Fourier Transform over Adele Rings and its Application

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Contents

1 Adele rings over global fields

A global field refers to one of following fields:

1. Number field, i.e. finite extension of $\mathbb{Q}$.
2. Function field, i.e. finite extension of $\mathbb{F}_p(t)$, or equivalently, the function field of a geometrically integral curve over $\mathbb{F}_p$.

A local field refers to one of following field:

1. $\mathbb{R}$ or $\mathbb{C}$,
2. Finite extension of $\mathbb{Q}_p$,
3. $\mathbb{F}_q((t))$, where q is a power of a prime. (equivalent to say finite extension of $F_p((t))$)

Equivalence classes of valuation of global fields have been classified. Completion of global fields with respect to valuations give rises to local fields. Note that all local fields are non-discrete locally compact topological fields. It is remarkable that the converse of last statement is also true.

Let I be a index set. For each $i \in I$, K_i is a locally compact topological group with a compact open subgroup $O_i \subset K_i$. The restrict product of $\{K_i : i \in I\}$ is defined by

$$\prod_{i\in I}{}'(K_i, O_i) = \{(x_i)_{i\in I} : x_i \in O_i \text{ for all but finite many i}\}$$

The topology on this restricted product is given by specifying a neighborhood base of the identity 1. The basis consists of all sets of the form

$$\prod_{i\in I} G_i$$

where G_i is a neighborhood of 1 in K_i and $G_i = O_i$ for all but finite many $i \in I$. We called this topology as restrict product topology. Definitions of restrict product and topology lack motivations at first glance. We are going to appreciate the power of these definitions through out the paper. The following two lemmas will be useful.

Lemma 1.1. *The subspace topology of open neighborhood $\prod_{i\in I} G_i$, where G_i is a neighborhood of 1 in K_i and $G_i = O_i$ for all but finite many $i \in I$, is same as the product topology of $\{G_i, i \in I\}$.*

Lemma 1.2. *The restrict product space defined above is a locally compact topological group.*

1.1 Adele rings over number fields

Let K be a number field of degree n over $\mathbb{Q}$. All equivalence classes of valuations over K are classified as follow:

	Parameter	Standard valuation	Standard norm	completion
Archimedean	Real embedding $\sigma : K \to \mathbb{R}$	$\lvert x\rvert_{\mathbb{R}} = \lvert\sigma(x)\rvert_{\mathbb{R}}$	$\lVert x\rVert_{\mathbb{R}} = \lvert x\rvert_{\mathbb{R}}$	$\mathbb{R}$
∞ -places	Complex embedding $\sigma : K \to \mathbb{C}$	$\lvert x\rvert_{\mathbb{C}} = \lvert\sigma(x)\rvert_{\mathbb{C}}$	$\lVert x\rVert_{\mathbb{C}} = \lvert x\rvert_{\mathbb{C}}^2$	$\mathbb{C}$
Non-archimedean finite places	Pirme $\mathfrak{p}$ of $\mathcal{O}_K$ lying over rational prime p with ramification index e and inertial degree f	$\lvert x\rvert_{\mathfrak{p}} = p^{-\frac{1}{e}\mathrm{ord}_{\mathfrak{p}}(x)}$	$\lVert x\rVert_{\mathfrak{p}} = \lvert x\rvert_{\mathfrak{p}}^{ef} = \lvert\mathrm{Nm}(\mathfrak{p})\rvert^{-\mathrm{ord}_{\mathfrak{p}}(x)}$	$K_{\mathfrak{p}}/\mathbb{Q}_p$

The standard valuation of finite prime $\mathfrak{p}$ always ensures $|p|_{\mathfrak{p}} = p^{-1}$. Standard norms are more useful in the our discussion because it measures the change of volume by multiplying a number.

For each place v, denote the completion of K with respect to this place by K_v. For finite place v, denote the valuation ring of K_v by $\mathcal{O}_v$.

Definition The adele ring over K, denote $\mathbb{A}_K$, is define by

$$\mathbb{A}_K \coloneqq \mathbb{A}_\infty \times \mathbb{A}_f = \prod_{\infty\text{-place } v} K_v \times \prod_{\text{finite place } v}{}' (K_v, \mathcal{O}_v)$$

with restrict product topology. Addition and multiplication are given by addition and multiplication at each place. If K has r real embeddings and $2s$ complex embeddings, then

$$\mathbb{A}_K = \mathbb{R}^r \times \mathbb{C}^s \times \prod_{\text{prime } \mathfrak{p}}{}' (K_{\mathfrak{p}}, \mathcal{O}_{\mathfrak{p}})$$

Definition The idele group over K, denote $\mathbb{A}_K^\times$, is define by

$$\mathbb{A}_K^\times = \mathbb{A}_\infty^\times \times \mathbb{A}_f^\times = \prod_{\infty\text{-place } v} K_v^\times \times \prod_{\text{finite place } v}' (K_v^\times, \mathcal{O}_v^\times)$$

with restrict product topology. Multiplication is given by multiplication at each place. If K has r real embeddings and $2s$ complex embeddings, then

$$\mathbb{A}_K = \mathbb{R}^{\times r} \times \mathbb{C}^{\times s} \times \prod_{\text{prime } \mathfrak{p}}' (K_\mathfrak{p}^\times, \mathcal{O}_\mathfrak{p}^\times)$$

Note that the topology on $\mathbb{A}_K^\times$ is not the subspace topology from $\mathbb{A}_K$. In fact, for general topological ring R, if we endow $R^\times$, the group of all units in R, with subspace topology, then the inverse map $u \longmapsto u^{-1}$ is not always continuous and $R^\times$ is not necessary a topological group. Instead, we should endow $R^\times$ with the subspace topology induced from the map $R^\times \to R \times R$ given by $u \longmapsto (u, u^{-1})$. The restrict product topology on $\mathbb{A}_K^\times$ is the same as topology induced by this way.

Definition The standard norm on $\mathbb{A}_K^\times$ is defined by

$$\|.\|_{\mathbb{A}_K^\times} : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times \qquad \|(g_v)\|_{\mathbb{A}_K^\times} = \prod_v \|g_v\|_v$$

Theorem 1.3. *For all $x \in K^\times$, we have*

$$\|x\|_{\mathbb{A}_K^\times} = 1$$

Proof. Note that for $x \in K^\times$, $\|x\|_{\mathbb{A}_K^\times} = \|\mathrm{Nm}_\mathbb{Q}^K(x)\|_{\mathbb{A}_\mathbb{Q}^\times}$. Thus, it is enough to prove the statement for $K = \mathbb{Q}$, which is trivial. □

Therefore, the norm $\|.\|_{\mathbb{A}_K^\times}$ is a well defined multiplicative character on the quotient group $\mathbb{A}_K^\times / K^\times$.

In latter sections, we need to calculate integrals on $\mathbb{A}_K/K$ and $\mathbb{A}_K^\times/K^\times$. Thus, it would be helpful to find a fundamental domain for $\mathbb{A}_K/K$ and $\mathbb{A}_K^\times/K^\times$.

Theorem 1.4. *The fundamental domain of $\mathbb{A}_K/K$ is given by*

$$D = \mathbb{A}_\infty/\mathcal{O}_K \times \prod_{\text{finite place } v} \mathcal{O}_v$$

Furthermore, let $\alpha_1, ..., \alpha_n$ be an integral basis for K over $\mathbb{Q}$. Then the fundamental domain can be written as

$$D = \left(\prod_{i=1}^n [0,1)\alpha_i \right) \times \prod_{\text{finite place } v} \mathcal{O}_v$$

where α_i above are images of α_i inside the product of all ∞-places.

Proof. For any $g = (g_v) \in \mathbb{A}_K$, let $S = \{v : v \text{ is finite place and } g_v \notin \mathcal{O}_v\}$, then S is a finite set. By Chinese reminder theorem (or week approximate theorem), there exists $\alpha \in K$, such that $\alpha - g_v \in \mathcal{O}_v$ for all finite place v. Thus, $g - \alpha$ lies in D.

If $g = (g_v), h = (h_v) \in D$ represent same element in $\mathbb{A}_K$, then there exists $\alpha \in K$ such that $\alpha = g - h$. For each finite place v, $\alpha = g_v - h_v \in \mathcal{O}_v$. Thus, $\alpha \in \mathcal{O}_v$ for all finite places. This implies $\alpha \in \mathcal{O}_K$ and $g = h$. Therefore, D is a fundamental domain. □

Different from $\mathbb{A}_K/K$, the fundamental domain for $\mathbb{A}_K^\times/K^\times$ is a little difficult to describe. First, we define maps

$$\begin{tikzcd} & & \mathbb{A}_\infty^\times = \mathbb{R}^{\times r} \times \mathbb{C}^{\times s} \arrow[d, "\varphi"] & & \\ 1 \arrow[r] & \{\pm 1\}^r \times (S^1)^s \arrow[r, "i"] & \mathbb{R}^r \times \mathbb{R}^s_{>0} \times (S^1)^s \arrow[r, "\mathrm{Log}"] & \mathbb{R}^{r+s} \arrow[r] & 0 \end{tikzcd}$$

where

$$\varphi(x_1, ..., x_r, r_1 e^{i\theta_1}, ..., r_s e^{i\theta_s}) = (x_1, ..., x_r, r_1, ..., r_s, \theta_1, ..., \theta_s)$$
$$\mathrm{Log}(x_1, ..., x_r, r_1, ..., r_s, \theta_1, ..., \theta_s) = (\log|x_1|, ..., \log|x_r|, 2\log r_1, ..., 2\log r_s)$$

It's easy to see that φ is a isomorphism and diffeomorphism and the sequence is split exact. Thus, there exists an isomorphism and diffeomorphism:

$$\mathbb{A}_\infty^\times \cong \mathbb{R}^r \times \mathbb{R}^s_{>0} \times (S^1)^s \cong \{\pm 1\}^r \times (S^1)^s \times \mathbb{R}^{r+s}$$

The group of unit $\mathcal{O}_K^\times$ is a subgroup of $\mathbb{A}_\infty^\times$. We can consider the action of $\mathcal{O}_K^\times$ on $\mathbb{A}_\infty^\times$ by multiplication. The corresponding actions on $\{\pm 1\}^r \times (S^1)^s \times \mathbb{R}^{r+s}$ are translations by vectors. The Dirichlet unit theorem tells us that $\mathcal{O}_K^\times = \mu(K) \times \mathbb{Z}^{r+s-1}$, where $\mu(K) \subset \{\pm 1\}^r \times (S^1)^s$ is the group of all roots of unity in $\mathcal{O}_K$ and $\mathbb{Z}^{r+s-1}$ is a lattice inside $\mathbb{R}^{r+s}$ (In fact, its real span generates the hyperplane $y_1 + \cdots y_{r+s} = 0$).

If we choose a basis $u_1, ..., u_{r+s-1}$ for this lattice and Let $\mathbf{v} = (\underbrace{1, ..., 1}_{r}, \underbrace{2, ..., 2}_{s})$ be a vector in $\mathbb{R}^{r+s}$. Then $\{u_1, ..., u_{r+s-1}, \mathbf{v}\}$ forms a new coordinate. Under this coordinate, the norm map $\|.\|_\infty$ has a clear formula, i.e. for $x_\infty = (\underbrace{\cdots\cdots}_{\{\pm 1\}^r \times (S^1)^s}, \underbrace{n_1, ..., n_{r+s-1}}_{u_1,...,u_{r+s-1}}, \underbrace{r}_{\mathbf{v}})$,

$$\|x_\infty\|_\infty = e^{nr}$$

where $n = r + 2s = [K : \mathbb{Q}]$. Therefore, we have an exact sequence and isomorphism:

$$0 \longrightarrow \{\pm 1\}^r \times (S^1)^s \times \mathbb{R}^{r+s-1} \longrightarrow \mathbb{A}_\infty \xrightarrow{\|\cdot\|_\infty} \mathbb{R}_{>0} \longrightarrow 1$$

$$\mathbb{A}_\infty = \{\pm 1\}^r \times (S^1)^s \times \mathbb{R}^{r+s-1} \times \mathbb{R}_{>0}$$

where $\mathbb{R}^{r+s-1}$ is the real span of units. The a fundamental domain for the action of $\mathcal{O}^\times$ on $\mathbb{A}_\infty^\times$ is given by

$$\left(\{\pm 1\}^r \times (S^1)^s\right)/\mu(K) \times \prod_{i=1}^{r+s-1} [0, 1)u_i \times \mathbb{R}\mathbf{v}$$

Now we describe a fundamental domain for $\mathbb{A}_K^\times/K^\times$.

