CHAPTER 3

FOURIER SERIES REPRESENTATION OF PERIODIC SIGNALS

Content

3.0 Introduction

- Eigenfunction property of LTI systems
- Representation of continuous-time and discrete-time periodic signals — Fourier Series
- Frequency spectrum of periodic signals
- Responses of LTI systems to periodic signals

3.1 The Response of LTI Systems to Complex Exponentials

- Defining two Functions $H(s)$ and $H(z)$ respectively:

$$H(s)=\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \qquad H(z)=\sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

Here $H(s)$ and $H(z)$ are functions of the complex variable s or z .

- A continuous-time LTI system with the impulse response $h(t)$ responds to a complex exponential input $e^{s_0 t}$, here s_0 is a complex constant by

$$y(t)=e^{s_0 t}*h(t)=\int_{-\infty}^{\infty} h(\tau)e^{s_0(t-\tau)}d\tau = e^{s_0 t}\int_{-\infty}^{\infty} h(\tau)e^{-s_0\tau}d\tau = H(s_0)e^{s_0 t}$$

- A discrete-time LTI system with the impulse response $h[n]$ responds to a complex exponential input z_0^n , here z_0 is a complex constant by

$$y[n]=z_0^n * h[n]=\sum_{k=-\infty}^{\infty} h[k]z_0^{n-k} = z_0^n\sum_{k=-\infty}^{\infty} h[k]z_0^{-k} = H(z_0)z_0^n$$

3.1 The Response of LTI Systems to Complex Exponentials

- Eigenfunction (特征函数): an input signal for which the system output is a constant times the input.

  For continuous-time systems: $e^{s_0 t} \rightarrow H(s_0)e^{s_0 t}$

  For discrete-time systems: $z_0^n \rightarrow H(z_0)z_0^n$

- e^{st} is the eigenfunction of continuous-time LTI systems. z^n is the eigenfunction of discrete-time LTI systems. This is referred to as the eigenfunction property of LTI systems.

- If the input to an LTI system is represented as a linear combination of complex exponentials as:

  $$x(t) = \sum_k a_k e^{s_k t} \quad \text{or} \quad x[n] = \sum_k a_k z_k^{\,n}$$

  From eigenfunction property and superposition property, the output will be:

  $$y(t) = \sum_k a_k H(s_k) e^{s_k t} \quad \text{or} \quad y[n] = \sum_k a_k H(z_k) z_k^{\,n}$$

3.2 Fourier Series Representation of Continuous-Time Periodic Signals

3.2.1 Complex Exponential Fourier Series (指数型傅里叶级数)

Given periodic $x(t)$ with fundamental period T , its complex exponential Fourier series is :

$$x(t)=\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \omega_0 = 2\pi / T$$

Fundamental frequency （基频）

Coefficients a_k is a complex function of $k\omega_0$.

$a_1 e^{j\omega_0 t} + a_{-1} e^{-j\omega_0 t}$: fundamental component or the first harmonic component
（基波分量）（一次谐波分量）

$a_N e^{jN\omega_0 t} + a_{-N} e^{-jN\omega_0 t}$： the Nth harmonic components（ N次谐波分量）

By taking use of orthogonal property of functions $e^{jk\omega_0 t}, k=0,\pm 1,\cdots$ we can derive the equation to determine the Fourier series coefficients:

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$

3.2 Fourier Series Representation of Continuous-Time Periodic Signals

synthesis equation:
（综合式）

$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t}=\sum_{k=-\infty}^{\infty}a_k e^{jk(2\pi/T)t}$$

analysis equation:
（分析式）

$$a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt=\frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt$$

Fourier series coefficients or *spectral coefficients* of *x*(*t*): $\{a_k\}$
（傅里叶级数系数）（频谱系数）

magnitude spectrum: $|a_k|$ *phase spectrum*: $\sphericalangle a_k$
（幅度频谱）（相位频谱）

constant component or *dc* of *x*(*t*):
（常数或直流分量）

$$a_0=\frac{1}{T}\int_T x(t)dt$$

3.2.2 Trigonometric Fourier Series (三角型傅里叶级数)

$$x(t) = a_0 + 2\sum_{k=1}^{\infty}\left[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t\right]$$

or

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

The relationships between a_k and B_k, C_k, A_k, ϑ_k are:

$$a_k = B_k + jC_k \qquad a_k = A_k e^{j\theta_k}$$

Note: $\{a_k\}$ are generally complex, while B_k, C_k, A_k, ϑ_k are real, so trigonometric FS is for real periodic signals.

Example 3.1

Consider a real periodic signal *x*(*t*), with fundamental frequency 2π, that is expressed in the complex exponential Fourier series as

$$x(t)=\sum_{k=-3}^{+3} a_k e^{jk2\pi t}$$

where $a_0=1,\quad a_1=a_{-1}=1/4,\quad a_2=a_{-2}=1/2,\quad a_3=a_{-3}=1/3$

Use the trigonometric form to express the signal *x*(*t*).

Sol: Collecting the harmonic components which have same frequency, we have

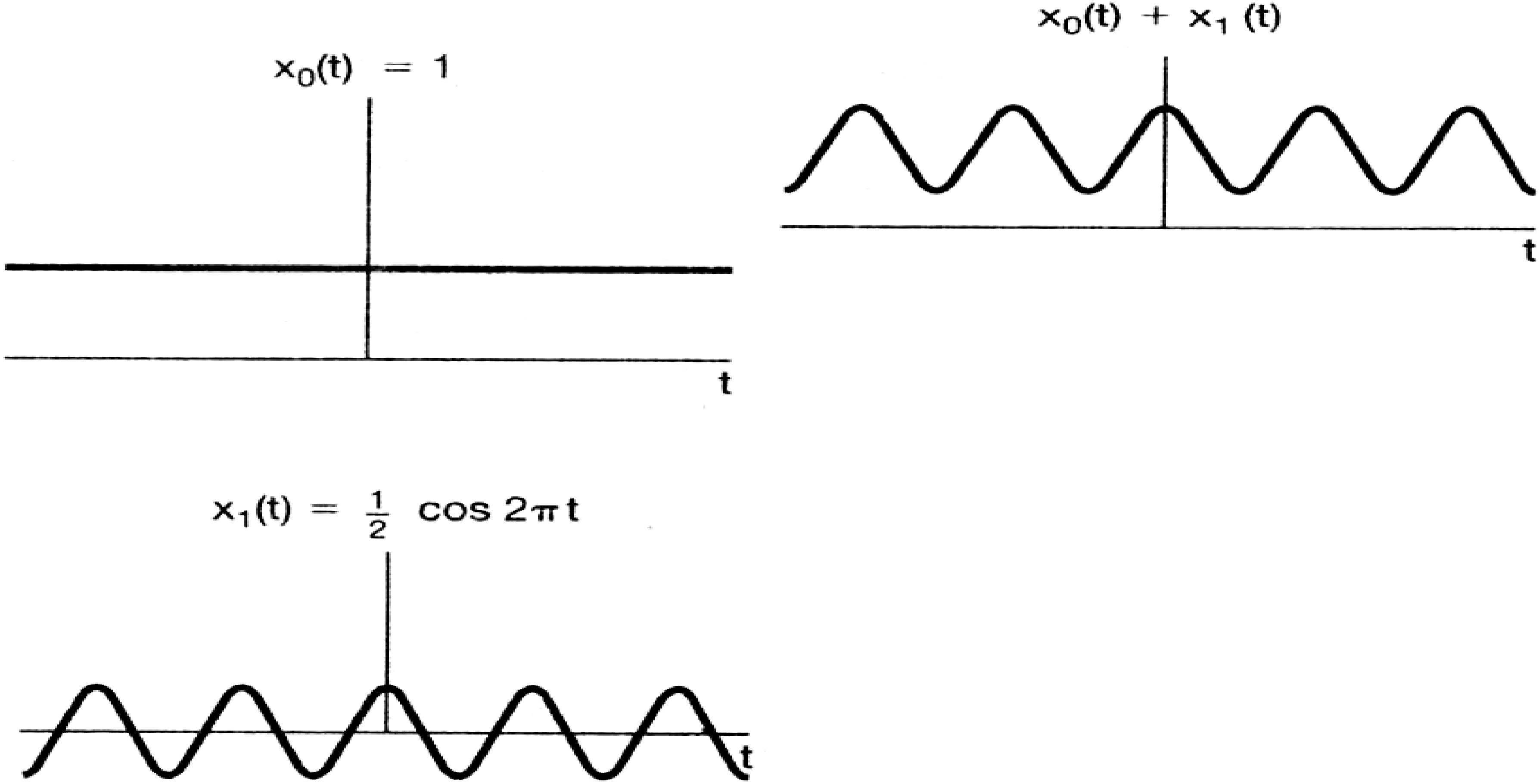
$$x(t)=1+\frac{1}{4}\left(e^{j2\pi t}+e^{-j2\pi t}\right)+\frac{1}{2}\left(e^{j4\pi t}+e^{-j4\pi t}\right)+\frac{1}{3}\left(e^{j6\pi t}+e^{-j6\pi t}\right)$$

