# CHAPTER 6

# THE LAPLACE TRANSFORM

# Content

Content

## 6.0 Introduction

- The Laplace transform is a generalization of the continuous-time Fourier transform. It provides us with a representation for signals as linear combinations of complex exponentials of the form $e^{st}$ with $s = \sigma + j\omega$ .

- With Laplace transform, we expand the applications in which Fourier transform can or can not be used.

- Relationships between the Laplace transform and the continuous-time Fourier transform.

## 6.1.1 Introduction of The Laplace Transform

For some signals which is not absolutely integrable, we can *preprocess* them by multiplying with a real exponential $e^{-\sigma t}$ and then calculate the Fourier transform of the product as:

$$\int_{-\infty}^{\infty}\left[x(t)e^{-\sigma t}\right]e^{-j\omega t}dt$$

Let $s = \sigma + j\omega$, and using $X(s)$ to denote this integral, we obtain

$$\boxed{X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt}$$

The *Laplace transform* of $x(t)$

The Laplace transform is a *generalization* of the Fourier transform with the exponential function taking the form $e^{st}$; the Fourier transform is a *special case* of the Laplace transform when $\sigma = 0$.

## 6.1.2 Examples

### Example 6.1

Consider the signal $x(t) = e^{-\alpha t}u(t)$.

$$\text{Sol}: X(s) = \int_{-\infty}^{\infty} e^{-\alpha t}u(t)e^{-st}dt = \int_{0}^{+\infty} e^{-(s+\alpha)t}dt$$

$$= -\frac{1}{s+\alpha}e^{-(s+\alpha)t}\bigg|_0^{\infty} = -\frac{1}{s+\alpha}e^{-(s+\alpha)\lim\limits_{t\to\infty} t} + \frac{1}{s+\alpha}$$

For convergence, we require that Re{$s + \alpha$} > 0, or Re{$s$} > $-\alpha$

Thus,

$$\boxed{X(s) = \frac{1}{s+\alpha}, \quad \mathrm{Re}\{s\} > -\alpha}$$

*region of convergence* (*ROC*)

(收斂域)

## Example 6.2

Consider the signal $x(t) = -e^{-\alpha t}u(-t)$.

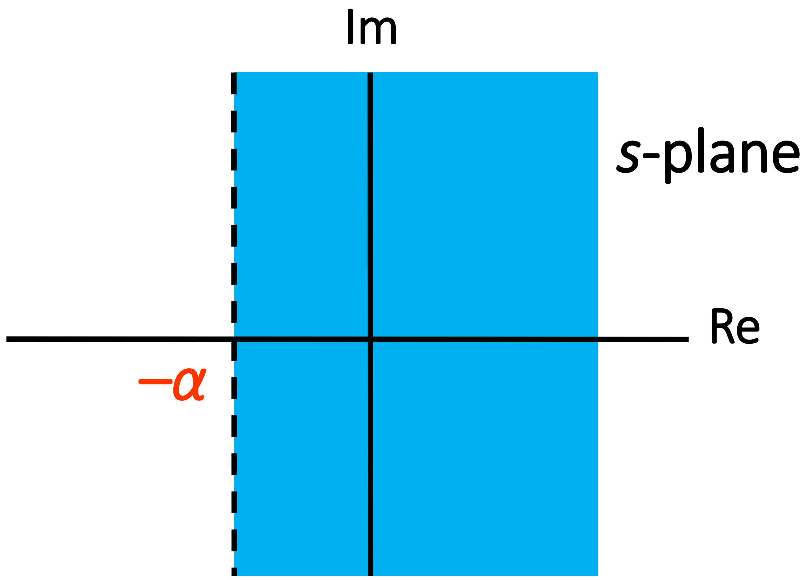
$$\text{Sol}: X(s) = -\int_{-\infty}^{\infty} e^{-\alpha t} e^{-st} u(-t)dt = -\int_{-\infty}^{0} e^{-(s+\alpha)t} dt$$

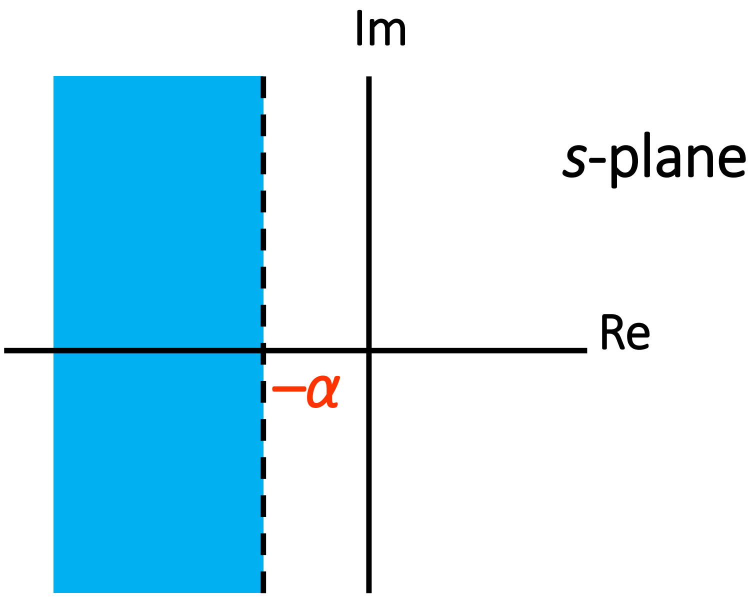
$$= \frac{1}{s+\alpha} e^{-(s+\alpha)t} \Big|_{-\infty}^{0} = \frac{1}{s+\alpha} - \frac{1}{s+\alpha} e^{-(s+\alpha)\lim\limits_{t\to-\infty} t}$$

For convergence, we require that Re{$s + \alpha$} < 0, or Re{$s$} < $-\alpha$

Thus,

$$\boxed{-e^{-\alpha t}u(-t) \xleftrightarrow{LT} \frac{1}{s+\alpha}, \quad \mathrm{Re}\{s\} < -\alpha}$$




*ROC* for Example 6.1

*ROC* for Example 6.2

## Example 6.3

Consider the signal $x(t) = e^{-2t}u(t) + e^{-t}(\cos 3t)u(t)$.

Sol: From Euler's relation, we can write

$$x(t) = \left[ e^{-2t} + \frac{1}{2}e^{-(1-3j)t} + \frac{1}{2}e^{-(1+3j)t} \right] u(t)$$

$$X(s) = \int_{-\infty}^{\infty} e^{-(1-3j)t} u(t) e^{-st} dt = \int_{0}^{\infty} e^{-(s+1-3j)t} dt$$

$$= -\frac{1}{s+1-3j} e^{-(s+1-3j)t} \Big|_{0}^{\infty} = -\frac{1}{s+1-3j} e^{-(s+1-3j)\lim\limits_{t\to\infty} t} + \frac{1}{s+1-3j}$$

Thus, $e^{-(1-3j)t}u(t) \xleftrightarrow{LT} \dfrac{1}{s+(1-3j)}, \ \mathrm{Re}\{s\} > -1,$

$$e^{-(1+3j)t}u(t) \xleftrightarrow{LT} \frac{1}{s+(1+3j)}, \ \mathrm{Re}\{s\} > -1, \qquad e^{-2t}u(t) \xleftrightarrow{LT} \frac{1}{s+2}, \ \mathrm{Re}\{s\} > -2.$$

Consequently,

$$e^{-2t}u(t)+e^{-t}(\cos 3t)u(t)\xleftrightarrow{LT}\frac{1}{s+2}+\frac{1}{2}\left(\frac{1}{s+(1-3j)}\right)+\frac{1}{2}\left(\frac{1}{s+(1+3j)}\right)$$

$$=\frac{2s^2+5s+12}{(s^2+2s+10)(s+2)},\quad \mathrm{Re}\{s\}>-1$$

Other useful LT pairs:

$$\boxed{\begin{aligned}&\cos(\omega t)u(t)\xleftrightarrow{LT}\frac{s}{s^2+\omega^2},\quad \mathrm{Re}\{s\}>0\\ &\sin(\omega t)u(t)\xleftrightarrow{LT}\frac{\omega}{s^2+\omega^2},\quad \mathrm{Re}\{s\}>0\end{aligned}}$$

# 6.1 The Laplace Transform

- Generally, the Laplace transform is *rational*, i.e., it is a ratio of polynomials in the complex variable *s*:

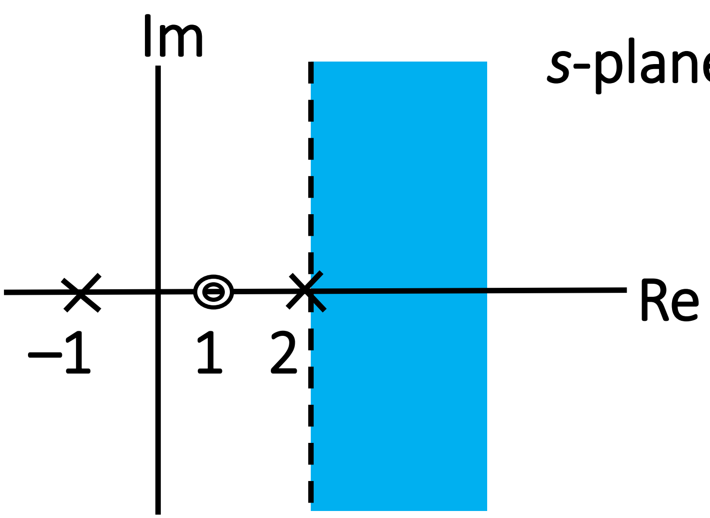
$$\boxed{X(s)=\frac{N(s)}{D(s)}}$$

- The roots of *N*(*s*) are referred to as the *zeros* (零点) of *X*(*s*) ; and the roots of *D*(*s*) are referred to as the *poles* (极点) of *X*(*s*).
- The representation of *X*(*s*) through its poles and zeros in the *s*-plane is referred to as the *pole-zero plot* (极零图)of *X*(*s*).
- Marking the locations of the roots of *N*(*s*) and *D*(*s*) in the *s*-plane and indicating the *ROC* provides a convenient *pictorial way of describing the Laplace transform*.
- Except for a scale factor, a complete specification of a rational Laplace transform consists of the pole-zero plot of the transform, together with its *ROC*.

