Machine Learning

Deep Learning: Backpropagation Algorithm

Acknowledge: The slides are partially referred to coursera online machine learning course by Prof. Andrew Ng, BYU Data Mining Lab, CS-478-Machine learning, and NYU machine learning course. All copyrights owned by original authors. (i.e. http://u.cs.biu.ac.il/~89-570/)

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Gradient Descent

Network parameters $\theta = \{w_1, w_2, \cdots, b_1, b_2, \cdots\}$

Starting Parameters $\theta^0 \longrightarrow \theta^1 \longrightarrow \theta^2 \longrightarrow$ ......

$$\nabla L(\theta) = \begin{bmatrix} \partial L(\theta)/\partial w_1 \\ \partial L(\theta)/\partial w_2 \\ \vdots \\ \partial L(\theta)/\partial b_1 \\ \partial L(\theta)/\partial b_2 \\ \vdots \end{bmatrix}$$

$Compute\ \nabla L(\theta^0)$ $\qquad \theta^1 = \theta^0 - \eta \nabla L(\theta^0)$

$Compute\ \nabla L(\theta^1)$ $\qquad \theta^2 = \theta^1 - \eta \nabla L(\theta^1)$

Millions of parameters ......

To compute the gradients efficiently, we use ***backpropagation***.

Chain Rule

Case 1

$$y = g(x) \qquad z = h(y)$$

$$\Delta x \to \Delta y \to \Delta z \qquad\qquad \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

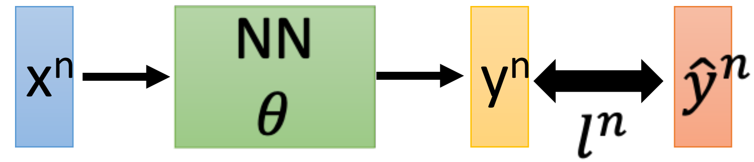
Case 2

$$x = g(s) \qquad y = h(s) \qquad z = k(x, y)$$

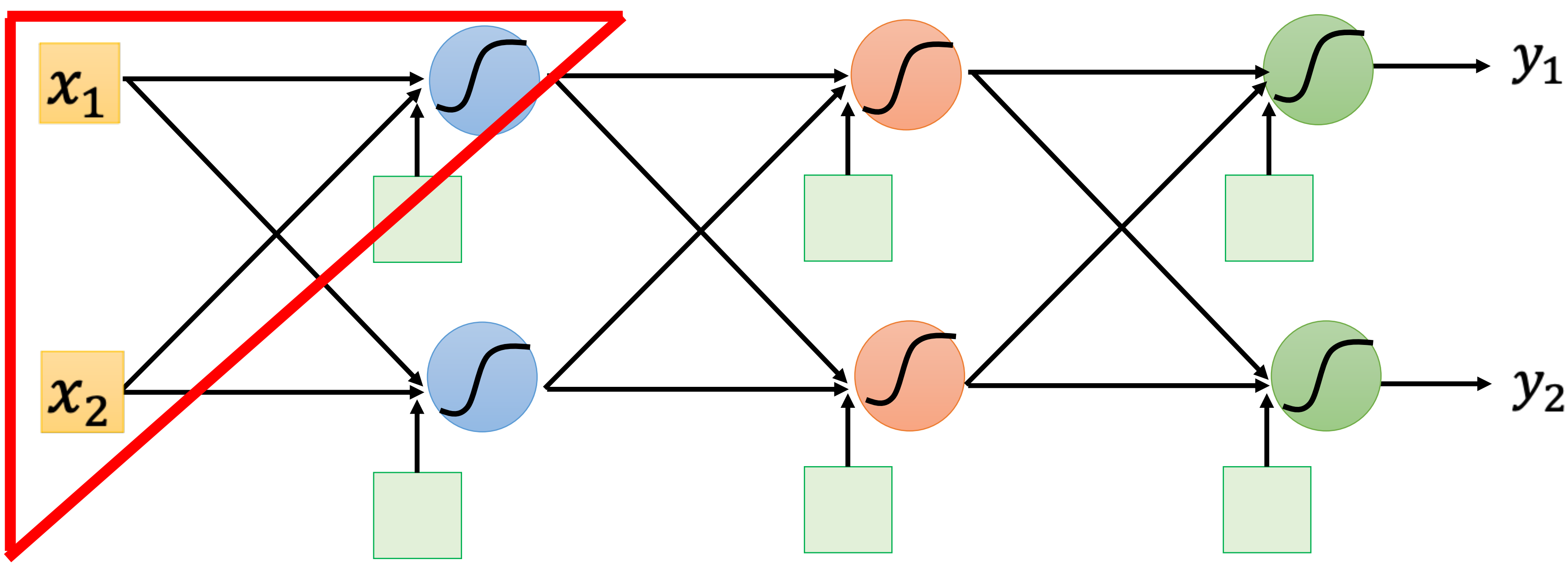
$$\begin{tikzcd} & \Delta x \arrow[dr] & \\ \Delta s \arrow[ur] \arrow[dr] & & \Delta z \\ & \Delta y \arrow[ur] & \end{tikzcd}$$

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}$$

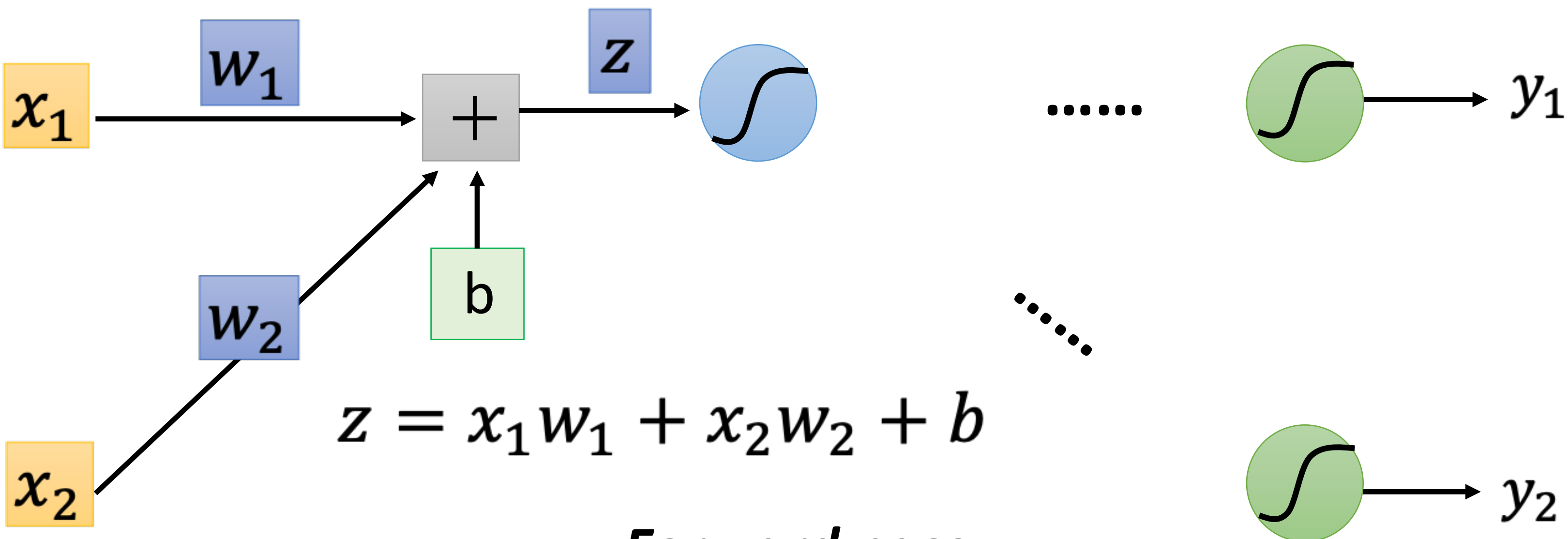
Backpropagation



$$L(\theta) = \sum_{n=1}^{N} l^n(\theta) \quad \Rightarrow \quad \frac{\partial L(\theta)}{\partial w} = \sum_{n=1}^{N} \frac{\partial l^n(\theta)}{\partial w}$$



