# Gaudin model, Center theorems, and Vertex algebras

Wenqing Wei

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**Abstract**

This paper is based on a one-semester lecture at UC Berkeley in Spring 2023, taught by Edward Frenkel. I incorporate some additional background in physics to make the context more self-contained. In this paper, we aim to introduce the center theorem regarding $U(\widehat{\mathfrak{g}}_-)$ which is motivated by the Gaudin model. First, we state and prove the classical analogue of the center theorem. Then we develop the theory of vertex algebra and reformulate the center theorem within the context of affine Kac-Moody vertex algebras. At last, we briefly discuss the link between the center theorem and opers.

# Contents

# 1 Hamiltonian Mechanics

Let's consider a classical system consisting of a particle moving in 1-dimensional space with force $F(x)$ exerting on it. The "state" of this particle at time $t$, in classical mechanics, refer to its position $x$ and velocity $v$. Hence, the space of all possible states is $\mathbb{R}^2 = \{(x,v)|x,v \in \mathbb{R}\}$, called the phase space. By Newton's second law, the time-evolution of this system is described by

$$\begin{cases} \dfrac{d}{dt}x(t) = v(t) \\ m\dfrac{d}{dt}v(t) = F(x(t)) \end{cases} \qquad \begin{cases} x(0) = x_0 \\ v(0) = v_0 \end{cases} \tag{1.1}$$

By the existence and uniqueness theorem of ODE, for each pair $(x_0, v_0) \in \mathbb{R}^2$, there exists a unique solution $(x(t), v(t))$. Therefore, the motion of this particle can be viewed as a curve of the phase space,

Let $V(x) = -\int_{-\infty}^{x} F(x)dx$, then $\frac{d}{dx}V(x) = -F(x)$. The total energy in physics is defined by

$$E(x,v) = \frac{1}{2}mv^2 + V(x)$$

**Lemma 1.1** (Energy conservation law)**.** *If $(x(t), v(t))$ is the solution of ODE 1.1, then $E(x(t), v(t))$ is a constant.*

*Proof.*

$$\begin{aligned}\frac{d}{dt}E(x(t),v(t)) &= mv(t)v'(t) + \left.\frac{dV}{dt}\right|_{x(t)} \cdot x'(t)\\ &= mv(t)v'(t) - F(x(t))v(t)\\ &= 0\end{aligned}$$

□

Now we consider $x, v$ as independent variables. Then $E(x,v)$ is a function on the phase space. The energy conservation law implies that the time-evolution on the phase space should be a level set of function $E$, i.e. if the starting point of a particle is $(x_0, v_0)$, then the motion of this particle is a part of the level set:

$$\{(x,v)|E(x,v) = E(x_0,v_0)\}$$

Note that

$$v = \frac{1}{m}\frac{\partial}{\partial v}E(x,v) \qquad F(x) = -\frac{\partial}{\partial x}E(x,v)$$

To find out the time-parameterization $(x(t), v(t))$ of this curve, we should rewrite the equation 1.1 as

$$\begin{cases}\frac{d}{dt}x(t) = \frac{1}{m}\frac{\partial}{\partial v}E(x(t),v(t)) \\ m\frac{d}{dt}v(t) = -\frac{\partial}{\partial x}E(x(t),v(t))\end{cases} \qquad \begin{cases}x(0) = x_0 \\ v(0) = v_0\end{cases} \tag{1.2}$$

The coefficient $m$ seems to be annoying, so we replace velocity $v$ by momentum $p = mv$. Now the phase space changes to $\mathbb{R}^2 = \{(x,p)|x,p \in \mathbb{R}\}$, energy expression change to (we replace $E$ by $H$, meaning a Hamiltonian function)

$$H = \frac{1}{2m}p^2 + V(x)$$

and equation 1.3 changes to

$$\begin{cases}\frac{dx}{dt} = \frac{\partial H}{\partial p} \\ \frac{dp}{dt} = -\frac{\partial H}{\partial x}\end{cases} \qquad \begin{cases}x(0) = x_0 \\ v(0) = v_0\end{cases} \tag{1.3}$$

We can generalize this reformulation to higher dimension. For example, consider system of $N$ planets in 1-dimension space interfering with each other by gravity. Let $x_1, ..., x_N$ and $p_1, ..., p_N$ be position and momentum of these N planets. Then the phase space of this system is $\mathbb{R}^{2N} = \{(\mathbf{x},\mathbf{p})|\mathbf{x},\mathbf{p} \in \mathbb{R}^N\}$ and the total energy of this system is given by

$$H = \sum_{i=1}^{N} \frac{1}{2m}p_i^2 + V(x_1, ..., x_N)$$

The time-evolution of this system is described by ODE:

$$\begin{cases}\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i} \\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}\end{cases} \qquad \begin{cases}\mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{v}(0) = \mathbf{v}_0\end{cases} \tag{1.4}$$

Therefore, we can temporarily hold the believe that for arbitrary classical physics system, the phase space is $\mathbb{R}^{2n} = \{(\mathbf{x}, \mathbf{p}) | \mathbf{x}, \mathbf{p} \in \mathbb{R}^n\}$ and the time-evolution of this system is controlled by a function (Hamiltonian)

$$H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$$

such that the time-evolution of this system is described by ODE:

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial p_i} \\ \dfrac{dp_i}{dt} = -\dfrac{\partial H}{\partial x_i} \end{cases} \qquad \begin{cases} \mathbf{x}(0) = \mathbf{x}_0 \\ \mathbf{v}(0) = \mathbf{v}_0 \end{cases} \tag{1.5}$$

In the next section, we will find a geometric interpretation of ODE 1.5 which would upgrade our interpretation towards classical systems.

# 2 Symplectic manifolds and Poisson brackets

Let $V$ be a finite dimensional vector space over $\mathbb{R}$. A quadratic form $w : V \times V \longrightarrow \mathbb{R}$ is called a symplectic form if

1. $\omega$ is non-degenerate, i.e. the induced map $V \longrightarrow V^*$ is an isomorphism,
2. $\omega$ is anti-symmetric, i.e. $\omega(v, w) = -\omega(w, v)$ for any $v, w \in V$.

For example, let $V = \mathbb{R}^{2n}$ with basis $x_1, ..., x_n, y_1, ..., y_n$. Define a quadratic form $\omega : V \times V \longrightarrow \mathbb{R}$ with matrix representation $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, i.e. for $1 \leq i, j \leq n$,

$$\omega(x_i, x_j) = 0 \qquad \omega(x_i, y_j) = \delta_{i,j} \qquad \omega(y_i, y_j) = 0 \qquad \omega(y_i, x_j) = -\delta_{i,j}$$

It's easy to verify that this is a symplectic form. Moreover, this symplectic form can also be written as

$$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$

**Proposition 2.1.** *Let $\omega$ be a symplectic form on an $m$-dimensional vector space. Then $V$ is even dimensional, i.e. $m = 2n$ for some integer $n$ and there exists a basis $x_1, ..., x_n, y_1, ..., y_n$ of $V$, such that*

$$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$

Let $M$ be a smooth manifold. A symplectic form on $M$ is a closed 2-form $\omega$ on $M$ which is non-degenerate at every point, i.e. $d\omega = 0$ and for each $p \in M$, $\omega_p$ is a symplectic form on $T_pM$.

**Example 2.1.** *The most fundamental example of symplectic manifolds are cotangent bundle. Let $M$ be an arbitrary manifold. We can construct a natural one-form $\tau$ on $T^*M$. Let $\pi : T^*M \longrightarrow M$ be the natural projection. For each point $(p, \varphi) \in T^*M$, where $p \in M$ and $\varphi \in T_p^*M$, the projection $\pi$ induces a map $\pi_p^* : T_p^*M \longrightarrow T_{(p,\varphi)}(T^*M)$. Define*

$$\tau_{p,\varphi} = \pi_p^*(\varphi)$$

*Explicitly, Let $x_M^1, ..., x_M^n$ be a coordinate of a neighborhood of $M$ and $x_{T^*M}^1, ..., x_{T^*M}^n, dx_{T^*M}^1, ..., dx_{T^*M}^n$ be the induced coordinate of a neighborhood of $T^*M$, i.e. $x_M^i = x_{T^*M}^i$, $i = 1, ..., n$. Now for each point $(p.\varphi) = (a_1, ..., a_n, b_1, ..., b_n)$,*

$$\begin{aligned}\tau_{(p,\varphi)} = \pi_p^*(\varphi) &= \pi_p^*(b_1 dx_M^1 + \cdots + b_n dx_M^n) \\ &= b_1 d(x_M^1 \circ \pi) + \cdots + b_n d(x_M^n \circ \pi) \\ &= b_1 dx_{T^*M}^1 + \cdots + b_n dx_{T^*M}^n\end{aligned}$$

*If we rewrite this local coordinate by $x^1, ..., x^n, y^1, ..., y^n$, then the one-form $\tau$ can be expressed as $\sum_{i=1}^n y^i dx^i$.*

*Define $\omega = -d\tau$. Locally $\omega$ can be expressed as*

$$\omega = -d\left(\sum_{i=1}^n y^i dx^i\right) = -\sum_{i=1}^n dy^i \wedge dx^i = \sum_{i=1}^n dx^i \wedge dy^i$$

*Thus, $\omega$ is a symplectic form. We have constructed a natural symplectic form on any cotangent bundle.*

**Example 2.2.** *Let $(M, g)$ be a Riemann manifold. The Riemann metric induces an isomorphism of vector bundles $TM \xrightarrow{\sim} T^*M$. We can endow $TM$ with a symplectic structure by pulling back the natural symplectic form on $T^*M$. Explicitly, assume $x^1, ..., x^n, y^1 = \frac{\partial}{\partial x^1}, ..., y^n = \frac{\partial}{\partial x^n}$ is a local coordinate on $TM$, and $g_{ij} = \delta_{i,j}$ Then the symplectic form is in form of*

$$\sum_{i=1}^n dx^i \wedge dy^i$$

*In last section, we argue that phase space of a classical system is given by $\mathbb{R}^{2n} = \{(\mathbf{x}, \mathbf{p}) | \mathbf{x}, \mathbf{p} \in \mathbb{R}^n\}$, where $\mathbf{p}$ (momentum) is derivatives of $\mathbf{x}$ (position) by definition. Therefore, we can view phase space as a tangent bundle over the space of position. Moreover, by our construction above, we can equipped a natural symplectic form on the phase space.*

We can always choose a nice coordinate for a neighborhood of symplectic manifold so that the symplectic form can be expressed in standard form.

**Theorem 2.2** (Darboux)**.** *Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold. For any $p \in M$, there exists a smooth coordinate $(x_1, ..., x_n, y_1, ..., y_n)$ centered at $p$ such that*

$$\omega = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$

*in a neighborhood of $p$.*

*Proof.* See [7], Theorem 22.13. □

Since $\omega$ is non-degenerate, it defines an isomorphism $\widehat{\omega} : TM \longrightarrow T^*M$ such that for any vector fields $X, Y$

$$\widehat{\omega}(X)Y = \omega(X, Y) \in C^\infty(M)$$

For any smooth function $f \in C^\infty(M)$, we define the Hamiltonian vector field of $f$ to be the smooth vector field $X_f$:

$$X_f = \widehat{\omega}^{-1}(df)$$

Hence, for any smooth vector field $Y$,

$$\omega(X_f, Y) = df(Y) = Y(f)$$

**Lemma 2.3.** *Let $(M, \omega)$ be a symplectic manifold of dimension $2n$. $(x_1, ..., x_n, y_1, ..., y_n)$ is a local coordinate such that $\omega$ can be expressed in standard form. Then for any $f \in C^\infty(M)$,*

$$X_f = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial y_i}$$

*Proof.* Assume $X_f = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} + b_i \frac{\partial}{\partial y_i}$

$$-b_i = \omega(X_f, \frac{\partial}{\partial x_i}) = df\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial f}{\partial x_i}$$

$$a_i = \omega(X_f, \frac{\partial}{\partial y_i}) = df\left(\frac{\partial}{\partial y_i}\right) = \frac{\partial f}{\partial y_i}$$

Thus,

$$X_f = \sum_{i=1}^{n} \frac{\partial f}{\partial y_i}\frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial}{\partial y_i}$$

□

For $f, g \in C^\infty(M)$, we define their Poisson bracket $\{f, g\} \in C^\infty(M)$ by

$$\{f, g\} = \omega(X_f, X_g)$$

on the standard coordinate, the Poisson bracket is given by

$$\{f, g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial g}{\partial x_i}\frac{\partial f}{\partial y_i}$$

**Proposition 2.4.** *For $f, g, h \in C^\infty(M)$,*

1. $\{f, g\} = -\{g, f\}$,
2. $\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0$,
3. $\{fg, h\} = \{f, h\}g + f\{g, h\}$.

*Proof.* See [7], Proposition 22.19. □

In fact, three properties in proposition 2.4 defines a so-called Poisson structure on the commutative algebra $C^\infty(M)$.

**Definition** A commutative algebra $A$ with a bilinear form $\{.\} : A \times A \longrightarrow A$ is called a Poisson algebra if for any $a, b, c \in A$,

1. $\{a, b\} = -\{b, a\}$,
2. $\{\{a, b\}, c\} + \{\{b, c\}, a\} + \{\{c, a\}, b\} = 0$,
3. $\{ab, c\} = \{a, c\}b + a\{b, c\}$.

Recall that given a smooth vector field $X$ on a smooth manifold $M$, the flow $\Phi_X(x,t)$ on $M$ attaches each point $x_0$ with a curve $\Phi_X(x_0,t)$ that is defined in a neighborhood of $t=0$, such that $\Phi(x_0,t)=x_0$ and

$$\frac{d}{dt}\Phi(x_0,t)=X_{\Phi(x_0,t)}$$

For fixed $x_0 \in M$, $\Phi(x_0,t)$ is called an integral curve of $X$.

**Definition** Let $(M,\omega)$ be a symplectic manifold of dimension $2n$ and $f$ be a smooth function on $M$. The Hamiltonian flow generated by $f$ is the flow associated to the vector field $X_f$.

**Proposition 2.5.** *Let $(x_1,...,x_n,y_1,...,y_n)$ be a standard coordinate for a neighborhood of $(M,\omega)$ and $f$ be a smooth function on $M$. If $(x(t),y(t))$ is an integral curve of $X_f$, then it satisfies ODE:*

$$\begin{cases} \dfrac{d}{dt}x_i(t)=\dfrac{\partial f}{\partial y_i}(x(t),y(t)) \\ \dfrac{d}{dt}y_i(t)=-\dfrac{\partial f}{\partial x_i}(x(t),y(t)) \end{cases} \tag{2.1}$$

*Proof.* The vector field $X_f$ is

$$\left(\frac{\partial f}{\partial y_1},...,\frac{\partial f}{\partial y_n},-\frac{\partial f}{\partial x_1},...,-\frac{\partial f}{\partial x_n}\right)$$

The tangent vector of the integral curve is

$$\left(\frac{dx_1(t)}{dt},...,\frac{dx_n(t)}{dt},\frac{dy_1(t)}{dt},...,\frac{dy_n(t)}{dt}\right)$$

These two vector fields should be equal, which derives the ODE. $\square$

In last section, we have demonstrate how to describe the time-evolution of a classical physics system as a curve in phase space. For instance, if the phase space is $\mathbb{R}^{2n}=\{(\mathbf{x},\mathbf{p})|\mathbf{x}\in\mathbb{R}^n,\mathbf{p}\in\mathbb{R}^n\}$ and the system is controlled by Hamiltonian $H$, which is a smooth function on $\mathbb{R}^{2n}$, then curve of time-evolution is determined by ODE:

$$\begin{cases} \dfrac{dx_i(t)}{dt}=\dfrac{\partial H}{\partial p_i} \\ \dfrac{dp_i(t)}{dt}=-\dfrac{\partial H}{\partial x_i} \end{cases} \qquad \begin{cases} x(0)=x_0 \\ p(0)=p_0 \end{cases}$$

We endow $\mathbb{R}^{2n}$ (a tangent bundle) with the natural symplectic structure. Then by proposition 2.5, the curve of time-evolution is exactly an integral curve with respect to Hamiltonian $H$. Therefore, to study classical physics system (of finite degree of freedom), it is mathematically equivalent to study the Hamiltonian flow on a symplectic manifold $M$ with respect to a fixed Hamiltonian $H\in C^\infty(M)$.

Classical physics quantities, such as position, velocity, and energy, can all be viewed as functions on the phase space. Generally, a classical observable refers to a smooth function on the phase space, which represents certain physical quantity at a given point. The whole system is controlled by one classical observable, the Hamiltonian $H$. If $f$ is a classical observable, then we are interested in how $f$ varies with respect to time. Here is a nice result.

**Proposition 2.6.** *Let $(M, \omega)$ be a symplectic manifold of dimension $2n$. $H$ is a smooth function (Hamiltonian) on $M$. Let $\gamma(t)$ be an integral curve with respect to $H$. Then for any $f \in C^\infty(M)$,*

$$\frac{df \circ \gamma(t)}{dt} = \{f, H\}(\gamma(t))$$

*Proof.* Let $(x_1, ..., x_n, y_1, ..., y_n)$ be a standard coordinate. Assume $\gamma(t) = (x(t), y(t))$, then $\frac{dx_i(t)}{dt} = \frac{\partial H}{dy_i}$, $\frac{dy_i(t)}{dt} = -\frac{\partial H}{dx_i}$, $i = 1, ..., n$

$$\begin{aligned}
\frac{df \circ \gamma(t)}{dt} &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{dx_i(t)}{dt} + \frac{\partial f}{\partial y_i}\frac{dy_i(t)}{dt} \\
&= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial H}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial H}{\partial x_i} \\
&= \{f, H\}(\gamma(t))
\end{aligned}$$

□

**Definition** Let $M, \omega$ be a symplectic manifold with Hamiltonian $H$. $f \in C^\infty(M)$ is called conserved quantity if $f \circ \gamma(t)$ is constant for any integral curve $\gamma(t)$ with respect to $H$.

**Corollary 2.7.** *$f \in C^\infty(M)$ is a conserved quantity if and only if*

$$\{f, H\} = 0$$

*In particular, the Hamiltonian $H$ is itself a conserved quantity.*

# 3 Quantum mechanics

Quantum mechanics is one of the greatest invention in physics in the twentieth century. It describes our world in a significantly distinct way. Let's still consider a system with one particle. In classical mechanics, we describe its state by position $x$ and momentum $p$. Once we know the Hamiltonian, then the time-evolution of this particle can be uniquely determined. However, in quantum world, we can never precisely determine classical quantities, such as position. Instead, what we can determine is the possibility of this particle appearing at certain position. Let's assume this particle lies in $\mathbb{R}$. The probability density of its position is given by a complex valued function $f(x) \in L^2(\mathbb{R})$, such that

$$\int_{\mathbb{R}} |f(x)|^2 dx = 1$$

We then say that this particle is at state $f(x)$. Hence, the phase space for quantum single particle system is $L^2(\mathbb{R})$- a Hilbert space.

We have claimed that in quantum world, a particle can appear at everywhere. However, when physicist conduct experiment, they do obtain stable data like position, momentum, and energy. How to explain the meaning of these data? Let's take position as an example. Intuitively, since the position of this particle follows a probability distribution, the outcome of measurement should equal to the mean value of this random variable, i.e.

$$x_{\text{measure}} = \int_{\mathbb{R}} x|f(x)|^2 dx$$

Let $\langle . \rangle$ be the standard inner product on $L^2(\mathbb{R})$, i.e. for any $f(x), g(x) \in L^2(\mathbb{R})$,

$$\langle f, g \rangle = \int_{\mathbb{R}} \overline{f(x)} g(x) dx$$

In this paper, we always assume that inner product on Hilbert space is anti-linear on the first factor.

Define an operator $\widehat{x} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ by

$$\widehat{x}(f(x)) \coloneqq x \cdot f(x)$$

This operator is not well-defined for all elements in $L^2(\mathbb{R})$. Let's ignore this problem temporarily. Then

$$x_{\text{measure}} = \langle \widehat{x}(f), f \rangle = \langle f, \widehat{x}(f) \rangle$$

We should consider $\widehat{x}$, a self-adjoint operator, as the quantum analogue of classical observable $x$. In classical system, measurement of position is done by evaluating the function $x : \mathbb{R}^2 \longrightarrow \mathbb{R}$ at a particular point. On the other hand, in quantum system, measurement of position is done by applying the operator $\widehat{x}$ to the state function. Other classical observables also have quantum analogues. For example, the quantum analogue of momentum $p$ is the operator $\widehat{p} = i\hbar \frac{\partial}{\partial x} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$, where $\hbar$ is a constant. In classical case, the Poisson bracket of $x$ and $p$ is given by

$$\{x.p\} = 1$$

The quantum analogue of Poisson bracket is the Lie bracket of operators. For example, in quantum case, the commutator of $\widehat{x}$ and $\widehat{p}$ is given by

$$[\widehat{x}, \widehat{p}] = -i\hbar x \frac{\partial}{\partial x} + i\hbar \frac{\partial}{\partial x} \circ x = i\hbar$$

Let $\mathbb{1}$ be a classical observable which maps every state to 1. We assume that the quantum analogue of $\mathbb{1}$ is the identity operator Id. Then we have an equation

$$i\hbar \widehat{\{x, p\}} = [\widehat{x}, \widehat{p}]$$

Now, we have enough motivation to state axioms of a quantum system. A quantum system should obey following axioms:

**Axiom 1.** *A Hilbert space $\mathbb{H}$ over $\mathbb{C}$ is associated to this quantum system. Sometimes we just represent this quantum system by $\mathbb{H}$.*

**Axiom 2.** *A state of this quantum system refers to a unit vector in $\mathbb{H}$. Two unit vectors $\psi_1, \psi_2$ represent same state if and only if there exists $z \in \mathbb{C}$ such that $\psi_1 = z\psi_2$. Generally speaking, we say any nonzero vector $v \in \mathbb{H}$ is a state, representing the state $\frac{v}{\|v\|}$. Hence, the phase space is $\mathbb{PH} = \mathbb{H}/\mathbb{C}^\times$*

**Axiom 3.** *For any two unit vectors $\psi_1, \psi_2 \in \mathbb{H}$, the transition probability between $\psi_1$ and $\psi_2$ is given by*

$$\delta(\psi_1, \psi_2) = |\langle \psi_1, \psi_2 \rangle|$$

**Axiom 4.** *A quantum observable refers to a self-adjoint operator $A : D \longrightarrow \mathbb{H}$, where $D$ is a dense subspace of $\mathbb{H}$.*

Axioms we introduced above are mathematical setting of a quantum system and no physics get involved. Now we introduce quantization of a classical system and state physical axioms of a quantum system.

**Definition** Quantization of a classical system $M$ to a quantum system $\mathbb{H}$ is an $\mathbb{R}$-linear assignment:

$$\widehat{\ }: \text{classical observables} \longrightarrow \text{ quantum observables} \qquad f \longrightarrow \widehat{f}$$

such that:

- $\widehat{\mathbb{1}} = \text{Id}$
- For any classical observables $f$ and $g$, $[\widehat{f}, \widehat{g}] = i\hbar\widehat{\{f, g\}}$.

**Axiom 5.** *There is a distinguished quantum observable (Hamiltonian) $H$. The time-evolution of a state $\psi \in \mathbb{H}$ satisfies*

$$i\hbar\frac{d}{dt}\psi(t) = H\psi(t)$$

*where $\psi(.) : (-\epsilon, \epsilon) \longrightarrow \mathbb{H}$, $\psi(0) = \psi$, is the time-evolution of state $\psi$.*

For example, if we consider the quantum system $L^2(\mathbb{R}^3)$ with Hamiltonian $H$ and $f(x)$ is a state, then the time-evolution of this state is given by a function $f(x, t)$ such that

$$i\hbar\frac{d}{dt}f(x, t) = Hf(x, t)$$

The next two axioms describe the rule of measurement:

**Axiom 6.** *Let $\Phi$ be a quantum observable. Measuring this quantum system with respect to $\Phi$ would obtain a real number $\lambda$. Moreover, immediately after the measurement, this quantum system will be in a state $\psi \in \mathbb{H}$ which is the eigenvector of $\Phi$ with eigenvalue $\lambda$, i.e.*

$$\Phi(\psi) = \lambda \cdot \psi$$

In other words, the outcome of measurement is always a eigenvalue of the quantum observable, which is always real due to the self-adjointness. After the measurement, system stays in an eigenstate (eigenvector) of this quantum observable.

**Axiom 7.** *Let $\Phi$ be a quantum observable. A unit-norm state $\psi$ can be expressed as linear combination of unit-norm eigenstates of $\Phi$*

$$\psi = \sum_{j \in I} c_j \psi_j$$

*Then after applying the measurement $\Phi$, the outcome is a random value in $\{c_i | i \in I\}$. The probability of obtaining $c_i$ and staying in state $\psi_i$ is*

$$\frac{\|c_i\|^2}{\sum_{j \in I} \|c_j\|^2}$$

The quantization of classical Hamiltonian is of most important. For instance, the classical Hamiltonian (energy) of 1-dimensional single particle system is $H = \frac{1}{2m}p^2 + V(x)$. It's quantization is given by

$$\widehat{H} = \frac{1}{2m}\widehat{p}^2 + \widehat{V(x)} = -\frac{\hbar^2}{2m}\frac{d^2}{d^2x} + V(x)$$

where operator $\widehat{V(x)}$ is multiplication by $V(x)$. Then eigenvalues of $\widehat{H}$ are exactly all possible measurement of energy. If $E$ is an eigenvalue of $\widehat{H}$, then the corresponding state $\psi(x) \in L^2(\mathbb{R})$ should satisfy

$$\widehat{H}\psi = E\psi \to -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{d^2x} + V(x)\psi(x) = E\psi(x)$$

This is the 1-dimensional Schrodinger equation. If the system is at this eigenstate, then by axiom 11.2, the system would stay at this state. In this case, the time-evolution is given by $\psi(t) = \psi(x) \cdot \varphi(t)$ such that

$$i\hbar\frac{d}{dt}\psi(t) = \widehat{H}\psi(t) = E\psi(t)$$

Thus $\varphi(t) = e^{-\frac{i}{\hbar}E}$ and $\psi(t) = \psi(x) \cdot e^{-\frac{i}{\hbar}E}$.

Recall that for classical system, we are interested in observable $f$ that is a conserved quantity , or equivalent $\{f, H\} = 0$. After quantization, we have $[\widehat{f}, \widehat{H}] = 0$, i.e. operators $\widehat{f}$ and $\widehat{H}$ commute with each other. In mathematics, two self-adjoint operators $A$ and $B$ commute with each other implies that they can be simultaneously diagonalized. In physics, it implies that these two quantum observables can be measured simultaneously. On the other hand, if two operator $A$ and $B$ do not commute, then we can never measure these two quantities correctly at one time. For example, $\widehat{x}$ and $\widehat{p}$ cannot be measured simultaneously. Therefore, it is of great interests for both physicists and mathematicians to find a large space of mutually commuting operators (including the Hamiltonian) for a quantum system and find their common eigenvectors.

In summary,

| | Classical system | Quantum system |
|---|---|---|
| Phase space | Symplectic manifold $M$ | Hilbert space $\mathbb{H}$ |
| Observables | Smooth function on $M$ | Self-adjoint operators on $\mathbb{H}$ |
| Bracket | $\{f, g\}$ | $[\widehat{f}, \widehat{g}]$ |
| Time-evolution | $\frac{df}{dt} = \{f, H\}$ | $i\hbar\frac{d\psi(t)}{dt} = H\psi(t)$ |

At last, Let's discuss one example, the harmonic oscillator. A classical model of harmonic oscillator is described by a ball attached to a spring oscillating from left to right. The Hamiltonian (total energy) of this system is

$$H = \frac{1}{2m}p^2 + \frac{1}{2}kx^2$$

An analogues of this model is a single particle system with Hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{1}{2}m\omega^2x^2$$

If we consider the quantization of this system, then the Hamiltonian becomes a operator:

$$\widehat{H} = \frac{1}{2m}\widehat{p}^2 + \frac{1}{2}m\omega^2\widehat{x}^2$$

For simplicity, we assume $m = \omega = \hbar = 1$, then

$$\widehat{H} = \frac{1}{2}\left(\widehat{x}^2 + \widehat{p}^2\right) \qquad [\widehat{x}, \widehat{p}] = i$$

We are interested in eigenvalues and eigenvectors of $\widehat{H}$.

$$\widehat{H} = \frac{1}{2}\left(\widehat{x}^2 + \widehat{p}^2\right) = \widehat{H} = \frac{1}{2}\left((\widehat{x} - i\widehat{p})\left(\widehat{x} + i\widehat{p}\right) - i[\widehat{x}, \widehat{p}]\right)$$

Define operators

$$a = \frac{1}{\sqrt{2}}(\widehat{x} + i\widehat{p}) \qquad a^\dagger = \frac{1}{\sqrt{2}}(\widehat{x} - i\widehat{p}) \qquad N = a^\dagger a$$

Then $\widehat{H} = a^\dagger a + \frac{1}{2} = N + \frac{1}{2}$. Hence, it is equivalent to consider eigenvalues and eigenvectors of operator $N$. Since $N$ is self-adjoint, all eigenvalues of $N$ is non negative. We denote a unit-norm eigenvector of $N$ with eigenvalue $\lambda \geq 0$ by $|\lambda\rangle$.

**Lemma 3.1.** *1. $a$ and $a^\dagger$ are adjoint to each other.*

*2. $[a, a^\dagger] = 1$, $[N, a] = -a$, $[N, a^\dagger] = a^\dagger$.*

*3. The Lie algebra generated by $a, a^\dagger, Id$ is isomorphic to Heisenberg Lie algebra $\mathfrak{h}_3$*

Consider the action of $a$ and $a^\dagger$ on an eigenvector $|\lambda\rangle$:

$$Na^\dagger|\lambda\rangle = a^\dagger N|\lambda\rangle + [N, a^\dagger]|\lambda\rangle = (\lambda + 1)a^\dagger|\lambda\rangle$$
$$Na|\lambda\rangle = aN|\lambda\rangle + [N, a]|\lambda\rangle = (\lambda - 1)a|\lambda\rangle$$

Therefore, starting with a state $|\lambda\rangle$, the operator $a^\dagger$ lifts this state by 1 while the operator $a$ lower this state by 1.

$$\langle\lambda|N|\lambda\rangle = \langle\lambda|a^\dagger a|\lambda\rangle = \lambda$$

Thus, $\|a|\lambda\rangle\| = \sqrt{\lambda}$ and

$$|\lambda - 1\rangle = \frac{1}{\sqrt{\lambda}}a|\lambda\rangle \qquad |\lambda\rangle = \frac{1}{\sqrt{\lambda}}a^\dagger|\lambda - 1\rangle$$

Now Let $|0\rangle$ be the state with eigenvalue 0. Then $a|0\rangle = 0$, $|0\rangle$ corresponds to a function $\psi(x) \in L^2(\mathbb{R})$ satisfying

$$(x + \frac{d}{dx})\psi(x) = 0 \to \psi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$$

For any $n \in \mathbb{Z}_{n\geq 0}$,

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle$$

Let $V$ be the vector space generated by $\{|n\rangle | n \in \mathbb{Z}_{n\geq 0}\}$. $V$ is a dense subspace of $L^2(\mathbb{R})$ invariant under the action of $a$ and $a^\dagger$. Therefore, $V$ is a representation of Heisenberg Lie algebra $\mathfrak{h}_3$. There is a simpler way to realize this representation. Note that the Lie algebra generated by operators $x$, $\frac{d}{dx}$, and 1 on $\mathbb{C}[x]$ is also isomorphic to $\mathfrak{h}_3$. Hence, we have an isomorphism of $\mathfrak{h}_3$-representations:

$$\mathbb{C}[x] \longrightarrow V \qquad \frac{x^n}{n!} \longmapsto |n\rangle$$

# 4 Symmetry of classical systems and Quantum systems

In mathematics, we say a object has certain symmetry if a group can act on it, i.e. this object admits a group action. In most case, we deal with symmetry that occurs as group representations, i.e. a group acts on a vector space. There is a wealth of symmetry in physics. For example, consider a classical single particle system in 3-dimensional space. We describe the state of this particle by $\mathbb{R}^6 = \{(\mathbf{x}, \mathbf{p}) | \mathbf{x} \in \mathbb{R}^3, \mathbf{p} \in \mathbb{R}^3\}$, where $\mathbf{x} = (x_1, x_2, x_3)$ is a coordinate system we choose for position. We can also choose a different coordinate system for position, for instance, rotating and translating the origin coordinate system. In this case, the phase space also undergoes a transformation $\mathbb{R}^6 \longrightarrow \mathbb{R}^6$. This gives us a representation of rotation group (translation group, or the group generated by these two groups) on the phase space $\mathbb{R}^6$.

In general, $G$-symmetry of a classical system $M$ is a group homomorphism

$$G \longrightarrow \mathrm{Aut(M)}$$

However, not all such homomorphisms are meaningful in physics. Frequently, the group action is a consequence of coordinate changes. If we are interested in the value of a classical observable $f$, then we prefer those coordinate change $g : M \longrightarrow M$ such that $f(gx) = f(x)$ for all $x \in M$. Because in this case we can measure the value of $f$ at $x$ on any charts given by these $G$-action without changing $f$. Generally, we prefer those group actions that preserve the symplectic form.

Next, we introduce symmetry on a quantum system $\mathbb{H}$. Apparently, it is a group action on the space of state $\mathbb{PH}$. A physically meaningful $G$-action should preserve the transition probability, i.e. for any $\psi_1, \psi_2 \in \mathbb{PH}$ and $g \in G$

$$\delta(g\psi_1, g\psi_2) = \delta(\psi_1, \psi_2) \tag{4.1}$$

we call a bijection $g : \mathbb{PH} \longrightarrow \mathbb{PH}$ unitary if the equation 4.1 holds. The space of all unitary transforms on $\mathbb{PH}$ is denoted by $U(\mathbb{PH})$.

**Definition** A symmetry on a quantum system $\mathbb{H}$ is a continuous projective unitary representation of $\mathbb{H}$, i.e. a homomorphism

$$G \longrightarrow U(\mathbb{PH})$$

For more details about the continuous condition, see [8], Definition 3.5.

**Lemma 4.1.** *If $\rho : G \longrightarrow U(\mathbb{PH})$ is a continuous unitary representation, then induced homomorphism*

$$\rho' : G \longrightarrow U(\mathbb{PH})$$

*is a projective unitary representation.*

Therefore, the study of symmetry on quantum system is inextricably linked to the unitary representation of Hilbert space.

In most case, symmetry on quantum system is obtained by quantizing symmetry of a classical system. Let's still take 3-dimensional one particle system as an example, the quantum system associated to it is given by $L^2(\mathbb{R}^3)$. $G$ is a group of coordinate change on $\mathbb{R}^3$ (like rotation group $SO(3)$, or translation group $\mathbb{R}^3$), then $G$ also acts on the function space over $\mathbb{R}^3$ by

$$G \times L^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^3) \qquad g \cdot f \longrightarrow f \cdot g^{-1}$$

It's easy to check that this is a unitary representation.

Now suppose $G$ is a Lie group, the unitary action of $G$ on a Hilbert space $\mathbb{H}$ usually induces an action of its Lie algebra $\mathfrak{g}$ on $\mathbb{H}$ (or a dense subspace). The unitary condition implies that for any $g \in \mathfrak{g}$,

$$\langle gv, w\rangle + \langle v, gw\rangle = 0 \qquad \text{for all } v, w \in \mathbb{H}$$

Hence, $g \in \mathfrak{g}$ acts on $\mathbb{H}$ as an anti-symmetry operator. Multiply $i$ gives

$$\langle igv, w\rangle = \langle v, igw\rangle \qquad \text{for all } v, w \in \mathbb{H}$$

$ig$ is a self-adjoint operator, or in other words, a quantum observable! Conversely, if $P$ is a quantum observable, and the operator

$$e^{-iPt} = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!} P^n$$

is well-defined on a dense subset of $\mathbb{H}$, then it induces a unitary representation

$$\mathbb{R} \longrightarrow U(\mathbb{H}) \qquad t \longmapsto e^{-itP}$$

We call such a representation as one-parameter unitary group on $\mathbb{H}$.

**Theorem 4.2 (Stone′s theorem).** *There is a one-to-one correspondence between the strongly continuous one-parameter unitary groups on $\mathbb{H}$ and the self-adjoint operators on $\mathbb{H}$.*

Therefore, the relation between unitary representations and quantum observables is given by:

$$\begin{tikzcd}
\text{Lie groups' unitary representations} \arrow[dr, Rightarrow, "\frac{d}{dt}"] & & \text{Quantum observables} \arrow[ll, Rightarrow, "\exp(-i\cdot *)"'] \\
& \text{Lie algebras' unitary representations} \arrow[ur, Rightarrow, "i\cdot"] &
\end{tikzcd}$$

The Stone's theorem tells us that every quantum observable correspondences to one kind of symmetry. In following examples, we will see to which symmetries do position, momentum, and energy correspond.