- About the *infinity* (无穷远点): In general, if the order of $D(s)$ exceeds the order of $N(s)$ by $k$, $X(s)$ will have $k$ zeros at infinity. Similarly, if the order of $N(s)$ exceeds the order of $D(s)$ by $k$, $X(s)$ will have $k$ poles at infinity.

Example 6.4

Consider the signal $x(t)=\delta(t)-\frac{4}{3}e^{-t}u(t)+\frac{1}{3}e^{2t}u(t).$




Pole-zero plot and ROC

$$X(s)=1-\frac{4}{3}\frac{1}{s+1}+\frac{1}{3}\frac{1}{s-2}, \quad \mathrm{Re}\{s\}>2$$

$$=\frac{(s-1)^2}{(s+1)(s-2)}, \quad \mathrm{Re}\{s\}>2$$

## 6.2 The Region of Convergence For Laplace Transforms

- Property 1: The *ROC* of $X(s)$ consists of strips parallel to the $j\omega$-axis in the $s$-plane.

- Property 2: For rational Laplace transforms, the *ROC* ***does not*** contain any poles.

- Property 3: If $x(t)$ is of *finite duration* and is absolutely integrable, then the *ROC is the entire s-plane*.

Example 6.5

Let $$x(t) = e^{-\alpha t}\left[u(t) - u(t-T)\right]$$

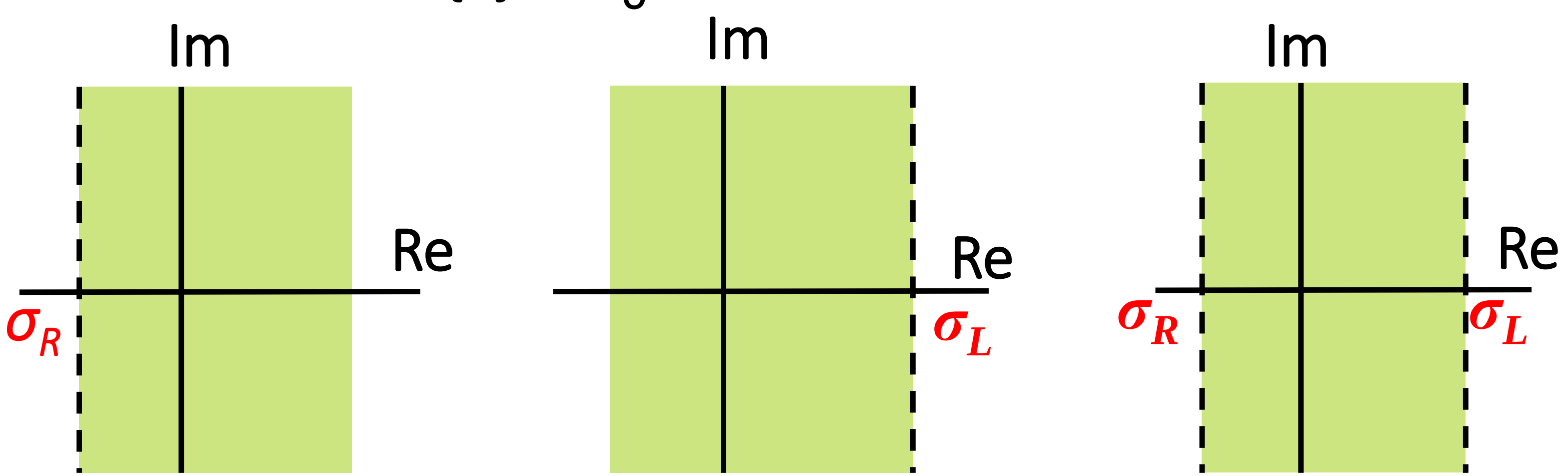
Sol: $$X(s) = \int_0^T e^{-\alpha t} e^{-st} dt = \frac{1}{s+\alpha}\left[1 - e^{-(s+\alpha)T}\right]$$

$$e^{-(s+\alpha)T} = \sum_{n=0}^{\infty} \frac{(-1)^n\left[(s+\alpha)T\right]^n}{n!}$$

The pole at $s = -\alpha$ is removable!

## 6.2 The Region of Convergence For Laplace Transforms

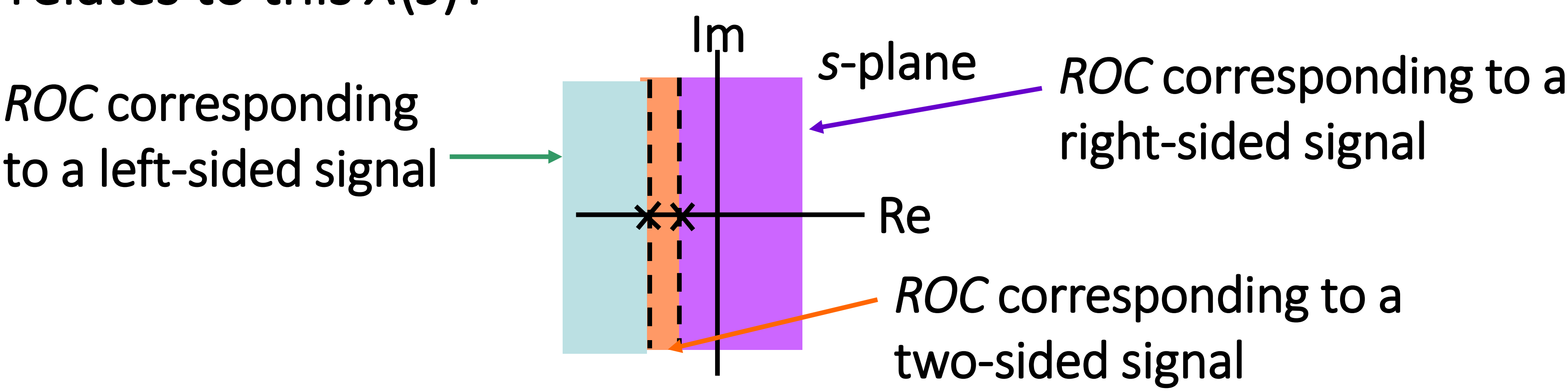
- Property 4: If $x(t)$ is *right sided*, and if the line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC, then all values of $s$ for which $\mathrm{Re}\{s\} > \sigma_0$ will also be in the ROC; and the ROC of a right-sided signal is a *right-half plane*.
- Property 5: If $x(t)$ is *left sided*, and if the line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC, then all values of $s$ for which $\mathrm{Re}\{s\} < \sigma_0$ will also be in the ROC; and the ROC of a left-sided signal is a *left-half plane*.
- Property 6: If $x(t)$ is *two sided*, and if the line $\mathrm{Re}\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a *strip* in the $s$-plane that includes the line $\mathrm{Re}\{s\} = \sigma_0$.



## 6.2 The Region of Convergence For Laplace Transforms

- Property 7: If the Laplace transform $X(s)$ of $x(t)$ is rational, then its *ROC* is bounded by poles or extends to infinity. In addition, *no poles* of $X(s)$ are contained *in the ROC*.
- Property 8: If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right sided, the *ROC* is the region in the $s$-plane to the right of the rightmost pole. If $x(t)$ is left sided, the *ROC* is the region in the $s$-plane to the left of the leftmost pole.

Example 6.6 Let $X(s)=\dfrac{1}{(s+1)(s+2)}$, how many possible *ROC*s relates to this $X(s)$?



## 6.3 The Inverse Laplace Transform

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}\{X(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{j\omega t}d\omega$$

Multiplying both sides by $e^{\sigma t}$ leading to $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(s)e^{st}d\omega$

Changing the variable of this integration from *ω* to *s* and using the fact that *σ* is constant, so that *ds = jdω*.

Thus, the basic *inverse Laplace transform* equation is:

$$\boxed{x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds}$$

- The inverse Laplace transform equation states that *x(t) can be represented as a weighted integral of complex exponentials*.
- The formal evaluation of the integral for a general *X*(*s*) requires the use of contour integration(围线积分) in the complex plane.
- For the class of *rational* transforms, the inverse Laplace transform can be determined by using the technique of partial-fraction expansion.

Example 6.7 Let $X(s)=\dfrac{1}{(s+1)(s+2)}, \quad -2<\mathrm{Re}\{s\}<-1.$

Sol: Performing the partial-fraction expansion, we obtain

$$X(s)=\frac{1}{(s+1)}-\frac{1}{(s+2)}$$

$$-e^{-t}u(-t)\xleftrightarrow{LT}\frac{1}{s+1}, \quad \mathrm{Re}\{s\}<-1$$

$$e^{-2t}u(t)\xleftrightarrow{LT}\frac{1}{s+2}, \quad \mathrm{Re}\{s\}>-2$$

$$x(t)=-e^{-t}u(-t)-e^{-2t}u(t)\xleftrightarrow{LT}\frac{1}{(s+1)(s+2)}, \quad -2<\mathrm{Re}\{s\}<-1$$

What if $X(s)=\dfrac{1}{s(s+1)(s+2)(s+4)}, \quad -2<\mathrm{Re}\{s\}<-1.$ ?