From Euler's relation

$$x(t)=1+\frac{1}{4}\cdot 2\cos 2\pi t+\frac{1}{2}\cdot 2\cos 4\pi t+\frac{1}{3}\cdot 2\cos 6\pi t$$

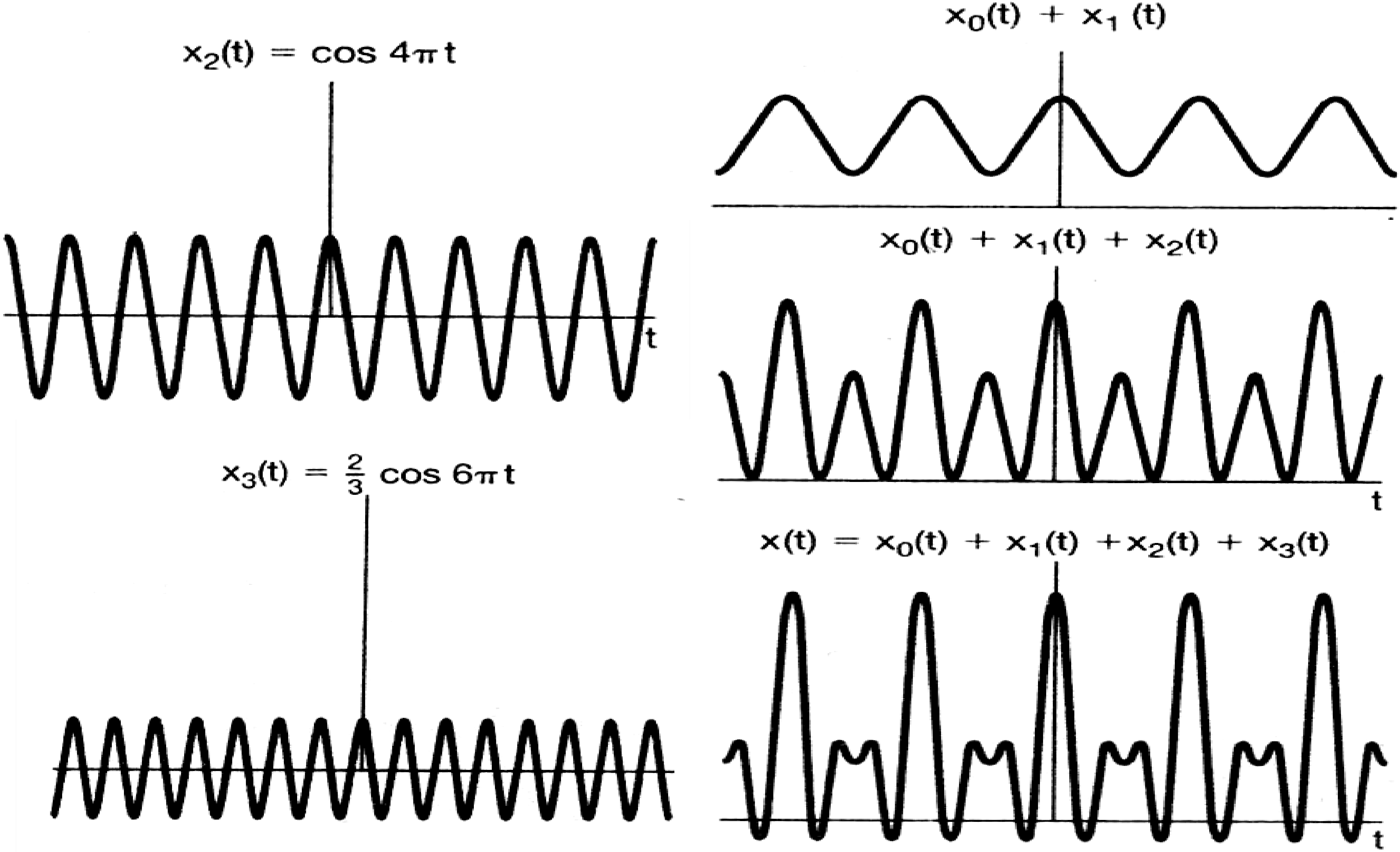
$$=1+\frac{1}{2}\cos 2\pi t+\cos 4\pi t+\frac{2}{3}\cos 6\pi t$$

3.2 Fourier Series Representation of Continuous-Time Periodic Signals



Construction of the signal $x(t)$ as a linear combination of the harmonically related sinusoidal signals

3.2 Fourier Series Representation of Continuous-Time Periodic Signals



Construction of the signal $x(t)$ as a linear combination of the harmonically related sinusoidal signals

Example 3.2

Consider the signal

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos(2\omega_0 t + \frac{\pi}{4})$$

Draw the graphs of magnitude and phase spectrums of *x*(*t*).

Sol: Complex exponential form can be easily obtained as

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \frac{1}{2}e^{j(\pi/4)}e^{j2\omega_0 t} + \frac{1}{2}e^{-j(\pi/4)}e^{-j2\omega_0 t}$$

Thus, the Fourier series coefficients for this example are :

$$a_0 = 1,$$

$$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j = \frac{\sqrt{5}}{2}e^{-j\arctan(1/2)},$$

$$a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j = \frac{\sqrt{5}}{2}e^{j\arctan(1/2)},$$

$$a_2 = \frac{1}{2}e^{j(\pi/4)},$$

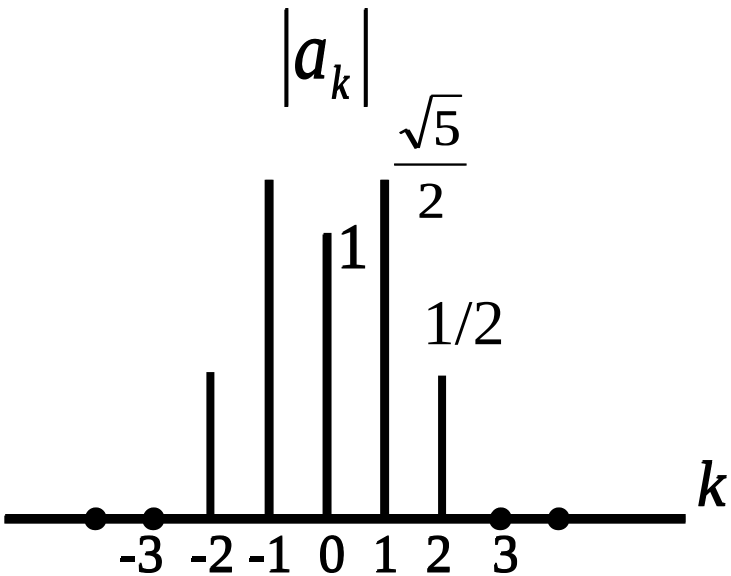
$$a_{-2} = \frac{1}{2}e^{-j(\pi/4)},$$

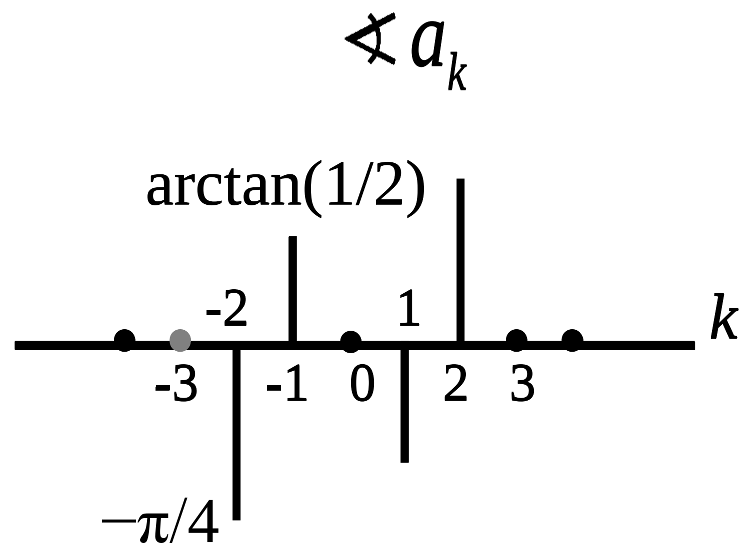
$$a_k = 0, \quad |k| > 2.$$

3.2 Fourier Series Representation of Continuous-Time Periodic Signals

$$a_0 = 1, \quad a_1 = \frac{\sqrt{5}}{2} e^{-j\arctan(1/2)}, \quad a_{-1} = \frac{\sqrt{5}}{2} e^{j\arctan(1/2)},$$

