# CHAPTER 4

# THE CONTINUOUS-TIME FOURIER TRANSFORM

# Content

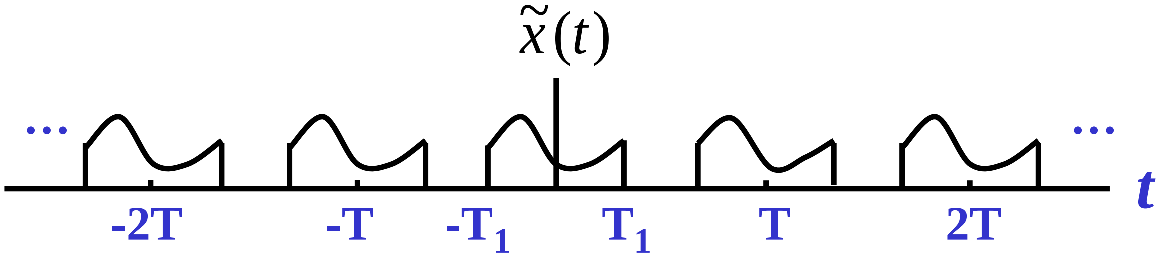
Content

## 4.0 Introduction

- Representation of continuous-time aperiodic signals as linear combination of complex exponentials — Inverse Fourier Transform
- Frequency spectrum of aperiodic signals — Fourier Transform
- Applications of continuous-time Fourier transform

# 4.1 Representation of Aperiodic Signals: The Continuous-Time Fourier Transform

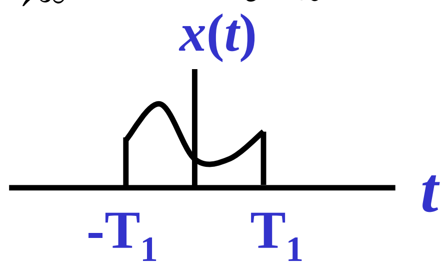
## 4.1.1 Fourier Transform and Inverse Fourier Transform




$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}dt$$

$$Ta_k = \int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}dt$$

As $T \to \infty$ ,
$$\lim_{T\to\infty} Ta_k = \int_{-\infty}^{\infty} x(t)e^{-jk\omega_0 t}dt = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$




Use $X(j\omega)$ to denote this integral, then we have:

*Frequency spectrum* of $x(t)$ →
$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt}$$
← *Fourier transform* of $x(t)$

# 4.1 Representation of Aperiodic Signals: The Continuous-Time Fourier Transform

Since $X(j\omega) = \lim_{T\to\infty} Ta_k = \lim_{\omega_0\to 0} 2\pi \frac{a_k}{\omega_0}$,

$X(j\omega)$ is actually spectrum-density function(频谱密度函数).

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \left[ \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt \right] e^{jk\omega_0 t}$$

$$= \frac{\omega_0}{2\pi} \sum_{k=-\infty}^{\infty} \left[ \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt \right] e^{jk\omega_0 t}$$

As $T \to \infty$ , $\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega}$ ← *Inverse Fourier transform*

*analysis* equation: $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

*synthesis* equation: $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$

Fourier Transform Pair

# 4.1 Representation of Aperiodic Signals: The Continuous-Time Fourier Transform

Comparing the synthesis equations in:

$$CTFS: \quad x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad CTFT: \quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$\boxed{a_k \Leftrightarrow \frac{X(j\omega)d\omega}{2\pi}}$$

This fact means: Aperiodic signals can also be decomposed as linear combination of infinite numbers of complex exponentials, which occur at a continuum of frequencies, but have amplitudes infinitesimally small — approaching zero!

$\left|X(j\omega)\right|$ indicate the relative amplitudes of all components, and angle $\sphericalangle X(j\omega)$ indicate the phases of all components.

# 4.1 Representation of Aperiodic Signals: The Continuous-Time Fourier Transform

An useful relationship: $a_k = \frac{1}{T} X(j\omega)\Big|_{\omega = k\omega_0}$

Where, $\tilde{x}(t) \xleftrightarrow{FS} a_k$, $x(t) \xleftrightarrow{FT} X(j\omega)$ .

This relationship is also valid for aperiodic signals with unlimited duration!

- Convergence of Fourier Transforms

Dirichlet conditions:

1. $x(t)$ is absolutely integrable; that is $\int_{-\infty}^{\infty} |x(t)| \, dt < \infty$
2. $x(t)$ have a finite number of maxima and minima within any finite interval.
3. $x(t)$ have a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.

If impulse functions are permitted in the transform, some signals which are not absolutely integrable over an infinite interval, can also be considered to have Fourier transforms. This will be convenient in the discussion of Fourier methods.

# 4.1 Representation of Aperiodic Signals: The Continuous-Time Fourier Transform

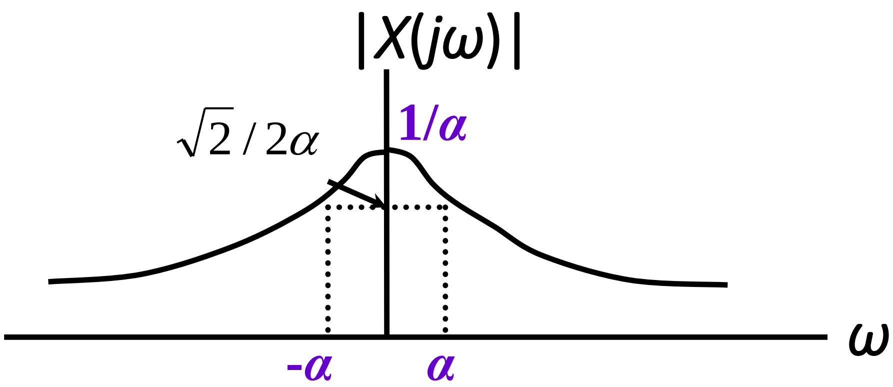
## 4.1.2 Examples

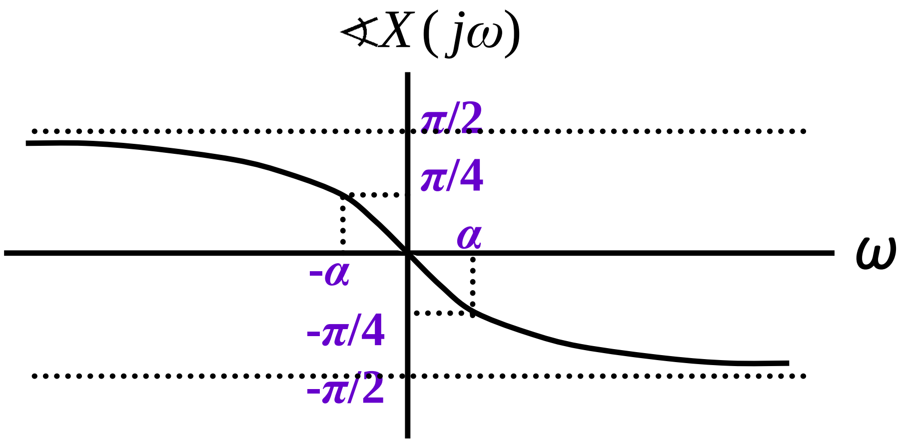
### Example 4.1

Consider the signal $x(t)=e^{-\alpha t}u(t) \quad \alpha>0$ .

$$X(j\omega)=\int_0^\infty e^{-\alpha t}e^{-j\omega t}dt=-\frac{1}{\alpha+j\omega}e^{-(\alpha+j\omega)t}\Big|_0^\infty=\frac{1}{\alpha+j\omega},\quad \alpha>0$$

$$|X(j\omega)|=\frac{1}{\sqrt{\alpha^2+\omega^2}},\quad \measuredangle X(j\omega)=-\tan^{-1}\left(\frac{\omega}{\alpha}\right)$$

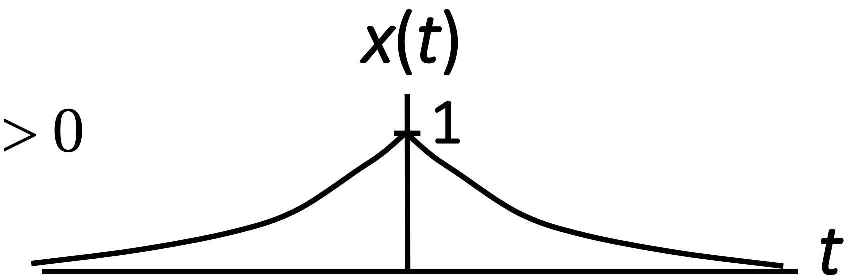


# 4.1 Representation of Aperiodic Signals: The Continuous-Time Fourier Transform

Example 4.2

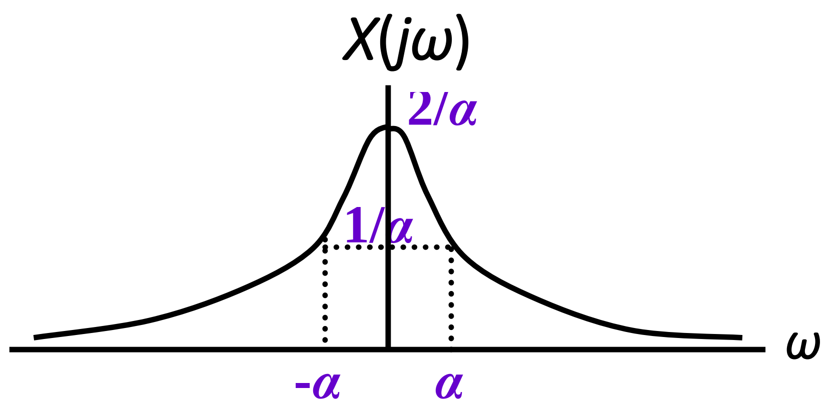
Consider the signal $x(t) = e^{-\alpha|t|}, \alpha > 0$




Sol: The Fourier transform of the signal is

$$X(j\omega) = \int_{-\infty}^{\infty} e^{-\alpha|t|} e^{-j\omega t} dt = \int_{-\infty}^{0} e^{\alpha t} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-\alpha t} e^{-j\omega t} dt$$

$$= \frac{1}{\alpha - j\omega} + \frac{1}{\alpha + j\omega} = \frac{2\alpha}{\alpha^2 + \omega^2}$$



## 4.1 Representation of Aperiodic Signals: The Continuous-Time Fourier Transform

Example 4.3

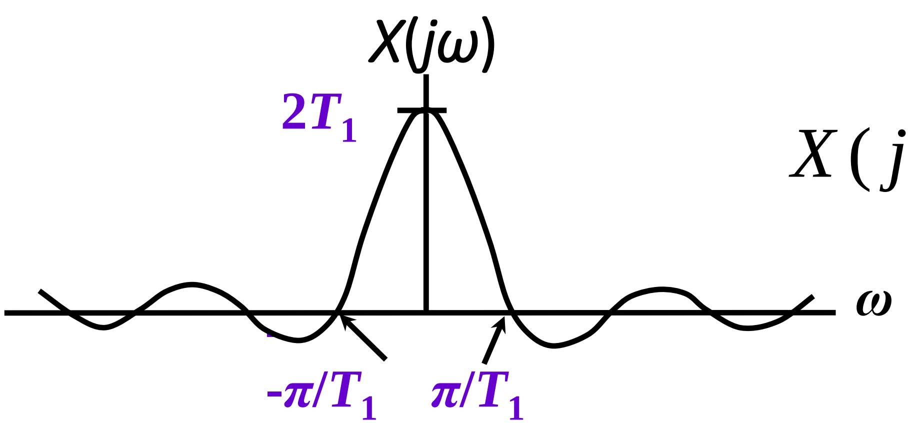
Determine the Fourier transform of the unit impulse $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

That is, the unit impulse has a Fourier transform consisting of equal contribution at all frequencies. This spectrum is referred to as white-spectrum.

Example 4.4

Consider the rectangular pulse signal $x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases}$




$$X(j\omega) = \int_{-T_1}^{T_1} e^{-j\omega t} dt = \frac{2\sin\omega T_1}{\omega}$$

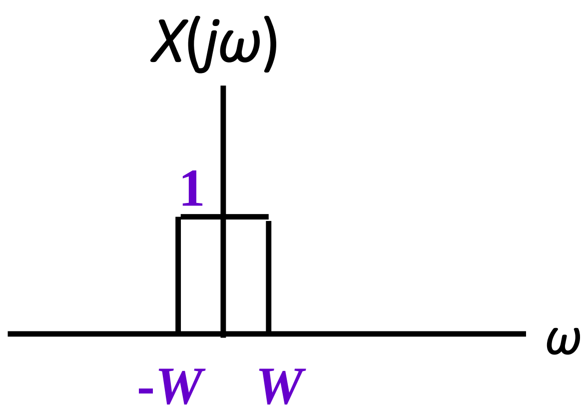
Example 4.5

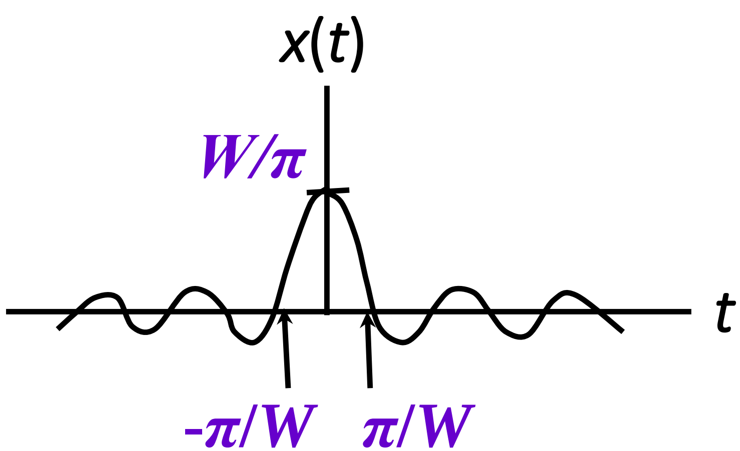
Consider the signal $x(t)$ whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$$