Theorem 1.5. *The fundamental domain of $\mathbb{A}_K^\times/K^\times$ is the union of $m = |Cl(K)|$ components all isomorphic to*

$$D = \mathbb{A}_\infty^\times/\mathcal{O}_K^\times \times \prod_{\textit{finite place } v} \mathcal{O}_v^\times$$

Specifically, there exists $\alpha_1 = 1, ..., \alpha_m \in \mathbb{A}_K^\times / K^\times$ such that $\alpha_1 D, ..., \alpha_m D$ is a fundamental domain for $\mathbb{A}_K/K$. Moreover, if we use above notations, then

$$D = \left(\{\pm 1\}^r \times (S^1)^s\right) / \mu(K) \times \prod_{i=1}^{r+s-1} [0,1)u_i \times \mathbb{R}\mathbf{v} \times \prod_{\textit{finite place } v} \mathcal{O}_v^\times$$

Proof. Construct following map:

$$\mathbb{A}_K^\times / K^\times \longrightarrow \mathrm{Cl}(K) \qquad (a_\infty, a_\mathfrak{p}) \longmapsto \prod_\mathfrak{p} \mathfrak{p}^{\mathrm{ord}_\mathfrak{p} a_\mathfrak{p}}$$

This map is a well-defined surjection. D is exactly the fundamental domain for the preimage of $1 \in \mathrm{Cl}(K)$. □

In particular, the fundamental domain for $\mathbb{A}_\mathbb{Q}/\mathbb{Q}$ is $[0,1) \times \prod_p \mathbb{Z}_p$ and the fundamental domain for $\mathbb{A}_\mathbb{Q}^\times/\mathbb{Q}^\times$ is $(0, +\infty) \times \prod_p \mathbb{Z}_p^\times$.

1.2 Adele rings over function fields

Since there are two ways to define a function field, namely, a finite extension of $\mathbb{F}_p(t)$ or function field of a geometrically integral curve, we have two ways to define adele rings over function fields. Although the outcome is the same, it would have different descriptions regarding self dual Haar measures and Fourier transforms. Therefore, we would state both of the definition in this section. Note that the key difference between two definitions of a function field is whether you choose an embedding of $\mathbb{F}_p(t)$ to the function field.

Let's start with the a function field K which is a finite extension of $\mathbb{F}_p(t)$. Let $\mathcal{O}_K$ be the integral closure of $\mathbb{F}_p[t]$ in K.Then $\mathcal{O}_K$ is a Dedekind domain. All valuation of K are non-archimedean. However, the equivalence classes of valuations on K are not one to one corresponds to the maximal prime of $\mathcal{O}_K$. Consider $\mathbb{F}_p(t)$ as an example. All equivalence classes of valuations are classified as follow:

	Parameter	Standard norm	Completion
Nonarchimedean	Irreducible polynomial $f \in \mathbb{F}[t]$	$\|x\|_f = \|\mathrm{Nm}(f)\|^{-\mathrm{ord}_f(x)} = p^{-\deg(f)\cdot \mathrm{ord}_f(x)}$	$\mathbb{F}_q((f)) \quad q = p^{\deg(f)}$
	$\frac{1}{t}$	$\|x\|_{\frac{1}{t}} = \|\mathrm{Nm}(\frac{1}{t})\|^{-\mathrm{ord}_{\frac{1}{t}}(x)} = p^{\deg(x)}$	$\mathbb{F}_p((\frac{1}{t}))$

Actually, this phenomenon can be better understand from algebraic geometry's viewpoint. Let X be a nonsingular projecitve curve with function field K. It is known that all closed points of X one to one correspond to valuation rings in K and hence equivalence classes of valuations. Let $t \in K$ be a transcendent element over $\mathbb{F}_p$. Let R and R' be the integral closure of $\mathbb{F}_p[t]$ and $\mathbb{F}_p[t^{-1}] \subset K$. Then R and R' are Dedekind domains and X is covered by $\mathrm{Spec}(R)$ and $\mathrm{Spec}(R')$. Therefore, every valuation can be viewed as a maximal ideal of a subring of K. However, it is not possible to view all valuations as maximal ideals of one of subrings.

For each place (valuation) v, denote the completion of K with respect to this place as K_v. Denote the valuation ring of K_v by $\mathcal{O}_v$.

Definition The adele ring over K, denote $\mathbb{A}_K$, is define by

$$\mathbb{A}_K = \prod_v{}' (K_v, \mathcal{O}_v)$$

with restrict product topology. Addition and multiplication are given by addition and multiplication at each place.

Definition The idele group over K, denote $\mathbb{A}_K^\times$, is define by

$$\mathbb{A}_K^\times = \prod_v{}' (K_v^\times, \mathcal{O}_v^\times)$$

with restrict product topology. Multiplication is given by multiplication at each place.

We wish to define norm maps on local fields K_v so that the product formula holds for idele over the function field. Let v be a valuation of K that lying over valuation w of $\mathbb{F}_p(t)$ (here we need to use the condition that K is the finite extension of $\mathbb{F}_p(t)$). Then there is a finite extension $K_v/\mathbb{F}_p(t)_w$. Define the norm map $\|.\|_v$ on K_v by

$$\|x\|_v = \|\mathrm{Nm}_{\mathbb{F}_p(t)_w}^{K_v}(x)\|_w$$

Definition The standard norm on $\mathbb{A}_K^\times$ is defined by

$$\|.\|_{\mathbb{A}_K^\times} : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times \qquad \|(g_v)\|_{\mathbb{A}_K^\times} = \prod_v \|g_v\|_v$$

Theorem 1.6. *For all $x \in K^\times$, we have*

$$\|x\|_{\mathbb{A}_K^\times} = 1$$

Proof. Note that for $x \in K^\times$, $\|x\|_{\mathbb{A}_K^\times} = \|\mathrm{Nm}_{\mathbb{F}_p(t)}^K(x)\|_{\mathbb{A}_{\mathbb{F}_p(t)}^\times}$. Thus, it is enough to prove the statement for $K = \mathbb{F}_p(t)$, which is trivial. □

Now we repeat our construction by geometric language. This time, we begin with a geometrically integral curve X over $\mathbb{F}_p$. It has function field K, which is a extension of $\mathbb{F}_p$ with transcendental degree 1. Let $\mathcal{O}$ be the structure sheaf of X. For each closed point $x \in X$, Let $\mathcal{O}_x$ be the stalk of $\mathcal{O}$ at x, $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_x$, $k_x = \mathcal{O}_x/\mathfrak{m}_x$ be the residue field. Then $\mathcal{O}_x$ is a discrete valuation ring of K and k_x is a finite extension of $\mathbb{F}_p$. Let $\widehat{\mathcal{O}}_x$ be the completion of $\mathcal{O}_x$, i.e.

$$\widehat{\mathcal{O}}_x = \varprojlim \mathcal{O}_x/\mathfrak{m}_x^n$$

Let $\widehat{K}_x = K \otimes_{\mathcal{O}_x} \widehat{\mathcal{O}}_x$. $\widehat{K}_x$ is also the fraction field of $\widehat{\mathcal{O}}_x$.

Denote the set of all closed points in X by $|X|$.

Definition The adele ring over geometrically integral curve X, denote $\mathbb{A}_X$, is define by

$$\mathbb{A}_X = \prod_{x\in|X|}{}' (\widehat{K}_x, \widehat{\mathcal{O}}_x)$$

with restrict product topology. Addition and multiplication are given by addition and multiplication at each place.

Definition The idele group over geometrically integral curve X, denote $\mathbb{A}_X^\times$, is define by

$$\mathbb{A}_X^\times = \prod_{x\in|X|}{}' (\widehat{K}_x^\times, \widehat{\mathcal{O}}_x^\times)$$

with restrict product topology. Multiplication is given by multiplication at each place.

Define the norm map $\|.\|_x$ on $\widehat{K}_x$ by

$$\|f\|_x = |k_x|^{-\mathrm{ord}_x(f)} = p^{-[k_x:\mathbb{F}_p]\cdot \mathrm{ord}_x(f)}$$

Definition The standard norm on $\mathbb{A}_X^\times$ is defined by

$$\|.\|_{\mathbb{A}_X^\times} : \mathbb{A}_X^\times \longrightarrow \mathbb{C}^\times \qquad \|(g_x)\|_{\mathbb{A}_X^\times} = \prod_{x\in|X|} \|g_x\|_x$$

Theorem 1.7. *For all $f \in K^\times$, we have*

$$\|f\|_{\mathbb{A}_X^\times} = 1$$

Proof. This is equivalent to the fact that the degree of principal divisors are zero. □

It is quit difficult to describe the fundamental domain of $\mathbb{A}_X/K$ and $\mathbb{A}_X^\times/K^\times$. They are not simply $\prod_x \widehat{\mathcal{O}}_x$ and $\prod_x \widehat{\mathcal{O}}_x^\times$ as one might guess. We quote a result from [1]:

Theorem 1.8. *Let $D = \sum_x a_x \cdot x$ be a divisor of degree greater than $2g-2$. Then*

$$\mathbb{A}_X = K + \prod_{x\in|X|} \pi_x^{-a_i} \widehat{\mathcal{O}}_x$$

2 Integration and Fourier transform

The following theorem is the foundation of this section:

Theorem 2.1. *Let G be a locally compact topological group, then up to a scalar there exists a unique left(right) invariant measure dg such that for any $h \in G$ and compact supported function f,*

$$\int_G f(hg)dg = \int_G f(g)dg \qquad \left(\int_G f(g)dg = \int_G f(gh)dg\right)$$

In particular, all local fields $\mathbb{F}$ are locally compact topological groups ($(\mathbb{F}, +)$ and $(\mathbb{F}^\times, \times)$). In this chapter, we delve into integration and Fourier transform on local fields.

2.1 Integration and Fourier transform over Archimedean fields

Archimedean fields refer to either $\mathbb{R}$ or $\mathbb{C}$. We are particularly familiar with the integration and Fourier theory over $\mathbb{R}$. The measure dx we use frequently in analysis course is exactly additive invariant. It satisfies $d(ax) = |a| \cdot dx = \|a\|_\mathbb{R} \cdot dx$ for any $a \in \mathbb{R}$. Therefore, we can define a multiplicative invariant measure on $\mathbb{R}$ by $d^\times x = \frac{dx}{\|x\|_\mathbb{R}}$.

Recall that the Fourier transform for function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is defined by

$$\widehat{f}(x) = \int_\mathbb{R} f(y)e^{2\pi ixy}dx = \int_\mathbb{R} f(y)e_\mathbb{R}(xy)dx$$

where $e_{\mathbb{R}}(x) = e^{2\pi i x}$. This transform is a well-defined operator on Schwartz space $\mathcal{S}(\mathbb{R})$, i.e.

$$\mathcal{S}(\mathbb{R}) = \left\{ f : \mathbb{R} \longrightarrow \mathbb{C} \text{ is smooth : for any } n, m \in \mathbb{Z}_{>0}, \ \sup_{x\in\mathbb{R}}\{x^n f^{(m)}(x)\} \text{ is bounded.} \right\}$$

Moreover, the Fourier Transform over $\mathbb{R}$ satisfies $\widehat{\widehat{f}}(x) = f(-x)$ for f in $\mathcal{S}(\mathbb{R})$.

For arbitrary local field $\mathbb{F}$, no matter Archimedean or non-archimedean, the Fourier theory over $\mathbb{F}$ is a fixed additive character $e_F : \mathbb{F} \to \mathbb{C}^\times$ and a Fourier transform

$$\widehat{f}(x) = \int_{\mathbb{F}} f(y) e_{\mathbb{F}}(xy) dx$$

such that for a set of functions $\mathcal{S}(\mathbb{F})$, the Fourier transform is a well-defined operator on $\mathcal{S}(\mathbb{F})$ such that $\widehat{\widehat{f}}(x) = f(-x)$ for any $f \in \mathbb{F}$.

Now we need to determinate dz, $d^\times z$, $e_{\mathbb{C}}(z)$, and Fourier transform over $\mathbb{C}$. Note that $\mathbb{C} \cong \mathbb{R}^2$ as abelian groups and local compact topology spaces. Thus, $dx_1 \wedge dx_2$ is an additive invariant measure on $\mathbb{C}$. We define

$$\begin{aligned}
&dz = 2dx_1 \wedge dx_2 \\
&d^\times z = \frac{dz}{\|z\|_{\mathbb{C}}} \\
&e_{\mathbb{C}}(z) = e_{\mathbb{R}}(\mathrm{Tr}(z)) = e^{2\pi i (z+\overline{z})} = e^{4\pi i \mathrm{Re}(z)}
\end{aligned}$$

where $\mathrm{Tr}(z) = \mathrm{Tr}_{\mathbb{R}}^{\mathbb{C}}(z)$ is the trace map. Here we normalize the measure by multiplying a coefficient 2 so that $\widehat{\widehat{f}}(x) = f(-x)$ holds.

