CHAPTER 5

THE DISCRETE-TIME FOURIER TRANSFORM

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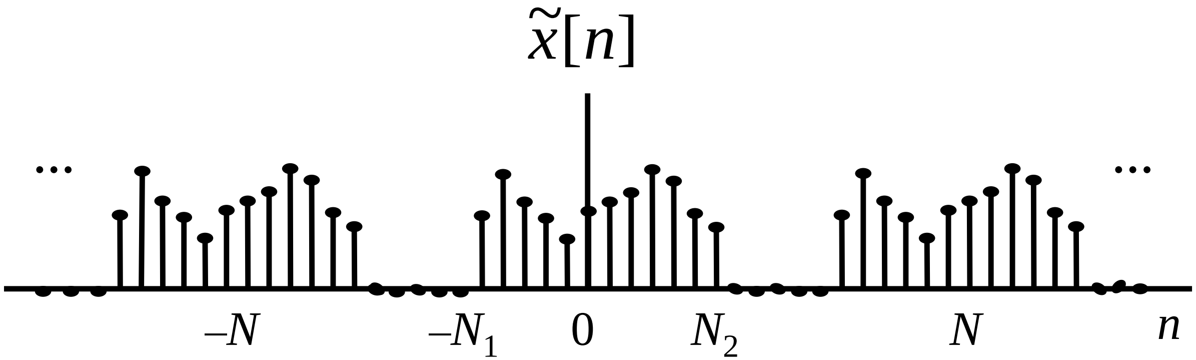
5.0 Introduction

- Discrete-time Fourier transform (DTFT) and inverse Fourier transform
- Application of DTFT in discrete-time LTI systems analysis
- Similarities and differences between continuous-time and discrete-time Fourier transforms

5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

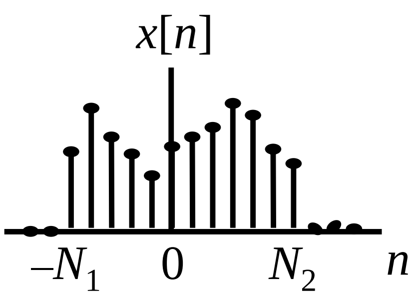
5.1.1 Fourier Transform and Inverse Fourier Transform



$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

As $N \to \infty$,



$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} \tilde{x}[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk(2\pi/N)n}$$

$$= \frac{1}{N} \sum_{n=-\infty}^{\infty} x[n] e^{-jk(2\pi/N)n}$$

Defining a function $$\boxed{X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}}$$

We see that $a_k = \frac{1}{N} X(e^{jk\omega_0})$

discrete-time Fourier transform

5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

Taking the new expression for a_k into synthesis equation yields

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

As $N \to \infty$,

$$\boxed{x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega}$$

discrete-time inverse Fourier transform

$$\omega = k\omega_0 = k \cdot \frac{2\pi}{N}, \quad As \quad N \to \infty,$$

$$for\ k = 0, \qquad k\omega_0 = 0$$

$$for\ k = N-1, \quad k \cdot \frac{2\pi}{N} \to 2\pi;$$

$$for\ k = -\frac{N}{2} (N\ even), \quad k \cdot \frac{2\pi}{N} = -\pi$$

$$for\ k = \frac{N}{2} - 1, \qquad k \cdot \frac{2\pi}{N} \to \pi$$

5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

- The synthesis equation is an integration over a *finite interval*.

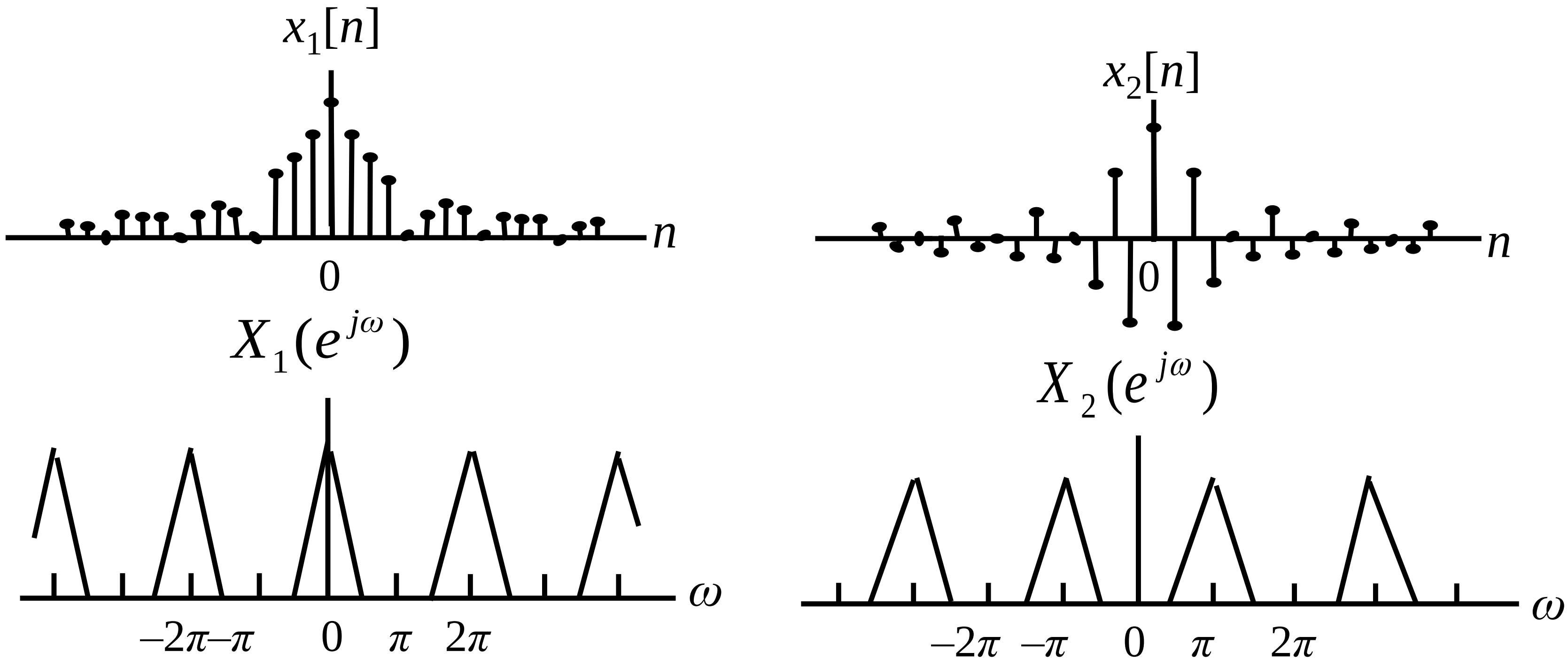
Two differences between CTFT and DTFT !

- As $N \to \infty$, $k\omega_0 \to \omega$, a continuous variable, so $X(e^{j\omega})$ is continuous as function of ω, and *periodic* with period 2π.

- From $a_k = \frac{1}{N} X(e^{j\omega})\Big|_{\omega=k\omega_0}$, a_k is proportional to the samples of $X(e^{j\omega})$, and ω_0 is the sampling period.

5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

Question: In discrete time what values of frequency can be referred to as high or low frequencies ($X(e^{j\omega})$ is periodic)?



- In discrete time,

Low frequencies are the values of ω near *even multiple of π* ;
high frequencies are those values of ω near *odd multiples of π*.

5.1 Representation of Aperiodic Signals: The Discrete-Time Fourier Transform

- Additional Information about frequencies in CTFT and DTFT:

$$x_p(t) = x(t) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

Applying FT on both sides yields

$$X_p(j\Omega) = \sum_{n=-\infty}^{\infty} x(nT)e^{-j\Omega \cdot nT}$$, Ω is analog frequency

Comparing with the analysis equation in DTFT leads to

$$\boxed{X(e^{j\omega})\Big|_{\omega=\Omega T} = X_p(j\Omega)}$$ $(\mathbf{I})$, ω is digital frequency

✓ $\omega = \Omega T$ tells that the unit of digital frequency ω is *rad* rather than *rad/sec*, which is the unit of analog frequency Ω.