**Example 4.1.** *Let $\mathbb{H}$ be a quantum system with Hamiltonian $H$. The time evolution gives arise to a unitary representation*

$$\mathbb{R} \longrightarrow U(\mathbb{H}) \qquad t \longmapsto T_t$$

*where*

$$T_t(\psi) = \psi(t) \qquad \textit{for any } \psi \in \mathbb{H}$$

*The derivative is a operator $T$,*

$$T(\psi) = \frac{d}{dt}\psi(t) = -\frac{i}{\hbar} H\psi$$

*so the associated quantum observable is $iT = \frac{1}{\hbar}H$, the Hamiltonian.*

**Example 4.2.** *Let $\mathbb{H} = L^2(\mathbb{R}^n)$. $\mathbb{R}^n$ acts on $\mathbb{H}$ by translation and rotation:*

1. $\alpha : \mathbb{R}^n \longrightarrow U(\mathbb{H})$. *For $k \in \mathbb{R}^n$ and $f(x) \in \mathbb{H}$,*
$$\left(\alpha(k)f\right)(x) = f(x - \hbar k)$$

2. $\beta : \mathbb{R}^n \longrightarrow U(\mathbb{H})$. *For $k \in \mathbb{R}^n$ and $f(x) \in \mathbb{H}$,*
$$\left(\beta(k)f\right)(x) = f(x)e^{-ikx}$$

*By taking derivatives and multiplying $i$, we can find that observables associated to $\alpha$ are exactly momentum operators:*

$$\widehat{p}_i = -i\hbar\frac{\partial}{\partial x_i} \qquad i = 1, ..., n$$

*and observables associated to $\beta$ are exactly position operators:*

$$\widehat{x}_i = x_i \qquad i = 1, ..., n$$

*Moreover, if we define the Fourier transform on $L^2(\mathbb{R}^n)$ by*

$$\left(\mathcal{F}(f)\right)(x) = \int_{\mathbb{R}^n} f(y)e^{ikxy}dy$$

*then the Fourier transform is a morphism between this two representation, i.e.*

$$\begin{tikzcd} L^2(\mathbb{R}^n) \arrow[r, "\mathcal{F}"] \arrow[d, "\beta(k)"'] & L^2(\mathbb{R}^n) \arrow[d, "\alpha(k)"] \\ L^2(\mathbb{R}^n) \arrow[r, "\mathcal{F}"] & L^2(\mathbb{R}^n) \end{tikzcd}$$

**Example 4.3.** *Let $\mathbb{H} = L^2(\mathbb{R}^n)$ and $\widehat{x}_k$, $\widehat{p}_k$, $Id$ be position, momentum, and identity operators, $k = 1, ..., n$. Define $X_k = -i\widehat{x}_k$, $P_k = -i\widehat{p}_k$, and $\mathbb{1} = -i \cdot Id$, $k = 1, ..., n$, then (assume $\hbar = 1$)*

$$[X_i, \mathbb{1}] = 0 \qquad [P_j, \mathbb{1}] = 0 \qquad [X_i, P_j] = \delta_{i,j} \cdot \mathbb{1} \qquad i, j = 1, ..., n$$

*hence, the Lie algebra $span_{\mathbb{C}}\langle X_i, Y_j, \mathbb{1}\rangle$ is isomorphic to Heisenberg Lie algebra $\mathfrak{h}_n$. We can exponent it to a unitary representation of Heisenberg group.*

# 5 XXX model

In this section, we introduce the XXX model in physics, which serves as a motivation for the Gaudin model.

The simplest quantum system in quantum mechanics is the model of spin $\frac{1}{2}$ particles. The state space of one spin $\frac{1}{2}$ particle is just $\mathbb{C}^2$ with standard inner product. Hence, any self-adjoint operator on this quantum system must be a linear combination of

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} \qquad \sigma_x = \begin{pmatrix}0 & 1\\1 & 0\end{pmatrix} \qquad \sigma_y = \begin{pmatrix}0 & -i\\i & 0\end{pmatrix} \qquad \sigma_z = \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$$

$\sigma_x, \sigma_y, \sigma_z$ are called Pauli matrices. $U(2)$ represents the set of all unitary operators on $\mathbb{C}^2$. Since scalar multiplications are identity operator on the space of states, the group of symmetry of this quantum system should be $SU(2)$. It's easy to check that $i\sigma_x, i\sigma_y, i\sigma_z$ form a basis of Lie algebra $\mathfrak{su}(2)$, which agree with the correspondence between symmetry and quantum observables we discuss in last section.

Now we consider the quantum system of $N$ spin $\frac{1}{2}$ particles evenly placed on a circle. The associated Hilbert space $\mathbb{H}$ is $(\mathbb{C}^2)^{\otimes N}$. This is because if the system is in a state where $k^{\text{th}}$ particle is at state $v_k \in \mathbb{C}^2$, then the whole system is in a state denoted by $(v_1, ..., v_N)$. The space of state should be a vector space generated by linearly independent elements $(v_1, ..., v_N)$, $v_i \in \mathbb{C}^2$, which is exactly $(\mathbb{C}^2)^{\otimes N}$.

If $A$ is an operator on $\mathbb{C}^2$, then define an operator $A^{(k)}$ on $\mathbb{H}$ by

$$A^{(k)} = 1 \otimes \cdots \otimes \overset{k^{\text{th}}}{A} \otimes \cdots 1$$

Let's assume that only adjacent particles interfere with each other. Then physicists formulate a model for this quantum system with Hamiltonian in form of:

$$H = \sum_{i=1}^{N} J_x \sigma_x^{(k)} \sigma_x^{(k+1)} + J_y \sigma_y^{(k)} \sigma_y^{(k+1)} + J_z \sigma_z^{(k)} \sigma_z^{(k+1)}$$

where $J_x, J_y, J_z$ are constants. if $J_x \neq J_y \neq J_z$, the model is called the Heisenberg XYZ model; in the case of $J_x = J_y \neq J_z$, the model is called the Heisenberg XXZ model; if $J_x = J_y = J_z$, the model is called the Heisenberg XXX model.

# 6 Gaudin model

We are finally ready for our discussion about Gaudin model. Actually, it was discussed on Frenkel's second lecture. Reader should keep in mind that the Gaudin model is the generalization of the XXX model.

Let $\mathfrak{g}$ be a $d$-dimensional simple Lie algebra over $\mathbb{C}$ and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$.

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \qquad \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$$

For $\lambda \in \mathfrak{h}^*$, denote the Verma module with highest weight $\lambda$ by $V_\lambda$, i.e.

$$V_\lambda = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C} \cdot \mathbb{1}_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}\mathbb{1}_\lambda = U(\mathfrak{n}_-) \cdot \mathbb{1}_\lambda$$

where $\mathfrak{h}$ acts on $\mathbb{1}_\lambda$ by $\lambda$ and $\mathfrak{n}_+$ annihilates $\mathbb{1}_\lambda$.

Let $n$ be a positive integer. We denote $\lambda_1, ..., \lambda_n \in \mathfrak{h}^*$. Consider the space

$$V = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$$

We should view $V_{\lambda_i}$ as a quantum system of one particle and hence $V$ is the quantum system of these $n$ particles. For $A \in \mathfrak{g}$, denote $A^{(i)} = 1 \otimes \cdots \otimes \overset{i^{\text{th}}}{A} \otimes \cdots \otimes 1$ as an operator on $V$.

**Definition**[**Gaudin Hamiltonians**] Let $z_1, ..., z_n$ be a collection of distinct complex numbers. Let $J_1, ..., J_d$ be a basis of $\mathfrak{g}$ and $J^1, ..., J^d$ be the dual basis with respect to a non-degenerate invariant bilinear form on $\mathfrak{g}$. The Gaudin Hamiltonians are following elements of the algebra $U(\mathfrak{g})^{\otimes n}$:

$$H_i = \sum_{j\neq i}\sum_{a=1}^{d}\frac{J_a^{(i)}\cdot J^{a(j)}}{z_i - z_j} \qquad i = 1,\cdots n$$

We can view Gaudin Hamiltonians as operators on $V$. They are indeed "Hamiltonians" as we will show later that they commute with each other.

**Lemma 6.1.** *Let $W$ be a $d$-dimensional vector space with a non-degenerate bilinear form $\langle\cdot,\cdot\rangle$. Let $e_1, ..., e_d$ be a basis of $W$ and $e^1, ..., e^d$ be its dual basis. Then the element*

$$\sum_{i=1}^{d} e_i \otimes e^i \in W^{\otimes 2}$$

*is independent of the choice of the basis, i.e. Let $w_1, ..., w_d$ be another basis with dual basis $w^1, ..., w^d$, then*

$$\sum_{i=1}^{d} e_i \otimes e^i = \sum_{i=1}^{d} w_i \otimes w^i$$

*Proof.* There are natural isomorphisms

$$V\otimes V \xrightarrow{\sim} V^*\otimes V \xrightarrow{\sim} \mathrm{End}(V)$$

$$v\otimes w \longmapsto \langle v, \_\rangle \otimes w \longmapsto \langle v, \_\rangle \cdot w$$

Thus, the image of $\sum_{i=1}^{d} e_i \otimes e^i$ is given by

$$\sum_{i=1}^{d}\langle e_i, \_\rangle e^i$$

This is exactly the identity map in $\mathrm{End}(V)$. Therefore, $\sum_{i=1}^{d} e_i \otimes e^i$ is independent of the choice of the basis. □

By lemma 6.1, we can conclude that Gaudin Hamiltonians we constructed is independent of the choice of the basis. Therefore, in some cases, we can directly assume that $J_1, ..., J_d$ is an orthonormal basis, i.e. $J^a = J_a$ for $a = 1, ..., d$.

**Proposition 6.2.** *Let $H_1, ..., H_n$ be Gaudin Hamiltonians we constructed above. Then*

1. *$[H_i, A_{diag}] = 0$ for any $A \in \mathfrak{g}$, where $A_{diag} = \sum_{i=1}^{n} A^{(i)}$ is the diagonal action.*
2. *$[H_i, H_j] = 0$ for any $i \neq j$.*

*Proof.* 1.

$$\begin{aligned}[H_i, A_{\mathrm{diag}}] &= \sum_{a=1}^{d}\sum_{k=1}^{n}\sum_{j\neq i}\frac{1}{z_i - z_j}[J_a^{(i)}J^{a(j)}, A_{\mathrm{diag}}^{(k)}] \\ &= \sum_{a=1}^{d}\sum_{j\neq i}\frac{1}{z_i - z_j}[J_a, A]^{(i)}J^{a(j)} + J_a^{(i)}[J^a, A]^{(j)}\end{aligned}$$

Therefore, it suffices to prove that:

$$[J_a, A]^{(i)} J^{a(j)} + J_a^{(i)} [J^a, A]^{(j)} = 0 \qquad \text{for any } A \in \mathfrak{g} \tag{6.1}$$

and it suffices to prove this equation for $A = J^b$, $b = 1, ..., d$. Assume

$$[J^a, J^b] = C_c^{ab} J^c$$

Then $C_c^{ab} = -C_c^{ba}$ for any $a, b, c \in \{1, ..., d\}$. Let's express $[J^b, J_a]$ by basis $J_1, ..., J_d$:

$$\begin{aligned}
[J^b, J_a] &= \sum_{c=1}^{d} \kappa([J^b, J_a], J^c) J_c \\
&= \sum_{c=1}^{d} -\kappa(J_a, [J^b, J^c]) J_c \\
&= \sum_{c=1}^{d} \sum_{l=1}^{d} -\kappa(J_a, C_l^{bc} J^l) J_c \\
&= \sum_{c=1}^{d} -C_a^{bc} J_c
\end{aligned}$$

Then

$$\begin{aligned}
[J_a, A]^{(i)} J^{a(j)} + J_a^{(i)} [J^a, A]^{(j)} &= \sum_{c=1}^{d} C_a^{bc} J_c^{(i)} J^{a(j)} + \sum_{c=1}^{d} C_c^{ab} J_a^{(i)} J^{c(j)} \\
&= \sum_{c=1}^{d} (C_c^{ba} + C_c^{ab}) J_a^{(i)} J^{c(j)} = 0
\end{aligned}$$

Therefore, $[H_i, A_{\text{diag}}] = 0$ for any $A \in \mathfrak{g}$.

2. For $i \neq j$, denote

$$(ij) = \sum_{a=1}^{d} \frac{J_a^i J^{a(j)}}{z_i - z_j}$$

then

$$[H_i, H_j] = \left[ \sum_{p \neq i} (ip), \sum_{q \neq j} (jq) \right] = \sum_{p \neq i} \sum_{q \neq j} [(ip), (jq)]$$

If $i, j, p, q$ are four distinct number, then $[(ip), (jq)] = 0$. Hence, we only need to sum over $p = q \neq i, j$, $q = i, p \neq j$, $p = jq \neq i$, $q = i, p = j$

$$[H_i, H_j] = [(ij), (ji)] + \sum_{l \neq i,j} [(il), (jl)] + [(il), (ji)] + [(ij), (jl)]$$

Firstly,

$$[(ij), (ji)] = -\frac{1}{(z_i - z_j)^2} \left[ \sum_a J_a^{(i)} J^{a(j)}, \sum_b J_b^{(j)} J^{b(i)} \right]$$

As we have prove, the definition of $\sum_a J_a^{(i)} J^{a(j)}$ is independent of the choice of basis. Thus, by changing basis from $J_1, ..., J_d$ to $J^1, ..., J^d$, we have

$$\sum_a J_a^{(i)} J^{a(j)} = \sum_b J_b^{(j)} J^{b(i)}$$

So $[(ij), (ji)] = 0$.

Secondly,

$$\begin{aligned}
[(il), (ji)] &= \sum_a \sum_b \frac{1}{(z_i - z_l)(z_j - z_i)} [J_a^{(i)} J^{a(l)}, J_b^{(j)} J^{b(i)}] \\
&= \frac{1}{(z_i - z_l)(z_j - z_i)} \sum_a \sum_b [J_a, J^b]^{(i)} J_b^{(j)} J^{a(l)} \\
&= \frac{-1}{(z_i - z_l)(z_j - z_i)} \sum_a \sum_b J_a^{(i)} J_b^{(j)} [J^a, J^b]^{(l)}
\end{aligned}$$

The last equation is obtained by using equation 6.1. Similarly,

$$[(ij), (jl)] = \frac{-1}{(z_j - z_l)(z_i - z_j)} \sum_a \sum_b J_a^{(i)} J_b^{(j)} [J^a, J^b]^{(l)}$$

Thus,

$$\begin{aligned}
[(il), (ji)] + [(ij), (jl)] &= \frac{-1}{z_i - z_j} \left( \frac{1}{z_j - z_l} - \frac{1}{z_i - z_l} \right) \sum_a \sum_b J_a^{(i)} J_b^{(j)} [J^a, J^b]^{(l)} \\
&== \frac{-1}{(z_i - z_l)(z_j - z_l)} \sum_a \sum_b J_a^{(i)} J_b^{(j)} [J^a, J^b]^{(l)}
\end{aligned}$$

At last,

$$[(il), (jl)] = \frac{1}{(z_i - z_l)(z_j - z_l)} \sum_a \sum_b J_a^{(i)} J_b^{(j)} [J^a, J^b]^{(l)}$$

Therefore,

$$[(il), (ji)] + [(ij), (ji)] + [(il), (jl)] = 0$$

We conclude that $[H_i, H_j] = 0$.

□

**Remark 6.1.** *It seems that the commutativity of Gaudin Hamiltonians relies on the commutation relation of the finite dimensional Lie algebra $\mathfrak{g}$. However, we will discover in next section that the essence of this commutativity lies in commutation relation of a infinite dimensional Lie algebra associated to $\mathfrak{g}$.*

As we mention before, Gaudin Hamiltonians should be viewed as operators acting on a quantum system $V = V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}$. Therefore, our goal is to diagonalize Gaudin Hamiltonians simultaneously, i.e. find their common eigenvectors (eigenstates).

Denote

$$|0\rangle = v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n}$$

then $|0\rangle$ is the highest weight of this quantum system.

**Lemma 6.3.** $|0\rangle$ *is a common eigenvectors of Gaudin Hamiltonians.*

*Proof.* Let
$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$
be a Cartan decomposition of $\mathfrak{g}$. We fix a basis $J_1, ..., J_d$ of $\mathfrak{g}$ such that they also form a basis of $\mathfrak{n}_\pm$ and $\mathfrak{h}$. Then its dual basis $J^1, ..., J^d$ satisfies

$$J^a \in \begin{cases} \mathfrak{n}_- & \text{if } J_a \in \mathfrak{n}_+ \\ \mathfrak{n}_+ & \text{if } J_a \in \mathfrak{n}_- \\ \mathfrak{h} & \text{if } J_a \in \mathfrak{h} \end{cases}$$

Thus, if $J_a \in \mathfrak{n}_\pm$, then $J_a^{(i)} \cdot J^{a(j)}$ annihilates $|0\rangle$. If $J_a \in \mathfrak{h}$, then $J^a \in \mathfrak{h}$ and $J_a^{(i)} \cdot J^{a(j)}$ acts on $|0\rangle$ by multiplying a scalar. Therefore, $|0\rangle$ is an eigenvector for all Gaudin Hamiltonians. $\square$

Now we introduce a new operator which is able to combine information about Gaudin Hamiltonian all together.

**Definition** For $A \in \mathfrak{g}$, define

$$A(z) = \sum_{i=1}^{n} \frac{A^{(i)}}{z - z_i}$$

$A(z)$ can be viewed as an element in $U(\mathfrak{g})^{\otimes n} \otimes R$, where $R$ can be each of the following:

1. The ring of meromorphic function on $\mathbb{C}$ with poles only at $z_1, ..., z_n$;
2. The ring of meromorphic function on $\mathbb{CP}$ with poles only at $z_1, ..., z_n$;
3. $\mathbb{C}\left[(z - z_i)^{-1}\right]_{i=1,...,n}$, a subring of $\mathbb{C}(z)$.

Define

$$S(z) = \frac{1}{2}\sum_{a=1}^{d} J_a(z) J^a(z)$$

**Proposition 6.4.**
$$S(z) = \sum_{i=1}^{n} \frac{Cas^{(i)}}{(z - z_i)^2} + \sum_{i=1}^{n} \frac{H_i}{z - z_i}$$

*Proof.*

$$\begin{aligned} S(z) &= \frac{1}{2}\sum_{a=1}^{d} J_a(z) J^a(z) \\ &= \frac{1}{2}\sum_{a=1}^{d} \left(\sum_{i=1}^{n} \frac{J_a^{(i)}}{z - z_i}\right)\left(\sum_{j=1}^{n} \frac{J^{a(j)}}{z - z_j}\right) \\ &= \sum_{a=1}^{n}\sum_{i=1}^{n} \frac{1}{2}\frac{J_a^{(i)} J^{a(i)}}{(z - z_i)^2} + \frac{1}{2}\sum_{1 \le i \ne j \le n} \frac{J_a^{(i)} J^{a(j)}}{(z - z_i)(z - z_j)} \end{aligned}$$

where

$$\sum_{1\le i\neq j\le n} \frac{J_a^{(i)}J^{a(j)}}{(z-z_i)(z-z_j)} = \sum_{1\le i\neq j\le n} \frac{J_a^{(i)}J^{a(j)}}{(z_i-z_j)}\left(\frac{1}{z-z_i}-\frac{1}{z-z_j}\right)$$
$$= \sum_{i=1}^{n}\frac{1}{z-z_i}\sum_{j\neq i}\frac{J_a^{(i)}J^{a(j)}}{(z_i-z_j)} + \sum_{j=1}^{n}\frac{1}{z-z_j}\sum_{i\neq j}\frac{J^{a(j)}J_a^{(i)}}{(z_j-z_i)}$$
$$= 2\sum_{i=1}^{n}\frac{H_i}{z-z_i}$$

Thus,

$$S(z) = \sum_{i=1}^{n}\frac{\mathrm{Cas}^{(i)}}{(z-z_i)^2} + \sum_{i=1}^{n}\frac{H_i}{z-z_i}$$

□

Apparently, if $v \in V$ is an eigenvector of all Gaudin Hamiltonian, then for any $z_0 \in \mathbb{C}\setminus\{z_1,...,z_n\}$, $v$ is an eigenvector of $S(z_0)$. The converse also holds

**Proposition 6.5.** *Let $v$ be an element of $V$. Then $v$ is an eigenvector of all Gaudin Hamiltonians if and only if for any $z_0 \in \mathbb{C}\setminus\{z_1,...,z_n\}$, $v$ is an eigenvector of $S(z_0)$.*

*Proof.* We only need to prove the if part. Suppose that for any $z_0 \in \mathbb{C}\setminus\{z_1,...,z_n\}$, $v$ is an eigenvector of $S(z_0)$. Let $e_0 = v, e_1, ..., e_m$ be linearly independent elements such that

$$H_i v = \sum_{j=0}^{m} a_{ij}e_j \qquad i = 1,...,n$$

Since Casimir elements act on $V$ by scalar multiplication, the assumption implies that there exists a complex valued function $\mu(z)$ defined on $\mathbb{C}\setminus\{z_1,...,z_n\}$ such that

$$\left(\sum_{i=1}^{n}\frac{H_i}{z-z_i}\right)\cdot v = \sum_{j=0}^{m}\left(\sum_{i=1}^{n}\frac{a_{ij}}{z-z_i}\right)e_j = \mu(z)\cdot v$$

Thus,

$$\mu(z) = \sum_{i=1}^{n}\frac{a_{i0}}{z-z_i} \text{ and } \sum_{i=1}^{n}\frac{a_{ij}}{z-z_i} = 0 \text{ for } j = 1,...,m$$

Therefore, $a_ij = 0$ for any $1 \le i \le n, 1 \le j \le m$ and $H_i v = a_{i0}v$, i.e. $v$ is an eigenvector for all Gaudin Hamiltonians.

□

Bethe ansatz is a method to obtain more common eigenvectors of Gaudin Hamiltonians. The idea of Bethe ansatz is to produce new eigenvectors by applying certain operators (field operators in quantum field theory) to $|0\rangle$. We take $\mathfrak{g} = \mathfrak{sl}_2$ as an example. Let $E, F, H$ be the standard basis for $\mathfrak{sl}_2$. Denote

$$F(w) = \sum_{i=1}^{n}\frac{F^{(i)}}{z-z_i}$$

Choose m complex numbers $w_1, ...w_m \in \mathbb{C}\setminus\{z_1,...,z_n\}$. The Bethe anzatz vector is defined by

$$|w_1\cdots w_m\rangle = F(w_1)\cdots F(w_m)|0\rangle \in V$$

**Proposition 6.6** ([1]). *The action of $S(z)$ on a Bethe anzata vector is given by*

$$S(z)|w_1\cdots w_m\rangle = s_m(z)|w_1\cdots w_m\rangle + \sum_{j=1}^{m}\frac{f_j}{z-w_j}|w_1\cdots, w_{j-1}, z, w_{j+1},\cdots, w_m\rangle$$

*where $s_m(z)$ is a function on $z$, and*

$$f_j = \sum_{i=1}^{n}\frac{\lambda_i}{w_j-z_i} - \sum_{k\neq j}\frac{2}{w_j-z_k} \qquad j=1,...,m$$

As a consequence,

**Corollary 6.7.** *The Bethe anzata vector $|w_1\cdots w_m\rangle$ is n eigenvector of all Gaudin Hamiltonians if and only if*

$$\sum_{i=1}^{n}\frac{\lambda_i}{w_j-z_i} - \sum_{k\neq j}\frac{2}{w_j-z_k} = 0 \qquad j=1,...,m$$

For $\mathfrak{g} = \mathfrak{sl}_2$, it is possible to work out the expression for $S(z)|w_1\cdots w_m\rangle$ by direct computation, even though the calculation is tedious. Nevertheless, this method of direct calculation is helpless for higher dimensional Lie algebras, where situation becomes more complex. In [1], authors introduce a method to construct Bethe vectors by conformal blocks, which is also discussed in this course.

# 7 Reformulation of the commutativity of Gaudin Hamiltonians

In this section, we will give a new perspective towards the commutativity of Gaudin Hamiltonians. From now on, suppose we haven't proved that Gaudin Hamiltonians mutually commute. In last section, we obtain an equation

$$S(z) = \sum_{i=1}^{n}\frac{\mathrm{Cas}^{(i)}}{(z-z_i)^2} + \sum_{i=1}^{n}\frac{H_i}{z-z_i}$$

**Lemma 7.1.** *The following are equivalent:*

*(1) Gaudin Hamiltonians mutually commute,*

*(2) $[S(z), S(z')] = 0$ for any $z, z' \in \mathbb{C}\setminus\{z_1, ..., z_n\}$,*

*(3) $[S(z), \partial_z^n S(z)] = 0$ for any $z \in \mathbb{C}\setminus\{z_1, ..., z_n\}$ and $n \geq 0$*

*(4) $[S(z), \partial_z^n S(z)] = 0$ for all $n \geq 0$, where we consider $S(z), \partial_z^n S(z)$ as elements in algebra $U(\mathfrak{g})^{\otimes n}\otimes_{\mathbb{C}}\mathbb{C}(z)$.*

*Proof.* It's obvious that (1) $\Rightarrow$ (2). To show (2) $\Rightarrow$ (1), we can use similar method in the proof of proposition 6.5. When $|z'-z|$ is small enough, we have Taylor expansion:

$$S(z') = \sum_{n=0}^{\infty}\frac{1}{n!}(z'-z)^n\partial_z^n S(z)$$

Therefore, (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4). □

**Remark 7.1.** *The last condition in lemma 7.1 entirely depends on the algebraic structure.*

Let $\widehat{\mathfrak{g}}_-$ be the Lie algebra $\mathfrak{g} \otimes z^{-1}\mathbb{C}[z^{-1}]$ with Lie bracket defined by

$$[A \otimes f(z), B \otimes g(z)] = [A, B] \otimes f(z)g(z)$$

for any $A, B \in \mathfrak{g}$ and $f(z), g(z) \in z^{-1}\mathbb{C}[z^{-1}]$. For $a \in \mathbb{C}$, we can translate the indeterminate $z$ by $a$ to get a Lie algebra homomorphism:

$$\widehat{\mathfrak{g}}_- \longrightarrow \mathfrak{g} \otimes (z-a)^{-1}\mathbb{C}[(z-a)^{-1}] \qquad A \otimes f(z) \longmapsto A \otimes f(z-a)$$

Denote $\mathfrak{g} \otimes (z-a)^{-1}\mathbb{C}[(z-a)^{-1}]$ by $\widehat{\mathfrak{g}}_-^a$. Now Let $a = z_1, ..., z_n$ and take the direct sum of $\widehat{\mathfrak{g}}_-^a$, we obtain a Lie algebra homomorphism

$$\widehat{\mathfrak{g}}_- \longrightarrow \widehat{\mathfrak{g}}_-^{z_1} \oplus \cdots \oplus \widehat{\mathfrak{g}}_-^{z_n} \qquad A \otimes f(z) \longmapsto (A \otimes f(z-z_i), ..., A \otimes f(z-z_n))$$

This Lie algebra homomorphism can be lifted to an algebra homomorphism between their universal enveloping algebras.

**Lemma 7.2.** *Let $\mathfrak{g}_1, ..., \mathfrak{g}_l$ be Lie algebras over field $k$. The Lie algebra homomorphism*

$$\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_l \longrightarrow U(\mathfrak{g}_1) \otimes_k \cdots \otimes_k U(\mathfrak{g}_l) \qquad (g_1, ..., g_l) \longmapsto \sum_{i=1}^{l} 1 \otimes \cdots \otimes \overset{i^{th}}{g_i} \otimes \cdots \otimes 1$$

*induces an algebra isomorphism*

$$U(\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_l) \xrightarrow{\sim} U(\mathfrak{g}_1) \otimes_k \cdots \otimes_k U(\mathfrak{g}_l)$$

Therefore, we obtain an algebra homomorphism

$$\Phi : U(\widehat{\mathfrak{g}}_-) \longrightarrow U(\widehat{\mathfrak{g}}_-^{z_1}) \otimes_k \cdots \otimes_k U(\widehat{\mathfrak{g}}_-^{z_n}) \qquad A \otimes f(z) \longmapsto \sum_{i=1}^{n} 1 \otimes \cdots \otimes A \otimes \overset{i^{\text{th}}}{f}(z-z_i) \otimes \cdots \otimes 1$$

For $A \in \mathfrak{g}$ and integer $k < 0$, denote $A \otimes z^k$ by $A_k$. The image of $A_k$ under $\Phi$ is given by

$$\Phi(A_k) = \sum_{i=1}^{n} 1 \otimes \cdots \otimes \left(A \otimes \overset{i^{\text{th}}}{(z-z_i)^k}\right) \otimes \cdots \otimes 1$$

In particular,

$$\widehat{A}(z) \coloneqq \Phi(A_{-1}) = \sum_{i=1}^{n} 1 \otimes \cdots \otimes \overset{i^{\text{th}}}{\frac{A}{z-z_i}} \otimes \cdots \otimes 1$$

Recall that in last section we define an element $A(z) \in U(\mathfrak{g})^{\otimes n}[\frac{1}{z-z_i}]_{i=1,...,n}$, which looks similar to $\widehat{A}(z)$. In fact, we have algebra homomorphism

$$\Psi : U(\widehat{\mathfrak{g}}_-^{z_1}) \otimes_k \cdots \otimes_k U(\widehat{\mathfrak{g}}_-^{z_n}) \longrightarrow U(\mathfrak{g})^{\otimes n} \otimes \left[\frac{1}{z-z_i}\right]_{i=1,...,n}$$

$$(A_1 \otimes f_1(z-z_1)) \otimes \cdots \otimes (A_n \otimes f_n(z-z_n)) \longmapsto A_1 \otimes \cdots A_n \otimes (f_1(z-z_1) \cdot f_n(z-z_n))$$

Under this algebra homomorphism, the image of $\widehat{A}(z)$ is exactly $A(z)$. Moreover, if we define

$$\widehat{S}(z) = \frac{1}{2}\sum_{d=1}^{n} \widehat{J}_a(z)\widehat{J}^a(z)$$

Then the image of $\widehat{S}(z)$ is exactly the $S(z)$ we defined in last section.

Let $\mathcal{S} = \frac{1}{2}\sum_{a=1}^{d} J_{a,-1}J^a_{-1}$. Since $\Phi$ is an algebra homomorphism,

$$\begin{aligned}\Phi(\mathcal{S}) &= \frac{1}{2}\sum_{a=1}^{d} \Phi(J_{a,-1})\Phi(J^a_{-1}) \\ &= \frac{1}{2}\sum_{a=1}^{d} J_a(z)J^a(z) = \widehat{S}(z)\end{aligned}$$

At last, we can lift the derivative $\partial_z$ on $U(\mathfrak{g})^{\otimes n}[\frac{1}{z-z_i}]_{i=1,\dots,n}$ to derivatives on $U(\widehat{\mathfrak{g}}_-)$ and $U(\widehat{\mathfrak{g}}_-^{z_1}) \otimes_k \cdots \otimes_k U(\widehat{\mathfrak{g}}_-^{z_n})$:

$$T = \partial_z : U(\widehat{\mathfrak{g}}_-) \longrightarrow U(\widehat{\mathfrak{g}}_-) \qquad A_n \longmapsto nA_{n-1}$$

$$\widehat{\partial}_z : \bigotimes_{i=1}^{n} U(\widehat{\mathfrak{g}}_-^{z_i}) \longrightarrow \bigotimes_{i=1}^{n} U(\widehat{\mathfrak{g}}_-^{z_i}) \qquad \widehat{\partial}_z = \sum_{i=1}^{n} 1 \otimes \cdots \otimes \partial_z \otimes \cdots \otimes 1$$

Then we have a commutative diagram

$$\begin{tikzcd} U(\widehat{\mathfrak{g}}_-) \arrow[r, "\Phi"] \arrow[d, "T"'] & \bigotimes_{i=1}^{n} U(\widehat{\mathfrak{g}}_-^{z_i}) \arrow[r, "\Psi"] \arrow[d, "\widehat{\partial}_z"'] & U(\mathfrak{g})^{\otimes n}\left[\frac{1}{z-z_i}\right]_{i=1,\dots,n} \arrow[d, "\partial_z"'] \\ U(\widehat{\mathfrak{g}}_-) \arrow[r, "\Phi"] & \bigotimes_{i=1}^{n} U(\widehat{\mathfrak{g}}_-^{z_i}) \arrow[r, "\Psi"] & U(\mathfrak{g})^{\otimes n}\left[\frac{1}{z-z_i}\right]_{i=1,\dots,n} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S} \arrow[r, mapsto] \arrow[d, mapsto] & \widehat{S}(z) \arrow[r, mapsto] & S(z) \arrow[d, mapsto] \\ T^n\mathcal{S} \arrow[rr, mapsto] & & \partial_z^n S(z) \end{tikzcd}$$

Recall that our goal is to prove that $S(z)$ and $\partial_z^n S(z)$ commute for all $n \geq 0$. Since $\Phi$ and $\Psi$ are algebra homomorphisms, it suffices to prove the following statement:

$$\mathcal{S} \text{ and } T^n\mathcal{S} \text{ commute for all } n \geq 0$$

To be honest, we haven't actually simplified this problem at all. But definitely, this statement is more fundamental and elegant. Actually, through out this lecture, we can prove following theorems (rough version):

**Theorem 7.3.** *$\{T^n\mathcal{S}\}_{n\geq 0}$ are algebraically independent in $U(\widehat{\mathfrak{g}}_-)$, and they mutually commute.*

**Theorem 7.4.** *There exists elements $S_i \in U(\widehat{\mathfrak{g}}_-)$, ord$(S_i) = d_i + 1$, deg$(S_i) = d_i + 1$, $i = 1, ..., r$, with $S_1 = \mathcal{S}$, such that $\{T^{n_i}S_i\}_{n_i\geq 0, i=1,\dots,r}$ are algebraically independent in $U(\widehat{\mathfrak{g}}_-)$ and mutually commute.*

We don't explain what $d_i$ and $S_i$ here. Instead, we will make this theorem more and more precise through out this note.

# 8 Deformation Quantization

In last section, we discover that the commutativity of Gaudin Hamiltonians derives from the commutativity of the subalgebra $\mathbb{C}[T^n\mathcal{S}]$ in $U(\widehat{\mathfrak{g}}_-)$. More generally, we claim that there exists a larger commutative subalgebra in $U(\widehat{\mathfrak{g}}_-)$. However, this theorem (which is called the center theorem of quantum version ) is not easy to prove. To motivate its prove, we should first study and prove its classical analogue. At the end of this section, we would state what this classical version of center theorem is.

Recall that for a classical system, observables are smooth functions on a symplectic manifold. The set of all classical observables is a commutative algebra (pointwise multiplication and addition) with a Poisson structure, i.e. a Poisson algebra. For a quantum system, observables are self-adjoint operators on a Hilbert space. The set of all quantum observables is an associative algebra. The key step in the process of quantization is to construct a linear map between the Poisson algebra and the associate algebra such that Poisson bracket and Lie bracket commute with quantization map.

$$\widehat{\ }: \text{ Poisson algebra } \longrightarrow \text{ Associative algebra}$$
$$\{f,g\} \longmapsto \frac{1}{i\hbar}[\widehat{f},\widehat{g}]$$

Therefore, we are interested in following questions

1. Given a Poisson algebra, can we quantize it?

2. Given an associative algebra, can we dequantize it?

One way to achieve these goals is so called deformation quantization. Let $A$ be a commutative algebra over field $k$. A formal deformation of $A$ is an associative $k[[\hbar]]$-algebra structure on $A[[\hbar]]$ (here we view $\hbar$ as an indeterminate) such that the nature map $A[[\hbar]]/(\hbar) \longrightarrow A$ is an algebra isomorphism. In other words, a formal deformation of $A$ is a bilinear map

$$\star : A[[\hbar]] \times A[[\hbar]] \longrightarrow A[[\hbar]]$$

such that

1. ($k[[\hbar]]$-algebra) for any $\alpha = \sum_{i\geq 0} a_i\hbar^i$ and $\beta = \sum_{j\geq 0} b_j\hbar^j$, $a_i, b_j \in A$,
$$\alpha \star \beta = \sum_{i,j\geq 0} a_i \star b_j \hbar^{i+j}$$

2. (Associativity) for any $f, g, h \in A$, $(f \star g) \star h = f \star (g \star h)$.

3. (Isomorphism) for any $f, g \in A$
$$f \star g = f \cdot g + B_1(f,g)\hbar + B_2(f,g)\hbar^2 + \cdots$$
where $B_n(,) : A \times A \longrightarrow A$ is a linear map.

Let $(A[[\hbar]], \star)$ be a formal deformation of $A$. Let $B_i : A \times A \longrightarrow A$ be the map defined above, for any $f, g, h \in A$,

$$(f\star g)\star h = fgh+(B_1(fg,h)+B_1(f,g)h)\hbar+(B_2(fg,h)+B_1(B_1(f,g),h)+B_2(f,g)h)\hbar^2+\mathcal{O}(\hbar^3)$$

$f\star(g\star h) = fgh+(B_1(f,gh)+fB_1(g,h))\hbar+(B_2(f,gh)+B_1(f,B_1(g,h))+fB_2(g,h))\hbar^2+\mathcal{O}(\hbar^3)$

Thus

$$B_1(fg,h) + B_1(f,g)h = B_1(f,gh) + fB_1(g,h) \tag{8.1}$$

$$B_2(fg,h) + B_1(B_1(f,g),h) + B_2(f,g)h = B_2(f,gh) + B_1(f,B_1(g,h)) + fB_2(g,h) \tag{8.2}$$

Denote the anti-symmetric part of $B_1$ by $B_1^- : A \times A \longrightarrow A$, i.e.

$$B_1^-(f,g) = \frac{1}{2}(B_1(f,g) - B_1(g,f))$$

**Lemma 8.1.** *$B_1^-$ is a Poisson bracket on $A$.*

*Proof.* We need to prove: for any $f, g, h \in A$

$$B_1^-(f,gh) = B_1^-(f,g)h+gB_1^-(f,h) \qquad B_1^-(B_1^-(f,g),h)+B_1^-(B_1^-(g,h),f)+B_1^-(B_1^-(h,f),g) = 0$$

By permuting variables in equation 8.1, we have

$$B_1(fg,h) + B_1(f,g)h = B_1(f,gh) + fB_1(g,h) \tag{1}$$

$$B_1(fg,h) + B_1(g,f)h = B_1(g,fh) + gB_1(f,h) \tag{2}$$

$$B_1(hg,f) + B_1(g,h)f = B_1(g,fh) + gB_1(h,f) \tag{3}$$

we can obtain the Poisson relation by taking (1) − (2) + (3). By permuting $f, g, h$ in equation 8.2 and sum up all six equations, we can obtain the Jacobin identity. □

The bilinear form $B_1^-$ arises naturally when we compare the commutator of $f \star g$ and $g \star f$

$$[f,g] = f \star g - g \star f = 2B_1^-(f,g)\hbar + \mathcal{O}(\hbar^2)$$

We define a Poisson bracket

$$\{f,g\} = 2B_1^-(f,g)$$

then

$$[f,g] = \{f,g\}\hbar + \mathcal{O}(\hbar^2)$$

This relation is very similar to the relation we want in the quantization. Now we can roughly answer our first question.