Backpropagation



$$z = x_1 w_1 + x_2 w_2 + b$$

$$\frac{\partial l}{\partial w} = ? \quad \frac{\partial z}{\partial w} \frac{\partial l}{\partial z}$$

(Chain rule)

Forward pass:

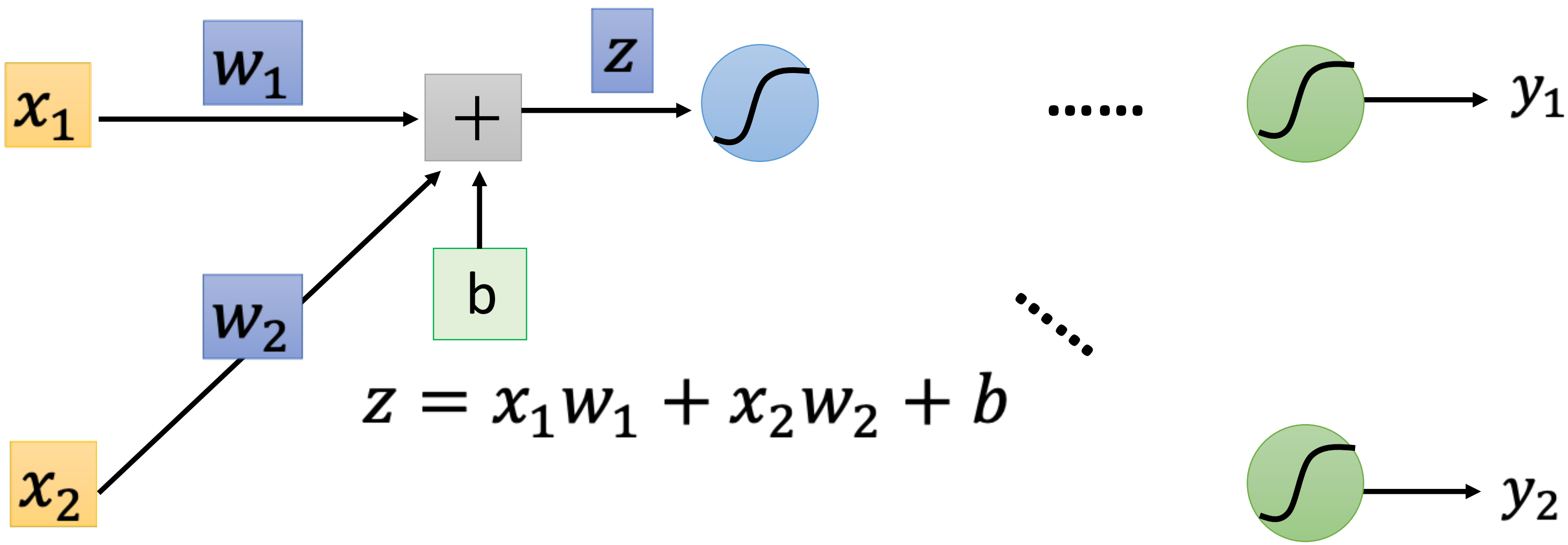
Compute $\partial z / \partial w$ for all parameters

Backward pass:

Compute $\partial l / \partial z$ for all activation function inputs z

Backpropagation – Forward pass

Compute $\partial z / \partial w$ for all parameters

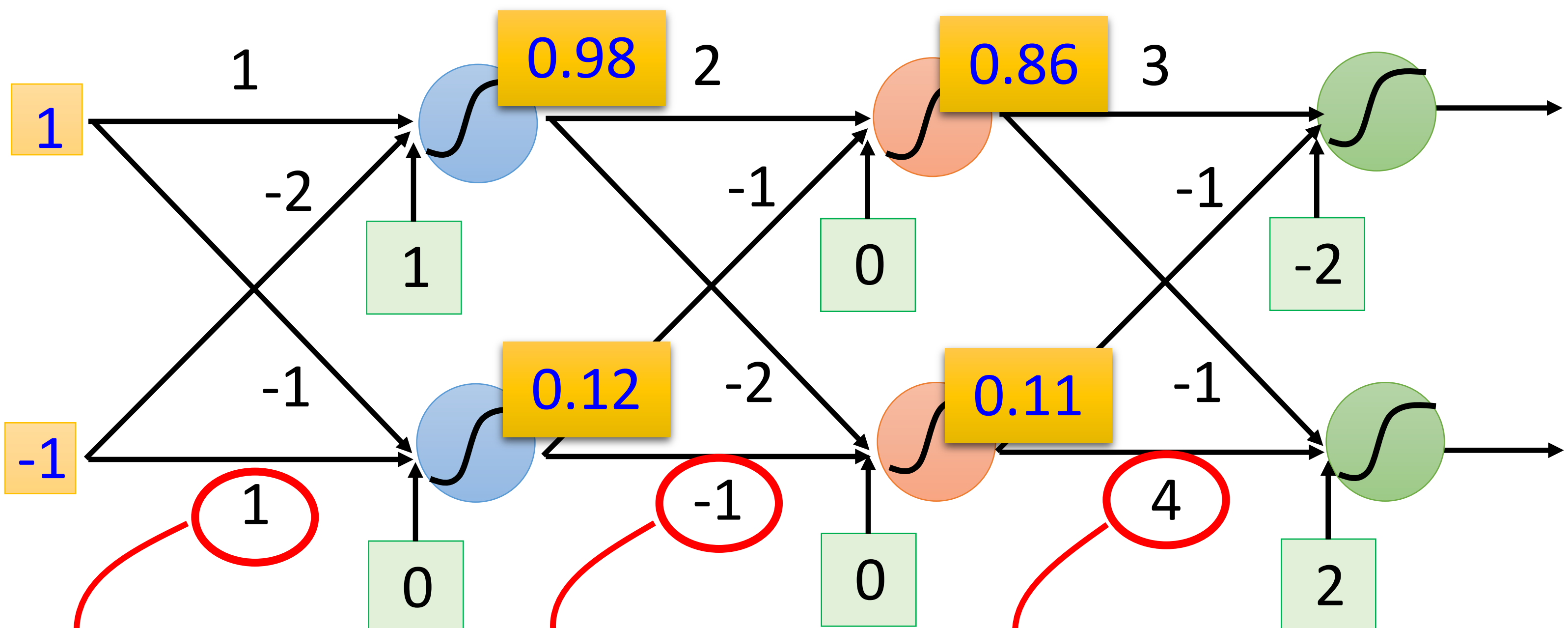


$$z = x_1 w_1 + x_2 w_2 + b$$

$$\left.\begin{aligned} \partial z / \partial w_1 &=? \;\; x_1 \\ \partial z / \partial w_2 &=? \;\; x_2 \end{aligned}\right\} \text{The value of the input connected by the weight}$$

