练习十一.

1. 通解为 $u(x,t)=f(x-at)+g(x+at)$

由边界条件

$f(-at)+g(at)=\phi(t)$

$f(2x)+g(0)=\psi(t)$

$\therefore f(\eta)=\psi(-\frac{\eta}{2a})-g(0)$

$g(\eta)=\phi(\frac{\eta}{a})-\psi(\frac{\eta}{2a})+g(0)$

$\therefore u(x,t)=\psi(\frac{t}{2}-\frac{x}{2a})+\phi(\frac{x+at}{a})-\psi(\frac{x+at}{2a})$

2. 对t取拉氏变换.

$L[u(x,t)] = U(x,s)$.

则 $\frac{a^2 d^2 U(x,s)}{dx^2} - sU = 0$

$U(0,s) = \frac{u_0}{s}$

$U(x,0) = 0$

通解为 $U(x,s) = C_1 e^{-\frac{\sqrt{s}}{a}x} + C_2 e^{\frac{\sqrt{s}}{a}x}$

由有界知 $C_2 = 0$

$U(0,s) = C_1 = \frac{u_0}{s}$

$\therefore U(x,s) = \frac{u_0}{s} e^{-\frac{\sqrt{s}}{a}x}$

取拉氏逆变换

$u(x,t) = u_0 * \frac{x}{2a\sqrt{\pi} t^{\frac{3}{2}}} e^{-\frac{x^2}{4a^2 t}}$

三、作拉氏变换

$$L[u(x,t)]=U(x,s)$$

则
$$\begin{cases} a^2\dfrac{d^2U}{ds^2}-s^2U=0 \\ U(0,s)=\dfrac{A\omega}{s^2+\omega^2},\quad \lim\limits_{x\to\infty}U(x,s)\le \bar{M} \end{cases}$$

通解为 $U(x,s)=C_1e^{\frac{s}{a}x}+C_2e^{-\frac{s}{a}x}$

由有界 $C_1=0$

$$U(0,s)=C_2=\frac{A\omega}{s^2+\omega^2}$$

$$\therefore U(x,s)=\frac{A\omega}{s^2+\omega^2}\cdot e^{-\frac{s}{a}x}$$

$$\therefore u(x,t)=A\sin\omega\left(t-\frac{x}{a}\right)\quad \left(t>\frac{x}{a}\right)$$

$$\therefore u(x,t)=\begin{cases} 0 & 0<t<\dfrac{x}{a} \\ A\sin\omega\left(t-\dfrac{x}{a}\right) & t>\dfrac{x}{a} \end{cases}$$

练习十二.

1. 对x进行傅式变换

记 $F[u(x,t)]=\bar{u}(\lambda,t)$.

则 $\dfrac{d\bar{u}(\lambda,t)}{dt}=-a^2\lambda^2\bar{u}(\lambda,t)$

$\bar{u}(\lambda,0)=\pi[\delta(\lambda-1)+\delta(\lambda+1)]$

解得 $\bar{u}(\lambda,t)=C_1e^{-a^2\lambda^2t}$

代入得 $C=\pi[\delta(\lambda-1)+\delta(\lambda+1)]$

$\therefore\ \bar{u}(\lambda,t)=\pi[\delta(\lambda-1)+\delta(\lambda+1)]e^{-a^2\lambda^2t}$

取逆变换得 $u(x,t)=\cos t*\dfrac{1}{\sqrt{4\pi a^2t}}e^{-\frac{x^2}{4a^2t}}$

$$2,\ u(x,y,z,t)=\frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}(y+t\sin\theta\sin\varphi)(z+t\cos\varphi)\sin\theta\, d\theta\, d\varphi\right)$$

$$+\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi}(x+t\sin\theta\cos\varphi)(z+\cos\varphi)\sin\theta\, d\theta\, d\varphi$$

$$=\frac{\partial}{\partial t}\left(\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} yz\sin\theta\, d\theta\, d\varphi\right)+\frac{t}{4\pi}\int_0^{2\pi}\int_0^{\pi} xz\sin\theta\, d\theta\, d\varphi$$

$$=yz+xzt$$

$$3,\ u(x,y,t)=\frac{1}{2\pi a}\frac{\partial}{\partial t}\left[\int_0^{at}\int_0^{2\pi}\frac{(x+\rho\cos\theta)^3+(x+\rho\cos\theta)^2(y+\rho\sin\theta)}{\sqrt{a t^2-\rho^2}}\rho\, d\theta\, d\rho\right.$$

$$=x^2(x+y)+a^2t^2(3x+y)$$