Definition For $f \in \mathcal{S}(\mathbb{C})$, the Fourier transform is defined by

$$\widehat{f}(x) = \int_{\mathbb{C}} f(y) e_{\mathbb{C}}(xy) dx$$

We know that traditional Fourier transform over $\mathbb{R}^2$ is in form of

$$\widehat{f}^{\mathbb{R}^2}(x, y) = \int\int_{\mathbb{R}^2} f(a, b) \cdot e^{2\pi i (ax+by)} da \wedge db$$

If we consider $f(z)$ as $f(x + yi) = f(x, y)$, then The Fourier transform over $\mathbb{C}$ can be expressed by Fourier transform over $\mathbb{R}^2$:

$$\begin{aligned}
\widehat{f}(z) &= \int_{\mathbb{C}} f(z') e_{\mathbb{C}}(zz') dz' \\
&= 2 \int\int_{\mathbb{R}^2} f(a, b) \cdot e^{2\pi i (\mathrm{Tr}((a+bi)(x+yi)))} da \wedge db \\
&= 2 \int\int_{\mathbb{R}^2} f(a, b) \cdot e^{4\pi i (ax-by)} da \wedge db \\
&= 2\widehat{f}^{\mathbb{R}^2}(2x, -2y)
\end{aligned}$$

Hence,

$$
\begin{aligned}
\widehat{\widehat{f}}(z) &= \widehat{\widehat{f}}(x+yi) \\
&= 2\int\int_{\mathbb{R}^2} \widehat{f}(a,b)\cdot e^{4\pi i(ax-by)} da\wedge db \\
&= 4\int\int_{\mathbb{R}^2} \widehat{f}^{\mathbb{R}^2}(2a,-2b)\cdot e^{4\pi i(ax-by)} da\wedge db \\
&= \int\int_{\mathbb{R}^2} \widehat{f}^{\mathbb{R}^2}(a,b)\cdot e^{2\pi i(ax+by)} da\wedge db \\
&= f(-x,-y) \\
&= f(-z)
\end{aligned}
$$

Therefore, the operator we defined is indeed a Fourier transform over $\mathbb{C}$.

2.2 Integration over non-Archimedean local fields

Let $\mathbb{F}$ be a local field which is a separable finite extension of $\mathbb{F}_0$ with ramification index f and inertial degree e, where $\mathbb{F}_0$ is either $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$. Let $\mathcal{O}$ be the valuation ring of $\mathbb{F}$, π be a generator of the unique maximal ideal $\mathfrak{m}$ and k be the residue field $\mathcal{O}/\mathfrak{m}$. The trace map $\mathrm{Tr}_{\mathbb{F}_0}^{\mathbb{F}}$ defines a non-degenerated bilinear form

$$\mathrm{Tr}:\mathbb{F}\times\mathbb{F}\longrightarrow\mathbb{F}_0 \qquad (x,y)\longmapsto \mathrm{Tr}_{\mathbb{F}_0}^{\mathbb{F}}(xy)$$

Define $\mathcal{O}^\wedge=\{\alpha\in\mathbb{F}:\mathrm{Tr}(\alpha\mathcal{O})\in\mathcal{O}_0\}$. It is known that $\mathcal{O}^\wedge$ is a fractional ideal and the difference $\mathcal{D}=(\mathcal{O}^\wedge)^{-1}$ is an ideal of $\mathcal{O}$

$$[\mathcal{O}^\wedge:\mathcal{O}]=[\mathcal{O}:\mathcal{D}]=\mathrm{Nm}(\mathcal{D})$$

Denote the additive invariant measure on $\mathbb{F}$ by μ or dx. We normalize it so that the measure of $\mathcal{O}$ is

$$\mathrm{Vol}(\mathcal{O})=(\mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}}$$

It seems more natural to normalize the measure so that $\mathcal{O}$ has measure 1. However, we will see later that our choice of the normalized measure is a self-dual measure, which guarantees that $\widehat{\widehat{f}}(x)$ and $f(-x)$ are equal ($\widehat{f}$ is the Fourier transform of f), rather than differ by a scalar. Moreover, it also guarantees that the fundamental domain of $\mathbb{A}_K/K$ has measure 1.

We choose $\alpha_1,...,\alpha_{|k|}\in\mathcal{O}$ as representatives of k, i.e. $\alpha_i\not\equiv\alpha_j \mod \pi\mathcal{O}$ for $1\le i,j\le |k|$. Then,

$$\mathcal{O}=\bigcup_{i=1}^{n}\alpha_i+\pi\mathcal{O}$$

and $(\alpha_j-\alpha_i)+(\alpha_i+\pi\mathcal{O})=\alpha_j+\pi\mathcal{O}$. Since μ is additive invariant, we have $\mu(\alpha_i+\pi\mathcal{O})=\frac{1}{|k|}$ for $i=1,...,|k|$. It is easy to generalize this result to

$$\mu(a+\pi^n\mathcal{O})=|k|^{-n} \qquad \text{for any } n\in\mathbb{Z} \text{ and } \alpha\in\mathbb{F}$$

Lemma 2.2. *Any nonempty open subset of $\mathbb{F}$ can be written as disjoint union of countable many open subset of form $a+\pi^n\mathcal{O}=|k|^{-n}$ for some $n\in\mathbb{Z}$ and $\alpha\in\mathbb{F}$*

Thus, for locally constant function f on $\mathbb{F}$, it can be written as

$$f(x) = \sum_{i=1}^{\infty} c_i \cdot \mathbb{1}_{U_i}(x)$$

where $c_i \in \mathbb{C}$ and U_i are disjoint open sets of form $a + \pi^n \mathcal{O}$. The integral of f over $\mathbb{F}$ exists if $\sum_{i=1}^{\infty} |c_i| \cdot \mu(U_i)$ converge. In this case,

$$\int_{\mathbb{F}} f(x)dx = \sum_{i=1}^{\infty} c_i \cdot \mu(U_i)$$

The function we are interested in, such as, $\|x\|_{\mathbb{F}}^s$ and $e_{\mathbb{F}}(x)$, are all locally constant functions. Thus, this class of function is already large enough for our investigation. In particular, if f is locally constant compact supported function, then the integral always exists.

Note that $\mu(\pi\mathcal{O}) = |k|^{-1}\mu(\mathcal{O}) = \|\pi\|_{\mathbb{F}}\mu(\mathcal{O})$. This can be generalize to

$$\mu(a\mathcal{O}) = \|a\|_{\mathbb{F}} \cdot \mu(\mathcal{O}) \qquad \text{for any } \alpha \in \mathbb{F}^{\times}$$

Thus,

$$d(ax) = \|a\|_{\mathbb{F}} \cdot dx \qquad \text{for any } \alpha \in \mathbb{F}^{\times} \tag{2.1}$$

Now we study the multiplicative invariant measure $d^{\times}x$ on $\mathbb{F}^{\times}$. We normalize it so that the measure of $\mathcal{O}^{\times}$ is

$$\mathrm{Vol}(\mathcal{O}^{\times}) = (\mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}}$$

By equation 2.1 the measure $\frac{dx}{\|x\|_{\mathbb{F}}}$ is also a multiplicative invariant measure and $\int_{\mathcal{O}^{\times}} \frac{dx}{\|x\|_{\mathbb{F}}} = \frac{|k|-1}{|k|}(\mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}}$. Thus, we have a equation

$$d^{\times}x = \frac{1}{1-|k|^{-1}} \cdot \frac{dx}{\|x\|_{\mathbb{F}}}$$

Example 2.1.

For $s \in \mathbb{C}$, $Re(s) > -1$

$$\int_{\mathcal{O}\setminus\{0\}} \|x\|_{\mathbb{F}}^s dx = \sum_{n=0}^{\infty} \int_{\pi^n\mathcal{O}^{\times}} \|x\|_{\mathbb{F}}^s dx = \sum_{n=0}^{\infty} |k|^{-ns}\mu(\pi^n\mathcal{O}^{\times}) = \frac{1-|k|^{-1}}{1-|k|^{-1-s}}(Nm(\mathcal{D}))^{-\frac{1}{2}}$$

For $s \in \mathbb{C}$, $Re(s) > 0$

$$\int_{\mathcal{O}\setminus\{0\}} \|x\|_{\mathbb{F}}^s d^{\times}x = \sum_{n=0}^{\infty} \int_{\pi^n\mathcal{O}^{\times}} \|x\|_{\mathbb{F}}^s d^{\times}x = \sum_{n=0}^{\infty} |k|^{-ns} \int_{\mathcal{O}^{\times}} 1 d^{\times}x = \frac{1}{1-|k|^{-s}}(Nm(\mathcal{D}))^{-\frac{1}{2}}$$

Let $\alpha_1, ..., \alpha_n$ be an integral basis of $\mathbb{F}$ over $\mathbb{F}_0$. Then there are isomorphisms of abelian grouops, which are also homeomorphisms of locally compact spaces.

$$\begin{tikzcd}
\mathbb{F} \arrow[r, "\sim"] & \mathbb{F}_0\alpha_1 \oplus \cdots \oplus \mathbb{F}_0\alpha_n \arrow[r, "\sim"] & \mathbb{F}^n \\
\mathcal{O} \arrow[r, "\sim"] \arrow[u, hook] & \mathcal{O}_0\alpha_1 \oplus \cdots \oplus \mathcal{O}_0\alpha_n \arrow[r, "\sim"] \arrow[u, hook] & \mathcal{O}_0^n \arrow[u, hook]
\end{tikzcd}$$

Let $dx_1, ..., dx_n$ be the normalized measure on $\mathbb{F}_0$, then $dx_1 \wedge \cdots \wedge dx_n$ is also an additive invariant measure.

$$\int_{\mathcal{O}} 1 \cdot dx_1 \wedge \cdots \wedge dx_n = \int_{\mathcal{O}_0} 1 \cdot dx_1 \cdots \int_{\mathcal{O}_0} 1 \cdot dx_n = 1$$

Therefore,

$$dx = (\mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}} \cdot dx_1 \wedge \cdots \wedge dx_n$$

2.3 Fourier transform over $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$

In this section, we focus on the Fourier transform for $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$, which serves as foundation for Fourier transform over arbitrary local fields.

The additive character for $\mathbb{Q}_p$ is defined by

$$e_{\mathbb{Q}_p}(x) = e^{-2\pi i \lfloor x \rfloor}$$

where $\lfloor . \rfloor : \mathbb{Q}_p \longrightarrow \mathbb{R}$ is defined by

$$\lfloor a_{-n}p^{-n} + \cdots + a_{-1}p^{-1} + a_0 + \cdots \rfloor = a_{-n}p^{-n} + \cdots + a_{-1}p^{-1} \in \mathbb{R}$$

More naturally, $\lfloor . \rfloor$ is given by the composition

$$\mathbb{Q}_p \twoheadrightarrow \mathbb{Q}_p/\mathbb{Z}_p \xleftarrow{\sim} \mathbb{Z}[\tfrac{1}{p}]/\mathbb{Z} \hookrightarrow \mathbb{R}/\mathbb{Z}$$

Apparently $e_{\mathbb{Q}_p}(x)$ is a locally constant function (but is not compactly supported).

Definition Let $f : \mathbb{Q}_p \longrightarrow \mathbb{C}$ be a local constant compact supported function, then the Fourier transform of f is defined by

$$\widehat{f}(x) = \int_{\mathbb{Q}_p} f(y) e_{\mathbb{Q}_p}(xy) dy$$

To verify the operator we defined above is indeed a Fourier transform, we need to show that $\widehat{\widehat{f}}(x) = f(-x)$. Since f is a finite sum of characteristic functions, we only need to verify it for characteristic functions. Specifically, Let's compute the Fourier transform of $\mathbb{1}_{\mathbb{Z}_p}$.

$$\begin{aligned} \widehat{\mathbb{1}}_{\mathbb{Z}_p}(x) &= \int_{\mathbb{Q}_p} \mathbb{1}_{\mathbb{Z}_p} \cdot e_{\mathbb{Q}_p}(xy) dy \\ &= \int_{\mathbb{Z}_p} e^{-2\pi i \lfloor xy \rfloor} dy \end{aligned}$$

Note that if $x \in \mathbb{Z}_p$, then $xy \in \mathbb{Z}_p$, $\widehat{\mathbb{1}}_{\mathbb{Z}_p}(x) = 1$. If $x \notin \mathbb{Z}_p$, assume $\mathrm{ord}_p(x) = -n$, then it is not hard to prove the following lemma:

Lemma 2.3. *For each $b = b_n p^{-n} + \cdots + b_1 p^{-1}$, $b_i \in \{0, 1, ..., p-1\}$, there exists a $y_0 \in \mathbb{Z}_p$ such that $\lfloor xy \rfloor = b$. The set of all $y \in \mathbb{Z}_p$ such that $\lfloor xy \rfloor = b$ is $y_0 + p^n\mathbb{Z}_p$.*

Therefore,

$$\begin{aligned}\int_{\mathbb{Z}_p} e^{-2\pi i \lfloor xy \rfloor} dy &= \sum_{b_1=0}^{p-1} \cdots \sum_{b_n=0}^{p-1} \int_{p^n\mathbb{Z}_p} e^{-2\pi i (b_n p^{-n} + \cdots + b_1 p^{-1})} dy \\ &= \mu(p^n\mathbb{Z}_p) \cdot \prod_{k=1}^{n} \left(\sum_{j=0}^{p-1} e^{-2\pi i p^{-k} j} \right)\end{aligned}$$

But when $k = 1$, $\sum_{j=0}^{p-1} e^{-2\pi i p^{-1} j} = 0$. Thus, $\widehat{\mathbb{1}}_{\mathbb{Z}_p}(x) = 0$ for $x \notin \mathbb{Z}_p$. We can conclude that:

$$\widehat{\mathbb{1}}_{\mathbb{Z}_p}(x) = \mathbb{1}_{\mathbb{Z}_p}(x)$$

$\widehat{\mathbb{1}}_{\mathbb{Z}_p}(x)$ is invariant under Fourier transform, just like $e^{-\pi x^2}$ for real case. We can utilize this fact to find functional equations of L-functions.