$$a_2 = \frac{1}{2} e^{j(\pi/4)}, \quad a_{-2} = \frac{1}{2} e^{-j(\pi/4)}$$



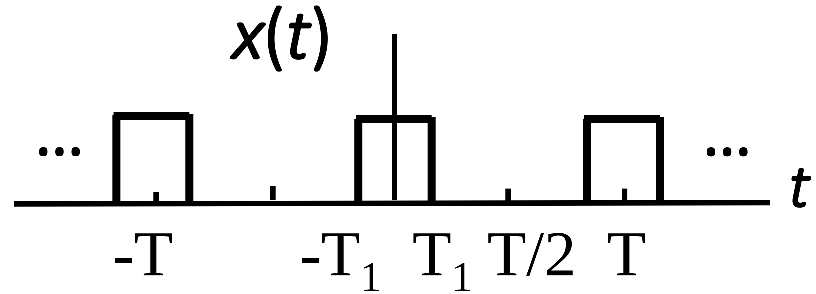


Plots of the magnitude and phase of Fourier coefficients of *x*(*t*)

Example 3.3

The periodic square wave is defined over one period as:

$$x(t)=\begin{cases}1, & |t|<T_1 \\ 0, & T_1<|t|<T/2\end{cases}$$



Do some research into its frequency spectrum.

Sol:
$$a_0=\frac{1}{T}\int_{-T_1}^{T_1}dt=\frac{2T_1}{T} \longleftarrow \text{Duty Circle (占空比)}$$

$$a_k=\frac{1}{T}\int_{-T_1}^{T_1}e^{-jk\omega_0 t}dt=-\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1}=\frac{2\sin\left(k\omega_0 T_1\right)}{k\omega_0 T}=\frac{\sin\left(k\omega_0 T_1\right)}{k\pi}, k\neq 0$$

$$\boxed{a_k=\frac{\sin\left(k\pi\frac{2T_1}{T}\right)}{k\pi}}$$

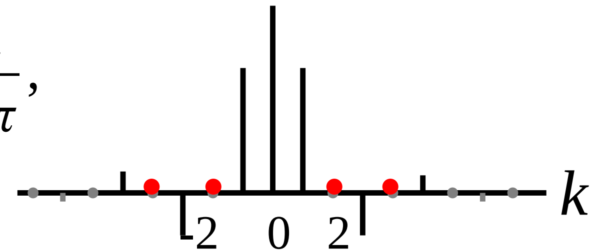
3.2 Fourier Series Representation of Continuous-Time Periodic Signals

$$a_k = \frac{\sin\left(k\pi \frac{2T_1}{T}\right)}{k\pi}$$

For T=4T_1, duty circle=50%, the coefficients are:

$$a_0 = \frac{1}{2}, \quad a_1 = a_{-1} = \frac{1}{\pi}, \quad a_3 = a_{-3} = -\frac{1}{3\pi}, \quad a_5 = a_{-5} = \frac{1}{5\pi},$$

$$\vdots$$

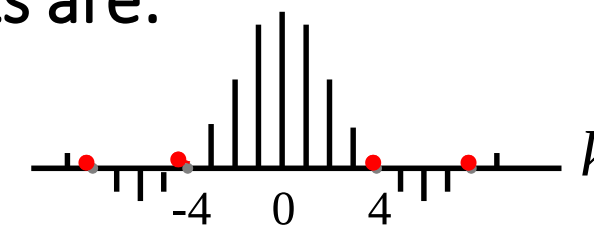


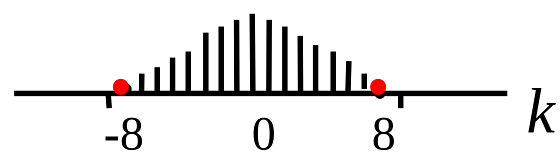
For T = 8T_1, duty circle=25%, coefficients are:

$$a_0 = \frac{1}{4}, \quad a_1 = a_{-1} = \frac{\sqrt{2}}{2\pi}, \quad a_2 = a_{-2} = \frac{1}{2\pi},$$

$$a_3 = a_{-3} = \frac{\sqrt{2}}{6\pi}, \quad a_4 = a_{-4} = 0, \quad a_5 = a_{-5} = -\frac{\sqrt{2}}{10\pi},$$

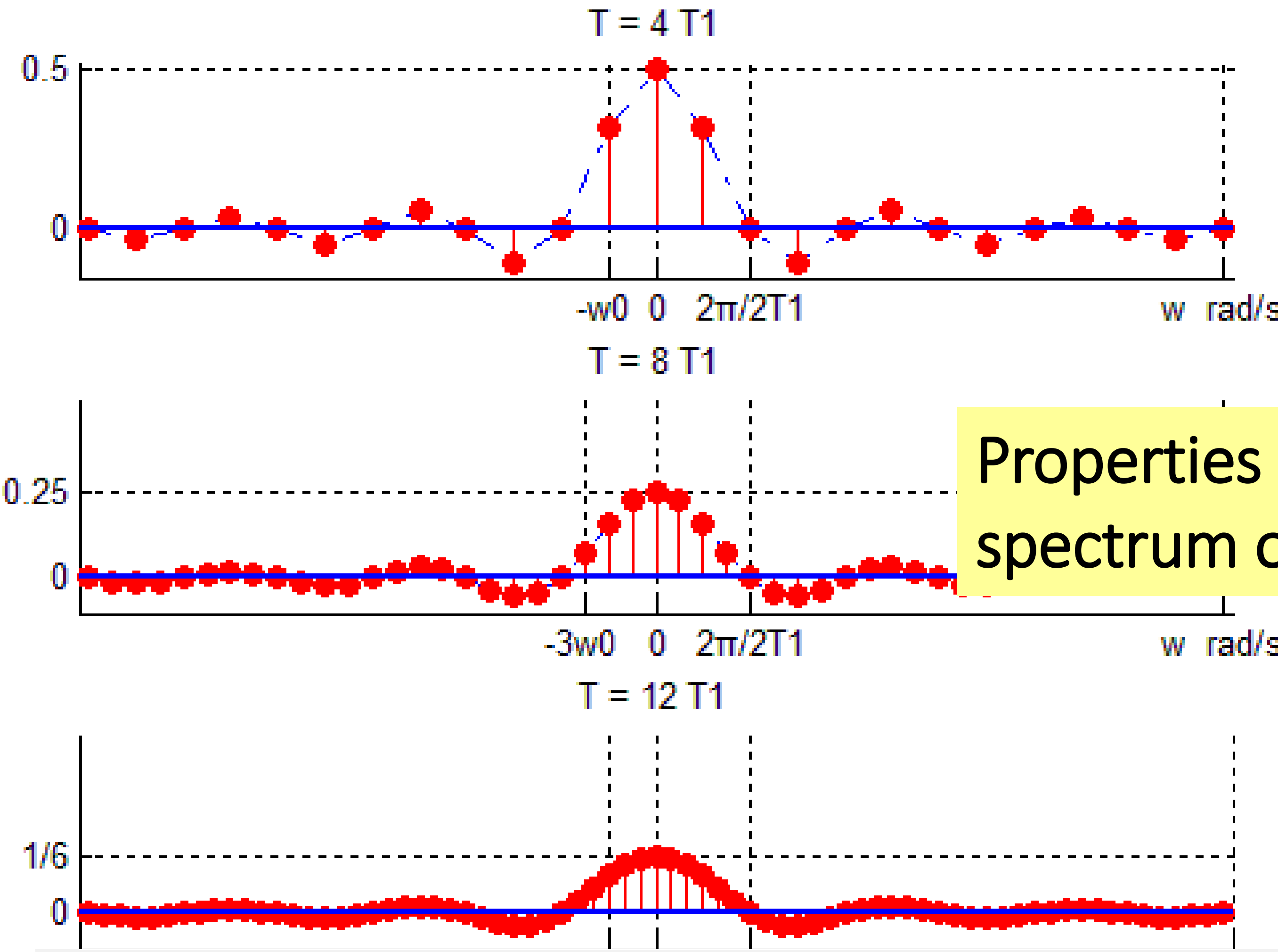
$$\vdots$$





Plots of the Fourier Series coefficients a_k for the periodic square wave with T_1 fixed and for several values of T:
(a) T=4 T_1; (b) T=8 T_1; (c) T=16 T_1.

