练习九.

1. 当$\lambda<0$时 $X(x)=Ae^{-\sqrt{-\lambda}x}+Be^{\sqrt{-\lambda}x}$

$\because X(-\pi)=X(\pi) \quad X'(-\pi)=X'(\pi)$

$\therefore \begin{cases} A=B \\ A+B=0 \end{cases} \Rightarrow A=B=0$ 无非平凡解

当$\lambda=0$时, $X(x)=Ax+B$

$\therefore A=0 \quad X_0(x)=B_0$

当$\lambda>0$时, $X(x)=A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x$

$\therefore \begin{cases} A\cos\sqrt{\lambda}\pi-B\sin\sqrt{\lambda}\pi=A\cos\sqrt{\lambda}\pi+B\sin\sqrt{\lambda}\pi \\ A\sin\sqrt{\lambda}\pi+B\cos\sqrt{\lambda}\pi=-A\sin\sqrt{\lambda}\pi+B\cos\sqrt{\lambda}\pi \end{cases}$

$\therefore \sin\sqrt{\lambda}\pi=0$

$\lambda=\lambda_n=n^2 \quad (n=1.2\cdots)$

$\therefore$ 特征值 $\lambda_n=n^2 \quad (n=0.1.2\cdots)$

特征函数为 $X_n=a_n\cos nx+b_n\sin nx$

2. 设 $x=e^t$，则 $t=\ln x$

$\therefore y_x = y_t \cdot \frac{1}{x}$, $y_{xx} = y_{tt} \cdot \frac{1}{x^2} - \frac{1}{x^2} y_t$.

代入原方程，

$y_{tt} + \lambda y = 0 \qquad y(0) = y(1) = 0.$

$\therefore \lambda_n = (n\pi)^2$. $y_n(t) = B_n \sin n\pi t$

$\therefore y_n(x) = B_n \sin(n\pi \ln x)$

$\therefore$ 固有函数系为 $\{\sin(n\pi \ln x)\}$

$\because \int_1^e \frac{1}{x} y_n(x) y_m(x)\, dx$.

$= \int_0^1 y_n(t) y_m(t)\, dt$

$= \int_0^1 \sin n\pi t \sin m\pi t\, dt = \begin{cases} 0, & m \neq n. \\ \frac{1}{2}, & m = n \end{cases}$

解计

练习十.

1. $u(x,t)=\dfrac{\phi(x-at)+\phi(x+at)}{2}+\dfrac{1}{2a}\int_{x-at}^{x+at}-k\phi'(\alpha)\,d\alpha$

$=\dfrac{k+a}{2a}\phi(x-at)+\dfrac{a-k}{2a}\phi(x+at)$

2. $u(x,t)=\dfrac{\sin(x-at)+\sin(x+at)}{2}+\dfrac{1}{2a}\int_{x-at}^{x+at}\alpha^2\,d\alpha$

$=\cancel{\sin(x-at)}\ \sin x\cos at+x^2t+\frac{1}{3}a^2t^3$

3. $u(x,t)=\dfrac{(x-at)+(x+at)}{2}+\dfrac{1}{2a}\int_{x-at}^{x+at}\sin\alpha\,d\alpha+\dfrac{1}{2a}\int_0^t\int_{x-a(t-\tau)}^{x+a(t-\tau)}a\tau+\xi\,d\xi\,d\tau$

$=x+\frac{1}{a}\sin x\sin at+\frac{1}{2}xt^2+\frac{1}{6}at^3$