Lemma 2.4. *Let $f : \mathbb{Q}_p \longrightarrow \mathbb{C}$ be a locally constant compact supported function, then*

1. $\widehat{af}(x) = \frac{1}{\|a\|_{\mathbb{Q}_p}} \widehat{f}\left(\frac{x}{a}\right)$
2. $\widehat{(f+a)}(x) = e_{\mathbb{Q}_p}(-xa) \cdot \widehat{f}(x)$

where $af(x) = f(xa)$ and $(f + a)(x) = f(x + a)$.

Applying lemma, we can obtain:

$$\begin{aligned}\widehat{\mathbb{1}}_{a+p^n\mathbb{Z}_p}(x) &= (\widehat{\mathbb{1}_{p^n\mathbb{Z}_p} - a})(x) \\ &= e_{\mathbb{Q}_p}(ax) \cdot \widehat{\mathbb{1}}_{p^n\mathbb{Z}_p}(x) \\ &= e_{\mathbb{Q}_p}(ax) \cdot \widehat{p^{-n}\mathbb{1}}_{\mathbb{Z}_p}(x) \\ &= e_{\mathbb{Q}_p}(ax) p^{-n} \cdot \widehat{\mathbb{1}}_{\mathbb{Z}_p}(p^n x) \\ &= e_{\mathbb{Q}_p}(ax) p^{-n} \cdot \mathbb{1}_{p^{-n}\mathbb{Z}_p}(x)\end{aligned}$$

Thus, for any $a \in \mathbb{Q}_p$ and integer n,

$$\begin{aligned}\widehat{\widehat{\mathbb{1}}}_{a+p^n\mathbb{Z}_p}(x) &= \int_{\mathbb{Q}_p} e_{\mathbb{Q}_p}(ay) p^{-n} \cdot \widehat{\mathbb{1}}_{p^{-n}\mathbb{Z}_p}(y) \cdot e_{\mathbb{Q}_p}(xy) dy \\ &= p^{-n} \int_{p^{-n}\mathbb{Z}_p} e_{\mathbb{Q}_p}((x+a)y) dy \\ &= \int_{\mathbb{Z}_p} e_{\mathbb{Q}_p}\left(p^{-n}(x+a)y\right) dy \\ &= \mathbb{1}_{\mathbb{Z}_p}(p^{-n}(x+a)) \\ &= \mathbb{1}_{-a+p^n\mathbb{Z}_p}(x)\end{aligned}$$

Therefore, all characteristic functions satisfies $\widehat{\widehat{f}}(x) = f(-x)$. We conclude that:

Theorem 2.5. *Let $f : \mathbb{Q}_p \longrightarrow \mathbb{C}$ be a local constant compact supported function. Then $\widehat{\widehat{f}}(x) = f(-x)$.*

Now Let's repeat our discussion for $\mathbb{F}_p((t))$. The additive character for $\mathbb{F}_p((t))$ is defined by

$$e_{\mathbb{F}_p((t))}(x) = e^{\frac{2\pi i}{q}\cdot \mathrm{Res}(x)}$$

where $\mathrm{Res} : \mathbb{F}_p((t)) \longrightarrow \mathbb{F}_p$ is defined by

$$\mathrm{Res}(a_{-n}t^{-n} + \cdots + a_{-1}t^{-1} + a_0 + \cdots) = a_{-1}$$

Apparently $e_{\mathbb{F}_p((t))}(x)$ is also a locally constant function (not compactly supported).

Definition Let $f : \mathbb{F}_p((t)) \longrightarrow \mathbb{C}$ be a local constant compact supported function, then the Fourier transform of f is defined by

$$\widehat{f}(x) = \int_{\mathbb{F}_p((t))} f(y) e_{\mathbb{F}_p((t))}(xy)dy$$

Again, Let's compute the Fourier transform of $\mathbb{1}_{\mathbb{F}_p[[t]]}$.

$$\begin{aligned}\widehat{\mathbb{1}}_{\mathbb{F}_p[[t]]}(x) &= \int_{\mathbb{F}_p((t))} \mathbb{1}_{\mathbb{F}_p[[t]]} \cdot e_{\mathbb{Q}_p}(xy)dy \\ &= \int_{\mathbb{F}_p[[t]]} e^{\frac{2\pi i}{p}\cdot \mathrm{Res}(xy)}dy\end{aligned}$$

If $x \in \mathbb{F}[[t]]$, then $xy \in \mathbb{F}[[t]]$, $\widehat{\mathbb{1}}_{\mathbb{F}[[t]]}(x) = 1$. If $x \notin \mathbb{F}[[t]]$, assume $\sum_{k\geq -n} a_n t^n$, where $a_{-n} \neq 0$, then we can prove that:

Lemma 2.6. *For arbitrary $b \in \mathbb{F}_p$ and $b_0, ..., b_{n-2} \in \mathbb{F}_p$, there exists a unique b_{n-1} such that $y_0 = \sum_{i=0}^{n-1} b_i t^i \in \mathbb{F}[[t]]$ satisfies $Res(xy) = b$ and for all $y \in y_0 + t^n\mathbb{F}_p[[t]]$, $Res(xy) = b$.*

Therefore,

$$\begin{aligned}\int_{\mathbb{F}_p[[t]]} e^{\frac{2\pi i}{p}\cdot \mathrm{Res}(xy)}dy &= \sum_{b_0\in\mathbb{F}_p} \cdots \sum_{b_{n-1}\in\mathbb{F}_p} \int_{t^n\mathbb{F}_p[[t]]} e^{\frac{2\pi i}{p}\cdot \sum_{p+q=-1} a_p b_q}dy \\ &= \sum_{b_0\in\mathbb{F}_p} \cdots \sum_{b_{n-2}\in\mathbb{F}_p} \int_{t^n\mathbb{F}_p[[t]]} \sum_{k\in\mathbb{F}_p} e^{\frac{2\pi i}{p}\cdot k}dy\end{aligned}$$

$\sum_{j=0}^{p-1} e^{-2\pi i p^{-1} j} = 0$ implies that $\widehat{\mathbb{1}}_{\mathbb{Z}_p}(x) = 0$ for $x \notin \mathbb{F}_p[[t]]$. We can conclude that:

$$\widehat{\mathbb{1}}_{\mathbb{F}_p[[t]]}(x) = \mathbb{1}_{\mathbb{F}_p[[t]]}(x)$$

By similar calculation as $\mathbb{Q}_p$ case, we can find that for any $a \in \mathbb{F}_p((t))$ and $n \in \mathbb{Z}$:

$$\widehat{\widehat{\mathbb{1}}}_{a+p^n\mathbb{Z}_p}(x) = \mathbb{1}_{-a+p^n\mathbb{Z}_p}(x)$$

Therefore, all characteristic functions satisfies $\widehat{\widehat{f}}(x) = f(-x)$. We conclude that:

Theorem 2.7. *Let $f : \mathbb{F}_p((t)) \longrightarrow \mathbb{C}$ be a local constant compact supported function. Then $\widehat{\widehat{f}}(x) = f(-x)$.*

2.4 Fourier transform over non-archimedean local fields

In last Let $\mathbb{F}$ be a local field which is a separable finite extension of $\mathbb{F}_0$ with ramification index f and inertial degree e, where $\mathbb{F}_0$ is either $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. Let $\mathcal{O}$ be the valuation ring of $\mathbb{F}$, π be a generator of the unique maximal ideal $\mathfrak{m}$ and k be the residue field $\mathcal{O}/\mathfrak{m}$.

Define the additive character for $\mathbb{F}$ by

$$e_{\mathbb{F}}(x) = e_{F_0}(\mathrm{Tr}(x))$$

where $\mathrm{Tr} = \mathrm{Tr}_{F_0}^{F}$ is the trace map.

Definition Let $f : \mathbb{F} \longrightarrow \mathbb{C}$ be a local constant compact supported function, then the Fourier transform of f is defined by

$$\widehat{f}(x) = \int_{\mathbb{F}} f(y) e_{\mathbb{F}}(xy) dy$$

Similarly, to verify the operator we defined is Fourier transform, we only need to check that $\widehat{\widehat{f}}(x) = f(-x)$ holds for characteristic functions. Let's compute the Fourier transform of $\mathbb{1}_{\mathcal{O}}$.

Let $\alpha_1, ..., \alpha_n$ be an integral basis of $\mathbb{F}$ over $\mathbb{F}_0$. For $x \in \mathbb{F}$,

$$\begin{aligned}
\widehat{\mathbb{1}}_{\mathcal{O}}(x) &= \int_{\mathbb{F}} \mathbb{1}_{\mathcal{O}} \cdot e_{\mathbb{F}}(xy) dy \\
&= \int_{\mathcal{O}} e_{\mathbb{F}_0}(\mathrm{Tr}(xy)) dy \\
&= (\mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}} \cdot \int_{\mathcal{O}_0} \cdots \int_{\mathcal{O}_0} e_{\mathbb{F}_0}(\mathrm{Tr}(x(y_1\alpha_1 + \cdots y_n\alpha_n))) dy_1 \wedge \cdots \wedge dy_n \\
&= (\mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}} \cdot \prod_{i=1}^{n} \int_{\mathcal{O}_0} e_{\mathbb{F}_0}(y_i \mathrm{Tr}(x\alpha_i)) dy_i \\
&= (\mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}} \cdot \prod_{i=1}^{n} \mathbb{1}_{\mathcal{O}_0}(\mathrm{Tr}(x\alpha_i))
\end{aligned}$$

Therefore, $\widehat{\mathbb{1}}_{\mathcal{O}}(x) = (\mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}}$ if and only if $\mathrm{Tr}(x\alpha_i) \in \mathcal{O}_0$ for all $i = 1, ..., n$. This is equivalent to $\mathrm{Tr}(x\mathcal{O}) \in \mathcal{O}_0$, i.e. $x \in \mathcal{O}^{\wedge}$. Therefore,

$$\widehat{\mathbb{1}}_{\mathcal{O}}(x) = \mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}} \mathbb{1}_{\mathcal{O}^{\wedge}}(x)$$

This result is surprising. The characteristic function for $\mathbb{1}_{\mathcal{O}_0}$ is invariant under the Fourier transform. However, it is not the case for general setting, no matter how you choose your Harr measure.

Lemma 2.8. *Let $f : \mathbb{F} \longrightarrow \mathbb{C}$ be a locally constant compact supported function, then*

1. $\widehat{af}(x) = \frac{1}{\|a\|_{\mathbb{F}}} \widehat{f}\left(\frac{x}{a}\right)$
2. $\widehat{(f + a)}(x) = e_{\mathbb{F}}(-xa) \cdot \widehat{f}(x)$

where $af(x) = f(xa)$ and $(f + a)(x) = f(x + a)$.