**Definition** Given a Poisson algebra $(A, \{,\})$ over a field $k$, a deformation quantization of $A$ is an $k[[\hbar]]$-algebra structure on $A[[\hbar]]$, i.e. a $k[[\hbar]]$-linear star product $\star$ on $A[[\hbar]]$, such that for any $f, g \in A$,

1. $f \star g = fg + \mathcal{O}(\hbar)$
2. $[f,g] = f \star g - g \star f = \hbar\{f,g\} + \mathcal{O}(\hbar^2)$

If a Poisson algebra $(A, \{,\})$ has a deformation quantization $(A[[\hbar]], \star)$, then the natural inclusion can be viewed as a quantization:

$$\iota : A \hookrightarrow A[[i\hbar]]/\hbar^2 \qquad \{f,g\} \longrightarrow \tfrac{1}{i\hbar}[f,g]$$

You might wonder the existence of non-trivial deformation quantization for a given Poisson algebra. A famous non-trivial deformation quantization is the Moyal product on the Poisson algebra $C^\infty(\mathbb{R}^{2n})$. See [2] for the construction.

Formal quantization also gives a partial answer to the second question. If an associative algebra $\widetilde{A}$ is isomorphic to $(A[[\hbar]], \star)$, where $A$ is a commutative algebra ( the product on $A$ is induced by product on $A[[\hbar]]/\hbar \xrightarrow{\sim} A$), then we have shown in lemma 8.1 that the bilinear form $B_1^-(,)$ defines a Poisson bracket on $A$. Then $A$ with this Poisson bracket can be viewed as dequantization of $\widetilde{A}$.

More generally, let $A_\hbar$ be an associative $k[\hbar]$-algebra such that for any $x \in A_\hbar$, $\hbar \cdot x = 0$ if and only if $x = 0$. Define

$$A_0 = A_\hbar/\hbar A_\hbar$$

Suppose $A_0$ with induced algebra structure is a commutative algebra, then we can define a bilinear map $\{,\} : A_0 \times A_0 \longrightarrow A_0$ as follow: for any $a = \widetilde{a} + (\hbar), b = \widetilde{b} + (\hbar) \in A_0$,

$$\widetilde{a}\widetilde{b} - \widetilde{b}\widetilde{a} \equiv ab - ba = 0 \mod \hbar$$

Thus, there exists a $c \in A_\hbar$ such that $\widetilde{a}\widetilde{b} - \widetilde{b}\widetilde{a} = \widetilde{c}\hbar$. Define $\{a, b\} = c = \widetilde{c} + (\hbar)$.

**Lemma 8.2.** *$\{,\}$ is a Poisson bracket on $A_0$.*

*Proof.* First, we need to show that $\{,\}$ is well-defined. Suppose $a, b \in A_0$ and $a = \widetilde{a}_1 + (\hbar) = \widetilde{a}_2 + (\hbar), b = \widetilde{b}_1 + (\hbar) = \widetilde{b}_2 + (\hbar)$, then there exists $\widetilde{c}, \widetilde{d} \in A_\hbar$, such that $\widetilde{a}_1 = \widetilde{a}_2 + \widetilde{c}\hbar, \widetilde{b}_1 = \widetilde{b}_2 + \widetilde{d}\hbar$. Note that every Lie algebra bracket belongs to $(\hbar)$.

$$\begin{aligned}
\widetilde{a}_1\widetilde{b}_1 - \widetilde{b}_1\widetilde{a}_1 &= (\widetilde{a}_2 + \widetilde{c}\hbar)(\widetilde{b}_2 + \widetilde{d}\hbar) - (\widetilde{b}_2 + \widetilde{d}\hbar)(\widetilde{a}_2 + \widetilde{c}\hbar) \\
&= \widetilde{a}_2\widetilde{b}_2 - \widetilde{b}_2\widetilde{a}_2 + \Big([\widetilde{c}, \widetilde{b}_2] + [\widetilde{a}_2, \widetilde{d}]\Big)\hbar + \mathcal{O}(\hbar^2) \\
&= \widetilde{a}_2\widetilde{b}_2 - \widetilde{b}_2\widetilde{a}_2 + \mathcal{O}(\hbar^2)
\end{aligned}$$

Therefore, $\{a, b\}$ is well-defined. Suppose $a = \widetilde{a} + (\hbar)$, $b = \widetilde{b} + (\hbar)$, $c = \widetilde{c} + (\hbar)$, then

$$\begin{aligned}
\{a, bc\} &= \hbar^{-1}(\widetilde{a}\widetilde{b}\widetilde{c} - \widetilde{b}\widetilde{c}\widetilde{a}) \\
&= \hbar^{-1}\Big((\widetilde{a}\widetilde{b} - \widetilde{b}\widetilde{a})\widetilde{c} + \widetilde{b}(\widetilde{a}\widetilde{c} - \widetilde{c}\widetilde{a})\Big) \\
&= \{a, b\}c + b\{a, c\}
\end{aligned}$$

and

$$[[\widetilde{a}, \widetilde{b}], \widetilde{c}] = [\{a, b\}\hbar, \widetilde{c}] + \mathcal{O}(\hbar^3) = \{\{a, b\}, c\}\hbar^2 + \mathcal{O}(\hbar^3)$$

Therefore, the Jacobin identity of Lie bracket on $A_\hbar$ leads to the Jacobin identity of $\{,\}$. $\square$

Now we can roughly answer the second question. Given an associative $k[\hbar]$-algebra $A_\hbar$ satisfying:

1. $\mathrm{Ann}(\hbar) = 0$
2. $A_0 = A_\hbar/\hbar A_\hbar$ is a commutative algebra.

We can define a Poisson structure on $A_0$.

Let $A$ be a filtered algebra over field $k$ such that the Lie bracket has degree $-1$, i.e. there exists $k$-vector space $A_0 \subset A_1 \subset \cdots$, such that

1. $A = \bigcup_{n=0}^{\infty} A_n$,

2. $A_i \cdot A_j \subset A_{i+j}$ for any $i, j \geq 0$.

3. $[A_i, A_j] \subset A_{i+j-1}$ for any $i, j \geq 0$, $A_{-1} = 0$.

We can define a Ress algebra by

$$A^{\sim} = \bigoplus_{n=0}^{\infty} A_n \hbar^n$$

Elements in $A^{\sim}$ are in form of $\sum_{i=1}^{m} a_i \hbar^i$, where $a_i \in A_i$, and the multiplication on $A^{\sim}$ is given by

$$a_n \hbar^n \cdot a_m \hbar^m = (a_n \cdot a_m) \hbar^{n+m}$$

Thus, $A^{\sim}$ is a $\mathbb{C}[\hbar]$ $\mathbb{C}[\hbar]$-module and $\mathrm{Ann}(\hbar) = 0$. Define

$$A_0 \coloneqq A^{\sim} / \hbar A^{\sim}$$

then

$$A_0 = \bigoplus_{n=0}^{\infty} A_n \hbar^n \Bigg/ \bigoplus_{n=1}^{\infty} A_{n-1} \hbar^n = \bigoplus_{n=0}^{\infty} A_n \hbar^n / A_{n-1} \hbar^n \cong \bigoplus_{n=0}^{\infty} A_n / A_{n-1} = \mathrm{gr}(A)$$

for any $a_n \in A_n$, $a_m \in A_m$,

$$[a_n \hbar^n, a_m \hbar^m] = [a_n, a_m] \hbar^{n+m} \in A_{n+m-1} \hbar^{n+m} \quad \Rightarrow \quad [a_n \hbar^n, a_m \hbar^m] = 0$$

Therefore, $A_0$ is a commutative algebra. By above construction, we can define a Poisson structure on $A_0$.

In conclusion, if $A$ is a filtered algebra such that the Lie bracket has degree $-1$, then we can define a Poisson structure on the graded algebra $\mathrm{gr}(A)$.

**Example 8.1.** *Let $\mathfrak{g}$ be a Lie algebra and $A = U(\mathfrak{g})$. $U(\mathfrak{g})$ has a filtration:*

$$U_{\leq n}(\mathfrak{g}) = span_{\mathbb{C}} \langle X_1 \cdots X_k | X_i \in \mathfrak{g}, k \leq n \rangle \qquad n \geq 0$$

*Under this filtration, the Lie bracket has degree $-1$. Note that*

$$[X, X_1 \cdots X_n] = \sum_{i=1}^{n} X_1 \cdots [X, X_i] \cdots X_n \in U_{\leq n}(\mathfrak{g})$$

*This implies $[\mathfrak{g}, U_{\leq n}(\mathfrak{g})] \subset U_{\leq n}(\mathfrak{g})$ and generally $[U_{\leq n}(\mathfrak{g}), U_{\leq m}(\mathfrak{g})] \subset U_{\leq n+m-1}(\mathfrak{g})$. Thus, the graded algebra associated to $U(\mathfrak{g})$ has a Poisson structure. It is known that*

$$gr(U(\mathfrak{g})) \cong S(\mathfrak{g})$$

*where $S(\mathfrak{g})$ is the symmetric algebra generated by $\mathfrak{g}$. Thus, $S(\mathfrak{g})$ is a Poisson algebra. For $X, Y \in \mathfrak{g}$, denote their image in $S(\mathfrak{g})$ by $\overline{X}, \overline{Y}$. Then*

$$\{\overline{X}, \overline{Y}\} = \{X\hbar, Y\hbar\} = \hbar^{-1}[X, Y]\hbar^2 = [X, Y]\hbar = \overline{[X, Y]}$$

*Therefore, the Poisson bracket on $S(\mathfrak{g})$ is the same as the Lie bracket on $U(\mathfrak{g})$.*

Now we can state the center theorem of classical version:

**Theorem 8.3.** *There exists elements $\overline{S}_i \in S(\widehat{\mathfrak{g}}_-)$, $ord(\overline{S}_i) = d_i + 1$, $deg(\overline{S}_i) = d_i + 1$, $i = 1, ..., r$, with $\overline{S}_1 = \overline{\mathfrak{S}}$, such that $\{T^{n_i} S_i\}_{n_i \geq 0, i=1,...,r}$ are algebraically independent in $S(\widehat{\mathfrak{g}}_-)$ and they mutually Poisson commute.*

# 9 Center of $U(\mathfrak{g})$

When we state theorem 7.4 and theorem 8.3, there is one thing we haven't explain. What is $S_1, ..., S_l$? Where do they come from? Actually, they are derived from the center of $U(\mathfrak{g})$. Therefore, in this section, we are going to determine the center of $U(\mathfrak{g})$ for simple Lie algebra $\mathfrak{g}$.

Let $r$ be the rank of $\mathfrak{g}$ and $d_1 \leq \cdots \leq d_r$ be exponents of $\mathfrak{g}$ (for the definition of exponents, see [9], Appendix A). Our main theorem is:

**Theorem 9.1.** *The center of $U(\mathfrak{g})$ is generated by $r$ homogeneous algebraically independent elements $S_1, ..., S_r$ such that $S_i$ has degree $d_i + 1$ and $S_1$ is the Casimir element, i.e.*

$$Z(\mathfrak{g}) = \mathbb{C}[S_1, ..., S_r]$$

The proof of this theorem involves two theorems:

**Theorem 9.2** (**Harish – Chandra Isomorphism**). *There is an algebra isomorphism between the center $Z(\mathfrak{g})$ and $S(\mathfrak{h})^W$.*

**Theorem 9.3** (**Chevally**). *Let $\Gamma$ be a finite reflation group ($w^2 = 1$ for any $w \in \Gamma$) acting on a $n$-dimensional vector space $V$ over field $k$, Char $k = 0$. Then the algebra of invariant polynomials over $V$ is generated by $n$ algebraically independent homogeneous polynomials $P_1, ..., P_n$, i.e.*

$$\mathit{Fun}(V)^\Gamma = S(V^*)^\Gamma \cong k[P_1, ..., P_n]$$

In this section, we focus on the construction of Harish-Chandra isomorphism. Reader can find more details about Chevalley's theorem in [3].

**Definition** Let $V$ be a finite dimensional vector space over a field $k$. A polynomial function on $V$ is a function $f : V \longrightarrow k$ that is given by an element in $S(V^*)$ (the symmetric algebra of $V^*$). To be specific, for $\varphi_1 \cdots \varphi_k \in S(V^*)$, where $\varphi_i \in V^*$, the polynomial function on $V$ is given by

$$\varphi_1 \cdots \varphi_k : V \longrightarrow k \qquad v \longmapsto \varphi_1(v) \cdots \varphi_k(v)$$

Denote the space of all polynomials on $V$ by Fun(V). Then there is an algebra isomorphism Fun(V) $\cong S(V^*)$

Now suppose $G$ is a group acting on $V$, then it induces an action of $G$ on Fun($V$), given by

$$(g \cdot f)(v) = f(g^{-1}v) \qquad \text{for any } g \in G, v \in V$$

Since Fun($V$) $\cong S(V^*)$, the group $G$ also acts on $S(V^*)$. Explicitly, Let $\varphi_1 \cdot \varphi_k \in S(V^*)$ be a homogeneous polynomial, then

$$g \cdot (\varphi_1 \cdots \varphi_k) = (\varphi_1 \circ g^{-1}) \cdot (\varphi_k \circ g^{-1}) \qquad \text{for any } g \in G$$

Suppose $V$ is a vector space over $\mathbb{R}$ and $G$ is a Lie group with Lie algebra $\mathfrak{g}$. If $G$ acts on $V$ smoothly, then it also induce an action of $\mathfrak{g}$ on Fun($V$) and $S(V^*)$. The action on $S(V^*)$ is given by

$$g \cdot (\varphi_1 \cdots \varphi_k) = -\sum_{i=1}^{k} \varphi_1 \cdots (\varphi_i \circ g) \cdots \varphi_k \qquad \text{for any } g \in \mathfrak{g}$$

Let $\mathrm{Fun}(V)^G$ be the space of all $G$-invariant polynomials, i.e.$g \cdot f = f$ for any $g \in G$. If $G$ is a Lie group, let $\mathrm{Fun}(V)^{\mathfrak{g}}$ be the space of all $\mathfrak{g}$-invariant polynomials, i.e. $g \cdot f = 0$ for any $g \in \mathfrak{g}$. If $G$ is a connected Lie group, then

$$\mathrm{Fun}(V)^G = \mathrm{Fun}(V)^{\mathfrak{g}}$$

Let's back to our discussion on Harish-Chandra isomorphism. Consider the space $\mathrm{Fun}(\mathfrak{g}) = S(\mathfrak{g}^*)$. Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}_0$ such that the complexification of $\mathfrak{g}_0$ is $\mathfrak{g}$. The group $G$ acts on the vector space $\mathfrak{g}$ by adjoint action. It induces actions of $G$ and $\mathfrak{g}$ on $\mathrm{Fun}(\mathfrak{g})$ and $S(\mathfrak{g}^*)$

We are interested in the invariant subspace $S(\mathfrak{g}^*)^{\mathfrak{g}}$. We can construct several functions on $\mathfrak{g}$ that belongs to this space. Let $\pi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. Define

$$\chi_\pi : \mathfrak{g} \longrightarrow \mathbb{C} \qquad \chi_\pi(g) = \mathrm{tr}(\pi(g))$$

It's obvious that $\chi_\pi$ is linear, so $\chi_\pi \in \mathfrak{g}^* \subset \mathrm{Fun}(\mathfrak{g})$. Define

$$\chi_{\pi,k} : \mathfrak{g} \longrightarrow \mathbb{C} \qquad \chi_{\pi,k}(g) = (\chi_\pi(g))^k = (\mathrm{tr}(\pi(g)))^k$$

$\chi_{\pi,k}$ is a power of a linear functional, so $\chi_{\pi,k} \in \mathrm{Fun}(\mathfrak{g})$.

**Lemma 9.4.** *For all $k \geq 1$, $\chi_{\pi,k} \in$ Fun$(\mathfrak{g})^{\mathfrak{g}}$.*

*Proof.* For any $h, g \in \mathfrak{g}$,

$$h \cdot \chi_{\pi,k}(g) = -k\,\mathrm{tr}(\pi(g))^{k-1} \cdot \mathrm{tr}(\pi([h,g])) = 0$$

Thus, $\chi_{\pi,k} \in \mathrm{Fun}(\mathfrak{g})$. ☐

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Then polynomials on $\mathfrak{g}$ can be restricted to $\mathfrak{h}$. We obtain an algebra homomorphism

$$\mathrm{Fun}(\mathfrak{g}) \longrightarrow \mathrm{Fun}(\mathfrak{h})$$

We are curious about the image of $\mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}}$ under this restriction. It should also be a space of certain invariant polynomial functions on $\mathfrak{h}$. Recall that the Weyl group acts on $\mathfrak{h}^*$ and $\mathfrak{h}$. We claim that the image is contained exactly $\mathrm{Fun}(\mathfrak{h})^W$ and the map $\mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}} \longrightarrow \mathrm{Fun}(\mathfrak{h})^W$ is an isomorphism of vector spaces.

Let $\Delta$ be the root system of $\mathfrak{g}$ and $\Delta_+$ be the set of all positive roots. Let $W$ be the Weyl group of $\mathfrak{g}$. For any $\alpha \in \Delta$, the reflection by $\alpha$ is denoted by $w_\alpha \in W$, i.e. for any $\beta \in \mathfrak{g}^*$,

$$w_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$$

Identifying $\mathfrak{h}^*$ with $\mathfrak{h}$ by Killing form, it also induces an action of Weyl group on $\mathfrak{h}$. For $\alpha \in \Delta$ and $h \in \mathfrak{h}$,

$$w_\alpha(h) = h - \alpha(h) h_\alpha$$

Assume $\{e_\alpha, f_\alpha, h_\alpha\}$ is a $\mathfrak{sl}_2$-stripe associated to $\alpha \in \Delta$, then for any $h \in \mathfrak{g}$

$$w_\alpha(h) = \exp(\mathrm{ad}_{e_\alpha}) \exp(-\mathrm{ad}_{f_\alpha}) \exp(\mathrm{ad}_{e_\alpha}) h$$

In other words, the action of Weyl group on $\mathfrak{h}$ is given by composition of adjoint actions of $G$. Hence, if $f \in \mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}} = \mathrm{Fun}(\mathfrak{g})^G$, then the function $f|_{\mathfrak{h}}$ is invariant under the action of Weyl group.

**Proposition 9.5.** *The map*

$$Res : Fun(\mathfrak{g})^{\mathfrak{g}} \longrightarrow Fun(\mathfrak{h})^W$$

*is an isomorphism.*

*Proof.* The injectivity is easy. If $f \in \mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}}$ that vanish on $\mathfrak{h}$, then $f = 0$ because every element in $\mathfrak{g}_0$ conjugates to an element in $\mathfrak{h}$.

To prove the surjectivity, we claim that $\mathrm{Fun}(\mathfrak{h})^W$ is generated by $\chi_{\pi,k}$ as $\pi$ ranges over all finite dimensional irreducible representations of $\mathfrak{g}$ and $k \geq 1$. Let $k$. Here we briefly sketch the proof:

1. The space $\mathrm{Fun}(\mathfrak{h}) = S(\mathfrak{h}^*)$ can be linearly spanned by

$$\{\lambda^k \in \mathfrak{h}^* | \lambda \text{ is a dominant integral weight}, k \geq 1\}$$

2. Let $A : S(\mathfrak{h}^*) \longrightarrow S(\mathfrak{h}^*)^W$ be the arrange map. Then the space $S(\mathfrak{h}^*)$ is spanned by $\{A\lambda^k \in \mathfrak{h}^* | \lambda \text{ is a dominant integral weight}, k \geq 1\}$.

3. Let $\lambda$ be a dominant integral weight and $\pi$ be the irreducible representation with highest weight $\lambda$, then $\chi_{\pi,k} - A\lambda^k$ is linear combination of power of dominant weights lower than $\lambda$. Such kind of weights are finite. Thus, by induction, we can prove the claim.

□

Therefore, we obtain an algebra isomorphism

$$\mathrm{Fun}(\mathfrak{g})^{\mathfrak{g}} = S(\mathfrak{g}^*)^{\mathfrak{g}} \xrightarrow{\sim} S(\mathfrak{h}^*)^W = \mathrm{Fun}(\mathfrak{h})^W$$

Since the Killing form on $\mathfrak{g}$ is non-degenerate, we can identify $\mathfrak{g}$, $\mathfrak{h}$, $S(\mathfrak{g})$, and $S(\mathfrak{h})$ with $\mathfrak{g}^*$, $\mathfrak{h}^*$, $S(\mathfrak{g}^*)$, and $S(\mathfrak{h}^*)$. Under this identification, let's figure out how $\mathfrak{g}$ acts on $S(\mathfrak{g})$, how $W$ acts on $S(\mathfrak{h})$ and how the isomorphism between $S(\mathfrak{g})^{\mathfrak{g}}$ and $S(\mathfrak{h})^W$ formulated.

Let $g_1 \cdots g_k \in S(\mathfrak{g})$ be a homogeneous element and $\mathfrak{g} \in S(\mathfrak{g})$. The action of $g$ on $g_1 \cdots g_k$ can be traced by diagram

$$\begin{tikzcd} S(\mathfrak{g}) \arrow[r, "\sim"] \arrow[d, "g"] & S(\mathfrak{g}^*) \arrow[d, "g"] \\ S(\mathfrak{g}) \arrow[r, "\sim"] & S(\mathfrak{g}^*) \end{tikzcd} \qquad \begin{tikzcd} g_1 \cdots g_k \arrow[r, mapsto] \arrow[d, mapsto] & g_1^* \cdots g_k^* \arrow[d, mapsto] \\ ? \arrow[r, mapsto] & -\sum_{i=1}^{k} g_1^* \cdots (g_i^* \circ \mathrm{ad}_g) \cdots g_k^* \end{tikzcd}$$

Note that for any $x \in \mathfrak{g}$,

$$g_i^* \circ \mathrm{ad}_g(x) = g_i^*([g, x]) = \kappa(g_i, [g, x]) = -\kappa([g, g_i], x) = (-[g, g_i])^*(x)$$

Thus

$$-\sum_{i=1}^{k} g_1^* \cdots (g_i^* \circ \mathrm{ad}_g) \cdots g_k^* = \sum_{i=1}^{k} g_1^* \cdots [g, g_i]^* \cdots g_k^*$$

Therefore,

$$g \cdot (g_1 \cdots g_k) = \sum_{i=1}^{k} g_1 \cdots [g, g_i] \cdots g_k = \{g, g_1 \cdots g_k\}$$

the induce action of $\mathfrak{g}$ on $S(\mathfrak{g})$ is exactly the Poisson bracket action.

Next we consider the action of Weyl group on $S(\mathfrak{h})$. For $h_1 \cdots h_k \in S(\mathfrak{h})$ and $w \in W$,

$$\begin{tikzcd} S(\mathfrak{h}) \arrow[r, "\sim"] \arrow[d, "g"'] & S(\mathfrak{h}^*) \arrow[d, "g"'] \\ S(\mathfrak{h}) \arrow[r, "\sim"] & S(\mathfrak{h}^*) \end{tikzcd} \qquad \begin{tikzcd} h_1 \cdots h_k \arrow[r, mapsto] \arrow[d, mapsto] & h_1^* \cdots h_k^* \arrow[d, mapsto] \\ ? \arrow[r, mapsto] & (h_1^* \circ w^{-1}) \cdots (h_k^* \circ w^{-1}) \end{tikzcd}$$

For $x \in \mathfrak{h}$,

$$h_i^* \circ w^{-1}(x) = h_i^*(w^{-1}x) = \kappa(h_i, w^{-1}x) = \kappa(wh_i, x) = (wh_i)^*(x)$$

Therefore,

$$w \cdot (h_1 \cdots h_k) = (wh_1) \cdots (wh_k)$$

which is the natural action of $W$ on $S(\mathfrak{h})$.

At last, let's figure out how the isomorphism between $S(\mathfrak{g})^{\mathfrak{g}}$ and $S(\mathfrak{h})^W$ is formulated.

$$\begin{tikzcd} S(\mathfrak{g}) \arrow[r] \arrow[d] & \mathrm{Fun}(\mathfrak{g}) = S(\mathfrak{g}^*) \arrow[d] \\ S(\mathfrak{h}) \arrow[r] & \mathrm{Fun}(\mathfrak{h}) = S(\mathfrak{h}^*) \end{tikzcd} \qquad \begin{tikzcd} g \arrow[r, mapsto] \arrow[d, mapsto] & g^* \arrow[d, mapsto] \\ ? \arrow[r, mapsto] & g^*|_{\mathfrak{h}} \end{tikzcd}$$

Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^{\perp}$ be the orthogonal decomposition of $\mathfrak{g}$ with respect to the Killing form and $\pi : \mathfrak{g} \longrightarrow \mathfrak{h}$ be the projection. For any $h \in \mathfrak{h}$,

$$g^*|_{\mathfrak{h}}(h) = g^*(h) = \kappa(g, h) = \kappa(\pi(g), h)$$

Therefore, the map $\pi : S(\mathfrak{g}) \longrightarrow S(\mathfrak{h})$ is induced by the orthogonal projection $\mathfrak{g} \longrightarrow \mathfrak{h}$ with respect to the Killing form.

In summary:

**Proposition 9.6.** *The orthogonal projection from $\mathfrak{g}$ to $\mathfrak{h}$ induces an algebra isomorphism*

$$S(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\sim} S(\mathfrak{h})^W$$

*where $S(\mathfrak{g})^{\mathfrak{g}}$ is the Poisson center and $S(\mathfrak{h})^W$ is the invariant subspace of natural $W$-action.*

Now we should point out the connection between the center of $U(\mathfrak{g})$ and what we discuss above. The projection map $\overline{\pi} : U(\mathfrak{g}) \longrightarrow S(\mathfrak{g})$ has a right inverse:

$$\iota : S(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \qquad X_1 \cdots X_k \longmapsto \frac{1}{k!} \sum_{\pi \in S_k} X_{\pi(1)} \cdots X_{\pi(k)}$$

i.e. $\overline{\pi} \circ \iota = \mathrm{Id}$.

**Proposition 9.7.** *The map $\iota$ is an isomorphism of vector spaces.*

*Proof.* Let $U(\mathfrak{g})_{\leq k}$, $k \geq 0$, be the filtration of $U(\mathfrak{g})$ we introduce in example 8.1 and $S_{\leq k}(\mathfrak{g}) = \pi\left(U(\mathfrak{g})_{\leq k}\right)$, $k \geq 0$, be a filtration of $S(\mathfrak{g})$. We prove that the $U(\mathfrak{g})_{\leq k}$ is contained in the image of $\iota$ by induction on $k$. When $k = 0, 1$, this is obvious. Suppose we have prove the case $k-1$. Now for case $k$, it suffices to prove that $X_1 \cdots X_k \in \iota(S(\mathfrak{g}))$ for any $X_1, ..., X_k \in \mathfrak{g}$. Note that for any $\pi \in S_k$,

$$X_{\pi(1)} \cdots X_{\pi(k)} \equiv X_1 \cdots X_k \mod S_{\leq k-1}(\mathfrak{g})$$

Hence,

$$\iota(X_1 \cdots X_k) \equiv X_1 \cdots X_k \mod S_{\leq k-1}(\mathfrak{g})$$

By induction, we can conclude that $X_1 \cdots X_k$ is contained in the image of $\iota$. Therefore, $\iota$ is an isomorphism.

□

Note that this map is only a vector space isomorphism, not an algebra isomorphism.

**Proposition 9.8.** *The map $\iota$ induces an isomorphism*

$$\iota : S(\mathfrak{g})^{\mathfrak{g}} \longrightarrow Z(\mathfrak{g})$$

*Proof.* For homogeneous element $X_1 \cdots X_k \in S(\mathfrak{g})$ and $g \in \mathfrak{g}$,

$$\begin{aligned}
[g, \iota(X_1 \cdots X_k)] &= \frac{1}{k!} \sum_{\pi \in S_k} [g, X_{\pi(1)} \cdots X_{\pi(k)}] \\
&= \sum_{i=1}^{k} \frac{1}{k!} \sum_{\pi \in S_k} X_{\pi(1)} \cdots [g, X_{\pi(i)}] \cdots X_{\pi(k)} \\
&= \iota\left(\{g, X_1 \cdots X_k\}\right)
\end{aligned}$$

Thus, for any $C \in S(\mathfrak{g})$,

$$[g, \iota(C)] = \iota(\{g, C\})$$

If $C \in S(\mathfrak{g})^{\mathfrak{g}}$, then $[g, \iota(C)] = 0$ for any $g \in \mathfrak{g}$, thus $C \in Z(\mathfrak{g})$. Conversely, if $D = \iota(C) \in Z(\mathfrak{g})$, then $\iota(\{g, C\}) = [g, D] = 0$ for any $g \in \mathfrak{g}$, thus $C \in S(\mathfrak{g})^{\mathfrak{g}}$. Therefore, $\iota$ is an isomorphism. □

**Corollary 9.9.** *If $X \in U_{\leq k}(\mathfrak{g})$, then*

$$\iota^{-1}(X) \equiv \pi(X) \mod S_{\leq k-1}(\mathfrak{g})$$

In brief, we have shown that

$$\begin{tikzcd}
& S(\mathfrak{g})^{\mathfrak{g}} \arrow[dl, "\iota"', "\sim"] \arrow[dr, "\pi", "\sim"'] & \\
Z(\mathfrak{g}) & & S(\mathfrak{h})^W
\end{tikzcd}$$

where $\pi$ is an algebra isomorphism and $\iota$ is a vector space isomorphism. The map $\mathrm{Res} \circ \iota^{-1}$ is merely an isomorphism of vector space between $Z(\mathfrak{g})$ and $S(\mathfrak{h})^W$. Hence, we haven't finished our proof.

In fact, there is a direct way to construct an algebra homomorphism from $Z(\mathfrak{g})$ to $S(\mathfrak{h})^W$. Let $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be a Cartan decomposition of $\mathfrak{g}$ and $e_1, ..., e_l$, $f_1, ..., f_l$, $h_1, ..., h_r$ be a standard basis for $\mathfrak{g}$. then by PBW theorem, $U(\mathfrak{g})$ and $S(\mathfrak{g})$ are linearly spanned by

$$f_1^{k_1} \cdots f_l^{k_l} h_1^{m_1} \cdots h_r^{m_r} e_1^{n_1} \cdots e_l^{n_l}$$

Thus, there are vector space decompositions

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (U(\mathfrak{g})\mathfrak{n}_+ + \mathfrak{n}_- U(\mathfrak{g})) \tag{9.1}$$

$$S(\mathfrak{g}) = S(\mathfrak{h}) \oplus (S(\mathfrak{g})\mathfrak{n}_+ + \mathfrak{n}_- S(\mathfrak{g})) \tag{9.2}$$

Let $\pi : S(\mathfrak{g}) \longrightarrow S(\mathfrak{h})$ be the map induced by orthogonal projection $\mathfrak{g} \longrightarrow \mathfrak{h}$. The Cartan decomposition is an orthogonal decomposition with respect to the killing form, i.e. $\kappa(e_i, \mathfrak{h}) = 0$, $\kappa(f_i, \mathfrak{h}) = 0$. Thus

$$\pi\left(f_1^{k_1} \cdots f_l^{k_l} h_1^{m_1} \cdots h_r^{m_r} e_1^{n_1} \cdots e_l^{n_l}\right) = \begin{cases} h_1^{m_1} \cdots h_r^{m_r} & \text{if } k_1, ..., k_l, n_1, ..., n_l = 0 \\ 0 & \text{else} \end{cases}$$

Therefore, the map $\pi : S(\mathfrak{g}) \longrightarrow S(\mathfrak{h})$ is exactly projection to the first factor in the decomposition 9.2.

**Warning 1.** *The projection $\gamma$ is not induced by orthogonal projection $\mathfrak{g} \longrightarrow \mathfrak{h}$. To compute this projection, you should express an element as a linear combination of*

$$f_1^{k_1} \cdots f_l^{k_l} h_1^{m_1} \cdots h_r^{m_r} e_1^{n_1} \cdots e_l^{n_l}$$

*and then wipe those terms which contain $e_i$ and $f_j$. For example, to compute the projection of $e_i f_i$, you should express it as*

$$e_i f_i = [e_i, f_i] + f_i e_i = 2h_i + f_i e_i$$

*and then wipe the term $f_i e_i$. Hence, $\gamma(e_i f_i) = 2h_i$.*

Let $\gamma : U(\mathfrak{g}) \longrightarrow U(\mathfrak{h}) = S(\mathfrak{h})$ be projection to the first factor in the decomposition 9.1.

$\gamma$ is an algebra homomorphism. By restricting to the center, we obtain an algebra homomorphism

$$\gamma : Z(\mathfrak{g}) \longrightarrow S(\mathfrak{h}) = \mathrm{Fun}(\mathfrak{h}^*)$$

Fix a $\gamma \in \mathfrak{h}^*$, we define a character $\chi_\lambda$ on $Z(\mathfrak{g})$ by

$$\chi_\lambda : Z(\mathfrak{g}) \longrightarrow \mathbb{C} \qquad \chi_\lambda(z) = \gamma(z)(\lambda)$$

**Lemma 9.10.** *Let $V_\lambda = U(\mathfrak{g})v_\lambda$ be any highest weight module of $\mathfrak{g}$ with highest weight $\lambda$. Then the action of $Z(\mathfrak{g})$ on $v_\lambda$ is given by*

$$z \cdot v_\lambda = \chi_\lambda(z) v_\lambda$$

*Proof.* The key point is $Z(\mathfrak{g}) \subset U(\mathfrak{h}) \oplus (U(\mathfrak{g})\mathfrak{n}_+ \cap \mathfrak{n}_- U(\mathfrak{g})) \subset U(\mathfrak{h}) \oplus U(\mathfrak{g})\mathfrak{n}_+$. Since $\mathfrak{n}_+$ annihilates $v_\lambda$, we have

$$z \cdot v_\lambda = \gamma(z) \cdot v_\lambda = \gamma(z)(\lambda) v_\lambda = \chi_\lambda(z) v_\lambda$$

□

We hope that the image of $\gamma$ lies in $S(\mathfrak{h})^W$. However, this is not the case.

