & -1 \\ 2 & s+1 \end{bmatrix}^{-1}\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} \frac{1}{s} \\ \frac{1}{s+1} \end{bmatrix} = \frac{1}{s^2+4s+5}\begin{bmatrix} s+1 & 1 \\ -2 & s+3 \end{bmatrix}\begin{bmatrix} \frac{5s+3}{s(s+1)} \\ \frac{3s+2}{s(s+1)} \end{bmatrix}$$

$$= \frac{1}{[(s+2)^2+1]s(s+1)}\begin{bmatrix} 5s^2+11s+5 \\ 3s^2+s \end{bmatrix}$$

Taking the inverse Laplace transform of $\vec{\boldsymbol{X}}_{zs}(s)$ yields

$$\vec{x}_{zs}(t)=\begin{bmatrix} e^{-2t}(-1.5\cos\ t+2.5\sin t)+1+0.5e^{-t} \\ e^{-2t}(\cos t+4\sin t)-e^{-t} \end{bmatrix}u(t)$$

Then the state variables are

$$\vec{x}(t)=\vec{x}_{zi}(t)+\vec{x}_{zs}(t)=\begin{bmatrix} e^{-2t}(-0.5\cos\ t+2.5\sin t)+1+0.5e^{-t} \\ e^{-2t}(2\cos t+3\sin t)-e^{-t} \end{bmatrix}u(t)$$

The output response

$$\vec{y}(t)=\vec{C}\,\vec{x}(t)=\begin{bmatrix} e^{-2t}(3.5\cos t+8.5\sin t)+1\text{-}1.5e^{\text{-}t} \\ e^{-2t}(5\ \cos t+\sin t)-2-3e^{-t} \\ e^{-2t}(-2.5\ \cos t-0.5\sin t)+1+1.5e^{-t} \end{bmatrix}u(t)$$

## 8.8 Discrete-Time Systems

A *p*-input *r*-output finite-dimensional linear time-invariant discrete-time system can be modeled by the state model:

$$\vec{x}[n+1] = \vec{A}\vec{x}[n] + \vec{B}\vec{v}[n]$$
$$\vec{y}[n] = \vec{C}\vec{x}[n] + \vec{D}\vec{v}[n]$$

The state vector $\vec{\boldsymbol{x}}[n]$ is the *N*-element column vector:

$$\vec{x}[n] = \begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_N[n] \end{bmatrix}$$

The input $\vec{\boldsymbol{v}}[n]$ and output $\vec{\boldsymbol{y}}[n]$ are the column vectors:

$$\vec{v}[n] = \begin{bmatrix} v_1[n] \\ v_2[n] \\ \vdots \\ v_p[n] \end{bmatrix}, \quad \vec{y}[n] = \begin{bmatrix} y_1[n] \\ y_2[n] \\ \vdots \\ y_r[n] \end{bmatrix}$$

The matrix $\vec{\boldsymbol{A}}$, $\vec{\boldsymbol{B}}$, $\vec{\boldsymbol{C}}$, and $\vec{\boldsymbol{D}}$ are $N \times N$, $N \times p$, $r \times N$, *and* $r \times p$ respectively.

# 8.9 Construction of State Models

For a single-input single-output LTI discrete-time system with the input/output difference equation:

$$y[n+N]+\sum_{i=0}^{N-1} a_i y[n+i]=\sum_{i=0}^{N-1} b_i v[n+i]$$

The system function is:

$$H(z)=\frac{\sum_{i=0}^{N-1} b_i z^i}{z^N+\sum_{i=0}^{N-1} a_i z^i}$$

Rewrite it as:

$$H(z)=H_1(z)H_2(z)=\frac{1}{z^N+\sum_{i=0}^{N-1} a_i z^i}\sum_{i=0}^{N-1} b_i z^i$$

Defining the state variables as

$$x_{i+1}[n]=z[n+i], \quad i=0,1,2,\ldots,N-1$$

Where $z[n]$ is the output of the first sub-system $H_1(z)$.