For any $a \in \mathbb{F}$ and $n \in \mathbb{Z}$, by applying lemma, we can obtain:

$$\begin{aligned}
\widehat{\mathbb{1}}_{a+\pi^n\mathcal{O}}(x) &= (\widehat{\mathbb{1}_{\pi^n\mathcal{O}} - a})(x) \\
&= e_{\mathbb{F}}(ax) \cdot \widehat{\mathbb{1}}_{\pi^n\mathcal{O}}(x) \\
&= e_{\mathbb{F}}(ax) \cdot \widehat{\pi^{-n}\mathbb{1}}_{\mathcal{O}}(x) \\
&= e_{\mathbb{F}}(ax)\|\pi\|_{\mathbb{F}}^{-n} \cdot \widehat{\mathbb{1}}_{\mathcal{O}}(\pi^n x) \\
&= \mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}}|k|^{-n} \cdot e_{\mathbb{F}}(ax) \cdot \mathbb{1}_{\pi^{-n}\mathcal{O}^\wedge}(x)
\end{aligned}$$

Since $\mathcal{O}^\wedge$ is a fraction ideal and $[\mathcal{O}^\wedge : \mathcal{O}] = \mathrm{Nm}(\mathcal{D}))$, we conclude that

$$\mathcal{O}^\wedge = \pi^{-l}\mathcal{O} \qquad l = \frac{\ln \mathrm{Nm}(\mathcal{D})}{\ln |k|}$$

So,

$$\begin{aligned}
\widehat{\widehat{\mathbb{1}}}_{a+\pi^n\mathcal{O}}(x) &= \mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}}|k|^{-n} \cdot \int_{\mathbb{F}} e_{\mathbb{F}}(ay) \cdot \mathbb{1}_{\pi^{-n}\mathcal{O}^\wedge}(y) \cdot e_{\mathbb{F}}(xy)dy \\
&= \mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}}|k|^{-n} \cdot \int_{\pi^{-n-l}\mathcal{O}} e_{\mathbb{F}}((x+a)y)dy \\
&= \mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}}|k|^{l} \cdot \int_{\mathcal{O}} e_{\mathbb{F}}(\pi^{-n-l}(x+a)y)dy \\
&= \mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}}|k|^{l} \cdot \widehat{\mathbb{1}}_{\mathcal{O}}(\pi^{-n-l}(x+a)) \\
&= \mathrm{Nm}(\mathcal{D}))^{-1}|k|^{l} \cdot \mathbb{1}_{\pi^{-l}\mathcal{O}}(\pi^{-n-l}(x+a)) \\
&= \mathbb{1}_{-a+\pi^n\mathcal{O}}
\end{aligned}$$

Therefore, all characteristic functions satisfies $\widehat{\widehat{f}}(x) = f(-x)$. We conclude that:

Theorem 2.9. *Let $f : \mathbb{F} \longrightarrow \mathbb{C}$ be a local constant compact supported function. Then $\widehat{\widehat{f}}(x) = f(-x)$.*

Now let X be a geometrically integral curve over $\mathbb{F}_p$ with function field K. $x \in |X|$. We need to establish the Fourier transform for locally constant function over $\widehat{K}_x$. In this case, we lost the field extension form a base field. Thus, to define an additive character, we need to introduce extra information. Let Ω be the differential sheaf over X. Its stalk at x is isomorphic to $k_x((\pi_x))d\pi_x$. The residue map is defined by

$$\text{Res: } \widehat{K}_x \times \Omega_x \longrightarrow k_x \qquad (f(\pi_x), g(\pi_x)d\pi_x) \longmapsto \text{coefficient of } \pi_x^{-1} \text{ of } f \cdot g$$

It is known that this map is independent of the choice of π_x.

Fix a $\omega \in \Gamma(X, \Omega)$, the additive character of $\widehat{K}_x$ is defined by

$$e_{\widehat{K}_x}(f) = e^{\frac{2\pi i}{p}\mathrm{Tr}_{\mathbb{F}_p}^{k_x}(\mathrm{Res}(f\omega))}$$

Let e_x be the order of ω at x, i.e. $\omega|_x \in \pi_x^{e_x}k_x[[\pi_x]]^\times d\pi_x$. We say ω is unramified at x if $e_x = 1$. It is know that ω only ramified at finite many closed points. We normalize the Harr measure at x such that the measure of $\widehat{\mathcal{O}}_x$ is $|k_x|^{-\frac{e_x}{2}}$.

Definition Let $f : \widehat{K}_x \longrightarrow \mathbb{C}$ be a local constant compact supported function, then the Fourier transform of f is defined by

$$\widehat{f}(g) = \int_{\widehat{K}_x} f(h) e_{\widehat{K}_x}(gh) dy$$

$$\widehat{\mathbb{1}}_{\widehat{\mathcal{O}}_x}(g) = \int_{\widehat{\mathcal{O}}_x} e_{\widehat{K}_x}(hg) dh = \int_{\widehat{\mathcal{O}}_x} e^{\frac{2\pi i}{p} \mathrm{Tr}^{k_x}_{\mathbb{F}_p}(\mathrm{Res}(hg\omega))} dh$$

If we identify $\widehat{K}_x \cong k_x((t))$, $\omega = \mu(t)dt$, then,

$$\widehat{\mathbb{1}}_{\widehat{\mathcal{O}}_x}(g) = \int_{k_x[[t]]} e^{\frac{2\pi i}{p} \mathrm{Tr}^{k_x}_{\mathbb{F}_p}(\mathrm{Res}(hg\mu))} dh = \begin{cases} |k_x|^{-\frac{e_x}{2}}, & \mathrm{ord}_x(g) + e_x \geq 0 \\ 0, & \mathrm{ord}_x(g) + e_x < 0 \end{cases}$$

Thus, we have

$$\widehat{\mathbb{1}}_{\widehat{\mathcal{O}}_x} = |k_x|^{-\frac{e_x}{2}} \cdot \mathbb{1}_{\pi_x^{-e_x}\widehat{\mathcal{O}}_x}$$

By similar calculation we can prove that $\widehat{\widehat{\mathbb{1}}}(x) = \mathbb{1}(-x)$ and this equation holds for all locally constant compact supported functions.

2.5 Integration over Adele rings

The following two tables summarize integration and Fourier transform theories over Local fields:

Integration

	Local fields $\mathbb{F}$	Additive Harr measures $dx_{\mathbb{F}}$	$\frac{d(ax)}{dx}$	Multiplicative Harr measures $d^{\times}x_{\mathbb{F}}$
Archimedean	$\mathbb{R}$	dx	$\|a\|_{\mathbb{F}}$	$d^{\times}x = \frac{dx}{\|x\|_{\mathbb{F}}}$
	$\mathbb{C}$	$dz = 2dx \wedge dy$		
Non-archimedean	$K/\mathbb{Q}_p$	$dx = (\mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}} \cdot dx_1 \wedge \cdots \wedge dx_n$		$d^{\times}x = \frac{1}{1-\|k\|^{-1}} \cdot \frac{dx}{\|x\|_{\mathbb{F}}}$
	$K/\mathbb{F}_p((t))$			

Fourier Transform

	Local fields $\mathbb{F}$	Additive characters	Fourier Transform	Allowed functions
Archimedean	$\mathbb{R}$	$e^{2\pi ix}$	$\widehat{f}(x) = \int_{\mathbb{F}} f(y) \cdot e_{\mathbb{F}}(xy)dx_{\mathbb{F}}$	Schwartz functions
	$\mathbb{C}$	$e^{2\pi i\mathrm{Tr}(z)}$		
Non-archimedean	$K/\mathbb{Q}_p$	$e^{-2\pi i\lfloor \mathrm{Tr}(x)\rfloor}$		Locally constant compact supported functions
	$K/\mathbb{F}_p((t))$	$e^{-2\pi i\mathrm{Res}(\mathrm{Tr}(x))}$		

There are several types of functions which we can define for all types of local fields. For example, Gaussian functions, which are "invariant" under Fourier Transform.

	Local fields $\mathbb{F}$	Gaussian functions
Archimedean	$\mathbb{R}$	$e^{-\pi x^2}$
	$\mathbb{C}$	$e^{-2\pi z\overline{z}}$
Non-archimedean	$K/\mathbb{Q}_p$	$\mathbb{1}_{\mathcal{O}}$
	$K/\mathbb{F}_p((t))$	

One might think: Is there a way to combine all these information into one.

Let K be a global field. $\mathbb{A}_K$ is a locally compact topological ring. Thus, additive invariant Harr measure exists on $\mathbb{A}_K$. We also know that locally $\mathbb{A}_K$ is a product space of local compact group. So, the question is: Is the product measure on all open neighborhoods forms a well-defined additive invariant measure on $\mathbb{A}_K$? The answer is Yes!

Definition An function $f : \mathbb{A}_K \longrightarrow \mathbb{C}$ is said to be factorizable Schwartz-Bruhat function if there exists functions $f_v : K_v \longrightarrow \mathbb{C}$ for all place v, such that

1. f_v is Schwartz function if v is ∞-place,
2. f_v is locally constant compact supported if v is finite place,
3. f_v is the characteristic function $\mathbb{1}_{\mathcal{O}}$ for all but finite many finite prime,
4. $$f = \prod_v f_v$$

An adelic function is called Schwartz-Bruhat function is it is a finite linear combination of factorizable Schwartz-Bruhat functions. The set of all Schwartz-Bruhat functions is denoted by $\mathcal{S}(\mathbb{A}_K)$.

For Schwartz-Bruhat function $f = \prod_v f_v$, it is defined on a standard open neighborhood of form .... Thus, we can compute its integral over $\mathbb{A}_K$ by

$$\int_{\mathbb{A}_K} f \cdot dx_{\mathbb{A}_K} = \prod_v \int_{K_v} f_v \cdot dx_v$$

Define the additive character over $\mathbb{A}_K$ to be

$$e_{\mathbb{A}_K}((g_v)) = \prod_v e_v(g_v)$$

Lemma 2.10. *For any $x \in K$, $e_{\mathbb{A}_K}(x) = 1$.*

Proof. If K is a number field,

$$\begin{aligned} e_{\mathbb{A}_K}(x) &= \prod_{\infty\text{-place } v} e_{K_v}(x) \cdot \prod_{\text{finite place } v} e_{K_v}(x) \\ &= e^{2\pi i(\mathrm{Tr}^K_{\mathbb{Q}}(x))} \cdot \prod_p \prod_{v|p} e^{-2\pi i \lfloor \mathrm{Tr}^{K_v}_{\mathbb{Q}_p}(x) \rfloor_p} \\ &= e^{2\pi i(\mathrm{Tr}^K_{\mathbb{Q}}(x) - \sum_p \lfloor \mathrm{Tr}^K_{\mathbb{Q}}(x) \rfloor_p)} \end{aligned}$$

Here we use the fact that for any rational prime p:

$$\mathrm{Tr}^K_{\mathbb{Q}}(x) = \sum_{w|p} \mathrm{Tr}^{K_w}_{\mathbb{Q}_p}(x)$$

Claim: for any $q \in \mathbb{Q}$,

$$q - \sum_p \lfloor q \rfloor_p \in \mathbb{Z}$$

This is because it belongs to $\mathbb{Z}_p$ for all rational primes. Therefore, we conclude that for number field K, $e_{\mathbb{A}_K}(x) = 1$ for all $x \in K$.

If K is a finite extension of $\mathbb{F}_p(t)$,

$$\begin{aligned} e_{\mathbb{A}_K}(x) &= \prod_v e_{K_v}(x) \\ &= \prod_f \prod_{v|f} e^{-2\pi i \lfloor \mathrm{Tr}^{K_v}_{\mathbb{Q}_p}(x) \rfloor_p} \\ &= e^{2\pi i(\mathrm{Tr}^K_{\mathbb{Q}}(x) - \sum_p \lfloor \mathrm{Tr}^K_{\mathbb{Q}}(x) \rfloor_p)} \end{aligned}$$

□

Therefore, $e_{\mathbb{A}_K}$ defines an additive character on $\mathbb{A}_K/K$.

If K is a function field, the result holds because the sum of residue of a meromorphic differential form is zero. See [1] for reference.

Theorem 2.11. *The fundamental domain of $\mathbb{A}_K/K$ has measure 1.*

Proof. We only prove for K being a number field, assume K has r real embeddings and $2s$ complex embeddings. Let $\alpha_1, ..., \alpha_n$ be an integral basis of K over $\mathbb{Q}$. Then the fundamental domain of $\mathbb{A}_K/K$ is

$$D = \prod_{i=1}^{n} [0,1)\alpha_i \times \prod_{\mathfrak{p}} \mathcal{O}_{\mathfrak{p}}$$

where .

$$\begin{aligned}
\int_D 1 \cdot dx_{\mathbb{A}_K} &= \int_{\prod_{i=1}^n [0,1)\alpha_i} dx_1 \cdots dx_r \cdot dz_1 \cdots dz_s \cdot \prod_{\mathfrak{p}} \int_{\mathcal{O}_{\mathfrak{p}}} 1 \cdot dx_{\mathfrak{p}} \\
&= 2^s \cdot \int_{\prod_{i=1}^n [0,1)\alpha_i} dx_1 \cdots dx_r \cdot dx^1 dy^1 \cdots dx^s dy^s \cdot \prod_{\mathfrak{p}} (\mathrm{Nm}(\mathcal{D}_{\mathfrak{p}}))^{-\frac{1}{2}} \\
&= 2^s |\Delta_K|^{-\frac{1}{2}} \cdot \mathrm{Vol}\left(\prod_{i=1}^n [0,1)\alpha_i \right) \\
&= 1
\end{aligned}$$

Here we use facts:

1. $$|\Delta_K| = \mathrm{Nm}(\mathcal{D}) = \prod_{\mathfrak{p}} \mathrm{Nm}(\mathcal{D}_{K_{\mathfrak{p}}/\mathbb{Q}_p})$$

2. $$\mathrm{Vol}\left(\prod_{i=1}^n [0,1)\alpha_i \right) = \frac{1}{2^s} \cdot \sqrt{\Delta_K}$$

□

Definition Let $f \in \mathcal{S}(\mathbb{A}_K)$. Then the Fourier transform of f is defined by

$$\widehat{f}(x) = \int_{\mathbb{A}_K} f(y) e_{\mathbb{A}_K}(xy) dy$$

In particular, if f is factorizable, $f = \prod_v f_v$, then

$$\widehat{f}(x) = \widehat{f}((x_v)) = \prod_v \int_{K_v} f_v(y_v) e_v(x_v y_v) dy_v$$

Theorem 2.12. *If $f \in \mathcal{S}(\mathbb{A}_K)$, then $\widehat{f} \in \mathcal{S}(\mathbb{A}_K)$. Moreover, $\widehat{\widehat{f}}(x) = f(-x)$*

Proof. we only need to prove it for factorizable $f \in \mathcal{S}(\mathbb{A}_K)$. Assume $f = \prod_v f_v$, then the Fourier transform of f is

$$\widehat{f} = \prod_v \widehat{f_v}$$

For ∞-place v, $\widehat{f_v}$ is still a Schwartz function. For finite place v, $\widehat{\mathbb{1}_{\mathcal{O}_v}} = \mathrm{Nm}(\mathcal{D}))^{-\frac{1}{2}} \cdot \mathbb{1}_{\mathcal{O}^\wedge}$. It is known that no matter number fields or function fields, there are only finite many valuations that are ramified. For unramified v, $\widehat{\mathbb{1}}_{\mathcal{O}_v} = \mathbb{1}_{\mathcal{O}_v}$. Thus, $\widehat{\mathbb{1}_{\mathcal{O}_v}} = \mathbb{1}_{\mathcal{O}_v}$ for all but finite many v. Therefore, $\widehat{f} \in \mathcal{S}(\mathbb{A}_K)$.

$$\widehat{\widehat{f}}((x_v)) = \prod_v \widehat{\widehat{f}}_v(x_v) = \prod_v f_v(-x_v)$$

Thus, $\widehat{\widehat{f}}(x) = f(-x)$. □

At last, Let's discuss the integration over idele group $\mathbb{A}_K^\times$. $\mathbb{A}_K^\times$ is locally compact group and hence it has a Harr measure exists. On standard neighborhood of form $\prod_v G_v$, where $G_v = \mathcal{O}_v^\times$ for all but finite many places, the Harr measure is exactly the product of Harr measures at each places.