Backpropagation – Forward pass

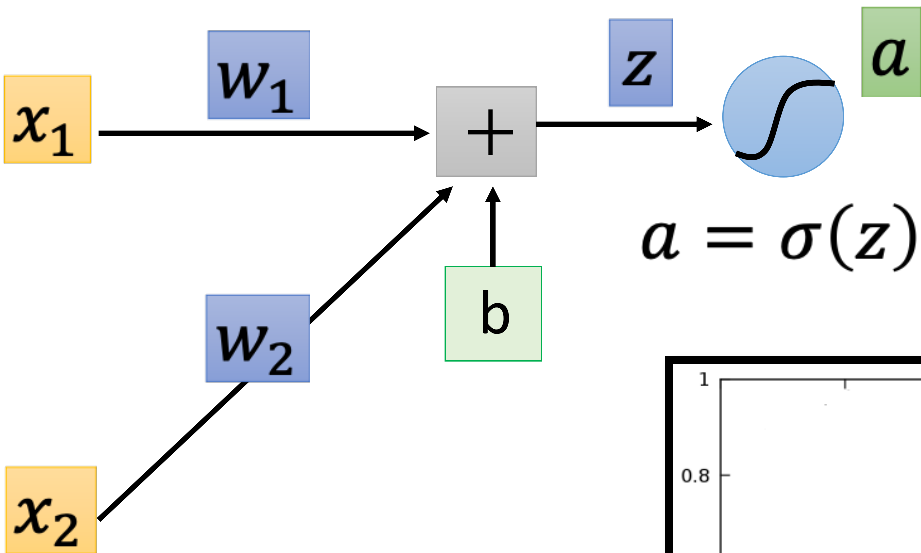
Compute $\partial z / \partial w$ for all parameters



$$\frac{\partial z}{\partial w} = -1 \qquad \frac{\partial z}{\partial w} = 0.12 \qquad \frac{\partial z}{\partial w} = 0.11$$

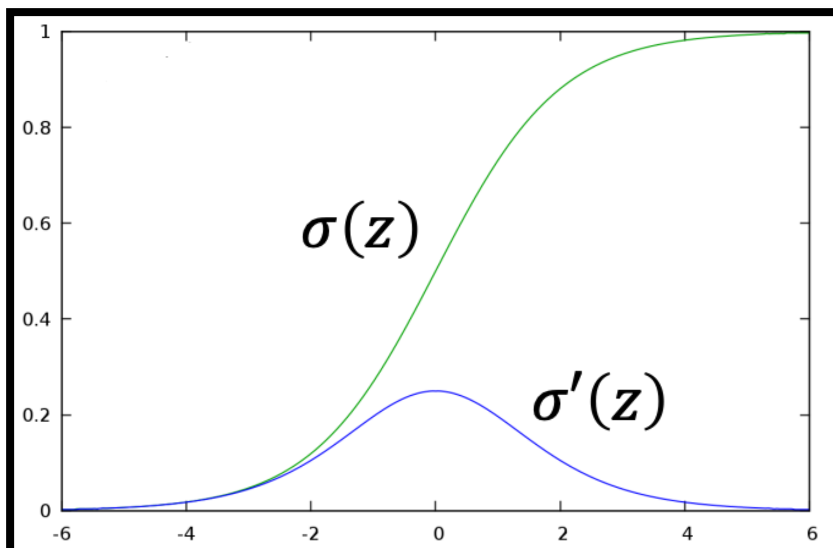
Backpropagation – Backward pass

Compute $\partial l / \partial z$ for all activation function inputs z



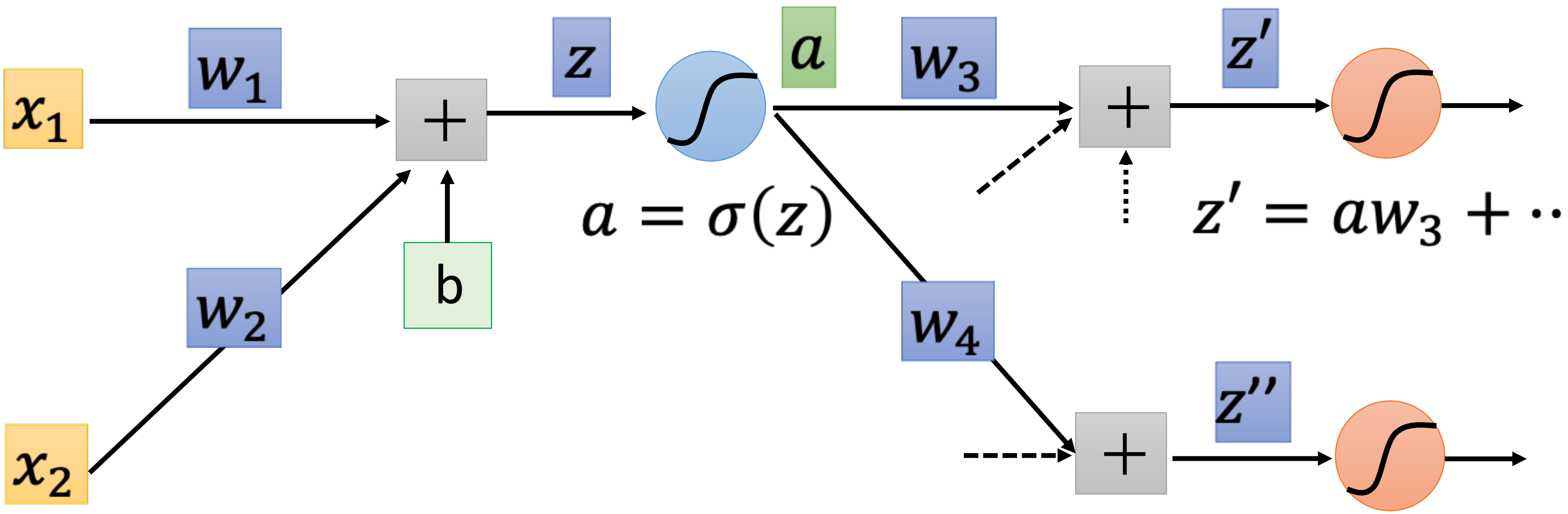
$$\frac{\partial l}{\partial z} = \frac{\partial a}{\partial z}\frac{\partial l}{\partial a}$$