3.2 Fourier Series Representation of Continuous-Time Periodic Signals



$$a_k = \frac{\sin\left(k\omega_0 T_1\right)}{k\pi}$$

$$k\omega_0 T_1 = m \cdot \pi$$

$$\Rightarrow k\omega_0 = \frac{m \cdot \pi}{T_1} = m\frac{2\pi}{2T_1}$$

Properties of frequency spectrum of periodic signals :

- Discrete
- Harmonic
- Convergent

Consider: If T fixed, $2T_1$ decreases (duty circle decreases), what are the changes of the spectrum?

3.2 Fourier Series Representation of Continuous-Time Periodic Signals

➢ Frequency bandwidth

✓ If the spectral function is a *Sinc* (or *Sa*), the frequency bandwidth is the width of the range of positive frequencies from zero to the first zero point of *Sinc*.

✓ In engineering, *3-dB* or *half-power* bandwidth is generally used, which is the width of the range of positive frequencies where a peak value at zero is attenuated to 0.707 the value at the peak.

➢ Two conclusions obtained from the frequency spectrum analysis of periodic square wave :

✓ The width of pulse in time domain is inverse proportional to the width of frequency band in frequency domain.

✓ Signals changing faster in time domain must have wider frequency band width. (Thus contain more high frequencies in the frequency band width.)

3.3 Convergence of the Fourier Series

- The Dirichlet conditions（狄里赫利条件）(sufficient not necessary conditions):

✓Condition 1: Over any period, *x*(*t*) must be *absolutely integrable*; that is $\int_T |x(t)| dt < \infty$

✓Condition 2: In any finite interval of time, *x*(*t*) is of bounded variation; that is, there are no more than a finite number of maxima and minima during any single period of the signal.

✓Condition 3: In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.

- If *x*(*t*) is continuous everywhere, then the series converges absolutely and uniformly; The infinite series equals *x*(*t*) at every continuity point and equals the average $0.5[x(t+0_+)+x(t+0_-)]$ at every discontinuity point.

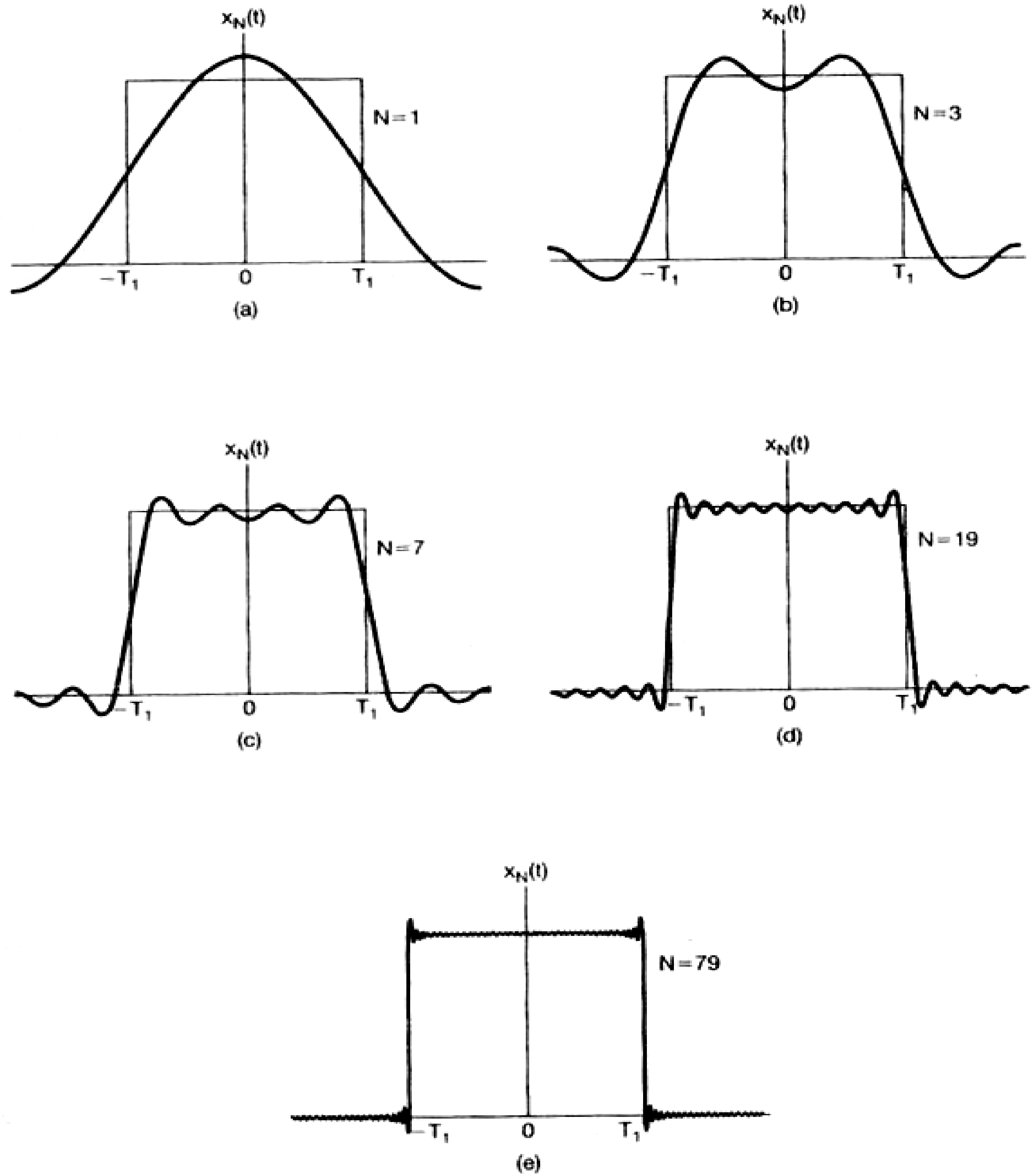
3.3 Convergence of the F

The smoother the sign
with a Fourier series w

Consider an approxim

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

Convergence of the Fourier series representation of a square wave:
an illustration of the Gibbs phenomenon (吉布斯现象).



3.4 Properties of Continuous-Time Fourier Series

3.4.1 Linearity

Let both $x(t)$ and $y(t)$ are periodic with period T, and

$$x(t) \xleftrightarrow{FS} a_k, \quad y(t) \xleftrightarrow{FS} b_k.$$

Then $$z(t) = Ax(t) + By(t) \xleftrightarrow{FS} c_k = Aa_k + Bb_k$$

3.4.2 Time Shifting

If $$x(t) \xleftrightarrow{FS} a_k,$$

Then $$x(t-t_0) \xleftrightarrow{FS} b_k = e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k$$

When a periodic signal is shifted in time, the magnitude spectrum remains unaltered. That is, $|b_k| = |a_k|$.

3.4.3 Time Reversal

If $x(t) \xleftrightarrow{FS} a_k,$

Then $x(-t) \xleftrightarrow{FS} a_{-k}$

If *x*(*t*) is even : $a_{-k} = a_k$ If *x*(*t*) is odd: $a_{-k} = -a_k$

3.4.4 Time Scaling

If *x*(*t*) is periodic with period *T*, and $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$

Then *x*(*αt*) is periodic with period *T/α*, $x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$

The Fourier coefficients for each of those components remain the same. However, the harmonic components change with the change in the fundamental frequency.