For number field K, the Harr measure at ∞ has more convenient expression. If we write $\mathbb{A}_K^\times = \mathbb{A}_\infty^\times \times \mathbb{A}_{\text{finite}}^\times$, $d^\times x = d^\times x_\infty \cdot d^\times x_{\text{finite}}$. Recall that we have isomorphism and diffeomorphism

$$\mathbb{A}_\infty^\times \cong \mathbb{R}^r \times \mathbb{R}_{>0}^s \times (S^1)^s \cong \{\pm 1\}^r \times (S^1)^s \times \mathbb{R}^{r+s}$$

The Harr measure $d^\times x_\infty$ can be expressed as (notations follow from chapter...):

$$\begin{aligned}
d^\times x_\infty &= d^\times x_1 \cdots d^\times x_r \cdot d^\times z_1 \cdots d^\times z_s \\
&= 2^s \frac{dx_1}{|x_1|} \cdots \frac{dx_r}{|x_r|} \cdot \frac{dx^1 dy^1}{|z_1|^2} \cdots \frac{dx^s dy^s}{|z_s|^2} \\
&= 2^s \frac{dx_1}{|x_1|} \cdots \frac{dx_r}{|x_r|} \cdot \frac{r_1 dr_1 d\theta_1}{r_1^2} \cdots \frac{r_s dr_s d\theta_s}{r_s^2} \\
&= 2^s \frac{dx_1}{|x_1|} \cdots \frac{dx_r}{|x_r|} \frac{dr_1}{r_1} \cdots \frac{dr_s}{r_s} \cdot d\theta_1 \cdots d\theta_s
\end{aligned}$$

Let $y_1, ..., y_r, y_{r+1}, ..., y_{r+s}$ be the coordinate of $\mathbb{R}^{r+s}$, then $|x_i| = e^{y_i}$ for $i = 1, ...r$, and $r_i = e^{\frac{1}{2} y_{r+i}}$ for $i = 1, ..., s$.

$$\begin{aligned}
d^\times x_\infty &= 2^s \frac{dx_1}{|x_1|} \cdots \frac{dx_r}{|x_r|} \frac{dr_1}{r_1} \cdots \frac{dr_s}{r_s} \cdot d\theta_1 \cdots d\theta_s \\
&= dy_1 \cdots dy_{r+s} \cdot d\theta_1 \cdots d\theta_s
\end{aligned}$$

Recall that the group $\mathcal{O}_K^\times$ acts on $\mathbb{A}_\infty^\times$. The fundamental domain of $\mathbb{A}_\infty^\times / \mathcal{O}_K^\times$ is given by

$$\left(\{\pm 1\}^r \times (S^1)^s\right) / \mu(K) \times \prod_{i=1}^{r+s-1} [0,1) u_i \times \mathbb{R}\mathbf{v}$$

Now consider a function f on $\mathbb{A}_\infty$, then integrate f on domain D can be express as

$$\int_D f d^\times x_\infty = \int_{(\{\pm 1\}^r \times (S^1)^s)/\mu(K)} d\theta_1 \cdots d\theta_s \int_{\prod_{i=1}^{r+s-1}[0,1)u_i \times \mathbb{R}\mathbf{v}} f \cdot dy_1 \cdots dy_{r+s}$$

If we choose $\{u_1, ..., u_{r+s-1}, \mathbf{v}\}$ as a coordinate, then the integral is given by

$$\int_D f d^\times x_\infty = nR_K \cdot \int_{(\{\pm 1\}^r \times (S^1)^s)/\mu(K)} d\theta_1 \cdots d\theta_s \underbrace{\int_0^1 \cdots \int_0^1}_{r+s-1} \int_{-\infty}^{+\infty} f \cdot dy_1 \cdots dy_{r+s-1} dr$$

where $n = r + 2s = [K : \mathbb{Q}]$ and R_K is called the regulator of K:

$$R_K = \frac{1}{n} \det(u_1, ..., u_{r+s-1}, \mathbf{v})$$

The definition of R_K is independent of the choices of $u_1, ..., u_{r+s-1}$. Recall that we have decomposition

$$\mathbb{A}_\infty = \{\pm 1\}^r \times (S^1)^s \times \mathbb{R}^{r+s-1} \times \mathbb{R}_{>0}$$

Where the last factor is $\|x\|_\infty$. Since $\|x\|_\infty = e^{nr}$, we have $d\|x\|_\infty = n\|x\|_\infty dr$, $d\|x\|_\infty^\times = ndr$. Thus,

$$\begin{aligned} d^\times x_\infty &= nR_K d\theta_1 \cdots \theta_s \cdot dy_1 \cdots dy_{r+s-1} dr \\ &= R_K d\theta_1 \cdots \theta_s \cdot dy_1 \cdots dy_{r+s-1} d^\times \|x\|_\infty \end{aligned}$$

Lemma 2.13. *If f is a function on $\mathbb{A}_\infty$ that its value only depends on $\|x\|_\infty$, then the integral can be express as*

$$\int_D f d^\times x_\infty = \frac{2^r (2\pi)^s R_K}{\mu_K} \cdot \int_{\mathbb{R}_{>0}} f(\|x\|_\infty) d^\times \|x\|_\infty$$

2.6 Adelic Poisson summation

In Fourier analysis over $\mathbb{R}$, for $f \in \mathcal{S}(\mathbb{R})$, we have

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \widehat{f}(n)$$

The key aspect in the proof of Poisson summation is the usage of Fourier expansion for a $\mathbb{Z}$-periodic function. $\mathbb{Z}$-periodic function f has Fourier expansion:

$$f(x) = \sum_{n \in \mathbb{Z}} \widehat{f}_n e_{\mathbb{R}}(nx)$$

where

$$\widehat{f}_n = \int_0^1 f(x) e_{\mathbb{R}}(-nx) dx = \int_{\mathbb{R}/\mathbb{Z}} f(x) e_{\mathbb{R}}(-nx) dx$$

It is natural to ask whether we have similar result regarding f over $\mathbb{A}_K$, being an analogue of $\mathbb{R}$, which is K-periodic, being an analogue of $\mathbb{Z}$-periodic. The answer is Yes!

Definition Let f be a function over $\mathbb{A}_K$. f is called periodic if for any $k \in K$,

$$f(x + k) = f(x)$$

A natural way to construct a periodic adelic function is to sum up all translation by elements in K. This method is valid for adelic Schwartz functions.

Proposition 2.14. *Let $f \in \mathcal{S}(\mathbb{A}_K)$. Then the sum*

$$h(x) = \sum_{\alpha \in K} f(x + \alpha)$$

converge absolutely and uniformly on compact subspace of $\mathbb{A}_K$. Hence, it defines

Proof. We only need to prove this result for $f \in \mathcal{S}(\mathbb{A}_K)$ that is factorizable. Let S be the set of all finite places of K such that f_v is not $\mathbb{1}_{\mathcal{O}_v}$. For any point $x = (x_v) \in \mathbb{A}_K$, Let S' be the set of all finite places v such that $g_v \notin \mathcal{O}_v$. Then for any $\alpha \in K$, the

If K is a function field, then f can be further decomposed into finite linear combination of characteristic functions of standard open sets. so we only need to consider

$$f' = \prod_{v\in S} \mathbb{1}_{a_v+\pi_v^{n_v}\mathcal{O}_v} \cdot \prod_{v\notin S} \mathbb{1}_{\mathcal{O}}$$

Similarly, if K is a function field, then f can be expressed as finite linear combination of form:

$$f' = f_\infty \cdot \prod_{v\in S} \mathbb{1}_{a_v+\pi_v^{n_v}\mathcal{O}_v} \cdot \prod_{v\notin S} \mathbb{1}_{\mathcal{O}}$$

We find out necessary conditions for $f'(x+\alpha) \neq 0$. For $v \notin S \cup S'$, $x_v + \alpha \in \mathcal{O}_v$ if and only if $\alpha \in \mathcal{O}_v$, i.e. $v(\alpha) \geq 0$. For $v \in S' \setminus S$, $x_v + \alpha \in O_v$ only if $v(\alpha) = v(x_v)$. For $v \in S$, $x_v + \alpha \in a_v + \pi_v^{n_v}\mathcal{O}_v$ only if $v(\alpha) \geq \min\{v(a_v), n_v\}$. In summary, For $\alpha \in K$ such that $f'(x+\alpha) \neq 0$, α should satisfies:

$$\begin{aligned} &v(\alpha) \geq 0 && \text{if } v \notin S \cup S' \\ &v(\alpha) = v(x_v) && \text{if } v \in S' \setminus S \\ &v(\alpha) \geq \min\{v(a_v), n_v\} && \text{if } v \in S \end{aligned}$$

For function field K, by Riemann-Roch theorem, the set of all $\alpha \in K$ satisfying these condition forms a finite dimensional vector space over $\mathbb{F}_p$, which is a finite set. Thus $h(x)$ converges for all $x \in \mathbb{A}_K$.

For number field K, there exists integer N, such that $f'(x+\alpha) \neq 0$ only if $\alpha \in \frac{1}{N}\mathcal{O}_K$. Thus,

$$|h(x+\alpha)| \leq \sum_{\alpha\in\frac{1}{N}\mathcal{O}_K} |f_\infty(x+\alpha)|$$

Assume $[K:\mathbb{Q}] = n$. The sum is taken over a n-dimensional $\mathbb{Z}$-lattice. Since $f \in \mathcal{S}(\mathbb{R}^n)$, this sum converge uniformly on compact sets. □

Definition For function field K, an adelic function f is said to be smooth if it is locally constant. For number field K, $\mathbb{A}_K = \mathbb{A}_\infty \times \mathbb{A}_f$. an adelic function f is said to be smooth function if for any point $x_0 \in \mathbb{A}_f$, there exists a neighborhood U and a smooth function $f^U : \mathbb{A} \longrightarrow \mathbb{C}$, such that for any $(x_\infty, x_f) \in \mathbb{A}_\infty \times U$, $f(x_\infty, x_f) = f^U(x_\infty)$.

Proposition 2.15. *Let f be an periodic smooth adelic function over $\mathbb{A}_K$. Then there exists an $h \in \mathcal{S}(\mathbb{A}_K)$, such that*

$$f(x) = \sum_{\alpha\in K} f(x+\alpha)$$

Proof. We only prove this proposition for number fields. Assume there exists an $h_0 \in \mathcal{S}(\mathbb{A}_K)$ such that for any $x \in \mathbb{A}_K$

$$\sum_{\alpha\in K} h_0(x+\alpha) = 1$$

Then, By taking $h(x) = f(x)h_0(x)$, we complete our proof by showing that $h(x)f(x) \in \mathcal{S}(\mathbb{A}_K)$. Thus, our fist task is to prove the existence of h_0.

Assume $[K:\mathbb{Q}] = n$, then the fundamental domain for $\mathbb{A}_K/K$ is $D = \prod_{i=1}^n [0,1)\alpha_i \times \prod_{\text{finite } v} \mathcal{O}_v$, where $\alpha_1, ..., \alpha_n$ is an integral basis of K.