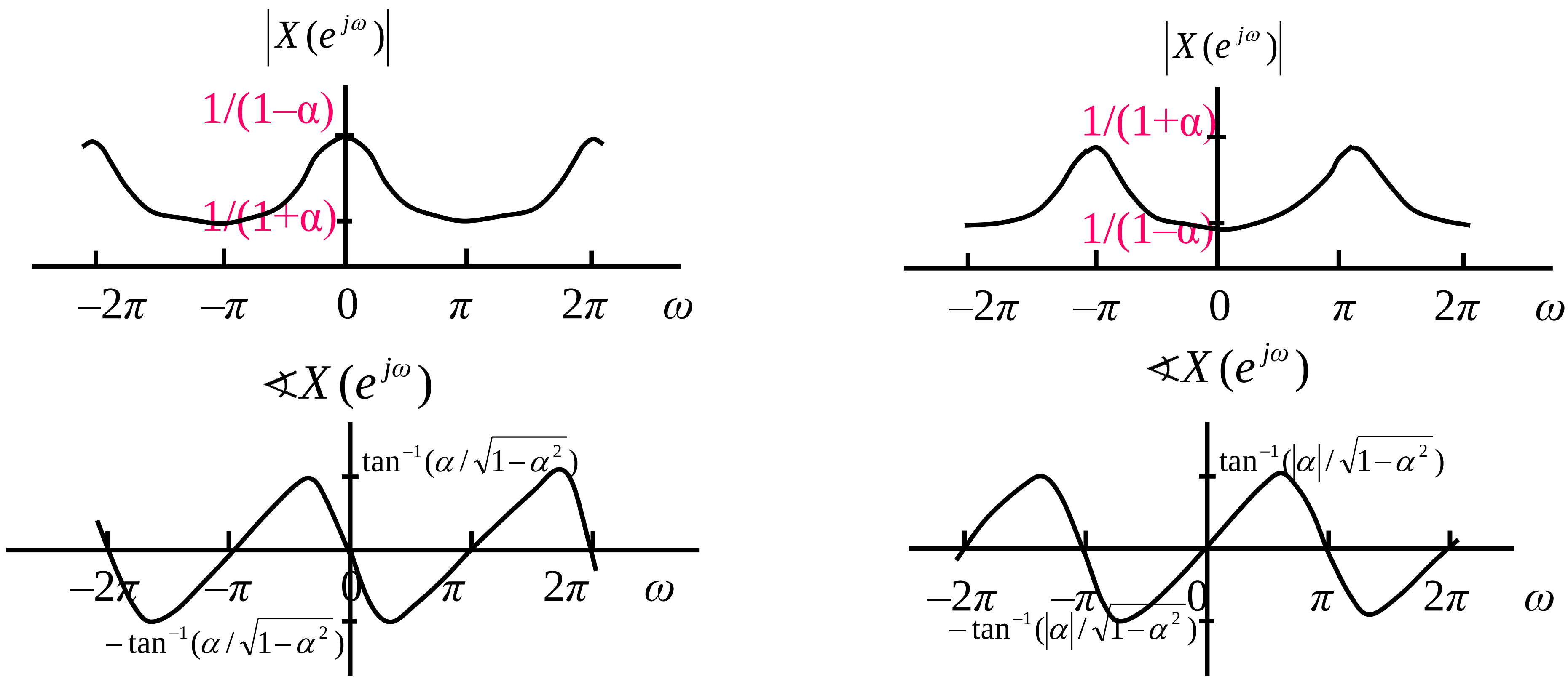
✓ Since $T = \frac{2\pi}{\Omega_s}$, ΩT maps Ω_s to 2π, and one period of $X_p(j\Omega)$ to an interval with length 2π (period).

✓ $\omega = \Omega T$ is called *frequency normalization*. Equation $(\mathbf{I})$ says $X(e^{j\omega})$ can be obtained by normalizing the frequency of $X_p(j\Omega)$.

5.1.2 Examples

Example 5.1 Consider the signal $x[n]=\alpha^n u[n], \qquad |\alpha|<1.$

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty}\alpha^n u[n]e^{-j\omega n}=\sum_{n=0}^{\infty}(\alpha e^{-j\omega})^n=\frac{1}{1-\alpha e^{-j\omega}}$$



$1>\alpha>0$ $\qquad$ $0>\alpha>-1$

Example 5.2

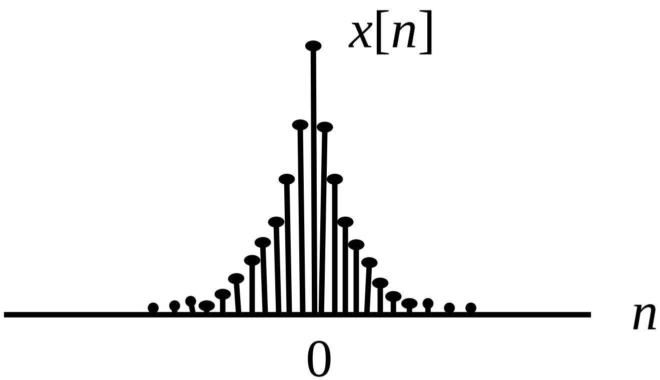
Consider the signal $x[n]=\alpha^{|n|}, \qquad |\alpha|<1.$

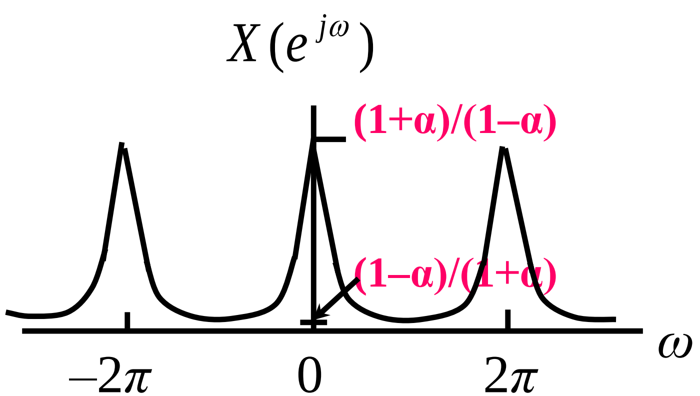
$$\text{Sol}: X(e^{j\omega})=\sum_{n=-\infty}^{\infty}\alpha^{|n|}e^{-j\omega n}=\sum_{n=0}^{\infty}\alpha^{n}e^{-j\omega n}+\sum_{n=-\infty}^{-1}\alpha^{-n}e^{-j\omega n}$$

$$X(e^{j\omega})=\sum_{n=0}^{\infty}(\alpha e^{-j\omega})^{n}+\sum_{m=1}^{\infty}(\alpha e^{j\omega})^{m}=\frac{1}{1-\alpha e^{-j\omega}}+\frac{\alpha e^{j\omega}}{1-\alpha e^{j\omega}}$$

$$=\frac{1-\alpha^2}{1-2\alpha\cos\omega+\alpha^2}$$

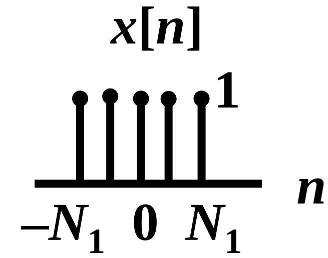
for $0<\alpha<1$,





Example 5.3

Consider the rectangular pulse $x[n]=\begin{cases}1, & |n|\le N_1\\ 0, & |n|>N_1\end{cases}$

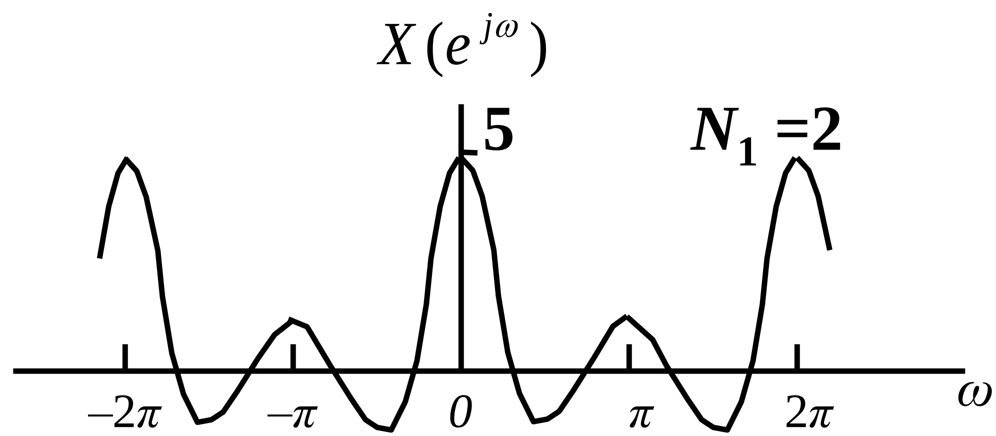


$$\text{Sol}: X(e^{j\omega})=\sum_{n=-N_1}^{N_1} e^{-j\omega n}=\frac{e^{j\omega N_1}(1-e^{-j\omega(2N_1+1)})}{1-e^{-j\omega}}$$

$$=\frac{e^{j\omega N_1}-e^{-j\omega N_1}e^{-j\omega}}{1-e^{-j\omega}}=\frac{e^{j\omega N_1}e^{j\omega(1/2)}-e^{-j\omega N_1}e^{-j\omega(1/2)}}{e^{j\omega(1/2)}-e^{-j\omega(1/2)}}$$

$$=\frac{2j\sin\omega(N_1+\frac{1}{2})}{2j\sin(\omega/2)}$$

$$\boxed{=\frac{\sin\omega(N_1+\frac{1}{2})}{\sin(\omega/2)}}$$



5.1.3 Convergence Issues of the Discrete-Time Fourier Transform

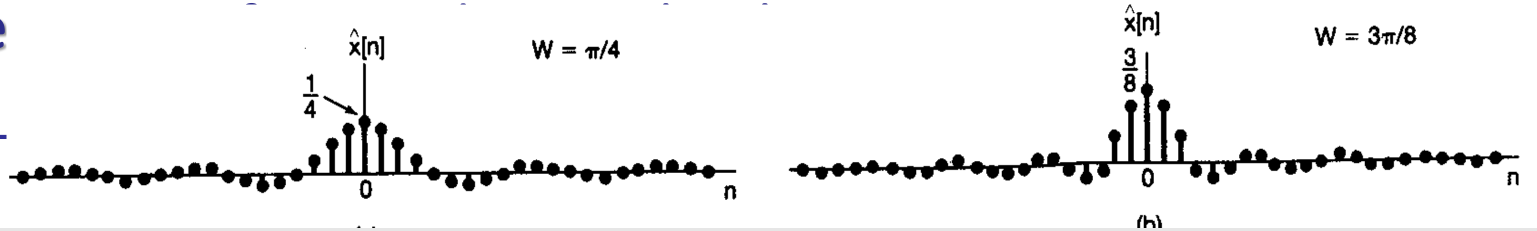
If $x[n]$ is an infinite duration signal, we must consider the question of convergence of the infinite summation in the analysis equation.