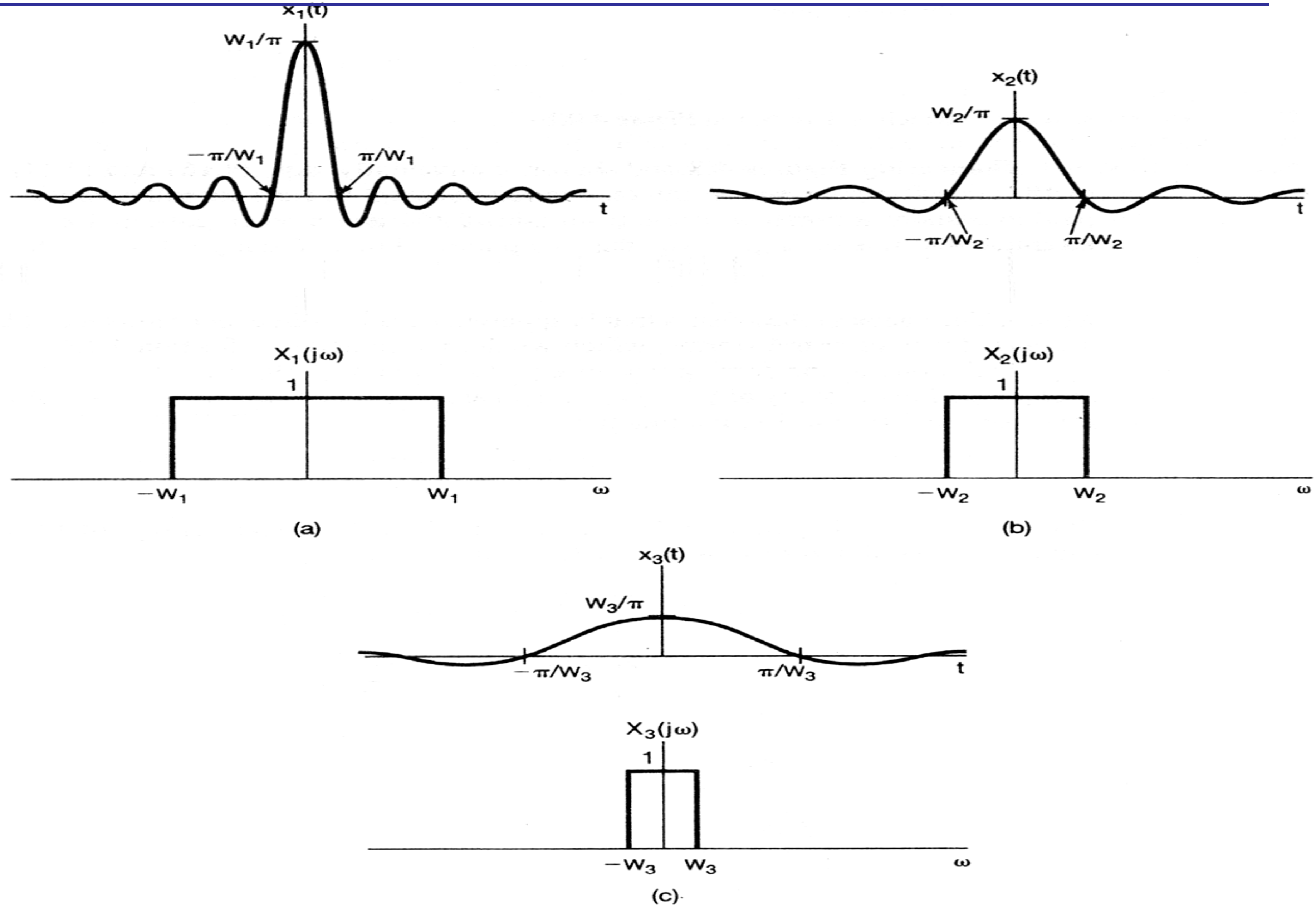
Sol: Using the synthesis equation, we can determine

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega = \frac{\sin Wt}{\pi t}$$



# 4.1 Representation of Aperiodic Signals: The Continuous-Time Fourier Transform



# 4.2 The Fourier Transform for Periodic Signals

$$\mathscr{F}\{e^{j\omega_0 t}\} = ?$$

From the analysis equation,

$$X(j\omega) = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} dt = \frac{1}{j(\omega_0 - \omega)} e^{j(\omega_0 - \omega)t} \Big|_{-\infty}^{\infty}$$ Does not converge !

We have obtained $\delta(t) \leftrightarrow 1$, so $\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 1 \cdot e^{j\omega t} d\omega$

$$2\pi\delta(t) = \int_{-\infty}^{\infty} 1 \cdot e^{j\omega t} d\omega$$

$$2\pi\delta(-\omega) = \int_{-\infty}^{\infty} e^{-ja\omega} da \quad \boxed{\Rightarrow 1 \xleftrightarrow{FT} 2\pi\delta(-\omega) = 2\pi\delta(\omega)}$$

$$2\pi\delta(\omega_0 - \omega) = \int_{-\infty}^{\infty} e^{-jt(\omega - \omega_0)} dt = \int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} dt \quad \boxed{\Rightarrow \mathscr{F}\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)}$$

The Fourier transform of the complex exponential signal $e^{j\omega_0 t}$ is an impulse located at $\omega = \omega_0$ with its area $2\pi$.

## 4.2 The Fourier Transform for Periodic Signals

For an arbitrary periodic signal $x(t)$, if it can be represented by Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

We can obtain its Fourier transform as

$$\boxed{X(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)}$$

The Fourier transform of a periodic signal with Fourier series coefficients $\{a_k\}$ can be interpreted as a train of impulses occurring at the harmonically related frequencies and for which the area of the impulse at the $k$th harmonic frequency $k\omega_0$ is $2\pi$ times the $k$th Fourier series coefficient $a_k$.

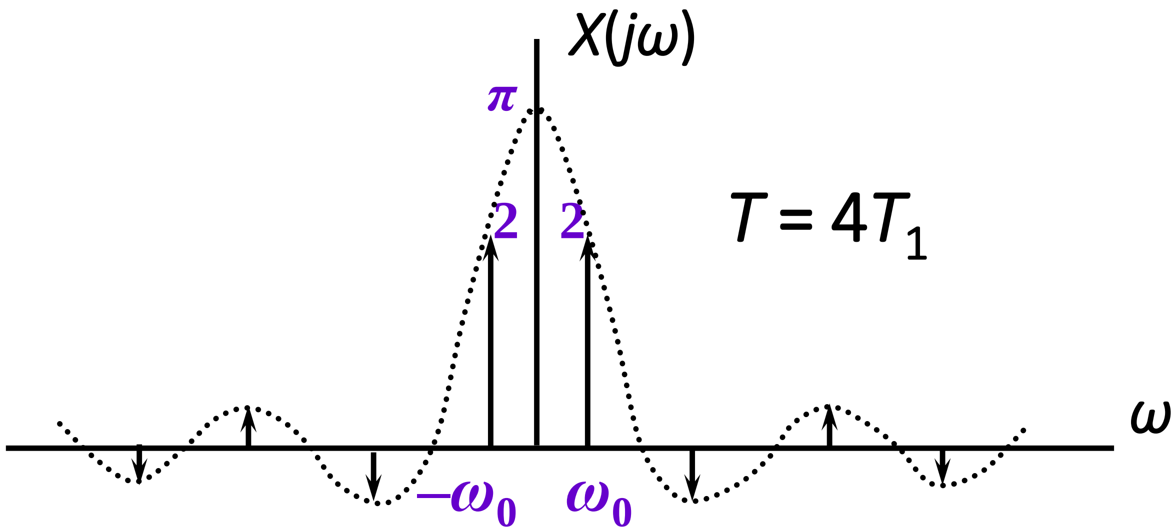
Example 4.6

Consider the periodic square wave with FS coefficients are

$$a_k = \frac{\sin k\omega_0 T_1}{k\pi}$$

Sol: From the formula its Fourier transform is

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi \cdot \frac{\sin k\omega_0 T_1}{k\pi} \delta(\omega - k\omega_0) = \sum_{k=-\infty}^{\infty} \frac{2\sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$$



Example 4.7

Find the Fourier Transforms of $x_1(t) = \sin\omega_0 t$ and $x_2(t) = \cos\omega_0 t$ .

Sol: The Fourier series coefficients for $x_1(t)$ are

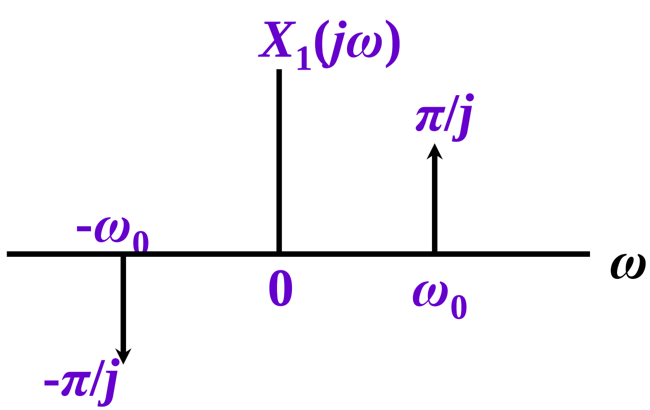
$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j}, \qquad a_k = 0, \quad k \neq \pm 1$$

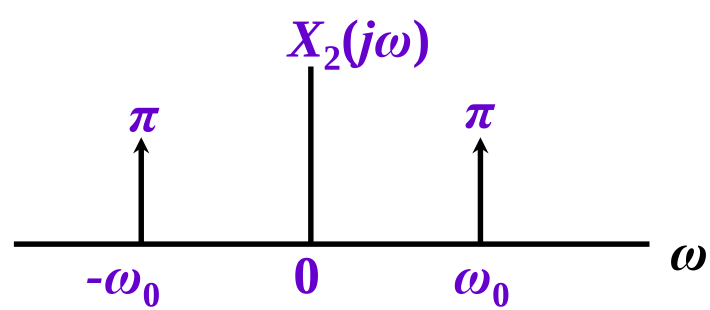
The Fourier series coefficients for $x_2(t)$ are

$$a_1 = a_{-1} = \frac{1}{2}, \qquad a_k = 0, \quad k \neq \pm 1$$

$$\sin\omega_0 t \longleftrightarrow j\pi[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)]$$

$$\cos\omega_0 t \longleftrightarrow \pi[\delta(\omega+\omega_0) + \delta(\omega-\omega_0)]$$



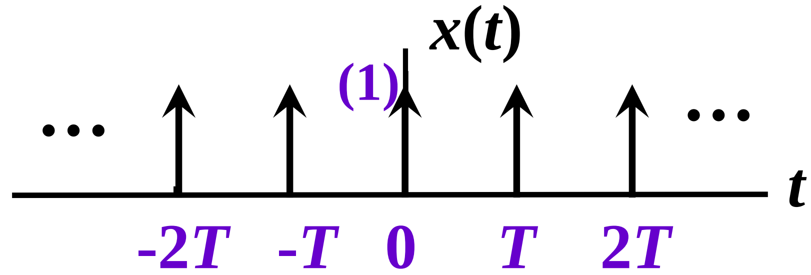



## 4.2 The Fourier Transform for Periodic Signals

Example 4.8

Consider the impulse train

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

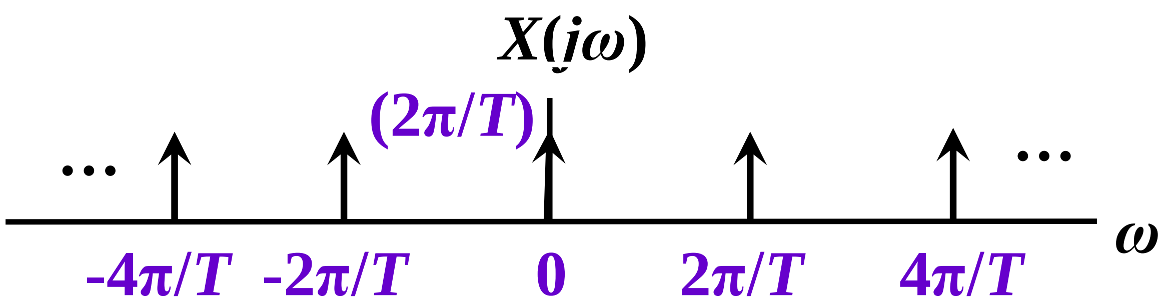



Sol: The Fourier series coefficients for this signal are

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T}$$

Thus, its Fourier transform is

$$X(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2k\pi}{T}) = \omega_0 \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$



# 4.3 Properties of The Continuous-Time Fourier Transform

## 4.3.1 Linearity

If $x(t) \xleftrightarrow{FT} X(j\omega)$ and $y(t) \xleftrightarrow{FT} Y(j\omega)$

then $ax(t) + by(t) \xleftrightarrow{FT} aX(j\omega) + bY(j\omega)$

## 4.3.2 Time Shifting

If $x(t) \xleftrightarrow{FT} X(j\omega)$,

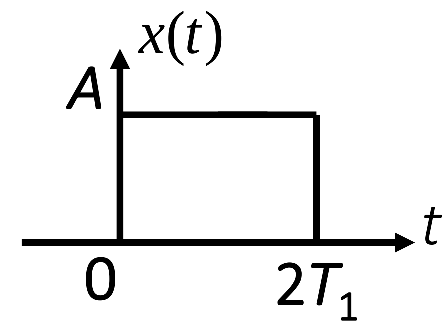
$$\sphericalangle\{e^{-j\omega t_0} X(j\omega)\} = \sphericalangle\{X(j\omega)\} - \omega t_0$$

then $x(t - t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(j\omega)$

A signal which is shifted in time does not have the magnitude of its Fourier transform altered. The effect of a time shift on a signal is to introduce into its transform a phase shift, namely, $-\omega t_0$.

Example 4.9

Determine the Fourier transform of *x*(*t*) shown in the figure.




Sol: Since

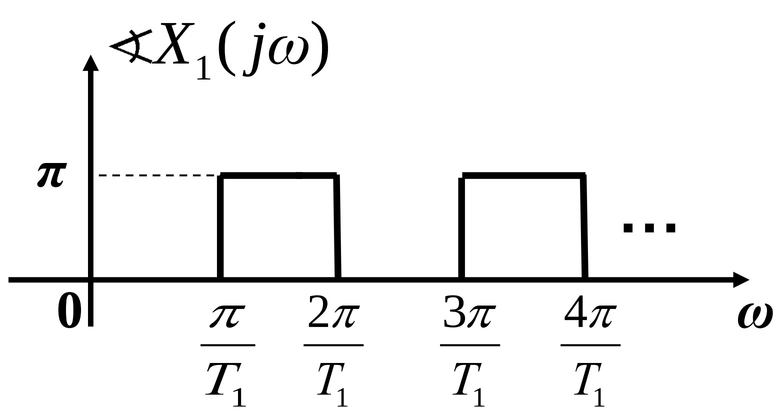
$$x_1(t) = u(t+T_1) - u(t-T_1) \leftrightarrow X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}$$

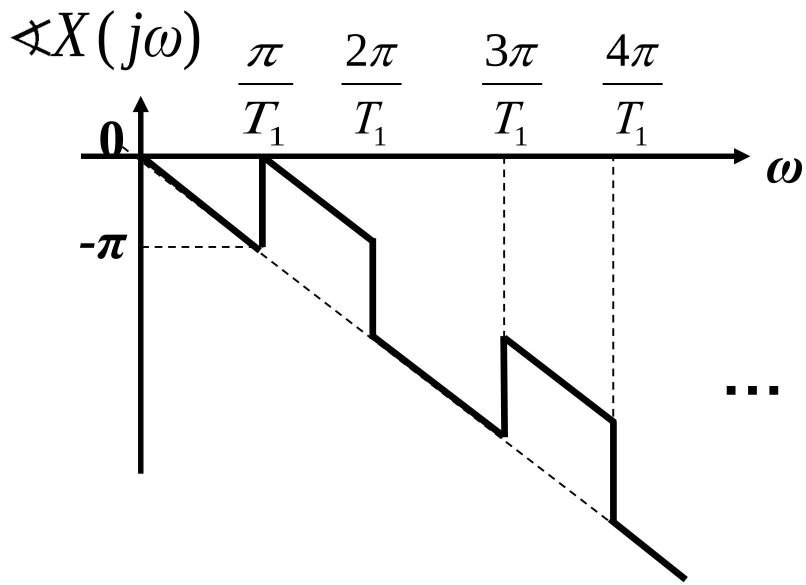
and $x(t) = Ax_1(t-T_1)$

From the time shifting property, we have

$$X(j\omega) = AX_1(j\omega)e^{-j\omega T_1} = \frac{2A\sin\omega T_1}{\omega}e^{-j\omega T_1}$$

$$\sphericalangle X(j\omega) = \sphericalangle X_1(j\omega) - \omega T_1$$



## 4.3.3 Conjugation and Conjugate Symmetry

If $x(t) \xleftrightarrow{FT} X(j\omega)$, then $x^*(t) \xleftrightarrow{FT} X^*(-j\omega)$

