温兆和. 10205501432. 数学基础作业4.

1、解：(1). $A^TA = \begin{pmatrix} 2 & -1 & -3 \\ -2 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -2 & 1 \\ -1 & 1 & -1 \\ -3 & -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 14 & -2 & 0 \\ -2 & 6 & -4 \\ 0 & -4 & 3 \end{pmatrix}$

$g_{11} = \sqrt{a_{11}} = \sqrt{14}$.

$g_{21} = \dfrac{a_{21}}{g_{11}} = -\dfrac{2}{\sqrt{14}}$. $g_{31} = \dfrac{a_{31}}{g_{11}} = 0$

$\cancel{g_{31} = \dfrac{a_{31}}{g_{11}} = 0}$. $g_{22} = \sqrt{a_{22} - g_{21}^2} = \sqrt{\dfrac{40}{7}}$

$g_{32} = \dfrac{a_{32} - g_{31}g_{21}}{g_{22}} = -2\sqrt{\dfrac{7}{10}}$.

$g_{33} = \sqrt{a_{33} - g_{31}^2 - g_{32}^2} = \sqrt{\dfrac{1}{5}}$.

故∃ $G = \begin{pmatrix} \sqrt{14} & 0 & 0 \\ -\dfrac{\sqrt{14}}{7} & 2\sqrt{70} & 0 \\ 0 & -\dfrac{2\sqrt{70}}{10} & \dfrac{\sqrt{5}}{5} \end{pmatrix}$ s.t. $A^TA = GG^T$.

(2). 设 A^TA 的特征值为 λ. 对应特征向量为 α.

A 的特征值为 λ_1. G 的特征值为 λ_2.

由于矩阵转置不改变矩阵的特征值：

A^T、G^T 的特征值也是 λ_1、λ_2.

由于 $A^TA\alpha = \lambda\alpha = GG^T\alpha$.

$\alpha^TA^TA\alpha = \lambda\alpha^T\alpha$. $\alpha^TGG^T\alpha = \lambda\alpha^T\alpha$.

$(A\alpha)^T(A\alpha) = \lambda\alpha^T\alpha.\quad (G^T\alpha)^T(G^T\alpha) = \lambda\alpha^T\alpha.$

$\lambda_1^2\alpha^T\alpha = \lambda\alpha^T\alpha = \lambda_2^2\alpha^T\alpha.$

故有 $\lambda = \lambda_1^2 = \lambda_2^2$.

由于 $\|A\|_2 = \sqrt{\lambda_{max}(A^TA)}$:

且根据上面的引理 $\lambda_{max}((A^TA)^TA^TA) = \lambda_{max}^2(A^TA)$

$= \lambda_{max}^2(G^TG)$.

故有 $\|A^TA\|_2 = \|A\|_2^2 = \|G\|_2^2$.

2、解：(1). $A^TA = \begin{pmatrix} -8 & -4 & -8 & -4 \\ 5 & 7 & 5 & 7 \\ 1 & 5 & 1 & 5 \end{pmatrix}\begin{pmatrix} -8 & 5 & 1 \\ -4 & 7 & 5 \\ -8 & 5 & 1 \\ -4 & 7 & 5 \end{pmatrix}$

$= \begin{pmatrix} 160 & -136 & -56 \\ -136 & 148 & 80 \\ -56 & 80 & 52 \end{pmatrix}$

$|\lambda E - A| = 0 \Rightarrow \lambda_1 = 324\quad \lambda_2 = 36.\quad \lambda_3 = 0.$

故 $\Sigma = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

分别求 $(\lambda E - A^TA)x = 0$ 基础解系得：$V = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{pmatrix}$