3.4.5 Multiplication

Let $x(t)$ and $y(t)$ are both periodic with period T, and

$$x(t) \xleftrightarrow{FS} a_k, \quad y(t) \xleftrightarrow{FS} b_k.$$

Then
$$x(t)y(t) \xleftrightarrow{FS} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

h_k is the convolution sum of a_k and b_k.

3.4.6 Conjugation and Conjugate Symmetry (共轭对称)

If $x(t) \xleftrightarrow{FS} a_k,$ Then $x^*(t) \xleftrightarrow{FS} a^*_{-k}$

- if $x(t)$ real, then a_k is *conjugate symmetric*, i.e. $a_{-k} = a_k^*$
- if $x(t)$ is real and even, then $a_k = a_{-k} = a_k^* = B_k$
- if $x(t)$ is real and odd, then $a_k = jC_k = a^*_{-k}$
- $x_e(t) \xleftrightarrow{FS} \mathrm{Re}\{a\}_k = B_k, \quad x_o(t) \xleftrightarrow{FS} j\,\mathrm{Im}\{a\}_k = jC_k$

3.4.7 Parseval's Relation for Continuous-Time Periodic Signals (帕色伐尔定理)

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Parseval's relation states that the total average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

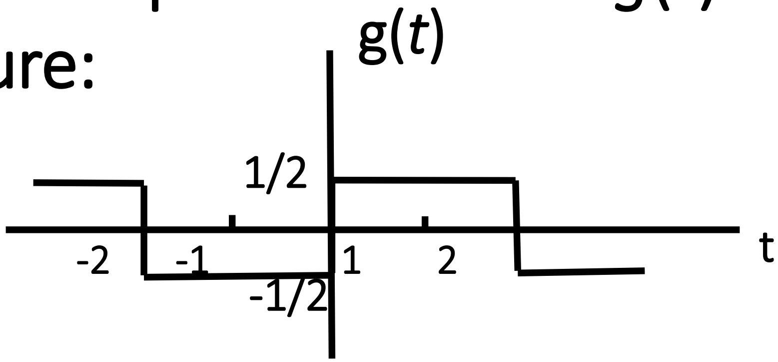
3.4.8 Differentiation

If $x(t) \xleftrightarrow{FS} a_k$

Then $\dfrac{dx(t)}{dt} \xleftrightarrow{FS} jk\omega_0 a_k$

Example 3.4

Determine the Fourier series representation of $g(t)$ which is shown in the following figure:



Sol: $g(t)= x(t-1)-1/2$, where $x(t)$ is the periodic square wave in Example 3.3, with T=4 and T_1=1, and $x(t) \xleftrightarrow{FS} a_k$.

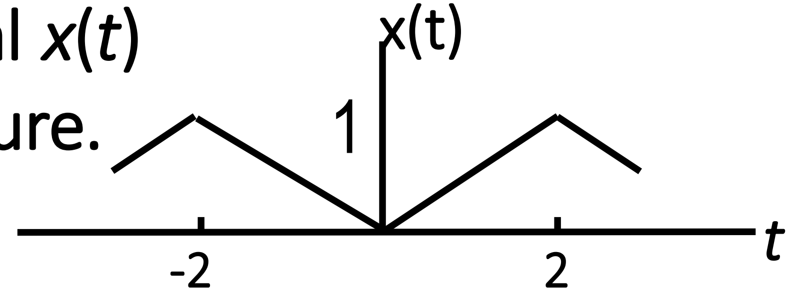
From time shifting property, $x(t-1) \xleftrightarrow{FS} a_k e^{-jk\pi/2}$

So the Fourier coefficients of $g(t)$ is

$$d_k = \begin{cases} a_k e^{-jk\pi/2} = \dfrac{\sin(k\pi/2)}{k\pi} e^{-jk\pi/2} & for \quad k \neq 0 \\ a_0 - \frac{1}{2} = 0 & for \quad k = 0 \end{cases}$$

Example 3.5

Consider the triangular wave signal $x(t)$ which is shown in the following figure.



Sol: The derivative of $x(t)$ is the signal $g(t)$ in Example 3.4.

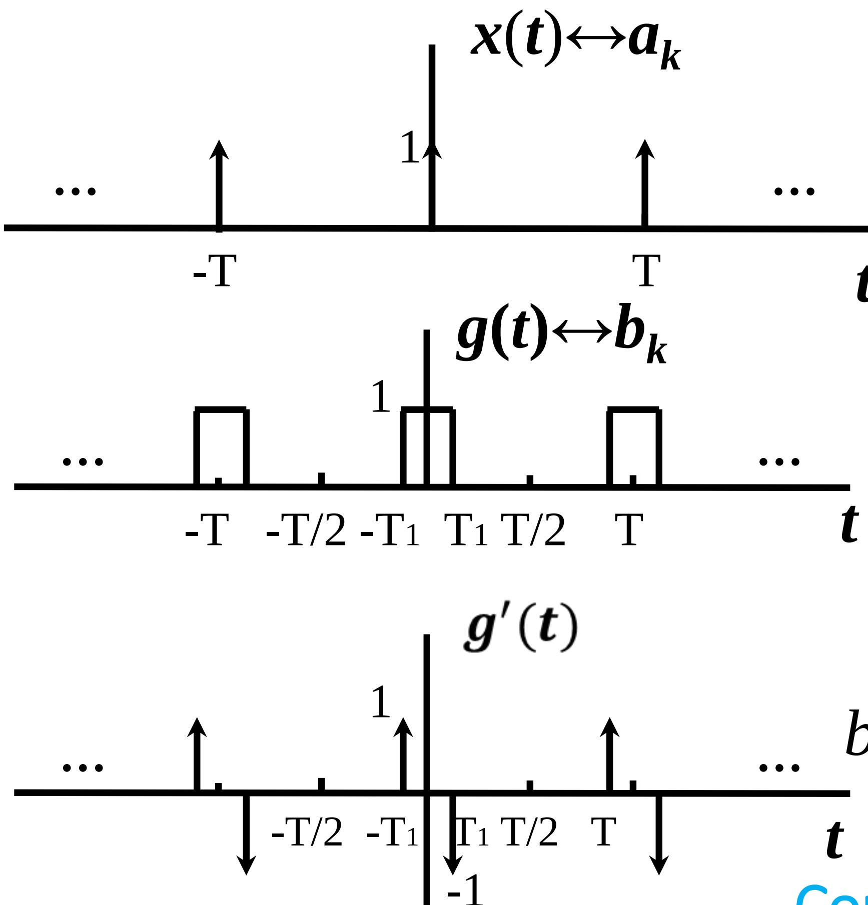
Denoting the coefficients of $g(t)$ by d_k and those of $x(t)$ by e_k, then we have: $d_k = jk(\pi/2)e_k$

Thus $$e_k = \frac{2d_k}{jk\pi} = \frac{2\sin(k\pi/2)}{j(k\pi)^2}e^{-jk\pi/2}, \quad k \neq 0$$

For $k = 0$, e_0 can be determined by finding the area under one period of $x(t)$ and dividing by the length of the period: $e_0 = \dfrac{1}{2}$

Example 3.6

Determine the Fourier series representation of the *impulse train*, which is periodic with period T and is expressed as :



$$\boxed{x(t)=\sum_{k=-\infty}^{\infty}\delta(t-kT)}$$

$$a_k=\frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jk2\pi t/T}dt=\frac{1}{T}$$

$$g'(t)=x(t+T_1)-x(t-T_1)$$

$$jk\omega_0 b_k=e^{jk\omega_0 T_1}a_k-e^{-jk\omega_0 T_1}a_k$$

$$b_k=\frac{2j\sin(k\omega_0 T_1)a_k}{jk\omega_0}=\frac{\sin(k\omega_0 T_1)}{k\pi},k\neq 0$$

Compare b_k with the result in Example 3.3!

Example 3.7

Giving the following facts about a signal *x*(*t*):

1. *x*(*t*) is a real signal;
2. *x*(*t*) is periodic with period *T*=4, and it has Fourier series coefficients a_k ;
3. $a_k = 0$ for $|k| > 1$;
4. The signal with Fourier coefficients $b_k = e^{-jk\pi/2} a_{-k}$ is odd;
5. $\frac{1}{4}\int_4 |x(t)|^2 dt = 1/2$.