- If *x*(*t*) is real, then $X(-j\omega) = X^*(j\omega)$

$$Re\{X(j\omega)\}=Re\{X(-j\omega)\}, Im\{X(j\omega)\}=-Im\{X(-j\omega)\}$$

even $|X(j\omega)| = |X(-j\omega)|$, $\sphericalangle X(j\omega) = -\sphericalangle X(-j\omega)$ odd

- If *x*(*t*) is both real and even, so is *X*(*jω*), i.e., *X*(*jω*)=Re{*X*(*jω*)}
- If *x*(*t*) is real and odd, *X(jω)* is purely imaginary and odd, i.e., *X*(*jω*)=*j*·*Im*{*X*(*jω*)}
- $$x(t) = x_e(t) + x_o(t)$$

$$x_e(t) \xleftrightarrow{FT} \mathrm{Re}\{X(j\omega)\} \qquad x_o(t) \xleftrightarrow{FT} j\,\mathrm{Im}\{X(j\omega)\}$$

Example 4.10

Consider again the Fourier transform evaluation of $x(t) = e^{-\alpha|t|}$

Sol: $$e^{-\alpha|t|} = e^{-\alpha t}u(t) + e^{\alpha t}u(-t) = 2\left[\frac{e^{-\alpha t}u(t) + e^{\alpha t}u(-t)}{2}\right] = 2Ev\left\{e^{-\alpha t}u(t)\right\}$$

and $$e^{-\alpha t}u(t) \xleftrightarrow{FT} \frac{1}{\alpha + j\omega}$$

From the symmetry properties of the Fourier transform, we have

$$X(j\omega) = 2\,\mathrm{Re}\left\{\frac{1}{\alpha + j\omega}\right\} = \frac{2\alpha}{\alpha^2 + \omega^2}$$

## 4.3.4 Differentiation and Integration

If $x(t) \xleftrightarrow{FT} X(j\omega)$, then $\dfrac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(j\omega)$

$$\frac{d^n x(t)}{dt^n} \xleftrightarrow{FT} (j\omega)^n X(j\omega)$$

Consequence: Strengthening the high-frequencies in the signal.

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{FT} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$$

Consequence: Strengthening the low-frequencies in the signal.

Example 4.11

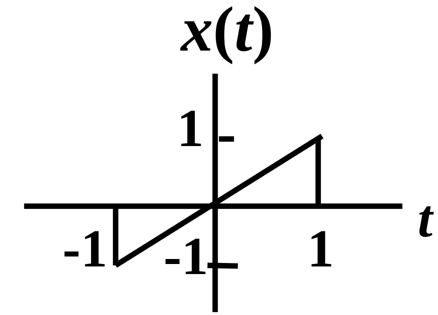
Determine the Fourier transform of the unit step $x(t) = u(t)$.

$$Sol: u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t) \qquad \text{sgn}(t) = \lim_{\alpha \to 0}\left[e^{-\alpha t}u(t) - e^{\alpha t}u(-t)\right](\alpha > 0)$$

$$\mathcal{F}\{\text{sgn}(t)\} = \lim_{\alpha \to 0}\left(\frac{1}{\alpha + j\omega} - \frac{1}{\alpha - j\omega}\right) = \frac{2}{j\omega} \qquad \boxed{X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)}$$

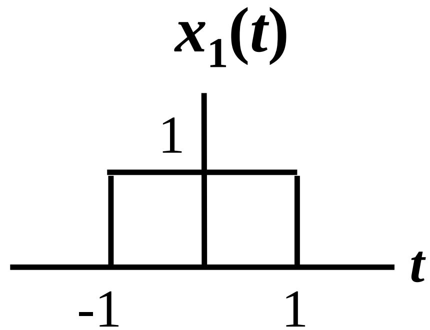
## Example 4.12

Calculate the Fourier transform $X(j\omega)$ for the signal $x(t)$ displayed in the figure:




Sol: Let

$$g(t)=\frac{dx(t)}{dt}=$$

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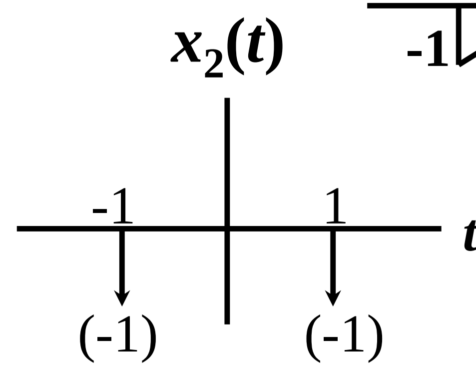



$$X_1(j\omega)=\left(\frac{2\sin\omega}{\omega}\right),\quad X_2(j\omega)=-e^{j\omega}-e^{-j\omega}$$

$$G(j\omega)=j\omega X(j\omega)=\left(\frac{2\sin\omega}{\omega}\right)-e^{j\omega}-e^{-j\omega}$$

$$X(j\omega)=\frac{2\sin\omega}{j\omega^2}-\frac{2\cos\omega}{j\omega}$$

## 4.3.5 Time and Frequency Scaling

If $x(t) \xleftrightarrow{FT} X(j\omega)$, then $x(at) \xleftrightarrow{FT} \frac{1}{|a|} X(\frac{j\omega}{a})$, $a$ is real

A linear scaling in time by a factor of *a* corresponds to a linear scaling in frequency by a factor of 1/*a*, and vice versa.

Specially, when *a* = -1, we have $x(-t) \xleftrightarrow{FT} X(-j\omega)$

## 4.3.6 Duality

If $x(t) \xleftrightarrow{FT} X(j\omega)$, then $X(jt) \xleftrightarrow{FT} 2\pi x(-\omega)$

Example 4.13

Using duality to find *G*(*jω*) of the signal $g(t) = \frac{2}{1+t^2}$.

Sol: From pair $x(t) = e^{-|t|} \xleftrightarrow{FT} X(j\omega) = \frac{2}{1+\omega^2}$

By duality property, we can write

$$g(t) = \frac{2}{1+t^2} \xleftrightarrow{FT} G(j\omega) = 2\pi e^{-|-\omega|} = 2\pi e^{-|\omega|}$$

Duality property shows that for any Fourier transform pair there is a *dual pair* with the time and frequency variables interchanged.

## Differentiation in Frequency-domain:

$$-jtx(t) \xleftrightarrow{FT} \frac{dX(j\omega)}{d\omega}$$

## Integration in Frequency-domain:

$$-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \xleftrightarrow{FT} \int_{-\infty}^{\omega} X(\eta)d\eta$$

## Frequency Shifting:

$$e^{j\omega_0 t}x(t) \xleftrightarrow{FT} X(j(\omega - \omega_0))$$

## 4.3.7 Parseval's Relation

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

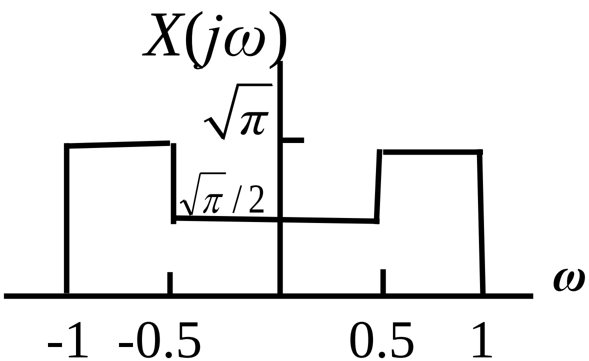
*energy-density spectrum*
(能量密度谱)

Parseval's relation says that this total energy may be determined either by computing energy per unit time ( $|x(t)|^2$ ) and integrating over all time or by computing the energy per unit frequency ( $|X(j\omega)|^2/2\pi$ ) and integrating over all frequencies.

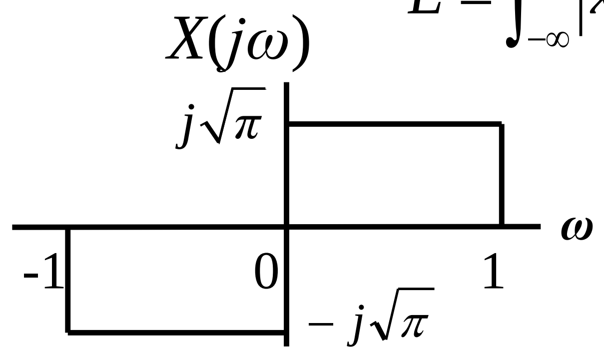
Example 4.14

For each of the Fourier transforms shown in figures (a) and (b), evaluate the following time-domain expressions:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt, \quad D = \frac{d}{dt} x(t)\Big|_{t=0}$$




(a)




(b)

Sol: $E_a = \frac{5}{8} \qquad E_b = 1$ (*Parseval's Relation*)

$$g(t) = \frac{d}{dt} x(t) \xleftrightarrow{FT} G(j\omega) = j\omega X(j\omega)$$

$$D = g(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(j\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} j\omega X(j\omega) d\omega$$

$$D_a = 0 \qquad D_b = -\frac{\sqrt{\pi}}{2\pi}$$

## 4.4 The Convolution Property

Consider the convolution integral: $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$

The Fourier transform of *y*(*t*) is:

$$Y(j\omega) = \mathcal{F}\{y(t)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt$$

Interchanging the order of integration and noting that *x*(*τ*) does not depend on *t*, we have $Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)\underbrace{\left[\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega t}dt\right]}_{e^{-j\omega\tau}H(j\omega)}d\tau$

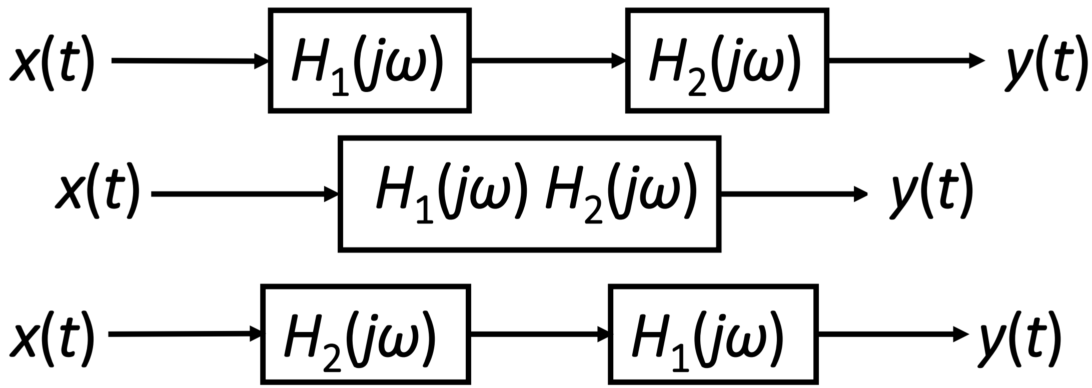
$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau = H(j\omega)\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau = H(j\omega)X(j\omega)$$

$$\boxed{y(t) = h(t) * x(t) \xleftrightarrow{FT} Y(j\omega) = H(j\omega)X(j\omega)}$$

The Fourier transform maps the convolution of two signals into the product of their Fourier transforms. *H*(*jω*), the frequency response, is the Fourier transform of the impulse response *h*(*t*). It captures the change in complex amplitude of the Fourier transform of the input at each frequency *ω*.

## 4.4 The Convolution Property

- The frequency response $H(j\omega)$ also can characterize an LTI system, just as its inverse transform, the unit impulse response $h(t)$ .




Three equivalent LTI systems. Here, each LTI system is represented by $H(j\omega)$

- $H(j\omega)$ cannot be defined for every LTI system.
- Since essentially all physical or practical signals satisfy the last two conditions in Dirichlet conditions, the condition of absolutely integrable becomes the determining factor which can guarantee the existence of the Fourier transform $H(j\omega)$ of $h(t)$. That is, only *a stable LTI system has a frequency response $H(j\omega)$*.

Example 4.15

Consider an integrator — that is, an LTI system specified by the equation

$$y(t)=\int_{-\infty}^{t}x(\tau)d\tau$$

Sol: Since $y(t)=x(t)*u(t)=\int_{-\infty}^{\infty}x(\tau)u(t-\tau)d\tau=\int_{-\infty}^{t}x(\tau)d\tau$

So the impulse response for this system is the unit step *u*(*t*).

The frequency response of the system is $H(j\omega)=\dfrac{1}{j\omega}+\pi\delta(\omega)$

Using the convolution property, we have

$$\begin{aligned}Y(j\omega)&=H(j\omega)X(j\omega)\\&=\frac{1}{j\omega}X(j\omega)+\pi X(j\omega)\delta(\omega)\\&=\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)\end{aligned}$$

Example 4.16

Find the response of an LTI system with $h(t)=e^{-at}u(t), \quad a>0$ to the input signal $x(t)=e^{-bt}u(t), \quad b>0$.

Sol: $X(j\omega)=\dfrac{1}{b+j\omega}, \; H(j\omega)=\dfrac{1}{a+j\omega} \Rightarrow Y(j\omega)=\dfrac{1}{(a+j\omega)(b+j\omega)}$

Expand $Y(j\omega)$ in a *partial-fraction expansion* (部分分式展开法).

When $b \neq a$， let $Y(j\omega)=\dfrac{A}{(a+j\omega)}+\dfrac{B}{(b+j\omega)} \quad =\dfrac{1}{b-a}\left[\dfrac{1}{a+j\omega}-\dfrac{1}{b+j\omega}\right]$

$$A=(a+j\omega)Y(j\omega)\Big|_{j\omega=-a}=\frac{1}{b+j\omega}\Big|_{j\omega=-a}=\frac{1}{b-a}$$

$$B=(b+j\omega)Y(j\omega)\Big|_{j\omega=-b}=\frac{1}{a+j\omega}\Big|_{j\omega=-b}=\frac{-1}{b-a}$$

$$y(t)=\frac{1}{b-a}\left[e^{-at}u(t)-e^{-bt}u(t)\right] \quad b\neq a$$

# 4.4 The Convolution Property

When *b* = *a*, $Y(j\omega) = \dfrac{1}{(a+j\omega)^2} = j\dfrac{d}{d\omega}\left[\dfrac{1}{(a+j\omega)}\right]$

From the differentiation in the frequency-domain property,

$$e^{-at}u(t) \xleftrightarrow{FT} \frac{1}{a+j\omega}$$

$$te^{-at}u(t) \xleftrightarrow{FT} j\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right] = \frac{1}{(a+j\omega)^2}$$

Consequently,

$$y(t) = te^{-at}u(t) \quad b = a$$

Example 4.17

Determine the response of an ideal low-pass filter to an input signal *x*(*t*) that has the form of a *sinc* function, $x(t)=\dfrac{\sin\omega_i t}{\pi t}$

Sol: The impulse response of the ideal low-pass filter is of a similar form: $h(t)=\dfrac{\sin\omega_c t}{\pi t}$

$$X(j\omega)=\begin{cases}1 & |\omega|\le\omega_i\\ 0 & elsewhere\end{cases},\qquad H(j\omega)=\begin{cases}1 & |\omega|\le\omega_c\\ 0 & elsewhere\end{cases}$$

Therefore, $Y(j\omega)=\begin{cases}1 & |\omega|\le\omega_0\\ 0 & elsewhere\end{cases}$ where $\omega_0 = \min(\omega_i, \omega_c)$.