$\Rightarrow \sigma'(z)$



Backpropagation – Backward pass

Compute $\partial l/\partial z$ for all activation function inputs z

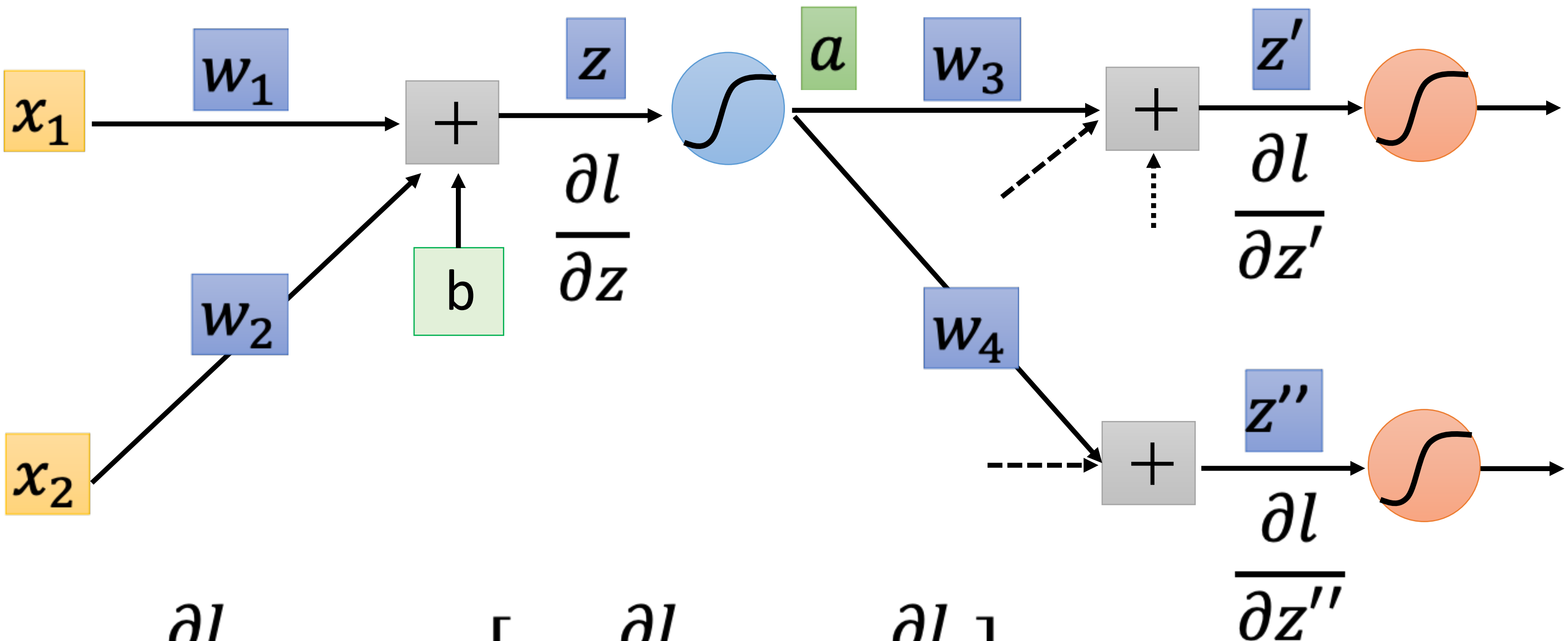


$$\frac{\partial l}{\partial z} = \frac{\partial a}{\partial z}\underline{\frac{\partial l}{\partial a}} \qquad \frac{\partial l}{\partial a} = \underbrace{\frac{\partial z'}{\partial a}}_{w_3}\underbrace{\frac{\partial l}{\partial z'}}_{?} + \underbrace{\frac{\partial z''}{\partial a}}_{w_4}\underbrace{\frac{\partial l}{\partial z''}}_{?}$$ (Chain rule)

Assumed it's known

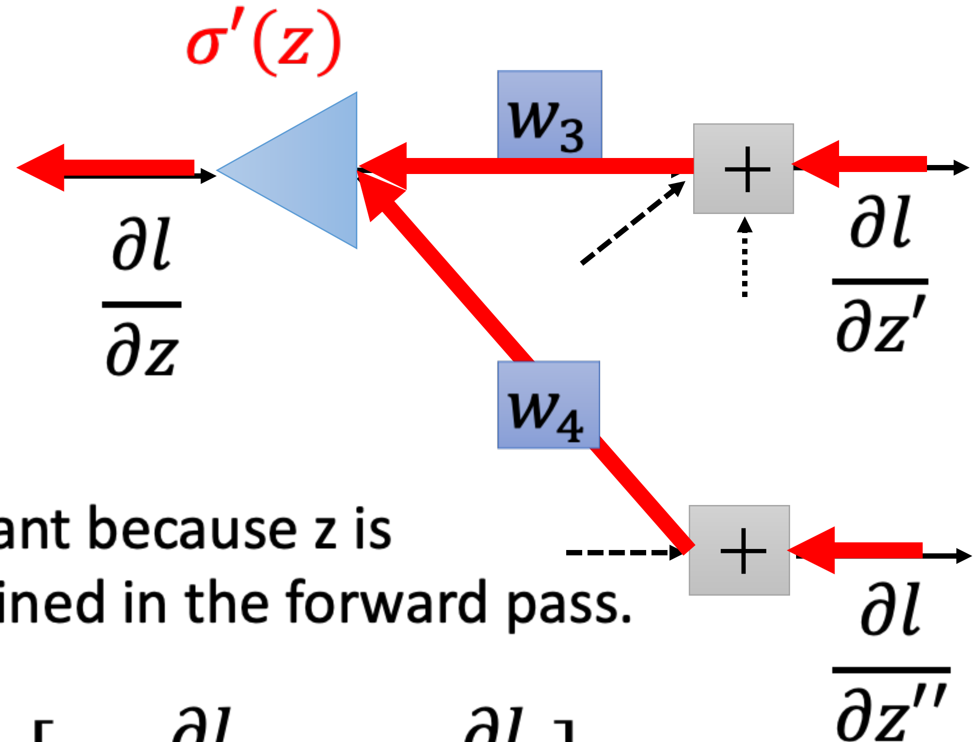
Backpropagation – Backward pass

Compute $\partial l / \partial z$ for all activation function inputs z



$$\frac{\partial l}{\partial z} = \sigma'(z) \left[w_3 \frac{\partial l}{\partial z'} + w_4 \frac{\partial l}{\partial z''} \right]$$