# 8.9 Construction of State Models

Then
$$x_1[n+1] = x_2[n]$$
$$x_2[n+1] = x_3[n]$$
$$\vdots$$
$$x_{N-1}[n+1] = x_N[n]$$
$$x_N[n+1] = -a_{N-1}x_N[n] - a_{N-2}x_{N-1}[n] - \cdots - a_0 x_1[n] + v[n]$$
$$y[n] = b_{N-1}x_N[n] + b_{N-2}x_{N-1}[n] + \cdots + b_1 x_2[n] + b_0 x_1[n]$$

Thus, the state model is:
$$\vec{x}[n+1] = \vec{A}\vec{x}[n] + \vec{B}v[n]$$
$$y[n] = \vec{C}\vec{x}[n] + Dv[n]$$

where

$$\vec{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{N-1} \end{bmatrix}, \quad \vec{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \vec{C} = [b_0 \quad b_1, \quad \cdots \quad b_{N-1}], \quad D = 0$$

## 8.10 Solution of State Equations

Consider the *p*-input *r*-output discrete-time system with the state model:

$$\vec{x}[n+1] = \vec{A}\vec{x}[n] + \vec{B}\vec{v}[n] \qquad (1)$$

$$\vec{y}[n] = \vec{C}\,\vec{x}[n] + \vec{D}\vec{v}[n] \qquad (2)$$

Setting *n*=0 in (1) gives $\vec{x}[1] = \vec{A}\vec{x}[0] + \vec{B}\vec{v}[0]$

Setting *n*=1 in (1) gives $\vec{x}[2] = \vec{A}\vec{x}[1] + \vec{B}\vec{v}[1]$

$$= \vec{A}\left[\vec{A}\vec{x}[0] + \vec{B}\vec{v}[0]\right] + \vec{B}\vec{v}[1]$$

$$= \vec{A}^2\vec{x}[0] + \vec{A}\vec{B}\vec{v}[0] + \vec{B}\vec{v}[1]$$

If this process is continued, for any integer value of $n \geq 1$,

$$\boxed{\vec{x}[n] = \vec{A}^n\vec{x}[0] + \sum_{i=0}^{n-1}\vec{A}^{n-i-1}\vec{B}\vec{v}[i], \qquad n \geq 1}$$

$$\vec{x}[n] = \vec{A}^n\vec{x}[0]u[n] + \sum_{i=0}^{n-1}\vec{A}^{n-i-1}\vec{B}\vec{v}[i]u[n-1]$$

# 8.10 Solution of State Equations

The right-hand side of the former equation is the state response resulting from initial state $\vec{\boldsymbol{x}}[0]$ and input $\vec{\boldsymbol{v}}\,[n]$ applied for $n \geq 0$. Note that if $\vec{\boldsymbol{v}}[n] = 0$ for $n \geq 0$ , then

$$\vec{x}[n] = \vec{A}^n \vec{x}[0], \qquad n \geq 0$$

It is seen that the state transition from initial state $\vec{\boldsymbol{x}}[0]$ to state $\vec{\boldsymbol{x}}[n]$ at time *n* (with no input applied) is equal to $\vec{\boldsymbol{x}}[0]$ times the matrix $\vec{A}^n$.

Therefore, in the discrete-time case the state-transition matrix is the matrix $\vec{A}^n$ .

# 8.10 Solution of State Equations

Taking the former equation into the output equation gives:

$$\vec{y}[n]=\vec{C}\vec{A}^{n}\vec{x}[0]+\sum_{i=0}^{n-1}\vec{C}\vec{A}^{n-i-1}\vec{B}\vec{v}[i]+\vec{D}\vec{v}[n], \qquad n\geq 1$$

Where the term $\vec{y}_{zi}[n]=\vec{C}\vec{A}^{n}\vec{x}[0], \qquad n\geq 0$

is the *zero-input* response, and the term

$$\vec{y}_{zs}[n]=\sum_{i=0}^{n-1}\vec{C}\vec{A}^{n-i-1}\vec{B}\vec{v}[i]+\vec{D}\vec{v}[n], \qquad n\geq 1$$

$$=\left[\vec{C}\vec{A}^{n-1}u[n-1]\vec{B}+\vec{D}\vec{\delta}[n]\right]*\vec{v}[n]$$

is the *zero-state* response.

With the sample response $\boxed{\vec{h}[n]=\begin{cases}\vec{D}, & n=0\\ \vec{C}\vec{A}^{n-1}\vec{B}, & n\geq 1\end{cases}}$

## 8.11 Solution via The z-Transform

Taking the z-transform of the vector difference equation gives :

$$z\vec{X}(z) - z\vec{x}[0] = \vec{A}\vec{X}(z) + \vec{B}\vec{V}(z)$$

Then

$$\vec{X}(z) = (z\vec{I} - \vec{A})^{-1} z\vec{x}[0] + (z\vec{I} - \vec{A})^{-1}\vec{B}\vec{V}(z)$$

Where $(z\vec{I} - \vec{A})^{-1} z$ is the z-transform of the state-transition matrix.

