Opers and Center of Affine Kac-Moody Vertex Algebras

Wei Wenqing

May 2023

Abstract

This paper studies the space of G-opers on a formal disk and the Feigin-Frenkel isomorphism. The first chapter discusses the equivalence among the space of projective connections, projective structures, and PGL_2-opers. In the second chapter, we establish the natural isomorphism between the space of G-opers and the direct sum of projective connections with powers of differential sheaves. The last chapter introduces the center of affine Kac-Moody vertex algebra and prove the commutativity of Segal-Sugawara operators at critical levels. Finally, we state the Feigin-Frenkel isomorphism and study its consequences.

Contents

1 PGL$_2$-Opers

To motivate the definition of G-oper for general Lie group, we first introduce the definition of $PGL_2(\mathbb{C})$-oper. The $PGL_2(\mathbb{C})$-opers are actually one of three different incarnations of one and the same object: the other two are projective connections and projective structures. We are going to introduce the projective connections first, then the projective structures. At last, we deduce the definition of $PGL_2(\mathbb{C})$-opers from projective structures.

1.1 Projective connections

In this section, we give the definition of projective connections on a formal disk and on a smooth algebraic curve.

Let X be a smooth algebraic curve, $x \in X(\mathbb{C})$. Denote the completion of the local ring at x by $\mathcal{O}_x$. From the knowledge of algebraic geometry, we know that there is an isomorphism $\varphi : \mathcal{O}_x \longrightarrow \mathbb{C}[[z]]$. However, this isomorphism is not canonical. The element z is a generator of the maximal ideal of $\mathcal{O}_x$. If we choose another generator w for this maximal ideal, then we get another isomorphism $\varphi' : \mathcal{O}_x \longrightarrow \mathbb{C}[[w]]$, and an isomorphism $\varphi' \circ \varphi^{-1} : \mathbb{C}[[z]] \longrightarrow \mathbb{C}[[w]]$. In this paper, We always say that $D_x = \text{Spec } \mathcal{O}_x$ is an abstract disk, $(\text{Spec}(\mathbb{C}[[z]]), \varphi)$ is a chart of D_x, $D = \text{Spec}(\mathbb{C}[[z]])$ is a formal disc, z is a formal coordinate, and $\varphi'\varphi^{-1} : \mathbb{C}[[z]] \longrightarrow \mathbb{C}[[w]]$ is a transition map between different charts. Since $\mathbb{C}[[w]] \longrightarrow \mathbb{C}[[z]]$ by sending w to z is an isomorphism, the transition maps can be viewed as an automorphism of $\mathbb{C}[[z]]$. If we give a (z)-adic topology on $\mathbb{C}[[z]]$ and only consider the continuous automorphisms (and hence transition maps), then the automorphism is uniquely determined by the image of z. Thus, $\text{Aut}(\mathbb{C}[[z]])$ can be viewed as a subset of $\mathbb{C}[[z]]$, $\text{Aut}(\mathbb{C}[[z]]) \cong (z(\mathbb{C}[[z]])^\times, \circ)$.

Let $D_x = \text{Spec}(\mathcal{O}_x)$ be an abstract disk, Ω_{D_x} is the $\mathcal{O}_x$-module of differentials, i.e., one-form on D_x. If $D_x \xrightarrow{\sim} \text{Spec}(\mathbb{C}[[z]])$ is a chart, then $\Omega_{D_x} \cong \mathbb{C}[[z]]dz$. Now we define the $\mathcal{O}_x$-module $\Omega^\lambda_{D_x}$ as the set of "λ-form", that is, things which look like $f(z)(dz)^\lambda$ on the chart $\text{Spec}(\mathbb{C}[[z]])$. Formally speaking, the $\mathcal{O}_x$-module $\Omega^\lambda_{D_x}$ is isomorphic to $\mathbb{C}[[z]](dz)^\lambda$ on the chart $\text{Spec}(\mathbb{C}[[z]])$. If $\text{Spec}(\mathbb{C}[[w]])$ is another chart with transition map $z = \mu(w)$, then the transition map of $\Omega^\lambda_{D_x}$ is intuitively given by:

$$\mathbb{C}[[z]](dz)^\lambda \longrightarrow \mathbb{C}[[w]](dw)^\lambda$$

$$f(z)(dz)^\lambda \longmapsto f(\mu(w))(\mu'(w))^\lambda(dw)^\lambda$$

A projective connection on D_x is a second order differential operator:

$$\rho : \Omega^{-\frac{1}{2}}_{D_x} \longrightarrow \Omega^{\frac{3}{2}}_{D_x}$$

such that the principal is 1 and the subprincipal is 0. Let's explain what this means. A second order differential operator has form $v_0(z)\partial_z^2 + v_1(z)\partial_z + v_2(z)$ on a given chart $\text{Spec}(\mathbb{C}[[z]])$, with action given by:

$$f(z)(dz)^{-\frac{1}{2}} \mapsto (v_0(z)f''(z) + v_1(z)f'(z) + v_2(z)f(z))(dz)^{\frac{3}{2}}$$

The principal symbol refers to $v_0(z)$ and the subprincipal symbol refers to $v_1(z)$. Thus, a projective connection on this chart is in form of

$$\partial_z^2 - v(z)$$

Suppose $\mathrm{Spec}(\mathbb{C}[[w]])$ is another chart with coordinate change given by $z = \mu(w)$, and the projective connection on this new chart has form $T(w) = v_0(w)\partial_w^2 + v_1(w)\partial_w - \widetilde{v}(w)$. The following diagram should commute:

$$\begin{tikzcd}
\mathbb{C}[[z]](dz)^{-\frac{1}{2}} \arrow[r, "\partial_z^2 - v(z)"] \arrow[d] & \mathbb{C}[[z]](dz)^{\frac{3}{2}} \arrow[d] \\
\mathbb{C}[[w]](dw)^{-\frac{1}{2}} \arrow[r, "T(w)"] & \mathbb{C}[[w]](dw)^{\frac{3}{2}}
\end{tikzcd}
\qquad
\begin{tikzcd}
f(z)(dz)^{-\frac{1}{2}} \arrow[r, mapsto] \arrow[d, mapsto] & (f''(z) - v(z)f(z))(dz)^{\frac{3}{2}} \arrow[d, mapsto] \\
f(\mu(w))(\mu'(w))^{-\frac{1}{2}}(dw)^{-\frac{1}{2}} \arrow[r, mapsto] & \bullet
\end{tikzcd}$$

Therefore, we should have an equation:

$$(v_0(w)\partial_w^2 + v_1(w)\partial_w - \widetilde{v}(w))(f(\mu(w))\mu'(w)^{-\frac{1}{2}}) = (f''(\mu(w)) - v(\mu(w))f(\mu(w)))\mu'(w)^{\frac{3}{2}}$$

Solve this equation, we get $v_0(w) = 1$, $v_1(w) = 0$, and

$$\widetilde{v}(w) = v(\mu(w))\mu'(w)^2 - \frac{1}{2}\{\mu, w\} \tag{1.1}$$

Where $\{\mu, w\}$ is called the Schwarzian derivative of $\mu(w)$, given by

$$\{\mu, w\} = \frac{\mu'''}{\mu'} - \frac{3}{2}\left(\frac{\mu''}{\mu'}\right)^2$$

Therefore, the condition on principal and subprincipal symbols are well-defined.

The notion of projective connections makes senses not only on the disk, but also on an arbitrary smooth algebraic curve X over $\mathbb{C}$. Let Ω_X be the differential sheaf of X. There exists line bundles whose square is Ω_X, we choose one and denote it by $\Omega_X^{\frac{1}{2}}$. A projective connection on X is a second order differential operator between the sheaves:

$$\rho : \Omega_X^{-\frac{1}{2}} \longrightarrow \Omega_X^{\frac{3}{2}}$$

such that the principal symbol is 1 and the subprincipal is 0.

1.2 Projective structures

In this section, we identify projective connections on X with a different kind of structure, called the projective structures.

A projective chart on X is a covering of X by analytic open subsets U_α, $\alpha \in \mathcal{A}$, together with local coordinates z_α and transition maps $z_\beta = f_{\alpha\beta}$ on the overlaps $U_\alpha \cap U_\beta$ given by Möbius transformations

$$f(z) = \frac{az + b}{cz + d} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C})$$

Two projective chart are called equivalent if their union is also a projective chart. The equivalent classes of projective chart are called projective structures.