**Lemma 9.11.** *Let $\lambda \in \mathfrak{h}^*$ and $\alpha$ be a simple root in $\Delta$ such that $m = \langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0}$. Let $V_\lambda = U(\mathfrak{g})v_\lambda$ be a highest weight module of $\mathfrak{g}$ with highest weight $\lambda$. Then the submodule in $V_\lambda$ generated by $f_\alpha^{m+1}$ is a highest weight module with highest weight $\lambda - (m+1)\alpha$.*

*Proof.* We only need to show that $\mathfrak{n}_+$ annihilates $f_\alpha^{m+1} v_\lambda$. For simple root $\beta$ not equal to $\alpha$, $[e_\beta, f_\alpha] = 0$, thus

$$e_\beta \cdot (f_\alpha^{m+1} v_\lambda) = f_\alpha^{m+1} e_\beta v_\lambda = 0$$

And

$$\begin{aligned} e_\alpha f_\alpha^{m+1} v_\lambda &= \sum_{i=0}^{m} f_\alpha^i [e_\alpha, f_\alpha] f_\alpha^{m-i} v_\lambda \\ &= 2 \sum_{i=0}^{m} f_\alpha^i h_\alpha f_\alpha^{m-i} v_\lambda \\ &= \sum_{i=0}^{m} (\lambda - (m-i)\alpha)(h_\alpha) f_\alpha^m v_\lambda \\ &= (m+1)(\lambda(h_\alpha) - m) f_\alpha^m v_\lambda = 0 \end{aligned}$$

Note that $\langle \lambda, \alpha \rangle = \lambda(h_\alpha)$. □

Therefore, if $\lambda$ is a dominant integral weight, then for every simple root $\alpha$, $V_\lambda = U(\mathfrak{g})v_\lambda$ contains a highest weight submodule with weight

$$\lambda - (\langle \lambda, \alpha \rangle + 1)\alpha = w_\alpha(\lambda) - \alpha$$

which is generated by $f_\alpha^{m+1} v_\lambda$, where $m = \langle \lambda, \alpha \rangle$. Consider the action of $z \in Z(\mathfrak{g})$ on this highest weight

$$z \cdot (f_\alpha^m v_\lambda) = f_\alpha^{m+1} z \cdot v_\lambda = \chi_\lambda(z) \cdot f_\alpha^m v_\lambda$$

Therefore, for dominant integral weight $\lambda$ and $z \in Z(\mathfrak{g})$, we have

$$\chi_{w_\alpha(\lambda)-\alpha}(z) = \chi_\lambda(z) \qquad \text{for any simple root } \alpha$$

$\gamma(z) = \chi_-(z)$ and $\chi_{w_\alpha(-)-\alpha}$ are two polynomial functions on $\mathfrak{h}^*$. Since they agree on an unbounded set (the set of all dominant integral weights), they must agree everywhere. We conclude that

**Proposition 9.12.** *For any $\lambda \in \mathfrak{h}^*$ and any simple root $\alpha$,*

$$\chi_{w_\alpha(\lambda)-\alpha} = \chi_\lambda$$

Therefore, we can not conclude that $\gamma(z) = \chi_-(z)$ is a $W$-invariant function over $\mathfrak{h}^*$. We should introduce a shift to obtain $W$-invariant functions. For any $\delta \in \mathfrak{h}^*$, define

$$\widetilde{\delta} : S(\mathfrak{h}) \longrightarrow S(\mathfrak{h}) \qquad f(\lambda) \longmapsto f(\lambda - \delta)$$

Let $\widetilde{\gamma} = \widetilde{\delta} \circ \gamma$, then

$$\widetilde{\gamma}(z)(\lambda) = \chi_{\lambda-\delta}(z) = \chi_{w_\alpha(\lambda-\delta)-\alpha}(z)$$
$$\widetilde{\gamma}(z)(w_\alpha(\lambda)) = \chi_{w_\alpha(\lambda)-\delta}(z)$$

We wish to let $\widetilde{\gamma}(\lambda) = \widetilde{\gamma}(w_\alpha(\lambda))$ for all simple roots $\alpha$. It suffices if

$$w_\alpha(\lambda) - \delta = w_\alpha(\lambda - \delta) - \alpha \qquad \text{for all simple root } \alpha$$

This is equivalent to

$$\langle \delta, \alpha \rangle = 1 \qquad \text{for all simple root } \alpha$$

It is known that

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$$

satisfies this condition. Therefore, we take $\delta = \rho$, $\widetilde{\gamma} = \widetilde{\rho} \circ \gamma$, then for any $z \in Z(\mathfrak{g})$, $\widetilde{\gamma}$ is a $W$-invariant function on $\mathfrak{h}^*$. We obtain an algebra homomorphism

$$\widetilde{\gamma} : Z(\mathfrak{g}) \longrightarrow S(\mathfrak{h})^W$$

Now we can finish our proof of Harish-Chandra theorem by showing that $\widetilde{\gamma}$ is an isomorphism.

$$\begin{tikzcd}
 & U(\mathfrak{g}) \arrow[dl, "\gamma"'] \arrow[d, "\widetilde{\gamma}"'] & S(\mathfrak{g}) \arrow[l, "\iota"'] \arrow[dr, "\pi"] & \\
S(\mathfrak{h}) \arrow[r, "\widetilde{\rho}"] & S(\mathfrak{h}) & & S(\mathfrak{h})
\end{tikzcd}
\qquad
\begin{tikzcd}
Z(\mathfrak{g}) \arrow[d, "\widetilde{\gamma}"'] \arrow[dr, "\pi\circ\iota^{-1}"] & S(\mathfrak{g})^{\mathfrak{g}} \arrow[l, "\iota"'] \arrow[d, "\pi"] \\
S(\mathfrak{h})^W & S(\mathfrak{h})^W
\end{tikzcd}$$

**Lemma 9.13.** *For $X \in U(\mathfrak{g})$,*

$$\widetilde{\gamma}(X) \equiv \pi \circ \iota^{-1}(X) \mod S_{\leq k-1}(\mathfrak{h})$$

*Proof.* Suppose $X = f_1^{k_1} \cdots f_l^{k_l} h_1^{m_1} \cdots h_r^{m_r} e_1^{n_1} \cdots e_l^{n_l}$, we shortly denote by $X = f^\alpha h^\beta e^\kappa$. Then

$$\widetilde{\gamma}(X) = \delta_{\alpha,0}\delta_{\kappa,0} \cdot \widetilde{\rho}(h^\beta) \equiv \delta_{\alpha,0}\delta_{\kappa,0} \cdot h^\beta \mod S_{\leq k-1}(\mathfrak{h})$$

On the other hand, by corollary 9.9,

$$\pi \circ \iota^{-1}(X) \equiv \pi(f^\alpha h^\beta e^\kappa) = \delta_{\alpha,0}\delta_{\kappa,0} \cdot h^\beta \mod S_{\leq k-1}(\mathfrak{h})$$

Therefore,

$$\widetilde{\gamma}(X) \equiv \pi \circ \iota^{-1}(X) \mod S_{\leq k-1}(\mathfrak{h})$$

□

By this lemma, we can conclude that the map $\gamma : Z(\mathfrak{g}) \longrightarrow S(\mathfrak{h})^W$ is an isomorphism. We are done for the proof of Harish-Chandra theorem.

# 10 Statement of the classical center theorem

In this section, we give a precise statement of the theorem 8.3.

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$. Define a Lie algebra

$$\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((t)) = \mathfrak{g}((t))$$

with the Lie bracket given by

$$[A \otimes f, B \otimes g] = [A, B] \otimes fg$$

for all $A, B \in \mathfrak{g}$ and $f, g \in \mathbb{C}((t))$. We define a topology on $\widehat{\mathfrak{g}}$ by taking $I_n = \mathfrak{g} \otimes t^n\mathbb{C}[[t]]$, $n \geq 0$, as a neighborhood basis of 0. Then every element in $\widehat{g}$ can be written as

$$\sum_{n\geq N} A_n \otimes t^n \qquad A_n \in \mathfrak{g}$$

Note that $\mathbb{C}((t)) = t^{-1}\mathbb{C}[t^{-1}] \oplus \mathbb{C}[[t]]$. Define two Lie subalegbras of $\widehat{\mathfrak{g}}$:

$$\widehat{\mathfrak{g}}_- = \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}] \qquad \widehat{\mathfrak{g}}_+ = \mathfrak{g} \otimes \mathbb{C}[[t]]$$

then $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_- \oplus \widehat{\mathfrak{g}}_+$.

For convenience, denote $A_n = A \otimes t^n$, where $A \in \mathfrak{g}$ and $n \in \mathbb{Z}$. Now we consider symmetric algebras $S(\widehat{\mathfrak{g}}), S(\widehat{\mathfrak{g}}_-)$, and $S(\widehat{\mathfrak{g}}_+)$. Define a topology on $S(\widehat{\mathfrak{g}})$ by taking ideals $I_n = (\mathfrak{g} \otimes t^n\mathbb{C}[[t]])$, $n \geq 0$, as a topological basis of 0. Then every element in $S(\widehat{\mathfrak{g}})$ can be expressed as

$$\sum_{N} \sum_{\substack{n_1+\cdot+n_k=N \\ n_i>M}} A^{(1)}_{n_1} \cdots A^{(k)}_{n_k} \tag{10.1}$$

where $M$ is a integer and $A^{(i)} \in \mathfrak{g}$. Thus, $S(\widehat{\mathfrak{g}})$ is topologically generated by $\{A_n | A \in \mathfrak{g}, n \in \mathbb{Z}\}$. Moreover, $S(\widehat{\mathfrak{g}})$ has a Poisson structure (construction given by deformation quantization, see Example 8.1 ). It's easy to verify that $\{I_n, I_m\} \subset I_{\min\{n,m\}}$. Hence, the Poisson bracket $\{,\}$ is continuous.

Let $J_1, ..., J_d$ be a basis of $\mathfrak{g}$ with a dual basis $J^1, ..., J^d$ with respect to an invariant bilinear form on $\mathfrak{g}$ (it does not matter which one we choose. However, in quantum case, the choice of invariant bilinear form matter)

Now we focus on the structure of $S(\widehat{\mathfrak{g}}_-)$. The induced topology on $S(\widehat{\mathfrak{g}}_-)$ is discrete. Thus, $S(\widehat{\mathfrak{g}}_-)$ is generated by $\{J_{a,-n} | a = 1, ..., d, n \in \mathbb{Z}\}$ as an algebra and spanned as a $\mathbb{C}$-vector space by

$$J_{a_1,-n_1} \cdots J_{a_k,-n_k}$$

where $a_1, ..., a_k \in \{1, ..., d\}$ and $n_1, ..., n_k \in \mathbb{Z}_{>0}$. For a homogeneous element $A \in S(\widehat{\mathfrak{g}}_-)$

$$A = A_{1,-n_1} \cdots A_{k,-n_k}$$

where $A_1, ..., A_k \in \mathfrak{g}$ and $n_1, ..., n_k \in \mathbb{Z}_{>0}$, we say the order of $A$ is $k$, denoted by $\mathrm{ord}(A)$, and the degree of $A$ is $n_1 + \cdots n_k$, denoted by $\deg(A)$. Then both order and degree define graded structures on $S(\widehat{\mathfrak{g}}_-)$.

The derivative $T : S(\widehat{\mathfrak{g}}_-) \longrightarrow S(\widehat{\mathfrak{g}}_-)$ is a linear map defined by $T = -\partial_t$ , i.e.

$$T(A_1 \otimes t^{n_1} \cdots A_k \otimes t^{n_k}) = -\sum_{i=1}^{k} A_1 \otimes t^{n_1} \cdots A_i \otimes \partial_t(t^{n_i}) \cdots A_k \otimes t^{n_k}$$

Now we define an algebra injection

$$\iota : S(\mathfrak{g}) \longrightarrow S(\widehat{\mathfrak{g}}_-) \qquad A_1 \cdots A_k \longmapsto A_{1,-1} \cdots A_{k,-1}$$

This morphism is order preserving. However, this morphism does not preserve Poisson bracket. For example, if $A, B \in \mathfrak{g}$,

$$\iota(\{A, B\}) = [A, B]_{-1} \qquad \{\iota(A), \iota(B)\} = \{A_{-1}, B_{-1}\} = [A, B]_{-2}$$

So, $\iota(\{A, B\}) \neq \{\iota(A), \iota(B)\}$.

By theorem, we have

$$Z(\mathfrak{g}) = \mathbb{C}[C_1, ..., C_r]$$

where $C_i$ is homogeneous of order $d_i + 1$, and $C_1$ is the Casimir element. Let $\overline{S_i}$ be the image of $C_i$ under the projection $U(\mathfrak{g}) \longrightarrow S(\mathfrak{g})$. Then $\overline{S_1}, ..., \overline{S_r}$ generate the Poisson center of $S(\mathfrak{g})$. Let $S_i = \iota(\overline{S_i}) \in S(\widehat{\mathfrak{g}}_-)$, $i = 1, ..., r$. Then $S_i$ is homogeneous of order $d_i + 1$ and degree $d_i + 1$. In particular,

$$S_i = \iota\left(\frac{1}{2}\sum_{a=1}^{d} J_a J^a\right) = \frac{1}{2}\sum_{a=1}^{d} J_{a,-1} J^a_{-1}$$

Now we are ready to state the classical version.

**Theorem 10.1.** *1. The algebra below is a maximal Poisson commutative algebra of $S(\widehat{\mathfrak{g}}_-)$.*

$$\mathbb{C}[T^n S_i]_{n\geq 0}^{i=1,...,r}$$

*2. $\{T^n S_i | n \geq 0, i = 1, ..., r\}$ are algebraically independent elements.*

*3. This algebra is the Poisson centralizer of $S_1$.*

It's natural to ask if this algebra is the Poisson center of $S(\widehat{\mathfrak{g}}_-)$, i.e. it Poisson commutes with all elements in $S(\widehat{\mathfrak{g}}_-)$. The answer is No! A counterexample is given by $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$, $S_1 = e_{-1}f_{-1} + f_{-1}e_{-1} + \frac{1}{2}h_{-1}^2$ and $e_{-1}$:

$$\begin{aligned}\{S_1, e_{-1}\} &= \{e_{-1}f_{-1}, e_{-1}\} + \{f_{-1}e_{-1}, e_{-1}\} + \frac{1}{2}\{h_{-1}^2, e_{-1}\} \\ &= -e_{-1}h_{-2} - h_{-2}e_{-1} + 2h_{-1}e_{-2} \\ &= 2(e_{-2}h_{-1} - h_{-2}e_{-1}) \neq 0\end{aligned}$$

This example also shows how distinct Poisson structures between $S(\mathfrak{g})$ and $S(\widehat{\mathfrak{g}}_-)$ are.

# 11 Proof for the classical center theorem I

The most important part of theorem 10.1 is to show that $\{T^n S_i | n \geq 0, i = 1, ..., r\}$ mutually Poisson commute. Apparently, we would not prove by direct calculation. We should develop a method to combine all $\{T^n S_i | n \geq 0, i = 1, ..., r\}$. Let's first consider

$$S_1 = \frac{1}{2}\sum_{a=1}^{d} J_{a,-1} J^a_{-1}$$

By calculation, one can find that

$$T^n S_1 = \frac{n!}{2}\sum_{a=1}^{d}\sum_{\substack{k+l=-n-2 \\ k,l\leq -1}} J_{a,k} J^a_l$$

For $A \in \mathfrak{g}$, if we define

$$A(z) = \sum_{n\leq -1} A_n z^{-n-1} \in S(\widehat{\mathfrak{g}}_-)[[z]]$$

then

$$\frac{1}{2}\sum_{a=1}^{d} J_a(z)J^a(z) = \sum_{n=0}^{\infty}\left(\frac{1}{2}\sum_{a=1}^{d}\sum_{\substack{k+l=-n-2\\k,l\leq -1}} J_{a,k}J^a_l\right)z^n$$
$$= \sum_{n=0}^{\infty}\left(\frac{1}{n!}T^n S_1\right)z^n$$

Everything seems perfect. Nevertheless, we will see later that it is more useful to include the negative part. Hence, for $A \in \mathfrak{g}$, we redefine

$$A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1} \in S(\widehat{\mathfrak{g}})[[z^{\pm}]]$$

Naturally, for $A, B \in \mathfrak{g}$, we would like to define the product of $A(z)$ and $B(z)$ by

$$A(z)B(z) = \sum_{n\in\mathbb{Z}}\left(\sum_{k+l=n} A_k B_l\right)z^{-n-2}$$

However, $\sum_{k+l=n} A_k B_l$ is not a well-defined element in $S(\widehat{\mathfrak{g}})$. (Recall that in 10.1, all subscripts are bounded below). To fix this problem, we should consider a larger algebra. For $n \geq 0$, define $I_n = (\mathfrak{g} \otimes t^n\mathbb{C}[[t]])$, i.e. an ideal of $S(\widehat{\mathfrak{g}})$ generated by $\mathfrak{g}\otimes t^n\mathbb{C}[[t]]$. Define

$$\widetilde{S}(\widehat{\mathfrak{g}}) := \varprojlim_{n} S(\widehat{\mathfrak{g}})/I_n$$

the $I$-adic completion of $S(\widehat{\mathfrak{g}})$. Every element in $\widetilde{S}(\widehat{\mathfrak{g}})$ can be expressed as

$$\sum_{n\geq 0} S_n \cdot A^{(n)}_n \qquad S_n \in S(\widehat{\mathfrak{g}}) \text{ and } A^{(n)} \in \mathfrak{g}$$

**Lemma 11.1.** *The Poisson bracket on $S(\widehat{\mathfrak{g}})$ can be lifted to $\widetilde{S}(\widehat{\mathfrak{g}})$.*

*Proof.* Consider two elements in $\widetilde{S}(\widehat{\mathfrak{g}})$:

$$A = \sum_{n\geq 0} S_n \cdot A^{(n)}_n \qquad A' = \sum_{n\geq 0} S'_n \cdot A'^{(n)}_n$$

Define the Poisson bracket by

$$\{A, A'\} = \sum_{n\geq 0}\sum_{m\geq 0}\{S_n A^{(n)}_n, S'_m A'^{(m)}_m\}$$

we need to verify that $\{A, A'\} \in \widetilde{S}(\widehat{\mathfrak{g}})$, i.e. for every $N \in \mathbb{Z}_{>0}$, $\{A, A'\} \mod I_N$ can be represented by an element in $S(\widehat{\mathfrak{g}})$. Note that

$$\begin{aligned} c_{n,m} :=& \{S_n A^{(n)}_n, S'_m A'^{(m)}_m\} \\ =& \{S_n, S'_m\}A^{(n)}_n A'^{(m)}_m + \{S_n, A'^{(m)}_m\}A^{(n)}_n S'_m \\ &+ \{A^{(n)}_n, S'_m\}S_n A'^{(m)}_m + [A^{(n)}, A^{(m)}]_{n+m} S_n S'_m \end{aligned}$$

If $\min\{n, m\} \geq N$, then $c_{n,m} \in I_N$. If $n < N$, then there exists a $N_n > N$ such that $\{S_n, A_m^{\prime(m)}\} \in I_N$ and thereby $C_{n,m} \in I_N$ for all $m > N_n$. Similarly, for any $m < N$, then there exists a $\widetilde{N}_m > N$ such that $C_{n,m} \in I_N$ for all $n > \widetilde{N}_m$. Thus,

$$\{A, A'\} \equiv \sum_{n\geq 0}^{N} \sum_{m\geq 0}^{N_n} c_{n,m} + \sum_{m\geq 0}^{N} \sum_{n\geq 0}^{\widetilde{N}_m} c_{n,m} \mod I_N$$

The right hand side is an element in $S(\widehat{g})$. Therefore, the Poisson bracket is well-defined. $\square$

We conclude that $\widetilde{S}(\widehat{\mathfrak{g}})$ is a topological Poisson algebra.

**Lemma 11.2.** *Let $A_1, ..., A_k \in \mathfrak{g}$, then*

$$A_1(z) \cdots A_k(z) = \sum_{n\in\mathbb{Z}} \left( \sum_{n_1+\cdots+n_k=n} A_{1,n_1} \cdots A_{k,n_k} \right) z^{-n-k}$$

*is a well-defined element in $\widetilde{S}(\widehat{\mathfrak{g}})[[z^{\pm 1}]]$*

*Proof.* We only need to prove that

$$A_n = \sum_{n_1+\cdots+n_k=n} A_{1,n_1} \cdots A_{k,n_k} \in \widetilde{S}(\widehat{\mathfrak{g}})$$

Note that for each $N \in \mathbb{Z}_{>0}$, there are only finite many $(n_1, ..., n_k) \in \mathbb{Z}^k$ such that $n_1 + \cdots + n_k = n$ and $n_i < N$ for $i = 1, ..., k$. Thus $A_n$ is congruent to an element in $S(\widehat{\mathfrak{g}})$ modulo $I_N$. Therefore, $A_n \in \widetilde{S}(\widehat{\mathfrak{g}})$. $\square$

In general, for $F(z), G(z) \in \widetilde{S}(\widehat{\mathfrak{g}})[[z^{\pm 1}]]$, $F(z)G(z)$ is not necessarily well-defined. So, $\widetilde{S}(\widehat{\mathfrak{g}})[[z^{\pm 1}]]$ is not an algebra. Still, we can construct a multiplicative morphism $\Phi_z$:

$$\Phi_z : S(\mathfrak{g}) \longrightarrow \widetilde{S}(\widehat{\mathfrak{g}})[[z^{\pm 1}]] \qquad A_1 \cdots A_k \longmapsto A_1(z) \cdots A_k(z)$$

By lemma 11.2, this map is well-defined. In particular, for Casimer element $\overline{S}_1 = \frac{1}{2} \sum_{a=1}^{d} J_a J^a$

$$\Phi_z(\overline{S}_1) = S_1(z) = \frac{1}{2} \sum_{a=1}^{d} J_a(z) J^a(z) = \frac{1}{2} \sum_{n\in\mathbb{Z}} \left( \sum_{a=1}^{d} \sum_{k+l=n} J_{a,k} J_l^a \right) z^{-n-2}$$

Let $\overline{S}_1, ..., \overline{S}_r$ be generators of Poisson center of $S(\mathfrak{g})$. Denote

$$S_i(z) := \Phi_z(\overline{S}_i) = \sum_{m\in\mathbb{Z}} s_{i,m} z^m$$

**Proposition 11.3.** *$s_{i,m}$ lies in the Poisson center of $\widetilde{S}(\widehat{\mathfrak{g}})$ for any $i = 1, ..., r$ and $m \in \mathbb{Z}$, i.e.*

$$\{s_{i,m}, J_{a,n}\} = 0 \qquad \textit{for any } n \in \mathbb{Z}, a = 1, ..., d$$

To prove this proposition, we need some preparations. Let $z, w$ be distinct indeterminates and

$$f(z) = \sum_{n\in\mathbb{Z}} f_n z^{-n-1} \in \widetilde{S}(\widehat{\mathfrak{g}})[[z^{\pm 1}]] \qquad g(w) = \sum_{n\in\mathbb{Z}} g_n z^{-n-1} \in \widetilde{S}(\widehat{\mathfrak{g}})[[w^{\pm 1}]]$$

Define their product and Poisson bracket by

$$\begin{aligned} f(z)g(z) &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} f_n g_m z^{-n-1} w^{-m-1} \\ \{f(z), g(w)\} &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \{f_n, g_m\} z^{-n-1} w^{-m-1} \end{aligned}$$

In particular, for $A, B \in \mathfrak{g}$, the Poisson bracket of $A(z)$ and $B(w)$ is given by

$$\begin{aligned} \{A(z), B(w)\} &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} \{A_n, B_m\} z^{-n-1} w^{-m-1} \\ &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} [A, B]_{n+m} z^{-n-1} w^{-m-1} \\ &= \sum_{N\in\mathbb{Z}} [A, B]_N w^{-N-1} \cdot \sum_{n\in\mathbb{Z}} z^{-n-1} w^n \\ &= [A, B](w) \cdot \delta(z-w) \end{aligned}$$

where $\delta(z-w)$ is defined by

$$\delta(z-w) \coloneqq \sum_{n\in\mathbb{Z}} z^n w^{-n-1} = \sum_{n\in\mathbb{Z}} w^n z^{-n-1} = \sum_{k+l=-1} z^k w^l$$

$\delta(z-w)$ should be viewed as a distribution on the line $z = w$. We would discuss more about formal distributions in section 18. Here we just show one important property of $\delta(z-w)$.

**Lemma 11.4.** *Let $R$ be a ring and $f(z) \in R[[z^{\pm 1}]]$, then*

$$f(z)\delta(z-w) = f(w)\delta(z-w)$$

*Proof.*

$$\begin{aligned} f(z)\delta(z-w) &= \sum_{n\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} a_n z^n z^m w^{-m-1} = \sum_{n\in\mathbb{Z}} a_n \sum_{k+l=n-1} z^k w^l = \sum_{n\in\mathbb{Z}} a_n w^n \sum_{k+l=-1} z^k w^l \\ &= f(w)\delta(z-w) \end{aligned}$$

□

**Proposition 11.5.** *Let $F_1(z), ..., F_k(z) \in \widetilde{S}(\widehat{\mathfrak{g}})[[z^{\pm 1}]]$ and $G_1(z), ..., G_l(z) \in \widetilde{S}(\widehat{\mathfrak{g}})[[w^{\pm 1}]]$. Suppose any product of $F_1(z), ..., F_k(z)$ belongs to $\widetilde{S}(\widehat{\mathfrak{g}})[[z^{\pm 1}]]$ and any product of $G_1(z), ..., G_l(z)$ belongs to $\widetilde{S}(\widehat{\mathfrak{g}})[[w^{\pm 1}]]$, then*

$$\{F_1(z)\cdots F_k(z), G_1(w)\cdots G_l(w)\} = \sum_{i=1}^{k}\sum_{j=1}^{l} \{F_i(z), G_j(w)\} \prod_{p\neq i} F_p(z) \prod_{q\neq j} G_q(w)$$

*In particular,*

$$\{F_1(z)\cdots F_k(z), G(w)\} = \sum_{i=1}^{k} \{F_i(z), G(w)\} \prod_{p\neq i} F_p(z)$$

*Proof.* We only need to prove the case when $l = 1$. Assume $F_i(z) = \sum_{n\in\mathbb{Z}} f_{i,n} z^{-n-1}$, $i = 1, ..., k$, and $G(z) = \sum_{n\in\mathbb{Z}} g_n z^{-n-1}$. Then

$$\begin{aligned}
\{F_1(z)\cdots F_k(z), G(w)\} &= \sum_{n,m\in\mathbb{Z}} \sum_{n_1+\cdots n_k=n} \{f_{1,n_1}\cdots f_{k,n_k}, g_m\} z^{-n-k} w^{-m-1} \\
&= \sum_{n,m\in\mathbb{Z}} \sum_{n_1+\cdots n_k=n} \sum_{i=1}^{k} f_{1,n_1}\cdots \{f_{i,n_i}, g_m\} \cdots f_{k,n_k} z^{-n-k} w^{-m-1} \\
&= \sum_{i=1}^{k} \sum_{n,m\in\mathbb{Z}} \sum_{n_1+\cdots n_k=n} f_{1,n_1}\cdots \widehat{f_{i,n_i}} \cdots f_{k,n_k} z^{-(n-n_i)-(k-1)} \cdot \{f_{i,n_i}, g_m\} z^{-n_i-1} w^{-m-1} \\
&= \sum_{i=1}^{k} \sum_{n\in\mathbb{Z}} \sum_{n_1+\cdots\widehat{n}_i+\cdots n_k=n} \prod_{j\neq i} f_{j,n_j} z^{-n-(k-1)} \cdot \sum_{n_i\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \{f_{i,n_i}, g_m\} z^{-n_i-1} w^{-m-1} \\
&= \sum_{i=1}^{k} \{F_i(z), G(w)\} \prod_{p\neq i} F_p(z)
\end{aligned}$$

□

**Corollary 11.6.** *Let $A_1, ..., A_k$, $B_1, ..., B_l \in \mathfrak{g}$, then*

$$\{A_1(z)\cdots A_k(z), B_1(w)\cdots B_l(w)\} = \Phi_w\left(\{A_1\cdots A_k, B_1\cdots B_l\}\right)\delta(z-w)$$

*Proof.* By lemma 11.2, $A_1(z), ..., A_k(z)$ and $B_1(w), ..., B_l(w)$ satisfy conditions in proposition 11.5. Thus,

$$\begin{aligned}
\{A_1(z)\cdots A_k(z), B_1(w)\cdots B_l(w)\} &= \sum_{i=1}^{k}\sum_{j=1}^{l} \{A_i(z), B_j(w)\} \prod_{p\neq i} A_p(z) \prod_{q\neq j} B_q(w) \\
&= \sum_{i=1}^{k}\sum_{j=1}^{l} [A_i, B_j](w)\delta(z-w) \prod_{p\neq i} A_p(z) \prod_{q\neq j} B_q(w) \\
&= \delta(z-w) \sum_{i=1}^{k}\sum_{j=1}^{l} [A_i, B_j](w) \prod_{p\neq i} A_p(w) \prod_{q\neq j} B_q(w) \\
&= \delta(z-w)\Phi_w\left(\{A_1\cdots A_k, B_1\cdots B_l\}\right)
\end{aligned}$$

□

**Corollary 11.7.** *For any $C, D \in S(\mathfrak{g})$,*

$$\{\Phi_z(C), \Phi_w(D)\} = \Phi_w\left(\{C, D\}\right)\delta(z-w)$$

*proof of proposition 11.3:* We only need to prove that for any $i = 1, ..., r$, $a = 1, ..., d$,

$$\{S_i(z), J_a(w)\} = \sum_{n,m\in\mathbb{Z}} \{s_{i,n}, J_{a,m}\} z^n w^{-m-1} = 0$$

By corollary 11.7,

$$\begin{aligned}
\{S_i(z), J_a(w)\} &= \{\Phi_z(\overline{S}_i), \Phi_w(J_a)\} \\
&= \Phi_w\left(\{\overline{S}_i, J_a\}\right)\delta(z-w) \\
&= 0
\end{aligned}$$

□

# 12 Proof for the classical center theorem II

The most significant result we obtain in last section is: for any $i = 1, ..., r$, $a = 1, ..., d$, $n, m \in \mathbb{Z}$,

$$\{s_{i,m}, J_{a,n}\} = 0$$

In particular, for any $i, j = 1, ..., r$, $n, m \in \mathbb{Z}$,

$$\{s_{i,m}, s_{j,n}\} = 0$$

In this section, we would use this result to complete the proof. Note that $S(\widehat{\mathfrak{g}}_-) \cong S(\widehat{g})/(\widehat{\mathfrak{g}}_+) \cong \widetilde{S}(\widehat{\mathfrak{g}})/(\widehat{\mathfrak{g}}_+)$. Let $\pi$ be the projection

$$\pi : \widetilde{S}(\widehat{\mathfrak{g}}) \longrightarrow \widetilde{S}(\widehat{\mathfrak{g}})/(\widehat{\mathfrak{g}}_+) \xrightarrow{\sim} S(\widehat{\mathfrak{g}}_-)$$

This projection is an algebra homomorphism. For instance,

$$s_{i,m} = \frac{1}{2} \sum_{k+l=-m-2} J_{a,k} J^a_l$$

For $m \geq 0$,

$$\pi(s_{1,m}) = \frac{1}{2} \sum_{\substack{k+l=-m-2 \\ k,l \leq -1}} J_{a,k} J^a_l = \frac{1}{m!} T^m \mathcal{S}$$

In fact, this follows from a general result

**Lemma 12.1.** $X \in S(\mathfrak{g})$ *is a homogeneous element,*

$$\Phi_z(X) = \sum_{n \in \mathbb{Z}} x_n z^n$$

*Then for* $m \geq 0$

$$\pi(x_m) = \frac{1}{m!} T^n \left(i(X)\right)$$

*Proof.* We only need to prove the case when $X = X_1 \cdots X_k$, $X_1, ..., X_k \in \mathfrak{g}$.

$$\begin{aligned}
\Phi_z(x) &= X_1(z) \cdots X_k(z) \\
&= \sum_{n \in \mathbb{Z}} \left( \sum_{n_1 + \cdots + n_k = n} X_{1,n_1} \cdots X_{k,n_k} \right) z^{-n-k} \\
&= \sum_{n \in \mathbb{Z}} \left( \sum_{n_1 + \cdots + n_k = -n-k} X_{1,n_1} \cdots X_{k,n_k} \right) z^{n}
\end{aligned}$$

Thus,

$$\pi(x_m) = \sum_{\substack{n_1 + \cdots + n_k = -m-k \\ n_1, ..., n_k \leq -1}} X_{1,n_1} \cdots X_{k,n_k}$$

On the other hand,

$$\begin{aligned}
\frac{1}{m!}T^n\left(i(X)\right) &= \frac{1}{m!}T^m(X_{1,-1}\cdots X_{k,-1}) \\
&= \frac{1}{m!}\sum_{\substack{n_1+\cdots n_k=m \\ n_1,\ldots,n_k\geq 0}} \begin{pmatrix} m \\ n_1\cdots n_k \end{pmatrix} (T^{n_i}X_{1,-1})\cdots(T^{n_k}X_{k,-1}) \\
&= \sum_{\substack{n_1+\cdots n_k=m \\ n_1,\ldots,n_k\geq 0}} \begin{pmatrix} m \\ n_1\cdots n_k \end{pmatrix} \frac{n_1!\cdots n_k!}{m!} X_{1,-1-n_1}\cdots X_{k,-1-n_k} \\
&= \sum_{\substack{n_1+\cdots+n_k=-m-k \\ n_1,\ldots,n_k\leq -1}} X_{1,n_1}\cdots X_{k,n_k}
\end{aligned}$$

Thus,

$$\pi(x_m) = \frac{1}{m!}T^n\left(i(X)\right)$$

□

Therefore, we have

$$\pi(s_{i,m}) = \frac{1}{m!}T^m(S_i)$$

We know that $\{s_{i,m}|m\in\mathbb{Z}, i=1,...,r\}$ mutually Poisson commute. If the projection map $\pi$ preserves Poisson structure, then we are done. However, it turn out that $\pi$ does not preserve Poisson structure. For example, take $A,B\in\mathfrak{g}$, such that $[A,B]\neq 0$, then

$$\pi(\{A_1,B_{-2}\}) = \pi([A,B]_{-1}) = [A,B]_{-1} \qquad \{\pi(A_1),\pi(B_{-2})\} = 0$$

In other words, $(\widehat{\mathfrak{g}}_+)$ is not a Poisson ideal of $\widetilde{S}(\widehat{\mathfrak{g}})$. Hence, we have to take more efforts to finish the proof.

Denote

$$s_{i,m} = s^-_{i,m} + s^+_{i,m}$$

such that $s^-_{i,m}\in\widetilde{S}(\widehat{\mathfrak{g}})$ and $\pi(s_{i,m}) = s^-_{i,m}$. Then $\pi(s^+_{i,m}) = 0$, $s^+_{i,m}\in(\widehat{\mathfrak{g}}_+)$.

$$\begin{aligned}
0 &= \{s_{i,m}, s_{j,n}\} \\
&= \{s^-_{i,m}, s^-_{j,n}\} + \{s^-_{i,m}, s^+_{j,n}\} + \{s^+_{i,m}, s^-_{j,n}\} + \{s^+_{i,m}, s^+_{j,n}\} \\
&= \{s^-_{i,m}, s^-_{j,n}\} + \{s_{i,m}, s^+_{j,n}\} + \{s^+_{i,m}, s_{j,n}\} - \{s^+_{i,m}, s^+_{j,n}\} \\
&= \{s^-_{i,m}, s^-_{j,n}\} - \{s^+_{i,m}, s^+_{j,n}\}
\end{aligned}$$

Here we use the fact that $s_{i,m}, s_{j,n}$ Poisson commute with any elements. Since $s^+_{i,n}, s^+_{j,m}\in(\widehat{\mathfrak{g}}_+)$, so is $\{s^+_{i,m}, s^+_{j,n}\}$. Thus, $\pi(\{s^+_{i,m}, s^+_{j,n}\}) = 0$. Therefore, we conclude that

$$\{s^-_{i,m}, s^-_{j,n}\} = 0$$

# 13 From classical version to quantum version

Let's briefly summarize what we achieve in the last three sections.

1. We construct an algebra homomorphism

$$\iota : S(\mathfrak{g}) \longrightarrow S(\widehat{\mathfrak{g}}_-) A \longmapsto A_{-1}$$

   and we claim that $\mathbb{C}[T^n \iota(\overline{S}_i)] = \mathbb{C}[T^n S_i]$ is a Poisson commutative subalgebra of $S(\widehat{\mathfrak{g}}_-)$.

2. To show this theorem, we construct a multiplicative homomorphism

$$\Phi_z : S(\mathfrak{g}) \longrightarrow \widetilde{S}(\widehat{\mathfrak{g}})[[z^{\pm 1}]] \qquad A \longmapsto A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1}$$

   and we prove that all coefficients of $\Phi_z(S_i) = S_i(z)$ lies in the Poisson center of $\widetilde{S}(\widehat{\mathfrak{g}})$.

3. Compositing the map $\Phi_z$ with a projection

$$\pi : \widetilde{S}(\widehat{\mathfrak{g}}) \longrightarrow S(\widehat{\mathfrak{g}}_-)$$

   we find that the non negative coefficient of $S_i(z)$ are exactly $\{T^n S_i\}$. By using the fact that all coefficients of $\Phi_z(S_i) = S_i(z)$ lies in the Poisson center of $\widetilde{S}(\widehat{\mathfrak{g}})$, we finally show that $T^n S_i$ commute with each other.

Consider the space

$$\mathcal{P} = \mathbb{C}[\partial_z^n A(z)]_{n \geq 0, A \in \mathfrak{g}}$$

It is easy to show that $\mathcal{P}$ is a subalgebra of $\widetilde{S}(\widehat{\mathfrak{g}})$ (That is, the product of two power series in $\mathcal{P}$ still has coefficients in $\widetilde{S}(\widehat{\mathfrak{g}})$). Then the morphism $\Phi_z$ actually is an algebra homomorphism from $S(\mathfrak{g})$ to $\mathcal{P}$. In fact, this map can be lift to an isomorphism of algebra between $S(\widehat{\mathfrak{g}}_-)$ and $\mathcal{P}$ .