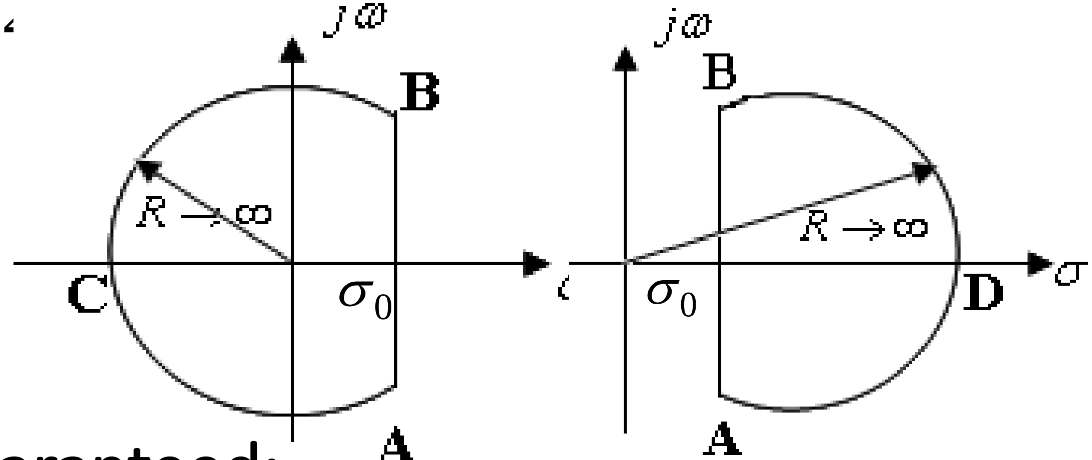
# 6.3 The Inverse Laplace Transform

➢ Use *Residue Theorem* to calculate contour integration:

$$x(t)=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}X(s)e^{st}ds=\frac{1}{2\pi j}\oint_{C}X(s)e^{st}ds=\sum_{i}^{n}\mathrm{Re}\,s_i$$

To make $\int_{BCA}X(s)e^{st}ds=0$

or $\int_{BDA}X(s)e^{st}ds=0$




Two conditions must be guaranteed:

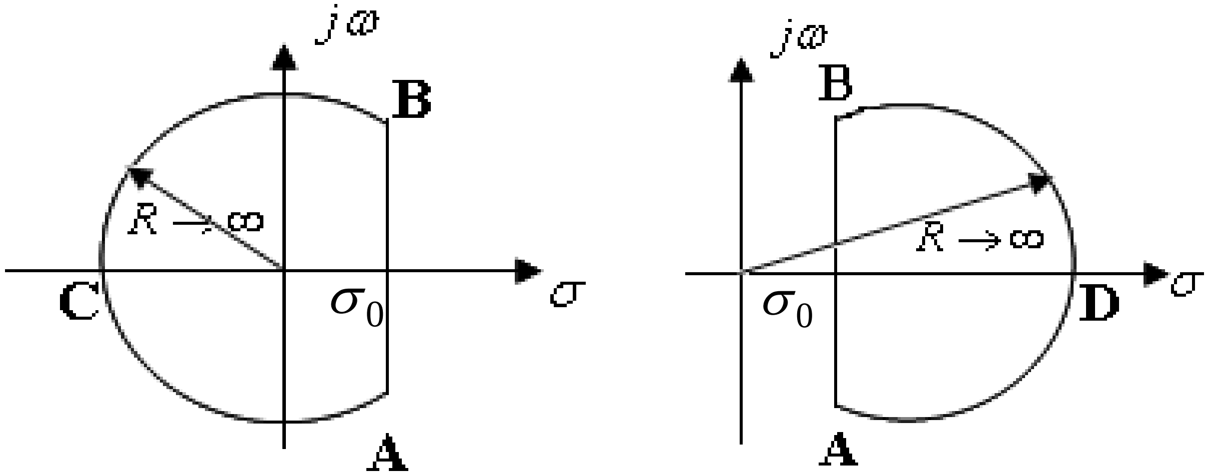
(1) As $|s|=R\to\infty$, $|X(s)|\to 0$ for all *s*;

(2) The exponent of $e^{st}$ has a real part less than $\sigma_0 t$, $\mathrm{Re}(st)=\sigma t<\sigma_0 t$

Condition (1) can be satisfied so long as the order of the denominator exceeds the order of the numerator;

Condition (2) requires that as $t>0$, $\sigma<\sigma_0$; as $t<0$, $\sigma>\sigma_0$.

## 6.3 The Inverse Laplace Transform




For $t > 0$, the contour is composed by the straight line *AB* and its left-side arc *BCA* ; for $t < 0$, the contour is composed by the straight line *AB* and its right-side arc *BDA* .

$$x(t) = \frac{1}{2\pi j}\oint_C X(s)e^{st}ds = \begin{cases} \displaystyle\sum_{left-side \quad poles} \operatorname{Re} s_l \,, \quad t > 0 \\ \displaystyle -\sum_{right-side \quad poles} \operatorname{Re} s_r \,, \quad t < 0 \end{cases}$$

Example 6.8 Let $X(s)=\dfrac{s+2}{s(s+3)(s+1)^2},\quad \mathrm{Re}\{s\}>0.$

Compute the $x(t)$ with contour integration method.

Sol: $X(s)$ has two first-order poles: $s_1=0,\quad s_2=-3.$

and a second-order pole: $s_{3,4}=-1.$

From the Residue Theorem,

$$x(t)=\mathrm{Re}\,s\left[X(s)e^{st},0\right]+\mathrm{Re}\,s\left[X(s)e^{st},-3\right]+\mathrm{Re}\,s\left[X(s)e^{st},-1\right]$$

$$\mathrm{Re}\,s\left[X(s)e^{st},0\right]=\left[sX(s)e^{st}\right]_{s=0}=\frac{s+2}{(s+3)(s+1)^2}e^{st}\bigg|_{s=0}=\frac{2}{3}$$

$$\mathrm{Re}\,s\left[X(s)e^{st},-3\right]=\left[(s+3)X(s)e^{st}\right]_{s=-3}=\frac{s+2}{s(s+1)^2}e^{st}\bigg|_{s=-3}=\frac{1}{12}e^{-3t}$$

$$\mathrm{Res}\left[X(s)e^{st},-1\right]=\frac{1}{1!}\frac{d}{ds}\left\{(s+1)^2X(s)e^{st}\right\}_{s=-1}=\frac{d}{ds}\left[\frac{s+2}{s(s+3)}e^{st}\right]_{s=-1}$$

$$=\left[-\frac{s^2+4s+6}{s^2(s+3)^2}e^{st}+t\frac{s+2}{s(s+3)}e^{st}\right]_{s=-1}=-\frac{1}{2}te^{-t}-\frac{3}{4}e^{-t}$$

$$Thus,\quad x(t)=\left[\frac{2}{3}+\frac{1}{12}e^{-3t}-\frac{1}{2}\left(t+\frac{3}{2}\right)e^{-t}\right]u(t)$$

What if $X(s)=\dfrac{s+2}{s(s+3)(s+1)^2},\quad -1<\mathrm{Re}\{s\}<0$ ?

Or $X(s)=\dfrac{s+2}{s(s+3)(s+1)^2},\quad \mathrm{Re}\{s\}<-3$ ?

then $x(t)=-\dfrac{2}{3}u(-t)+\left[\dfrac{1}{12}e^{-3t}-\dfrac{1}{2}\left(t+\dfrac{3}{2}\right)e^{-t}\right]u(t)$

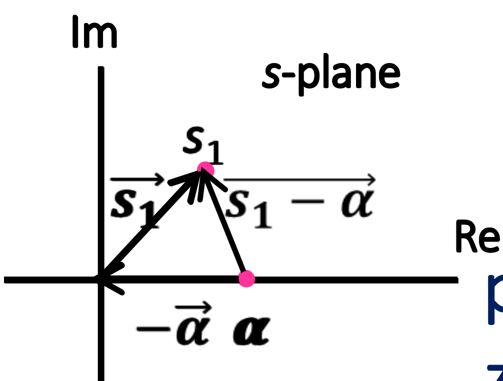
or $x(t)=-\left[\dfrac{2}{3}+\dfrac{1}{12}e^{-3t}-\dfrac{1}{2}\left(t+\dfrac{3}{2}\right)e^{-t}\right]u(-t)$

# 6.4 Geometric Evaluation of The Fourier Transform From The Pole-Zero Plot

A general rational Laplace transform has the form: $X(s)=\dfrac{N(s)}{D(s)}$

and it can be factored into the form: $X(s)=M\dfrac{\prod_{i=1}^{R}(s-\beta_i)}{\prod_{k=1}^{P}(s-\alpha_k)}$

where β$_i$, α$_j$ are zeros and poles of *X*(*s*), respectively.




$$X(j\omega)=M\frac{\prod_{i=1}^{R}(j\omega-\beta_i)}{\prod_{k=1}^{P}(j\omega-\alpha_k)}\qquad X(j\omega_1)=M\frac{\prod_{i=1}^{R}A_i\cdot e^{j\sum_{i=1}^{R}\theta_i}}{\prod_{k=1}^{P}B_k\cdot e^{j\sum_{k=1}^{P}\varphi_k}}$$

pole vectors (极点向量)

zero vectors (零点向量)

$$\boxed{|X(j\omega_1)|=M\frac{\prod_{i=1}^{R}A_i}{\prod_{k=1}^{P}B_k}}$$

$$\boxed{\measuredangle X(j\omega_1)=\sum_{i=1}^{R}\theta_i-\sum_{k=1}^{P}\varphi_k}$$

Complex plane representation of the vectors $\vec{s_1}$, $\vec{\alpha}$, and $\overrightarrow{s_1-\alpha}$ representing the complex numbers $s_1$, $\alpha$ and $s_1-\alpha$ respectively.