Proposition 1.1. *There is a bijection between the set of projective structures on X and the set of projective connection on X.*

Proof. The following result plays a key role in the proof:

$$\{\varphi, z\} \text{ is zero if and only if } \varphi(z) \text{ is a Möbius transformation.}$$

Given a projective structure $\{U_\alpha, z_\alpha, f_{\alpha\beta}\}$, we can define a projective connection by assigning the second order operator $\partial^2_{z_\alpha}$ on each chart U_α. For two charts U_α, U_β, $\partial^2_{z_\alpha}$ transforms to $\partial^2_{z_\beta} - \frac{1}{2}\{f_{\alpha\beta}, z_\alpha\}$. Since $f_{\alpha\beta}$ is a Möbius transformation, $\{f_{\alpha\beta}, z_\alpha\} = 0$. Thus, $\partial^2_{z_\alpha}$ transforms to $\partial^2_{z_\beta}$. Thus, this connection is globally defined.

Conversely, given a projective connection. On the chart U_α with coordinate z_α, assume the connection can be written as $\partial^2_{z_\alpha} - v_\alpha(z_\alpha)$. Consider the space of solutions of the differential equation

$$(\partial^2_{z_\alpha} - v_\alpha(z_\alpha))\phi(z_\alpha) = 0$$

on U_α. It has a two-dimensional space of solutions, spanned by $\phi_{1,\alpha}$ and $\phi_{2,\alpha}$. Choosing the cover to be fine enough, we may assume that ϕ_2 is never zero and the Wronskian of the two solutions is never zero. Define

$$\mu_\alpha = \frac{\phi_{1,\alpha}}{\phi_{2,\alpha}} : U_\alpha \longrightarrow \mathbb{C}$$

This map locally gives a complex coordinate w_β on $V_\beta \subset U_\alpha$. By calculation, one can find that the connection on the chart V_β can be expressed as $\partial^2_{w_\alpha}$. Therefore, the transition maps between $\{V_\beta\}$ must have zero Schwarzian derivative. THus, the transition maps between $\{V_\beta\}$ are Möbius transformations. This defines a projective structure on X. □

1.3 PGL_2-Opers

Now we rephrase the definition of projective structures in a coordinate independent way, which we can generalize to the general case.

Suppose X has a projective structure given by $\{U_\alpha, z_\alpha, f_{\alpha\beta} : \alpha, \beta \in \mathcal{A}\}$, where $f_{\alpha\beta}$ are Möbius transformations. For each pair $\alpha, \beta \in \mathcal{A}$, assume the transition map is defined by $A_{\alpha\beta} \in \mathrm{PGL}_2(\mathbb{C})$. We associate a constant map to the overlap $U_\alpha \cap U_\beta$

$$\widetilde{f}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathrm{PGL}_2(\mathbb{C})$$

whose image is just $A_{\alpha\beta}$. For each triple of charts $U_\alpha, U_\beta, U_\gamma$, the transition maps satisfy $f_{\alpha\gamma} = f_{\gamma\beta} \circ f_{\beta\alpha}$ on $U_\alpha \cap U_\beta \cap U_\gamma$, that is $A_{\alpha\gamma} = A_{\gamma\beta} \cdot A_{\beta\alpha}$. Hence, $\widetilde{f}_{\alpha\gamma} = \widetilde{f}_{\gamma\beta} \circ \widetilde{f}_{\beta\alpha}$. Thus, the datum of transition maps $\{\widetilde{f}_{\alpha\beta}\}$ define a flat $\mathrm{PGL}_2(\mathbb{C})$-bundle $\mathcal{F}$. Besides, this flat $\mathrm{PGL}_2(\mathbb{C})$-bundle has a flat connection ∇ which is just ∂_{z_α} on each trivialization $U_\alpha \times \mathrm{PGL}_2(\mathbb{C})$.

We have rephrased the datum of transition maps $\{f_{\alpha\beta}\}$ by a principal bundle. Now we need to rephrase the datum of coordinate $\{z_\alpha\}$ by some geometric objects.