**Proposition 13.1.** *The morphism $\Phi_z$ can be uniquely lifted to an algebra homomorphism $\Phi$ between $S(\widehat{\mathfrak{g}}_-)$ and $\mathcal{P}$, which is compatible with derivatives, i.e. $\Phi \circ T = \partial_z \circ \Phi$.*

$$\begin{tikzcd}
S(\widehat{\mathfrak{g}}_-) \arrow[rr, "\Phi"] & & \mathcal{P} \arrow[r, hook] & \widetilde{S}(\widehat{\mathfrak{g}})[[z^{\pm 1}]] \\
& S(\mathfrak{g}) \arrow[ul, "\iota"] \arrow[ur, "\Phi_z"'] & &
\end{tikzcd}$$

*Proof.* Since $A_{-n-1} = \frac{1}{n!} T^n A_{-1}$, we can only define

$$\Phi(A_{-n-1}) = \frac{1}{n!} \partial_z^n A(z) \qquad n \geq 0$$

$S(\widehat{\mathfrak{g}}_-)$ is an algebra generated by $\{A_{-n-1} | A \in \mathfrak{g} \text{ and } n \geq 0\}$. Therefore, This induce an algebra isomorphism between $S(\widehat{\mathfrak{g}}_-)$ and $\mathcal{P}$. □

Define $Y(, z)$ to be the composition of $\Phi$ and projection $\pi : \widetilde{S}(\widehat{\mathfrak{g}})[[z^{\pm 1}]] \longrightarrow S(\widehat{\mathfrak{g}}_-)$, i.e.

$$Y(, z) = \pi \circ \Phi : S(\widehat{\mathfrak{g}}_-) \longrightarrow S(\widehat{\mathfrak{g}}_-)[[z^{\pm 1}]]$$

**Proposition 13.2.** *For any $X \in S(\widehat{\mathfrak{g}}_-)$,*

$$Y(X,z) = e^{zT}X = \sum_{n\geq 0} \frac{1}{n!}(T^n X)z^n$$

*Proof.* For any $X, Y \in S(\widehat{\mathfrak{g}}_-)$, since $T$ is a derivative on $S(\widehat{\mathfrak{g}}_-)$, we can show that

$$e^{zT}(X \cdot Y) = \left(e^{zT}X\right)\left(e^{zT}Y\right)$$

Thus, we only need to show this proposition for $X = A_{-n-1}$, where $A \in \mathfrak{g}$ and $n \geq 0$. For $n = 0$,

$$Y(A_{-1}, z) = \sum_{n\geq 0} A_{-n-1}z^n = \sum_{n\geq 0} \frac{1}{n!}T^n A_{-1}z^n = e^{zT}A_{-1}$$

for arbitrary $n > 0$,

$$Y(A_{-n-1}, z) = \frac{1}{n!}Y(T^n A_{-1}, z) = \frac{1}{n!}\partial_z^n Y(A_{-1}, z) = \frac{1}{n!}\partial_z^n e^{zT}A_{-1} = e^{zT}A_{-n-1}$$

□

In summary, for commutative algebra $S(\widehat{\mathfrak{g}}_-)$, we define a map

$$\begin{tikzcd} S(\widehat{\mathfrak{g}}_-) \arrow[rr, "Y(,z)"] \arrow[dr, "\Phi"'] & & \widetilde{S}(\widehat{\mathfrak{g}}_-)[[z^{\pm 1}]] \\ & \widetilde{S}(\widehat{\mathfrak{g}})[[z^{\pm 1}]] \arrow[ur, "\pi"'] & \end{tikzcd}$$

satiefies

$$Y(X,z) = e^{zT}X \qquad X \in S(\widehat{\mathfrak{g}}_-)$$

In fact, an vector space endowed with such a structure is called a commutative vertex algebra, the simplest kind of vertex algebra. This kind of construction would also appear in the proof of quantum version of the center theorem, i.e.

$$\begin{tikzcd} U(\widehat{\mathfrak{g}}_-) \arrow[rr, "Y(,z)"] \arrow[dr, "\Phi"'] & & \mathrm{End}\left(\widetilde{U}(\widehat{\mathfrak{g}}_-)\right)[[z^{\pm 1}]] \\ & \widetilde{U}_\kappa(\mathfrak{g})[[z^{\pm 1}]] \arrow[ur, "\pi"'] & \end{tikzcd}$$

Starting from the next chapter, we will begin discussing the quantum version of center theorem. The quantum center theorem describes the center of affine Kac-Moody vertex algebra $V_\kappa(\mathfrak{g})$. This time, the algebra $U(\widehat{\mathfrak{g}}_-)$ is non commutative and hence it would takes more effort to study its vertex algebra structure. The vertex algebra $V_\kappa(\mathfrak{g})$ is isomorphic to $U(\widehat{\mathfrak{g}}_-)$ as $U(\widehat{\mathfrak{g}}_-)$-module. The center of this vertex algebra is isomorphic to a commutative subalgebra of $U(\widehat{\mathfrak{g}}_-)$. Therefore, we would obtain theorem 7.4 as a corollary.

We will provide a precise statement of the quantum center theorem. However, we are unable to provide a complete proof of this theorem as it involves much more sophisticated knowledge.

# 14 Central extension

In following discussion, we would frequently encounter Lie algebras with central extensions. Hence, in this section, let's try to explain why central extensions appear naturally in quantum case.

Recall that a symmetry of a quantum system $\mathbb{H}$ is a projective unitary representation

$$G \longrightarrow U(\mathbb{PH})$$

Apparently, $\mathbb{PH}$ is not a good space to study. So we ask if we can lift this $G$-action to a linear space, i.e. can we find a group homomorphism $\gamma$ such that the diagram below commutes

$$\begin{tikzcd} & & & G \arrow[dl, dashed, "\gamma"'] \arrow[d] & \\ 1 \arrow[r] & U(\mathbb{C}) \arrow[r] & U(\mathbb{H}) \arrow[r] & U(\mathbb{PH}) \arrow[r] & 1 \end{tikzcd}$$

Generally, the answer is No! For example, we know that the irreducible representation $V_{2n+1}$ of $SU(2)$ can not be restricted to a representation of $SO(3)$. Thus, there is no lift for this diagram

$$\begin{tikzcd} & & & SO(3) \arrow[d] & \\ 1 \arrow[r] & U(\mathbb{C}) \arrow[r] & U(V_{2n+1}) \arrow[r] & U(\mathbb{P}V_{2n+1}) \arrow[r] & 1 \end{tikzcd}$$

**Proposition 14.1.** *Let $G$ be a group and $T : G \longrightarrow U(\mathbb{PH})$ be a homomorphism. Then there exists a central extension $\widehat{G}$ of $G$ by $U(\mathbb{C})$ and a homomorphism $\widehat{T} : \widehat{G} \longrightarrow U(\mathbb{H})$, so that the following diagram commutes:*

$$\begin{tikzcd} 1 \arrow[r] & U(\mathbb{C}) \arrow[r] \arrow[d, "Id"'] & \widehat{G} \arrow[r, "\widehat{\pi}"] \arrow[d, "\widehat{T}"'] & G \arrow[r] \arrow[d, "T"'] & 1 \\ 1 \arrow[r] & U(\mathbb{C}) \arrow[r] & U(\mathbb{H}) \arrow[r, "\pi"] & U(\mathbb{PH}) \arrow[r] & 1 \end{tikzcd}$$

*Proof.* Let $\widehat{G}$ be the pull back of $\pi$ and $T$, i.e.

$$\widehat{G} = \{(\rho, g) | \rho \in U(\mathbb{H}), g \in G, \pi(\rho) = T(g)\}$$

Let $\widehat{\pi} : \widehat{G} \longrightarrow G$ and $\widehat{T} : \widehat{G} \longrightarrow U(\mathbb{H})$ be projections. It's easy to see that the kernel of $\widetilde{\pi}$ is

$$\{(\lambda \cdot \mathrm{Id}, 1) | \lambda \in U(\mathbb{C})\} \cong U(\mathbb{C})$$

$\square$

Moreover, if $G$ is a Lie group, then we can prove that $\widehat{G}$ is also a Lie group and the exact sequence

$$1 \longrightarrow U(\mathbb{C}) \longrightarrow \widehat{G} \longrightarrow G \longrightarrow 1$$

are smooth homomorphism between Lie groups. Take derivatives, we obtain a diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d, "\mathrm{Id}"'] & \widehat{\mathfrak{g}} \arrow[r, "\widehat{\pi}"] \arrow[d, "\widehat{T}"'] & \mathfrak{g} \arrow[r] \arrow[d, "T"'] & 0 \\ 0 \arrow[r] & \mathbb{R} \arrow[r] & \mathfrak{u}(\mathbb{H}) \arrow[r, "\pi"] & \mathfrak{u}(\mathbb{PH}) \arrow[r] & 0 \end{tikzcd}$$

Since $U(1)$ lies in the center of $\widehat{G}$, its Lie algebra $\mathbb{R}$ also lies in the center of $\widehat{\mathfrak{g}}$. Therefore,

$$0 \longrightarrow \mathbb{R} \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

is a central extension of Lie algebra. Therefore, in quantum case, we should study how central extension of a Lie algebra acts on the Hilbert space.

At last, Let's discuss what kinds of projective unitary representation can be directly lifted to an unitary representation. One possible case is when the central extension of group splits.

**Theorem 14.2.** *Let $G$ be a connected and simply connected finite dimensional Lie group with Lie algebra $\mathfrak{g}$ such that*

$$H^2(\mathfrak{g}, \mathbb{R}) = 0$$

*then every projective representation $T : G \longrightarrow U(\mathbb{PH})$ can be lifted to a unitary representation $T' : G \longrightarrow U(\mathbb{H})$.*

*Proof.* By proposition 14.1, this projective representation can be lifted to a unitary representation $\widehat{T} : \widehat{G} \longrightarrow U(\mathbb{H})$, where $\widehat{G}$ is a central extension of $G$. The corresponding Lie algebra extension is given by

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r] & \widehat{\mathfrak{g}} \arrow[r, "\widetilde{\pi}"] & \mathfrak{g} \arrow[r] \arrow[l, bend right, "\iota"'] & 0 \end{tikzcd}$$

Since $H^2(\mathfrak{g}, \mathbb{R}) = 0$, this exact sequence splits. Thus, there exists a Lie algebra homomorphism $\iota : \mathfrak{g} \longrightarrow \widehat{\mathfrak{g}}$ such that $\widetilde{\pi} \circ \iota = \mathrm{Id}$. Since $G$ is a simply connected Lie algebra, by Lie theorem, $\iota$ can be lift to a map $\widehat{\iota} : G \longrightarrow \widehat{G}$. The derivative of $\widehat{\pi} \circ \widehat{\iota}$ is $\widetilde{\pi} \circ \iota = \mathrm{Id}$, thus the map $\widehat{\pi} \circ \widehat{\iota}$ is identity on $G$. Therefore, the homomorphism $\widetilde{T} = \widehat{T} \circ \widehat{\iota}$ is a lift of $T$, i.e. $\pi \circ \widetilde{T} = T$.

$$\begin{tikzcd} 1 \arrow[r] & U(\mathbb{C}) \arrow[r] \arrow[d, "\mathrm{Id}"'] & \widehat{G} \arrow[r, "\widehat{\pi}"] \arrow[d, "\widehat{T}"'] & G \arrow[r] \arrow[d, "T"'] \arrow[l, bend right, "\widehat{\iota}"'] \arrow[dl, "\widetilde{T}"'] & 1 \\ 1 \arrow[r] & U(\mathbb{C}) \arrow[r] & U(\mathbb{H}) \arrow[r, "\pi"] & U(\mathbb{PH}) \arrow[r] & 1 \end{tikzcd}$$

□

We know that for semi-simple Lie algebra $\mathfrak{g}$, $H^2(\mathfrak{g}, \mathbb{R}) = 0$. Therefore, a projective unitary representation of $G$ can always be lifted to a unitary representation, providing that $G$ is semi-simple and simply connected.

# 15 Quantum field theory

Up to now, we only study physical systems with finite degree of freedom, such as the $N$-planets system. We can describe such kind of systems by symplectic manifolds and functions on them. In physics, we also encounter systems with infinite degree of freedom, called fields. Roughly speaking, given a manifold $M$ and a finite dimensional $k$-vector space $V$, a field on $M$ assigning each point on $M$ with a vector in $V$, i.e. a continuous (smooth, $L^2$, ...) map $M \longrightarrow V$. If $V = k$, then these fields are called scalar fields.

**Example 15.1.** *Heat (Temperature) in space $\mathbb{R}^3$ is a scalar field. It assigns each point with a value, representing heat density (temperature) at that point.*

**Example 15.2.** *Electronmagnetic fields* $\mathbf{E}$ *and* $\mathbf{B}$ *assign each point in* $\mathbb{R}^3$ *with vectors* $\vec{E}$ *and* $\vec{B}$.

Fields, just like position and momentum, change over the time. The time-evolution of a field is governed by PDE. For example, the time-evolution of heat is

$$\frac{d}{dt}H(x,t) = k^2 \cdot \Delta H(x,t)$$

The time evolution of electronmagnetic fields are controlled by

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \cdot \mathbf{E} = 4\pi\rho \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \times \mathbf{B} = J + \frac{\partial \mathbf{E}}{\partial t}$$

Why do we say fields have infinite dimensions of freedom? Let's illustrate this point. First, Let's consider finite systems, for instance, the $N$-planets system. Denote the set of these $N$ planets by $S$. The phase space for each planet is $\mathbb{R}^2 = \{(x,p)|x \in \mathbb{R}, p \in \mathbb{R}\}$. A state of the whole system is given by combination of each planet's state. Therefore, the space of all states is

$$\text{Phase space} = \prod_{x \in S} \mathbb{R}^2 = \text{Map}(S, \mathbb{R}^2) \tag{15.1}$$

Now, let's consider fields over manifold $M$ with values in $V$. We really should view $V$ as phase space of a finite system. To be specific, $V$ represents the phase space of single point. For instance, in temperature field, $V = \mathbb{R}$ is the space of all possible temperature values at one point. In magnetic fields, $V = \mathbb{R}^3$ is the space of all possible magnetic vectors at one point. Hence, a field records information of every point's state. The space of all possible states should be the product of all possible states at each point, i.e.

$$\text{Phase space} = \prod_{x \in M} V_x = \text{Map}(M, V) \tag{15.2}$$

Comparing the Phase space 15.1 and the Phase space 15.2, it's easy to see that finite systems are fields over finite sets. The degree of freedom of a system can be viewed as the index set of phase space as a product space. Hence, fields, with a manifold as index set, have infinite dimensions of freedom. The following chart compares finite systems and fields.

| | Finite systems | Fields |
|---|---|---|
| Index set | $i = 1, ..., n$ | all $x \in M$ |
| Phase space | $\mathbb{R}^{2n}$ | $\text{Map}(M,V)$ or $L^2(M,V)$ |
| Coordinates (basis of phase space) | $x_1, ..., x_n, p_1, ..., p_n$ | a linear basis of the phase space |
| Hamiltonian | $\sum_{i=1}^{n} H_i$ | $\int_M H_x dx$ |

Since fields are classical objects, it's natural to ask if we can construct a quantum theory for it? This question is itself very vague because hardly can we give a precise definition of what does a quantum theory means. Let's just hold the believe that what we construct below is a quantum theory. Recall that in the quantization of finite systems, for each index $i$, the coordinate $x_i$ should be quantized to a self-adjoint operator $\widehat{x_i}$. We define quantization of field similarly. For each index $x$, the coordinate $\psi(x) \in \mathrm{Map}(M, V)$ should be quantize to a self-adjoint operator, denoted by $\Psi(x)$. In other words, given a coordinate field $\psi : M \longrightarrow V$, we should attach a self-adjoint operator to each $x \in M$. Hence, $\psi$ is quantized to a operator-valued function

$$\Psi : M \longrightarrow \text{Self-adjoint operators on } \mathbb{H}$$

However, we find that in some cases this operator-valued function would be in form of $\delta(x - x_0)$, which is not well-defined as a function. Therefore, instead of quantizing $\psi$ to an operator-valued function, we should quantize it to an operator-valued distribution, i.e. a $\mathbb{R}$-linear map

$$\Psi : \mathcal{S}(M) \longrightarrow \text{Self- adjoint operators on } \mathbb{H}$$

Note that the space of self-adjoint operators on $\mathbb{H}$ is a $\mathbb{R}$-vector space, that's why we only consider $\mathbb{R}$-linear distributions. On the other hand, it will be more convenient to allow complex multiplication on distribution. Hence, generally, we view this distribution as a $\mathbb{R}$-linear map

$$\Phi : \mathcal{S} \longrightarrow \mathcal{O}(\mathbb{H})$$

and keep in mind that its values are self-adjoint operators ($\mathcal{O}(\mathbb{H})$ is the space of all operators on $\mathbb{H}$).

**Lemma 15.1.** *Let $V, W$ be two $\mathbb{C}$-vector spaces and $\varphi : V \longrightarrow W$ be a $\mathbb{R}$-linear map, then there exists a $\mathbb{C}$-linear map $\varphi^+$ and an anti $\mathbb{C}$-linear map $\varphi^-$ such that*

$$\varphi = \varphi^+ + \varphi^-$$

*Proof.*

$$\varphi^+(v) = \frac{1}{2}\left(\varphi(v) - i\varphi(iv)\right) \qquad \varphi^-(v) = \frac{1}{2}\left(\varphi(v) + i\varphi(iv)\right)$$

□

As a consequence, we can decompose the field operator $\Psi$ as

$$\Psi = \Psi^+ + \Psi^-$$

where $\Psi^+$ is a $\mathbb{C}$-linear map $\Psi^-$ is an anti $\mathbb{C}$-linear map.

In most case, we would define $\Psi$ on a basis $\{f_i\}_{i\in I}$ of $\mathcal{S}(M)$, i.e. we fix values of $\Psi^+(f_i)$ and $\Psi^-(f_i)$. Since $\Psi$ is linear, for any $f \in \mathcal{S}(M)$,

$$\begin{aligned}\Psi(f) &= (\Psi^+ + \Psi^-)\left(\sum_{i\in I}\langle f_i, f\rangle f_i\right)\\ &= \sum_{i\in I}\langle f_i, f\rangle \cdot \Psi^+(f_i) + \overline{\langle f_i, f\rangle} \cdot \Psi^-(f_i)\end{aligned}$$

For example, denote $\delta_{x_0} = \delta(x - x_0) \in \mathcal{D}(M)$, then the set $\{\delta_{x_0} | x_0 \in M\}$ can be viewed as a basis of $\mathcal{S}(M) \approx \mathcal{D}(M)$. For any $f \in \mathcal{S}(M)$,

$$f = \int_M \langle \delta_x, f\rangle \cdot \delta_x dx = \int_M f(x) \cdot \delta_x dx$$

So

$$\Psi(f) = \int_M f(x) \cdot \Psi^+(\delta_x) dx + \int_M \overline{f(x)} \cdot \Psi^-(\delta_x) dx$$

Define $\Psi^+(x) = \Psi^+(\delta_x)$ and $\Psi^-(x) = \Psi^-(\delta_x)$, then

$$\Psi(f) = \int_M f(x) \cdot \Psi^+(x) dx + \int_M \overline{f(x)} \cdot \Psi^-(x) dx$$

In particular, if $\Psi$ is (anti) $\mathbb{C}$-linear, then

$$\Psi(f) = \int_M f(x) \cdot \Psi(x) dx \qquad \left( \Psi(f) = \int_M \overline{f(x)} \cdot \Psi(x) dx \right)$$

By tradition, we write $\Psi(x)$ to represent a field when we know $\Psi$ is $\mathbb{C}$-linear or anti $\mathbb{C}$-linear. The function $\Psi(x)$ indeed contains all information about this field.

At last, let $\{f_i\}_{i \in I}$ and $\{g_j\}_{j \in J}$ be two basis of $\mathcal{S}(M)$, then the base change is given by

$$\Psi^+(f_i) = \sum_{j \in J} \langle g_j, f_i \rangle \Psi^+(g_j) \qquad \Psi^-(f_i) = \sum_{j \in J} \langle f_i, g_j \rangle \Psi^-(g_j)$$

**Remark 15.1.** *Quantization of classical finite systems can be viewed as transforming finite systems to fields. For example, in quantum theory, we describe the state of a particle by a probability distribution function $\varphi(x) \in L^2(\mathbb{R}^3)$, which can be viewed as a scalar field on $\mathbb{R}^3$.*

# 16 Second quantization

In this section, we introduce a basic model in quantum field theory. We want to describe a multi-identical particle system. The state of multi-identical particle system is constrained to the following axiom:

**Axiom 8.** *Let $\mathbb{H}$ be the state space of one particle system. A state of $N$ identical particles system is a one-dimensional subspace of $\mathbb{H}^{\otimes N}$ invariant under the action of $S_N$.*

There are two non isomorphic one-dimensional representations of $S_N$, trivial and sign representations. We say a state is bosonic if it is a trivial representation of $S_N$ and fermionic if it is a sign representation. In our discussion, we only consider bosonic case.

Recall that in quantum harmonic oscillator model for single particle, the space of energy states is spanned by $\{|n\rangle | n = 1, 2, ...\}$ so that each state $|n\rangle$ represents a unit-norm function $\varphi_n$ in $L^2(\mathbb{R})$. The state space for $N$-identical particles is a subspace of $\mathbb{H}^{\otimes N}$. Here we say a subspace because not all vectors in $\mathbb{H}^{\otimes N}$ represents a valid state. For example, Let $\psi_1, ..., \psi_N$ be distinct energy states, then

$$A = \psi_1 \otimes \psi_2 \otimes \cdots \otimes \psi_N$$

is not a valid state. This is because if we interchange two particle, 1 and 2 for instance, then the state becomes

$$B = \psi_2 \otimes \psi_1 \otimes \cdots \otimes \psi_N$$

Apparently, $A$ and $B$ are not equal in $\mathbb{H}^{\otimes N}$. Therefore, the correct state space should be $\left(\mathbb{H}^{\otimes N}\right)^{S_N}$, the invariant subspace under the action of $S_N$. For example, if particles in this system are at states $\psi_1, ..., \psi_N$, then the state of this system is represented by

$$|\psi_1, ..., \psi_N\rangle = \frac{1}{N!} \sum_{\pi \in S_N} \psi_{\pi(1)} \otimes \cdots \otimes \psi_{\pi(N)}$$

The space $\left(\mathbb{H}^{\otimes N}\right)^{S_N}$ is exactly spanned by these elements. $|\psi_1, ..., \psi_N\rangle$ and $|\psi_1', ..., \psi_N'\rangle$ represents the same states if and only if two multi-sets $\{\psi_1, ..., \psi_N\}$ and $\{\psi_1, ..., \psi_N\}$ are the same, i.e. numbers of state $\varphi_n$ in these two multi-set are equal for all $n \geq 1$. Therefore, we represent every energy state by $|N_1, N_2, ...\rangle$, where $N_n$ is the number of particles at state $|n\rangle$.

In quantum harmonic oscillator model for single particle, we define creation and annihilation operators that arise or lower the energy state of this particle. For $N$-particle system, it's natural to believe that we can still apply operators to arise and lower each particle's energy state. For example, define

$$a^{\dagger(i)} = 1 \otimes \cdots \otimes \overset{i^{\text{th}}}{a^{\dagger(i)}} \otimes \cdots \otimes 1$$

$$a^{(i)} = 1 \otimes \cdots \otimes \overset{i^{\text{th}}}{a^{(i)}} \otimes \cdots \otimes 1$$

to be operators that arise and lower the energy state of $i^{\text{th}}$ particle. However, these operators are invalid. Here are two ways to understand the failure. First, since particles are indistinguishable, it makes no senses to talk about one particular particle. Second, operators $a^{\dagger(i)}$ and $a^{(i)}$ simply don't commute with actions of $S_N$ and hence the image of $a^{\dagger(i)}$ and $a^{(i)}$ do not lie in $(\mathbb{H}^{\otimes N})^{S_N}$.

An improvement is to define creation and annihilation operators by

$$b_n^\dagger |N_1, ..., N_n, ...\rangle = C_n^\dagger |N_1, ..., N_n - 1, N_{n+1} + 1, ...\rangle$$

$$b_n |N_1, ..., N_n, ...\rangle = C_n |N_1, ..., N_{n-1} + 1, N_n - 1, ...\rangle$$

where $C_n^\dagger, C_n$ are constants. This definition seems to be reasonable because arising (and lowering) one particle's energy means increasing the number of particle at one state and decreasing the number of particle at another state. A drawback of this construction is that the space $(\mathbb{H}^{\otimes N})^{S_N}$ is not a highest weight module with respect to the operator $b_n^\dagger$ and $b_n$.

A key observation is that $b_n^\dagger$ and $b_n$ we construct above can be decomposed into two steps:

$$b_n^\dagger : |N_1, ..., N_n, ...\rangle \longmapsto |N_1, ..., N_n - 1, N_{n+1}, ...\rangle \longmapsto |N_1, ..., N_n - 1, N_{n+1} + 1, ...\rangle$$

$$b_n : |N_1, ..., N_n, ...\rangle \longmapsto |N_1, ..., N_{n-1}, N_n - 1, ...\rangle \longmapsto |N_1, ..., N_{n-1} + 1, N_n - 1, ...\rangle$$

This motivate us to define

$$a_n^\dagger |N_1, ..., N_n, ...\rangle = C_n^\dagger |N_1, ..., N_n + 1, ...\rangle$$

$$a_n |N_1, ..., N_n, ...\rangle = C_n |N_1, ..., N_{n-1}, N_n - 1, ...\rangle$$

Then $b_n^\dagger = a_{n+1}^\dagger + a_n$, $b_n = a_{n-1}^\dagger + a_n$, we lose no information if we only consider operators $a_n^\dagger$ and $a_n$. There is only one problem left, namely, $a_n^\dagger$ and $a_n$ do not maps $\left(\mathbb{H}^{\otimes N}\right)^{S_N}$ to itself!

$$a_n^\dagger : \left(\mathbb{H}^{\otimes N}\right)^{S_N} \longrightarrow \left(\mathbb{H}^{\otimes N+1}\right)^{S_{N+1}} \qquad a_n : \left(\mathbb{H}^{\otimes N}\right)^{S_N} \longrightarrow \left(\mathbb{H}^{\otimes N-1}\right)^{S_{N-1}}$$

To fix this problem, we can take the direct sum of all $\left(\mathbb{H}^{\otimes N}\right)^{S_N}$, then $a_n^\dagger$ and $a_n$ can be defined as operators on this space.

**Definition** Let $\mathbb{H}$ be a Hilbert space, the Fock space of $\mathbb{H}$ is defined by

$$\mathcal{F}(\mathbb{H}) = \bigoplus_{n=0}^{\infty} \left(\mathbb{H}^{\otimes n}\right)^{S_n}$$

where $\mathbb{H}^0 = \mathbb{C}$. Let $|N_1, N_2, ...\rangle$ be a state in $\mathcal{F}(\mathbb{H})$ defined above. Define

$$V = \operatorname{span}_{\mathbb{C}}\langle |N_1, N_2, ...\rangle | N_n \geq 0, \sum_{n\geq 0} N_n < \infty\rangle$$

$V$ is called a vacuum module.

**Definition** For $n \geq 0$, creation and annihilation operators $a_n^\dagger, a_n : V \longrightarrow V$ are defined by

$$a_n^\dagger |N_1, ..., N_n, ...\rangle = \sqrt{N_n + 1}|N_1, ..., N_n + 1, ...\rangle$$
$$a_n |N_1, ..., N_n, ...\rangle = \sqrt{N_n}|N_1, ..., N_{n-1}, N_n - 1, ...\rangle$$

We choose constants in this way so that following commutator relations hold:

**Proposition 16.1.**

$$[a_n, a_m] = 0 \qquad [a_n^\dagger, a_m^\dagger] = 0 \qquad [a_n, a_m^\dagger] = \delta_{n,m}$$

Denote the state $|0, 0, ...\rangle$ by $|0\rangle$. This state is called the vacuum vector in $V$.

**Lemma 16.2.**

$$|N_1, N_2, ...\rangle = \prod_{k\geq 0} \frac{1}{\sqrt{N_k!}} \left(a_k^\dagger\right)^{N_k} |0\rangle$$

Let $\mathfrak{h}_\infty$ be the Lie algebra generated by $a_n^\dagger$ and $a_n$, $n \geq 1$. The above lemma implies that $V$ is a highest weight module of $\mathfrak{h}_\infty$, generated by the vacuum vector $|0\rangle$. Similar to the single particle case, this module is isomorphic to a polynomial ring as $\mathfrak{h}_\infty$-modules.

**Proposition 16.3.** *Let $\mathbb{C}[x_1, x_2, ...]$ be a polynomial ring with countable many indeterminates. $\mathfrak{h}_\infty$ acts on $\mathbb{C}[x_1, x_2, ...]$ by*

$$a_n^\dagger = x_n \qquad a_n = \frac{\partial}{\partial x_n} \qquad n \geq 1$$

*Then there exists an $\mathfrak{h}_\infty$-module isomorphism between $\mathbb{C}[x_1, x_2, ...]$ and $V$, given by*

$$\mathbb{C}[x_1, x_2, ...] \longrightarrow V \qquad \frac{X_{n_1}^{\alpha_1} \cdots X_{n_k}^{\alpha_k}}{\sqrt{\alpha_1! \cdots \alpha_k!}} \longmapsto |..., \overset{i^{th}}{\alpha_i}, ...\rangle$$

As we have mention in last section, elements in $\mathbb{H} = L^2(\mathbb{R})$ can be viewed as scalar fields on $\mathbb{R}$. We should treat $\mathcal{F}(\mathbb{H})$ as the quantization of $\mathbb{H}$. That's why we say $\mathcal{F}(\mathbb{H})$ is second quantization.

Note that $\{\varphi(x)|n \geq 1\}$ is a basis of $\mathbb{H} = L^2(\mathbb{R})$. We should associate operator distributions to every one of them. Recalled that in last section, we have pointed out that to define a field operator, we only need to define the value of $\mathbb{C}$-linear part and anti $\mathbb{C}$-linear part on basis. Let $\Psi_n^\dagger$ be a $\mathbb{C}$-linear distribution and $\Psi_n$ be an anti $\mathbb{C}$-linear distribution defined by

$$\Psi_n^\dagger(\varphi_m) = \delta_{n,m} a_n^\dagger \qquad \Psi_n(\varphi_m) = \delta_{n,m} a_n$$

$\{\delta_x|x \in \mathbb{R}\}$ is also a basis of $L^2(\mathbb{R}) \approx \mathcal{D}(\mathbb{R})$. Then

$$\Psi_n^\dagger(x) = \Psi_n^\dagger(\delta_x) = \langle \varphi_n, \delta_x \rangle a_n^\dagger = \overline{\varphi}_n(x) a_n^\dagger \qquad \Psi_n(x) = \Psi_n(\delta_x) = \langle \delta_x, \varphi_n \rangle a_n = \varphi_n(x) a_n$$

The distribution are given by

$$\Psi_n^\dagger(f) = \int_{\mathbb{R}} f(x)\overline{\varphi_n(x)}dx \cdot a_n^\dagger \qquad \Psi_n(f) = \int_{\mathbb{R}} \overline{f(x)}\varphi_n(x)dx \cdot a_n$$

$\Psi_n^\dagger(x)$ and $\Psi_n(x)$ can be interpreted as operators that create and annihilate a particle with state $|n\rangle$ at position $x$. We define the quantization of coordinate $\varphi_n$ to be

$$\widehat{\varphi_n} = \Psi_n^\dagger + \Psi_n$$

Then

$$\widehat{\varphi_n}(f) = \int_{\mathbb{R}} f(x)\overline{\varphi_n(x)}dx \cdot a_n^\dagger + \int_{\mathbb{R}} \overline{f(x)}\varphi_n(x)dx \cdot a_n$$

Define

$$\Psi^\dagger = \sum_{n \geq 1} \Psi_n^\dagger \qquad \Psi = \sum_{n \geq 1} \Psi_n$$

then under basis $\{\varphi_n|n \geq 1\}$,

$$\Psi^\dagger(\varphi_n) = a_n^\dagger \qquad \Psi(\varphi_n) = a_n$$

under basis $\{\delta_x|x \in \mathbb{R}\}$,

$$\Psi^\dagger(x) = \sum_{n \geq 1} \varphi_n(x) a_n^\dagger \qquad \Psi(x) = \sum_{n \geq 1} \overline{\varphi_n(x)} a_n$$

$\Psi^\dagger(x)$ and $\Psi(x)$ can be interpreted as operators that create and annihilate a particle with arbitrary state at position $x$.

**Proposition 16.4.** *For any $f, g \in \mathcal{S}(\mathbb{R})$,*

$$[\Psi(f), \Psi(g)] = 0 \qquad [\Psi^\dagger(f), \Psi^\dagger(g)] = 0 \qquad [\Psi(f), \Psi^\dagger(g)] = \langle f, g \rangle$$

*Proof.*

$$\begin{aligned} [\Psi(f), \Psi^\dagger(g)] &= \sum_{n \geq 1} \sum_{m \geq 1} \langle f, \varphi_n \rangle \langle \varphi_m, g \rangle \left[a_n, a_m^\dagger\right] \\ &= \sum_{n \geq 1} \langle f, \varphi_n \rangle \langle \varphi_n, g \rangle \\ &= \langle f, g \rangle \end{aligned}$$

□

In particular, we have

$$[\Psi(x), \Psi^{\dagger}(x')] = \delta(x - x')$$

You may think this equation is a nonsense because $\delta(x-x')$ is not an operator, not even a function. We will see how to make this equation meaningful under the language of vertex algebra.

# 17 Formal distributions in one variable

Let $R$ be a ring. The following six objects are similar in notations, but have entirely different algebraic structures:

$$R[z] \quad R[z^{\pm 1}] \quad R(z) \quad R[[z]] \quad R((z)) \quad R[[z^{\pm 1}]]$$

Specifically,

$$\begin{aligned}
R[z] &= \left\{ \sum_{n=0}^{\infty} a_n z^n \middle| a_n \in R \text{ and } a_n = 0 \text{ for all but finite many } n \right\} \\
R[z^{\pm 1}] &= \left\{ \sum_{n=-\infty}^{\infty} a_n z^n \middle| a_n \in R \text{ and } a_n = 0 \text{ for all but finite many } n \right\} \\
R[[z]] &= \left\{ \sum_{n=0}^{\infty} a_n z^n \middle| a_n \in R \right\} \\
R((z)) &= \left\{ \sum_{n>N}^{\infty} a_n z^n \middle| a_n \in R \text{ and } N \in \mathbb{Z} \right\} \\
R[[z^{\pm 1}]] &= \left\{ \sum_{n=0}^{\infty} a_n z^n \middle| a_n \in R \right\}
\end{aligned}$$

and $R(z)$ is the localization of $R[z]$ with respect to all non zero-divisors. The algebraic structures of these six objects are classified as follow:

1. $R[z], R[z^{\pm 1}], R(z), R[[z]], R((z))$ are rings.
2. when $R$ is a field,
   (a) $R[z], R[z^{\pm 1}]$ are PID and $R[[z]]$ is a DVR,
   (b) $R(z)$ and $R((z))$ are fields.
3. $R[[z^{\pm 1}]]$ is not a ring. It is invalid to define the product by

$$\left( \sum_{n\in\mathbb{Z}} a_n z^n \right) \left( \sum_{m\in\mathbb{Z}} b_m z^m \right) = \sum_{n\in\mathbb{Z}} \left( \sum_{k+l=n} a_k b_l \right) z^n$$

because $\sum_{k+l=n} a_k b_l$ is a infinite sum. Instead, $R[[z^{\pm 1}]]$ is a $R[z^{\pm 1}]$-module, i.e., the product

$$\left( \sum_{N_1 \le n \le N_2} a_n z^n \right) \left( \sum_{m\in\mathbb{Z}} b_m z^m \right) = \sum_{n\in\mathbb{Z}} \left( \sum_{\substack{k+l=n \\ N_1 \le k \le N_2}} a_k b_l \right) z^n$$

is well-defined.

Elements in $R[[z^{\pm 1}]]$ are often called formal distributions. Let's explain why we call them distributions. Recall that distributions in real analysis refer to a $\mathbb{C}$-linear functional $\phi : \mathcal{S}(M) \longrightarrow \mathbb{C}$, where $\mathcal{S}(M)$ is the space of Schwartz functions on a manifold $M$. We often write $\phi$ as a function $\phi(x)$ so that

$$\phi(f) = \int_M \overline{\phi(x)} f(x) dx$$

no matter whether function $\phi(x)$ or this integral is well-defined (For example, $\phi(x) = \delta(x - x_0)$).