Backpropagation – Backward pass

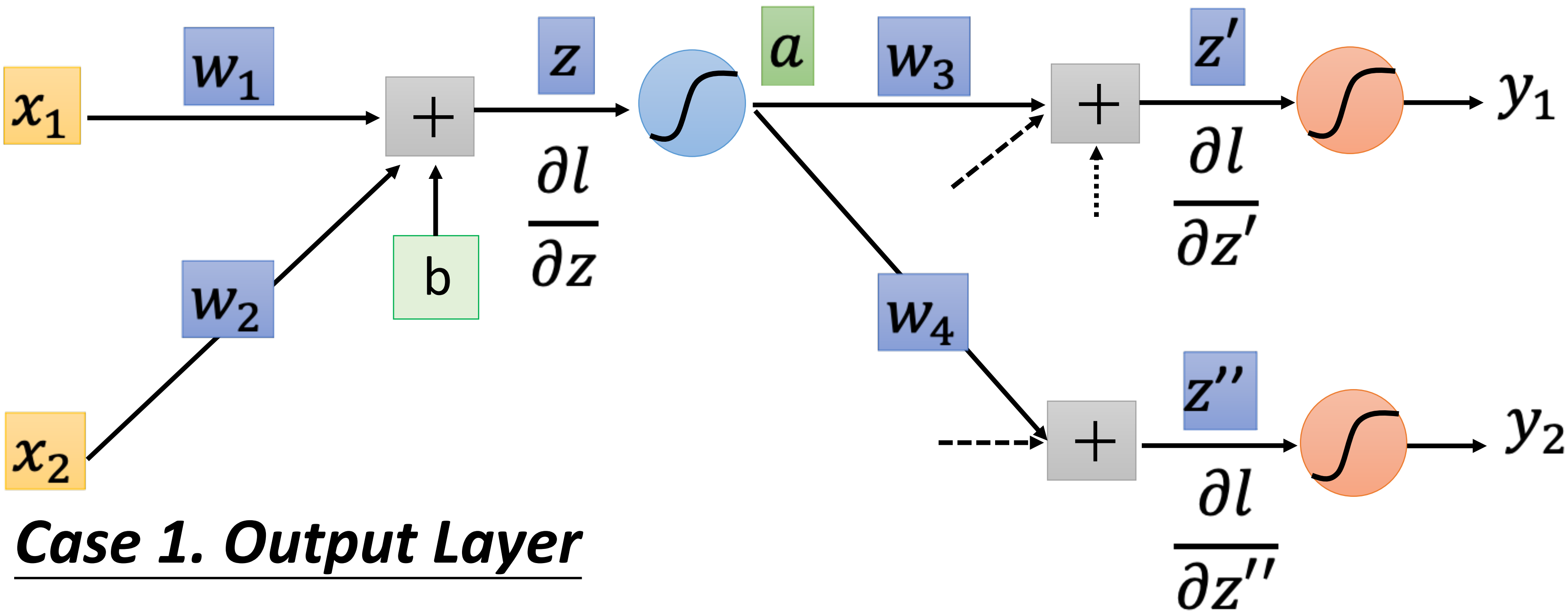


$\sigma'(z)$ is a constant because z is already determined in the forward pass.

$$\frac{\partial l}{\partial z} = \sigma'(z)\left[w_3 \frac{\partial l}{\partial z'} + w_4 \frac{\partial l}{\partial z''}\right]$$

Backpropagation – Backward pass

Compute $\partial l / \partial z$ for all activation function inputs z



Case 1. Output Layer

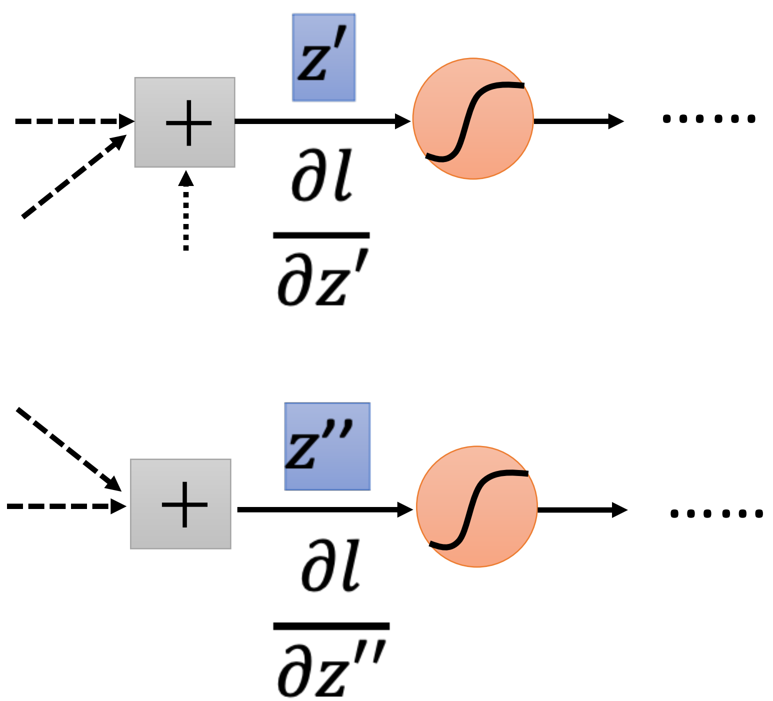
$$\frac{\partial l}{\partial z'} = \frac{\partial y_1}{\partial z'} \frac{\partial l}{\partial y_1} \qquad \frac{\partial l}{\partial z''} = \frac{\partial y_2}{\partial z''} \frac{\partial l}{\partial y_2}$$

Done!