The group $\mathrm{PGL}_2(\mathbb{C})$ acts on projective line $\mathbb{P}^1$ by matrix multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x : y] = [ax + by : cx + dy]$$

Let's form the associated bundle

$$\mathbb{P}^1_{\mathcal{F}} = \mathcal{F} \times_{\mathrm{PGL}_2(\mathbb{C})} \mathbb{P}^1$$

Suppose $\mathcal{F}$ can be trivialized on $U_\alpha \subset X$, then $\mathbb{P}^1_{\mathcal{F}}$ can also be trivialized on U_α. We can use the coordinate z_α to define a local section of $\mathbb{P}^1_{\mathcal{F}}|_{U_\alpha} = U_\alpha \times \mathbb{P}$. The section is defined by $s_\alpha : U_\alpha \longrightarrow \mathbb{P}^1$, $s(p) = [z_\alpha(p) : 1]$. These local sections $\{s_\alpha : \alpha \in \mathcal{A}\}$ actually define a global section on $\mathbb{P}^1_{\mathcal{F}}$. Because the coordinates transform between each other by exactly the same element in $\mathrm{PGL}_2(\mathbb{C})$ which we used as the transition function of our bundle on the overlapping open subsets. Therefore, the geometric object relating to coordinates $\{z_\alpha\}$ is a global section of the associated bundle $\mathbb{P}^1_{\mathcal{F}}$ which has non-vanishing derivative at all points with respect to the connection ∇.

Definition A $\mathrm{PGL}_2(\mathbb{C})$-oper on X is a triple $(\mathcal{F}, \nabla, s)$, where $\mathcal{F}$ is a flat $\mathrm{PGL}_2(\mathbb{C})$-bundle, ∇ is a connection on $\mathcal{F}$, and s is a section on the associated bundle $\mathbb{P}^1_{\mathcal{F}}$, which has non-vanishing derivative with respect to ∇.

We have seen that a projective connection/structure gives rise to a $\mathrm{PGL}_2(\mathbb{C})$-oper on X. Conversely, suppose a $\mathrm{PGL}_2(\mathbb{C})$-oper $(\mathcal{F}, \nabla, s)$ is given. We can use the section to define local coordinates for X. On the overlaps, the transition maps are given by the transition of $\mathcal{F}$, which are exactly the Möbius transformations. Hence, the $\mathrm{PGL}_2(\mathbb{C})$-oper can induce a projective structure on X.

Therefore, in this section, we prove the following three geometric objects are equivalent on smooth projective curves:

$$\boxed{\text{Projective Connections}} \longleftrightarrow \boxed{\text{Projective Structures}} \longleftrightarrow \boxed{\mathrm{PGL}_2(\mathbb{C})\text{-oper}}$$

We wish to generalize the concept of $\mathrm{PGL}_2(\mathbb{C})$-oper to more general Lie group G. The $\mathrm{PGL}_2(\mathbb{C})$-bundle with a connection should be generalize to a principal G-bundle with a connection. What about the section s on $\mathbb{P}^1_{\mathcal{F}}$? In order to generalize it, we need to give another description about $\mathbb{P}^1$. Note that $\mathrm{PGL}_2(\mathbb{C})$ acts on $\mathbb{P}^1$ transitively. Consider the stablizer of $[1 : 0]$, it is the subgroup B of all upper triangular matrices in $\mathrm{PGL}_2(\mathbb{C})$, which is a Borel subgroup of $\mathrm{PGL}_2(\mathbb{C})$. Thus, $\mathbb{P}^1$ can be identified with the homogeneous space $\mathrm{PGL}_2(\mathbb{C})/B$. An element of $\mathbb{P}^1$ can be regarded as a right coset of B in $\mathrm{PGL}_2(\mathbb{C})$. Hence, at every point $x \in X$, the value of a section (on $\mathbb{P}^1_{\mathcal{F}}$) at x actually gives a right coset in the fiber $\mathcal{F}_x$. Therefore, a section on $\mathbb{P}^1_{\mathcal{F}}$ gives a subbundle $\mathcal{F}_B$ in $\mathcal{F}$. Moverover, this subbundle $\mathcal{F}_B$ is a B-reduction. That is, it satisfies

$$\mathcal{F} = \mathcal{F}_B \times_B \mathrm{PGL}_2(\mathbb{C})$$

Now we have to interpret the non-vanishing condition on the derivative of our section. Recall that connection on fiber bundle is equivalent to parallel transport between fibers. Suppose $F \to E \to M$ is a fiber bundle, with connection ∇, and $s \in \Gamma(E)$. For $x \in M$, $s(x) = p$, we can construct a horizontal section $\widetilde{s}_x$ in a open neighborhood U of x by parallel transport p to its neighborhood. The derivative ∇s doesn't vanish at x roughly means that s doesn't intersect $\widetilde{s}_x$ in a small neighborhood of U, except at x. In our situation the section is replaced by a subbundle, but the idea is the same. In a neighborhood of $x \in X$, we can construct a horizontal subbundle $\mathcal{F}'_B$ by parallel transport the whole fiber $\mathcal{F}_{B,x}$ to its neighborhood (We call such a bundle is preserved by the connection ∇). Then the non-vanishing condition on the derivative roughly means that $\mathcal{F}_B$ doesn't intersect the bundle $\mathcal{F}'_B$ in a neighborhood of D, except at x. We will make this idea precise in the next chapter.