Claim: There exists a smooth function $h_\infty : \mathbb{R}^n \to \mathbb{R}$, such that

$$\sum_{\alpha \in \mathcal{O}_K} h_\infty(x+\alpha) = 1 \tag{2.2}$$

When $n = 1$, take $\mu(x)$ to be a increasing smooth function vanishing when $x \leq -1$ and is equal to 1 when $x \geq 0$. Then $h_\infty(x) = h_1(x) := \mu(x) - \mu(x-1)$ satisfies the condition.

For arbitrary n, take

$$h_\infty(x) = h_\infty\left(\sum_{i=1}^n x_i\alpha_i\right) := \prod_{i=1}^n h_1(x_i)$$

Thus, the function $h_0(x) = h_\infty(x_\infty) \cdot \prod_v \mathbb{1}_{\mathcal{O}_v}$ satisfies equation 2.2. Note that $h_0(x)$ is supported on D for both cases.

Next, we prove that $h(x) = f(x)h_0(x) \in \mathcal{S}(\mathbb{A}_K)$. By the definition of smoothness, there exist a cover $\{U_i\}$of D, such that each U_i is a product of $\mathbb{A}_\infty$ and a standard open set of $\mathbb{A}_f$, and the function $h(x)|_{U_i} = h^{U_i}(x_\infty)$ depends only on the ∞ part. Since D is compact, there exists a finite subcover $U_1, ..., U_m$ of D. Denote $U_i = \mathbb{A}_\infty \times V_i$. We can assume V_i are mutually disjoint. Therefore,

$$h(x) = \sum_{i=1}^m h(x)|_{U_i} = \sum_{i=1}^m h^{U_i}(x_\infty) \cdot \mathbb{1}_{U_i}$$

is a adelic Schwartz function. □

$$\begin{aligned}\int_{\mathbb{A}_K} f(x)dx &= \sum_{\alpha\in K}\int_{\mathbb{A}_K} f \cdot \mathbb{1}_{\alpha+D}dx = \sum_{\alpha\in K}\int_{\alpha+D} f(x)dx = \sum_{\alpha\in K}\int_D f(x+\alpha)dx \\ &= \int_{\mathbb{A}_K/K}\sum_{\alpha\in K} f(x+\alpha)dx\end{aligned}$$

Therefore, to compute the integral of function $f \in \mathcal{S}(\mathbb{A}_K)$ over $\mathbb{A}_K$, we can

Theorem 2.16. *Let $f : \mathbb{A}_K \longrightarrow \mathbb{C}$ be a smooth periodic function. Then,*

$$f(x) = \sum_{\alpha\in K} \widehat{f}_\alpha \cdot e(\alpha x)$$

where the sum converges absolutely for all $x \in \mathbb{A}_K$ and

$$\widehat{f}_\alpha = \int_{\mathbb{A}_K/K} f(x)e(-\alpha x)dx = \widehat{h}(\alpha) \qquad \text{for all } \alpha \in K$$

Here $h \in \mathcal{S}(\mathbb{A}_K)$ such that $f(x) = \sum_{\beta\in K} h(x+\beta)$ and $\widehat{h}$ is the Fourier transform of h.

Lemma 2.17. *Let K be a number field, then for non zero $\alpha \in K$,*

$$\int_{\mathbb{A}_K/K} e_{\mathbb{A}_K}(\alpha x)dx = 0$$

Proof. Let $\alpha_1, ...\alpha_n$ be an integral basis for K over $\mathbb{Q}$. Then the fundamental domain for $\mathbb{A}_K/K$ is $D = \prod_{i=1}^n [0,1)\alpha_i \times \prod_v \mathcal{O}_v$, thus

$$\begin{aligned}\int_{\mathbb{A}_K/K} e_{\mathbb{A}_K}(\alpha x)dx &= \int_{\prod_{i=1}^n [0,1)\alpha_i} e_\infty(\alpha x_\infty)dx_\infty \cdot \prod_v \int_{\mathcal{O}_v} e_v(\alpha x_v)dx_v \\ &= \int_{\prod_{i=1}^n [0,1)\alpha_i} e_\infty(\alpha x_\infty)dx_\infty \cdot \prod_v \widehat{\mathbb{1}}_{\mathcal{O}_v}(\alpha) \\ &= |\Delta|^{-\frac{1}{2}} \int_{\prod_{i=1}^n [0,1)\alpha_i} e_\infty(\alpha x_\infty)dx_\infty \cdot \prod_v \mathbb{1}_{\mathcal{O}^\wedge}(\alpha)\end{aligned}$$

Assume K has r real embeddings $\sigma_1, ..., \sigma_r$ and s complex embeddings $\tau_1, \overline{\tau}_1, ..., \tau_s, \overline{\tau}_s$. To compute the integral, it will be more convenient to use coordinates $\alpha_1, ..., \alpha_n$. The change of coordinates ϕ is given by a linear map

$$A = \begin{pmatrix} \sigma_1(\alpha_1) & \dots & \sigma_1(\alpha_n) \\ \vdots & \ddots & \vdots \\ \sigma_r(\alpha_1) & \dots & \sigma_r(\alpha_n) \\ \mathrm{Re}(\tau_1(\alpha_1)) & \dots & \mathrm{Re}(\tau_1(\alpha_n)) \\ \mathrm{Im}(\tau_1(\alpha_1)) & \dots & \mathrm{Im}(\tau_1(\alpha_n)) \\ \vdots & \ddots & \vdots \\ \mathrm{Re}(\tau_s(\alpha_1)) & \dots & \mathrm{Re}(\tau_s(\alpha_n)) \\ \mathrm{Im}(\tau_s(\alpha_1)) & \dots & \mathrm{Im}(\tau_s(\alpha_n)) \end{pmatrix}$$

$$\begin{array}{ccccc} \mathbb{A}_\infty = \mathbb{R}^r \times \mathbb{C}^s & \xrightarrow{\sim} & \mathbb{R}^n & \xleftarrow{\phi} & \mathbb{R}^n \\ (x_1, ..., x_r, z_1, ..., z_s) & \longmapsto & (x_1, ..., x_r, \mathrm{Re}(z_1), \mathrm{Im}(z_1), ...) & \longleftarrow & (y_1, ..., y_n) \end{array}$$

The aboslute value of the determinant of ϕ is $\frac{1}{2^s}|\Delta|^{\frac{1}{2}}$. Thus,

$$dx_\infty = 2^s dx_1 \cdots dx_r dx^1 dy^1 \cdots dx^s dy^s = |\Delta|^{\frac{1}{2}} dy_1 \cdots dy_n$$

Now we need to figure out how to $e_\infty(xy)$ for two elements $x, y \in \mathbb{A}_\infty$ with their y_i's coordinates. Under the standard coordinate,

$$e_\infty(xy) = \exp(2\pi i(x_1y_1 + \cdots x_ry_r + 2\mathrm{Re}(z_1)\mathrm{Re}(w_1) - 2\mathrm{Im}(z_1)\mathrm{Im}(w_1) + \cdots))$$

The number on the exponent is a bilinear form and under the standard coordinate it is given by matrix $B = \mathrm{diag}(\underbrace{1, ..., 1}_{r}, \underbrace{2, -2, ..., 2, -2}_{2s})$. Thus, under the basis $\alpha_1, ..., \alpha_n$ the number on the exponent is also a bilinear form given by matrix A^TBA. By calculation, one can find that the matrix is exactly $(\mathrm{Tr}(\alpha_i\alpha_j))$. For convenience, denote $A^TBA = (b_{ij})$. Now assume $\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n$. Then $\alpha \in \mathcal{O}_k^\wedge$ if and only if $c_j = \sum_{i=1}^n a_ib_{ij} \in \mathbb{Z}$ for all $j = 1, ..., n$.

$$\int_{\prod_{i=1}^{n}[0,1)\alpha_i} e_\infty(\alpha x_\infty)dx_\infty = |\Delta|^{\frac{1}{2}} \int_0^1 \cdots \int_0^1 \exp\left(2\pi i \left(\sum_{j=1}^{n} c_j y_j\right)\right) dy_1 \cdots dy_n$$
$$= |\Delta|^{\frac{1}{2}} \prod_{j=1}^{n} \int_0^1 e^{2\pi i c_j y_j} dy_j$$

Thus,

$$\int_{\mathbb{A}_K/K} e_{\mathbb{A}_K}(\alpha x)dx = \prod_{j=1}^{n} \int_0^1 e^{2\pi i c_j y_j} dy_j \cdot \prod_v \mathbb{1}_{\mathcal{O}^\wedge}(\alpha)$$

If there exists a non zero $c_j \in \mathbb{Z}$, then $\int_0^1 e^{2\pi i c_j y_j} dy_j = 0$. Otherwise, $\alpha \notin \mathcal{O}_K^\wedge$, then there exists a valuation v such that $\alpha \notin \mathcal{O}_v$. For both cases, we have

$$\int_{\mathbb{A}_K/K} e_{\mathbb{A}_K}(\alpha x)dx = 0$$

□

Proof. Assume K is a number field. Denote e by $e_{\mathbb{A}_K}$ for short.

$$\begin{aligned}
\widehat{f_\alpha} &= \int_{\mathbb{A}_K/K} f(x)e(-\alpha x)dx \\
&= \int_{\mathbb{A}_K/K} \sum_{\beta \in K} h(x+\beta)e(-\alpha x)dx \\
&= \sum_{\beta \in K} \int_{\mathbb{A}_K/K} h(x+\beta)e(-\alpha(x+\beta))dx \\
&= \sum_{\beta \in K} \int_{\beta+\mathbb{A}_K/K} h(x)e(-\alpha x)dx \\
&= \widehat{h}(\alpha) \qquad \text{for all } \alpha \in K
\end{aligned}$$

By similar argument in..., we know that there exists N such that $\widehat{f_\alpha}$ if and only if $\alpha \in \frac{1}{N}\mathcal{O}_K$. We first prove this result holds for $x = 0$. The method is very similar to the one we use in real analysis. Let $\alpha_1, ..., \alpha_n$ be an integral basis for K over $\mathbb{Q}$. consider the partial sum

$$\begin{aligned}
S_M &= \sum_{\substack{\alpha=\frac{1}{N}(t_1\alpha_1+\cdot+t_n\alpha_n) \\ |t_1|,...,|t_n|\le M}} \widehat{f_\alpha} \\
&= \int_{\mathbb{A}_K/K} \sum_{\substack{\alpha=\frac{1}{N}(t_1\alpha_1+\cdot+t_n\alpha_n) \\ |t_1|,...,|t_n|\le M}} f(x)e(-\alpha x)dx \\
&= \int_{\mathbb{A}_K/K} f(x) \sum_{t_1=-M}^{M} \cdots \sum_{t_n=-M}^{M} e\left(-\frac{1}{N}(t_1\alpha_1 + \cdot + t_n\alpha_n)x\right) dx \\
&= \int_{\mathbb{A}_K/K} f(x) \prod_{i=1}^{n} \sum_{t_i=-M}^{M} e\left(-\frac{1}{N}\alpha_i x\right)^{t_i} dx
\end{aligned}$$

Denote $e(x) = e^{2\pi i \gamma(x)}$, then

$$\begin{aligned} S_M &= = \int_{\mathbb{A}_K/K} f(x) \prod_{i=1}^{n} \sum_{t_i=-M}^{M} e\left(-\frac{1}{N}\alpha_i x\right)^{t_i} dx \\ &= \int_{\mathbb{A}_K/K} f(x) \prod_{i=1}^{n} \frac{\sin\big((2M+1)\pi\gamma(\frac{\alpha_i x}{N})\big)}{\sin\big(\pi\gamma(\frac{\alpha_i x}{N})\big)} dx \end{aligned}$$

By lemma..., we know that

$$f(0) = \int_{\mathbb{A}_K/K} \sum_{\substack{\alpha = \frac{1}{N}(t_1\alpha_1 + \cdot + t_n\alpha_n) \\ |t_1|,\ldots,|t_n| \le M}} f(0) e(-\alpha x) dx$$

Therefore, the difference is given by

$$S_M - f(0) = \int_{\mathbb{A}_K/K} (f(x) - f(0)) \prod_{i=1}^{n} \frac{\sin\big((2M+1)\pi\gamma(\frac{\alpha_i x}{N})\big)}{\sin\big(\pi\gamma(\frac{\alpha_i x}{N})\big)} dx$$

Since $f(x)$ is a smooth function, $S_M - f(0) \to 0$ as $M \to \infty$. Thus,

$$f(0) = \sum_{\alpha \in K} \widehat{f}_\alpha$$

For any $x_0 \in \mathbb{A}_K$, by applying above equation to function $g(x) = f(x + x_0)$, we can get

$$f(x_0) = g(0) = \sum_{\alpha \in K} \widehat{g}_\alpha = \sum_{\alpha \in K} \widehat{f}_\alpha e(\alpha x_0)$$

□

Corollary 2.18. *Let $f \in \mathcal{S}(\mathbb{A}_K)$. Then*

$$\sum_{\alpha \in K} f(\alpha) = \sum_{\alpha \in K} \widehat{f}(\alpha)$$

Corollary 2.19. *Let $f \in \mathcal{S}(\mathbb{A}_K)$. For any $y \in \mathbb{A}_K$*

$$\sum_{\alpha \in K} f(\alpha y) = \frac{1}{\|y\|_{\mathbb{A}_K}} \sum_{\alpha \in K} \widehat{f}\left(\frac{\alpha}{y}\right)$$

3 Application

3.1 Functional equations for Dedekind zeta functions

In this section, we are going to prove the functional equations for Dedekind zeta functions by the Fourier transform over adeles.