Now we study an example which we can view elements in $\mathbb{C}[[z^{\pm 1}]]$ as distributions. Let's consider distribution on functions space over $S^1$. By Fourier theory, every function or distribution on $S^1$ can be expressed as

$$\sum_{n\in\mathbb{Z}} a_n e^{in\theta}$$

Take $z = e^{i\theta}$, then every function or distribution can be expressed as

$$\sum_{n\in\mathbb{Z}} a_n z^n$$

Therefore, we can identify the space of all distributions on $S^1$ as a subspace of $\mathbb{C}[[z^{\pm 1}]]$ and we can identify $\mathbb{C}[z^{\pm 1}]$ as a subspace of the Schwartz space $\mathcal{S}(S^1)$. Let $\phi(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ be a distribution, $f(z) = \sum_{m\in\mathbb{Z}} b_n z^n$ be a test function. Then

$$\begin{aligned}
\phi(f) &= \int_0^{2\pi} \overline{\phi(\theta)} f(\theta) d\theta \\
&= \int_0^{2\pi} \left( \sum_{n\in\mathbb{Z}} a_n e^{-in\pi} \right) \left( \sum_{m\in\mathbb{Z}} b_m e^{im\pi} \right) d\theta \\
&= \int_{S^1} \left( \sum_{n\in\mathbb{Z}} a_n z^{-n} \right) \left( \sum_{m\in\mathbb{Z}} b_m z^m \right) \frac{dz}{iz} \\
&= \frac{1}{2\pi i} \int_{S^1} \left( \sum_{n\in\mathbb{Z}} 2\pi a_n z^{-n-1} \right) \left( \sum_{m\in\mathbb{Z}} b_m z^m \right) dz \\
&= \mathrm{Res}_z\ \phi'(z) f(z)
\end{aligned}$$

where $\phi'(z) = \sum_{n\in\mathbb{Z}} 2\pi a_n z^{-n-1}$. Thus, calculating the value of distribution by integration can be replaced by calculating residues. We prefer the later one because its definition is purely algebraic. Hence, for any $f \in \mathbb{C}[[z^{\pm 1}]]$, we identify it with a distribution $\widehat{f}$, defined by

$$\widehat{f} : \mathbb{C}[z^{\pm 1}] \longrightarrow \mathbb{C} \qquad g \longmapsto \mathrm{Res}_z\ f \cdot g$$

Generally, Let $k$ be a field and $R$ is a $k$-vector space. The residue map on $R[[z^{\pm 1}]]$ is defined by

$$\mathrm{Res}_z : R[[z^{\pm 1}]] \longrightarrow R \qquad \sum_{n\in\mathbb{Z}} a_n z^n \longmapsto a_{-1}$$

The Schwartz space of test function is $k[z^{\pm 1}]$. A $R$-valued distribution is a $k$-linear map between the Schwartz space $k[z^{\pm 1}]$ and $R$. Then the result below shows that every $R$-valued distribution is a residue map:

**Proposition 17.1.** *There is an isomorphism of $k$-vector space*

$$R[[z^{\pm 1}]] \longrightarrow Hom_k\big(k[z^{\pm 1}], R\big) \qquad f(z) \longrightarrow \widehat{f}$$

*where*

$$\widehat{f}(g) = Res_z\ f(z)g(z)$$

# 18 Formal distributions in two variables

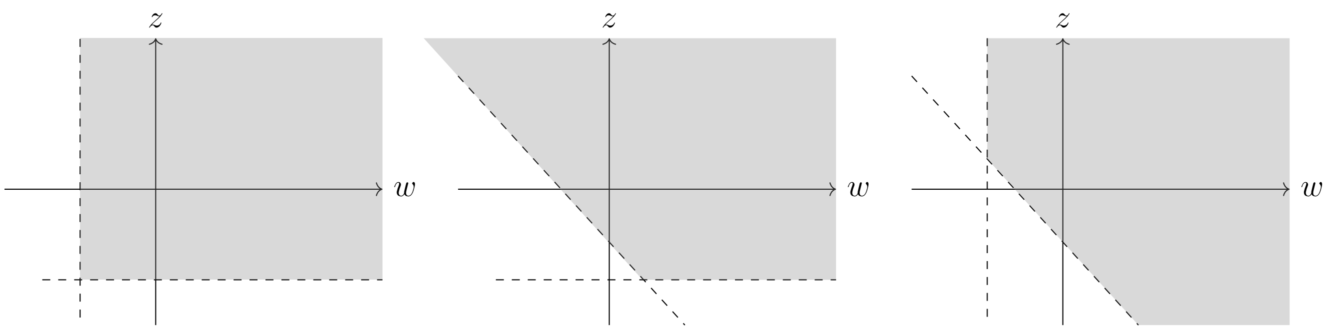
In this section, we study formal distribution in two variables, i.e. elements in

$$R[[z^{\pm 1}, w^{\pm 1}]]$$

To begin with, it is crucial to distinguish following rings:

$$R((z,w)) \qquad R((z))((w)) \qquad R((w))((z))$$

Their coefficients distribute in following patterns:




$R((z,w))$ $\qquad$ $R((z))((w))$ $\qquad$ $R((w))((z))$

Note that $R((z))((w)) \cap R((w))((z)) = R((z,w))$, we have inclusions

$$\begin{tikzcd}
 & R[[z^{\pm 1}, w^{\pm 1}]] & \\
R((z))((w)) \arrow[ur, hook, "i_{z,w}"] & & R((w))((z)) \arrow[ul, hook', "i_{w,z}"'] \\
 & R((z,w)) \arrow[ul, hook] \arrow[ur, hook] &
\end{tikzcd}$$

Where $i_{z,w}$ and $i_{w,z}$ is multiplicative. In particular, if $R$ is a field, then both $R((z))((w))$ and $R((w))((z))$ are fields. However, things become strange here. For example, $z - w \in R[[z,w]]$, so it belongs to both $R((z))((w))$ and $R((w))((z))$. The inverse of $z-w$ in these two fields is given by

$$R((z))((w)): \qquad \frac{1}{z-w}\bigg|_{|z|>|w|} = \frac{1}{z}\frac{1}{1-\frac{w}{z}} = \sum_{n\geq 0} z^{-n-1}w^n =: \delta_-(z-w)$$

$$R((w))((z)): \qquad \frac{1}{z-w}\bigg|_{|z|<|w|} = \frac{-1}{w}\frac{1}{1-\frac{z}{w}} = -\sum_{n\geq 0} w^{-n-1}z^n =: -\delta_+(z-w)$$

Therefore, in $R[[z^{\pm 1}, w^{\pm 1}]]$, we have

$$1 = (z-w)\delta_-(z-w) = (z-w)\left(-\delta_+(z-w)\right)$$

Define

$$\delta(z-w) \coloneqq \delta_-(z-w) + \delta_+(z-w)$$

Then

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} w^{-n-1}z^n = \sum_{n\in\mathbb{Z}} z^{-n-1}w^n = \sum_{k+l=-1} z^k w^l$$

$$(z-w)\delta(z-w) = 0$$

This is exactly what we prove in lemma 11.4. Similarly, we can consider the inverse of $(z-w)^{n+1}$ for $n \geq 0$,

$$R((z))((w)): \qquad \frac{1}{(z-w)^{n+1}} = \sum_{k\geq 0} \binom{n+k}{n} z^{-k-n-1}w^k$$

$$R((w))((z)): \qquad \frac{1}{(z-w)^{n+1}} = (-1)^{n+1}\sum_{k\geq 0} \binom{n+k}{n} w^{-k-n-1}z^k$$

By direct computation, we have

$$\frac{1}{n!}\partial_w^n \delta_-(z-w) = \sum_{k\geq 0} \binom{n+k}{n} z^{-k-n-1}w^k$$

$$\frac{1}{n!}\partial_w^n \delta_+(z-w) = (-1)^n\sum_{k\geq 0} \binom{n+k}{n} w^{-k-n-1}z^k$$

$$\frac{1}{n!}\partial_w^n \delta(z-w) = \sum_{k\in\mathbb{Z}} \binom{n+k}{n} z^{-n-1-k}w^k = (-1)^n\sum_{k\in\mathbb{Z}} \binom{n+k}{n} w^{-n-1-k}z^k$$

If we define an operator $\partial_w^{(n)} = \frac{1}{n!}\partial_w^n$, $n \geq 0$, then

$$\left.\frac{1}{(z-w)^{n+1}}\right|_{|z|>|w|} = \partial_w^{(n)}\delta_-(z-w)$$

$$\left.\frac{1}{(z-w)^{n+1}}\right|_{|z|<|w|} = -\partial_w^{(n)}\delta_+(z-w)$$

**Proposition 18.1.** *1.* $f(z)\delta(z-w) = f(w)\delta(z-w)$,

*2.* $(z-w)\partial_w^{(n+1)}\delta(z-w) = \partial_w^{(n)}\delta(z-w)$

*3.* $(z-w)^{n+1}\partial_w^{(n)}\delta(z-w) = 0$,

*4.* $\mathrm{Res}_z\, f(z)\delta(z-w) = f(w)$.

*Proof.* 1. We have proved in lemma 11.4.

2.

$$
\begin{aligned}
(z-w)\partial_w^{(n+1)}\delta(z-w) &= (z-w)\sum_{k\in\mathbb{Z}}\binom{n+k+1}{n+1}z^{-n-2-k}w^k \\
&= \sum_{k\in\mathbb{Z}}\left(\binom{n+k+1}{n+1}-\binom{n+k}{n+1}\right)z^{-n-1-k}w^k \\
&= \sum_{k\in\mathbb{Z}}\binom{n+k}{n}z^{-n-1-k}w^k \\
&= \partial_w^{(n)}\delta(z-w)
\end{aligned}
$$

3. $(z-w)^{n+1}\partial_w^{(n)}\delta(z-w) = (z-w)\delta(z-w) = 0$

4. $\mathrm{Res}_z\, f(z)\delta(z-w) = \mathrm{Res}_z\, f(w)\delta(z-w) = f(w)$

□

Now, Let's discuss how formal distributions with two variables arise naturally in complex analysis. What we are going to do is, roughly speaking, transforming theorems in complex analysis, such as Cauchy formula, into purely algebraic results. For $r \in \mathbb{R}$, $r > 0$, define $D_r = \{z \in \mathbb{C} | |z| < r\}$

1. $M$ : the space of all holomorphic functions defined on a punctured connected neighborhood of 0 which doesn't contain 0.

2. $M_r$ : the space of all holomorphic functions defined on the punctured disk $D_r \setminus \{0\}$.

3. $\mathcal{O}$ : the space of all holomorphic functions defined on a connected neighborhood of 0.

4. $\mathcal{O}_r$ : the space of all holomorphic functions defined on the disk $D_r$.

5. $\mathcal{O}^-$ : the space of all holomorphic functions defined on a connected neighborhood of $\infty \in \mathbb{CP}^1$ and vanishing at $z = \infty$

6. $\mathcal{O}_r^-$ : the space of all holomorphic functions defined on $\mathbb{CP}^1 \setminus D_r$ and vanishing at $z = \infty$

Then

$$
\varinjlim_{r>0} M_r = M \qquad \varinjlim_{r>0} \mathcal{O}_r = \mathcal{O} \qquad \varinjlim_{r>0} \mathcal{O}_r^- = \mathcal{O}^-
$$

Expand $f$ which belongs to any of these spaces at $z = 0$, then it is in form of

$$
f(z) = \sum_{n\in\mathbb{Z}} a_n z^n
$$

where $a_n = 0$ for $n < 0$ if $f \in \mathcal{O}$, and $a_n = 0$ for $n \geq 0$ if $f \in \mathcal{O}^-$.

Therefore, we can identify these function as elements in $\mathbb{C}[[z^{\pm 1}]]$. Specifically,

$$
M_r \hookrightarrow M \hookrightarrow \mathbb{C}[[z^{\pm 1}]] \qquad \mathcal{O}_r \hookrightarrow \mathcal{O} \hookrightarrow \mathbb{C}[[z]] \qquad \mathcal{O}_r^- \hookrightarrow \mathcal{O}^- \hookrightarrow z^{-1}\mathbb{C}[[z^{-1}]]
$$

We are interested in distributions over these function spaces and we wish to lift distributions over functions spaces to distribution over their according formal power series spaces. For example, let's fix a complex number $w_0 \in \mathbb{C} \setminus \{0\}$. Then for any $r > |w_0|$ we can construct distributions

$$\delta_{w_0} : M_r, \mathcal{O}_r \longrightarrow \mathbb{C} \qquad f \longmapsto f(w_0)$$

for $r < |w_0|$ we can construct distribution

$$\delta_{w_0} : \mathcal{O}_r^- \longrightarrow \mathbb{C} \qquad f \longmapsto f(w_0)$$

Let's first consider $\delta_{w_0}$ defined on $\mathcal{O}$. By Cauchy integral formula,

$$f(w_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - w_0} dz$$

where $C$ is a circle centered at $z = 0$ with radius $R$, $|w_0| < R < r$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $\mathbb{C}[[z]]$, then

$$\begin{aligned} f(w_0) &= \frac{1}{2\pi i} \oint_C \left( \sum_{n=0}^{\infty} a_n z^n \right) \left( \frac{1}{z - w_0} \bigg|_{|z|>|w_0|} \right) dz \\ &= \frac{1}{2\pi i} \oint_C \left( \sum_{n=0}^{\infty} a_n z^n \right) \left( \sum_{n=0}^{\infty} z^{-n-1} w_0^n \right) dz \\ &= \operatorname{Res}_z f(z) \cdot \delta_-(z - w_0) \end{aligned}$$

Naturally, we hope to lift this distribution to $\delta_{w_0} : \mathbb{C}[[z]] \longrightarrow \mathbb{C}$ by

$$\delta_{w_0} = \operatorname{Res}_z \delta_-(z - w_0)$$

However, this map is not well-defined. For example, if $f(z) = \sum_{n \geq 0} z^n$ and $w_0 = 1$, then $\delta_{w_0} = \infty$. Instead of considering $w_0$ as a complex number, we should treat it as an indeterminate. Therefore, we lift this distribution to $\delta_w : \mathbb{C}[[z]] \longrightarrow \mathbb{C}[[w]]$, given by

$$\delta_w = \operatorname{Res}_z \delta_-(z - w)$$

In general, Let $k$ be a field and $R$ is a $k$-algebra. $R[[w^{\pm 1}]]$-valued distribution on $k[z^{\pm 1}]$ is a $k$-linear map

$$\varphi : k[z^{\pm 1}] \longrightarrow R[[w^{\pm 1}]]$$

By proposition 17.1, every distribution of this kind can be represented by an element $\Psi(z, w) \in R[[w^{\pm 1}]][[z^{\pm 1}]] = R[[z^{\pm 1}, w^{\pm 1}]]$, such that

$$\varphi(f(z)) = \operatorname{Res}_z f(z) \Psi(z, w)$$

Therefore, the distribution $\delta_w$ is represented by $\delta_-(z - w)$.

Next, we consider $\delta_{w_0}$ on $\mathcal{O}^-$. Assume $f(z) = \sum_{n \leq -1} a_n z^n$, then $f\left(\frac{1}{z}\right)$ is a well-defined holomorphic function on $D_{\frac{1}{r}}$. By Cauchy integral formula,

$$\begin{aligned} f(w_0) = f\left(\frac{1}{w_0^{-1}}\right) &= \frac{1}{2\pi i} \oint_C f\left(\frac{1}{z}\right) \left( \frac{1}{z - w_0{}^{-1}} \bigg|_{|z|>|w_0{}^{-1}|} \right) dz \\ &= \frac{1}{2\pi i} \oint_C \left( \sum_{n \leq -1} a_n z^{-n} \right) \left( \sum_{n=0}^{\infty} z^{-n-1} w_0^{-n} \right) dz \\ &= \sum_{n \leq -1} a_n w_0^n \\ &= \operatorname{Res}_z f(z) \cdot \delta_+(z - w_0) \end{aligned}$$

Thus, $\delta_{w_0} = \mathrm{Res}_z\, \delta_+(z - w_0)$ and we can lift this distribution to a $w^{-1}\mathbb{C}[[w^{-1}]]$-valued distribution

$$\delta_w : z^{-1}\mathbb{C}[[z^{-1}]] \longrightarrow w^{-1}\mathbb{C}[[w^{-1}]] \qquad \delta_w = \mathrm{Res}_z\, \delta_+(z - w)$$

At last, we consider $\delta_{w_0}$ on $M_r$. In complex analysis, we learn that

$$f(w_0) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z - w_0}dz - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z - w_0}dz$$

where $C_1$ is a circle centered at $z = 0$ with radius $|w_0| < R < r$ and $C_2$ is a circle centered at $z = 0$ with radius $0 < r < |w_0|$.

Assume $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ in $\mathbb{C}[[z^{\pm1}]]$, then

$$\begin{aligned}
f(w_0) &= \frac{1}{2\pi i}\oint_{C_1} \left(\sum_{n\in\mathbb{Z}} a_n z^n\right) \frac{1}{z - w_0}\bigg|_{|z|>|w_0|} dz - \frac{1}{2\pi i}\oint_{C_2} \left(\sum_{n\in\mathbb{Z}} a_n z^n\right) \frac{1}{z - w_0}\bigg|_{|z|<|w_0|} dz \\
&= \frac{1}{2\pi i}\oint_{C_1} \left(\sum_{n\in\mathbb{Z}} a_n z^n\right) \delta_-(z - w_0)dz + \frac{1}{2\pi i}\oint_{C_2} \left(\sum_{n\in\mathbb{Z}} a_n z^n\right) \delta_+(z - w_0)dz \\
&= \mathrm{Res}_z\, f(z)\cdot\delta_-(z - w_0) + \mathrm{Res}_z\, f(z)\cdot\delta_+(z - w_0) \\
&= \mathrm{Res}_z\, f(z)\cdot\delta(z - w_0)
\end{aligned}$$

Thus, $\delta_{w_0} = \mathrm{Res}_z\, \delta(z - w_0)$ and we can lift this distribution to a $\mathbb{C}[[w^{\pm1}]]$-valued distribution

$$\delta_w : \mathbb{C}[[z^{\pm1}]] \longrightarrow \mathbb{C}[[w^{\pm1}]] \qquad \delta_w = \mathrm{Res}_z\, \delta(z - w)$$

It's easy to see that the integral method in $\mathcal{O}_r$ case and $\mathcal{O}_r^-$ case are both special forms of $M_r$ case. Hence, in our discussion below, we only focus on distributions over $M_r$.

More generally, we can consider distribution of form

$$\delta_w^n : M_r, \mathcal{O}_r, \mathcal{O}_r^- \longrightarrow \mathbb{C} \qquad f(z) \longmapsto \partial_z^n f(w_0)$$

The Cauchy integral formula implies

$$\begin{aligned}
\mathcal{M}: \qquad \partial_z^n f(w_0) &= \frac{n!}{2\pi i}\oint_{C_1} f(z)\frac{1}{(z - w_0)^{n+1}}\bigg|_{|z|>|w_0|} dz - \frac{1}{2\pi i}\oint_{C_2} f(z)\frac{1}{(z - w_0)^{n+1}}\bigg|_{|z|<|w_0|} dz \\
&= \mathrm{Res}_z\, f(z)\cdot\partial_w^n\delta_-(z - w) + \mathrm{Res}_z\, f(z)\cdot\partial_w^n\delta_+(z - w) \\
&= \mathrm{Res}_z\, f(z)\cdot\partial_w^n\delta(z - w)
\end{aligned}$$

Now we introduce a new function space

- $\overline{M_r}$ : the space of all holomorphic functions defined for all but finite many points on $D_r$.

$\overline{M_r}$ can also be viewed as a subset of $\mathbb{C}[[z^{\pm1}]]$,

$$M_r \hookrightarrow \overline{M_r} \hookrightarrow \mathbb{C}[[z^{\pm1}]]$$

Fix a $w_0 \in \mathbb{C}^\times$, $|w_0| < r$. For $f \in \overline{M_r}$, we'd like to find the Taylor expansion of $f$ at point $w_0$. It should be in form of

$$f(z) = \sum_{n\geq1} \frac{a_{-n}}{(z - w_0)^n} + \sum_{n\geq0} a_n(z - w_0)^n \tag{18.1}$$

We can obtain the coefficient $a_n$ by the residues theorem:

$$a_n = \frac{1}{2\pi i} \oint_C f(z) \cdot (z - w_0)^{-n-1} dz$$

where $C$ is a circle centered at $w_0$ and contained in $D_r$, with no poles inside the circle except $w_0$ itself. Now we need to cheat a little bit here. Suppose $f(z) = \sum_{n\in\mathbb{Z}} a_n z^n$ on $C$ and $|z| > |w_0|$ for all $z$ on $C$. Both of these two assumption are incorrect in general. For example, if $f$ has singularity at $z = w_0$, then this expansion can not be valid for all points on $C$. Nevertheless, we taking these two incorrect assumptions end up obtaining valid results. For $n \geq 0$,

$$a_n = \operatorname{Res} \partial_w^{(n)} \delta_-(z-w) f(z) \qquad a_{-n-1} = \operatorname{Res}_z (z-w)^n f(z) \qquad \left.\frac{1}{(z-w)^{n+1}}\right|_{|z|>|w_0|} = \partial_w^{(n)} \delta_-(z-w)$$

Equation 18.1 becomes

$$f(z) = \sum_{n\geq 0} \partial_w^{(n)} \delta_-(z - w_0) \cdot \operatorname{Res}_z (z - w_0)^n f(z) + \sum_{n\geq 0} (z - w_0)^n \cdot \operatorname{Res}_z \partial_w^{(n)} \delta_-(z - w_0) f(z)$$

**Proposition 18.2.** *Let $k$ be a field, Char $k = 0$, and $R$ be a $k$-algebra. Then*

1. *For any $f \in R[[z^{\pm 1}]]$,*

$$f(z) = \sum_{n\geq 0} (z - w)^n \operatorname{Res}_z \partial_w^{(n)} \delta_-(z - w) f(z)$$

2. *For any $f \in z^{-1}R[[z^{-1}]]$,*

$$f(z) = \sum_{n\geq 0} \partial_w^{(n)} \delta_-(z - w) \operatorname{Res}_z (z - w)^n f(z)$$

3. *For any $f(z) \in R[[z^{\pm 1}]]$,*

$$f(z) = \sum_{n\geq 0} \partial_w^{(n)} \delta_-(z - w) \cdot \operatorname{Res}_z (z - w)^n f(z) + \sum_{n\geq 0} (z - w)^n \cdot \operatorname{Res}_z \partial_w^{(n)} \delta_-(z - w) f(z) \tag{18.2}$$

*Proof.* 1. We only need to show that this equation holds for $f(z) = z^m$, $m \geq 0$

$$\begin{aligned}
&\sum_{n\geq 0} (z - w)^n \operatorname{Res}_z \partial_w^{(n)} \delta_-(z - w) z^m \\
=&\sum_{n\geq 0} (z - w)^n \operatorname{Res}_z \sum_{k\geq 0} \binom{n+k}{n} z^{-n-1-k+m} w^k \\
=&\sum_{n\geq 0} \binom{m}{n} (z - w)^n w^{m-n} = z^m
\end{aligned}$$

2. We only need to show that this equation holds for $f(z) = z^{-m}$, $m \geq 1$

$$
\begin{aligned}
&\sum_{n\geq 0} \partial_w^{(n)} \delta_-(z-w) \operatorname{Res}_z (z-w)^n z^{-m} \\
=&\sum_{n\geq 0}\sum_{k\geq 0} \binom{n+k}{n} z^{-n-1-k+m} \cdot \binom{n}{m-1} (-w)^{n-m+1} \\
=&\sum_{n\geq 0}\sum_{k\geq 0} (-1)^{k-m+1} \binom{n}{k}\binom{k}{m-1} z^{-n-1} w^{n-m+1} \\
=&\sum_{n\geq m-1}\sum_{k\geq m-1} (-1)^{k-m+1} \binom{n-m+1}{k-m+1}\binom{n}{m-1} z^{-n-1} w^{n-m+1} \\
=&\sum_{n\geq m-1} \left(\sum_{k\geq 0} (-1)^k \binom{n-m+1}{k}\right) \binom{n}{m-1} z^{-n-1} w^{n-m+1} \\
=&z^{-m}
\end{aligned}
$$

3. Suppose $f = f_- + f_+$, where $f_- \in z^{-1}R[[z^{-1}]]$ and $f_+ \in R[[z]]$, then

$$
\operatorname{Res}_z \partial_w^{(n)} \delta_-(z-w) f_- = 0 \qquad \operatorname{Res}_z (z-w)^n f_+ = 0 \qquad n \geq 0
$$

Therefore

$$
\begin{aligned}
&\sum_{n\geq 0} \partial_w^{(n)} \delta_-(z-w) \cdot \operatorname{Res}_z (z-w)^n f(z) + \sum_{n\geq 0} (z-w)^n \cdot \operatorname{Res}_z \partial_w^{(n)} \delta_-(z-w) f(z) \\
=&\sum_{n\geq 0} \partial_w^{(n)} \delta_-(z-w) \cdot \operatorname{Res}_z (z-w)^n f_-(z) + \sum_{n\geq 0} (z-w)^n \cdot \operatorname{Res}_z \partial_w^{(n)} \delta_-(z-w) f_+(z) \\
=&f_- + f_+ = f
\end{aligned}
$$

□

Let $f(z, w)$ be a formal distribution in two variables. Assume

$$
f(z, w) = \sum_{m\in\mathbb{Z}} f_m(z) w^m
$$

Then we can decompose $f_m(z)$ by equation 18.2,

$$
\begin{aligned}
f(z, w) &= \sum_{m\in\mathbb{Z}} \left( \sum_{n\geq 0} \partial_w^{(n)} \delta_-(z-w) \cdot \operatorname{Res}_z (z-w)^n f_m(z) + \sum_{n\geq 0} (z-w)^n \cdot \operatorname{Res}_z \partial_w^{(n)} \delta_-(z-w) f_m(z) \right) w^m \\
&= \sum_{n\geq 0} \partial_w^{(n)} \delta_-(z-w) \cdot \operatorname{Res}_z (z-w)^n f(z, w) + \sum_{m\in\mathbb{Z}} \left( \sum_{n\geq 0} (z-w)^n \cdot \operatorname{Res}_z \partial_w^{(n)} \delta_-(z-w) f_m(z) \right) w \\
&= \sum_{n\geq 0} \partial_w^{(n)} \delta_-(z-w) \cdot \operatorname{Res}_z (z-w)^n f(z, w) + g(z, w)
\end{aligned}
$$

where $g(z, w) \in R[[z, w^{\pm 1}]]$. Not that we can not interchange sum over $m \in \mathbb{Z}$ and sum over $n \geq 0$ in the second term because the product $\partial_w^{(n)} \delta_-(z-w) f(z-w)$ is not always well-defined. This decomposition is unique in some sense and we shall use it in next section.

# 19 Locality

In section 16, we have pointed out that the most important feature of field operators is that they commute with each other when evaluated at different point, i.e. for field operators $\Phi$ and $\Psi$,

$$[\Phi(x), \Psi(y)] = 0 \qquad \text{for } x \neq y \in M$$

We only study the case when $M$ is the complex plane without the origin, thus

$$[\Phi(z), \Psi(w)] = 0 \qquad \text{for } z \neq w \in \mathbb{C}^\times$$

If we consider $\Phi(z)$ and $\Psi(z)$ as elements in $R[[z^{\pm 1}]]$, where $R$ is an associative algebra over $\mathbb{C}$ (for example $End(\mathbb{H})$), then $[\Phi(z), \Psi(w)]$ becomes an element in $R[[z^{\pm 1}, w^{\pm 1}]]$. Then, we are interested in following question:

- Given a $f(z, w) \in R[[z^{\pm 1}, w^{\pm 1}]]$, what kinds of condition should $f(z, w)$ satisfies if for any $z_0, w_0 \in \mathbb{C}^\times$, $z_0 \neq w_0$, $f(z_0, w_0) = 0$, providing that $f(z_0, w_0)$ is well-defined.

We should give a algebraic answer to this analytic question. In last section, we tackle such kind of question by translating equations in complex analysis to equations in formal distributions. This time, we would use methodology in algebraic geometry to answer this question.

Let $A$ be an commutative ring and $M$ is a $A$-module. The support of $M$ is defined by

$$\text{Supp}(M) = \{\mathfrak{p} \in \text{Spec}A | M_\mathfrak{p} \neq 0\}$$

We should view $M$ as a sheaf on Spec $A$ and the support of $M$ are exactly those points which stalks of $M$ does not vanish. Define

$$\text{Ann}(M) = \{a \in A | a \cdot M = 0\}$$

$\text{Ann}(M)$ is an ideal of $A$.

**Lemma 19.1.** *If $M$ is finitely generated $A$-module, then*

$$V(Ann(M)) = Supp(M)$$

*where $V(I)$ is the set of all primes containing the ideal $I$.*

Back to our discussion, since $(z_0, w_0)$ above belongs to $\mathbb{C}^{\times 2}$, we should consider the space Spec $\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$. Then $R[[z^{\pm 1}, w^{\pm 1}]]$ is a $\mathbb{C}[z^{\pm 1}, w^{\pm 1}]$-module. Let $f$ be an element of $R[[z^{\pm 1}, w^{\pm 1}]]$ such that if $f(z_0, w_0)$ is well-defined for $(z_0, w_0) \in \mathbb{C}^{\times 2}$, $z_0 \neq w_0$, then $f(z_0, w_0) = 0$. We should view $(f)$ as a submodule of $R[[z^{\pm 1}, w^{\pm 1}]]$ supported on the line $z = w$, i.e.

$$\text{Supp}(f) = V(z - w)$$

Since $(f)$ is finitely generated, we have

$$V(\text{ann}(f)) = \text{Supp}(f) = V((z - w))$$

Thus, $(z - w) \in V(\text{Ann}(f)) = \sqrt{\text{Ann}(f)}$, there exists a $N \geq 1$ such that

$$(z - w)^N \cdot f = 0$$

Conversely, if $f \in R[[z^{\pm 1}, w^{\pm 1}]]$ such that there exists a integer $N \geq 0$, $(z-w)^N \cdot f = 0$, then apparently $f(z_0, w_0) = 0$ if it is well-defined for $z_0, w_0 \in \mathbb{C}^\times$, $z_0 \neq w_0$. Therefore,

**Definition** Let $a(z), b(w)$ be two formal distribution. We say $a(z)$ and $b(w)$ are mutually local if there exists an integer $N \geq 0$, such that

$$(z-w)^N \cdot [a(z), b(w)] = 0$$

Next, we aim to find out all formal distributions $f(z,w)$ such that $(z-w)^{n+1} \cdot f(z,w) = 0$ for some integer $n \geq 0$. Recall that

$$(z-w)^{n+1} \partial_w^{(n)} \delta(z-w) = 0$$

Thus, any finite linear combination of $\partial_w^{(n)} \delta(z-w)$ is annihilated by some power of $(z-w)$. We are going to show that the converse is also true.

Note that for any $f(z,w) \in R[[z^{\pm 1}, w^{\pm 1}]]$, we have shown in last section that

$$\begin{aligned} f(z,w) &= \sum_{n=0}^{\infty} \partial_w^{(n)} \delta_-(z-w) \operatorname{Res}_z (z-w)^n f(z-w) + \mathcal{O}(R[[z, w^{\pm 1}]]) \\ &= \sum_{n=0}^{\infty} \partial_w^{(n)} \delta(z-w) \operatorname{Res}_z (z-w)^n f(z-w) + \mathcal{O}(R[[z, w^{\pm 1}]]) \end{aligned}$$

The last equation is because ,$\partial_w^{(n)} \delta_+(z-w) \in R[[z, w^{\pm 1}]]$.

**Lemma 19.2.** *For any formal distribution $f(z,w) \in R[[z^{\pm 1}, w^{\pm 1}]]$, there exists unique $c_n(w) \in R[[w^{\pm 1}]]$, $n \geq 0$, and $g(z,w) \in R[[z, w^{\pm 1}]]$, such that*

$$f(z,w) = \sum_{n \geq 0} c_n(w) \partial_w^{(n)} \delta(z-w) + g(z,w)$$

*In particular, $c_n(w) = \operatorname{Res}_z (z-w)^n f(z,w)$.*

**Proposition 19.3.** *Let $f(z,w)$ be a formal distribution. Then $(z-w)^{N+1} f(z,w) = 0$, if and only if*

$$f(z,w) = \sum_{n=0}^{N} c_n(w) \partial_w^{(n)} \delta(z-w)$$

*where $c_n(w) = \operatorname{Res}_z (z-w)^n f(z,w)$.*

*Proof.* By lemma 19.2, we can assume that

$$f(z,w) = \sum_{n \geq 0} c_n(w) \partial_w^{(n)} \delta(z-w) + g(z,w)$$

Then

$$\begin{aligned} 0 = (z-w)^{N+1} f(z,w) &= \sum_{n \geq 0} c_n(w) (z-w)^{N+1} \partial_w^{(n)} \delta(z-w) + (z-w)^{N+1} g(z,w) \\ &= \sum_{n \geq N+1} c_n(w) \partial_w^{(n-N-1)} \delta(z-w) + (z-w)^{N+1} g(z,w) \\ &= \sum_{n \geq 0} c_{n+N+1}(w) \partial_w^{(n)} \delta(z-w) + (z-w)^{N+1} g(z,w) \end{aligned}$$

By the uniqueness of such a decomposition, we conclude that $c_n(w) = 0$ for all $n \geq N+1$ and $g(z,w) = 0$ □

From now on, we always write formal distributions in form of

$$f(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1} \qquad f(z,w) = \sum_{n,m\in\mathbb{Z}} a_{(m,n)} z^{-n-1} w^{-m-1}$$

Then

**Lemma 19.4.** *If*

$$f(z,w) = \sum_{k=0}^{N} c_k(w) \partial_w^{(k)} \delta(z-w)$$

*then*

$$f_{(n,m)} = \sum_{k=0}^{N} \binom{n}{k} c_{k,(n+m-k)}$$

*Proof.*

$$\begin{aligned}
f(z,w) &= \sum_{k=0}^{N} c_k(w) \partial_w^{(k)} \delta(z-w) \\
&= \sum_{k=0}^{N} \left( \sum_{l\in\mathbb{Z}} c_{k,(l)} w^{-l-1} \right) \left( \sum_{n\in\mathbb{Z}} \binom{n}{k} z^{-n-1} w^{n-k} \right) \\
&= \sum_{k=0}^{N} \sum_{n,l\in\mathbb{Z}} \binom{n}{k} c_{k,(l)} z^{-n-1} w^{n-k-l-1} \\
&= \sum_{k=0}^{N} \sum_{n,m\in\mathbb{Z}} \binom{n}{k} c_{k,(n+m-k)} z^{-n-1} w^{-m-1}
\end{aligned}$$

Thus,

$$f_{(n,m)} = \sum_{k=0}^{N} \binom{n}{k} c_{k,(n+m-k)}$$

□

# 20 Normal ordering and Locality

Recall that in the proof of classical version of the center theorem, we construct an algebra homomorphism:

$$S(\mathfrak{g}) \longrightarrow \mathcal{P} \hookrightarrow \widetilde{S}(\widehat{\mathfrak{g}})[[z^{\pm 1}]] \qquad A_1 \cdots A_n \longmapsto A_1(z) \cdots A_n(z)$$

For $A, B \in \mathfrak{g}$,

$$A(z)B(z) = \sum_{n\in\mathbb{Z}} \left( \sum_{k+l=n} A_k B_l \right) z^{-n-2}$$

This product in well-defined, i.e. $\sum_{k+l=n} A_k B_l$ is a well-defined element in $\widetilde{S}(\widehat{\mathfrak{g}})$, we use the fact that $S(\widehat{\mathfrak{g}})$ is an commutative algebra, so that

$$\sum_{k+l=n} A_k B_l = \sum_{\substack{k+l=n \\ k<0}} A_k B_l + \sum_{\substack{k+l=n \\ k\geq 0}} B_l A_k \in \widetilde{S}(\widehat{\mathfrak{g}})$$

In quantum case, we would construct an associative algebra $\widetilde{U}(\widehat{\mathfrak{g}})$ which is the completion of $U(\widehat{\mathfrak{g}})$ with respect to the $I$-adic topology (similar to $\widetilde{S}(\widehat{\mathfrak{g}})$) and we hope to construct a "multiplicative" map:

$$U(\mathfrak{g}) \longrightarrow \widetilde{U}(\widehat{\mathfrak{g}})[[z^{\pm 1}]]$$

such that $A \in \mathfrak{g}$ maps to $A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1}$. Intuitively, for any $A, B \in \mathfrak{g}$, we define the image of $A \cdot B$ to be

$$A(z)B(z) = \sum_{n\in\mathbb{Z}} \left( \sum_{k+l=n} A_k B_l \right) z^{-n-2} \tag{20.1}$$

However, this time the above method of interchanging two elements fails because $\widetilde{U}(\widehat{\mathfrak{g}})$ is not commutative. Hence, the product 20.1 is not well-defined!

One solution to this problem is to change the definition of product. Instead of taking product in usual sense, we define

$$: A(z)B(z) := \sum_{n\in\mathbb{Z}} \left( \sum_{\substack{k+l=n \\ k<0}} A_k B_l + \sum_{\substack{k+l=n \\ k\geq 0}} B_l A_k \right) z^{-n-2}$$

**Definition** The normal ordering is a rule of taking product of coefficients in two distribution. For $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ and $b(z) = \sum_{m\in\mathbb{Z}} b_m z^{-m-1}$, define

$$: a(z)b(w) := \sum_{n,m\in\mathbb{Z}} : a_n b_m : z^{-n-1} w^{-m-1}$$

where

$$: a_n b_m := \begin{cases} a_n b_m & n < 0 \\ b_m a_n & n \geq 0 \end{cases}$$

We say $R$ is a $I$-adic topological ring if it equips with $I$-adic topology defined by left ideals $I_0 \supset I_1 \supset \cdots$ such that

- $R$ is complete with respect to this topology.
- For any $x \in R$, $\mathrm{ad}_x : R \longrightarrow R$ is continuous with respect to this topology.

**Definition** A formal distribution $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1} \in R[[z^{\pm 1}]]$ is said to be $I$-adic if for each integer $n$, there exists $N$ such that $a_m \in I_n$ for all $m \geq N$.

**Proposition 20.1.** *Let $a(z)$ and $b(z)$ be two $I$-adic formal distributions. Then $: a(z)b(z) :$ is well-defined. Moreover, $: a(z)b(z) :$ is also $I$-adic.*

*Proof.* Assume $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ and $b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-1}$. Then

$$: a(z)b(z) := \sum_{n\in\mathbb{Z}} \left( \sum_{\substack{k+l=n \\ k<0}} a_k b_l + \sum_{\substack{k+l=n \\ k\geq 0}} b_l a_k \right) z^{-n-2}$$

For each $N>0$, since $a(z)$ and $b(z)$ are $I$-adic, there exists integers $N_1, N_2$, such that $a_n, b_m \in I_N$ for any $n \geq N_1$ $m \geq N_2$. Then

$$\sum_{\substack{k+l=n\\k<0}} a_k b_l + \sum_{\substack{k+l=n\\k\geq 0}} b_l a_k \equiv \sum_{\substack{k+l=n\\n<l<N_2}} a_k b_l + \sum_{\substack{k+l=n\\0\leq k<N_1}} b_l a_k \mod I_N$$

The right hand side is an element in $R$. Therefore, this product is well-defined.