2 Oper

2.1 Definition of G-opers

Let G be a simple algebraic group of adjoint type, B is a Borel subgroup and $N = [B, B]$ its unipotent radical, with corresponding Lie algebras $\mathfrak{n} \subset \mathfrak{b} \subset \mathfrak{g}$. Choose a Cartan subgroup $H \hookrightarrow B$ with Lie algebra $\mathfrak{h}$. Then $B = N \rtimes H$. $\mathfrak{g}$ is a simple Lie algebra. It has a Cartan decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

where $\mathfrak{n}_\pm = \bigoplus_{\alpha\in\Delta_+} \mathfrak{n}_{\pm\alpha}$ (here $\mathfrak{n}_+ = \mathfrak{n}$, and $\mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{b}$). Assume $\mathfrak{g}$ has dimension d and rank l. Choose generators $e_1, ..., e_l$ of $\mathfrak{n}_+$, $h_1, ..., h_l$ of $\mathfrak{h}$, $f_1, ..., f_l$ of n_- such that their commutators are determined by the Cartan matrix of $\mathfrak{g}$ (see Appendix A). Let $\alpha_1, ..., \alpha_l$ be the simple roots, $w_1, ..., w_l$ be the coweights, i.e. $\alpha_i(w_j) = \delta_{i,j}$.

Let $[\mathfrak{n}, \mathfrak{n}]^\perp \subset \mathfrak{g}$ be the orthogonal complement of $[\mathfrak{n}, \mathfrak{n}]$ with respect to Killing form on $\mathfrak{g}$. $[\mathfrak{n}, \mathfrak{n}] = \bigoplus_{\alpha\in\Delta_+,|\alpha|\geq 2} \mathfrak{n}_\alpha$, thus

$$[\mathfrak{n}, \mathfrak{n}]^\perp/\mathfrak{b} = \bigoplus_{i=1}^{l} \mathfrak{n}_{\alpha_i}$$

The Borel subgroup B acts on $[\mathfrak{n}, \mathfrak{n}]^\perp/\mathfrak{b}$ by adjoint action. $B = N \rtimes H$. Let's figure out how N and H act on $[\mathfrak{n}, \mathfrak{n}]^\perp/\mathfrak{b}$.

Lemma 2.1. *(1) N acts on $[\mathfrak{n}, \mathfrak{n}]^\perp/\mathfrak{b}$ as identity.*

(2) Assume $x \in H$, $x = \exp(h), h \in \mathfrak{h}$. For $\gamma = \sum_{i=1}^{l} \lambda_i f_i \in [\mathfrak{n}, \mathfrak{n}]^\perp/\mathfrak{b}$, $\lambda_i \in \mathbb{C}, i = 1, ..., l$, we have

$$x \cdot \lambda = \sum_{i=1}^{l} \lambda_i e^{-\alpha_i(h)} f_i$$

(3) H acts on the subset

$$\left\{ \sum_{i=1}^{l} \lambda_i f_i : \lambda_i \neq 0, i = 1, ..., l \right\}$$

simply transitive.

Proof. (1) For any $x = \exp(y) \in N, y \in \mathfrak{n}$, $\gamma = \sum_{i=1}^{l} \lambda_i f_i$,

$$\mathrm{Ad}_x\gamma = \exp(\mathrm{ad}_y)\gamma = (1 + \mathrm{ad}_y + \frac{1}{2}(\mathrm{ad}_y)^2 + \cdots)\gamma$$

For $n \geq 1$, $(\mathrm{ad}_y)^n\gamma \in \mathfrak{b}$. Thus, $\mathrm{Ad}_x\lambda = \lambda \mod \mathfrak{b}$. N acts on $[\mathfrak{n}, \mathfrak{n}]^\perp/\mathfrak{b}$ as identity.