Finally, the inverse Fourier transform of $Y(j\omega)$ is given by

$$y(t)=\begin{cases}\dfrac{\sin\omega_c t}{\pi t}=h(t) & if\ \ \omega_c\le\omega_i\\[2ex] \dfrac{\sin\omega_i t}{\pi t}=x(t) & if\ \ \omega_i\le\omega_c\end{cases}$$

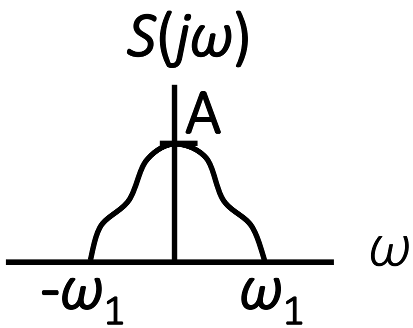
# 4.5 The Multiplication Property

$$r(t)=s(t)p(t)\xleftrightarrow{FT}R(j\omega)=\frac{1}{2\pi}\left[S(j\omega)*P(j\omega)\right]$$

***amplitude modulation property*** (幅度调制定理)

Example 4.18

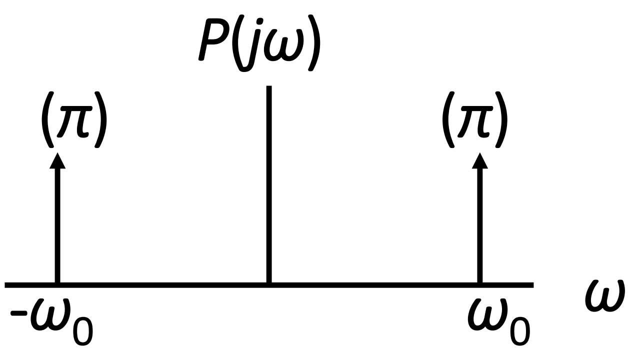
Let $s(t)$ be a signal whose spectrum $S(j\omega)$ is depicted in the following Figure. Also, consider the signal $p(t)=\cos\omega_0 t$,

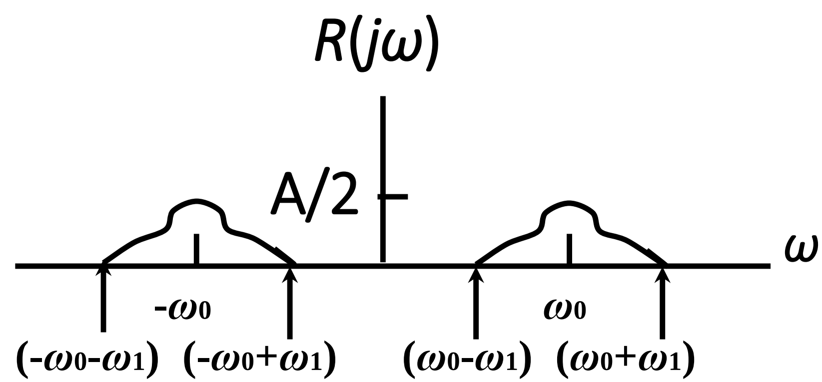



Sol: Since $P(j\omega)=\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$

From the multiplication property:

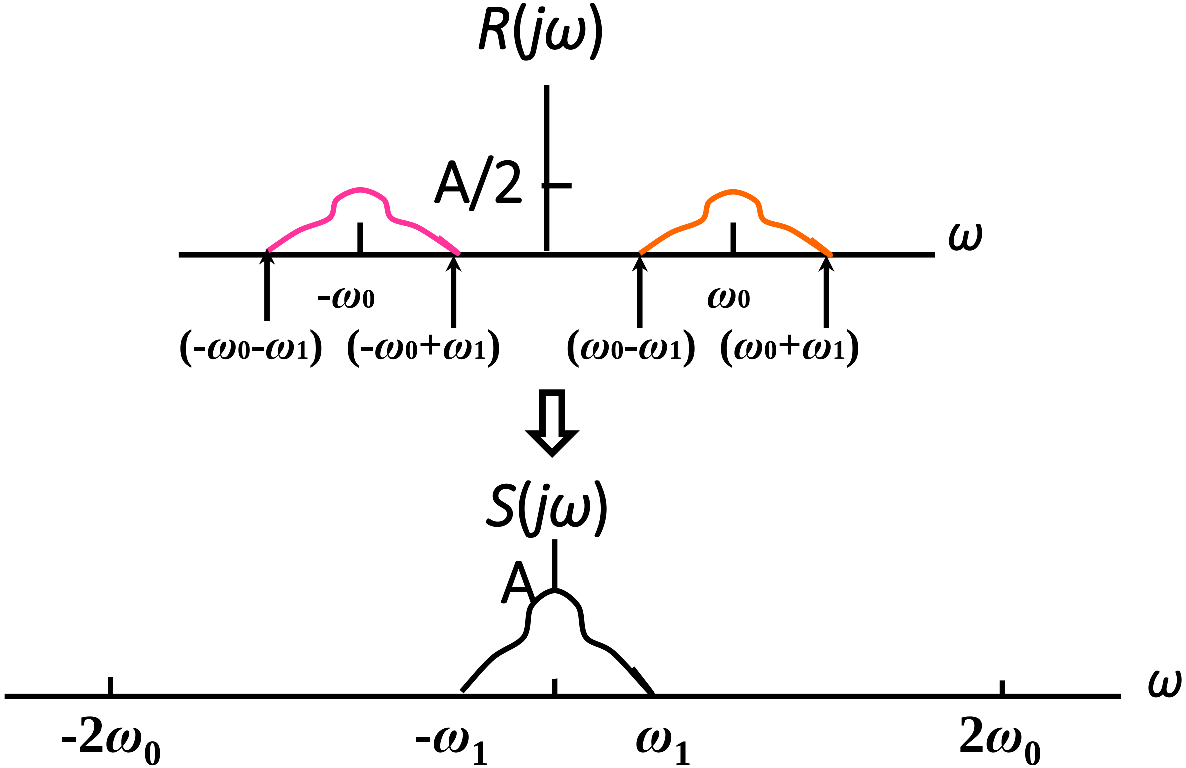
$$R(j\omega)=\frac{1}{2\pi}S(j\omega)*P(j\omega)=\frac{1}{2}S(j(\omega-\omega_0))+\frac{1}{2}S(j(\omega+\omega_0))$$



Example 4.19

Let us consider $r(t)$ as obtained in Example 4.18, and let $g(t) = r(t)p(t)$, we will show how to recover the modulated signal $s(t)$.



# 4.6 Systems Characterized By Linear Constant Coefficient Differential Equations

For a *stable* LTI system which is described by a linear constant-coefficient differential equation of the form :

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

$$\mathcal{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\} \quad \sum_{k=0}^{N} a_k \mathcal{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathcal{F}\left\{\frac{d^k x(t)}{dt^k}\right\}$$

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega)$$ (differentiation)

Equivalently, $$Y(j\omega)\left[\sum_{k=0}^{N} a_k (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k (j\omega)^k\right]$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$$ (Convolution)

- $H(j\omega)$ is a ratio of polynomials in $(j\omega)$, so it is a rational function.

Example 4.20

Consider a stable LTI system that is characterized by the differential equation $\frac{d^2y(t)}{dt^2}+4\frac{dy(t)}{dt}+3y(t)=\frac{dx(t)}{dt}+2x(t)$ , determine its impulse response.

Sol: The frequency response is

$$H(j\omega)=\frac{(j\omega)+2}{(j\omega)^2+4(j\omega)+3}$$

To determine the corresponding impulse response, we use the method of partial-fraction expansion:

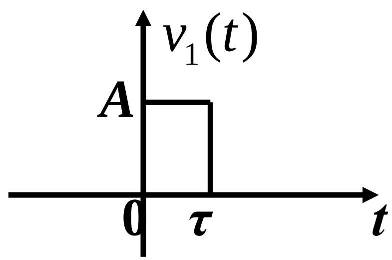
$$H(j\omega)=\frac{\frac{1}{2}}{j\omega+1}+\frac{\frac{1}{2}}{j\omega+3}$$

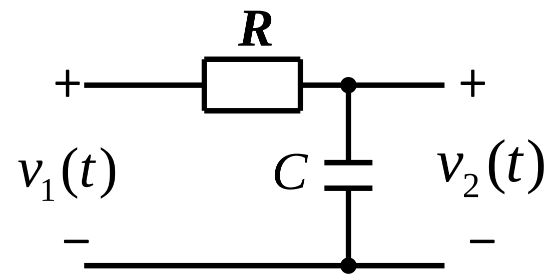
Thus, the impulse response is

$$h(t)=\frac{1}{2}e^{-t}u(t)+\frac{1}{2}e^{-3t}u(t)$$

Example 4.21

A *RC* circuit is as follows, $\boldsymbol{v_1(t)}$ is the input, determine $\boldsymbol{v_2(t)}$.



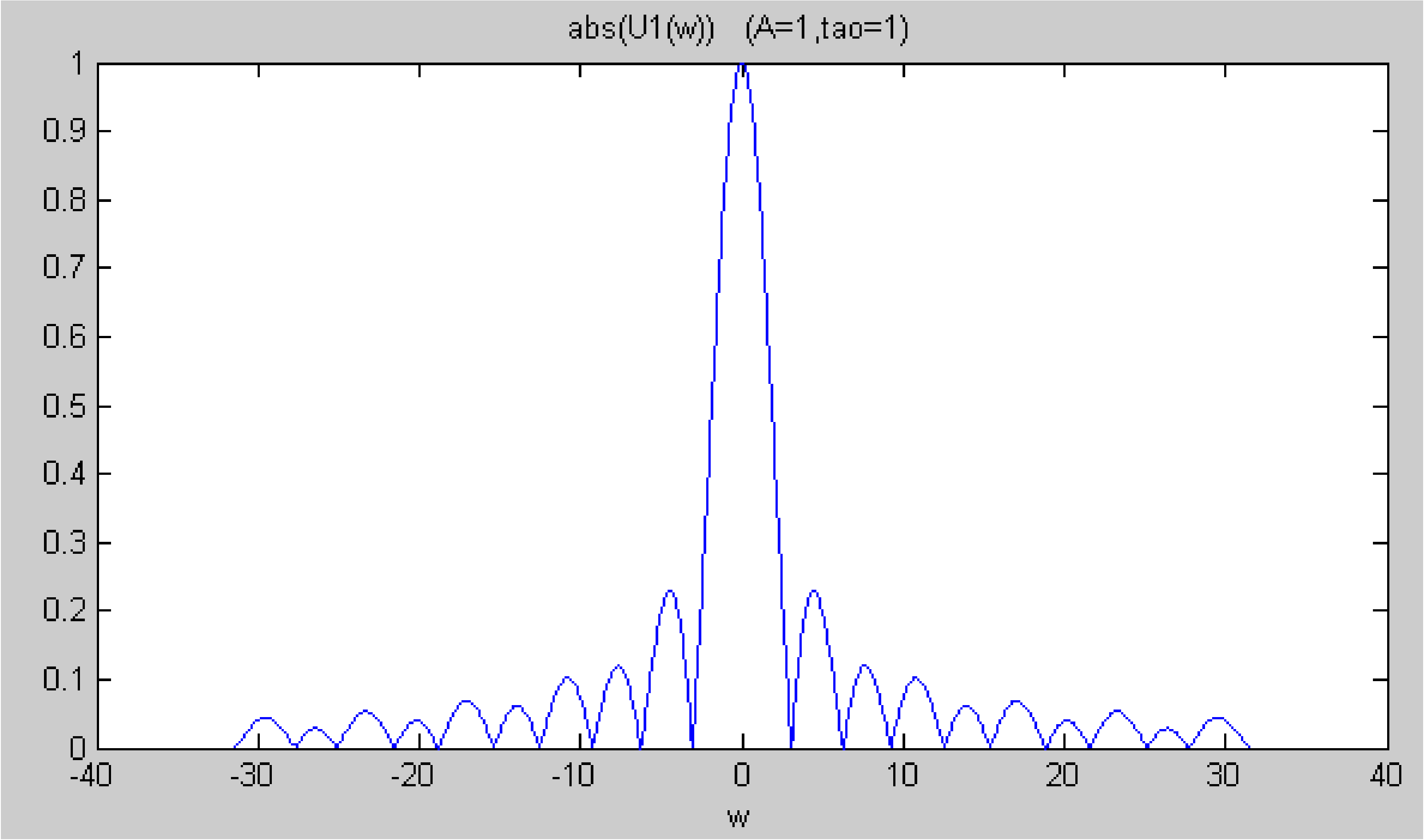

Sol: 1) Compute $V_1(j\omega)$

$$v_1(t) = A[u(t) - u(t-\tau)]$$

$$V_1(j\omega) = A[\pi\delta(\omega) + \frac{1}{j\omega} - \pi\delta(\omega)e^{-j\omega\tau} - \frac{e^{-j\omega\tau}}{j\omega}]$$

$$= \frac{A}{j\omega}(1 - e^{-j\omega\tau}) \quad = \frac{2A}{\omega}\sin\frac{\omega\tau}{2}e^{-j\frac{\omega\tau}{2}} = A\tau Sa(\frac{\omega\tau}{2})e^{-j\frac{\omega\tau}{2}}$$

# 4.6 Systems Characterized By Linear Constant Coefficient Differential Equations



# 4.6 Systems Characterized By Linear Constant Coefficient Differential Equations

2) Compute $H(j\omega)$

$$H(j\omega)=\frac{V_2(j\omega)}{V_1(j\omega)} = \frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}} = \frac{1}{1+j\omega RC} = \frac{\frac{1}{RC}}{\frac{1}{RC}+j\omega}$$

Let $\quad \frac{1}{RC}=\alpha=\frac{1}{\tau_0}$

$$H(j\omega)=\frac{\alpha}{\alpha+j\omega}=\frac{\alpha}{\sqrt{\alpha^2+\omega^2}}e^{-j\arctan\frac{\omega}{\alpha}}$$