Let K be a number field. The Gaussian functions over Let the global Gaussian function defined by product of local Gaussian functions:

$$\Phi(x) = \Phi((x_v)) = \prod_v G_v(x_v)$$

	Local fields $\mathbb{F}$	Gaussian functions G_v
Archimedean	$\mathbb{R}$	$e^{-\pi x^2}$
	$\mathbb{C}$	$e^{-2\pi z\overline{z}}$
Non-archimedean	$K/\mathbb{Q}_p$	$\mathbb{1}_{\mathcal{O}}$
	$K/\mathbb{F}_p((t))$	

If K is a number field with r real embeddings and s complex embeddings, then the global Gaussian function can be written as

$$\Phi(x) = \Phi((x_v)) = \prod_{i=1}^{r} e^{-\pi x_i^2} \cdot \prod_{i=1}^{s} e^{-2\pi z_i\overline{z_i}} \cdot \prod_{\mathfrak{p}} \mathbb{1}_{\mathcal{O}_\mathfrak{p}}(x_\mathfrak{p})$$

Then the Fourier transforms of Φ is given by

$$\widehat{\Phi}(x) = \widehat{\Phi}((x_v)) = |\Delta|^{-\frac{1}{2}} \prod_{i=1}^{r} e^{-\pi x_i^2} \cdot \prod_{i=1}^{s} e^{-2\pi z_i\overline{z_i}} \cdot \prod_{\mathfrak{p}} \mathbb{1}_{\mathcal{O}_\mathfrak{p}^\wedge}(x_\mathfrak{p})$$

Consider the Mellin transform of $\Phi(x)$ over $\mathbb{A}_K^\times$, it is the

$$\xi(z,\Phi) = \int_{\mathbb{A}_K^\times} \Phi(x)\|x\|_{\mathbb{A}_K^\times}^z d^\times x$$

Note that $\Phi(x)\|x\|_{\mathbb{A}_K^\times}^z$ is factorizable, so we can compute the integral at each place to get the $\xi(z)$:

$$\xi(z,\Phi) = \left(\int_{\mathbb{R}^\times} e^{-\pi x^2}|x|^z d^\times x\right)^r \cdot \left(\int_{\mathbb{C}^\times} e^{-2\pi w\overline{w}}|w|^{2z} d^\times w\right)^s \cdot \prod_{\mathfrak{p}} \int_{\mathcal{O}_\mathfrak{p}\setminus\{0\}} \|x_\mathfrak{p}\|_\mathfrak{p}^z d^\times x_\mathfrak{p}$$

By computation, we can obtain:

$$\Gamma_\mathbb{R}(z) = \int_{\mathbb{R}^\times} e^{-\pi x^2}|x|^z d^\times x = \frac{\Gamma(\frac{z}{2})}{\pi^{\frac{z}{2}}}$$
$$\Gamma_\mathbb{C}(z) = \int_{\mathbb{C}^\times} e^{-2\pi w\overline{w}}|w|^{2z} d^\times w = \frac{\Gamma(z)}{(2\pi)^{z-1}}$$
$$\int_{\mathcal{O}_\mathfrak{p}\setminus\{0\}} \|x_\mathfrak{p}\|_\mathfrak{p}^z d^\times x_\mathfrak{p} = \frac{\mathrm{Nm}(\mathcal{D}_\mathfrak{p})^{-\frac{1}{2}}}{1-\mathrm{Nm}(\mathfrak{p})^{-z}}$$

Therefore, for $\mathrm{Re}(z) > 1$, we have

$$\begin{aligned}\xi(z,\Phi) &= |\Delta|^{-\frac{1}{2}}(\Gamma_\mathbb{R}(z))^r \cdot (\Gamma_\mathbb{C}(z))^s \cdot \prod_{\mathfrak{p}} \frac{1}{1-\mathrm{Nm}(\mathfrak{p})^{-s}} \\ &= |\Delta|^{-\frac{1}{2}}(\Gamma_\mathbb{R}(z))^r \cdot (\Gamma_\mathbb{C}(z))^s \cdot L_K(z)\end{aligned}$$

Now we can compute ....

$$\begin{aligned}
\xi(z,\Phi) &= \sum_{\alpha\in K^\times}\int_{\alpha\cdot\mathbb{A}_K^\times/K^\times}\Phi(x)\|x\|_{\mathbb{A}_K^\times}^z d^\times x\\
&= \int_{\mathbb{A}_K^\times/K^\times}\sum_{\alpha\in K^\times}\Phi(\alpha x)\|x\|_{\mathbb{A}_K^\times}^z d^\times x\\
&= \int_{\substack{\mathbb{A}_K^\times/K^\times\\ \|x\|\le 1}}\sum_{\alpha\in K^\times}\Phi(\alpha x)\|x\|_{\mathbb{A}_K^\times}^z d^\times x + \int_{\substack{\mathbb{A}_K^\times/K^\times\\ \|x\|>1}}\sum_{\alpha\in K^\times}\Phi(\alpha x)\|x\|_{\mathbb{A}_K^\times}^z d^\times x
\end{aligned}$$

By Poisson summation formula, we have

$$\Phi(0)+\sum_{\alpha\in K^\times}\Phi(\alpha x)=\frac{\widehat{\Phi}(0)}{\|x\|}+\sum_{\alpha\in K^\times}\Phi\left(\frac{\alpha}{x}\right)$$

Thus,

$$\begin{aligned}
\int_{\substack{\mathbb{A}_K^\times/K^\times\\ \|x\|\le 1}}\sum_{\alpha\in K^\times}\Phi(\alpha x)\|x\|_{\mathbb{A}_K^\times}^z d^\times x &= \int_{\substack{\mathbb{A}_K^\times/K^\times\\ \|x\|\le 1}}\sum_{\alpha\in K^\times}\widehat{\Phi}\left(\frac{\alpha}{x}\right)\|x\|_{\mathbb{A}_K^\times}^z d^\times x + \int_{\substack{\mathbb{A}_K^\times/K^\times\\ \|x\|\le 1}}\left(\frac{\widehat{\Phi}(0)}{\|x\|}-\Phi(0)\right)\|x\|_{\mathbb{A}_K^\times}^z d^\times x\\
&= \int_{\substack{\mathbb{A}_K^\times/K^\times\\ \|x\|\ge 1}}\sum_{\alpha\in K^\times}\widehat{\Phi}(\alpha x)\|x\|_{\mathbb{A}_K^\times}^{1-z} d^\times x + \int_{\substack{\mathbb{A}_K^\times/K^\times\\ \|x\|\le 1}}\|x\|_{\mathbb{A}_K^\times}^{z-1}-\|x\|_{\mathbb{A}_K^\times}^z d^\times x
\end{aligned}$$

The area $\{x\in\mathbb{A}_K/K:\|x\|=1\}$ has measure zero, thus

$$\xi(z,\Phi)=\int_{\substack{\mathbb{A}_K^\times/K^\times\\ \|x\|\le 1}}\|x\|_{\mathbb{A}_K^\times}^{z-1}-\|x\|_{\mathbb{A}_K^\times}^z d^\times x+\int_{\substack{\mathbb{A}_K^\times/K^\times\\ \|x\|\ge 1}}\sum_{\alpha\in K^\times}\Phi(\alpha x)\|x\|_{\mathbb{A}_K^\times}^z+\widehat{\Phi}(\alpha x)\|x\|_{\mathbb{A}_K^\times}^{1-z}d^\times x$$

Let's compute the first term. By theorem..., the fundamental domain is given by $\alpha_1 D = D,\dots,\alpha_{h_K}D$, where $\alpha_1,\dots,\alpha_{h_K}\in\mathbb{A}_K^\times$ and $h_K=|\mathrm{Cl}(K)|$. So

$$\int_{\substack{\alpha_i D\\ \|x\|\le 1}}\|x\|^z d^\times x=\|\alpha_i\|^z\int_{\substack{D\\ \|x\|\le\|\alpha_i\|^{-1}}}\|x\|^z d^\times x$$

Thus, we only need to compute $\int_{\substack{D\\ \|x\|\le t}}\|x\|^z d^\times x$.

$$\begin{aligned}
\int_{\substack{D\\ \|x\|\le t}}\|x\|^z d^\times z &= \int_{\substack{D_\infty\\ \|x\|_\infty\le t}}\|x\|_\infty^z d^\times z_\infty\cdot\prod_{\mathfrak{p}}\int_{\mathcal{O}_\mathfrak{p}^\times}d^\times x_\mathfrak{p}\\
&= |\Delta|^{-\frac{1}{2}}\int_{\substack{D_\infty\\ \|x\|_\infty\le t}}\|x\|_\infty^z d^\times x_\infty\\
&= \frac{2^r(2\pi)^s R_K}{\mu_K\sqrt{|\Delta|}}\int_{0<\|x\|_\infty\le t}\|x\|_\infty^z d^\times\|x\|_\infty\\
&= \frac{2^r(2\pi)^s R_K}{\mu_K\sqrt{|\Delta|}}\int_0^t x^{z-1}dx\\
&= \frac{2^r(2\pi)^s R_K}{\mu_K\sqrt{|\Delta|}z}t^z
\end{aligned}$$

$$\int_{\substack{\alpha_i D\\ \|x\|\le 1}} \|x\|^z d^\times x = \|\alpha_i\|^z \int_{\substack{D\\ \|x\|\le \|\alpha_i\|^{-1}}} \|x\|^z d^\times x = \frac{2^r(2\pi)^s R_K}{\mu_K\sqrt{|\Delta|}z}$$

Thus,

$$\int_{\substack{\mathbb{A}_K^\times/K^\times\\ \|x\|\le 1}} \|x\|_{\mathbb{A}_K^\times}^{z-1} - \|x\|_{\mathbb{A}_K^\times}^{z} d^\times x = \frac{2^r(2\pi)^s h_K R_K}{\mu_K\sqrt{|\Delta|}}\left(\frac{1}{z-1}-\frac{1}{z}\right)$$

In summary, we have

$$\xi(z,\Phi) = \frac{2^r(2\pi)^s h_K R_K}{\mu_K\sqrt{|\Delta|}}\left(\frac{1}{z-1}-\frac{1}{z}\right) + \int_{\substack{\mathbb{A}_K^\times/K^\times\\ \|x\|\ge 1}} \sum_{\alpha\in K^\times} \Phi(\alpha x)\|x\|_{\mathbb{A}_K^\times}^{z} + \widehat{\Phi}(\alpha x)\|x\|_{\mathbb{A}_K^\times}^{1-z} d^\times x \tag{3.1}$$

The function on the right hand side is a meomorphic function on $\mathbb{C}$, thus, we conclude that the Dedekind zeta function $L_K(z)$ has analytic continuation on the whole plane.

To obtain the functional equation, note that the equation 3.1 is invariant under the transform:

$$z \to 1-z \qquad \Phi \to \widehat{\Phi}$$

Thus, we have equation

$$\xi(z,\Phi) = \xi(1-z,\widehat{\Phi})$$

By integrating at each place, we can obtain that

$$\xi(z,\widehat{\Phi}) = |\Delta|^{z-1}(\Gamma_\mathbb{R}(z))^r \cdot (\Gamma_\mathbb{C}(z))^s \cdot L_K(z)$$

Hence,

$$|\Delta|^{-\frac{1}{2}}(\Gamma_\mathbb{R}(z))^r \cdot (\Gamma_\mathbb{C}(z))^s \cdot L_K(z) = |\Delta|^{-z}(\Gamma_\mathbb{R}(1-z))^r \cdot (\Gamma_\mathbb{C}(1-z))^s \cdot L_K(1-z)$$

Take $\widehat{\xi}(z) = |\Delta|^{\frac{z-1}{2}}(\Gamma_\mathbb{R}(z))^r \cdot (\Gamma_\mathbb{C}(z))^s L_K(z)$, we conclude that:

Theorem 3.1. *$\widehat{\xi}(z)$ is a meomorphic function on the whole plane with only pole 0 and 1, with residue $\frac{2^r(2\pi)^s h_K R_K}{\mu_K\sqrt{|\Delta|}}$. Moreover, $\widehat{\xi}(z)$ satisfies functional equation:*

$$\widehat{\xi}(z) = \widehat{\xi}(1-z)$$

References

[1] Weil, André (1974). *Basic Number Theory.* Berlin, Heidelberg: Springer Berlin Heidelberg. doi:10.1007/978-3-642-61945-8.

[2]