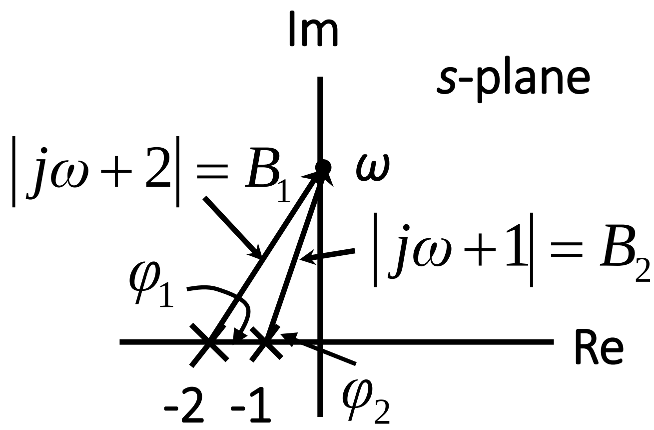
# 6.4 Geometric Evaluation of The Fourier Transform From The Pole-Zero Plot

Given $X(s)=\dfrac{1}{(s+1)(s+2)},\quad \mathrm{Re}\{s\}>-1$

*Geometrically*, we can write

$$|X(j\omega)|=\sqrt{\frac{1}{(\omega^2+2^2)(\omega^2+1^2)}}=\frac{1}{B_1\cdot B_2}$$

$$\measuredangle X(j\omega)=-\left(\tan^{-1}\frac{\omega}{2}+\tan^{-1}\omega\right)=-(\varphi_1+\varphi_2)$$



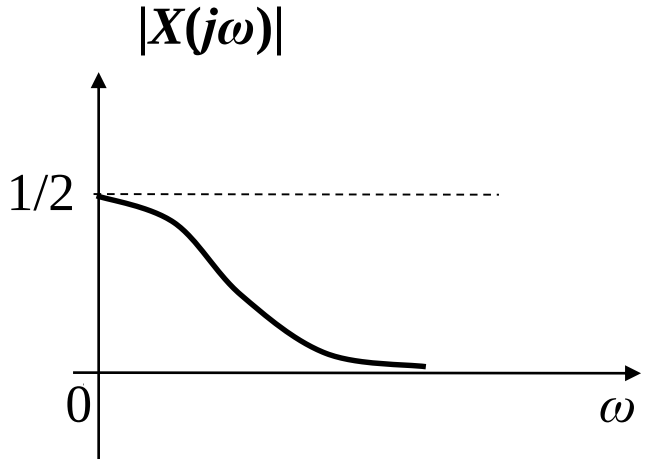

$\omega=0:\quad B_1=2,\ B_2=1,\ |X(j\omega)|=\dfrac{1}{2},\ \varphi_1=0,\varphi_2=0,\measuredangle X(j\omega)=0$

$$\omega\uparrow:\quad \begin{cases} B_1\uparrow, B_2\uparrow\Rightarrow |X(j\omega)|\downarrow \\ \varphi_1\uparrow,\varphi_2\uparrow\ (however,\ \varphi_1<\dfrac{\pi}{2},\varphi_2<\dfrac{\pi}{2})\Rightarrow \measuredangle X(j\omega)<0, |\measuredangle X(j\omega)|\uparrow \end{cases}$$

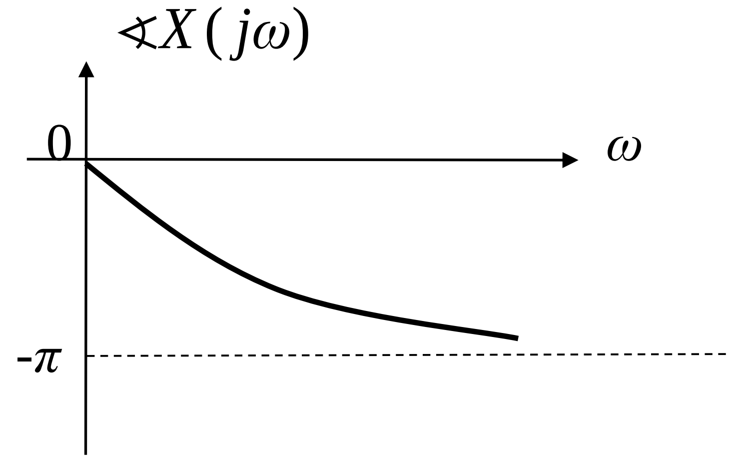
$\omega\to+\infty:$

$$B_1\to\infty, B_2\to\infty,\ |X(j\omega)|\to 0$$

$$\varphi_1\to\frac{\pi}{2},\varphi_2\to\frac{\pi}{2},\measuredangle X(j\omega)\to-\pi$$

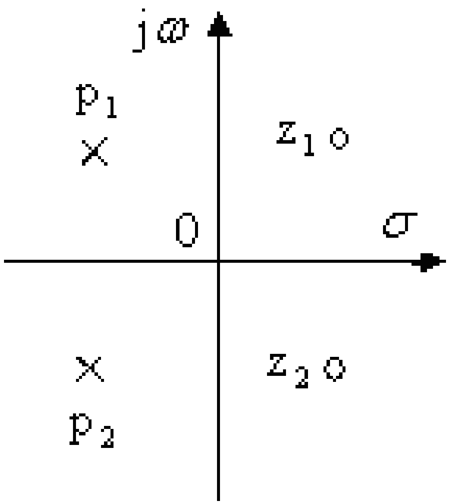


|X(jω)|
1/2
0
ω

∢X(jω)
0
ω
-π

All-pass function： A Laplace transform with all of its poles and zeros located on both sides of the *jω*- axis symmetrically. And all the poles are on the left of the *jω*- axis. All the zeros are on the right of the *jω*- axis.




$$p_1 = p_2^* = -z_2 = -z_1^*$$

The products of the magnitudes of all pole vectors

= The products of the magnitudes of all zero vectors

# 6.5 Properties of The Laplace Transform

## 6.5.1 Linearity

If $x_1(t) \leftrightarrow X_1(s), \quad ROC = R_1$ and $x_2(t) \leftrightarrow X_2(s), \quad ROC = R_2$

then $ax_1(t) + bx_2(t) \leftrightarrow aX_1(s) + bX_2(s), ROC \ \ containing \quad R_1 \cap R_2$

Note: *ROC* is at least the intersection of $R_1$ and $R_2$, which could be empty, also can be larger than the intersection.

## 6.5.2 Time Shifting

If $x(t) \leftrightarrow X(s), \quad ROC = R$

then $x(t - t_0) \leftrightarrow e^{-st_0} X(s), \quad ROC = R$

## 6.5.3 Shifting in the s-Domain

If $x(t) \leftrightarrow X(s), \quad ROC = R$

then $e^{s_0 t} x(t) \leftrightarrow X(s - s_0), \quad ROC = R + \mathrm{Re}\{s_0\}$

## 6.5.4 Time Scaling

If $x(t) \leftrightarrow X(s), \quad ROC = R$

then $x(\alpha t) \leftrightarrow \frac{1}{|\alpha|} X\left(\frac{s}{\alpha}\right), \quad ROC = \alpha R$

Special case: $x(-t) \leftrightarrow X(-s), \quad ROC = -R$

## 6.5.5 Conjugation

$$x^*(t) \leftrightarrow X^*(s^*), \quad ROC = R$$

If $x(t)$ is real: $X(s) = X^*(s^*)$

If $X(s)$ has a pole or zero at $s = s_0$ , then $X(s)$ also has a pole or zero at the complex conjugate point $s = s_0$*.

## 6.5.6 Convolution Property

If $x_1(t) \leftrightarrow X_1(s), \quad ROC = R_1$ and $x_2(t) \leftrightarrow X_2(s), \quad ROC = R_2$

then $x_1(t) * x_2(t) \leftrightarrow X_1(s) X_2(s), \ ROC \ \ containing \ R_1 \cap R_2$

## 6.5.7 Differentiation in the Time Domain

If $$x(t) \leftrightarrow X(s), \quad ROC = R$$

then $$\frac{dx(t)}{dt} \leftrightarrow sX(s), \quad ROC \ \ containing \ \ R$$

## 6.5. 8 Differentiation in the s-Domain

$$-tx(t) \leftrightarrow \frac{dX(s)}{ds}, \quad ROC = R$$

## 6.5.9 Integration in the Time Domain

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{X(s)}{s}, \quad ROC \ \ containing \ R \bigcap \left\{ \mathrm{Re}\left\{ s \right\} > 0 \right\}$$

## 6.5.10 The Initial- and Final-Value Theorems (初值和终值定理)

*Initial-value theorem* :
$$x(0^+) = \lim_{s\to\infty} sX(s)$$

*Note*: For rational $\boldsymbol{X(s)} = \frac{\boldsymbol{N(s)}}{\boldsymbol{D(s)}}$, If the order of $\boldsymbol{N(s)}$ is no less than the order of $\boldsymbol{D(s)}$, rewrite $X(s)$ as a sum of a polynomial of $s$ and $X_1(s)$, which has a lower order of numerator than that of denominator, then take $X_1(s)$ into the above limit on the right side.

*Final -value theorem* :
$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$$

To guarantee the existence of the final value, no poles of $X(s)$ lie on the right side of $j\omega$-axis and a first-order pole at the origin if there is pole on the $j\omega$-axis.

Example 6.9

Consider the signal $x(t)=u(t)-u(t-1)$.

Sol: As we know $u(t)\xleftrightarrow{LT}\frac{1}{s},\quad \mathrm{Re}\{s\}>0$

From the time shifting $u(t-1)\xleftrightarrow{LT}\frac{1}{s}e^{-s},\quad \mathrm{Re}\{s\}>0$

So that $X(s)=\frac{1}{s}-\frac{1}{s}e^{-s}=\frac{1-e^{-s}}{s},\quad ROC=entire\ s\ plane.$

Example 6.10

Determine the Laplace transform of $x(t)=te^{-\alpha t}u(t)$.

Sol: Since $e^{-\alpha t}u(t)\xleftrightarrow{LT}\frac{1}{s+\alpha},\quad \mathrm{Re}\{s\}>-\alpha$

From the differentiation in the s-domain

$$te^{-\alpha t}u(t)\xleftrightarrow{LT}-\frac{d}{ds}\left(\frac{1}{s+\alpha}\right)=\frac{1}{(s+\alpha)^2},\quad \mathrm{Re}\{s\}>-\alpha$$

# 6.5 Properties of The Laplace Transform

In fact, by repeated application of this property, we obtain

$$\boxed{\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t) \xleftrightarrow{LT} \frac{1}{(s+\alpha)^n}, \quad \mathcal{Re}\{s\} > -\alpha}$$

Example 6.11

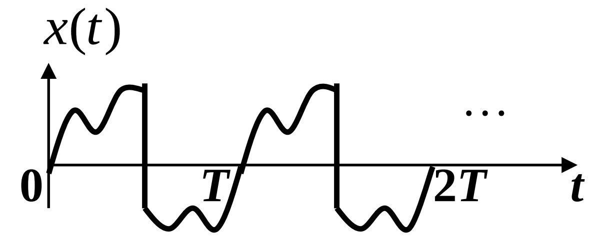
Use the initial-value theorem to determine the initial-value of

$$x(t) = e^{-2t}u(t) + e^{-t}(\cos 3t)u(t)$$

$$e^{-2t}u(t) + e^{-t}(\cos 3t)u(t) \xleftrightarrow{LT} \frac{2s^2+5s+12}{(s^2+2s+10)(s+2)}, \quad \mathrm{Re}\{s\} > -1$$

$$x(0^+) = \lim_{s\to\infty} sX(s) = \lim_{s\to\infty} \frac{2s^3+5s^2+12s}{s^3+4s^2+14s+20} = 2$$