Backpropagation – Backward pass

Compute $\partial l/\partial z$ for all activation function inputs z

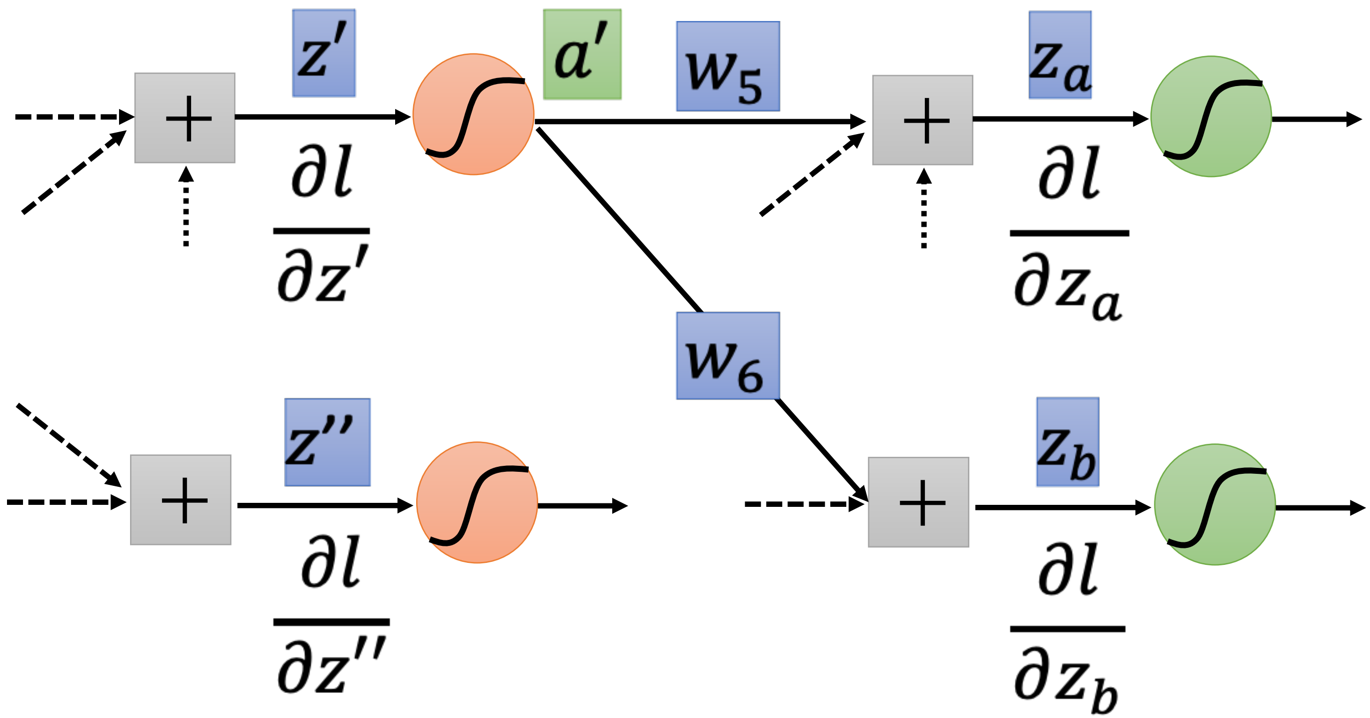
Case 2. Not Output Layer



Backpropagation – Backward pass

Compute $\partial l / \partial z$ for all activation function inputs z

Case 2. Not Output Layer

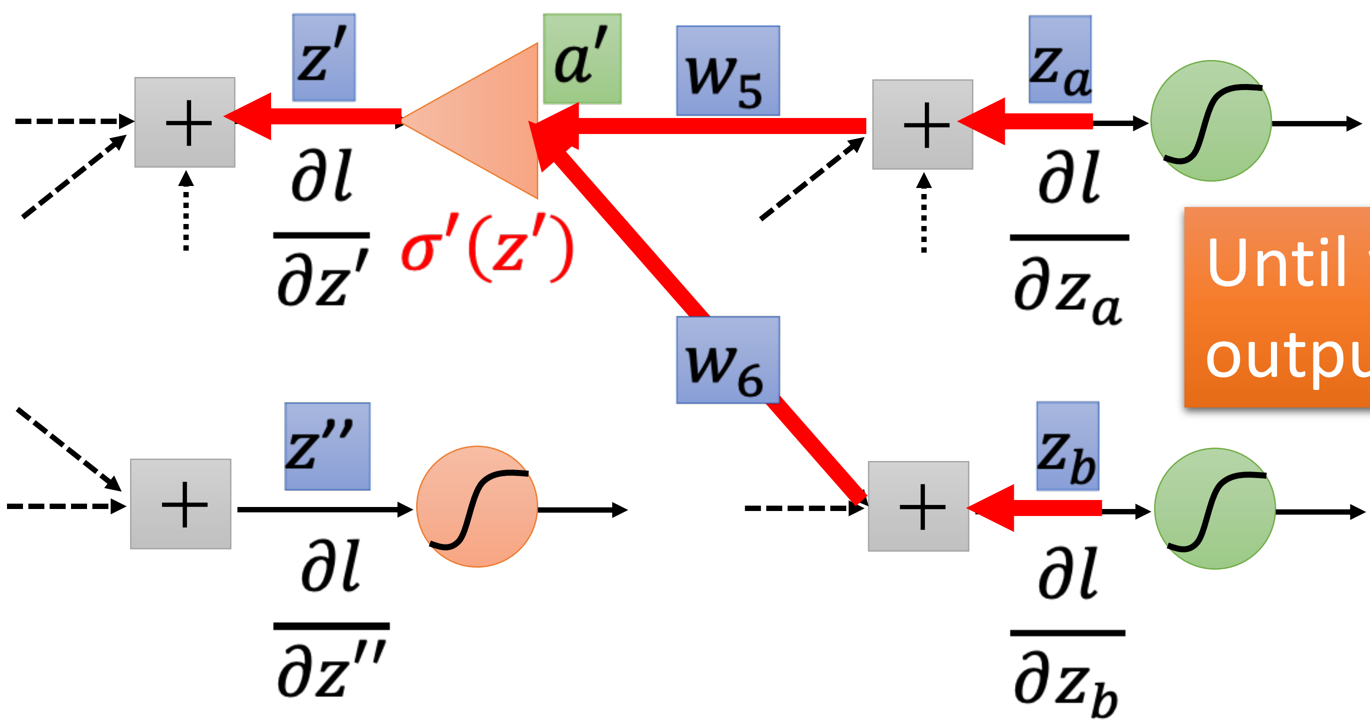


Backpropagation – Backward pass

Compute $\partial l/\partial z$ for all activation function inputs z

Case 2. Not Output Layer