- The analysis equation will converge either if $x[n]$ is absolutely summable; that is $\sum_{n=-\infty}^{\infty} |x[n]| < \infty$ or if $x[n]$ has finite energy, that is $\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$.
- In contrast to the situation for the analysis equation, there are generally *no convergence issues* associated with the *synthesis equation*.

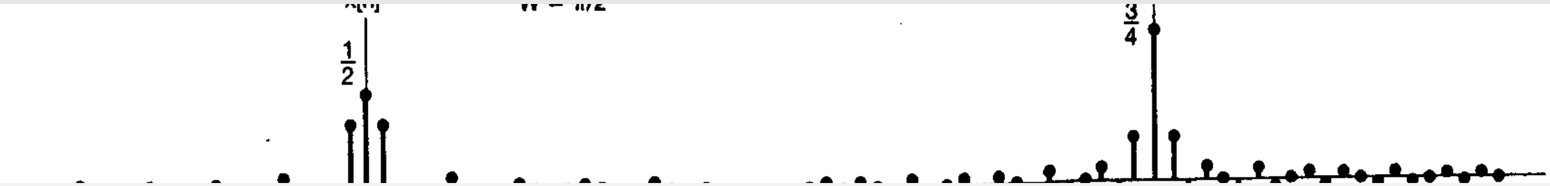
Consider the FT pair $\delta[n] \xleftrightarrow{FT} 1$

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} 1 \cdot e^{j\omega n} d\omega = \frac{\sin Wn}{\pi n}$$

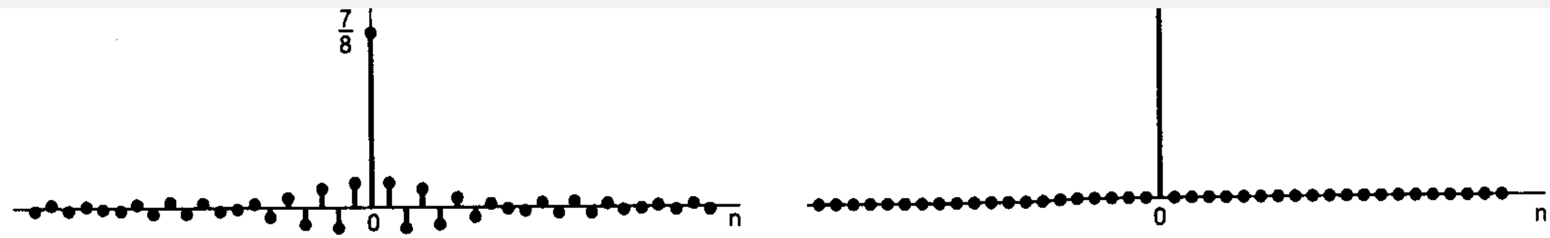
5.1 Represe Transform



- ✓ As *W* is increased, the oscillation frequency of the approximation also increase.



- ✓ As *W* is increased, the amplitude of these oscillations decreases relative to the magnitude of $\hat{x}[0]$, and the oscillations disappear entirely for *W* = π. No Gibbs phenomenon happens!



(e) (f)

Approximation to the unit sample using complex exponentials with frequencies $|\omega| \leq W$: (a) *W* = π/4; (b) *W* = 3π/8; (c) *W* = π/2; (d) *W* = 3π/4; (e) *W* = 7π/8; (f) *W* = π. Notice that for *W* = π, $\hat{x}[n] = \delta[n]$

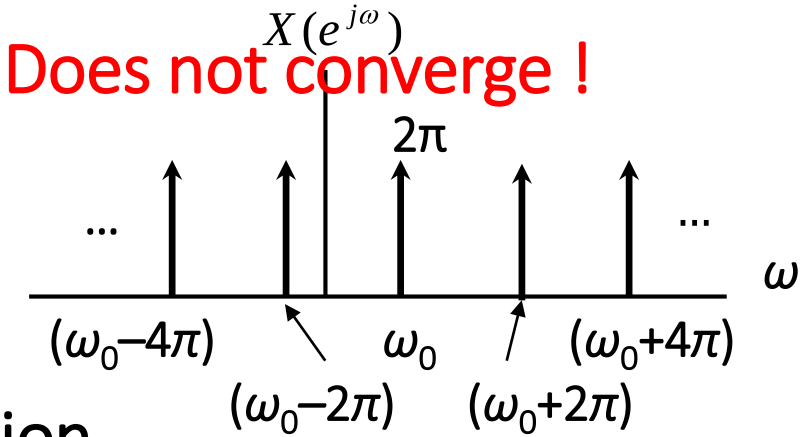
5.2 The Fourier Transform for Periodic Signals

First consider the Fourier transform of $x[n] = e^{j\omega_0 n}$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} e^{j\omega_0 n} e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{j(\omega_0-\omega)n}$$

Does not converge !

$$\boxed{X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi l)}$$



To check the validity of this expression,

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi l)e^{j\omega n}d\omega = e^{j(\omega_0+2\pi r)n} = e^{j\omega_0 n}$$

For an arbitrary periodic sequence $x[n]$ with period N and with the Fourier series representation $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$.

The Fourier transform is $$\boxed{X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)}$$

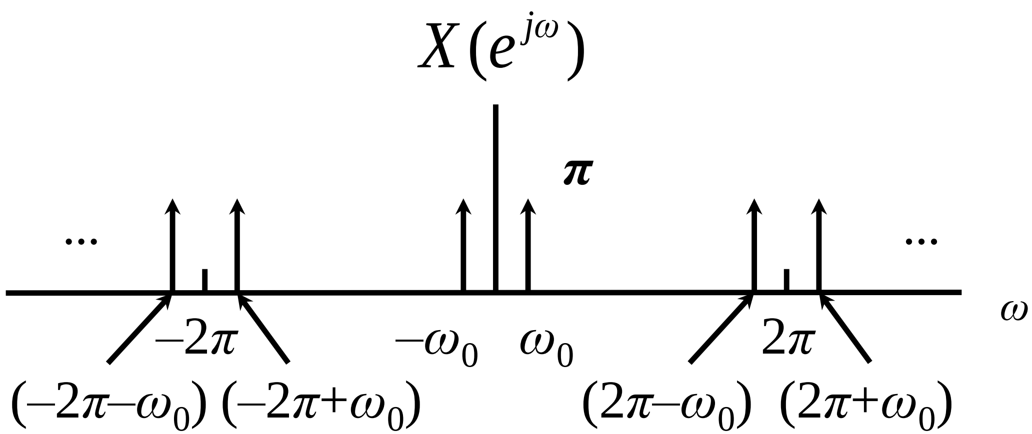
Example 5.4

Consider the periodic signal $x[n]=\cos\omega_0 n$, with $\omega_0=\dfrac{2\pi}{5}$

Sol: $$x[n]=\cos\omega_0 n=\frac{1}{2}e^{j\omega_0 n}+\frac{1}{2}e^{-j\omega_0 n}$$

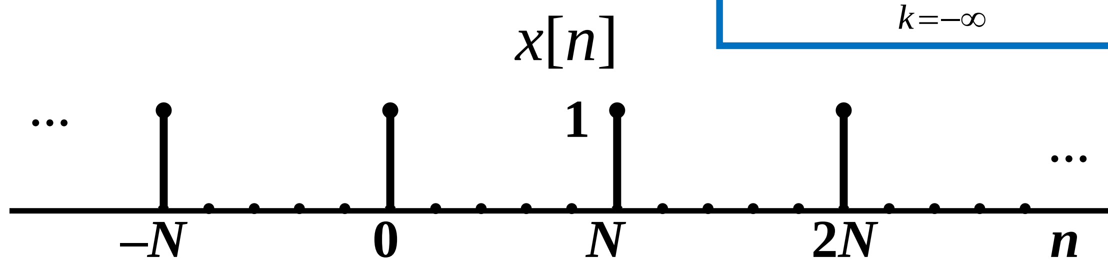
$$X(e^{j\omega})=\sum_{l=-\infty}^{\infty}\pi\delta(\omega-\frac{2\pi}{5}-2\pi l)+\sum_{l=-\infty}^{\infty}\pi\delta(\omega+\frac{2\pi}{5}-2\pi l)$$