Example 6.12 Determine the Laplace transform of the *causal periodic signal* $x(t)$ which is depicted in the figure:




$$\text{Sol}: x(t) = x_0(t) + x_0(t-T) + x_0(t-2T) + \cdots$$

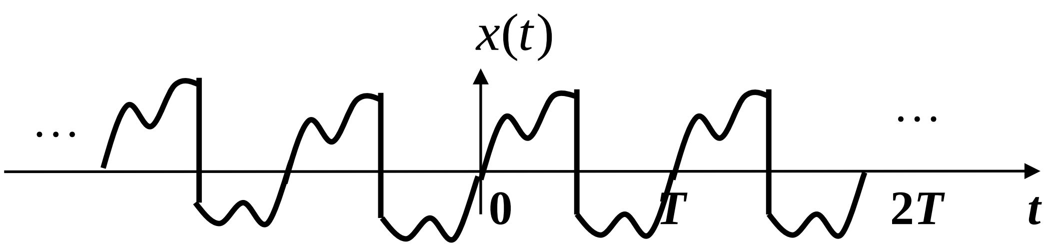
$$X(s) = X_0(s) + X_0(s)\cdot e^{-s\cdot T} + X_0(s)\cdot e^{-s\cdot 2T} + \cdots$$

$$= X_0(s)[1 + e^{-s\cdot T} + e^{-s\cdot 2T} + \cdots]$$

$$= X_0(s)\cdot \frac{1}{1-e^{-sT}} \quad \left(\mathrm{Re}\{s\} > 0\right) \quad \boxed{= \frac{X_0(s)}{1-e^{-sT}}}$$

*Consider:*

What's the LT of the periodic signal in the following figure?



# 6.6 Analysis And Characterization of LTI Systems Using Laplace Transform

- The Laplace transforms of the input and the output of an LTI system are related through multiplication by the Laplace transform of the impulse response of the system.

$$Y(s) = \boxed{H(s)}\, X(s)$$

System function、Transfer function

- The *ROC* associated with the system function for a causal system is a right-half plane.
  - An *ROC* to the right of the rightmost pole does not guarantee that a system is causal.
  - For a system with a *rational* system function, causality of the system is *equivalent to* the *ROC* being the right-half plane to the right of the rightmost pole.

Example 6.13

Consider a system with impulse response $h(t) = e^{-(1+2j)t}u(t)$.

Sol: Since $h(t) = 0$ for $t < 0$, this system is causal.

It's easy to find the system function: $H(s) = \dfrac{1}{s+1+2j}, \ \mathrm{Re}\{s\} > -1$

The $H(s)$ is rational and the *ROC* is to the right of the rightmost pole, consistent with our statement.

Example 6.14

Consider the system function $H(s) = \dfrac{e^{s}}{s+1}, \quad \mathrm{Re}\{s\} > -1$ irrational !

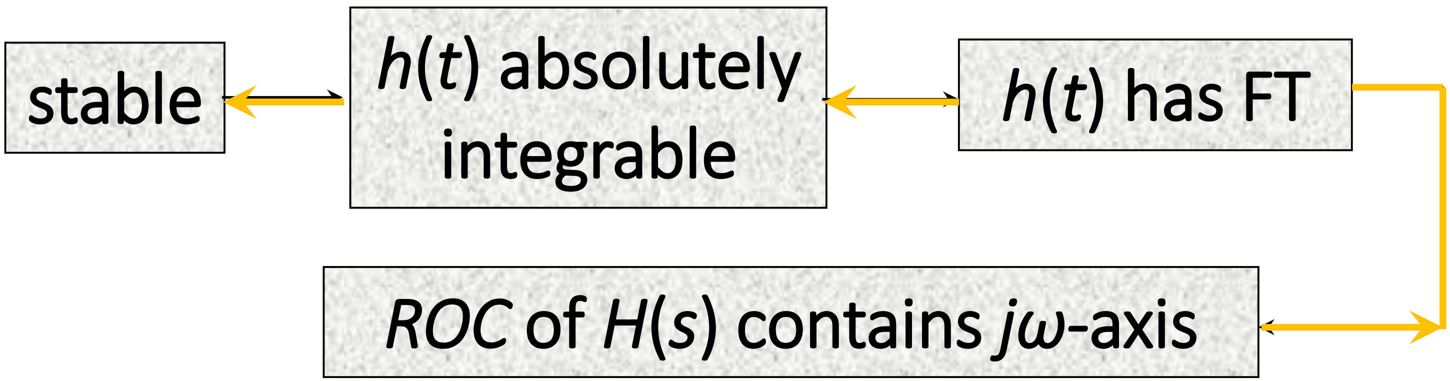
Sol: For this system, the *ROC* is to the right of the rightmost pole.

But the impulse response can be obtained as $h(t) = e^{-(t+1)}u(t+1)$

Obviously this system is not causal.

## 6.6 Analysis And Characterization of LTI Systems Using Laplace Transform

- An LTI system is stable *if and only if* the *ROC* of its system function $H(s)$ includes the $j\omega$-axis [i.e., $\text{Re}\{s\} = 0$].




- A causal system with rational system function $H(s)$ is stable if and only if all of the poles of $H(s)$ lie in the left-half of the $s$-plane —i.e., all of the poles have negative real parts.

- For an LTI system which is described by a linear constant-coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{d t^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{d t^k}$$

Its system function (transfer function) is:

$$\boxed{H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}}$$

The system function for a system specified by a differential equation is always rational.

# 6.6 Analysis And Characterization of LTI Systems Using Laplace Transform

## Example 6.15 Given the information about an LTI system:

1. The system is causal. 2. The $H(s)$ is rational and has only two poles, at $s = -2$ and $s = 4$. 3. If $x(t) = 1$, then $y(t) = 0$. 4. $\boldsymbol{h(0^+) = 4}$.

## Determine the system function of the system.

Sol: From fact 2, we write $$H(s) = \frac{p(s)}{(s+2)(s-4)} = \frac{p(s)}{s^2 - 2s - 8}$$

From fact 3, using eigenfunction property

$$x(t) = 1 = e^{0 \cdot t} \quad \Rightarrow \quad y(t) = H(0) \cdot e^{0 \cdot t} = H(0) = 0$$

$p(s)$ must have a root at $s = 0$ and thus is of the form $p(s) = sq(s)$.

From fact 4 and 1, $$\lim_{s \to \infty} sH(s) = \lim_{s \to \infty} \frac{s^2 q(s)}{s^2 - 2s - 8} = 4$$

The highest powers in $s$ in both the denominator and the numerator are identical, that is, $q(s)$ must be a constant. We let $q(s) = k$.

It's easy to find that $k = 4$. So that $$H(s) = \frac{4s}{(s+2)(s-4)}$$

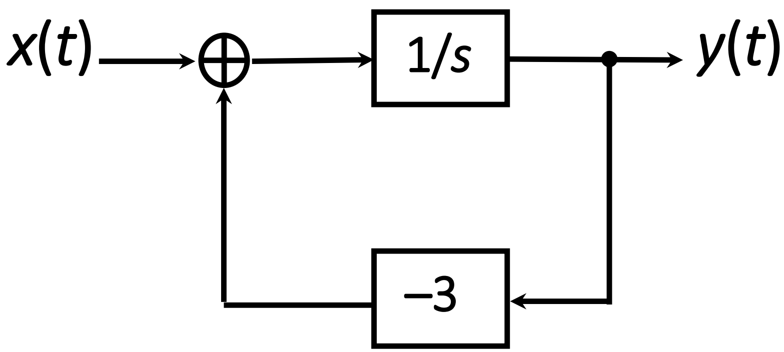
Example 6.16

Consider the causal LTI system with system function

$$H(s)=\frac{1}{s+3}$$

This system can also be described by the differential equation

$$\frac{dy(t)}{dt}+3y(t)=x(t)$$




Block diagram representation of the causal LTI system

$1/s$ is the system function of a system with impulse response $u(t)$, i.e., it is the system function of an integrator.

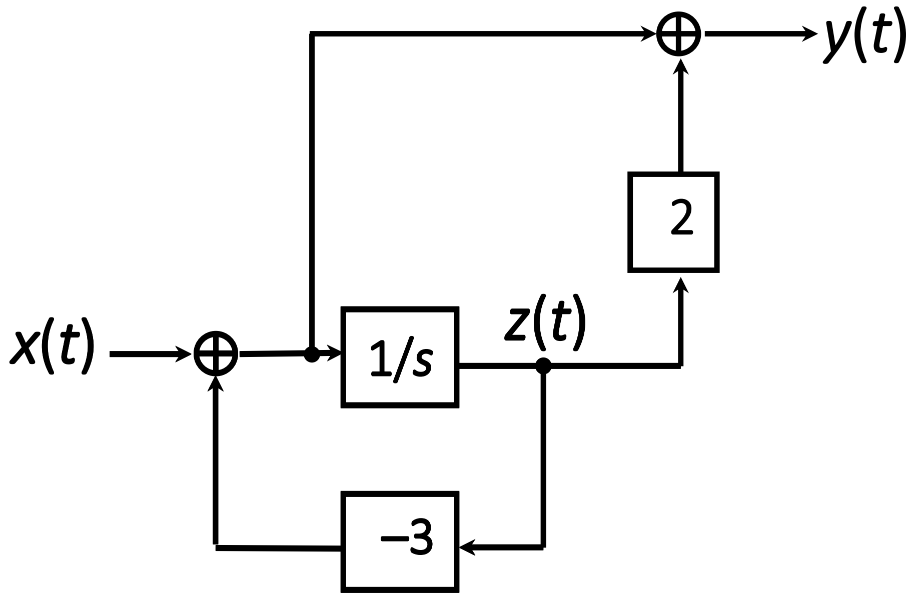
Example 6.17

Consider a causal LTI system with system function $H(s)=\dfrac{s+2}{s+3}$

$$H(s)=\underbrace{\left(\frac{1}{s+3}\right)}_{H_1(s)}\underbrace{(s+2)}_{H_2(s)}$$

Let $z(t)$ be the output of the first subsystem, $y(t)$ is the output of the overall system.