Determine the signal.

$$x(t) = -\cos(\pi t / 2)$$

or

$$x(t) = \cos(\pi t / 2)$$

3.5.1 Linear Combination of Harmonically Related Complex Exponentials

Given periodic $x[n]$ with fundamental period N , its Fourier series takes the form:

$$x[n]=\sum_{k} a_k e^{jk\omega_0 n}=\sum_{k} a_k e^{jk(2\pi/N)n} \qquad k=0,\pm 1,\pm 2,\cdots$$

Since

$$\begin{array}{ccccccc}
e^{j0(2\pi/N)n} & = & e^{jN(2\pi/N)n} & = & \cdots & = & e^{jrN(2\pi/N)n} \\
e^{j(2\pi/N)n} & = & e^{j(N+1)(2\pi/N)n} & = & \cdots & = & e^{j(rN+1)(2\pi/N)n} \\
e^{j2(2\pi/N)n} & = & e^{j(N+2)(2\pi/N)n} & = & \cdots & = & e^{j(rN+2)(2\pi/N)n} \\
\vdots & & \vdots & & & & \vdots \\
e^{j(N-1)(2\pi/N)n} & = & e^{j(2N-1)(2\pi/N)n} & = & \cdots & = & e^{j(rN+N-1)(2\pi/N)n}
\end{array}$$

3.5.1 Linear Combination of Harmonically Related Complex Exponentials

$\boldsymbol{e^{jk(2\pi/N)n}}$ is periodic thus there are only *N* distinct signals in

$$\left\{ e^{jk(2\pi/N)n}, \quad k = 0, \pm 1, \pm 2, \cdots \right\}$$

The summation need only include terms over a range of *N* successive values of *k*. We use $k = \langle N \rangle$ to indicate this. Then,

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

finite series

3.5.2 Determination of the Fourier Series Representation of a Periodic Signal

Multiplying both sides of the discrete-time Fourier series equation by $e^{-jr(2\pi/N)n}$ and summing over *N* terms, we obtain

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{j(k-r)(2\pi/N)n}$$

Interchanging the order of summation on the right side yields

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)(2\pi/N)n}$$

$$\sum_{n=\langle N\rangle} e^{j(k-r)(\frac{2\pi}{N})n} = \frac{1-\left[e^{j\frac{2\pi}{N}(k-r)}\right]^N}{1-e^{j\frac{2\pi}{N}(k-r)}} = \begin{cases} 0, & k \neq r+mN \\ N, & k = r+mN \end{cases}$$

3.5 Fourier Series Representation of Discrete-Time Periodic Signals

The Fourier series coefficients are determined by equation:

$$a_r = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jr(2\pi/N)n}$$

synthesis equation:

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

analysis equation:

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$$

periodic

$$a_k = a_{k+N}$$

Discreteness $\leftrightarrow$ Periodicity

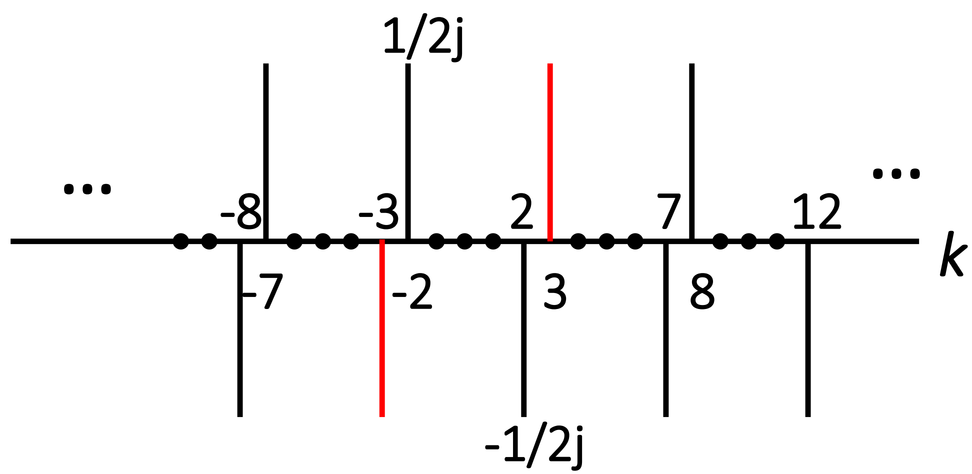
Example 3.8

Consider the signal $x[n]=\sin 3(2\pi/5)n$, draw the graph of coefficients.

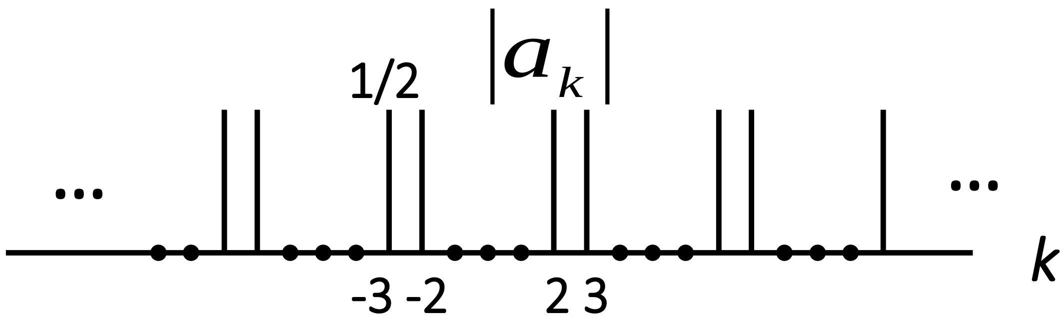
Sol: This signal is periodic with period $N = 5$.

$$x[n] = \frac{1}{2j} e^{j3(2\pi/5)n} - \frac{1}{2j} e^{-j3(2\pi/5)n}$$

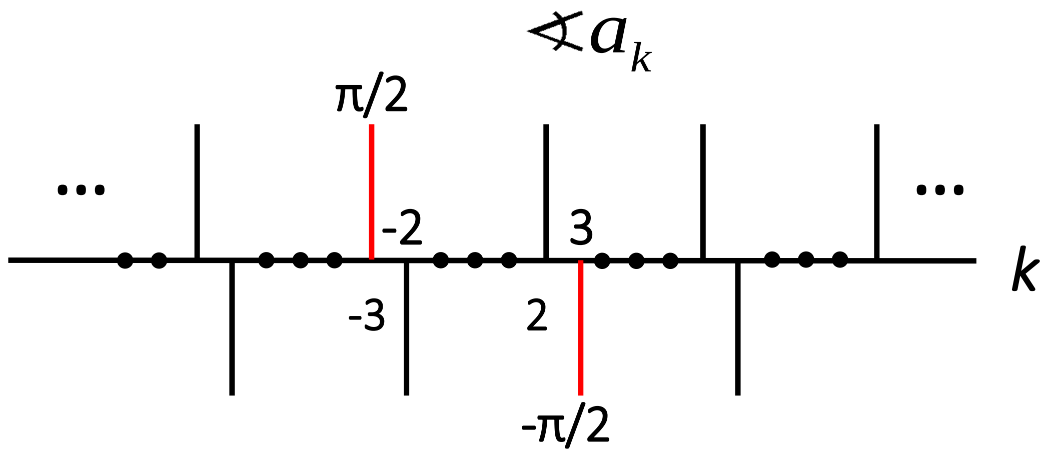
$$a_3 = \frac{1}{2j}, \quad a_{-3} = -\frac{1}{2j}$$



Fourier coefficients for $x[n]=\sin 3(2\pi/5)n$.