Thus $\vec{A}^n$ = inverse z-transform of $(z\vec{I} - \vec{A})^{-1} z$

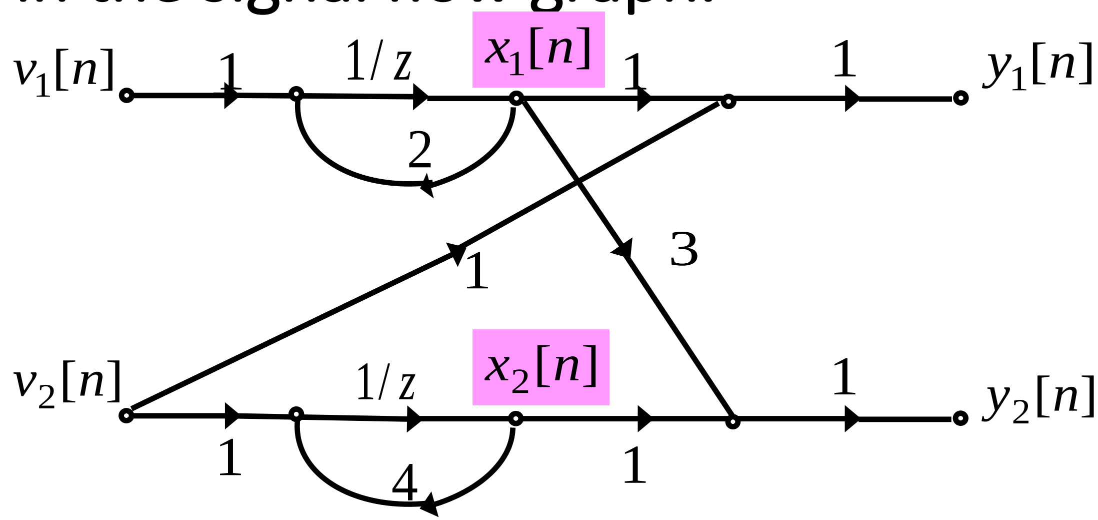
Taking the solution for state variable into the output equation to obtain:

$$\vec{Y}(z) = \vec{C}(z\vec{I} - \vec{A})^{-1} z\vec{x}[0] + [\vec{C}(z\vec{I} - \vec{A})^{-1}\vec{B} + \vec{D}]\vec{V}(z)$$

And the transfer function matrix $\vec{H}(z) = \vec{C}(z\vec{I} - \vec{A})^{-1}\vec{B} + \vec{D}$

## 8.11 Solution via The z-Transform

Example 8.8 Consider the two-input two-output two-dimensional system shown in the signal flow graph.




Construct the state equations and compute the state-transition matrix $\vec{A}^n$ and the transfer function matrix $\overrightarrow{\boldsymbol{H}}(z)$.

Sol: From the signal flow graph, we can construct the following equations:

$$\begin{cases} x_1[n+1] = 2x_1[n] + v_1[n] \\ x_2[n+1] = 4x_2[n] + v_2[n] \end{cases} \qquad \begin{cases} y_1[n] = x_1[n] + v_2[n] \\ y_2[n] = 3x_1[n] + x_2[n] \end{cases}$$

# 8.11 Solution via The z-Transform

Matrix form:

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix}$$

$$\begin{bmatrix} y_1[n] \\ y_2[n] \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1[n] \\ v_2[n] \end{bmatrix}$$

Thus,

$$\vec{A} = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \qquad \vec{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\vec{C} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \qquad \vec{D} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The state-transition matrix:

$$\vec{A}^n = \mathcal{Z}^{-1}\{(\vec{I} - z^{-1}\vec{A})^{-1}\} = \mathcal{Z}^{-1} \begin{bmatrix} \dfrac{1}{1-2z^{-1}} & 0 \\ 0 & \dfrac{1}{1-4z^{-1}} \end{bmatrix} = \begin{bmatrix} 2^n & 0 \\ 0 & 4^n \end{bmatrix} u(n)$$