Moreover, since the Lie bracket is continuous, their exists a $N' > N$, such that $[a_i, I_{N'}] \in I_N$ for all $i = 1, ..., N-1$. Since $b(z)$ is $I$-adic, there exists $N'' > N'$ such that $b_l \in I_{N'}$ for any $l > N''$. Then, for any $n \geq N'' + N$,

$$\sum_{\substack{k+l=n\\k<0}} a_k b_l + \sum_{\substack{k+l=n\\k\geq 0}} b_l a_k = \sum_{\substack{k+l=n\\k<N}} a_k b_l + \sum_{\substack{k+l=n\\0\leq k<N}} [b_l, a_k] + \sum_{\substack{k+l=n\\k\geq N}} b_l a_k \equiv 0 \mod I_N$$

Therefore, $:a(z)b(z):$ is also $I$-adic. □

We can define normal ordering for arbitrary number of formal distributions. For $a_1(z), ..., a_k(z) \in R[[z^{\pm 1}]]$, $a_i(z) = \sum_{n\in\mathbb{Z}} a_n^{(i)} z^{-n-1}$, we define

$$:a_1(z_1)\cdots a_k(z_k) := \sum_{n_1,\dots,n_k\in\mathbb{Z}} :a_{n_1}^{(1)}\cdots a_{n_k}^{(k)}: z_1^{-n_1-1}\cdots z_k^{-n_k-1}$$

where

$$:a_{n_1}^{(1)}\cdots a_{n_k}^{(k)} :=: a_{n_1}^{(1)}\left(:a_{n_2}^{(2)}\cdots a_{n_k}^{(k)}:\right):$$

Equivalently, if we define the normal ordering of two formal distributions $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$ and $b(z_1, ..., z_k) = \sum_{\alpha\in\mathbb{Z}^k} b_\alpha z^\alpha$ by

$$:a(z)b(z_1,...,z_k) := \sum_{n\in\mathbb{Z},\alpha\in\mathbb{Z}^k} :a_n b_\alpha: z^{-n-1} z^\alpha$$

where

$$:a_n b_\alpha := \begin{cases} a_n b_\alpha & n<0 \\ b_\alpha a_n & n\geq 0\end{cases}$$

Then

$$:a_1(z_1)\cdots a_k(z_k) :=: a_1(z_1)\left(:a_2(z_2)\cdots a_k(z_k):\right):$$

For example, if $a(z) = \sum_{n\in\mathbb{Z}} a_n z^{-n-1}$, $b(z) = \sum_{n\in\mathbb{Z}} b_n z^{-n-1}$, $c(z) = \sum_{n\in\mathbb{Z}} c_n z^{-n-1}$, then

$$:a_k b_m c_n := \begin{cases} a_k b_m c_n & k<0, m<0 \\ a_k c_n b_m & k<0, m\geq 0 \\ b_m c_n a_k & k\geq 0, m<0 \\ c_n b_m a_k & k\geq 0, m\geq 0\end{cases}$$

In general, we have

$$:a_{n_1}^{(1)}\cdots a_{n_k}^{(k)} := \prod_{\substack{i=1\\n_i<0}}^{k} a_{n_i}^{(i)} \prod_{\substack{i=k\\n_i\geq 0}}^{1} a_{n_i}^{(i)}$$

**Proposition 20.2.** *If $R$ is an $I$-adic topological ring and $a_1(z), ..., a_k(z)$ are $I$-adic formal distributions, then $: a_1(z) \cdots a_k(z) :$ is a well-defined $I$-adic formal distribution.*

*Proof.* Show by induction and proposition 20.1. □

The relation between locality and normal ordering is summarized in the following proposition:

**Proposition 20.3.** *The following are equivalent:*

1. $(z-w)^{N+1}[a(z), b(z)] = 0$

2. $$[a(z), b(w)] = \sum_{n=0}^{N} \partial_w^{(n)} \delta(z-w) \cdot c_n(w)$$

3. $$[a(z)_-, b(w)] = \sum_{n=0}^{N} \frac{1}{(z-w)^{n+1}}\bigg|_{|z|>|w|} c_n(w)$$
$$-[a(z)_+, b(w)] = \sum_{n=0}^{N} \frac{1}{(z-w)^{n+1}}\bigg|_{|z|<|w|} c_n(w)$$

4. $$a(z)b(w) = \sum_{n=0}^{N} \frac{1}{(z-w)^{n+1}}\bigg|_{|z|>|w|} c_n(w) + : a(z)b(w) :$$
$$b(w)a(z) = \sum_{n=0}^{N} \frac{1}{(z-w)^{n+1}}\bigg|_{|z|<|w|} c_n(w) + : a(z)b(w) :$$

**Corollary 20.4.** *If $a(z)$ and $b(z)$ are local to each other and*

$$[a(z), b(w)] = \sum_{k=0}^{N} \partial_w^{(k)} \delta(z-w) \cdot c_k(w)$$

*then*

$$[a_{(n)}, b_{(m)}] = \sum_{k=0}^{N} \binom{n}{k} c_{k,(n+m-k)}$$

# 21 Vertex algebras

We finally arrive at the definition of vertex algebra.

Let $V$ be a vector space, a formal distribution $f(z) \in \mathrm{End}V[[z^{\pm 1}]]$ is called a field if for any $v \in V$

$$f(z)v \in V((z))$$

A vertex algebra consists of following data:

1. (space of states) a vector space $V$,
2. (vacuum vector) a distinguished vector $|0\rangle \in V$,

3. (Translation operator) a linear operator $T : V \longrightarrow V$

4. (state-field correspondence) a linear map

$$Y(.,z) : V \longrightarrow \mathrm{End}V[[z^{\pm 1}]]$$

such that for any $A \in V$, $Y(A,z)$ is a field.

These data are subject to the following axioms:

1. (vacuum axiom) $Y(|0\rangle, z) = \mathrm{Id}$. Furthermore, for any $A \in V$, we have

$$Y(A,z)|0\rangle = A + \mathcal{O}(z)$$

In other words, $A_{(n)}|0\rangle = 0$ for all $n \geq 0$ and $A_{(-1)}|0\rangle = A$.

2. (transition axiom) For any $A \in V$,

$$[T, Y(A,z)] = \partial_z Y(A,z)$$

and $T|0\rangle = 0$

3. (locality axiom) All fields $Y(A,z)$ are local with respect to each other.

Let's try to understand the definition of vertex algebras through QFT.

1. The vector space $V$ with a distinguished vector $|0\rangle$ should be regarded as the space of states in quantum mechanics or quantum field theory. For example, in the second quantization, we construct the Fock space, which is a vector space spanned by states $|N_1, N_2, ...\rangle$. In particular, Fock space is a highest weight representation of $\mathfrak{h}_\infty$ generated by weight $|0\rangle$.

2. The state-field correspondence means that every vector (state) in $V$ corresponds to a field operator. To be specific, Let's think about quantizing fields over $S^1$ (You can also think about quantizing fields on $\mathbb{C}^\times$ in the context of string theory). In this case, the space of test function is $\mathbb{C}[z^{\pm 1}]$ and a field operator should be a linear map

$$\Psi : \mathbb{C}[z^{\pm 1}] \longrightarrow \mathrm{End}(V)$$

Now for each $A \in V$, $Y(A,z)$ is a formal distribution with coefficients in $\mathrm{End}(V)$. By proposition 17.1, we know that it induces a distribution

$$\widehat{Y}(A,z) : \mathbb{C}[z^{\pm 1}] \longrightarrow \mathrm{End}(V) \qquad f(z) \longmapsto \mathrm{Res}_z\, Y(A,z) \cdot f(z)$$

This is exactly a field operator. That's why we say every state (vector in $V$) corresponds to a field operator.

3. The field axiom is natural. In most case, to define the state-field correspondence, we first define an $I$-adic topological ring $R$ acting on $V$. The map $R \times V \longrightarrow V$ is defined to be continuous, where $V$ is usually endowed with discrete topology. Then

each $v \in V$ is annihilated by $I_n$ for all $n$ greater than some $N$. The state-field correspondence is defined by factoring through $R[[z^{\pm 1}]]$:

$$\begin{tikzcd} V \arrow[rr, "Y(,z)"] \arrow[dr, "\widetilde{Y}(,z)"'] & & \mathrm{End}(V)[[z^{\pm 1}]] \\ & R[[z^{\pm 1}]] \arrow[ur] & \end{tikzcd}$$

and we usually define $\widetilde{Y}(,z)$ so that its image are $I$-adic distributions in $R[[z^{\pm 1}]]$. Then naturally all vertex operators are fields.

4. The translation axiom seems to have no analogue in the quantum field theory we introduce. On the other hand, the translation operator is completely determined by the state-field correspondence as follow:

   **Lemma 21.1.** *For any $A \in V$, $TA = A_{-2}|0\rangle$*

   *Proof.*

   $$[T, Y(A,z)]|0\rangle = TY(A,z)|0\rangle - Y(A,z)T|0\rangle = T(A + \mathcal{O}(z)) = TA + \mathcal{O}(z)$$

   $$\partial_z Y(A,z)|0\rangle = \sum_{n\in\mathbb{Z}} -nA_{n-1}z^{-n-1}|0\rangle = A_{-2}|0\rangle + \mathcal{O}(z)$$

   Thus, $TA = A_{-2}|0\rangle$. $\square$

   Therefore, we can view $T$ as a distinguished operator on $V$. For our purpose, $T$ is always defined to be a derivative.

5. Locality is not strange to us now. In quantum field theory, it is common for two fields operators to commute with each other when they are evaluated at distinct points. For example,

   $$[\Psi(x), \Psi^\dagger(y)] = \delta(x-y)$$

   Similarly, in vertex algebra, we hope that any two field operators $Y(A,z)$, $Y(B,w)$ commute with each other when $z \neq w$. By our discussion in previous sections, it suffices to require that $Y(A,z)$ and $Y(B,w)$ are local as formal distributions. That's why we assume all fields are local to each other.

We present several properties of vertex algebras.

**Lemma 21.2.** *Let $H(z) \in R[[z]]$ and $f_0 \in R$. Then the ODE*

$$\frac{d}{dz}f(z) = H(z)\cdot f(z)$$

*has a unique solution with $f(z) \in R[[z]]$ and $f(0) = f_0$.*

**Proposition 21.3.** *Let $(V, |0\rangle, T, Y(.,z))$ be a vertex algebra and $A, B \in V$, then*

1. *$Y(A,z)|0\rangle = e^{zT}A$.*

2. *$e^{zT}Y(A,w)e^{-zT} = Y(A, w+z)$, where $(w+z)^{-n} = \partial_w^{(n)}\delta_+(z-(-w))$.*

*3.* $Y(A,z)B = e^{zT}Y(B,-z)A$

*Proof.* 1. Both $Y(A,z)|0\rangle$ and $e^{zT}A$ are solutions to the ODE

$$\frac{d}{dz}f(z) = T \cdot f(z)$$

with initial value $f(0) = A$. Thus, they are equal.

2.

$$\begin{aligned}
Y(A,w+z) &= \sum_{n<0} A_n(w+z)^{-n-1} + \sum_{n\geq 0} A_n(w+z)^{-n-1} \\
&= \sum_{n<0} A_n(w+z)^{-n-1} + \sum_{n\geq 0} A_n \sum_{k\geq 0} \binom{n}{k}(-1)^k w^{-n-1-k}z^k
\end{aligned}$$

Take $z = 0$, $Y(A,w+z)|_{z=0} = Y(A,w)$. Therefore, both $Y(A,w+z)$ and $e^{zT}Y(A,w)e^{-zT}$ are solutions to

$$\frac{d}{dz}f(z) = \frac{d}{dw}f(z) \qquad f(z) \in \mathrm{End}(V)[[w^{\pm 1}]][[z]]$$

Thus, they are equal.

3. $Y(A,z)$ and $Y(B,w)$ are local to each other, so there exists $N$ such that

$$(z-w)^N Y(A,z)Y(B,w) = (z-w)^N Y(B,w)Y(A,z)$$

evaluate at $|0\rangle$, we have

$$(z-w)^N Y(A,z)e^{wT}B = (z-w)^N Y(B,w)e^{zT}A = (z-w)^N e^{zT}Y(B,w-z)A$$

Since, $Y(B,w-z)$ is a field, $Y(B,w-z)A \in V((w-z))$. Take $N$ large enough so that $(z-w)^N Y(B,w-z)A \in V[[z-w]]$. Take $w = 0$ and divide $(z-w)^N$ on both sides, we obtains

$$Y(A,z)B = e^{zT}Y(B,-z)A$$

□

**Proposition 21.4.** *Let $A(z) \in EndV[[z^{\pm 1}]]$ be a field which is local to all field operators. Suppose there exists $B \in V$ such that*

$$A(z)|0\rangle = e^{zT}B$$

*then $A(z) = Y(B,z)$.*

*Proof.* Since $A(z)$ and $Y(B,w)$ local to each other, there exists $N > 0$ such that

$$\begin{aligned}
(z-w)^N A(z)Y(B,w)|0\rangle &= (z-w)^N Y(B,w)A(z)|0\rangle \\
(z-w)^N A(z)e^{wT}B &= (z-w)^N Y(B,w)e^{zT}B \\
&= (z-w)^N Y(B,w)Y(B,z)|0\rangle \\
&= (z-w)^N Y(B,z)Y(B,w)|0\rangle \\
&= (z-w)^N Y(B,z)e^{wT}B
\end{aligned}$$

we obtain the result by taking $w = 0$. □

**Proposition 21.5.** *For any $A \in V$, $Y(TA, z) = \partial_z Y(A, z)$.*

*Proof.* $\partial_z Y(A, z)$ is local with all vertex operators and

$$\partial_z Y(A, z)|0\rangle = [T, Y(A, z)]|0\rangle = e^{zT} TA = Y(TA, z)|0\rangle$$

By proposition 21.4, $Y(TA, z) = \partial_z Y(A, z)$. $\square$

For any $A, B \in V$, since $Y(A, z)$ and $Y(B, w)$ are local to each other, we have

$$[Y(A, z), Y(B, w)] = \sum_{n=0}^{N} c_n(w) \partial_w^{(n)} \delta(z - w) \tag{21.1}$$

where $c_n(w) = \mathrm{Res}_z (z - w)^n [Y(A, z), Y(B, w)]$. In general, for two fields $a(z), b(z)$ and $n \in \mathbb{Z}$, we define

$$a(z)_n b(z) = \mathrm{Res}_z\, a(z) b(w)(z - w)^n - b(w) a(z)(z - w)^n$$

when $n < 0$, the first $(z - w)^n$ is expanded as $|z| > |w|$ and the second one is expand as $|w| > |z|$. In particular,

$$a(w)_{-1} b(w) =: a(w) b(w) :$$

Hence, we can rewrite equation 21.2 as

$$[Y(A, z), Y(B, w)] = \sum_{n=0}^{N} Y(A, w)_n Y(B, w) \partial_w^{(n)} \delta(z - w) \tag{21.2}$$

**Proposition 21.6.** *For any $A, B \in V$,*

$$Y(A, w)_n Y(B, w) = Y(A_{(n)} B, w)$$

*Proof.* By proposition 21.4, it suffices to show that $Y(A, w)_n Y(B, w)|0\rangle = e^{wT} A_{(n)} B$.

$$\begin{aligned} Y(A, w)_n Y(B, w)|0\rangle &= \mathrm{Res}_z \left((z - w)^n Y(A, z) Y(B, w) - (z - w)^n Y(B, w) Y(A, z)\right)|0\rangle \\ &= \mathrm{Res}_z \left((z - w)^n Y(A, z) e^{wT} B - (z - w)^n Y(B, w) e^{zT} A\right) \\ &= \mathrm{Res}_z \left((z - w)^n e^{wT} Y(A, z - w) B\right) \\ &= e^{wT} A_{(n)} B \end{aligned}$$

$\square$

Therefore,

$$[Y(A, z), Y(B, w)] = \sum_{n=0}^{N} Y(A_{(n)} B, w) \partial_w^{(n)} \delta(z - w)$$

and

$$[A_{(n)}, B_{(m)}] = \sum_{n=0}^{N} \binom{n}{k} (A_{(k)} B)_{(m+n-k)}$$

**Definition** A vertex algebra $(V, |0\rangle, T, Y(., z))$ is said to be commutative if for any $A, B \in V$.

$$[Y(A, z), Y(B, w)] = 0$$

Let $A$ be a unitary commutative algebra with a derivative $T$, i.e. $T(ab) = T(a)b + aT(b)$. We can endow $A$ with a vertex algebra structure as follow

1. vacuum vector: $|0\rangle = 1$
2. Transition operator: $T$
3. state-field correspondence: for any $a \in A$,

$$Y(a, z) = e^{zT} \cdot a$$

   where $a \in \mathrm{End}A$ represents scalar multiplication by $a$.

It's easy to check that these data defines a commutative vertex algebra. In particular, the map $S(\widehat{\mathfrak{g}}_-) \longrightarrow \widetilde{S}(\widehat{\mathfrak{g}})[[z^{\pm 1}]] \longrightarrow S(\widehat{\mathfrak{g}}_-)[[z^{\pm 1}]]$ we constructed in section 13 defines a commutative vertex algebra on $S(\widehat{\mathfrak{g}}_-)$.

Conversely, given a commutative vertex algebra, we can construct a commutative associated algebra structure on it.

**Proposition 21.7.** *Let $(V, |0\rangle, T, Y(., z))$ be a commutative vertex algebra. Then the multiplication*

$$\cdot : V \times V \longrightarrow V \qquad (A, B) \longmapsto A_{(-1)}B$$

*defines an unitary commutative associated algebra structure on $V$ with derivative $T$ and unit $|0\rangle$*

# 22 Heisenberg vertex algebra

In this section, we introduce an example of non commutative vertex algebra. Recall that Fock space plays an important role in second quantization. We are going to define a vertex algebra structure on the Fock space. However, this time we construct the Fock space in a different way.

Let $\mathbb{C}[t^{\pm 1}]$ be the ring of test function. We consider it as a commutative Lie algebra. Define Heisenberg Lie algebra $\mathcal{H}$ to be a central extension of $\mathbb{C}[t^{\pm 1}]$,

$$0 \longrightarrow \mathbb{C} \cdot \mathbb{1} \longrightarrow \mathcal{H} \longrightarrow \mathbb{C}[t^{\pm 1}] \longrightarrow 0$$

with Lie bracket defined by

$$[f, g] = -\operatorname{Res}_z f dg \cdot \mathbb{1}$$

Denote $b_n = t^n \in \mathcal{H}$, then $\mathcal{H}$ is spanned by $\{b_n | n \in \mathbb{Z}\} \cup \{\mathbb{1}\}$. For $n, m \in \mathbb{Z}$,

$$[b_n, b_m] = n\delta_{n,-m} \cdot \mathbb{1}$$

Recall that we define another Heisenberg Lie algebra $\mathfrak{h}_\infty$ in... which is a Lie algebra generated by all creation and annihilation operators. There is an Lie algebra embedding $\mathfrak{h}_\infty \hookrightarrow \mathcal{H}$, given by

$$a_n \longmapsto \frac{1}{\sqrt{n}} b_n \qquad a_n^\dagger \longmapsto \frac{1}{\sqrt{n}} b_{-n} \qquad \mathrm{Id} \longmapsto \mathbb{1} \qquad n \geq 1$$

... To define a vertex algebra, we should consider an algebra bigger than $U(\mathcal{H})$. For $N \geq 0$, let $I_N$ be the left ideal of $U(\mathbb{H})$ generated by $b_N$. Consider the completion

$$\widetilde{U}(\mathcal{H}) = \varprojlim_N U(\mathcal{H})/I_N$$

Every element in $\widetilde{U}(\mathcal{H})$ can be expressed as

$$S + \sum_{n\geq 0} S_n b_n \qquad \text{where } S, S_n \in U(\mathcal{H})$$

Although we only quotient by left ideals, we can show that $\widetilde{U}(\mathcal{H})$ is an algebra with product induced from $U(\mathcal{H})$.

**Lemma 22.1.** *The product on $U(\mathcal{H})$ can be continuously lifted to a product on $\widetilde{U}(\mathcal{H})$.*

*Proof.* Let $A = \sum_{n\geq 0} S_n \cdot b_n$, $B = \sum_{n\geq 0} S'_n \cdot b'_n$. For each $N \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} A \cdot B &= \left(\sum_{n\geq 0} S_n \cdot b_n\right)\left(\sum_{m\geq 0} S'_m \cdot b'_m\right) \\ &= \left(\sum_{n\geq 0} S_n \cdot b_n\right)\left(\sum_{m=0}^{N-1} S'_m \cdot b'_m\right) \\ &= \sum_{m=0}^{N-1}\sum_{n\geq 0} S_n b_n S'_m b'_m \mod I_N \end{aligned}$$

For each $m \in \{0, ..., N-1\}$, there exists a $N_m > N$, such that for any $n > N_m$, $b_n$ commutes with $S'_m$. Hence, $S_n b_n S'_m b'_m = S_n S'_m b_n b'_m = S_n S'_m b'_m b_n \equiv 0 \mod I_N$. Thus

$$A \cdot B = \sum_{m=0}^{N-1}\sum_{n=0}^{N_m} S_n b_n S'_m b'_m \mod I_N$$

the right hand side is an element in $U(\mathcal{H})$. Therefore, $A \cdot B$ is a well-defined element in $\widetilde{U}(\mathcal{H})$. □

Therefore, we conclude that $\widetilde{U}(\mathcal{H})$ is an associative algebra. In our discussion, we only study representation of $\widetilde{U}(\mathcal{H})$ that $\mathbb{1}$ acts as identity. Hence, it is equivalent to study representation of Weyl algebra $\widetilde{\mathcal{H}} = \widetilde{U}(\mathcal{H})/(\mathbb{1} - 1)$.

Let $\mathcal{H}_+$ be the subalgebra of $\widetilde{\mathcal{H}}$ topologically generated by $b_n$, $n \geq 0$, and $\mathcal{H}_-$ be the subalgebra of $\widetilde{\mathcal{H}}$ generated by $b_n$, $n < 0$. Then, both $\mathcal{H}_-$ and $\mathcal{H}_+$ are commutative algebra and $\widetilde{\mathcal{H}} = \mathcal{H}_+ \otimes \mathcal{H}_-$. Let $\mathbb{C}$ be a trivial representation of $\mathcal{H}_+$. The Fock representation of $\widetilde{\mathcal{H}}$ is defined by

$$\pi = \mathrm{Ind}_{\mathcal{H}_+}^{\widetilde{\mathcal{H}}} \mathbb{C}$$

Since $\widetilde{\mathcal{H}} = \mathcal{H}_+ \otimes \mathcal{H}_-$,

$$\pi = \widetilde{\mathcal{H}} \otimes_{\mathcal{H}_+} \mathbb{C} \cong \mathcal{H}_- = \mathbb{C}[b_{-n}]_{n\geq 1}$$

It's easy to check that $b_n$ acts on $\pi$ by multiplication for $n \leq -1$, $b_0$ acts as zero map and $b_n$ acts on $\pi$ by $n\frac{\partial}{\partial b_{-n}}$ for $n \geq 1$. Hence, there is an isomorphism between $\pi$ and the Fock space we construct before:

$$V \longrightarrow \pi \qquad \prod_{k\in\mathbb{Z}_{\geq 0}} z_k^{n_k} \longrightarrow \prod_{k\in\mathbb{Z}_{\geq 0}} \left(\frac{1}{\sqrt{k}} b_{-k}\right)^{n_k}$$

This isomorphism is compatible with Lie algebra actions $\mathfrak{h}_\infty \hookrightarrow \mathcal{H}$.

Now we are going to define a $\mathbb{Z}_{\geq 0}$-graded vertex algebra structure on $\pi$.

1. (Grading) We define the degree of $\mathbb{C}$ in $\pi$ is zero. The degree of $b_{-n_1}\cdots b_{-n_k}$ is $n_1+\cdots+n_k$.
2. (Vacuum vector) $|0\rangle = 1 \in \mathbb{C}[b_{-n}]_{n\geq 1}$
3. (Translation operator) $T=-\partial_t$, i.e. $T$ is a derivative such that $T(b_{-n}) = nb_{-n-1}$. Or equivalently, as we will show later,

$$T = \frac{1}{2}\sum_{n\in\mathbb{Z}} b_n b_{-n-1} = \frac{1}{2}\sum_{k+l=-1} :b_k b_l: \in \widetilde{\mathcal{H}} \tag{22.1}$$

Here comes to the most sophisticated part: defining state-field correspondence. Since this correspondence is linear, it suffices to determine

$$Y(b_{-n_1}\cdots b_{-n_k}, z)$$

Let

$$b(z) = \sum_{b_n} z^{-n-1} \in \mathrm{End}(\pi)[[z^{\pm 1}]]$$

**Lemma 22.2.** *$b(z)$ is a field, $b(z)$ local to itself, and $[T, b(z)] = \partial_z b(z)$.*

*Proof.* It's easy to verify that $b(z)$ is a field.

$$\begin{aligned}
[b(z), b(w)] &= \sum_{n,m\in\mathbb{Z}} [b_n, b_m] z^{-n-1} w^{-m-1} \\
&= \sum_{n\in\mathbb{Z}} n z^{-n-1} w^{n-1} \\
&= \partial_w \delta(z-w)
\end{aligned}$$

Thus, $b(z)$ is local to itself.

Let's assume $T$ is defined by equation 22.1. We show that $[T, b_n] = -nb_{n-1}$:

$$\begin{aligned}
[T, b_n] &= \frac{1}{2}\sum_{m\in\mathbb{Z}} [b_m b_{-m-1}, b_n] \\
&= \frac{1}{2}\sum_{m\in\mathbb{Z}} [b_m, b_n] b_{-m-1} + b_m [b_{-m-1}, b_n] \\
&= \frac{1}{2}\sum_{m\in\mathbb{Z}} m\delta_{m,-n} b_{-m-1} + (-m-1)\delta_{m+1,n} b_m \\
&= -nb_{n-1}
\end{aligned}$$

Therefore, $[T, b_n] = -nb_{n-1}$. This also implies that $T = -\partial_t$. □

Define $Y(b_{-1}, z) = b(z)$. By proposition 21.5,

$$Y(b_{-k}, z) = \frac{1}{(k-1)!} Y(T^{k-1} b_{-1}, z) = \frac{1}{(k-1)!} \partial_w^{k-1} Y(b_{-1}, z) = \sum_{n\in\mathbb{Z}} \binom{-n-1}{k-1} b_n z^{-n-k}$$

How do we assign fields to products of $b_n$, such as $b_{-1}^2$? Intuitively, we would like to define

$$Y(b_{-1}^2, z) = b(z)^2 = \sum_{n\in\mathbb{Z}} \left(\sum_{k+l=n} b_k b_l\right) z^{-n-2}$$

However, the coefficient $\sum_{k+l=0} b_k b_l$ is not a well-defined element in $\mathrm{End}(\pi)$. One solution to this issue is to take normal ordering. Note that $\widetilde{\mathcal{H}}$ is an $I$-adic topological ring and $b(z)$ is an $I$-adic formal distribution. Thus, the following formal distribution is well-defined:

$$Y(b_{-1}^2, z) =: b(z)b(z):$$

Generally, we define

$$Y(b_{n_1}\cdots b_{n_k}, z) =: Y(b_{n_1}, z)\cdots Y(b_{n_k}, z):$$

by proposition 20.2, this is a well-defined formal distribution.

In summary, the vertex algebra structure on $\pi$ is defined by

1. (Vacuum vector) $|0\rangle = 1 \in \mathbb{C}[b_{-n}]_{n\geq 1}$,
2. (Translation operator) $T = -\partial_t$,
3. (state-field correspondence) $Y(|0\rangle, z) = \mathrm{Id}$

$$Y(b_{-n_1}\cdots b_{-n_k}, z) =: \partial_z^{(n_1-1)}b(z)\cdots\partial_z^{(n_k-1)}b(z):$$

Now we need to prove that these definitions satisfy axioms of a vertex algebra.

1. (Vacuum axiom) By definition, we have $Y(|0\rangle, z) = Id$. For $A = b_{-n_1}\cdots b_{-n_k}$,

$$\begin{aligned}
Y(A,z)|0\rangle &= \sum_{m\in\mathbb{Z}}\sum_{m_1+\cdots+m_k=m}\binom{-m_1-1}{n_1-1}\cdots\binom{-m_k-1}{n_k-1}\prod_{m_i<0} b_{m_i}\prod_{m_i\geq 0} b_{m_i}|0\rangle z^{-m-n_1-\cdots-n_k}\\
&= \sum_{m\in\mathbb{Z}}\sum_{\substack{m_1+\cdots+m_k=m\\ m_i<0}}\binom{-m_1-1}{n_1-1}\cdots\binom{-m_k-1}{n_k-1}\prod_{i=1}^{k} b_{m_i} z^{-m-n_1-\cdots-n_k}\\
&= \sum_{m\in\mathbb{Z}}\sum_{\substack{m_1+\cdots+m_k=m\\ m_i\leq -n_i}}\binom{-m_1-1}{n_1-1}\cdots\binom{-m_k-1}{n_k-1}\prod_{i=1}^{k} b_{m_i} z^{-m-n_1-\cdots-n_k}
\end{aligned}$$

Therefore, $Y(A,z)|0\rangle \in \mathrm{End}(V)[[z^{\pm 1}]]$ and $Y(A,z)|0\rangle|_{z=0} = \prod_{i=1}^{k} b_{-n_i} = A$.

2. (Translation axiom) We call a field $F(z)$ is perfect if $[T, F(z)] = \partial_z F(z)$. So, we need to show that all vertex operators are perfect. We have already prove that $b(z)$ is perfect, i.e. $[T, b_n] = -nb_{n-1}$. For $m \geq 1$,

$$\begin{aligned}
[T, Y(b_{-m}, z)] &= \sum_{n\in\mathbb{Z}}\binom{-n-1}{m-1}[T, b_n]z^{-n-m}\\
&= \sum_{n\in\mathbb{Z}}\binom{-n-1}{m-1} - nb_{n-1}z^{-n-m}\\
&= \sum_{n\in\mathbb{Z}}(-n-m)\binom{-n-1}{m-1}b_n z^{-n-m-1}\\
&= \partial_z Y(b_{-m}, z)
\end{aligned}$$

Thus $Y(b_{-m}, z)$ is perfect. We conclude that all vertex operators are perfect by following lemma

**Lemma 22.3.** *If $a(z)$ and $b(z)$ are perfect, then so is $:a(z)b(z):$.*

*Proof.* Since $a(z)$ is perfect, we have $[T, a_-(z)] = \partial_z a_-(z)$, $[T, a_+(z)] = \partial_z a_+(z)$

$$\begin{aligned}[T, :a(z)b(z):] &= [T, a_-(z)b(z)] + [T, b(z)a_+(z)] \\ &= Ta_-(z)b(z) - a_-(z)b(z)T + Tb(z)a_+(z) - b(z)a_+(z)T \\ &= [T, a_-(z)]b(z) + a_-(z)[T, b(z)] + [T, b(z)]a_+(z) + b(z)[T, a_+(z)] \\ &= \partial_z a_-(z)b(z) + a_-(z)\partial_z b(z) + \partial_z b(z)a_+(z) + b(z)\partial_z a_+(z) \\ &= \partial_z :a(z)b(z):\end{aligned}$$

□

3. (Locality axiom) We have shown that $b(z)$ is local to itself. By taking derivatives, we would know that $Y(b_{-n}, z)$ are local to each other for all $n \geq 1$. We conclude that all fields are local to each other by following lemma

   **Lemma 22.4.** *Let $a(z), b(z), c(z)$ be three fields that are local to each other, then $:a(z)b(z):$ is local to $c(z)$.*

   *Proof.* See [5], Lemma 2.3.4. □

Therefore, we conclude that $\pi$ is a vertex algebra.

It's easy to see that our proof can be generalized. The general version is so called Reconstruct Theorem. Reader can find the precise statement in [5], theorem 2.3.11.

# 23 Affine Kac-Moody vertex algebra

For the center theorem of classical version, we define a Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((t))$ acts on the space $S(\widehat{\mathfrak{g}}_-)$. We should view it as symmetry on the classical system $S(\widehat{\mathfrak{g}}_-)$. Now in quantum case, we quantize this symmetry, which should be a central extension of $\widehat{\mathfrak{g}}$ acting on a quantum system. This leads to our definition of affine Kac-Moody algebra $\widehat{\mathfrak{g}}_\kappa$.

**Definition** Let $\mathfrak{g}$ be a simple Lie algebra and $\kappa$ be an invariant bilinear form on $\mathfrak{g}$. The affine Kac-Moody algebra $\widehat{\mathfrak{g}}_\kappa$ is a central extension of $\widehat{\mathfrak{g}}$

$$0 \longrightarrow \mathbb{C} \cdot \mathbb{1} \longrightarrow \widehat{\mathfrak{g}}_\kappa \longrightarrow \widehat{\mathfrak{g}} \longrightarrow 0$$

with commutation relations: $[\mathbb{1}, \cdot] = 0$ (so $\mathbb{1}$ is central), and

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) - (\mathrm{Res}_t\, f dg)\kappa(A, B) \cdot \mathbf{1}$$

In particular,

$$[A_n, B_m] = [A, B]_{n+m} + n\delta_{n,-m}\kappa(A, B) \cdot \mathbb{1}$$

It is known that the space of invariant bilinear forms on $\mathfrak{g}$ is a one-dimensional space. Thus, different choices of $\kappa$ would only differ by a constant.

**Definition** A representation of $\widehat{\mathfrak{g}_\kappa}$ is called smooth if $\mathbb{1}$ acts as identity and for any $v \in V$ there exists an integer $N$ such that

$$\mathfrak{g} \otimes t^N \mathbb{C}[[t]] \cdot v = 0$$

in other words, $V$ is a discrete $\widehat{\mathfrak{g}_\kappa}$-module.

Every smooth $\widehat{\mathfrak{g}_\kappa}$- module is naturally a $U(\widehat{\mathfrak{g}_\kappa})$-module. It is also a module of a more larger algebra, as we are going to define below.

Let $I_N = U(\widehat{\mathfrak{g}_\kappa}) \cdot \mathfrak{g} \otimes t^N\mathbb{C}[[t]]$ be the left ideal of $U(\widehat{\mathfrak{g}_\kappa})$ generated by $\mathfrak{g} \otimes t^N\mathbb{C}[[t]]$. $\{I_n | n \geq 0\}$ defines an $I$-adic topology on $U(\widehat{\mathfrak{g}_\kappa})$.

**Lemma 23.1.** *For any $X \in U(\widehat{\mathfrak{g}_\kappa})$, $ad_X : U(\widehat{\mathfrak{g}_\kappa}) \longrightarrow U(\widehat{\mathfrak{g}_\kappa})$ is continuous with respect to the $I$-adic topology.*

*Proof.* We only need to prove the case when $X = (A_1 \otimes f_1) \cdots (A_k \otimes f_k)$. For any integer $N \geq 0$, we claim that there exists an integer $N' > N$, such that

$$[X, I_m] \subset I_N \qquad \text{for all } m > N'$$

Note that this is equivalent to

$$[X, \mathfrak{g} \otimes t^m\mathbb{C}[[t]]] \subset I_N \qquad \text{for any } m > N'$$

We show this claim by induction on $k$. When $k = 1$, $X = A \otimes f$, take $N' = \max N, N - \deg(f) + 1$, then for any $m > N'$ and $B \otimes g \in \mathfrak{g} \otimes t^m\mathbb{C}[[t]]$,

$$[X, B \otimes g] = [A \otimes f, B \otimes g] = [A, B] \otimes fg - \kappa(A, B) \operatorname{Res}_t f dg \cdot \mathbb{1} = [A, B] \otimes fg \in I_N$$

Assume we have proved the $k-1$ case. For $X = (A_1 \otimes f_1) \cdots (A_k \otimes f_k)$, assume $X' = (A_1 \otimes f_1) \cdots (A_{k-1} \otimes f_{k-1})$, then $X = X' \otimes (A_k \otimes f_k)$. By induction, there exists integers $N_1, N_2 > N$, such that

$$[A_k \otimes f_k, I_m] \subset I_N \text{ for any } m \geq N_1$$

$$[X', I_m] \subset I_{\max\{N, N_1\}} \text{ for any } m \geq N_2$$

Then, for any $m > \max\{N, N_1, N_2\}$ and $B \otimes g \in \mathfrak{g} \otimes t^m\mathbb{C}[[t]]$,

$$\begin{aligned}
[X, B \otimes g] &= [X', B \otimes g] \cdot A_k \otimes f_k + X'[A_k \otimes f_k, B \otimes g] \\
&= A_k \otimes f_k \cdot [X', B \otimes g] - [A_k \otimes f_k, [X', B \otimes g]] + X'[A_k \otimes f_k, B \otimes g] \\
&\in I_N
\end{aligned}$$

Therefore, the lemma holds. □

Define

$$\widetilde{U}(\widehat{\mathfrak{g}}_\kappa) = \varprojlim_N U(\widehat{\mathfrak{g}}_\kappa)/I_N$$

Then every element in $\widetilde{U}(\widehat{\mathfrak{g}}_\kappa)$ can be expressed as

$$S + \sum_{n \geq 0} S_n \cdot A_n^{(n)}$$

where $S, S_n \in U(\widehat{\mathfrak{g}}_\kappa)$ and $A^{(n)} \in \mathfrak{g}$.