Magnitude of the coefficients

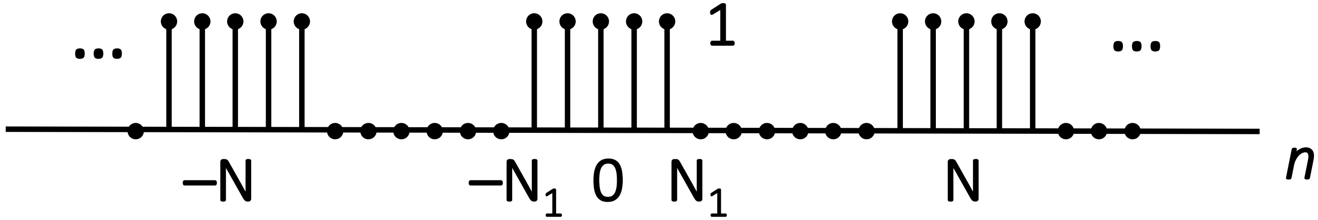


Phase of the coefficients

3.5 Fourier Series Representation of Discrete-Time Periodic Signals

Example 3.9

Consider the discrete-time periodic square wave:



$$a_0=\frac{1}{N}\sum_{n=-N_1}^{N_1}e^{-j0(\frac{2\pi}{N})n}=\frac{1}{N}\sum_{n=-N_1}^{N_1}1=\frac{2N_1+1}{N}=a_N=a_{-N}=a_{2N}=a_{-2N}=\cdots$$

$$a_k=\frac{1}{N}\sum_{n=-N_1}^{N_1}e^{-jk(\frac{2\pi}{N})n}\underline{\underline{m=n+N_1}}\frac{1}{N}\sum_{m=0}^{2N_1}e^{-jk(\frac{2\pi}{N})(m-N_1)}=\frac{1}{N}e^{jk(\frac{2\pi}{N})N_1}\sum_{m=0}^{2N_1}e^{-jk(\frac{2\pi}{N})m}$$

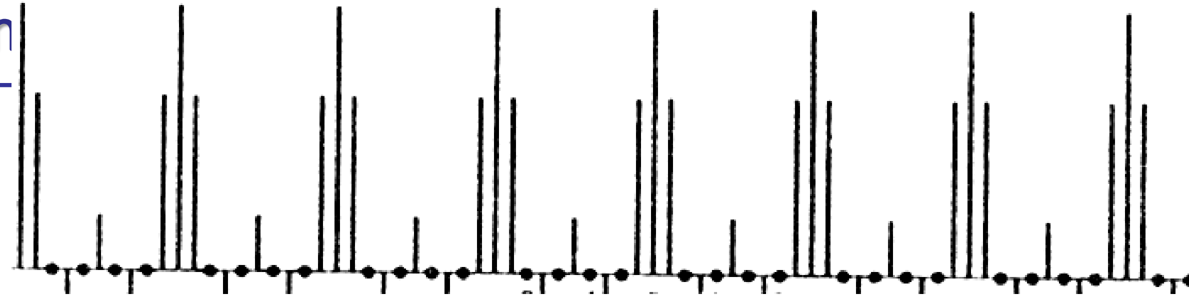
$$=\frac{1}{N}\frac{\sin\left[2k\pi\left(\frac{N_1+1/2}{N}\right)\right]}{\sin(\frac{k\pi}{N})},\qquad k\neq 0,\pm N,\pm 2N,\cdots$$

3.5 Fourier Series Represen

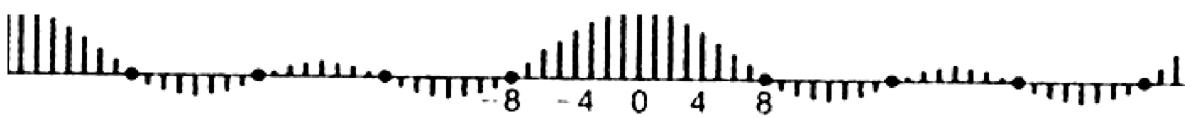
$$a_k = \frac{1}{N}\frac{\sin\left[2k\pi\left(\frac{N_1+1/2}{N}\right)\right]}{\sin(\frac{k\pi}{N})}$$

$$k \neq 0, \pm N, \pm 2N, \cdots$$

Fourier series coefficients for the periodic square wave of Example 3.9; plots of Na_k for $2N_1 + 1 = 5$ and
(a) $N = 10$;
(b) $N = 20$;
(c) $N = 40$



- In contrast to CTFS, there are no convergence issues with the DTFS in general, because any discrete-time periodic sequence $x[n]$ is completely specified by a *finite* number N of parameters. So for discrete-time square wave we will not observe Gibbs phenomenon as the number of FS terms increases.



(c)

3.6 Properties of Discrete-Time Fourier Series

3.6.1 Multiplication

Let $x[n]$ and $y[n]$ are both periodic with period N, and

$$x[n] \xleftrightarrow{FS} a_k, \qquad y[n] \xleftrightarrow{FS} b_k$$

Then $$x[n]y[n] \xleftrightarrow{FS} d_k = \sum_{l=\langle N \rangle} a_l b_{k-l} \longleftarrow \textit{periodic convolution}$$

3.6.2 First Difference

If $x[n] \xleftrightarrow{FS} a_k$, then $x[n]-x[n-1] \xleftrightarrow{FS} \left(1-e^{-jk(2\pi/N)}\right)a_k$

3.6.3 Parseval's Relation for Discrete-Time Periodic Signals

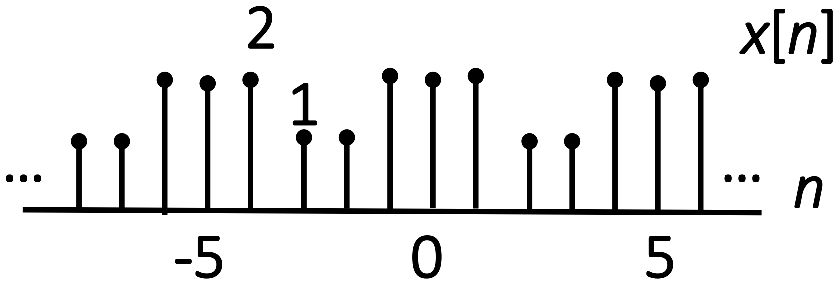
$$\frac{1}{N}\sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

$|a_k|^2$ is the average power in the kth harmonic component of $x[n]$.

Parseval's relation states that the average power in a periodic signal equals the sum of the average powers in all of its harmonic components.

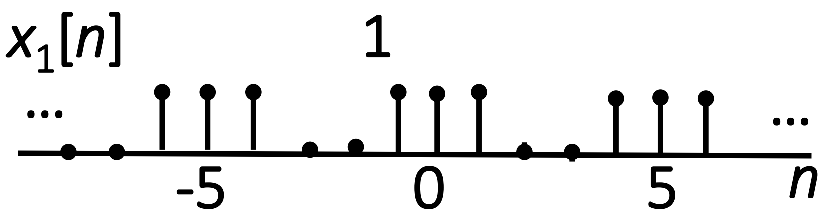
Example 3.10

Find the Fourier series coefficients a_k of following *x*[*n*]:

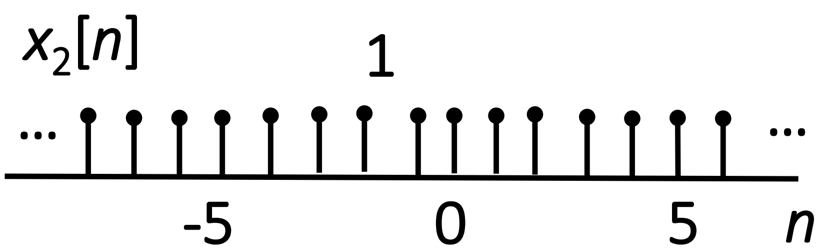


$$\begin{cases} \dfrac{1}{N}\dfrac{\sin\left[2k\pi(N_1+1/2)/N\right]}{\sin(k\pi/N)}, & k \neq 0, \pm N\cdots \\ (2N_1+1)/N, & k = 0, \pm N\cdots \end{cases}$$

Sol: Representing *x*[*n*] as a sum of the square wave $x_1[n]$ and the dc sequence $x_2[n]$ and letting $x_1[n] \leftrightarrow b_k$, $x_2[n] \leftrightarrow c_k$, then $a_k = b_k + c_k$