(2)

$$\mathrm{ad}_h \cdot f_i = -\alpha_i(h) f_i$$

Thus,

$$\mathrm{Ad}_x\gamma = \exp(\mathrm{ad}_h)\gamma = \sum_{i=1}^{l}\sum_{i=0}^{+\infty} \frac{\mathrm{ad}_h^n}{n!}\lambda_i f_i = \sum_{i=1}^{l}\sum_{i=0}^{+\infty} \frac{(-\alpha_i(h))^n}{n!}\lambda_i f_i = \sum_{i=1}^{l} e^{-\alpha_i(h)}\lambda_i f_i$$

(3) We can see from (2) that for any $\gamma = \sum_{i=1}^{l} \lambda_i f_i$, there exists a unique $x \in H$, namely, $x = \exp\Big(\sum_{i=1}^{l} \ln(\lambda_i) w_i\Big)$, such that $x \cdot \gamma = \sum_{i=1}^{l} f_i$. Therefore, the action of H is simply transitive.

$\square$

Denote $\mathbf{O} = \Big\{\sum_{i=1}^{l} \lambda_i f_i : \lambda_i \neq 0, i = 1, ..., l\Big\}$. The Lemma above actually shows that $\mathbf{O}$ is an open B-obit, and is actually a H-torsor.

Let X be a smooth curve and $x \in X(\mathbb{C})$. In the last chapter, we argue that a G-opers on X or D_x, should be a principal G-bundle with a connection, and a B-reduction bundle satisfies some non-vanishing conditions on the derivative. Let $\mathcal{F}$ be a principal G-bundle, ∇ be a connection on $\mathcal{F}$, and $\mathcal{F}_B$ be a B-reduction. We define the relative position of ∇ and $\mathcal{F}_B$ (i.e. the failure of ∇ to preserve $\mathcal{F}_B$) as follow.

Locally, choose any flat connection ∇' on $\mathcal{F}$ that preserves $\mathcal{F}_B$ (Such connection exists! For example, choose a local trivialization $U \times G$ such that the subbundle $\mathcal{F}_B$ is just $U \times B$, then the fat connection $\nabla_{\partial_z} = \partial_z$ obviously preserves $\mathcal{F}_B$). Take the difference $\nabla - \nabla'$. The difference of two connections can be viewed as a $\mathfrak{g}$-value section of Ω_X. Thus, $\nabla - \nabla'$ is a local section of $\mathfrak{g}_{\mathcal{F}} \otimes \Omega_X$. We project it onto $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B} \otimes \Omega_X$. The resulting local section of $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B} \otimes \Omega_X$ is independent of the choice ∇'. These local sections patch together to define a global $(\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B}$-valued one-form on X. Denote this global section by $\nabla/\mathcal{F}_B$. We say $\mathcal{F}_B$ is *transversal* to ∇ if the one-form $\nabla/\mathcal{F}_B$ takes values in $\mathbf{O}_{\mathcal{F}_B} \subset (\mathfrak{g}/\mathfrak{b})_{\mathcal{F}_B}$.

Definition a G-oper on X is a triple $(\mathcal{F}, \nabla, \mathcal{F}_B)$, where $\mathcal{F}$ is a principal G-bundle on X, ∇ is a connection on $\mathcal{F}$ and $\mathcal{F}_B$ is a B-reduction of $\mathcal{F}$ such that $\mathcal{F}_B$ is transversal to ∇. Denote the space of all G-opers on X (D_x) by $\mathrm{Op}_G(X)$ ($\mathrm{Op}_G(D_x)$).

Let's see what these conditions means locally. Suppose the principal bundle $\mathcal{F}$ can be trivialized on an analytic disc $U \subset X$ with coordinate z. Let $U \times G$ be the trivialization such that $U \times B$ is the bundle $\mathcal{F}_B$ (we call such kind of trivialization as B-preserved trivialization). Then, the connection ∇ can be written as

$$\partial_z + A(z)$$

where $A(z) \in \mathfrak{g}$ for all $z \in U$ (In appendix **??**, we explain why a connection on a principal bundle can be written as a differential operator). Take $\nabla' = \partial_z$ to be the connection that preserve $\mathcal{F}_B$. Then $\nabla - \nabla' = A(z)dz \in \mathfrak{g} \otimes \Omega_U$. By the oper condition:

$$A(z) = \sum_{i=1}^{l} \phi_i(z) f_i \mod \mathfrak{b}$$

where $\phi_i(z)$ doesn't vanish on U, $i = 1, ..., l$. Thus, there exists a function $\mathfrak{b}(z)$ with values in $\mathfrak{b}$, such that