3) Compute $V_2(j\omega)$

$$V_2(j\omega) = V_1(j\omega)H(j\omega)$$

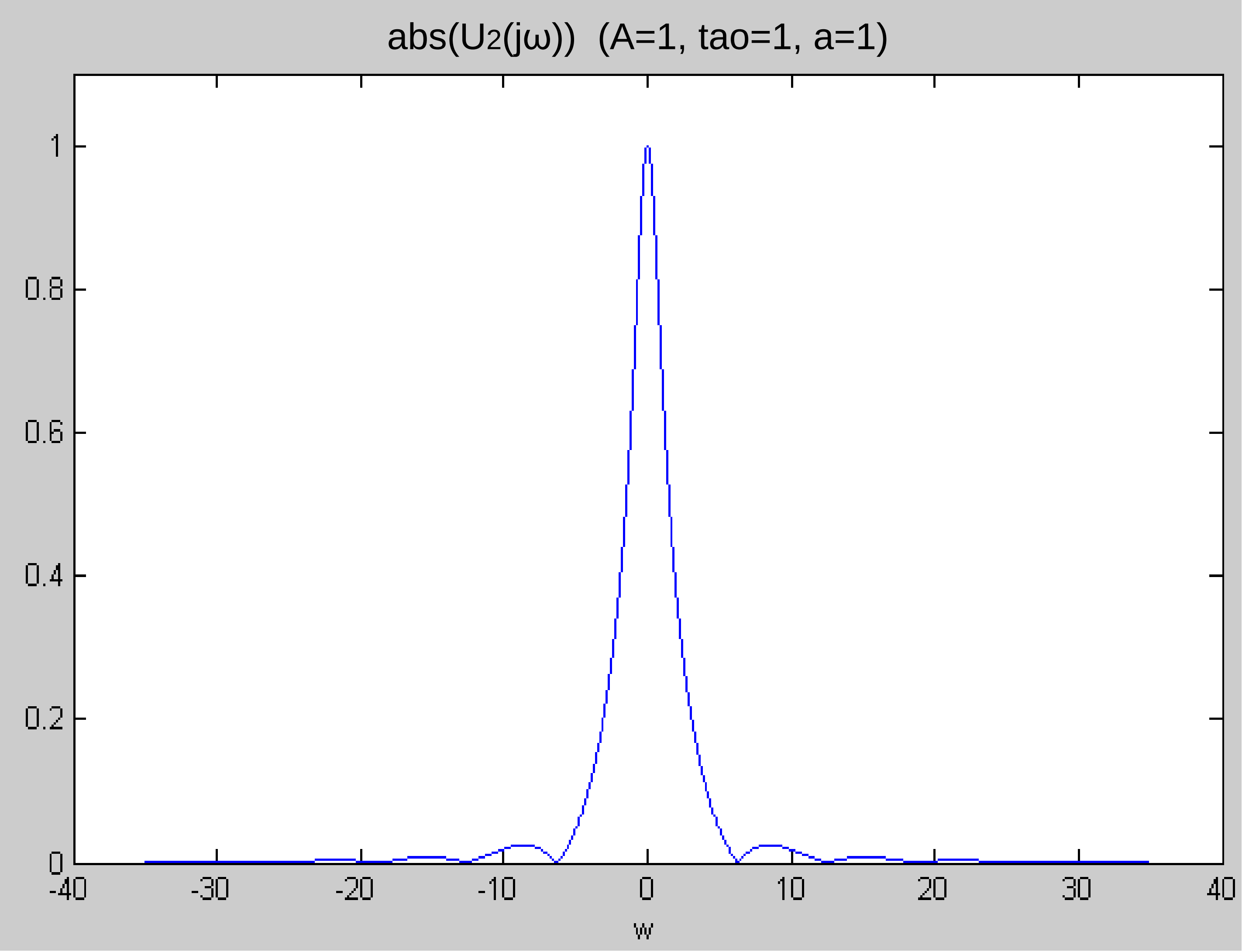
$$= A\tau Sa(\frac{\omega\tau}{2})e^{-j\frac{\omega\tau}{2}} \cdot \frac{\alpha}{\sqrt{\alpha^2+\omega^2}} e^{-j\arctan\frac{\omega}{\alpha}}$$

$$= \frac{\alpha A\tau}{\sqrt{\alpha^2+\omega^2}} Sa(\frac{\omega\tau}{2})e^{-j(\frac{\omega\tau}{2}+\arctan\frac{\omega}{\alpha})}$$

$$|V_2(j\omega)| = \frac{\alpha A\tau}{\sqrt{\alpha^2+\omega^2}}\left|Sa(\frac{\omega\tau}{2})\right| = \frac{2\alpha A\left|\sin(\frac{\omega\tau}{2})\right|}{\omega\sqrt{\alpha^2+\omega^2}}$$

$$\varphi_2(\omega) = \begin{cases} -(\frac{\omega\tau}{2}+\arctan\frac{\omega}{\alpha}), \sin(\frac{\omega\tau}{2}) > 0 \\ \pm\pi-(\frac{\omega\tau}{2}+\arctan\frac{\omega}{\alpha}), \sin(\frac{\omega\tau}{2}) < 0 \end{cases}$$

# 4.6 Systems Characterized By Linear Constant Coefficient Differential Equations



# 4.6 Systems Characterized [...] Equations

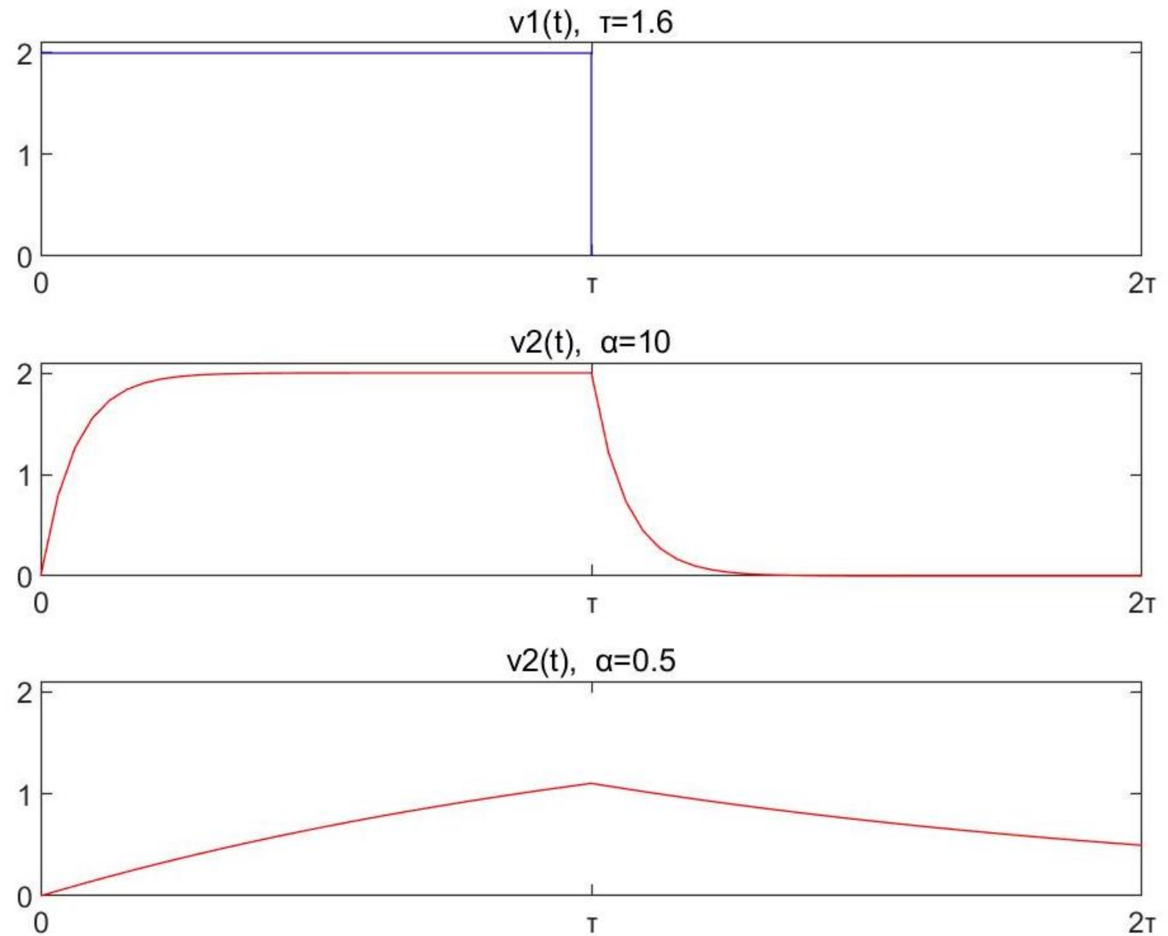
4) Compute $v_2(t) = \mathscr{F}$

$$V_2(j\omega) = A(1 - e^{-j\omega\tau}$$

$$= \frac{A(1 - e^{-j\omega\tau}}{j\omega}$$

$$\therefore v_2(t) = A[u(t) - u$$

$$= A(1 - e^{-\alpha t}$$




Rising and falling characteristics in time domain:

$v_1(t): t = 0, \tau$ Change very fast (jump)—Abundant high frequencies

$v_2(t): t = 0, \tau$ Change slowly, need a period of time to rise or fall

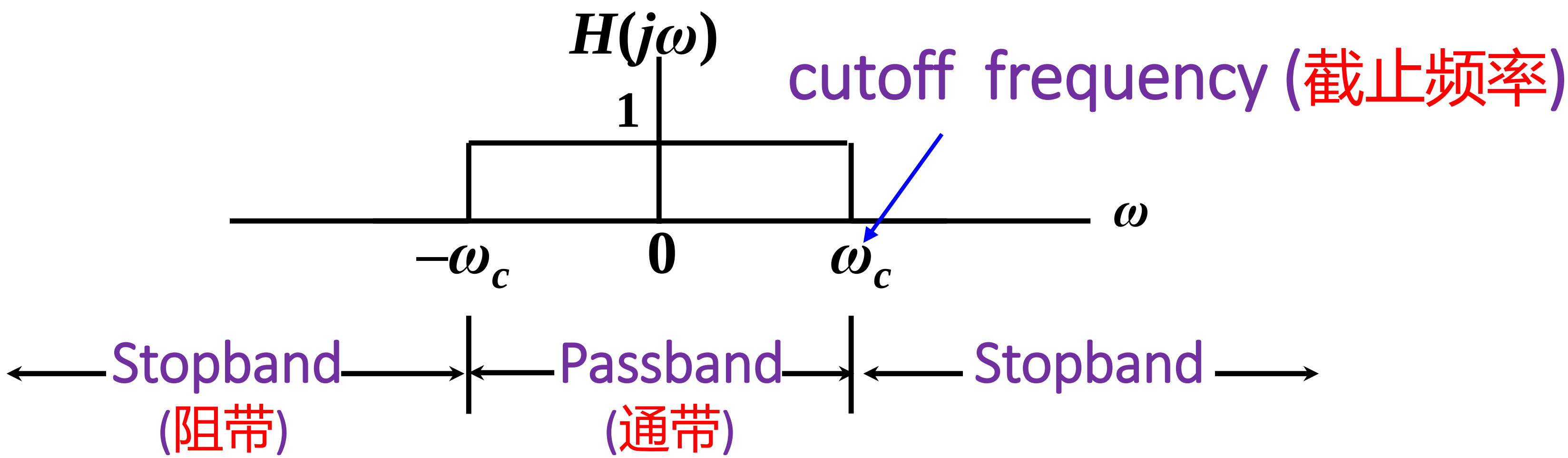
— High frequencies are attenuated

# 4.7 Frequency-Selective Filters

## 4.7.1 Introduction to Ideal Frequency-Selective Filters

- *Filtering*: a process in which the relative amplitudes of the frequency components in a signal are changed or some frequency components are eliminated entirely.
- *Frequency-selective filters* (频选滤波器): systems that are designed to pass some frequencies essentially undistorted and significantly attenuate or eliminate others.
- Types of frequency-selective filters
  - ✓ low-pass filter (低通滤波器)
  - ✓ high-pass filter (高通滤波器)
  - ✓ band-pass filter (带通滤波器)
  - ✓ band-stop filter (带阻滤波器)

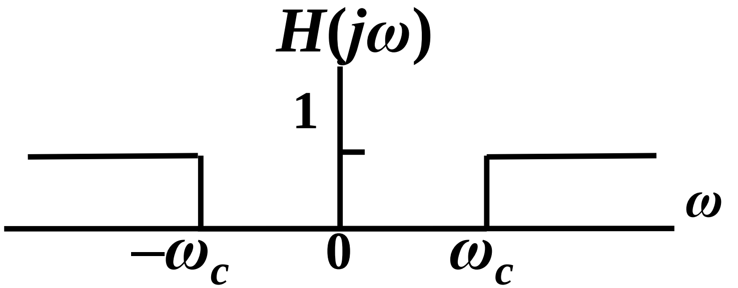
The frequency responses of a *zero-phase ideal* low-pass filter, a *zero-phase ideal* high-pass filter, a *zero-phase ideal* band-pass filter and a *zero-phase ideal* band-stop filter are illustrated in the following figures, respectively:

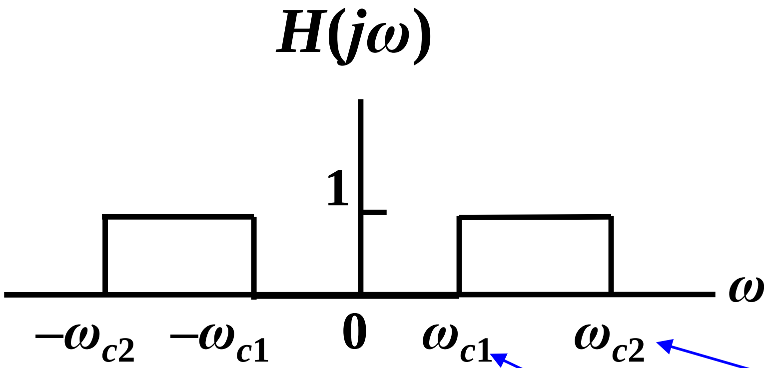


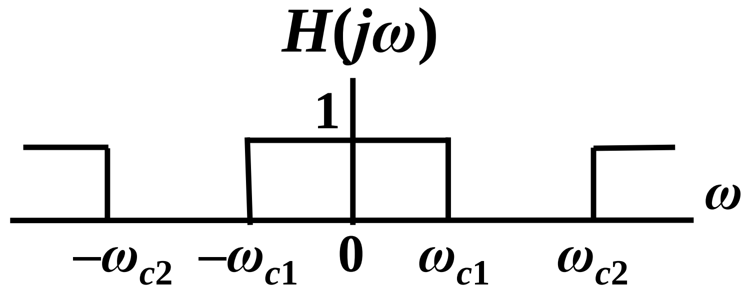

$$H(j\omega)=\begin{cases}1, & |\omega|\le\omega_c\\ 0, & |\omega|>\omega_c\end{cases}$$

# 4.7 Frequency-Selective Filters

$$H(j\omega)=\begin{cases}1, & |\omega|\geq\omega_c\\ 0, & |\omega|<\omega_c\end{cases}$$




$$H(j\omega)=\begin{cases}1, & \omega_{c_2}\geq|\omega|\geq\omega_{c_1}\\ 0, & |\omega|<\omega_{c_1}\ \text{or}\ |\omega|>\omega_{c_2}\end{cases}$$