Or $$X(e^{j\omega})=\pi\delta(\omega-\frac{2\pi}{5})+\pi\delta(\omega+\frac{2\pi}{5}),\qquad -\pi\le\omega<\pi$$



Example 5.5

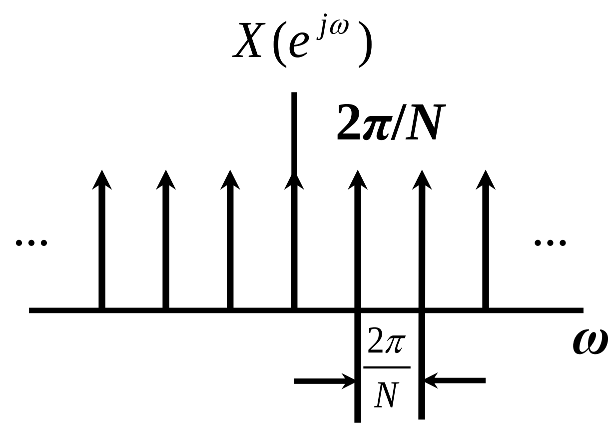
Consider the periodic sample train $$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN]$$



Sol: $$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$$

Choosing the interval of summation as 0 ≤ n ≤ N-1, we have

$$a_k = \frac{1}{N}$$



Then

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$

5.3.1 Periodicity of the Discrete-Time Fourier Transform

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

5.3.2 Linearity

If $x_1[n] \xleftrightarrow{FT} X_1(e^{j\omega})$ and $x_2[n] \xleftrightarrow{FT} X_2(e^{j\omega})$

then $ax_1[n]+bx_2[n] \xleftrightarrow{FT} aX_1(e^{j\omega})+bX_2(e^{j\omega})$

5.3.3 Time Shifting and Frequency Shifting

If $x[n] \xleftrightarrow{FT} X(e^{j\omega})$

then $x[n-n_0] \xleftrightarrow{FT} e^{-j\omega n_0}X(e^{j\omega})$ and $e^{j\omega_0 n}x[n] \xleftrightarrow{FT} X(e^{j(\omega-\omega_0)})$

5.3.4 Conjugation and Conjugate Symmetry

If $x[n] \xleftrightarrow{FT} X(e^{j\omega})$ then $x^*[n] \xleftrightarrow{FT} X^*(e^{-j\omega})$

- For real valued *x*[*n*],

  $X(e^{j\omega}) = X^*(e^{-j\omega})$ ⇨ $\mathrm{Re}\{X(e^{j\omega})\}$、$|X(e^{j\omega})|$ are even

  $\mathrm{Im}\{X(e^{j\omega})\}$、$\sphericalangle X(e^{j\omega})$ are odd

  $x_e[n] \xleftrightarrow{FT} \mathrm{Re}\{X(e^{j\omega})\}$ $\quad x_o[n] \xleftrightarrow{FT} j\,\mathrm{Im}\{X(e^{j\omega})\}$

5.3.5 Differencing and Accumulation

First-difference: $x[n]-x[n-1] \xleftrightarrow{FT} (1-e^{-j\omega})X(e^{j\omega})$

Accumulation: $\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{FT} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$

5.3.6 Time Reversal

$x[-n] \xleftrightarrow{FT} X(e^{-j\omega})$

5.3.7 Time Expansion

$$x_{(k)}[n] = \begin{cases} x[n/k], & n \text{ is a multiple of } k \\ 0, & n \text{ is not a multiple of } k \end{cases}$$

If $x[n] \xleftrightarrow{FT} X(e^{j\omega})$

then $x_{(k)}[n] \xleftrightarrow{FT} X(e^{jk\omega})$ $k:$ a positive integer

5.3.8 Differentiation in Frequency

$nx[n] \xleftrightarrow{FT} j\frac{dX(e^{j\omega})}{d\omega}$

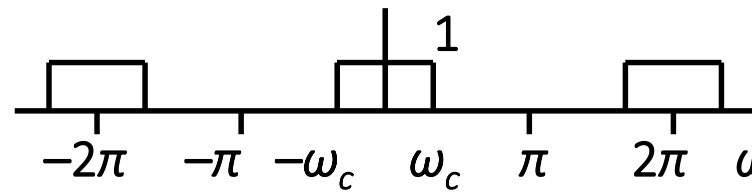
5.3.9 Parseval's Relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 \, d\omega$$

Example 5.6

The frequency response of a discrete-time low-pass filter with cutoff frequency ω_c is illustrated in the figure:

$H_l(e^{j\omega})$

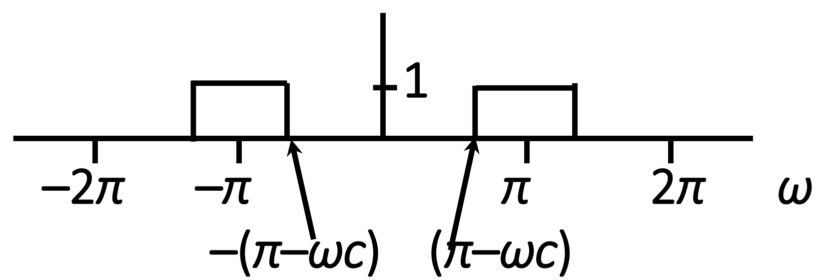


If we shift $H_l(e^{j\omega})$ by one-half period, i.e., by π, we obtain

$$H_h(e^{j\omega}) = H_l(e^{j(\omega-\pi)})$$

By frequency-shifting property,

$H_l(e^{j(\omega-\pi)})$



$$h_h[n] = h_l[n]e^{j\pi n} = (-1)^n h_l[n]$$

Example 5.7

Determine the Fourier transform of the unit step *u*[*n*].

Sol: Since $$u[n]=\sum_{m=-\infty}^{n}\delta[m]$$

and $$g[n]=\delta[n]\xleftrightarrow{FT}G(e^{j\omega})=1$$

Thus, $$\mathcal{F}\{u[n]\}=\frac{1}{1-e^{-j\omega}}G(e^{j\omega})+\pi G(e^{j0})\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

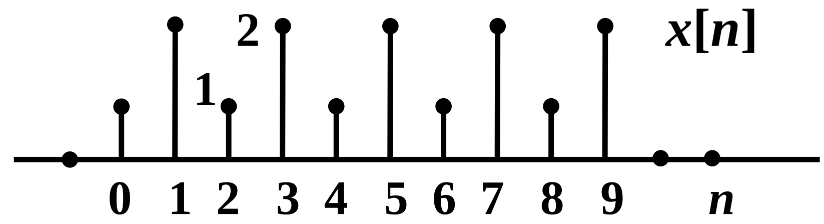
$$=\frac{1}{1-e^{-j\omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)$$

$$\boxed{u[n]\xleftrightarrow{FT}\frac{1}{1-e^{-j\omega}}+\pi\sum_{k=-\infty}^{\infty}\delta(\omega-2\pi k)}$$

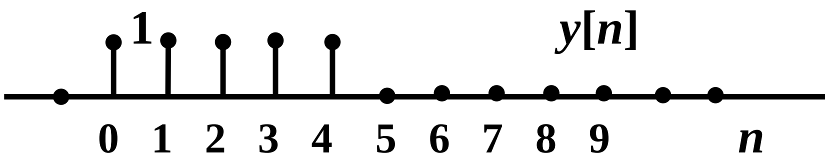
5.3 Properties of The Discrete-Time Fourier Transform

Example 5.8

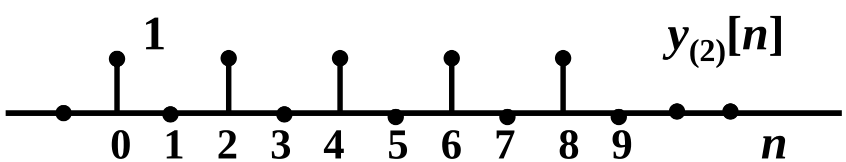
Consider the sequence $x[n]$ which is illustrated in the figure:



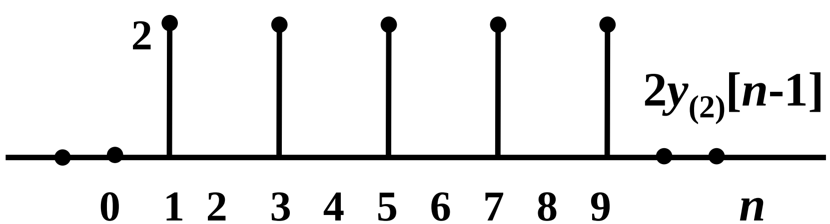
$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$



$$Y(e^{j\omega}) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}$$



$$y_{(2)}[n] \xleftrightarrow{FT} e^{-j4\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$



$$2y_{(2)}[n-1] \xleftrightarrow{FT} 2e^{-j5\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$

$$X(e^{j\omega}) = e^{-j4\omega}\left(1+2e^{-j\omega}\right)\left(\frac{\sin(5\omega)}{\sin(\omega)}\right)$$

5.4 The Convolution Property

If $y[n]=x[n]*h[n]$, then $Y(e^{j\omega})=X(e^{j\omega})H(e^{j\omega})$

- The *convolution property* represents that the Fourier transform of the response of an LTI system to a non-periodic input are simply the Fourier transform of the input multiplied by the system's frequency response evaluated at the corresponding frequencies.
- The convolution property maps the convolution operation of two time signals to the multiplication operation of their Fourier transforms.
- The frequency response $H(e^{j\omega})$ captures the change in complex amplitude of the Fourier transform of the input at each frequency ω.