$u_1 = \frac{1}{18}AV_1$ #

$$= \frac{1}{18}\begin{pmatrix} -8 & 5 & 1 \\ -4 & 7 & 5 \\ -8 & 5 & 1 \\ -4 & 7 & 5 \end{pmatrix}\begin{pmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$u_2 = \frac{1}{6}Av_2 = \frac{1}{6}\begin{pmatrix} -8 & 5 & 1 \\ -4 & 7 & 5 \\ -8 & 5 & 1 \\ -4 & 7 & 5 \end{pmatrix}\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

由于Σ第三、四行均为0，故U第三、四列可随意构造。

设 $U' = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix}$ 则 $V' = -\frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ -1 & 2 & -2 \end{pmatrix}$

s.t. $A = U'\Sigma V'^T$

(2). 由于A^TA有两个非0特征值，故 $\text{rank}(A) = 2$.

当 $j=1,2$，$u_j = \frac{1}{\sigma_j}Av_j$，故 $u_1, u_2 \in R(A)$

又 $\because \text{rank}(A) = 2$ 且 $u_1 = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$ 与 $u_2 = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$ 线性无关

故 $R(A) = \text{span}\{(1,1,1,1)^T, (1,-1,1,-1)^T\}$.

由于 ~~$Ax=0$~~ ~~A~~ $=0$. ~~且~~ $Ax=0 \Leftrightarrow A^TAx=0$

故 $N(A) = N(A^TA)$.

又∵ $\lambda_3=0$，故 $A^TAx=0$ 的基础解系就是 $(\lambda_3E-A^TA)x=0$ 的基础解系，解出即 v_3.

故 $N(A)=\text{span}\{(1,2,-2)\}$.

$\|A\|_2=\sqrt{\lambda_{max}(A^TA)}=18$

$\|A\|_F=\sqrt{Tr(A^TA)}=\sqrt{324+36}=6\sqrt{10}$

(3). 当 $k=0$：A_k 只能是零矩阵.

$\gamma_0=\|A^T\|_2=18$

当 $k=1$：由于 $A=U\Sigma V^T$

$A^T=V\Sigma^TU^T$.

此时 $$A_k=\frac{1}{3}\times\frac{1}{2}\begin{pmatrix}2&2&1\\-2&1&2\\-1&2&-2\end{pmatrix}\begin{pmatrix}18&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}1&1&1&1\\1&-1&1&-1\\1&1&1&1\\1&-1&1&-1\end{pmatrix}$$

$$=3\begin{pmatrix}2&2&2&2\\-2&-2&-2&-2\\-1&-1&-1&-1\end{pmatrix}$$

$\gamma_1=\sigma_2=6$.

当 $k\geq 2$. 仅当 $M=A^T$ 时 $\|A^T-M\|_2$ 取最小值 0. 故 $A_k=A^T$. $\gamma_k=0$

综

3. 解：(1). 由 $A=U\Sigma V^T=U\Sigma V^{-1}$.

$A^{-1}=V\Sigma^{-1}U^{-1}=V\Sigma^{-1}U^T$.（$U$、$V$ 是正交阵）

由于 $\Sigma=diag(\sigma_1,\dots\sigma_n)$

故 $\Sigma^{-1}=diag(\sigma_1^{-1},\dots\sigma_n^{-1})$

(2). 由于Q是正交阵：设 $Q = U\Sigma V^T = U\Sigma V^{-1}$.

则 $Q^{-1} = V\Sigma^{-1}U^T = Q^T = V\Sigma^T U^T$

$\Sigma^{-1} = \Sigma \Rightarrow$ 对 $\forall i = \{1, \dots, n\}$; 有 $\sigma_i = 1$

故 $\Sigma = I$. $Q = UV^T$. (U, V是正交阵)

(3). 由于 $A = QBQ^T$ 且 Q是正交阵.

$A^TA = QB^TQ^TQBQ^T = QB^TBQ^T = QB^TBQ^{-1}$

故 A^TA 与 B^TB 相似.

由于相似矩阵特征值相同且 A、B的奇异值为 A^TA、B^TB 特征值的算术平方根.

故 A、B 奇异值相同;