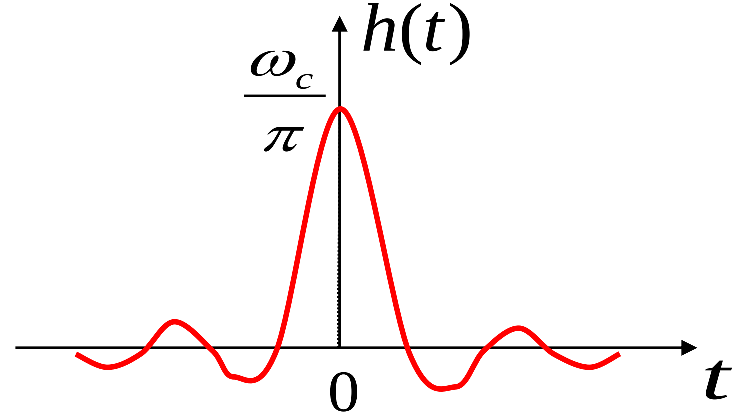
upper cutoff frequency
(上截止频率)

lower cutoff frequency
(下截止频率)

# 4.7 Frequency-Selective Filters

- The impulse response of the ideal low-pass filter is:

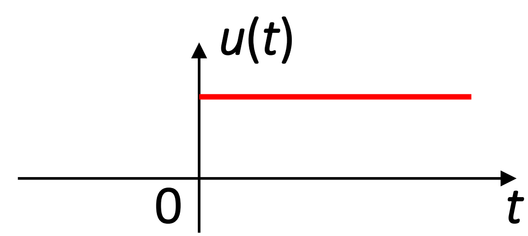
$$h(t) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t} d\omega = \frac{\sin \omega_c t}{\pi t}$$




- The step response is:

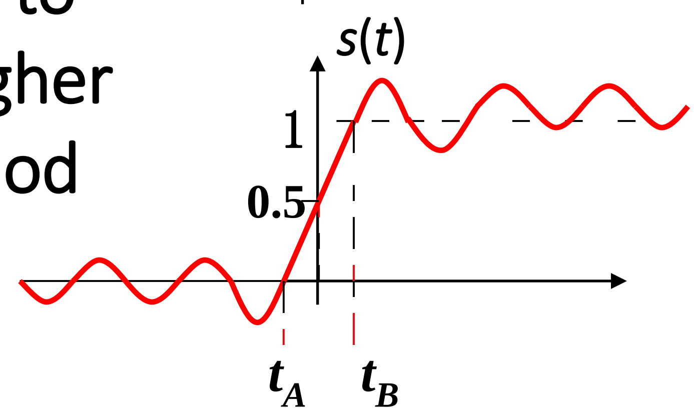
$$s(t) = \frac{1}{2} + \frac{1}{\pi} Si(\omega_c t)$$

Here, $Si(\omega_c t) = \int_0^{\omega_c t} \frac{\sin x}{x} dx$




Different from the input signal — $u(t)$, output signal $s(t)$ needs a period of time to rise from 0 to 1, because frequencies higher than $\omega_c$ in $u(t)$ are rejected. And the period

$$t_r = t_B - t_A = \frac{3.84}{\omega_c}$$



- The frequency response of the ideal high-pass filter can be represented in terms of the frequency response of the low-pass filter as:

$$H_h(j\omega)=\begin{cases}1, & |\omega|\geq\omega_c \\ 0, & |\omega|<\omega_c\end{cases}=1-\begin{cases}1, & |\omega|\leq\omega_c \\ 0, & |\omega|>\omega_c\end{cases}=1-H_l(j\omega)$$

Thus, the impulse response of the ideal high-pass filter is:

$$h(t)=\delta(t)-\frac{\sin\omega_c t}{\pi t}$$

Clearly *h*(*t*) is not causal, so the ideal high-pass filter cannot be implemented.

## 4.7.2 Realizable Systems and Binding Characteristic of $H(j\omega)$

- A physically realizable system must have its impulse response $h(t)$ satisfy $h(t) = h(t)\cdot u(t)$ (sufficient and necessary condition) – Time Domain condition
- Imposing causality on the system restricts the $H(j\omega)$ in significant ways. The *Paley-Wiener Criterion* says if $|H(j\omega)|^2$ is integrable, i.e., $\int_{-\infty}^{\infty} |H(j\omega)|^2 d\omega < \infty$, it can be proved that the necessary condition for a realizable $|H(j\omega)|$ (causal system) is:

$$\int_{-\infty}^{\infty} \frac{\left|\ln|H(j\omega)|\right|}{1+\omega^2} d\omega < \infty \qquad \text{(I)}$$

✓To satisfy above condition, $H(j\omega)$ cannot be zero in any band of frequencies (although it can be zero at a finite number of frequencies).

✓ The decay rate of $|H(j\omega)|$ cannot be greater than that of exponentials.

Consider the causality of system with $|\boldsymbol{H}(\boldsymbol{j\omega})| = \boldsymbol{e}^{-|\boldsymbol{\omega}|}$ ,

$$\lim_{B\to\infty}\int_{-B}^{B}\frac{\left|\ln|H(j\omega)|\right|}{1+\omega^2}d\omega=\lim_{B\to\infty}\int_{-B}^{B}\frac{\left|\ln e^{-|\omega|}\right|}{1+\omega^2}d\omega=\lim_{B\to\infty}\int_{-B}^{B}\frac{|\omega|}{1+\omega^2}d\omega$$

$$=2\times\lim_{B\to\infty}\int_{0}^{B}\frac{\omega}{1+\omega^2}d\omega=\lim_{B\to\infty}\ln(1+\omega^2)\Big|_0^B\to\infty$$

This shows that exponential magnitude response $|\boldsymbol{H}(\boldsymbol{j\omega})| = \boldsymbol{e}^{-|\boldsymbol{\omega}|}$ does not satisfy the equation (**I**). Consequently, a system with a magnitude response function which decays faster than exponential must be non-causal so that cannot be implemented.

- It can be proved that a magnitude response function $|H(j\omega)|$ that is composed of rational polynomials can satisfy equation (**I**).

## 4.7 Frequency-Selective Filters

- Because of the causality restriction ($h(t)=h(t)\cdot u(t)$) there is some kind of mutually binding character between the real and imaginary parts or the magnitude and phase of $H(j\omega)$. Specifically, if let

$$H(j\omega)=H_R(j\omega)+jH_I(j\omega)$$

then

$$H_R(j\omega)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H_I(j\eta)}{\omega-\eta}d\eta \quad \text{(II)}$$

$$H_I(j\omega)=-\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{H_R(j\eta)}{\omega-\eta}d\eta \quad \text{(III)}$$

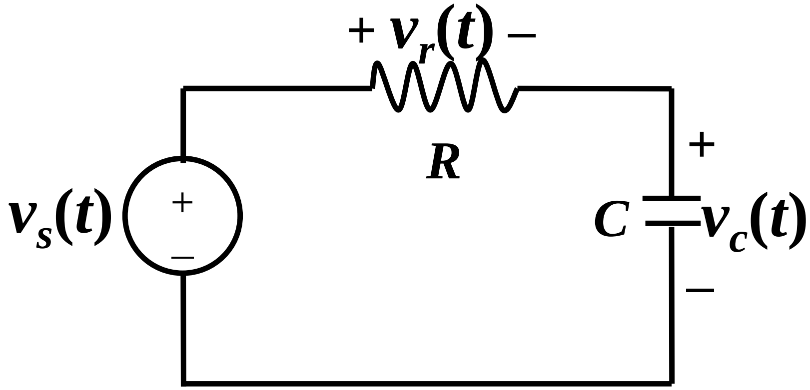
Equations (II) and (III) are referred to as *Hilbert Transform pair*.

*Conclusion*: The real and imaginary parts of the transform of a real, causal impulse response $h(t)$ can be determined from one another using the *Hilbert Transform*. (Problems 4.47 and 4.48) So do the phase and logarithm of the magnitude of $H(j\omega)$.

## 4.7.3 A Real Simple *RC* Low-pass Filter with rational $H(j\omega)$

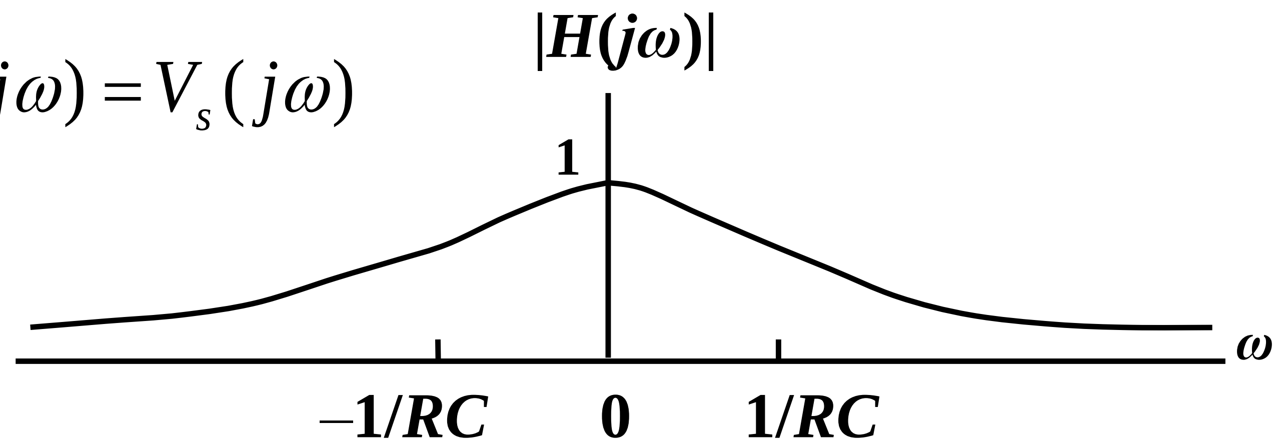
Input: source voltage $v_s(t)$;

Output: capacitor voltage $v_c(t)$




$$RC\frac{dv_c(t)}{dt} + v_c(t) = v_s(t)$$

$$RCj\omega V_c(j\omega) + V_c(j\omega) = V_s(j\omega)$$



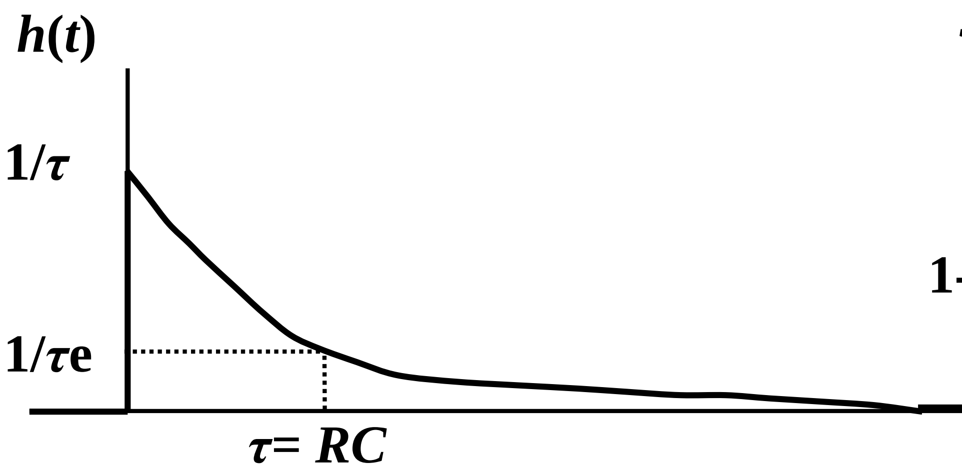

$$H_{lp}(j\omega) = \frac{V_c(j\omega)}{V_s(j\omega)}$$

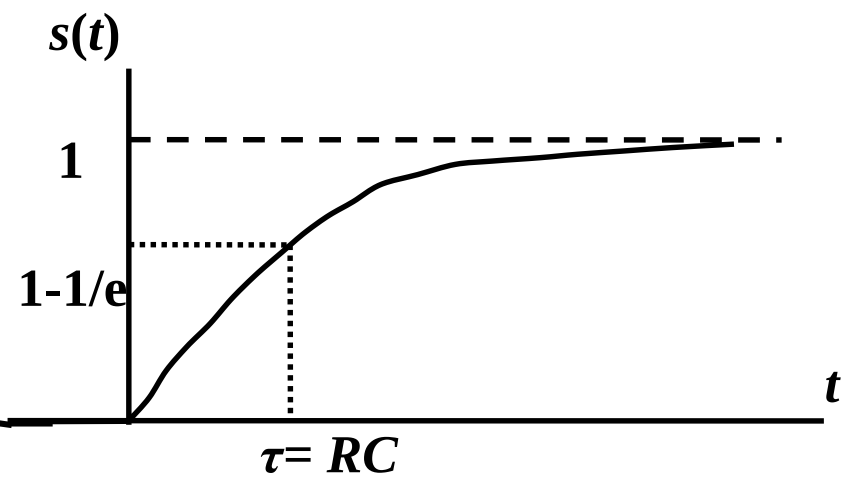
$$= \frac{1}{1+RCj\omega} = \frac{1/RC}{1/RC + j\omega}$$

# 4.7 Frequency-Selective Filters

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

$$s(t) = \left[1 - e^{-t/RC}\right] u(t)$$



## 4.7.4 A Real Simple *RC* High-pass Filter with rational $H(j\omega)$

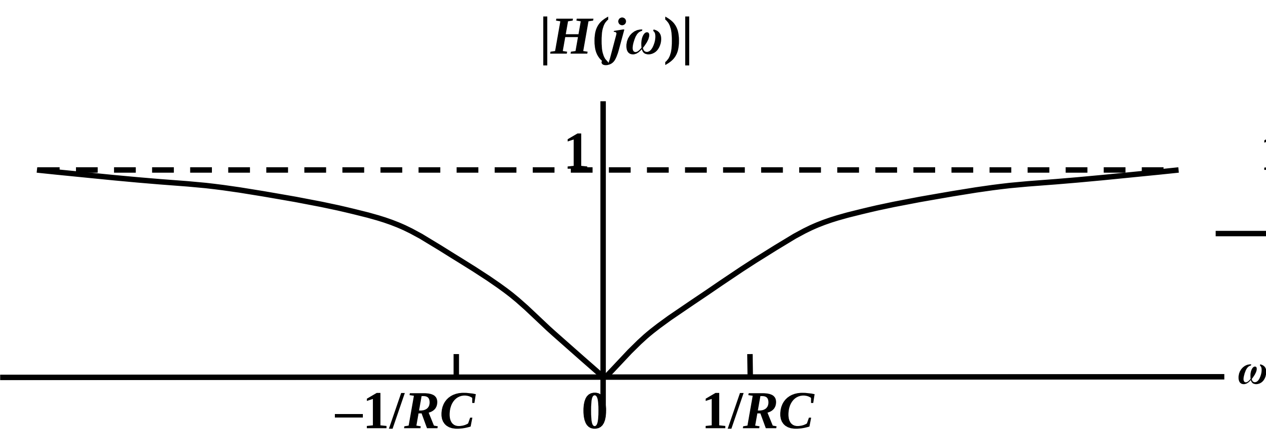
Input: source voltage $v_s(t)$; Output: resistor voltage $v_r(t)$

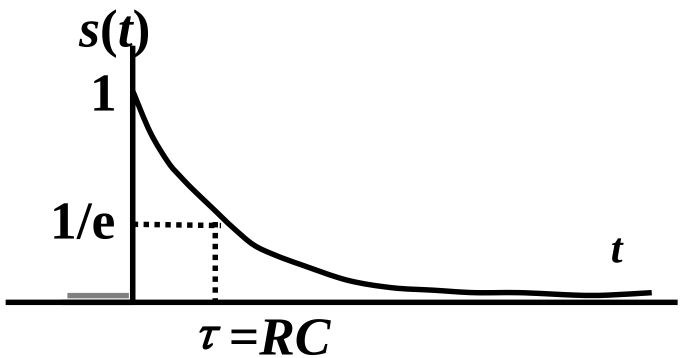
$$RC\frac{dv_r(t)}{dt}+v_r(t)=RC\frac{dv_s(t)}{dt}$$

$$H_{hp}(j\omega)=\frac{j\omega RC}{1+j\omega RC}=1-\frac{1/RC}{1/RC+j\omega}$$

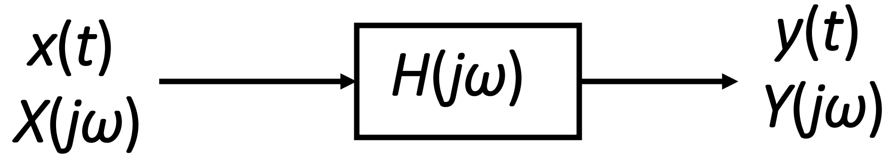
$$s(t)=e^{-t/RC}u(t)$$

Non-ideal high-pass filter



# 4.8 Transmission without Distortion




If the waveform of $y(t)$ is different from that of $x(t)$, distortion occurs.