Example 5.8

Consider an LTI system with sample response $h[n] = \delta[n - n_0]$.

The frequency response is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

Thus, for any input *x*[*n*], the Fourier transform of the output is

$$Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

Consequently,

$$y[n] = x[n - n_0]$$

$$H(e^{j\omega}) = e^{-j\omega n_0} \Rightarrow$$

Magnitude = 1 at all frequencies

phase = $-\omega n_0$

Example 5.9

Consider an LTI system with sample response $h[n]=\alpha^n u[n], |\alpha|<1$

The input to this system is $x[n]=\beta^n u[n], |\beta|<1$. The output *y*[*n*] = ?

$$\text{Sol}: H(e^{j\omega})=\frac{1}{1-\alpha e^{-j\omega}},\quad X(e^{j\omega})=\frac{1}{1-\beta e^{-j\omega}}$$

$$Y(e^{j\omega})=H(e^{j\omega})X(e^{j\omega})=\frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$$

If $\alpha\neq\beta,$

$$Y(e^{j\omega})=\frac{A}{1-\alpha e^{-j\omega}}+\frac{B}{1-\beta e^{-j\omega}}$$

$$A=\frac{\alpha}{\alpha-\beta},\quad B=-\frac{\beta}{\alpha-\beta}$$

$$y[n]=\frac{\alpha}{\alpha-\beta}\alpha^n u[n]-\frac{\beta}{\alpha-\beta}\beta^n u[n]=\frac{1}{\alpha-\beta}\left\{\alpha^{n+1}-\beta^{n+1}\right\}u[n]$$

5.4 The Convolution Property

If $\alpha = \beta$ $$Y(e^{j\omega}) = \left(\frac{1}{1-\alpha e^{-j\omega}}\right)^2$$

$$(Y(e^{j\omega}) = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right))$$

$$\alpha^n u[n] \xleftrightarrow{FT} \frac{1}{1-\alpha e^{-j\omega}}$$

$$n\alpha^n u[n] \xleftrightarrow{FT} j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

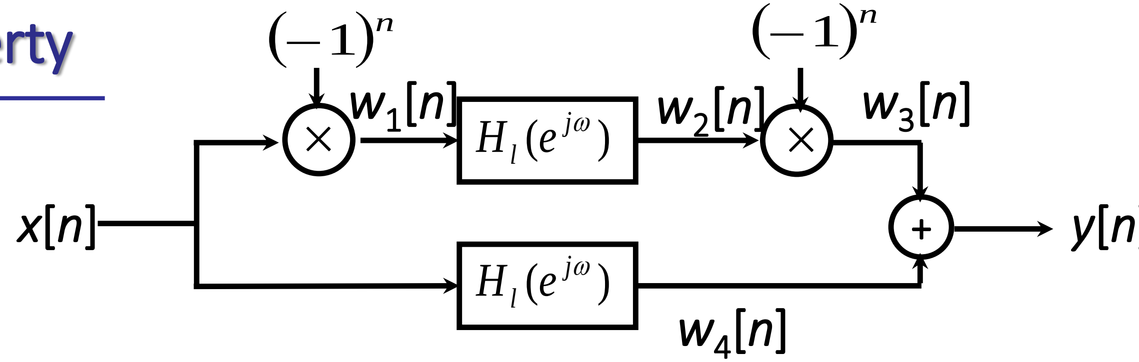
$$(n+1)\alpha^{n+1} u[n+1] \xleftrightarrow{FT} je^{j\omega}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$y[n] = (n+1)\alpha^n u[n]$$

5.4 The Convolution Property

Example 5.10

Consider the system



What is the frequency response of the overall system?

Where $H_l(e^{j\omega})$ is an ideal low-pass filter with cutoff frequency $\pi/4$ and unity gain in the passband.

Sol: The key step: $(-1)^n = e^{j\pi n}$

Then $\quad w_1[n] = e^{j\pi n}x[n] \longleftrightarrow W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$

Since, $\quad w_3[n] = e^{j\pi n}w_2[n] \longleftrightarrow W_3(e^{j\omega}) = W_2(e^{j(\omega-\pi)})$

$$= H_l(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)})$$

$$= H_l(e^{j(\omega-\pi)})X(e^{j\omega})$$

$$W_4(e^{j\omega}) = H_l(e^{j\omega})X(e^{j\omega})$$

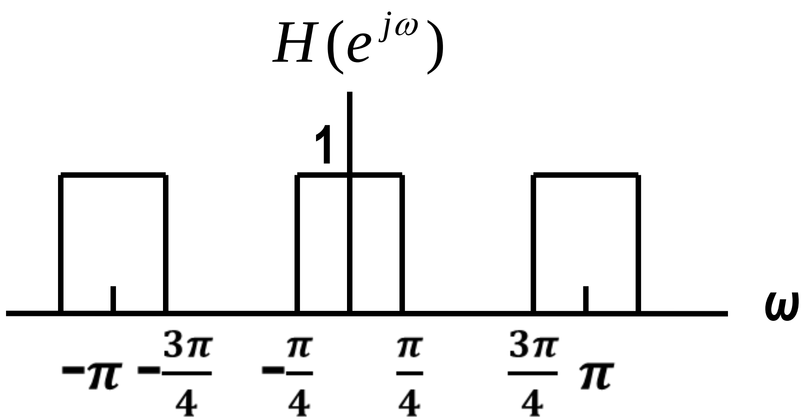
Consequently,

$$Y(e^{j\omega}) = W_3(e^{j\omega}) + W_4(e^{j\omega}) = \left[H_l(e^{j(\omega-\pi)}) + H_l(e^{j\omega}) \right] X(e^{j\omega})$$

From the convolution property, the overall system has the frequency response:

$$H(e^{j\omega}) = H_l(e^{j(\omega-\pi)}) + H_l(e^{j\omega})$$

$$= H_h(e^{j\omega}) + H_l(e^{j\omega})$$



Ideal **band-stop** *filter*

Stopband: $\pi/4 < |\omega| < 3\pi/4$

5.5 The Multiplication Property

Consider the Fourier transform of $\boldsymbol{y[n] = x_1[n]x_2[n]}$, where $\boldsymbol{x_1[n] \leftrightarrow X_1(e^{j\omega})}$, $\boldsymbol{x_2[n] \leftrightarrow X_2(e^{j\omega})}$.

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x_1[n]x_2[n]e^{-j\omega n}$$

Since

$$x_1[n] = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_2[n]\left\{\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta\right\}e^{-j\omega n}$$

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\underbrace{\left[\sum_{n=-\infty}^{\infty} x_2[n]e^{-j(\omega-\theta)n}\right]}_{X_2(e^{j(\omega-\theta)})}d\theta$$

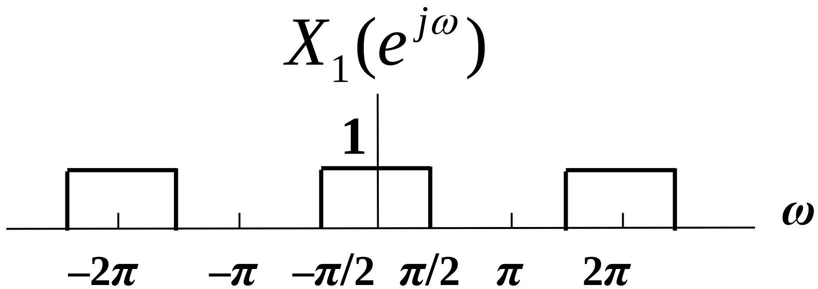
$$\boxed{Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta}$$

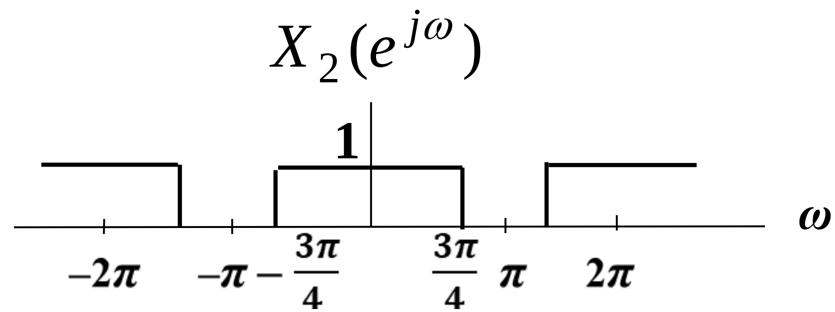
periodic convolution

Example 5.11

Find the Fourier transform $X(e^{j\omega})$ of a signal *x*[*n*] which is the product of $x_1[n]=\dfrac{\sin(\pi n/2)}{\pi n}$ and $x_2[n]=\dfrac{\sin(3\pi n/4)}{\pi n}$.