The transfer function matrix:

$$\vec{H}(z) = \vec{C}(z\vec{I} - \vec{A})^{-1}\vec{B} + \vec{D}$$

$$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} z-2 & 0 \\ 0 & z-4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{z-2} & 1 \\ \dfrac{3}{z-2} & \dfrac{1}{z-4} \end{bmatrix}$$

Example(13.15): A two-input two-output LTI system has the transfer function matrix

$$\vec{H}(s)=\begin{bmatrix}\dfrac{1}{s+1} & 0\\ \dfrac{1}{s+1} & \dfrac{1}{s+2}\end{bmatrix}$$

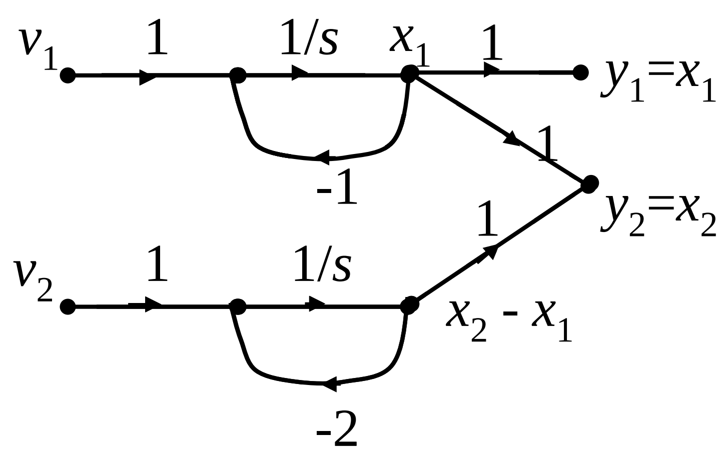
Find the state model of the system with the state variables defined to be $x_1(t) = y_1(t)$, $x_2(t) = y_2(t)$ , where $y_1(t)$ is the first system output and $y_2(t)$ is the second system output.

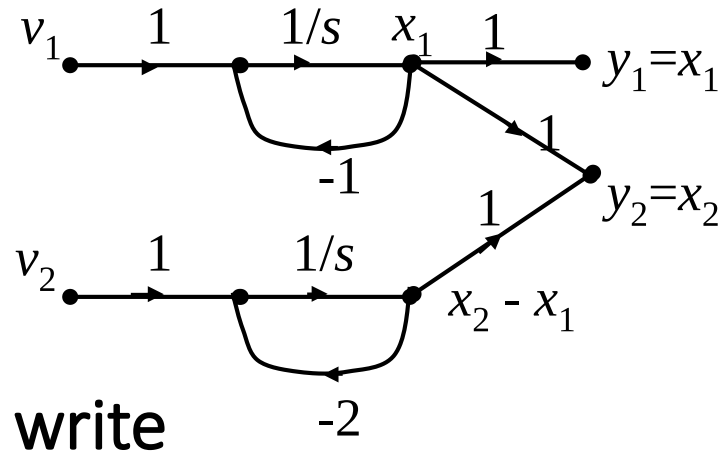
Sol: Suppose $v_1(t)$ and $v_2(t)$ are the inputs, from the transfer function matrix, we have

$$\left.\frac{Y_1(s)}{V_1(s)}\right|_{V_2=0}=\frac{1}{s+1},$$

$$\left.\frac{Y_2(s)}{V_1(s)}\right|_{V_1=0}=\frac{1}{s+1},\quad \left.\frac{Y_2(s)}{V_2(s)}\right|_{V_1=0}=\frac{1}{s+2}$$

Drawing the diagram step by step:






From the signal flow graph, we can write

$$\dot{x}_1 = -x_1 + v_1 \qquad y_1 = x_1$$

$$\dot{x}_2 - \dot{x}_1 = -2(x_2 - x_1) + v_2 \qquad y_2 = x_2$$

Matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

# 8.12 SUMMARY

- Concepts of state model, state variable, state equation, output equation, state transition matrix;
- Construction methods of state model for both continuous- and discrete-time systems;
- Time domain solutions of state model for both continuous- and discrete-time LTI systems;
- Laplace transform solution of state model for continuous-time LTI systems;
- Z-transform solution of state model for discrete-time LTI systems.

# Homework

13.1 13.4 13.7 13.15

13.16 13.22 13.23