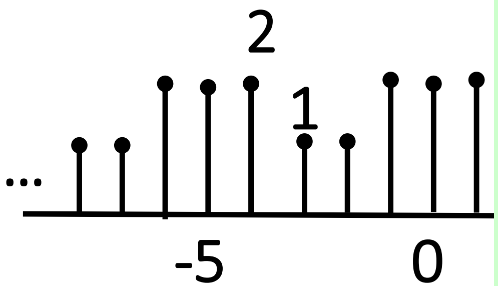
$$b_k = \begin{cases} \dfrac{1}{5}\dfrac{\sin(3k\pi/5)}{\sin(k\pi/5)}, & k \neq 0, \pm 5\cdots \\ 3/5, & k = 0, \pm 5\cdots \end{cases}$$



$$c_k = 1, \quad k = 0, \pm 5, \pm 10, \cdots$$

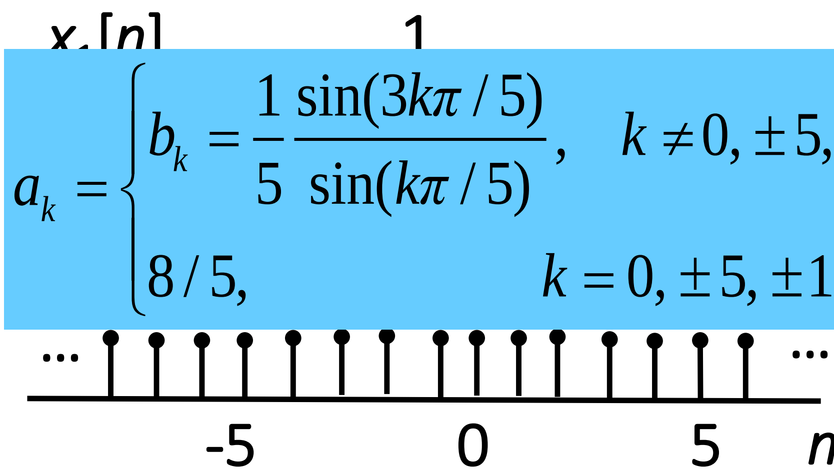
Example 3.10

Find the Fourier series coefficients a_k of following $x[n]$:



$$c_k = \frac{1}{5}\sum_{n=0}^{4} x_2[n] e^{-jk\frac{2\pi}{5}n} = \frac{1}{5}\frac{1-\left(e^{-jk\frac{2\pi}{5}}\right)^5}{1-e^{-jk\frac{2\pi}{5}}} = 1, \quad k = 0, \pm 5 \cdots$$

Sol: Representing $x[n]$ as a sum of the square wave $x_1[n]$ and the dc sequence $x_2[n]$ and letting $x_1[n] \leftrightarrow b_k$, $x_2[n] \leftrightarrow c_k$, then $a_k = b_k + c_k$



$$a_k = \begin{cases} b_k = \frac{1}{5}\frac{\sin(3k\pi/5)}{\sin(k\pi/5)}, & k \neq 0, \pm 5, \pm 10, \cdots \\ 8/5, & k = 0, \pm 5, \pm 10, \cdots \end{cases} = \begin{cases} \frac{1}{5}\frac{\sin(3k\pi/5)}{\sin(k\pi/5)}, & k \neq 0, \pm 5 \cdots \\ 3/5, & k = 0, \pm 5 \cdots \end{cases}$$

$$c_k = 1, \quad k = 0, \pm 5, \pm 10, \cdots$$

Example 3.11

Giving the following facts about a sequence *x*[*n*]:

1. *x*[*n*] is periodic with period *N* = 6.
2. $\sum_{n=0}^{5} x[n] = 2$ 3. $\sum_{n=2}^{7} (-1)^n x[n] = 1$
4. *x*[*n*] has the minimum power per period among the set of signals satisfying the preceding three conditions.

Determine the sequence *x*[*n*].

$$x[n] = a_0 + a_3 e^{jn\pi} = (1/3) + (1/6)(-1)^n$$

3.7 Fourier Series and LTI Systems

- Reviewing eigenfunction property of LTI systems:

$$x(t)=e^{st}\rightarrow y(t)=H(s)e^{st},\ -\infty<t<\infty$$

$$x[n]=z^{n}\rightarrow y[n]=H(z)z^{n},\ -\infty<n<\infty$$

system function (系统函数)

- If Re{s} = 0, $s = j\omega$, $H(j\omega)=\int_{-\infty}^{\infty}h(t)e^{-j\omega t}dt$
- If |z| = 1, $z=e^{j\omega}$, $H(e^{j\omega})=\sum_{n=-\infty}^{\infty}h[n]e^{-j\omega n}$

frequency response (频率响应)

- If the input signal of an LTI system is periodic, ...

$$x_T(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t}$$

$$x_N[n]=\sum_{k=\langle N\rangle}a_k e^{jk(2\pi/N)n}$$

$$\boxed{H(j\omega)\,/\,H(e^{j\omega})}\rightarrow$$

$$y(t)=\sum_{k=-\infty}^{\infty}a_k H(jk\omega_0)e^{jk\omega_0 t}$$

$$y[n]=\sum_{k=\langle N\rangle}a_k H(e^{jk\frac{2\pi}{N}})e^{jk\frac{2\pi}{N}n}$$

3.7 Fourier Series and LTI Systems

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t} \qquad y[n] = \sum_{k=\langle N \rangle} a_k H(e^{j2k\pi/N}) e^{jk(2\pi/N)n}$$

- This results show that the response of an LTI system to a periodic signal is still *periodic*, and the output signal has same period as that of the input signal; furthermore, relationships of FS coefficients between input and output are as

$$x_T(t) \leftrightarrow a_k \rightarrow y_T(t) \leftrightarrow b_k = a_k H(jk\omega_0)$$ *Steady response*

$$x_N[n] \leftrightarrow a_k \rightarrow y_N[n] \leftrightarrow b_k = a_k H(e^{jk(2\pi/N)})$$ (稳态响应)

- The effect of the LTI system is to modify individually each of the Fourier coefficients of the input through multiplication by the value of the frequency response at the corresponding frequency.

Example 3.12

Input periodic signal $x(t)=\sum_{k=-3}^{3} a_k e^{jk2\pi t}$ as in example 3.1 to an LTI system with impulse response $h(t)=e^{-t}u(t)$, determine the Fourier series coefficients b_k of the output $y(t)$.

Sol: First compute the frequency response,

$$H(j\omega)=\int_0^{\infty} e^{-\tau}e^{-j\omega\tau}d\tau=-\frac{1}{1+j\omega}e^{-\tau}e^{-j\omega\tau}\Bigg|_0^{\infty}=\frac{1}{1+j\omega}$$

Since $b_k = a_k H(jk2\pi)$

$$a_0=1,\quad a_1=a_{-1}=1/4,$$

$$a_2=a_{-2}=1/2,$$

$$a_3=a_{-3}=1/3$$

$$b_0=a_0\cdot\frac{1}{1+j0}=1,$$

$$b_1=\frac{1}{4}\left(\frac{1}{1+j2\pi}\right),\qquad b_{-1}=\frac{1}{4}\left(\frac{1}{1-j2\pi}\right),$$

$$b_2=\frac{1}{2}\left(\frac{1}{1+j4\pi}\right),\qquad b_{-2}=\frac{1}{2}\left(\frac{1}{1-j4\pi}\right),$$

$$b_3=\frac{1}{3}\left(\frac{1}{1+j6\pi}\right),\qquad b_{-3}=\frac{1}{3}\left(\frac{1}{1-j6\pi}\right).$$

3.7 Fourier Series and LTI Systems

$$y[n] = r\cos\left(\frac{2\pi}{N}n + \theta\right)$$

Example 3.13

Consider an LTI system with sample response $h[n] = \alpha^n u[n], -1 < \alpha < 1$, and the input $x[n] = \cos\left(\frac{2\pi}{N}n\right)$, find the steady-response *y*[*n*].

Sol: First compute the frequency response:

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n = \frac{1}{1-\alpha e^{-j\omega}}$$

Writing *x*[*n*] as

$$x[n] = \frac{1}{2}e^{j(2\pi/N)n} + \frac{1}{2}e^{-j(2\pi/N)n}$$

$$y[n] = \frac{1}{2}H(e^{j2\pi/N})e^{j(2\pi/N)n} + \frac{1}{2}H(e^{-j2\pi/N})e^{-j(2\pi/N)n}$$

$$= \frac{1}{2}\left(\boxed{\frac{1}{1-\alpha e^{-j2\pi/N}}}\right)e^{j(2\pi/N)n} + \frac{1}{2}\left(\boxed{\frac{1}{1-\alpha e^{j2\pi/N}}}\right)e$$

$re^{j\theta} =$ $\quad$ $= re^{-j\theta}$

3.8 SUMMARY

- Eigenfunction property of LTI systems;
- Fourier series representations for both continuous-time and discrete-time periodic signals—i.e., weighted sum of harmonically related complex exponentials that share a common period;
- Different characteristics of signals reflected in their Fourier series coefficients (properties of Fourier series);
- Steady response of LTI systems – when input signal is periodic.

Homework

3.22 (d) in (a)、(c)　　3.25

3.27　　3.28 (b) (c)

3.31　　3.34 (a) (b)