Compute $\partial l/\partial z$ recursively

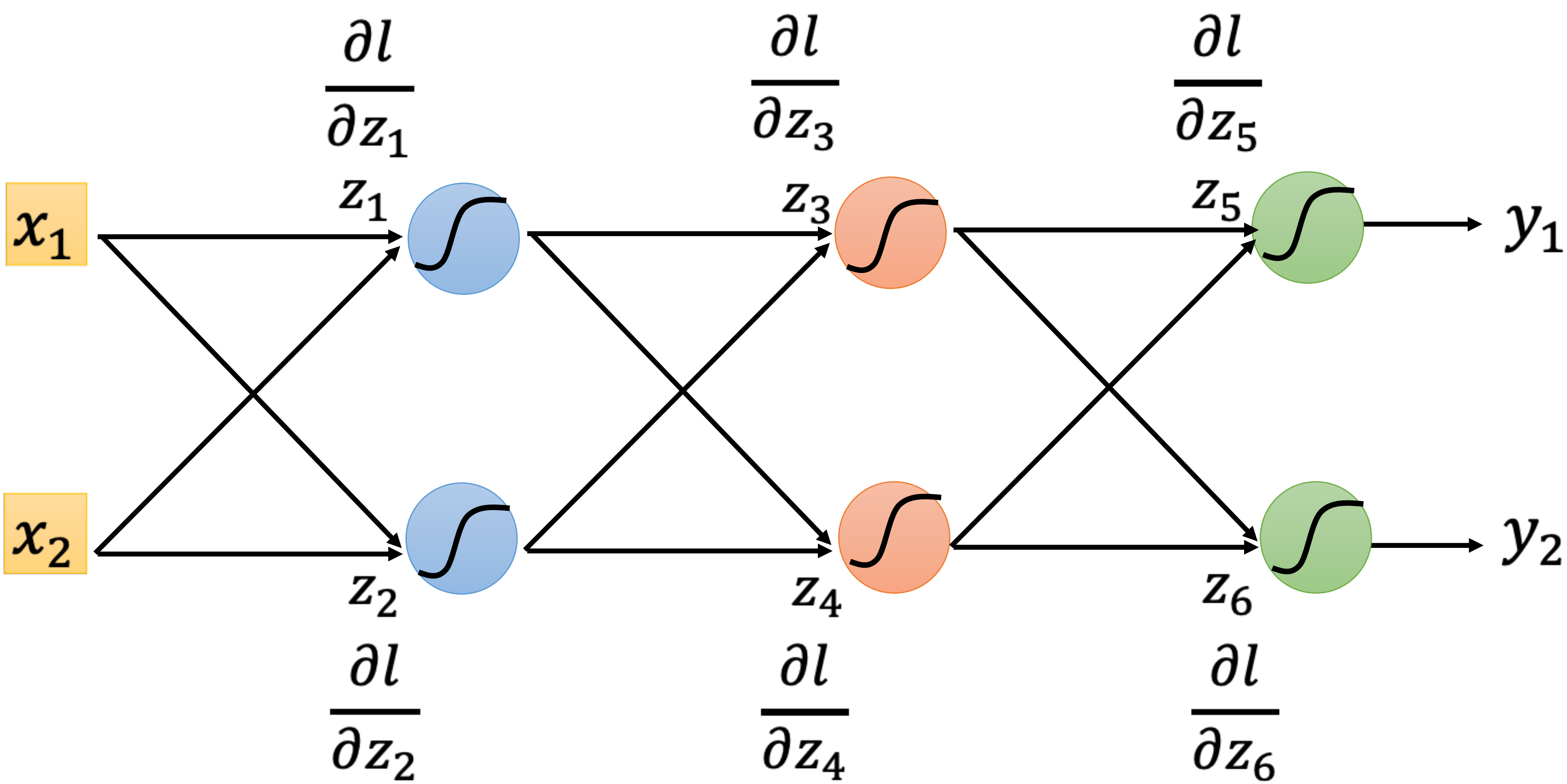


Until we reach the output layer ……

Backpropagation – Backward Pass

Compute $\partial l / \partial z$ for all activation function inputs z

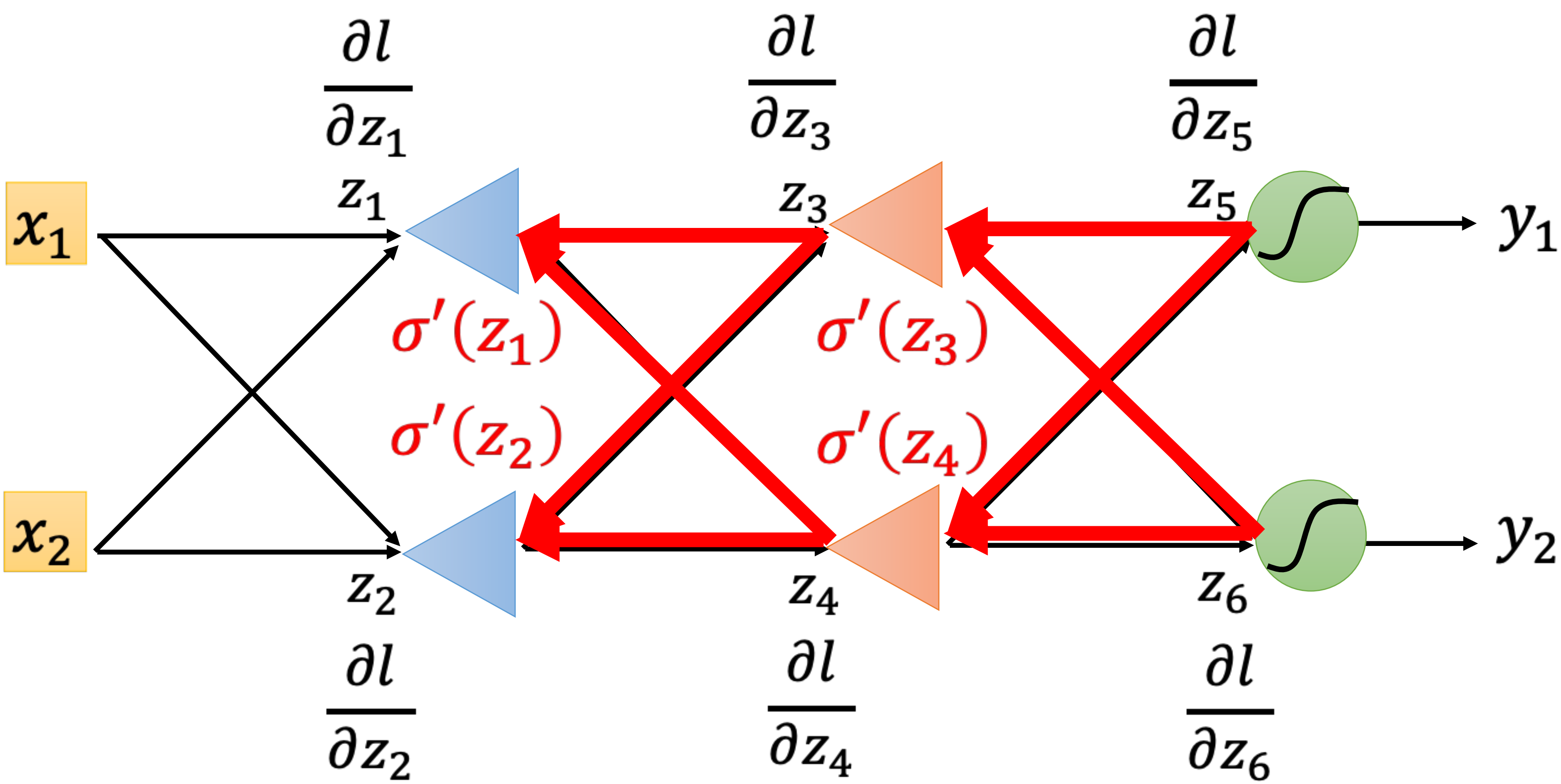
Compute $\partial l / \partial z$ from the output layer



Backpropagation – Backward Pass

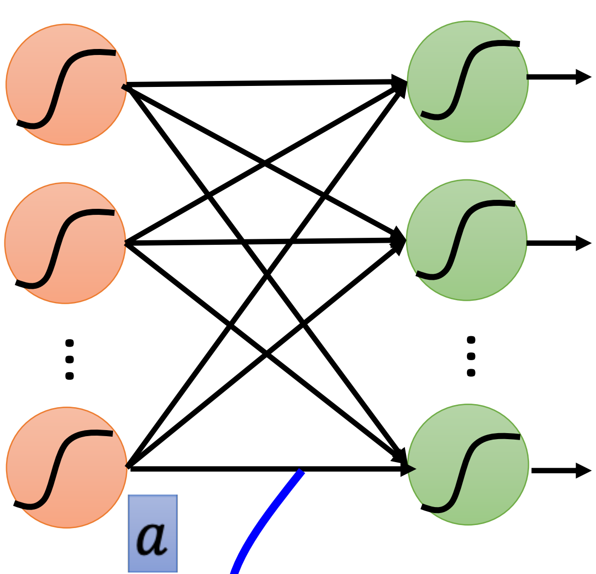
Compute $\partial l/\partial z$ for all activation function inputs z