$$y(t)=\frac{dz(t)}{dt}+2z(t)$$



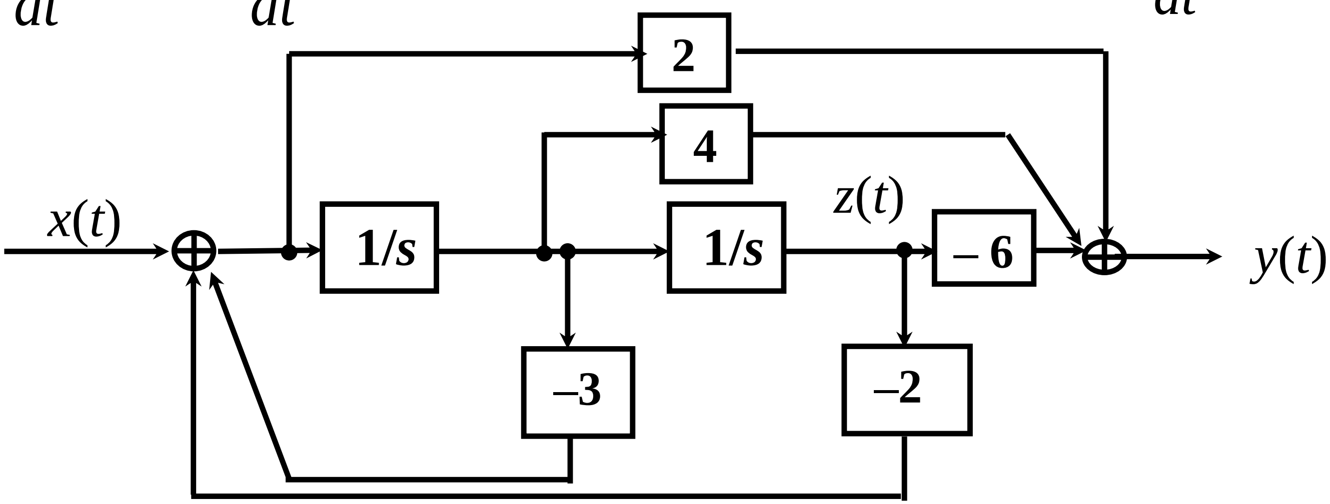

Block diagram representation for the system

Example 6.18 Consider a second-order LTI system with system function

$$H(s)=\frac{2s^2+4s-6}{s^2+3s+2}$$

$$H(s)=\underbrace{\left(\frac{1}{s^2+3s+2}\right)}_{H_1(s)}\underbrace{\left(2s^2+4s-6\right)}_{H_2(s)}$$

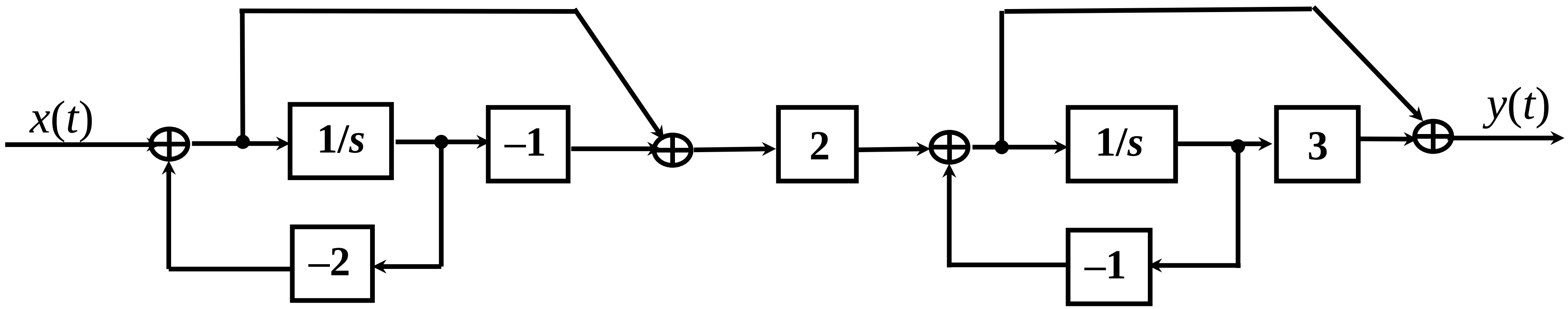
$$H_1(s):\quad \frac{d^2z(t)}{dt^2}+3\frac{dz(t)}{dt}+2z(t)=x(t) \qquad H_2(s):\quad y(t)=2\frac{d^2z(t)}{dt^2}+4\frac{dz(t)}{dt}-6z(t)$$




Direct-form (直接型) representation for the system

# 6.7 System Function Algebra And Block Diagram Representations

$$H(s)=\left(2\cdot\frac{s-1}{s+2}\right)\left(\frac{s+3}{s+1}\right)$$




Cascade-form (级联型) representation

What's the cascade-form representation if

$$H(s)=\left(\frac{2s-2}{s+2}\right)\left(\frac{s+3}{s+1}\right)$$

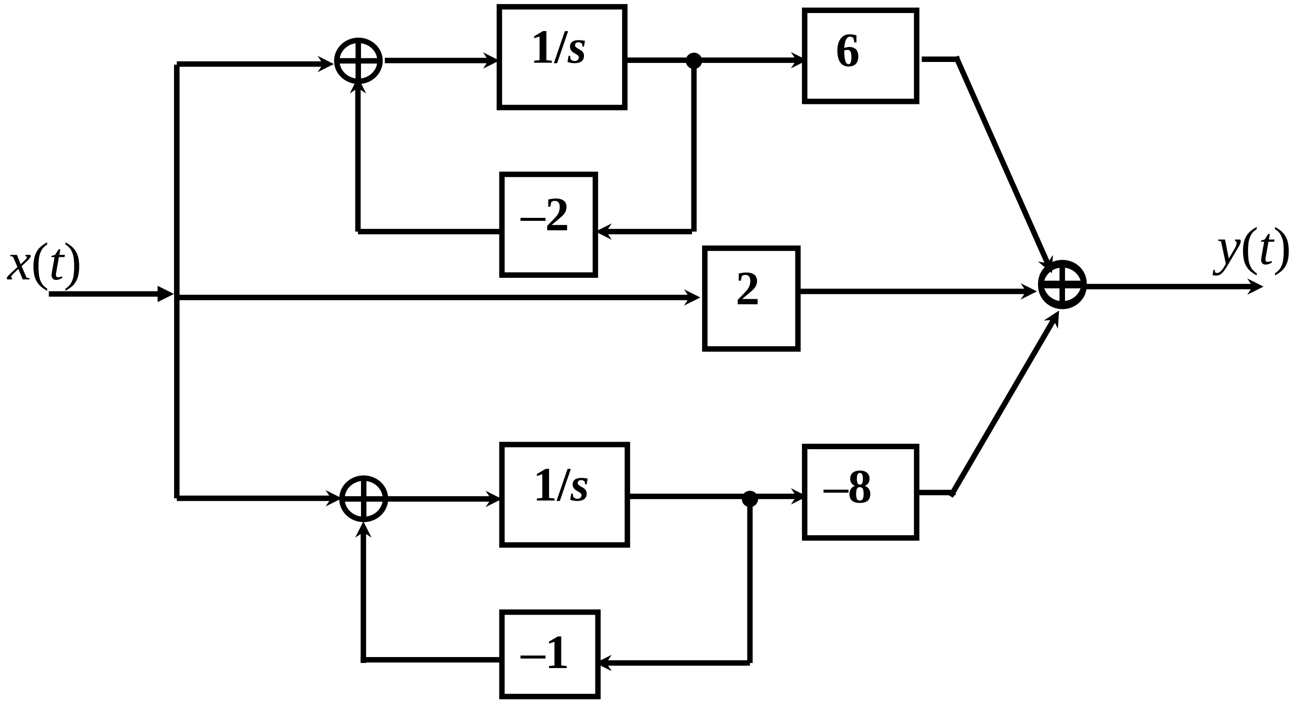
or

$$H(s)=2\left(\frac{s+3}{s+2}\right)\left(\frac{s-1}{s+1}\right)$$

?