- Linear systems can only introduce *linear distortion*, in which there isn't new frequencies generated.
- We can see from $\boldsymbol{Y(j\omega) = |H(j\omega)||X(j\omega)|e^{j(argH(j\omega)+argX(j\omega))}}$ that linear distortions include *magnitude distortion* and *phase distortion*.

In the case of $y(t)=Kx(t-t_0)$, $x(t)$ is transmitted without distortion.

Applying Fourier transform we have $Y(j\omega)=KX(j\omega)e^{-j\omega t_0}$

$$H(j\omega)=Ke^{-j\omega t_0} \qquad or \qquad \begin{cases} |H(j\omega)|=K \\ \measuredangle H(j\omega)=-\omega t_0 \end{cases}$$

# 4.8 Transmission without Distortion

Suppose $x(t) = E_1 \sin \omega_0 t + E_2 \sin 2\omega_0 t$

Then

$$y(t) = KE_1 \sin(\omega_0 t + \varphi_1) + KE_2 \sin(2\omega_0 t + \varphi_2)$$

$$= KE_1 \sin \omega_0 (t + \frac{\varphi_1}{\omega_0}) + KE_2 \sin 2\omega_0 (t + \frac{\varphi_2}{2\omega_0})$$

To guarantee undistorted,

$$\frac{\varphi_1}{\omega_0} = \frac{\varphi_2}{2\omega_0} = const \quad = -t_0$$

Consider non-periodic inputs case, we can write without loss of generality that

$$\varphi(\omega) = -\omega t_0 = \sphericalangle H(j\omega)$$

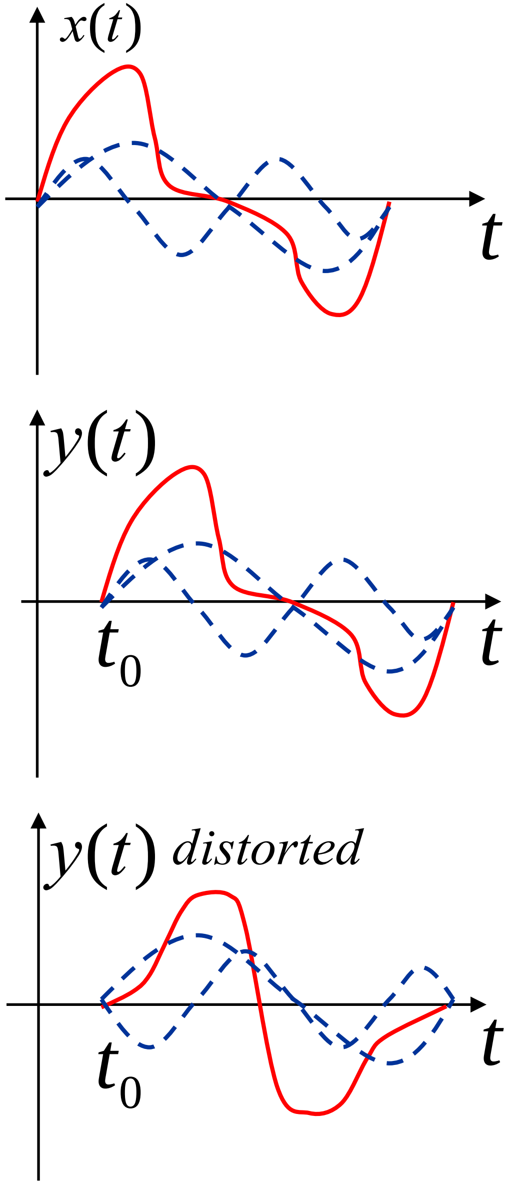
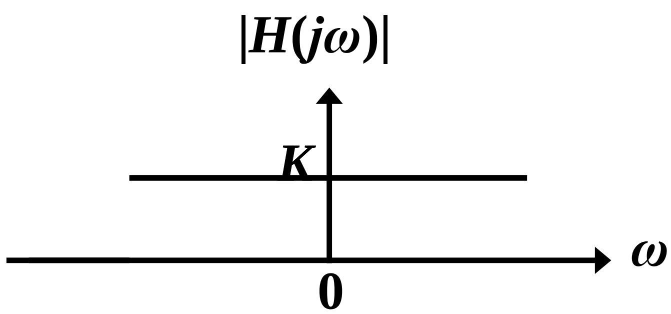
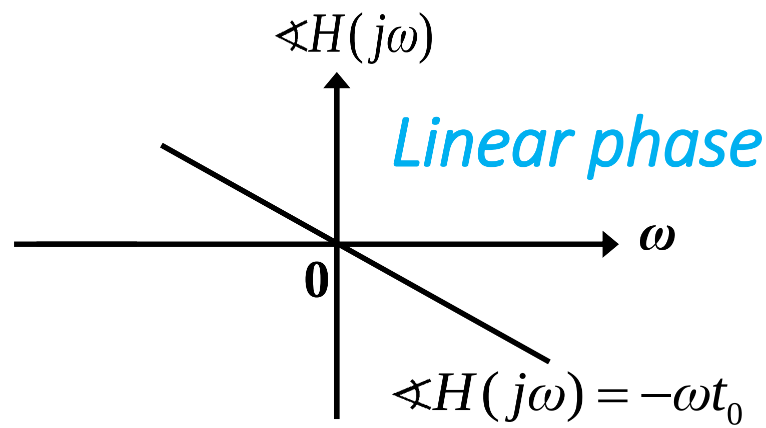



Illustration of phase distortion

# 4.8 Transmission without Distortion

- Conclusions: a linear system that can transmit signals applying to it as input without distortion must have a constant magnitude response and a phase response directly proportional to frequencies.




Linear phase shifts lead to very simple and easily understood change in a signal.

$$h(t) = K\delta(t - t_0)$$

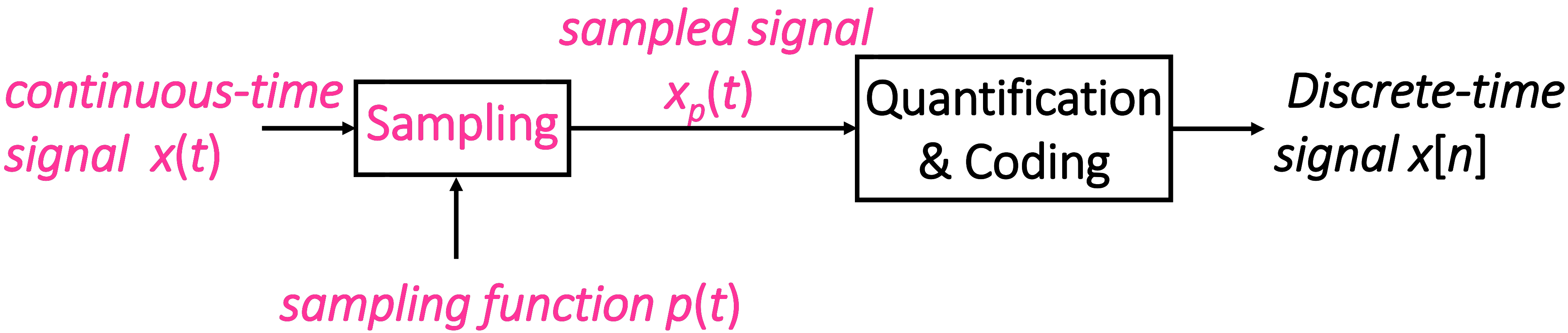
*Group Delay*: $\tau(\omega) = -\dfrac{d\{\sphericalangle H(j\omega)\}}{d\,\omega}$

Systems with this impulse response is ideal!

More about this refer to Section 6.2.

## 4.9 Sampling

Question: Why do we need to study *sampling*?

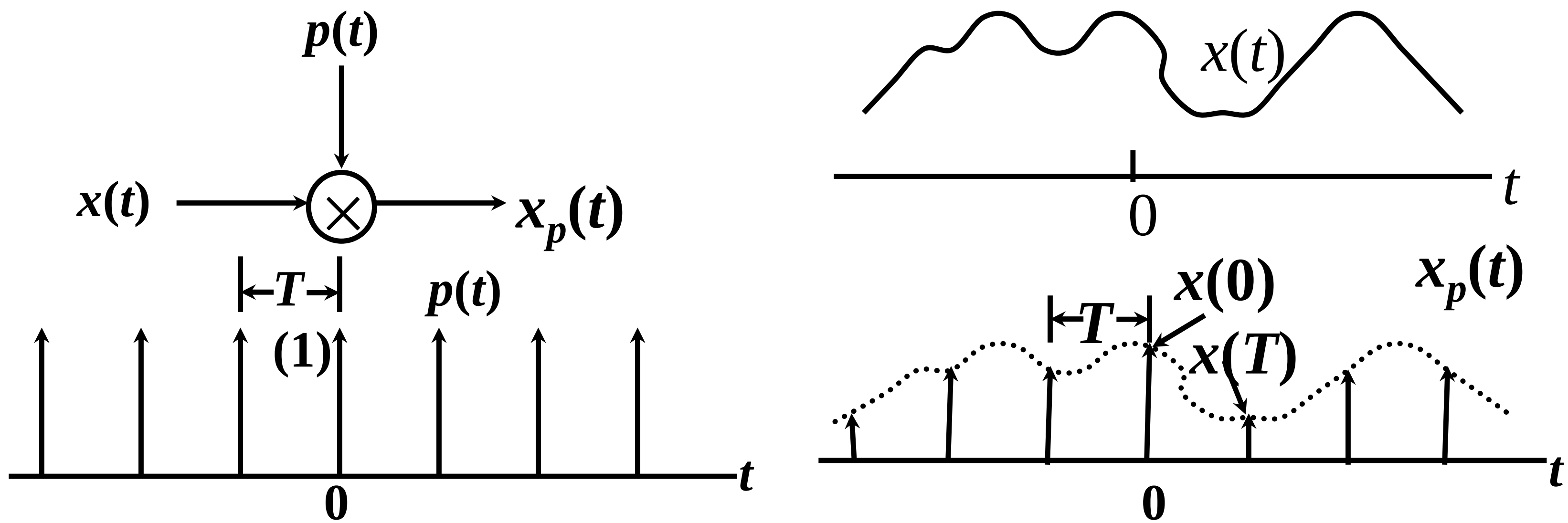



To make sure that $x[n]$ keep most information in $x(t)$, or $x(t)$ can recovered from $x[n]$, two questions need to be answered:

1. What is the Fourier transform of $x_p(t)$? How is it related to that of $x(t)$ ?

2. How to recover $x(t)$ from $x_p(t)$?

## 4.9.1 Impulse-train sampling (冲激串采样)



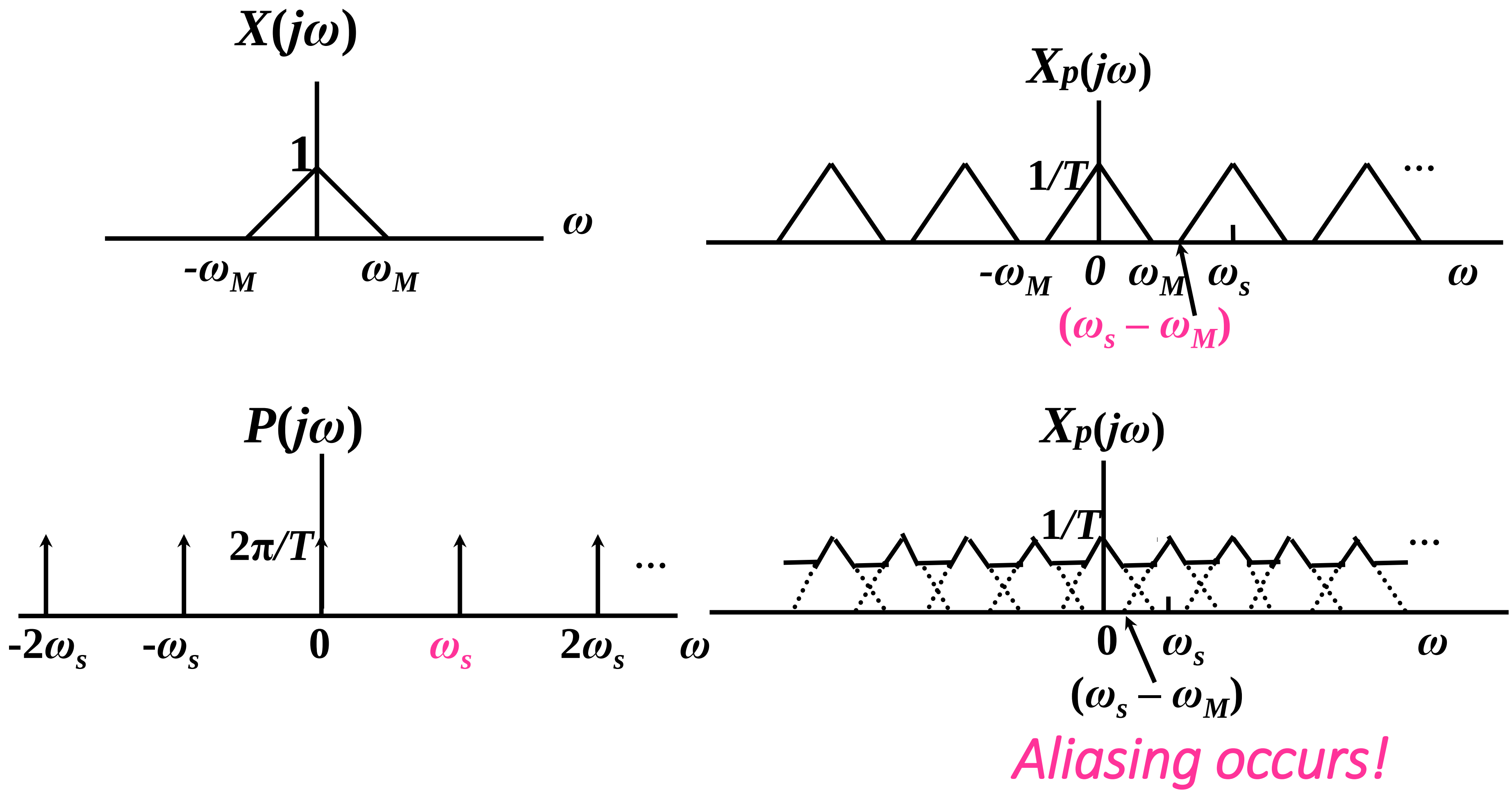

Mechanism of Impulse-train sampling

$$x_p(t) = x(t)\cdot\sum_{n=-\infty}^{\infty}\delta(t-nT) \quad X_p(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * P(j\omega)\right] \quad P(j\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - k\omega_s)$$

$$\boxed{X_p(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))}$$

Answer for Q1: $X_p(j\omega)$ is a periodic function of $\omega$ consisting of a superposition of shifted replicas of $X(j\omega)$, scaled by $\frac{1}{T}$.