Sol:

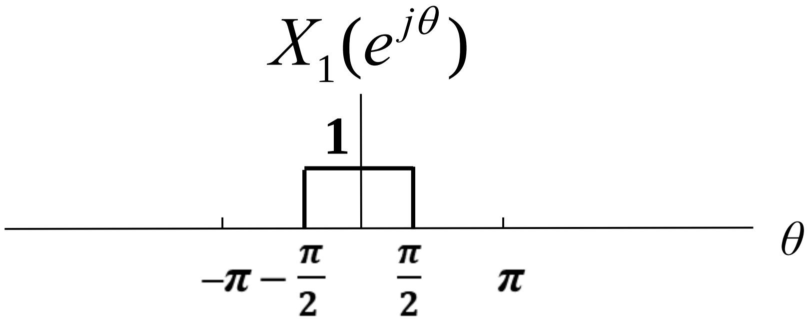




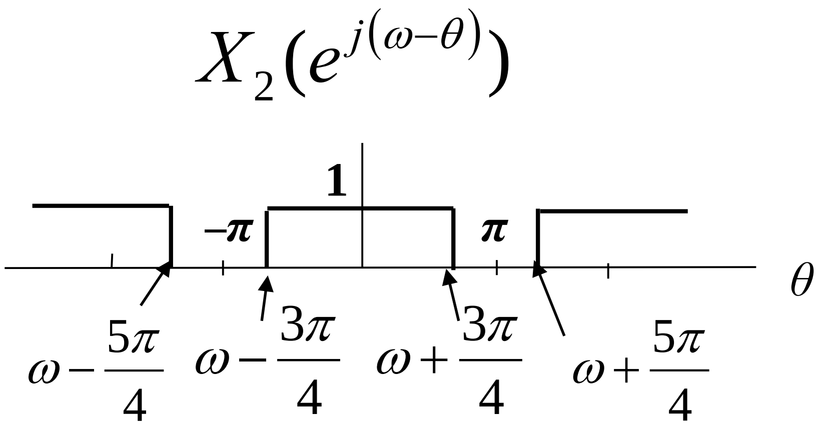
From the multiplication property,

$$X(e^{j\omega})=\frac{1}{2\pi}\int_{-\pi}^{\pi}X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

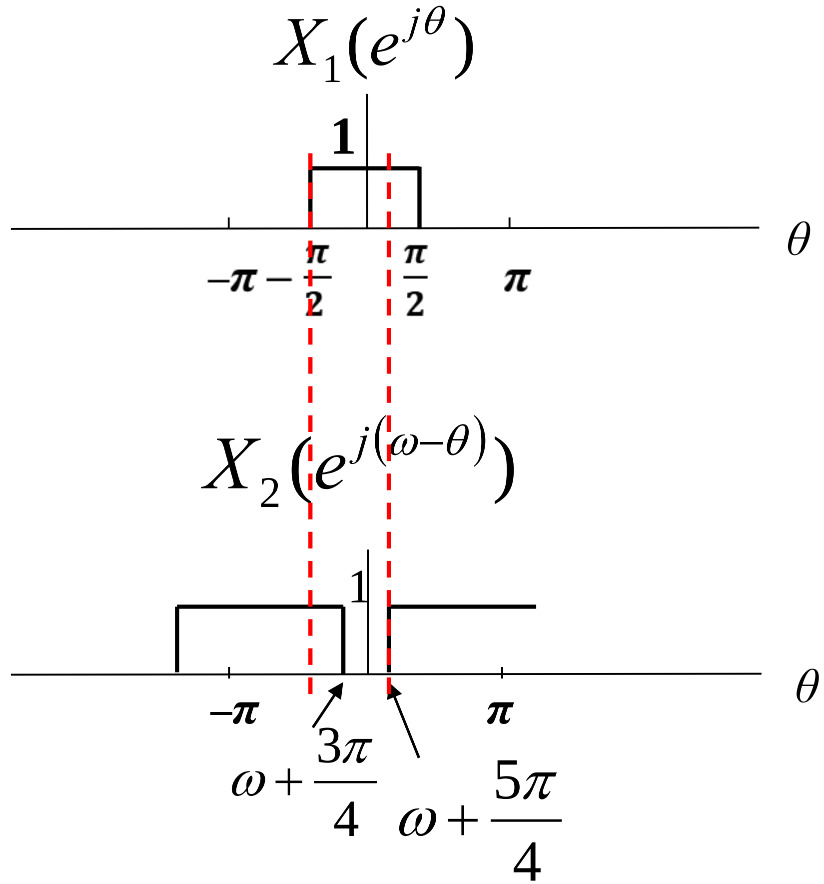
5.5 The Multiplication Property



$$when \quad \omega = 0$$



5.5 The Multiplication Property

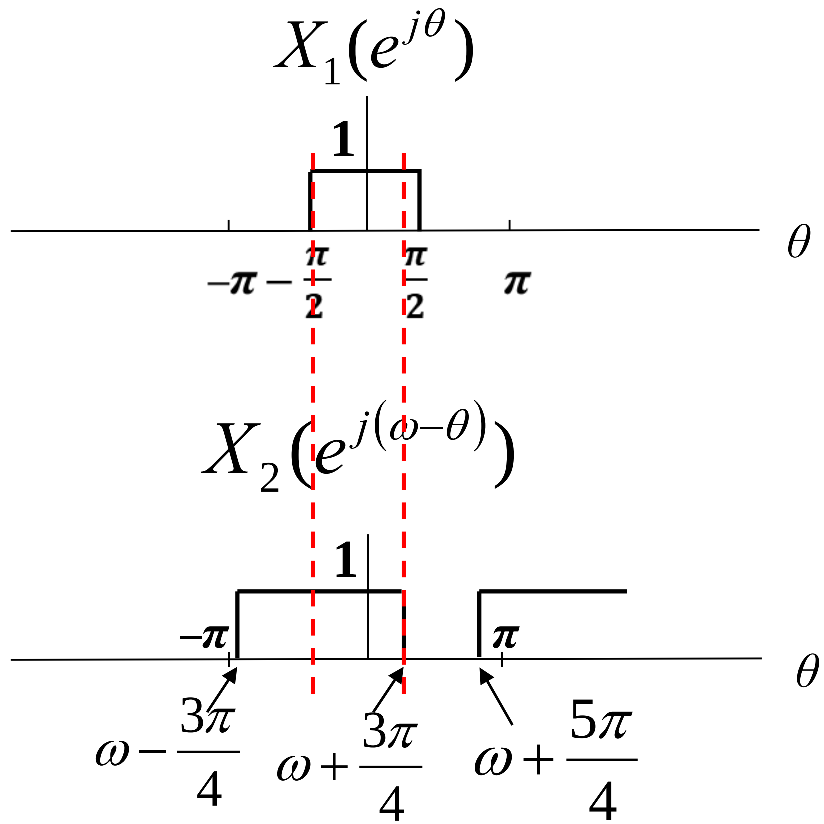


$$For \quad -\pi \le \omega < -\frac{3\pi}{4},$$

$$\left(\omega+\frac{5\pi}{4}<\frac{\pi}{2}\right)$$

$$X\left(e^{j\omega}\right)=\frac{1}{2\pi}\left[\int_{-\frac{\pi}{2}}^{\omega+\frac{3\pi}{4}} d\theta+\int_{\omega+\frac{5\pi}{4}}^{\frac{\pi}{2}} d\theta\right]=\frac{1}{2\pi}\cdot\frac{\pi}{2}=\frac{1}{4}$$

5.5 The Multiplication Property

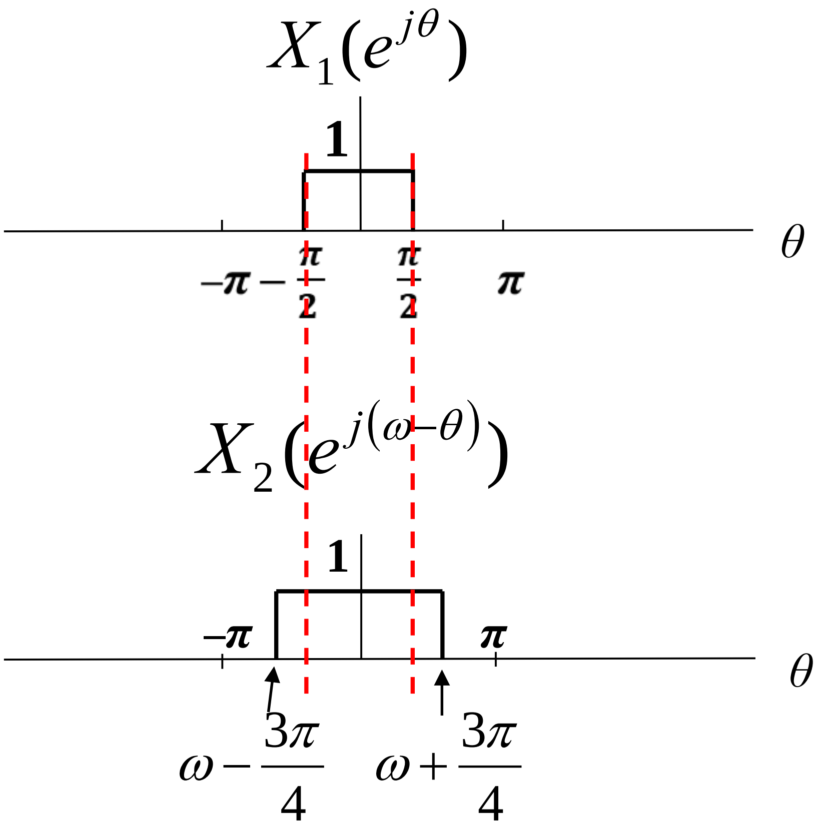


$$For \quad -\frac{3\pi}{4} \le \omega < -\frac{\pi}{4},$$

$$\left(\omega+\frac{5\pi}{4} \ge \frac{\pi}{2}, \omega+\frac{3\pi}{4} < \frac{\pi}{2}\right)$$

$$X\left(e^{j\omega}\right)=\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\omega+\frac{3\pi}{4}} d\theta=\frac{1}{2\pi}\left(\omega+\frac{5\pi}{4}\right)=\frac{\omega}{2\pi}+\frac{5}{8}$$