Compute $\partial l/\partial z$ from the output layer

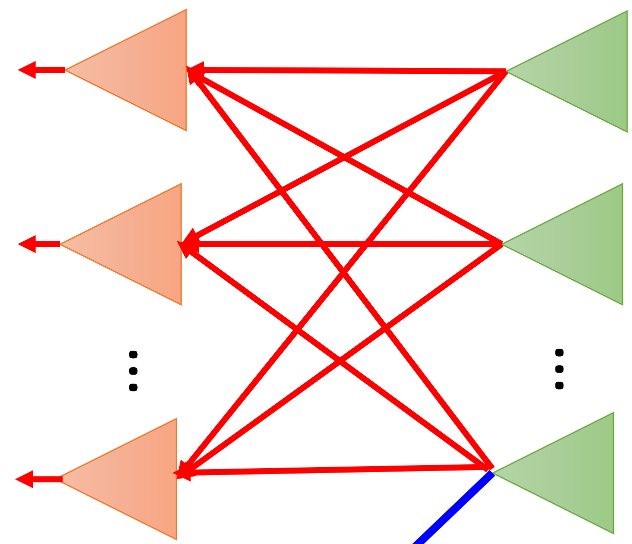


Backpropagation – Summary

Forward Pass



Backward Pass

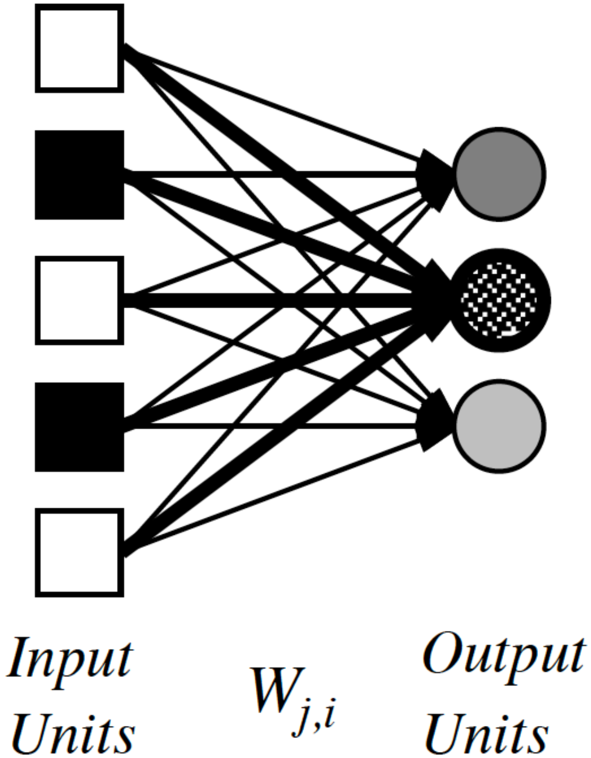


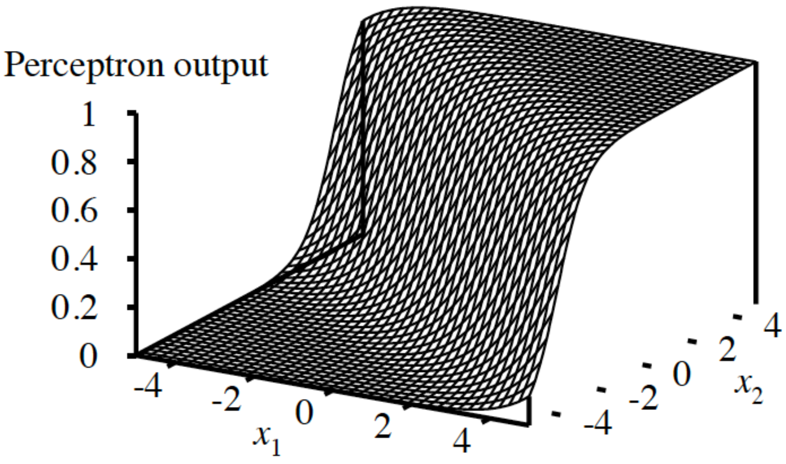
$$\frac{\partial z}{\partial w} = a \quad \times \quad \frac{\partial l}{\partial z} \quad = \frac{\partial l}{\partial w}$$

for all w

Single-layer perceptrons

- Output units all operate separately an no shared weights
- Adjusting weights moves the location, orientation, and steepness of cliff





Perceptron learning

Learn by adjusting weights to reduce error on training set

The squared error for an example with input $\mathbf{x}$ and true output y is

$$E = \frac{1}{2}Err^2 \equiv \frac{1}{2}(y - h_{\mathbf{W}}(\mathbf{x}))^2 \ ,$$

Perform optimization search by gradient descent:

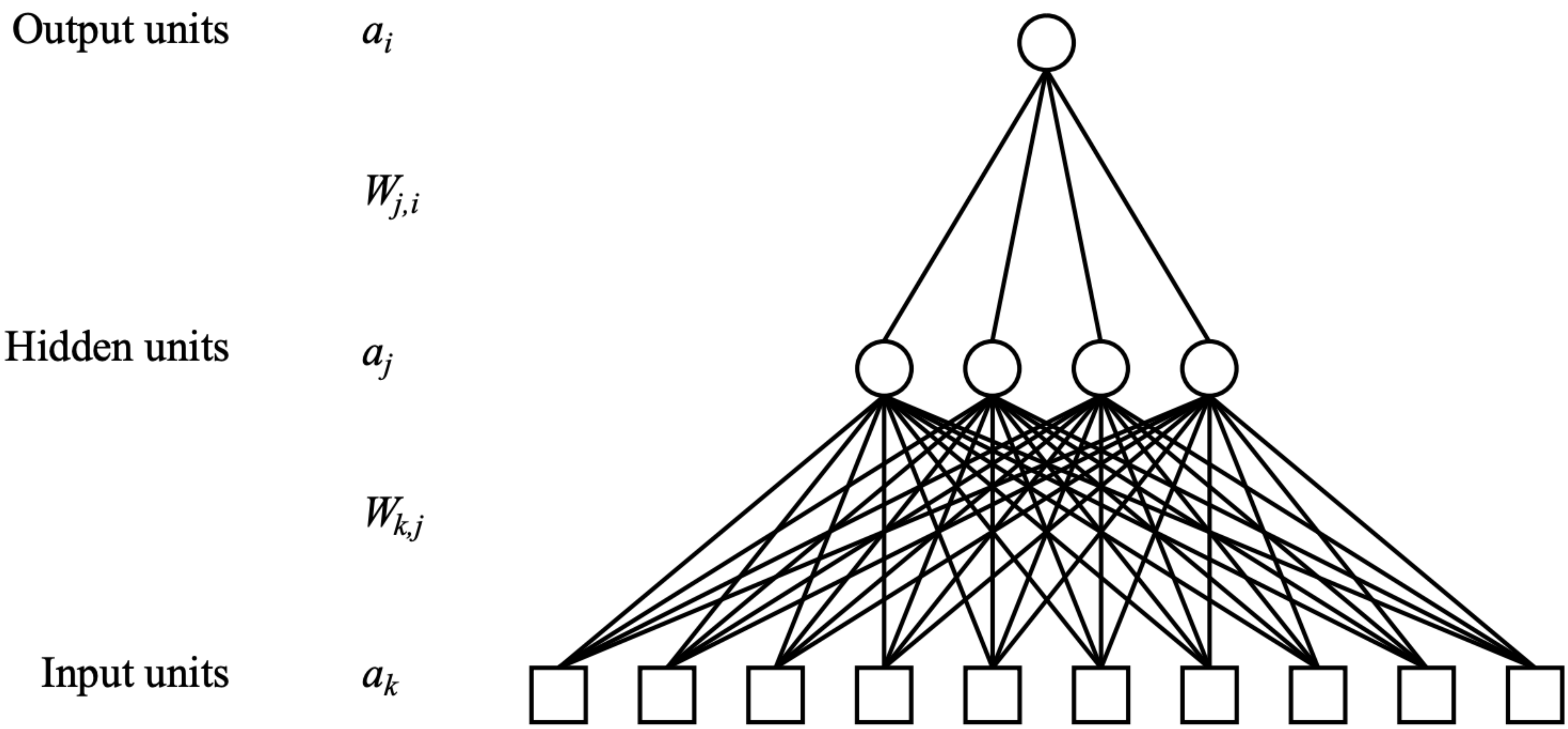
$$\begin{aligned}\frac{\partial E}{\partial W_j} &= Err \times \frac{\partial Err}{\partial W_j} = Err \times \frac{\partial}{\partial W_j}\left(y - g(\Sigma_{j=0}^{n} W_j x_j)\right) \\ &= -Err \times g'(in) \times x_j\end{aligned}$$

Simple weight update rule:

$$W_j \leftarrow W_j + \alpha \times Err \times g'(in) \times x_j$$

Multilayer perceptrons

- Layers are usually fully connected
- Numbers of hidden units typically chosen by hand



Back-propagation learning

Output layer: same as for single-layer perceptron,

$$W_{j,i} \leftarrow W_{j,i} + \alpha \times a_j \times \Delta_i$$

where $\Delta_i = Err_i \times g'(in_i)$

Hidden layer: **back-propagate** the error from the output layer:

$$\Delta_j = g'(in_j) \sum_i W_{j,i} \Delta_i \ .$$

Update rule for weights in hidden layer:

$$W_{k,j} \leftarrow W_{k,j} + \alpha \times a_k \times \Delta_j \ .$$