# 6.7 System Function Algebra And Block Diagram Representations

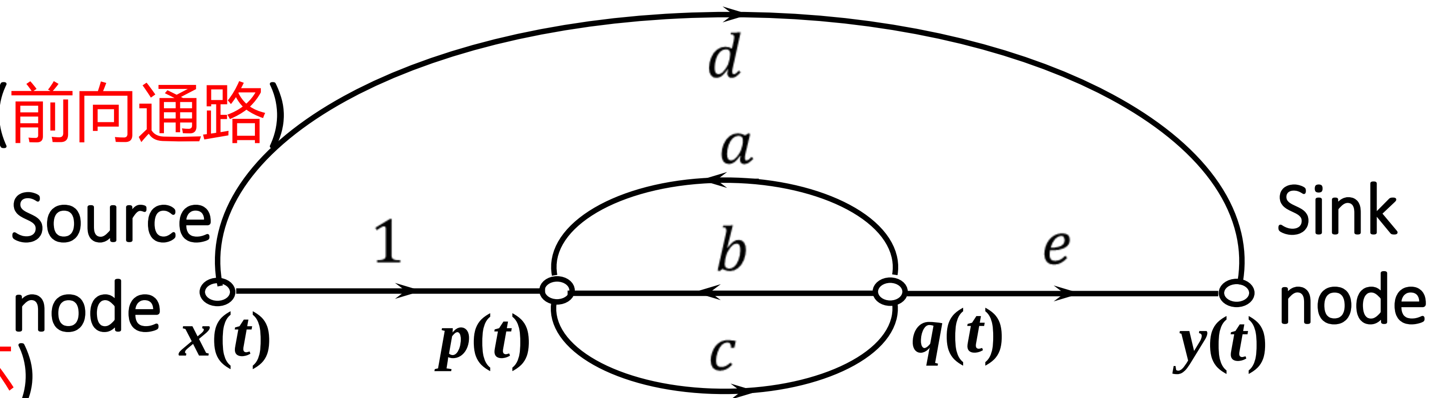
$$H(s)=\frac{6}{s+2}+2-\frac{8}{s+1}$$




Parallel-form (并联型) representation

# 6.8 Signal Flow Graph (信号流图) Representations

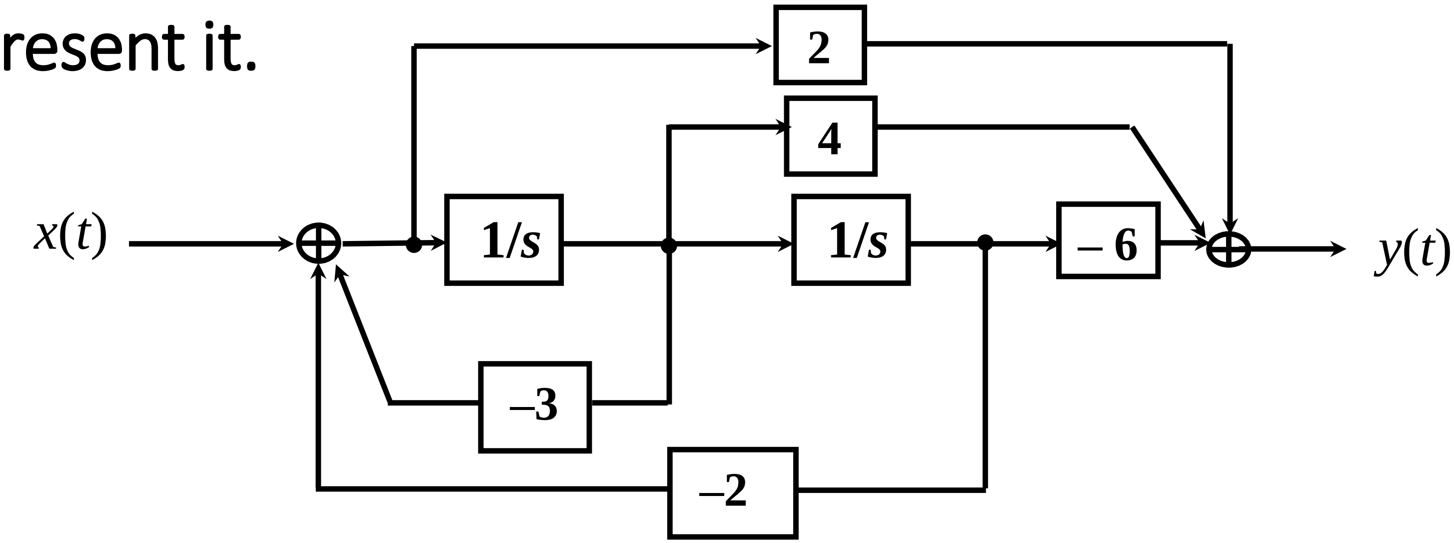
Formally, a *signal flow graph* is a network of directed branches that connect at nodes. Associated with each node is a variable or node value.

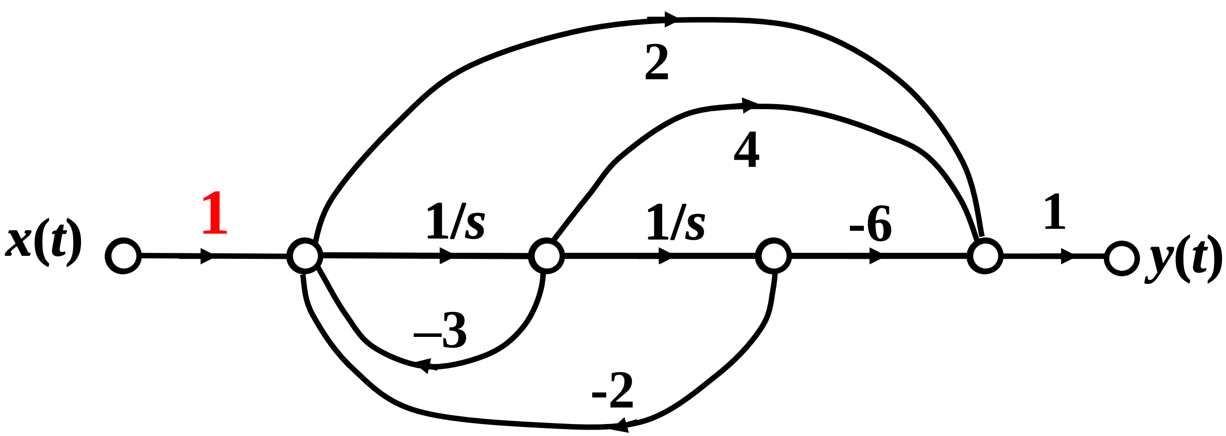



- forward path (前向通路)
- loop (环)
- self-loop (自环)
- *Source nodes* (源结点): nodes that have only outgoing branches (出支路), which are used to represent the injection of external inputs or signal sources into a graph.
- *Sink nodes* (汇结点): nodes that have only entering branches (入支路), which are used to extract outputs from a graph.
- mixed node (混合结点)

## Example 6.19

Consider again the system in example 6.18. Use signal flow graph to represent it.




$$H(s)=\frac{2s^2+4s-6}{s^2+3s+2}$$

# 6.8 Signal Flow Graph (信号流图) Representations

**Mason's Formula：(梅森公式)**

$$H = \frac{1}{\Delta}\sum_k G_k \Delta_k$$

Manson's formula is used to calculate the transfer value (transfer function) between any source node and sink node (or mixed node) in a signal flow graph.

Where $\Delta = 1 - \sum_i L_i + \sum_{i,j} L_i L_j - \sum_{i,j,k} L_i L_j L_k + \cdots$

is the graph determinant (特征行列式) .

$\boldsymbol{L_i}$ is the gain of each loop.

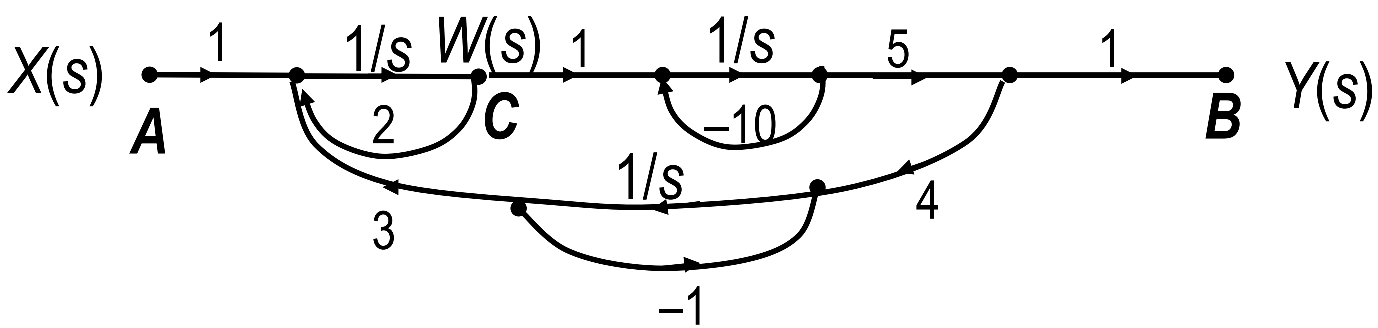
$\boldsymbol{L_i L_j}$ is the product of the gains of two loops which have no shared nodes and branches .

$\boldsymbol{G_k}$ is the gain of the *k*-th forward path between the source node and the sink node (or mixed node).

$\boldsymbol{\Delta_k}$ is the graph determinant of the left graph after remove the *k*-th forward path.

## Example 6.20

Compute the transfer functions between nodes A and B, and nodes A and C in the following signal flow graph.

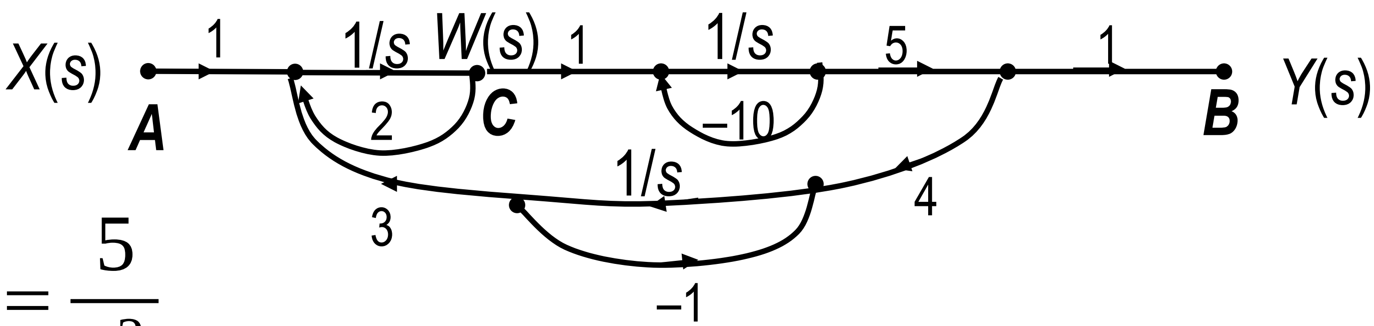



$$\Delta = 1 - \left[\frac{2}{s} - \frac{10}{s} - \frac{1}{s} + \frac{60}{s^3}\right] + \left[\frac{2}{s}\left(-\frac{10}{s}\right) + \frac{2}{s}\left(-\frac{1}{s}\right) + \left(-\frac{10}{s}\right)\left(-\frac{1}{s}\right)\right] - \frac{2}{s}\left(-\frac{10}{s}\right)\left(-\frac{1}{s}\right)$$

$$= 1 + \frac{9}{s} - \frac{12}{s^2} - \frac{80}{s^3}$$

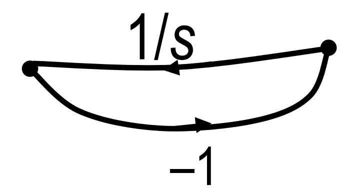
# 6.8 Signal Flow Graph (信号流图) Representations

A to B:




$$G_1 = \frac{5}{s^2}$$

After removing $G_1$, the left graph is:



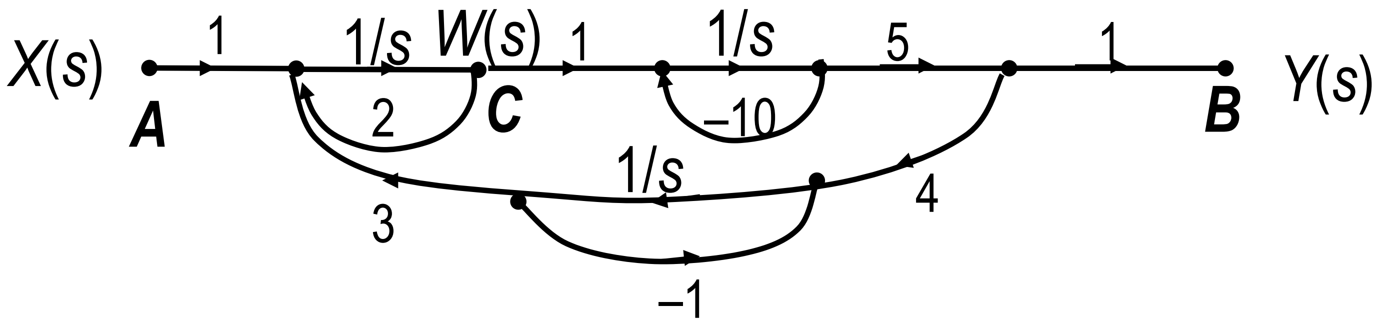

$$\Delta_1 = 1 + \frac{1}{s}$$

Thus,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{5}{s^2}\left(1+\frac{1}{s}\right)}{1+\frac{9}{s}-\frac{12}{s^2}-\frac{80}{s^3}} = \frac{5s+5}{s^3+9s^2-12s-80}$$

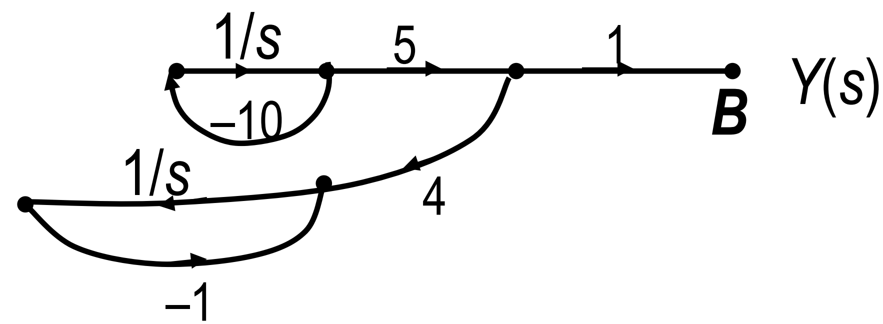
# 6.8 Signal Flow Graph (信号流图) Representations

A to C:




$$G_1 = \frac{1}{s}$$

After removing $G_1$ , the left graph is:




$$\Delta_1 = 1 - \left[\left(-\frac{10}{s}\right) + \left(-\frac{1}{s}\right)\right] + \left[\left(-\frac{10}{s}\right)\left(-\frac{1}{s}\right)\right] = 1 + \frac{11}{s} + \frac{10}{s^2}$$

Thus, $$H(s) = \frac{W(s)}{X(s)} = \frac{s^2 + 11s + 10}{s^3 + 9s^2 - 12s - 80}$$

# 6.9 The Unilateral Laplace Transform

## 6.9.1 Introduction of the Unilateral Laplace Transform

*Bilateral* Laplace transform: (双边拉普拉斯变换)

$$X(s)=\int_{-\infty}^{\infty}x(t)e^{-st}dt$$

*Unilateral* Laplace transform: (单边拉普拉斯变换)

$$\mathcal{X}(s)=\int_{0^-}^{\infty}x(t)e^{-st}dt$$

- The lower limit of integration, $\mathbf{0}^-$, signifies that we include in the interval of integration any impulses or higher order singularity functions (奇异函数) concentrated at $t$ = 0.
- The bilateral transform depends on the entire signal from $\boldsymbol{t}=-\infty$ to $\boldsymbol{t}=+\infty$, whereas the unilateral transform depends only on the signal from $\boldsymbol{t}=\mathbf{0}^-$ to $\infty$.
- The bilateral transform and the unilateral transform of a causal signal are identical.
- The *ROC* for the unilateral transform is always a right-half plane

## 6.9 The Unilateral Laplace Transform

- The evaluation of the inverse unilateral Laplace transforms is also the same as for bilateral transforms, with the constraint that the *ROC* for a unilateral transform must always be a right-half plane.

Example 6.21 Consider the signal $x(t) = e^{-\alpha(t+1)}u(t+1)$.

Sol: The bilateral transform *X*(*s*) can be obtained from Example 6.1 and the time-shifting property:

$$X(s) = \frac{e^{s}}{s+\alpha}, \ \mathrm{Re}\{s\} > -\alpha$$

By contrast, the unilateral transform is

$$\mathcal{X}(s) = \int_{0^-}^{\infty} e^{-\alpha(t+1)}u(t+1)e^{-st}dt = \int_{0^-}^{\infty} e^{-\alpha}e^{-t(s+\alpha)}dt = e^{-\alpha}\frac{1}{s+\alpha} \quad \mathrm{Re}\{s\} > -\alpha$$

We could recognize $\mathcal{X}(s)$ as the bilateral transform of *x*(*t*)*u*(*t*).

Since $x(t)u(t) = e^{-\alpha}e^{-\alpha t}u(t)$, Thus $\mathcal{X}(s) = e^{-\alpha}\frac{1}{s+\alpha}, \quad \mathrm{Re}\{s\} > -\alpha$

# 6.9 The Unilateral Laplace Transform

Example 6.22 Consider the unilateral LT $\mathcal{X}(s)=\dfrac{1}{(s+1)(s+2)}$

Determine the corresponding $x(t)$.

Sol: For the unilateral transform, the ROC must be the right-half plane to the right of the rightmost pole of $\mathcal{X}(s)$.

In this case, the ROC consists of all points $s$ with $\text{Re}\{s\} > -1$.

Thus $$x(t)=(e^{-t}-e^{-2t})u(t)$$

✓unilateral Laplace transform provide us with information about signals only for $\boldsymbol{t > 0^-}$ .

Example 6.23 Calculate the inverse of the unilateral LT $\mathcal{X}(s)=\dfrac{s^2-3}{s+2}$.

Sol: $$\mathcal{X}(s)=-2+s+\frac{1}{s+2}$$

Taking inverse transforms of each term results in

$$x(t)=-2\delta(t)+\delta'(t)+e^{-2t}u(t)$$

## 6.9.2 Properties of the Unilateral Laplace Transform

✓ Time scaling: $$x(at) \xleftrightarrow{UL} \frac{1}{a} X(\frac{s}{a}), \quad a > 0$$

✓ Convolution: assuming that $x_1(t)$ and $x_2(t)$ are identically zero for $t < 0$.

$$x_1(t) * x_2(t) \xleftrightarrow{UL} X_1(s) X_2(s)$$

✓ Differentiation in the time domain :

$$\frac{d}{dt} x(t) \xleftrightarrow{UL} sX(s) - x(0^-)$$

$$\boxed{\frac{d^n}{dt^n} x(t) \xleftrightarrow{UL} s^n X(s) - \sum_{k=0}^{n-1} s^{n-k-1} x^{(k)}(0^-)}$$

Proof of this property for first-derivative of *x*(*t*):

$$\mathcal{UL}\left\{\frac{dx(t)}{dt}\right\} = \int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = \int_{0^-}^{\infty} e^{-st} dx(t)$$

$$= x(t)e^{-st}\Big|_{0^-}^{\infty} + s\int_{0^-}^{\infty} x(t)e^{-st} dt$$

$$= sX(s) - x(0^-)$$

Similarly, the unilateral Laplace transform of second-derivative of *x*(*t*) can be obtained by repeating using the property:

$$\mathcal{UL}\left\{\frac{d^2x(t)}{dt^2}\right\} = s(sX(s) - x(0^-)) - x'(0^-)$$

$$= s^2X(s) - sx(0^-) - x'(0^-).$$

## 6.9.2 Solving Differential Equations Using the Unilateral Laplace Transform

Example 6.24

Consider the causal system characterized by the differential equation

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=x(t)$$

with initial conditions $\boldsymbol{y}(\mathbf{0}^-)=\boldsymbol{\beta}, \boldsymbol{y}'(\mathbf{0}^-)=\boldsymbol{\gamma}$ and input signal $\boldsymbol{x}(\boldsymbol{t})=\boldsymbol{\alpha}\boldsymbol{u}(\boldsymbol{t})$. Determine the output signal $y(t)$.

Sol: Applying the unilateral LT to both sides of the differential equation yields

$$s^2Y(s)-sy(0^-)-y'(0^-)+3sY(s)-3y(0^-)+2Y(s)=X(s)$$

or equivalently, $s^2Y(s)-\beta s-\gamma+3sY(s)-3\beta+2Y(s)=\dfrac{\alpha}{s}$

Thus, we obtain $Y(s)=\dfrac{\beta s^2+(\gamma+3\beta)s+\alpha}{s(s+1)(s+2)}$

$$Y(s)=\underbrace{\frac{\beta(s+3)}{(s+1)(s+2)}+\frac{\gamma}{(s+1)(s+2)}}_{zero-input\ response}+\underbrace{\frac{\alpha}{s(s+1)(s+2)}}_{zero-state\ response}$$

Conclusion: The unilateral Laplace transform is of *considerable value* in analyzing causal systems which are specified by linear constant-coefficient differential equations with *nonzero initial conditions* (i.e., systems that are not initially at rest).

# 6.10 SUMMARY

- The bilateral and unilateral Laplace transforms;
- The properties of the *ROC* of LT and the relationship between the *ROC* and the poles;
- Methods to calculate the inverse Laplace transform;
- The properties of the bilateral and unilateral Laplace transforms (note the similarities and the differences);
- Significance of the poles and zeros of LT in characterizing continuous-time signals and systems;
- The computations of the zero-state response and the zero-input response by Laplace transform.
- The block diagram and signal flow graph representations of continuous-time LTI systems.

# Homework

9.21 (a) (d) (g) (j)    9.22 (a) (c) (e) (g)

9.23    9.24    9.25 (b) (d) (f)    9.27

9.31    9.38    9.40