5.5 The Multiplication Property

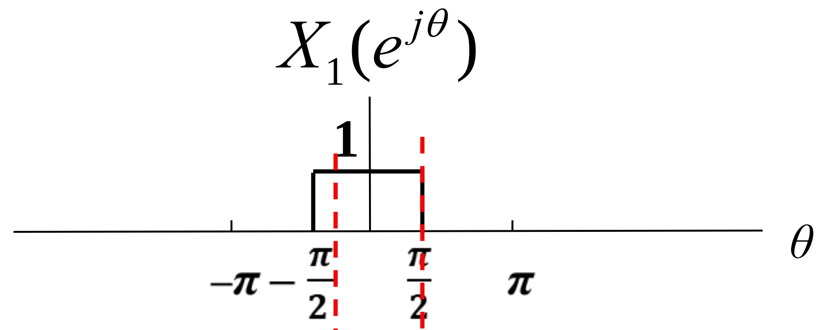


$$For \quad -\frac{\pi}{4} \le \omega < \frac{\pi}{4},$$

$$\left(\omega+\frac{3\pi}{4} \ge \frac{\pi}{2},\ \omega-\frac{3\pi}{4} < -\frac{\pi}{2}\right)$$

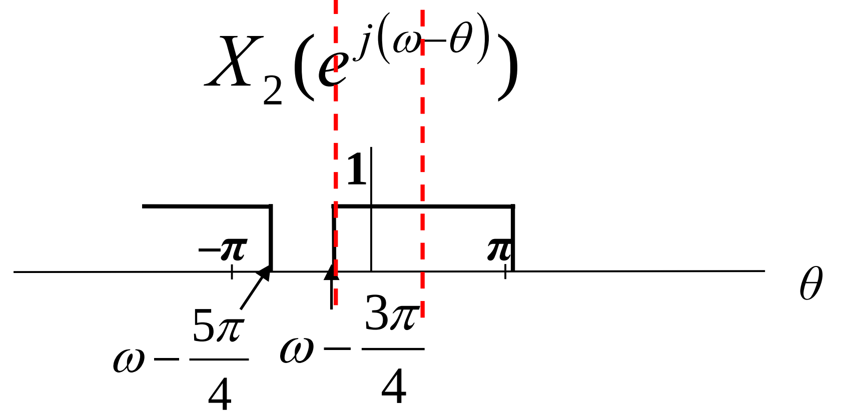
$$X\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta = \frac{1}{2\pi}\cdot\pi = \frac{1}{2}$$

5.5 The Multiplication Property



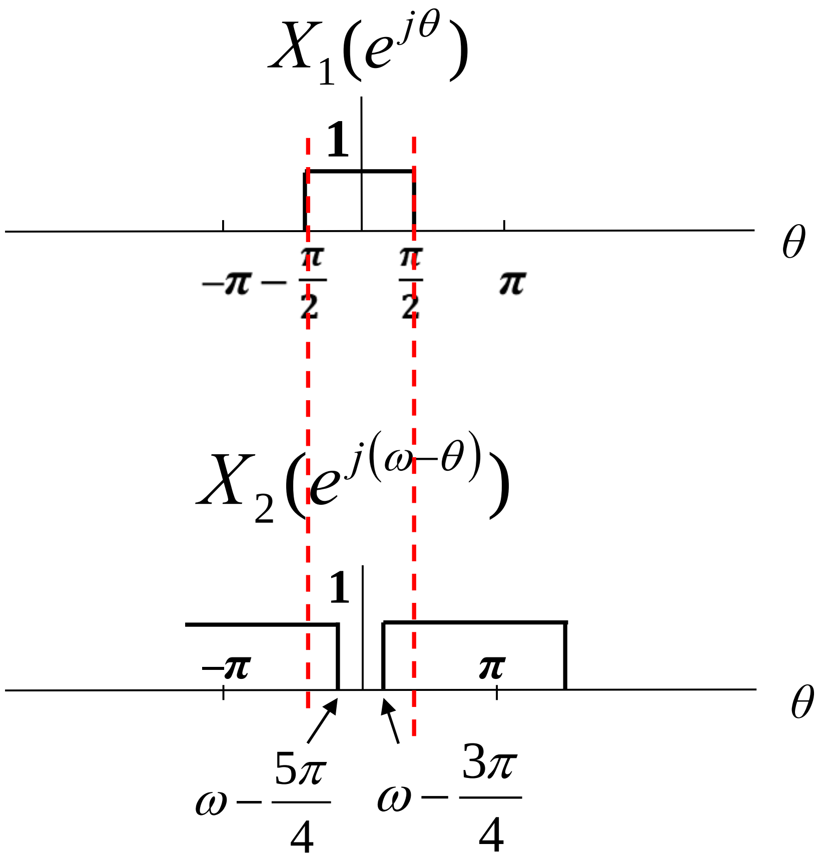
$$For \quad \frac{\pi}{4} \le \omega < \frac{3\pi}{4},$$

$$\left(\omega - \frac{3\pi}{4} \ge -\frac{\pi}{2}, \omega - \frac{5\pi}{4} < -\frac{\pi}{2}\right)$$



$$X\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{\omega-\frac{3\pi}{4}}^{\frac{\pi}{2}} d\theta = \frac{1}{2\pi}\left(\frac{\pi}{2} - \omega + \frac{3\pi}{4}\right) = -\frac{\omega}{2\pi} + \frac{5}{8}$$