# 4.9 Sampling




Effect in the frequency domain of sampling in the time domain

## 4.9.2 Sampling Theorem (采样定理)

Let $x(t)$ be a band-limited signal with $X(j\omega)=0$ for $|\omega|>\omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, n=0, ±1, ±2, ..., if

$$\omega_s > 2\omega_M$$

where

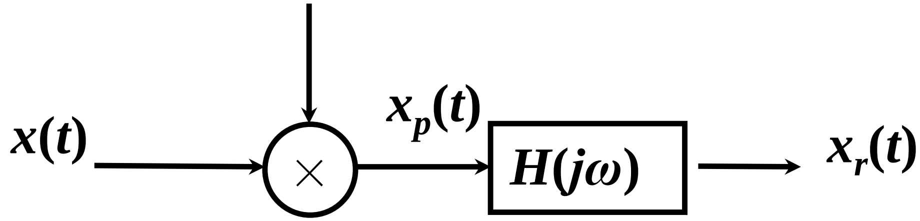
$$\omega_s = \frac{2\pi}{T}$$

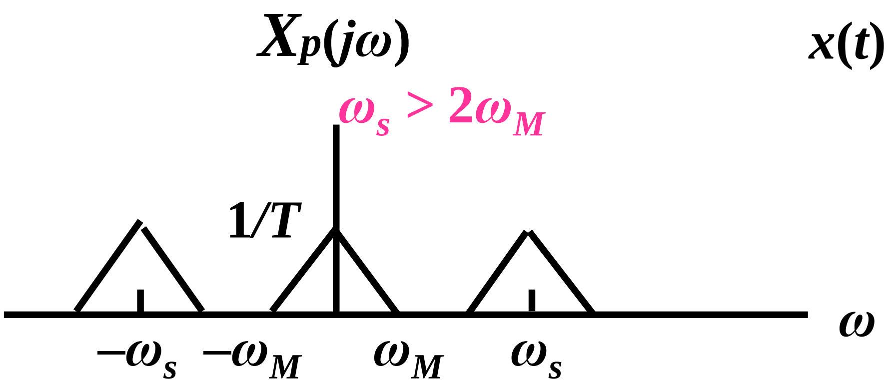
***Nyquist frequency***
(奈奎斯特频率)

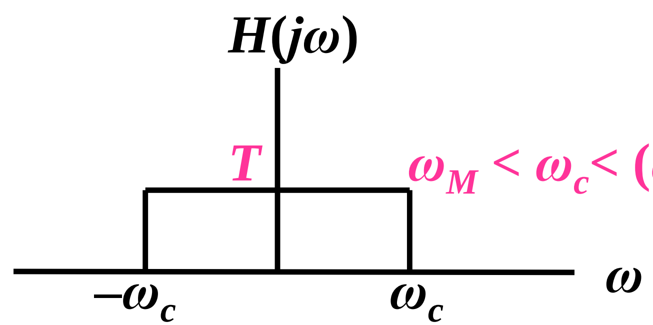
Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal low-pass filter with gain $T$ and cutoff frequency greater than $\omega_M$ and less than $\omega_s - \omega_M$. The resulting output signal will exactly equal $x(t)$.

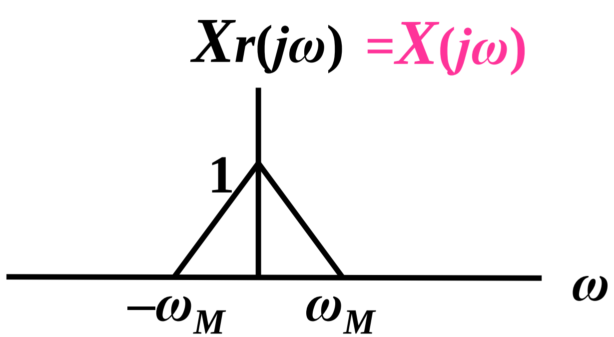
# 4.9 Sampling

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



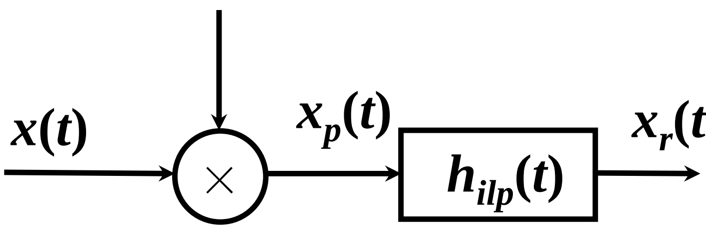

Exact recovery of a continuous-time signal from its samples using an ideal low-pass filter

## 4.9.3 Reconstruction of A Signal From Its Samples Using Interpolation

➢*Interpolation* (插值): a procedure in which the fitting (拟合) of a continuous-time signal to a set of sample values. It is commonly used for reconstruction of a function from samples.

➢**Ways to interpolate: zero-order hold interpolation、 linear interpolation、 band-limited interpolation …**

$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$$




$$x_r(t)=x_p(t)*h(t)=\sum_{n=-\infty}^{\infty}x(nT)h(t-nT)$$

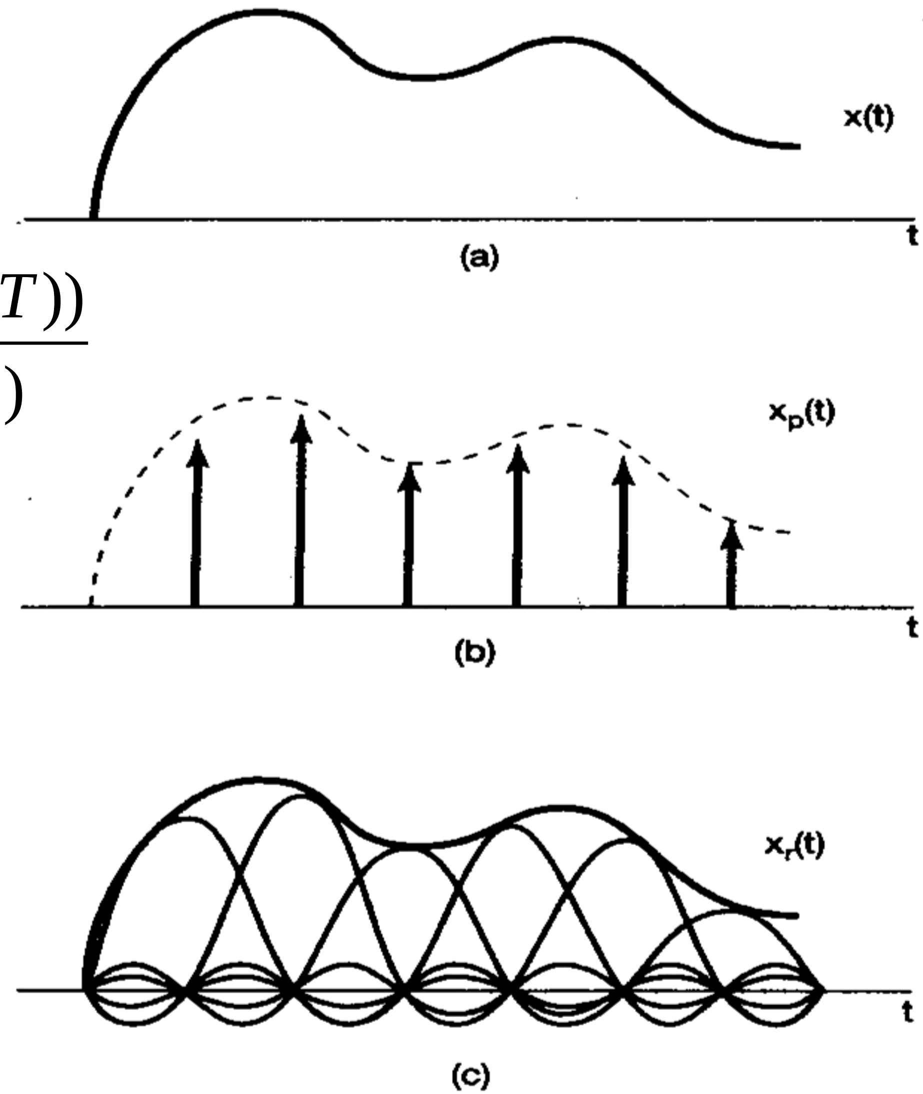
$$h(t)=\frac{\omega_c T\sin(\omega_c t)}{\pi\omega_c t}$$

$$\boxed{x_r(t)=\sum_{n=-\infty}^{\infty}x(nT)\frac{\omega_c T}{\pi}\frac{\sin(\omega_c(t-nT))}{\omega_c(t-nT)}}$$

*Interpolation formula*

# 4.9 Sampling

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{\omega_c T}{\pi} \frac{\sin(\omega_c (t - nT))}{\omega_c (t - nT)}$$




Ideal band-limited interpolation using the *sinc* function with $\boldsymbol{\omega_c = \frac{\omega_s}{2}}$

## 4.9.4 The Effect of Under-Sampling（欠采样): Aliasing(混叠)

$$x(t) = \cos \omega_0 t,$$

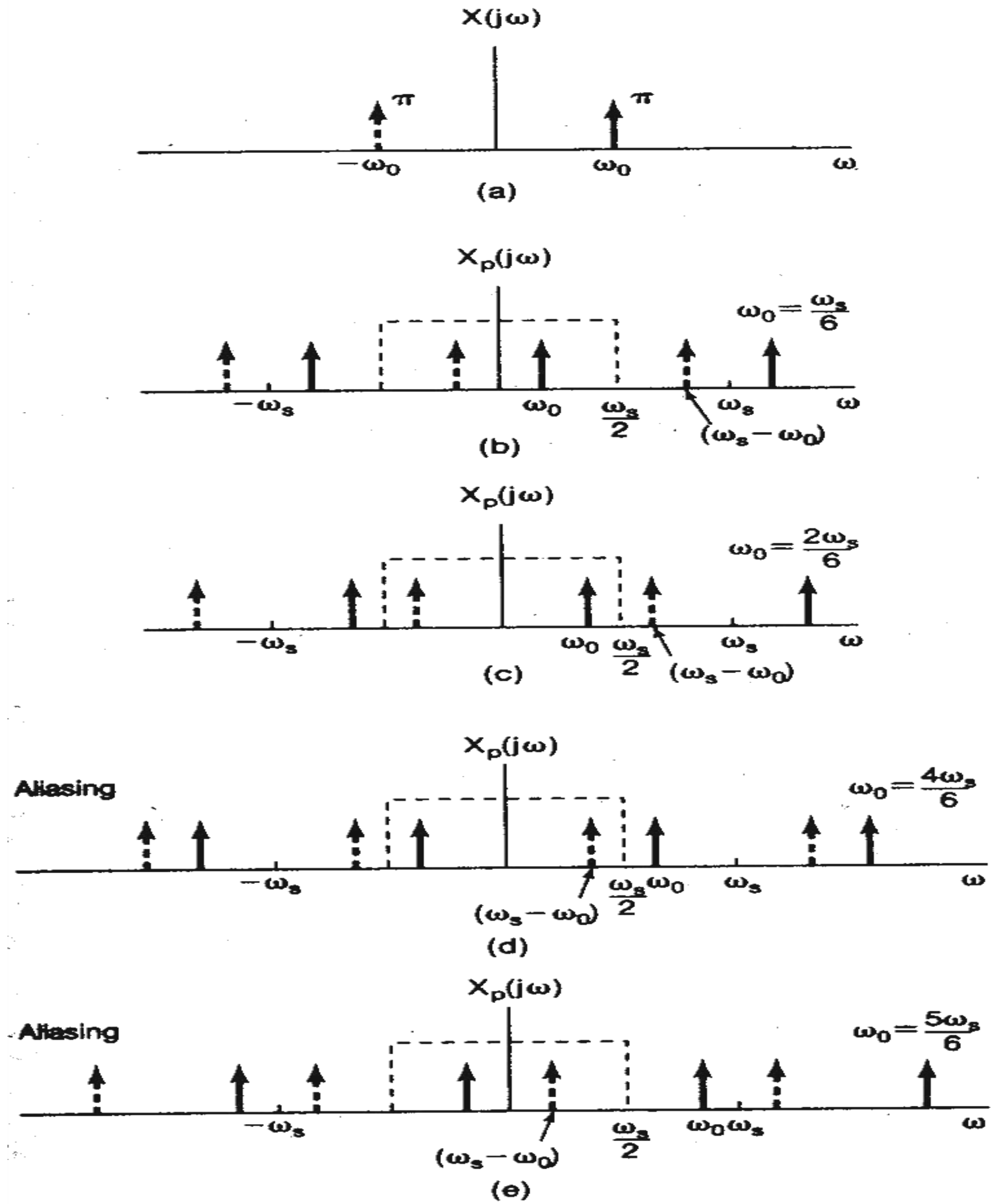
$$x(t) = \cos 1000t$$

is sampled with

$$\omega_s = 1500 rad/s$$

and $\omega_s = 1200 rad/s$

respectively.



# 4.10 SUMMARY

- The Fourier transform for both non-periodic and periodic continuous-time signals;
- The properties of the Fourier transform (relationships between characteristics of a continuous-time signal in time and frequency domain);
- Fourier analysis (Frequency domain analysis) for continuous-time LTI systems including both characteristics of systems and responses to some input signals;
- Frequency response and the way to obtain it;
- Continuous-time signals’ sampling and their reconstruction.

# Homework

4.21 (b)(c) (d) (h)     4.22 (a) (b) (c) (d)

4.24     4.28 (a) 、(i)(iv)(vi)(viii) in (b)

4.32 (b) (c)     4.34

4.35     7.22     7.23