**Lemma 23.2.** *The product on $U(\widehat{\mathfrak{g}}_\kappa)$ can be continuously lifted to a product on $\widetilde{U}(\widehat{\mathfrak{g}}_\kappa)$. Moreover, for any $X \in \widetilde{U}(\widehat{\mathfrak{g}}_\kappa)$, $ad_X$ is continuous.*

*Proof.* Consider two elements in $\widetilde{U}(\widehat{\mathfrak{g}}_\kappa)$:

$$A = \sum_{n\geq 0} S_n \cdot A_n^{(n)} \qquad A' = \sum_{n\geq 0} S'_n \cdot A_n^{\prime(n)}$$

Define the product by

$$A \cdot A' = \sum_{n\geq 0}\sum_{m\geq 0} S_n A_n^{(n)} \cdot S'_m A_m^{\prime(m)}$$

we need to verify that $A \cdot A' \in \widetilde{U}(\widehat{\mathfrak{g}}_\kappa)$, i.e. for every $N \in \mathbb{Z}_{>0}$, $A \cdot A' \mod I_N$ can be represented by an element in $U(\widehat{\mathfrak{g}}_\kappa)$.

$$A \cdot A' \equiv \sum_{m=0}^{N-1}\sum_{n\geq 0} S_n A_n^{(n)} \cdot S'_m A_m^{\prime(m)} \mod I_N$$

For each $m = \{0, ..., N-1\}$, since $\mathrm{ad}_{S'_m A_m^{\prime(m)}}$ is continuous, there exists a $N_m > N$, such that for any $n > N_m$, $[S'_m A_m^{\prime(m)}, I_n] \subset I_N$. Hence, for any $n > N_m$, $A_n^{(n)} \cdot S'_m A_m^{\prime(m)} = [A_n^{(n)}, S'_m A_m^{\prime(m)}] + S'_m A_m^{\prime(m)} \cdot A_n^{(n)} \equiv 0 \mod I_N$. Thus

$$A \cdot A' \equiv \sum_{m=0}^{N-1}\sum_{n=0}^{N_m} S_n A_n^{(n)} \cdot S'_m A_m^{\prime(m)} \mod I_N$$

The right hand side is an element in $U(\widehat{\mathfrak{g}}_\kappa)$. Therefore, this product is well-defined. $\square$

Now we are going to endow $V_\kappa(\mathfrak{g})$ with a $Z_{\geq 0}$-graded vertex algebra structure. It is defined as follow:

1. (Vacuum vector) $|0\rangle = v_k$.
2. (Translation operator) $T = -\partial_t$, or equivalently, if $J_1, ..., J_d$ is a basis for $\mathfrak{g}$ and $J^1, ..., J^d$ is its dual basis with respect to $\kappa$, then

$$T = \frac{1}{2}\sum_{a=1}^{d}\sum_{n\in\mathbb{Z}} J_{a,n} J^a_{-n-1} = \frac{1}{2}\sum_{a=1}^{d}\sum_{k+l=-1} : J_{a,k} J^a_l : \in \widetilde{U}_\kappa(\widehat{\mathfrak{g}}) \tag{23.1}$$

3. (Vertex operator) Define

$$\begin{tikzcd} V_k(\mathfrak{g}) \arrow[rr, "Y(,z)"] \arrow[dr, "\widetilde{Y}(,z)"'] & & \mathrm{End}(V_k(\mathfrak{g}))[[z^{\pm 1}]] \\ & \widetilde{U}_\kappa(\widehat{\mathfrak{g}})[[z^{\pm 1}]] \arrow[ur, "\pi"] & \end{tikzcd}$$

where $\widetilde{Y}(,z)$ is defined by $\widetilde{Y}(v_k, z) = \mathrm{Id}$

$$\widetilde{Y}(A_{-1}v_k, z) = A(z) = \sum_{n\in\mathbb{Z}} A_n z^{-n-1} \qquad \text{for any } A \in \mathfrak{g}$$

and generally for $A_1, ..., A_k \in \mathfrak{g}$,

$$Y(A_{1,-n_1} \cdots A_{k,-n_k} v_k, z) =: \partial_z^{(n_1)} A_1(z) \cdots \partial_z^{(n_k)} A_k(z) :$$

Since $A(z)$ is $I$-adic formal distribution for all $A \in \mathfrak{g}$, by proposition 20.2, this vertex operator map is well-defined.

Since $V_\kappa(\mathfrak{g})$ is spanned by elements of form

$$J_{a_1,-n_1}\cdots J_{a_k,-n_k}v_k$$

Hence, by reconstruction theorem, we only need to verify that

1. Vacuum axiom, that is $J_a(z)|0\rangle = J_a + \mathcal{O}(z)$ for $a = 1, ..., d$. This is obvious.
2. Translation axiom, that is $[T, J_a(z)] = \partial_z J_a(z)$ for $a = 1, ..., d$. Let's first show that two definition are equivalent. Suppose $T$ is defined by equation 23.1. Then in $\widetilde{U}_\kappa(\widehat{\mathfrak{g}})$,

$$\begin{aligned}
[T, J_{b,m}] &= \frac{1}{2}\sum_{a=1}^{d}\sum_{n\in\mathbb{Z}}[J_{a,n}J^a_{-n-1}, J_{b,m}] \\
&= \frac{1}{2}\sum_{a=1}^{d}\sum_{n\in\mathbb{Z}}[J_{a,n}, J_{b,m}]J^a_{-n-1} + J_{a,n}[J^a_{-n-1}, J_{b,m}] \\
&= \frac{1}{2}\sum_{a=1}^{d}\sum_{n\in\mathbb{Z}}[J_a, J_b]_{n+m}J^a_{-n-1} + n\kappa(J_a, J_b)\delta_{n,-m}J^a_{-n-1} + J_{a,n}[J^a, J_b]_{m-n-1} - (n+1)\kappa(J^a, J_b \\
&= \frac{1}{2}\sum_{a=1}^{d}\left(\sum_{n\in\mathbb{Z}}[J_a, J_b]_{n+m}J^a_{-n-1} + J_{a,n+m}[J^a, J_b]_{-n-1}\right) - \sum_{a=1}^{d} m\kappa(J_a, J_b)J^a_{m-1} - mJ_{a,m-1}
\end{aligned}$$

In $U(\mathfrak{g})$, we have

$$\sum_{a=1}^{d}[J_a, J_b]J^a + J_a[J^a, J_b] = 0 \text{ and } J_b = \sum_{a=1}^{d}\kappa(J_b, J_a)J^a$$

Thus,

$$[J_a, J_b]_{n+m}J^a_{-n-1} + J_{a,n+m}[J^a, J_b]_{-n-1} = 0 \text{ and } \sum_{a=1}^{d}\kappa(J^a, J_b)J^a_{m-1} = J_{b,m-1}$$

Therefore, $[T, J_{b,m}] = -mJ_{b,m-1}$. This implies that $T$ is indeed equal to $-\partial_t$ and $[T, J_b(z)] = \partial_z J_b(z)$.

3. Locality, that is $J_a(z)$ and $J_b(w)$ should be mutually local for all $a, b = 1, ..., d$.

$$\begin{aligned}
[J_a(z), J_b(w)] &= \sum_{n,m\in\mathbb{Z}}[J_{a,n}, J_{b,m}]z^{-n-1}w^{-m-1} \\
&= \sum_{n,m\in\mathbb{Z}}\left([J_a, J_b])n + m + n\delta_{n,-m}\kappa(J_a, J_b)\right)z^{-n-1}w^{-m-1} \\
&= \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}[J_a, J_b]_n + k\delta_{k,k-n}\kappa(J_a, J_b)z^{-k-1}w^{-n+k-1} \\
&= \sum_{n\in\mathbb{Z}}[J_a, J_b]_n w^{-n-1}\left(\sum_{k\in\mathbb{Z}}z^{-k-1}w^k\right) + \sum_{k\in\mathbb{Z}}k\kappa(J_a, J_b)z^{-k-1}w^{k-1} \\
&= [J_a, J_b](w)\delta(z-w) + \kappa(J_a, J_b)\partial_w\delta(z-w)
\end{aligned}$$

Thus, $J_a(z)$ and $J_b(w)$ mutually commute for any $a, b = 1, ..., d$.

In conclusion, $V_\kappa(\mathfrak{g})$ is a vertex algebra. We call it an affine Kac-Moody vertex algebra.

# 24 Center of Vertex algebra

Recall that our goal is to construct a large commutative subalgebra of $U(\widehat{\mathfrak{g}}_-)$. We have already endowed $U(\widehat{\mathfrak{g}}_-)$ with a new structure, vertex algebra structure. It is natural to ask: Does commutativity of vertex operators somehow indicate commutativity of elements in $U(\widehat{\mathfrak{g}}_-)$? The answer is Yes! In this section, we would find that the center of $V_\kappa(\mathfrak{g})$ as a vertex algebra gives arise to a commutative subalgebra of $U(\widehat{\mathfrak{g}}_-)$.

**Definition** Let $(V, |0\rangle, T, Y(,z))$ be a vertex algebra. The center of this vertex algebra is defined by

$$\mathfrak{Z}(V) \coloneqq \{A \in V | [Y(A,z), Y(B,w)] = 0 \text{ for any } B \in V\}$$

**Lemma 24.1.**

$$\begin{aligned}\mathfrak{Z}(V) &= \{A \in V | B_{(n)}A = 0 \textit{ for any } B \in V \textit{ and } n \geq 0\} \\ &= \{A \in V | A_{(n)}B = 0 \textit{ for any } B \in V \textit{ and } n \geq 0\} \\ &= \{A \in V | Y(A,z) \in \textit{End}(V)[[z]] \textit{ for any } B \in V\}\end{aligned}$$

*Proof.* By locality condition, for any $A, B \in V$,

$$[Y(A,z), Y(B,w)] = \sum_{n \geq 0} Y(A_{(n)}B, w)\partial_w^{(n)}\delta(z-w)$$

By the uniqueness of this decomposition, $[Y(A,z), Y(B,w)] = 0$ if and only if $Y(A_{(n)}B, w) = 0$ for all $n \geq 0$, or equivalently, $A_{(n)}B = 0$ for all $n \geq 0$. Therefore, $A$ is central if and only if $A_{(n)}B = 0$ ($B_{(n)}A = 0$) for any $B \in V$ and $n \geq 0$. $\square$

**Proposition 24.2.** $\mathfrak{Z}(V)$ *is a commutative vertex algebra.*

*Proof.* First, we need to verify that $T$ and $A_{(n)}$ are well-defined maps on $\mathfrak{Z}(V)$ for any $A \in \mathfrak{Z}(v)$ and $n \in \mathbb{Z}$. For any $B \in \mathfrak{Z}(V)$, $Y(B,z) \in \mathrm{End}(V)[[z]]$, so $Y(TB,z) = \partial_z Y(B,z) \in \mathrm{End}(V)[[z]]$, thus $TB \in \mathfrak{Z}(V)$. If $n \geq 0$, then $A_{(n)}B = 0 \in \mathfrak{Z}(V)$. If $n < 0$, then

$$\begin{aligned}Y(A_{(n)}B, w) &= \mathrm{Res}_z \left(Y(A,z)Y(B,w)\partial_w^{(n-1)}\delta_-(z-w) - Y(B,w)Y(A,z)\partial_w^{(n-1)}\delta_+(z-w)\right) \\ &= \mathrm{Res}_z \left(Y(A,z)Y(B,w)\partial_w^{(n-1)}\delta_-(z-w)\right) \\ &\in \mathrm{End}(V)[[w]]\end{aligned}$$

Thus, $A_{(n)}B \in \mathfrak{Z}(V)$. Therefore, $(\mathfrak{Z}(V), |0\rangle, T, Y(,z))$ is a well-defined vertex subalgebra. By definition, for any $A, B \in \mathfrak{Z}(V)$,

$$[Y(A,z), Y(B,w)] = 0$$

Thus, $(\mathfrak{Z}(V), |0\rangle, T, Y(,z))$ is a commutative vertex algebra. $\square$

Now we focus on the center of affine Kac-Moody vertex algebra $V_\kappa(\mathfrak{g})$. Denote

$$\mathfrak{Z}_\kappa(\mathfrak{g}) \coloneqq \mathfrak{Z}(V_\kappa(\mathfrak{g}))$$

**Lemma 24.3.** $\mathfrak{Z}_\kappa(\mathfrak{g}) = V_\kappa(\mathfrak{g})^{\mathfrak{g}[[t]]}$.

*Proof.* For any $X \in \mathfrak{Z}_\kappa(\mathfrak{g})$ and $A \in \mathfrak{g}$, $A_{(n)}X = 0$ for all $n \geq 0$, thus $X \in V_\kappa(\mathfrak{g})^{\mathfrak{g}[[t]]}$.

Conversely, if $X \in V_\kappa(\mathfrak{g})^{\mathfrak{g}[[t]]}$, then for any $A \in \mathfrak{g}$ and $k \geq 0$, $A_k X = 0$. For any $n, m \in \mathbb{Z}$,

$$[A_n, X_{(m)}] = \sum_{k=0}^{n} \binom{n}{k} (A_k X)_{m+n-k} = 0$$

Furthermore, this implies that for any $Y \in V_\kappa(\mathfrak{g})$, $[Y_{(n)}, X_{(m)}] = 0$ for any $n, m \in \mathbb{Z}$. Thus,

$$[Y(Y, z), Y(X, w)] = 0 \text{ for any } Y \in V_\kappa(\mathfrak{g})$$

This implies $X \in \mathfrak{Z}_\kappa(\mathfrak{g})$. Therefore, $\mathfrak{Z}_\kappa(\mathfrak{g}) = V_\kappa(\mathfrak{g})^{\mathfrak{g}[[t]]}$. □

This lemma enables us to identify $\mathfrak{Z}_\kappa(\mathfrak{g})$ with the algebra of $\widehat{\mathfrak{g}}_\kappa$-endomorphism of $V_\kappa(\mathfrak{g})$. Indeed, a $\widehat{\mathfrak{g}}_\kappa$-endomorphism $\varphi$ of $V_\kappa(\mathfrak{g})$ is uniquely determined by $\varphi(v_\kappa)$. If $\varphi(v_\kappa) = \alpha$, then $0 = \varphi(J_a^n v_\kappa) = J_a^n \varphi(v_\kappa) = J_a^n \alpha$. Thus $\alpha$ is $\mathfrak{g}[[t]]$-invariant. Conversely, if $\alpha \in \mathfrak{g}[[t]]$, then the map $\varphi : V_\kappa(\mathfrak{g}) \longrightarrow V_\kappa(\mathfrak{g})$ sending $v_\kappa$ to $\alpha$ induces a $\widehat{\mathfrak{g}}_\kappa$-linear transformation. Thus, we obtain an isomorphism

$$\Phi : V_\kappa(\mathfrak{g})^{\mathfrak{g}[[t]]} \longrightarrow \mathrm{End}_{\widehat{\mathfrak{g}}_\kappa}(V_\kappa(\mathfrak{g}))$$

Let $\alpha_1, \alpha_2 \in V_\kappa(\mathfrak{g})^{\mathfrak{g}[[t]]}$, $e_1, e_2$ are their corresponding endomorphisms. Then

$$e_1 \circ e_2(v_\kappa) = e_1(\alpha_2) = e_1((\alpha_2)_{(-1)} v_\kappa) = (\alpha_2)_{(-1)} \circ e_1(v_\kappa) = (\alpha_2)_{(-1)} \alpha_1$$

Since $V_\kappa(\mathfrak{g})^{\mathfrak{g}[[t]]} = \mathfrak{Z}(\widehat{\mathfrak{g}})$ is a commutative vertex algebra, $(\alpha_2)_{(-1)}\alpha_1 = (\alpha_1)_{(-1)}\alpha_2$. Thus $\Phi((\alpha_1)_{(-1)}\alpha_2) = \Phi(\alpha_1) \circ \Phi(\alpha_2)$, $\Phi$ is an isomorphism of commutative algebra. In particular, $\mathrm{End}_{\widehat{\mathfrak{g}}_\kappa}(V_\kappa(\mathfrak{g}))$ is a commutative algebra!

Finally, recall that $V_\kappa(\mathfrak{g})$ is isomorphic to $U(\widehat{\mathfrak{g}}_-)$. Thus we have an injection

$$\mathfrak{Z}(\widehat{\mathfrak{g}}) \hookrightarrow U(\widehat{\mathfrak{g}}_-)$$

This is an embedding of algebra. Thus, $\mathfrak{Z}(\widehat{\mathfrak{g}})$ can be viewed as a commutative subalgebra of $U(\widehat{\mathfrak{g}}_-)$.

# 25 Segal-Sugawara operators and the center theorem

One non-trivial fact regarding $\mathfrak{Z}_\kappa(\mathfrak{g})$ is that $\mathfrak{Z}_\kappa(\mathfrak{g})$ is trivial for all but exactly one invariant form $\kappa_c$, i.e.

$$\mathfrak{Z}_\kappa(\widehat{g}_-) = \mathbb{C}|0\rangle \text{ for all } \kappa \neq \kappa_c$$

$$\mathfrak{Z}_{\kappa_c}(\widehat{g}_-) \supsetneq \mathbb{C}|0\rangle$$

We call $\kappa_c$ the critical level of $\mathfrak{g}$. In this section, we would construct some operators associated to the Casimir element, called Segal-Sugawara operators, which belong to the center of affine Kac-Moody vertex algebra exactly when $\kappa = \kappa_c$.

Let $\kappa_0$ be a fixed invariant form, $\kappa_{\mathfrak{g}}$ be the Killing form, and $\kappa$ be an arbitrary invariant form on $\mathfrak{g}$. For any two invariant forms on $\mathfrak{g}$, $\kappa_1$ and $\kappa_2$ are, the value $\frac{\kappa_2(A,B)}{\kappa_1(A,B)}$ is a constant for any $A, B \in \mathfrak{g}$. We denote this constant by $\frac{\kappa_2}{\kappa_1}$. Let $J_1, ..., J_d$ be a basis of $\mathfrak{g}$ and $J^1, ..., J^d$ be the dual basis with respect to $\kappa_0$. Define

$$S = \frac{1}{2} \sum_{a=1}^{d} J_{a,-1} J^a_{-1} |0\rangle \in V_{\kappa_c}(\mathfrak{g})$$

and denote

$$S(z) = Y(S,z) = \sum_{n\in\mathbb{Z}} S_n z^{-n-2}$$

then $S_{(n)} = S_{n-1}$, $n \in \mathbb{Z}$. By definition, we have

$$S(z) = \frac{1}{2}\sum_{a=1}^{d} : J_a(z)J^a(z) = \frac{1}{2}\sum_{a=1}^{d}\sum_{n\in\mathbb{Z}}\left(\sum_{k+l=n} : J_{a,k}J^a_l :\right) z^{-n-2}$$

In particular, when $\kappa_0 = \kappa$, $S_0$ is exactly the translation operator $T$. We'd like to know how $S_m$ acts on $V_\kappa(\mathfrak{g})$.

**Proposition 25.1.** *For any $A \in \mathfrak{g}$, $[S_m, A_n] = -\frac{\kappa+\frac{1}{2}\kappa_{\mathfrak{g}}}{\kappa_0} nA_{m+n}$*

*Proof.* We only need to prove this equation for basis elements $J_1, ..., J_d$. We have

$$[J_{a,n}, S_{(m)}] = \sum_{k=0}^{n}\binom{n}{k}(J_{a,k}S|0\rangle)_{(m+n-k)} \tag{25.1}$$

$$\begin{aligned}
J_{a,k}S =&\frac{1}{2}\sum_{b=1}^{d} J_{a,k}J_{b,-1}J^b_{-1}\\
=&\frac{1}{2}\sum_{b=1}^{d} J_{b,-1}J^b_{-1}J_{a,k} + J^b_{-1}[J_a,J_b]_{k-1} + J_{b,-1}[J_a,J^b]_{k-1} + [J^b,[J_b,J_a]]_{k-2}\\
&+\kappa([J_a,J_b],J^b)(k-1)\delta_{k-1,1} + \kappa(J_a,J_b)k\delta_{k,1}J^b_{-1} + \kappa(J_a,J^b)k\delta_{k,1}J_{b,-1}
\end{aligned}$$

When $k \geq 3$, $J_{a,k}S|0\rangle = 0$. When $k = 2$,

$$J_{a,2}S|0\rangle = \frac{1}{2}\sum_{b=1}^{d}\kappa([J_a,J_b],J^b)|0\rangle = \kappa\left(J_a, \frac{1}{2}\sum_{b=1}^{d}[J_b,J^b]\right)$$

In $U(\mathfrak{g})$, we have

$$\frac{1}{2}\sum_{b=1}^{d}[J_b,J^b] = \frac{1}{2}\sum_{b=1}^{d} J_bJ^b - \frac{1}{2}J^bJ_b = \mathrm{Cas} - \mathrm{Cas} = 0$$

Thus, $J_{a,2}S|0\rangle = 0$. When $k = 1$,

$$J_{a,1}S|0\rangle = \frac{1}{2}\sum_{b=1}^{d}[J^b,[J_b,J_a]]_{-1} + \kappa(J_a,J_b)J^b_{-1} + \kappa(J_a,J^b)J_{b,-1}$$

Note that

$$J_a = \sum_{b=1}^{d}\kappa_0(J_a,J_b)J^b = \frac{\kappa_0}{\kappa}\sum_{b=1}^{d}\kappa(J_a,J_b)J^b$$

$$J_a = \sum_{b=1}^{d}\kappa_0(J_a,J^b)J_b = \frac{\kappa_0}{\kappa}\sum_{b=1}^{d}\kappa(J_a,J^b)J_b$$

$$\frac{1}{2}\sum_{b=1}^{d}[J^b,[J_b,J_a]] = \frac{1}{2}\sum_{b=1}^{d}\mathrm{ad}_{J^b}\mathrm{ad}_{J_b}(J_a) = \mathrm{ad}_{\mathrm{Cas}}(J_a)$$

Here $U(\mathfrak{g})$ acts on $\mathfrak{g}$ by adjoint action. The Casimir element should act by multiplying a scalar:

$$\mathrm{ad}_{\mathrm{Cas}} = \frac{1}{\dim\mathfrak{g}}\cdot\mathrm{tr}(\mathrm{Cas}) = \frac{1}{2d}\sum_{b=1}^{d}\mathrm{tr}\big(J_bJ^b\big) = \frac{1}{2d}\sum_{b=1}^{d}\kappa_{\mathfrak{g}}(J_b,J^b) = \frac{\kappa_{\mathfrak{g}}}{2\kappa_0}$$

Thus,

$$J_{a,1}S|0\rangle = \frac{\kappa+\frac{1}{2}\kappa_{\mathfrak{g}}}{\kappa_0}$$

When $k=0$,

$$\begin{aligned}
J_{a,0}S|0\rangle &= \frac{1}{2}\sum_{b=1}^{d} J^b_{-1}[J_a,J_b]_{-1} + J_{b,-1}[J_a,J^b]_{-1} + [J^b,[J_b,J_a]]_{-2}\\
&= \frac{1}{2}\sum_{b=1}^{d}[J_a,J_b]_{-1}J^b_{-1} + J_{b,-1}[J_a,J^b]_{-1} + [J^b,[J_a,J_b]]_{-2} + [J^b,[J_b,J_a]]_{-2}\\
&= \sum_{b=1}^{d}[J_a,J_b]_{-1}J^b_{-1} + J_{b,-1}[J_a,J^b]_{-1}\\
&= 0
\end{aligned}$$

In summary, by equation 25.1, we have

$$[J_{a,n},S_{(m)}] = \frac{\kappa+\frac{1}{2}\kappa_{\mathfrak{g}}}{\kappa_0}n(J_{a,-1})_{m+n-1} = \frac{\kappa+\frac{1}{2}\kappa_{\mathfrak{g}}}{\kappa_0}nJ_{a,m+n-1}$$

In other words,

$$[S_m,J_{a,n}] = -\frac{\kappa+\frac{1}{2}\kappa_{\mathfrak{g}}}{\kappa_0}nJ_{a,m+n}$$

□

**Remark 25.1.** *What we prove are commutator relations between $S_m$ and $A_n$ as operators on $V_\kappa(\mathfrak{g})$. However, we claim that it also implies same commutator relations between $S_m$ and $A_n$ as elements in $\widetilde{U}_\kappa(\mathfrak{g})$. A complete proof can be found in [5], Proposition 4.2.2.*

Define

$$\kappa_c = -\frac{1}{2}\kappa_{\mathfrak{g}}$$

$\kappa_c$ is called the critical value of $\mathfrak{g}$. The proposition 25.1 implies that if $\kappa=\kappa_c$, then for any $m\in\mathbb{Z}$, $S_m$ commutes with $J_{a,n}$ for all $a=1,...,d$ and $n\in\mathbb{Z}$. Therefore, we conclude that

1. $\mathbb{C}[S_m]_{m\in\mathbb{Z}}$ lies in the center of $\widetilde{U}_{\kappa_c}(\mathfrak{g})$,
2. $\mathbb{C}[S_m]_{m\le -2}|0\rangle$ lies in $\mathfrak{Z}_{\kappa_c}(\mathfrak{g})$.
3. $\mathbb{C}[S_m]_{m\le -2}|0\rangle$ is a commutative subalgebra of $U(\widehat{\mathfrak{g}}_-)$

It is also interesting to study the case when $\kappa \neq \kappa_c$. Denote

$$\widetilde{S} = \frac{\kappa_0}{\kappa - \kappa_c} S \qquad \widetilde{S}(z) = Y(\widetilde{S}, z) = \sum_{n\in\mathbb{Z}} \widetilde{S}_n z^{-n-2}$$

Then

$$[\widetilde{S}_m, A_n] = -nA_{m+n} \qquad A \in \mathfrak{g}$$

Therefore, both operators $\mathrm{ad}_{\widetilde{S}_m}$ acting on $\widetilde{U}_\kappa(\mathfrak{g})$ and operators $\widetilde{S}_m$, $m \geq -1$, acting on $V_\kappa(\mathfrak{g})$ are the same as operators $L_m = -t^{m+1}\frac{d}{dt}$. Furthermore,

**Proposition 25.2.** *$[\widetilde{S}_n, \widetilde{S}_m] = (n-m)\widetilde{S}_{n+m} + \frac{n^3-n}{12} c_\kappa \delta_{n,-m}$ where $c_\kappa = \frac{\kappa}{\kappa-\kappa_c} \cdot \dim \mathfrak{g}$.*

*Proof.* Assume $\mu_\kappa = \frac{\kappa_0}{\kappa-\kappa_c}$, then $\widetilde{S} = \mu_\kappa S$

$$[\widetilde{S}_{(n)}, \widetilde{S}_{(m)}] = \sum_{k=0}^{n} \binom{n}{k} (\widetilde{S}_{(k)}\widetilde{S}|0\rangle)_{(m+n-k)}$$

$$\begin{aligned}
\widetilde{S}_{(k)}\widetilde{S}|0\rangle &= \frac{\mu_\kappa}{2} \cdot \widetilde{S}_{k-1} \sum_{a=1}^{d} J_{a,-1}J^a_{-1}|0\rangle \\
&= \frac{\mu_\kappa}{2} \left( \sum_{a=1}^{d} [\widetilde{S}_{k-1}, J_{a,-1}]J^a_{-1} + J_{a,-1}[\widetilde{S}_{k-1}, J^a_{-1}] \right) |0\rangle \\
&= \frac{\mu_\kappa}{2} \left( \sum_{a=1}^{d} J_{a,k-2}J^a_{-1} + J_{a,-1}J^a_{k-2} \right) |0\rangle \\
&= \frac{\mu_\kappa}{2} \left( \sum_{a=1}^{d} J^a_{-1}J_{a,k-2} + J_{a,-1}J^a_{k-2} + \frac{\kappa}{\kappa_0} d\delta_{k-2,1} \right) |0\rangle
\end{aligned}$$

When $k \geq 4$ and $k = 2$, $\widetilde{S}_{(4)}\widetilde{S}|0\rangle = 0$. When $k = 3$, $\widetilde{S}_{(3)}\widetilde{S}|0\rangle = \frac{1}{2}c_k|0\rangle$. When $k = 1$,

$$\widetilde{S}_{(1)}\widetilde{S}|0\rangle = \frac{\mu_\kappa}{2} \left( \sum_{a=1}^{d} J^a_{-1}J_{a,-1} + J_{a,-1}J^a_{-1} \right) |0\rangle = 2\widetilde{S}$$

When $k = 0$,

$$\begin{aligned}
\widetilde{S}_{(0)}\widetilde{S}|0\rangle &= \frac{\mu_\kappa}{2} \left( \sum_{a=1}^{d} J^a_{-1}J_{a,-2} + J_{a,-1}J^a_{-2} \right) |0\rangle \\
&= \frac{\mu_\kappa}{2} \left( \sum_{a=1}^{d} J_{a,-2}J^a_{-1} + J_{a,-1}J^a_{-2} \right) |0\rangle \\
&= T\widetilde{S}
\end{aligned}$$

Since $Y(T\widetilde{S}, z) = \partial_z Y(\widetilde{S}, z)$, $(T\widetilde{S})_{(m+n)} = -(m+n)\widetilde{S}_{(m+n-1)}$. Thus

$$\begin{aligned}
[\widetilde{S}_{(n)}, \widetilde{S}_{(m)}] &= (T\widetilde{S})_{(m+n)} + n(2\widetilde{S})_{m+n-1} + \binom{n}{3}\frac{c_\kappa}{2}\delta_{m+n-3,-1} \\
&= (n-m)\widetilde{S}_{(m+n-1)} + \binom{n}{3}\frac{c_\kappa}{2}\delta_{m+n,2}
\end{aligned}$$

Therefore,

$$[\widetilde{S}_n, \widetilde{S}_m] = (n-m)\widetilde{S}_{n+m} + \frac{n^3-n}{12} c_\kappa \delta_{n,-m} \tag{25.2}$$

□

Equations 25.2 are exactly the defining relations of a Virasoro algebra. On the other hand, the Virasoro algebra indeed acts on this vertex algebra $V_\kappa(\mathfrak{g})$ as a consequence of change of coordinate. For more discussion, see [9], [4], Chapter3, Section5.

At last, let's state the center theorem:

**Theorem 25.3.** *There exists elements $S_i \in V_{\kappa_c}(\mathfrak{g})$, ord$(S_i) = d_i + 1$, deg$(S_i) = d_i + 1$, $i = 1, ..., r$, such that*

$$\mathfrak{Z}_{\kappa_c}(\mathfrak{g}) = \mathbb{C}[T^{n_i} S_i]_{n_i \geq 0, i=1,\ldots,d}$$

# 26 What's next?

The complete proof of theorem 25.3 requests quit amount of knowledge. In this section, we just give a brief introduction to the next step of this theorem. The main reference of this section is [9]. The whole proof of the center theorem can be found in Frenkel's book *Langlands correspondence for Loop group* .

The vertex algebra we define is actually coordinate dependent! To illustrate this point, let's think about a slightly unmotivated question: how to define an affine Kac-Moody vertex algebra bundle over a compact Riemann surface $X$? To be specific, an affine Kac-Moody vertex algebra bundle over $X$ is a vector bundle over $X$ such that each fiber is isomorphic to $V_\kappa(\mathfrak{g})$. We don't have to consider such a big problem now. Let's first consider how to attach an affine Kac-Moody algebra to one point on $X$.

We consider this compact Riemann surface as a projective smooth curve over $\mathbb{C}$. Let $x$ be a closed point on $X$. Let $\mathcal{O}_x$ be the completion of local ring (stalk of the structure sheaf) at $x$ . $\mathcal{K}_x$ is the completion of the function field $K(X)$ with respect to the valuation given by $\mathcal{O}_x$, which is a discrete valuation ring. From the knowledge of algebraic geometry, we know that if we choose an isomorphism $\mathcal{O}_x \cong \mathbb{C}[[t]]$, then $\mathcal{K}_x \cong \mathbb{C}((t))$. The affine Kac-Moody algebra $\widehat{\mathfrak{g}}_{\kappa,x}$ is defined to be the central extension of $\mathfrak{g} \otimes \mathcal{K}_x$ (the residue is coordinate independent). The vacuum $\widehat{\mathfrak{g}}_{\kappa,x}$-module is

$$V_\kappa(\mathfrak{g})_x = \mathrm{Ind}_{\mathfrak{g}\otimes\mathcal{O}_x \oplus \mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_{\kappa,x}} \mathbb{C}v_\kappa$$

The algebra $\mathfrak{Z}_\kappa(\mathfrak{g})_x$ is defined to be

$$\mathrm{End}_{\widehat{\mathfrak{g}}_{\kappa,x}} V_\kappa(\mathfrak{g})_x = (V_\kappa(\mathfrak{g})_x)^{\mathfrak{g}\otimes\mathcal{O}_x}$$

We know that this algebra is isomorphic to the center of affine Kac-Moody vertex algebra if we fix an isomorphism $\mathcal{O}_x \cong \mathbb{C}[[t]]$.

Up to now, what we have achieved is just translating our construction of affine Kac-Moody algebra into a more geometric revision. Next we are going to introduce another geometric object, "Opers", which turn out to have incredible relation with the center $\mathfrak{Z}_\kappa(\mathfrak{g})_x$.

**Definition** Let $G$ be a simply connected Lie group with a Borel subgroup $B$ and Lie algebra $\mathfrak{g}$. A $G$-oper on a space $X$ is a triple $(\mathcal{F}, \nabla, \mathcal{F}_B)$, where $\mathcal{F}$ is a principal $G$-bundle on $X$, $\nabla$ is a connection on $\mathcal{F}$ and $\mathcal{F}_B$ is a $B$-reduction of $\mathcal{F}$ such that $\mathcal{F}_B$ is transversal to $\nabla$. Denote the space of all $G$-opers on $X$ $(D_x)$ by $\mathrm{Op}_G(X)$ $(\mathrm{Op}_G(D_x))$.

Then the striking result is

**Theorem 26.1** (B.Feigin and E. Frenkel). *For finite dimensional simple Lie algebra $\mathfrak{g}$, the algebra $\mathfrak{Z}_{\kappa_c}(\mathfrak{g})_x$ is isomorphic to the algebra of functions on the space $\mathrm{Op}_{^LG}(D_x)$ of $^LG$-opers on the abstract disk $D_x$.*

If we choose a coordinate for the disk $D_x$, i.e. fix an isomorphism $D_x \cong \mathrm{Spec}\ \mathbb{C}[[z]]$, then

**Theorem 26.2.** *For finite dimensional simple Lie algebra $\mathfrak{g}$, the algebra $\mathfrak{Z}_{\kappa_c}(\mathfrak{g})$ is isomorphic to the algebra* $\mathrm{Fun}\ \mathrm{Op}_{^LG}(D)$ *in a* $(\mathrm{Der}\mathcal{O}, \mathcal{A}ut\mathcal{O})$*-equivalent way.*

where the action of $\mathrm{Der}\mathcal{O}$ (Virasoro algebra) describes how expressions of opers and elements in $\mathfrak{Z}_{\kappa_c}(\mathfrak{g})$ change under the change of coordinate.

We can explicitly describe the space of opers. On a given chart, it is given by

$$\mathrm{Op}_{^LG}(D) \cong \mathbb{C}[[z]]^{\oplus r}$$

Thus, each oper is represented by $(v_1(z), ..., v_r(z))$,

$$v_i(t) = \sum_{n<0} v_{i,n} z^{-n-1}$$

Then there is an isomorphism

$$\mathrm{Fun}\ \mathrm{Op}_{^LG}(D) \cong \mathbb{C}[v_{i,n}]_{i=1,\dots,r,n<0}$$

Here $v_{i,n}$ is the linear functional on $\mathrm{Op}_{^LG}(D)$ that the takes the $(-n-1)^{\mathrm{th}}$ coefficient of $v_i(z)$. We can show that the action of $\mathrm{Der}\mathcal{O}$ on $\mathbb{C}[v_{i,n}]_{i=1,\dots,r,n<0}$ is given by

$$L_n \cdot v_{1,m} = \begin{cases} (n-m+1)v_{1,m+n} & m+n \le -1 \\ -\frac{1}{2}(n^3-n) & m+n = 1 \\ 0 & \text{otherwise} \end{cases} \qquad L_n \cdot v_{i,m} = \begin{cases} (d_i(n+1)-m)v_{i,m+n} & m+n \le -1 \\ 0 & \text{otherwise} \end{cases}$$

Specially, we have $L_{-1} \cdot v_{i,m} = -m \cdot v_{i,m}$ for all $i = 1, ..., r$ and $m \le -1$.

For $\mathfrak{Z}_{\kappa_c}(\mathfrak{g})$, the operator $L_{-1} = -\partial_z$ is the same as the translation operator $T$ on $V_{\kappa_c}(\mathfrak{g})$. Since the isomorphism between $\mathfrak{Z}_{\kappa_c}(\mathfrak{g})$ and $\mathbb{C}[v_{i,n}]_{i=1,\dots,r,n<0}$ is $\mathrm{Der}\mathcal{O}$-equivalent, we get

$$L_{-1}^m S_i \longmapsto L_{-1}^m v_{i,-1}$$

Assume $Y(S_i v_\kappa, z) = \sum_{n\le 0} S_{i,(n)} z^{-n-1}$, then $L_{-1}^m S_i v_\kappa = T^m S_i v_\kappa = m! S_{i,(-m-1)}$ (recall that $\mathfrak{Z}_{\kappa_c}(\mathfrak{g})$ is a commutative vertex algebra). Besides, $L_{-1}^m v_{i,-1} = m! v_{i,-m-1}$, thus

$$S_{i,(-m-1)} \longrightarrow v_{i,-m-1}$$

Therefore, we obtain that

$$\mathfrak{Z}_{\kappa_c}(\mathfrak{g}) = \mathbb{C}[S_{i,(n)}]_{i=1,\dots,r;n<0} = \mathbb{C}[T^{n_i} S_i v_{\kappa_c}]_{n_i \ge 0; i=1,\dots,r}$$

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