5.5 The Multiplication Property

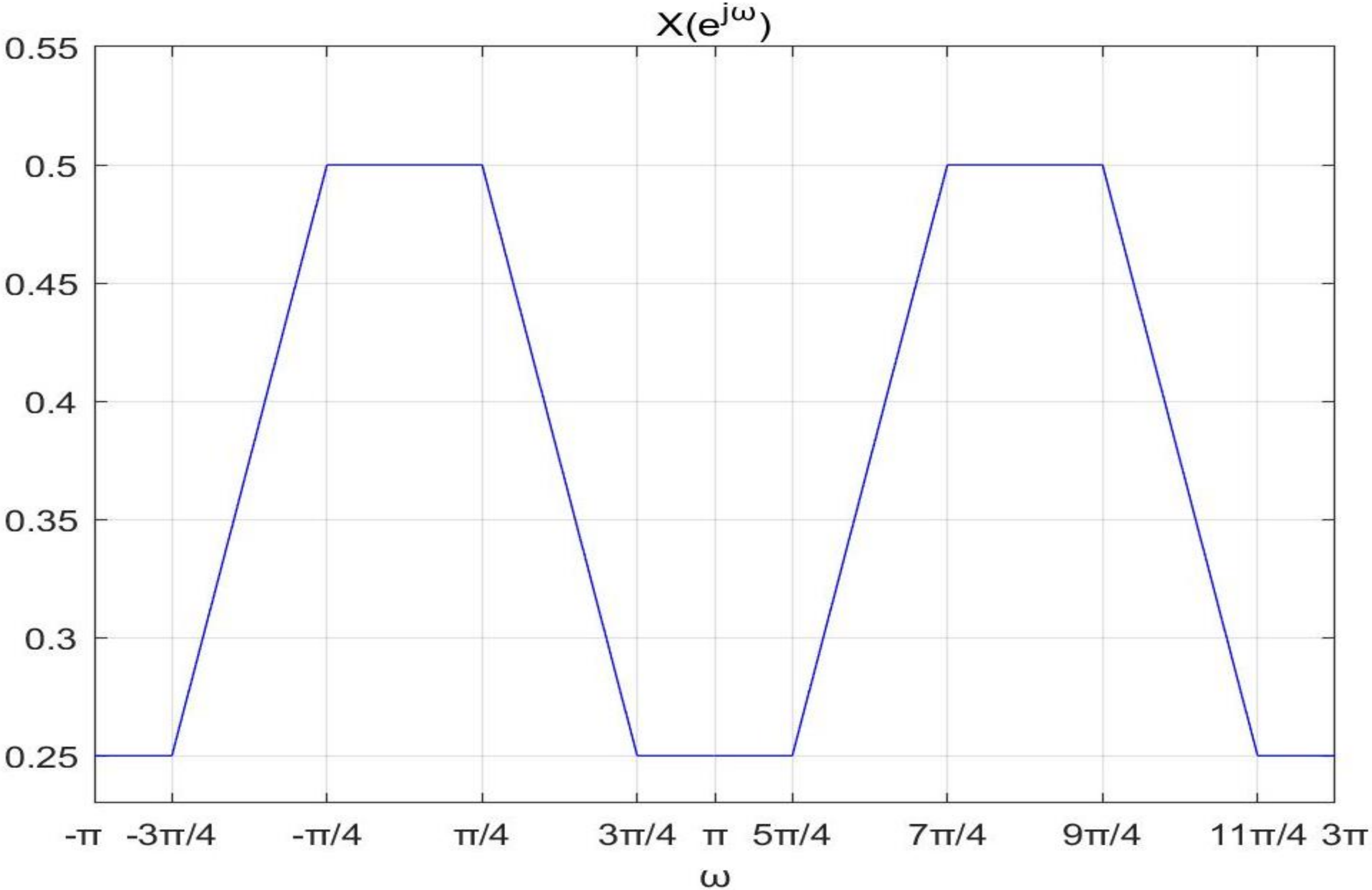


$$For \quad \frac{3\pi}{4} \le \omega < \pi,$$

$$\left(\omega - \frac{5\pi}{4} \ge -\frac{\pi}{2}\right)$$

$$X\left(e^{j\omega}\right) = \frac{1}{2\pi}\left[\int_{-\frac{\pi}{2}}^{\omega-\frac{5\pi}{4}} d\theta + \int_{\omega-\frac{3\pi}{4}}^{\frac{\pi}{2}} d\theta\right] = \frac{1}{2\pi}\cdot\frac{\pi}{2} = \frac{1}{4}$$

5.5 The Multiplication Property



5.6 Systems Characterized By Linear Constant-Coefficient Difference Equations

A general *N*th-order difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega})$$

$$\boxed{H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}}$$

- $H(e^{j\omega})$ is a ratio of polynomials in the variable $e^{-j\omega}$.
- Coefficients of the *numerator polynomial* = Coefficients appearing on the *right side* of the difference equation.
- Coefficients of the *denominator polynomial* = Coefficients appearing on the *left side* of the difference equation.

Example 5.13

Consider a causal LTI system that is characterized by the difference equations $y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n]$, and let the input to this system be $x[n]=(1/4)^n u[n]$. Determine the output *y*[*n*].

Sol:
$$Y(e^{j\omega})=H(e^{j\omega})X(e^{j\omega})=\left[\frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}}\right]\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right]$$

$$=\left[\frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}\right]\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right]=\frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)^2}$$

The form of the partial-fraction expansion in this case is

$$Y(e^{j\omega})=\frac{B_{11}}{1-\frac{1}{4}e^{-j\omega}}+\frac{B_{12}}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2}+\frac{B_{21}}{1-\frac{1}{2}e^{-j\omega}}$$

5.6 Systems Characterized By Linear Constant-Coefficient Difference Equations

$$B_{12} = \left[\left(1-\frac{1}{4}e^{-j\omega}\right)^2 Y(e^{j\omega})\right]\Bigg|_{e^{-j\omega}=4} = -2,$$

$$B_{21} = \left[\left(1-\frac{1}{2}e^{-j\omega}\right) Y(e^{j\omega})\right]\Bigg|_{e^{-j\omega}=2} = 8,$$

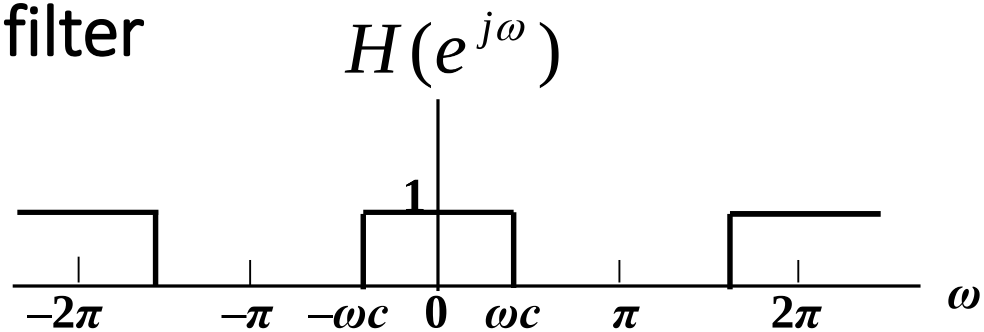
$$B_{11} = (-4)\left[\frac{d}{d(e^{-j\omega})}\left(1-\frac{1}{4}e^{-j\omega}\right)^2 Y(e^{j\omega})\right]\Bigg|_{e^{-j\omega}=4} = -4.$$

So that,
$$Y(e^{j\omega}) = \frac{-4}{1-\frac{1}{4}e^{-j\omega}} + \frac{-2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^2} + \frac{8}{1-\frac{1}{2}e^{-j\omega}}$$

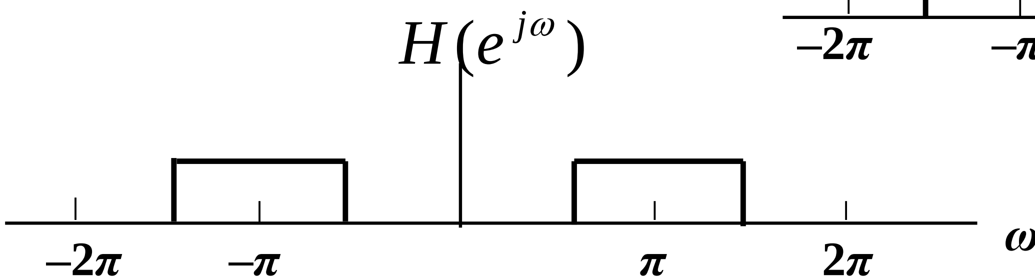
Consequently,
$$y[n] = \left\{-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right\}u[n]$$

5.7 Discrete-Time Frequency-Selective Filters

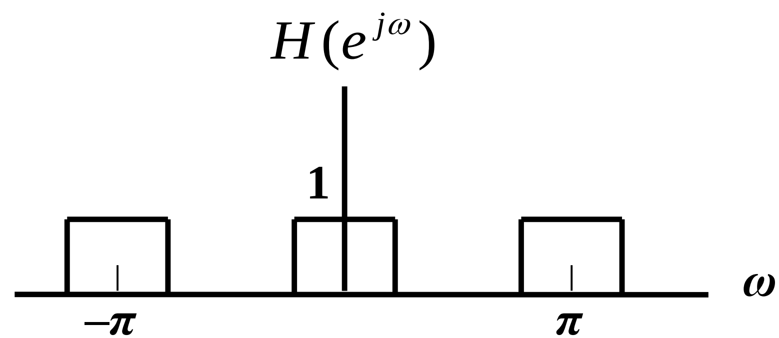
Ideal low-pass filter



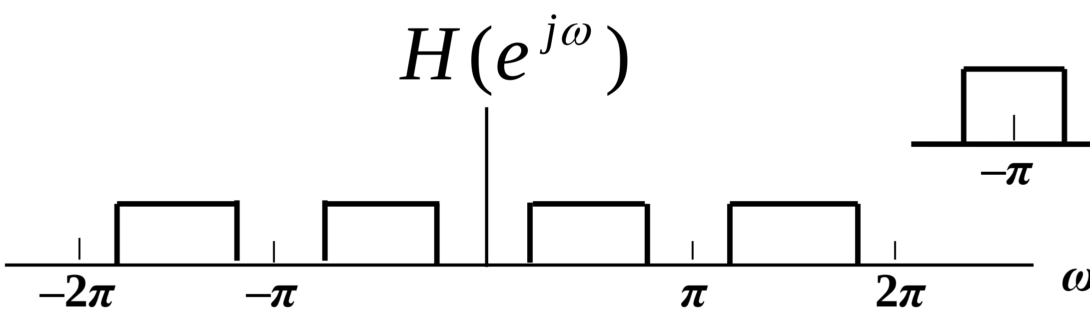
Ideal high-pass filter



Ideal band-stop filter



Ideal band-pass filter



5.8 SUMMARY

- The Fourier transform for non-periodic and periodic discrete-time signals;
- The differences between the DTFT and the CTFT (Especially the periodicity of DTFT);
- The properties of the Fourier transform (relationships between characteristics of a discrete-time signal in time and frequency domain);
- Fourier analysis (Frequency domain analysis) for discrete-time LTI systems including both characteristics of systems and responses to some input signals;
- Frequency response and the way to obtain it;
- Sampling of discrete-time signals.

Homework

5.21 (a) (g) (h) (i)   5.22 (a) (b) (d) (f)

5.26   5.29 (i) in (a)、(ii) in (b)

5.30 (a) 、(ii) (iii) in (b)、(ii) in (c)