温兆和 10205501432 数学基础作业 9.

1. 解：(1). $L(X,\theta)=\frac{1}{\theta^n}\exp\{-\frac{\sum_1^n(x_i-c)}{\theta}\}I\{诸\ x_i\geq c\}$

$=\frac{1}{\theta^n}\exp\{-\frac{\sum_1^n(x_i-c)}{\theta}\}I\{x_1\geq c\}$

欲使 L 最大化，从使 c 尽可能大，但 $c\leq x_1$

故 $\hat{c}=x_1$

又：$\ln L=-n\ln\theta-\frac{\sum_1^n(x_i-c)}{\theta}$

$\frac{d\ln L}{d\theta}=-\frac{n}{\theta}+\frac{\sum_1^n(x_i-c)}{\theta^2}$.

令 $\frac{d\ln L}{d\theta}=0$ 得：$\theta n=\sum_1^n(x_i-c)$

故 $\theta=\frac{\sum_1^n(x_i-c)}{n}=\bar{x}-c$,

由最大似然估计的不变性，$\hat{\theta}=\bar{x}-\hat{c}=\bar{x}-x_1$

(2). $E(X)=\int_c^{+\infty}\frac{t}{\theta}e^{\frac{c-t}{\theta}}dt$

$=-te^{\frac{c-t}{\theta}}\Big|_c^{+\infty}+\int_c^{+\infty}e^{\frac{c-t}{\theta}}dt$

$=c+\theta$.

$E(X^2)=\int_c^{+\infty}\frac{t^2}{\theta}e^{\frac{c-t}{\theta}}dt$

$=-t^2e^{\frac{c-t}{\theta}}\Big|_c^{+\infty}+\int_c^{+\infty}2te^{\frac{c-t}{\theta}}dt$

$=c^2+2\theta(c+\theta)$

$=c^2+2c\theta+2\theta^2$.

$=(c+\theta)^2+\theta^2$,

由：$E(X^2)=(E(X))^2+\theta^2$,

得 $\theta=\sqrt{E(X^2)-(E(X))^2}=\sqrt{Var(X)}$

$c=E(X)-\sqrt{Var(X)}$.

用样本参数代替总体参数得:

$$\hat{\theta} = \sqrt{s^2} = s, \quad \hat{c} = \bar{x} - s$$

其中 $\bar{x}$ 为样本均值, s 为样本标准差

2. 解: (1). $L(X;\theta) = \frac{1}{\theta^n}\left(\prod x_i\right)^{\frac{1-\theta}{\theta}}$

$$\frac{dL}{d\theta} = -n\theta^{-n-1}\left(\prod x_i\right)^{\frac{1}{\theta}-1} - \frac{1}{\theta}\left(\prod x_i\right)^{\frac{1}{\theta}-1}\ln\left(\prod x_i\right)\frac{1}{\theta^2}$$

令 $\frac{dL}{d\theta} = 0$, 得: $\hat{\theta} = -\frac{1}{n}\ln\left(\prod x_i\right)$

$$= -\frac{1}{n}\sum_{i=1}^{n}\ln X_i$$

(2) 证: $E(\hat{\theta}) = -\frac{1}{n}\sum E(\ln X_i)$

$$E(\ln \cancel{X}x) = \int_0^1 \frac{\ln x}{\theta} x^{\frac{1}{\theta}-1}\,dx$$

$$= x^{\frac{1}{\theta}}\ln x\Big|_0^1 - \int_0^1 x^{\frac{1}{\theta}-1}\,dx$$

$$= \theta x^{\frac{1}{\theta}}\Big|_1^0 = -\theta.$$

故 $E(\hat{\theta}) = -\frac{1}{n}\sum E(\ln X_i)$

$$= -\frac{1}{n}\cdot(-n\theta) = \theta.$$

故 $\hat{\theta}$ 是 θ 的无偏估计

3. 解: $p(X|\mu) = \frac{1}{(\sqrt{2\pi}\sigma)^n}\exp\left\{\frac{\sum(x_i-\mu)^2}{-2\sigma^2}\right\}$

$h(X,\mu) = p(X|\mu)\pi(\mu)$

$$= \frac{1}{(\sqrt{2\pi}\sigma)^n\sqrt{2\pi\sigma_\mu^2}}\exp\left\{\frac{\sum(x_i-\mu)^2}{-2\sigma^2} + \frac{(\mu-\mu_0)^2}{-2\sigma^2}\right\}$$

$$= \frac{1}{(\sqrt{2\pi}\sigma)^n\sqrt{2\pi\sigma_\mu^2}}\exp\left\{-\left(\frac{\sigma_\mu^2\sum(x_i-\mu)^2+\sigma^2(\mu-\mu_0)^2}{2\sigma^2\sigma_\mu^2}\right)\right\}$$

$$= \frac{1}{(\sqrt{2\pi\sigma^2})^n \sqrt{2\pi\sigma_\mu^2}} \exp\left\{-\left(\frac{\left(\mu - \frac{\sigma^2\mu_0 + \sigma_\mu^2\sum x_i}{\sigma^2 + n\sigma_\mu^2}\right)^2}{\frac{2\sigma^2\sigma_\mu^2}{\sigma^2 + n\sigma_\mu^2}}\right)\right\}$$

令 $\mu' = \frac{\mu_0\sigma^2 + \sigma_\mu^2 \sum x_i}{\sigma^2 + n\sigma_\mu^2}$ $\quad \sigma'^2 = \frac{\sigma^2\sigma_\mu^2}{\sigma^2 + n\sigma_\mu^2}$

则 $\pi(\mu|X) = \frac{h(X,\mu)}{\int h(X,\mu)\,d\mu}$

$$= \frac{\exp\left\{-\frac{(\mu-\mu')^2}{2\sigma'^2}\right\}}{\int_{-\infty}^{+\infty} \exp\left\{-\frac{(\mu-\mu')^2}{2\sigma'^2}\right\} d\mu}$$

$$= \frac{\frac{1}{\sqrt{2\pi\sigma'^2}} \exp\left\{-\frac{(\mu-\mu')^2}{2\sigma'^2}\right\}}{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi\sigma'^2}} \exp\left\{-\frac{(\mu-\mu')^2}{2\sigma'^2}\right\} d\mu}$$

$$= \frac{1}{\sqrt{2\pi\sigma'^2}} \exp\left\{-\frac{(\mu-\mu')^2}{2\sigma'^2}\right\}$$

故：μ 的后验分布为：$N\left(\frac{\mu_0\sigma^2 + \sigma_\mu^2\sum x_i}{\sigma^2 + n\sigma_\mu^2}, \frac{\sigma^2\sigma_\mu^2}{\sigma^2 + n\sigma_\mu^2}\right)$

4. 解：$P(X|\lambda) = \frac{\lambda^{\sum x_i}}{\pi x_i!} e^{-n\lambda}$

$h(X,\lambda) = P(X|\lambda)\pi(\lambda)$

$$= \frac{\lambda^{\sum x_i}}{\pi x_i!} e^{-n\lambda} \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda}$$

$$= \frac{\beta^\alpha}{\pi x_i!\,\Gamma(\alpha)} \lambda^{\alpha+\sum x_i - 1} e^{-(n+\beta)\lambda}$$

故 $\pi(\lambda|x) = \dfrac{h(x, \lambda)}{\int_0^{+\infty} h(x, \lambda)\, d\lambda}$

$$= \frac{\left((n+\beta)^{\alpha+\sum x_i} / \Gamma(\alpha+\sum x_i)\right) h(x, \lambda)}{\left((n+\beta)^{\alpha+\sum x_i} / \Gamma(\alpha+\sum x_i)\right) \int_0^{+\infty} h(x, \lambda)\, d\lambda}$$

$$= \frac{(n+\beta)^{\alpha+\sum x_i}}{\Gamma(\alpha+\sum x_i)} \lambda^{\alpha+\sum x_i - 1} e^{-(n+\beta)\lambda}.$$

故 $\pi(\lambda|x)$ 为 $Ga(\alpha+\sum x_i,\ n+\beta)$

其期望为 $\dfrac{\alpha+\sum x_i}{n+\beta}$

故 $\hat{\lambda} = \dfrac{\alpha+\sum x_i}{n+\beta}.$