$$\nabla_{\partial_z} = \partial_z + \sum_{i=1}^{l} \phi_i(z) f_i + b(z)$$

The above discussion is based on an analytic open disc. We can translate the result to the formal disk $D_x = \mathrm{Spec}(\mathcal{O})$. That is, given a chart $\mathcal{O} = \mathrm{Spec}(\mathbb{C}[[z]])$, and a B-preserved trivialization of $\mathcal{F}$, the connection ∇ can be written as

$$\nabla_{\partial_z} = \partial_z + \sum_{i=1}^{l} \phi_i(z) f_i + b(z)$$

where $\phi_i(z) \in \mathbb{C}[[z]]$ and $\phi_i(0) \neq 0$, $i = 1, ..., l$, $b(z) \in b[[z]]$.

2.2 Opers on formal disk

In this section, we study the space of all opers on the formal disk. In order to let the gauge transformation make sense on the formal disk, we start with an analytic open disk U with coordinate z and trivial principal G-bundle $\mathcal{F} = U \times G$. Assume this trivialization is B-preserved. Then, in last section we conclude that the connection ∇ appearing in the G-oper can be written as

$$\nabla_{\partial_z} = \partial_z + \sum_{i=1}^{l} \phi_i(z) f_i + b(z) \tag{2.1}$$

where $\phi_i(z)$ doesn't vanish on U, $i = 1, ..., l$ and $\mathfrak{b}(z)$ is a $\mathfrak{b}$-valued function. Suppose now we have another B-preserved trivialization for $\mathcal{F}$. Then the transition map is given by a B-valued function function $B(z) : U \longrightarrow B$, such that

$$U \times G \text{ (old)} \longrightarrow U \times G \text{ (new)} \qquad (z, g) \longmapsto (z, B(z) \cdot g)$$

The transitions of connections follow the gauge transformations. Thus, over the new trivialization, the connection is given by

$$\nabla_{\partial_z} = \partial_z + B(z) \left(\sum_{i=1}^{l} \phi_i(z) f_i + b(z) \right) B(z)^{-1} - B^{-1}(z) \partial_z B(z)$$

Notice that $B = N \rtimes H$, $B(z)$ can be written as $N(z)H(z)$, where $N(z)$ is a N-valued function and $H(z)$ is a H-valued function. For any connection (2.1), there exists a unique H-valued function $H(z)$, namely

$$H(z) = \prod_{i=1}^{l} \check{w}_i(\phi_i(z))$$

such that the corresponding gauge transformation brings our connection (2.1) to

$$\nabla_{\partial_z} = \partial_z + \sum_{i=1}^{l} f_i + v(z)$$

where $v(z)$ is a $\mathfrak{b}$-valued function. Denote

$$\widetilde{\mathrm{Op}}_G(U) = \left\{ \partial_z + \sum_{i=1}^{l} f_i + v(z) : v(t) \text{ is a } \mathfrak{b}\text{-valued} \right\}$$

Then

$$\mathrm{Op}_G(U) = \widetilde{\mathrm{Op}}_G(U) \Big/ N\text{-valued gauge equivalence}$$

Now we can make sense all the discussion above on the formal disk Spec($C[[z]]$). Given a B-preserved trivialization of $\mathcal{F}$, the connection can be written as

$$\nabla_{\partial_z} = \partial_z + \sum_{i=1}^{l} \phi_i(z) f_i + b(z)$$

where $\phi_i(z) \in \mathbb{C}[[z]]$ and $\phi_i(0) \neq 0$, $i = 1, ..., l$, $b(z) \in b[[z]]$. The B-valued gauge transformation is now given by elements in formal Taylor series group $B[[z]]$, whose Lie algebra is $\mathfrak{b}[[z]] = \mathfrak{b} \otimes \mathbb{C}[[z]]$. We first normalize the coefficients before f_i, $i = 1, ..., l$. The previous method to normalize the coefficients doesn't work here, because the homomorphism $\check{w}_i : \mathbb{C}^\times \to H$ doesn't take power series as inputs. We need to generalize the cocharacters to the formal formal Taylor series group.

Proposition 2.2. *(a) the exponential map*

$$\exp : \mathbb{C}[[z]] \longrightarrow (\mathbb{C}[[z]])^\times \qquad f(z) \longmapsto \sum_{n\geq 0} \frac{1}{n!}(f(z))^n$$

is well-defined and surjective with kernal $\{2\pi n i : n \in \mathbb{Z}\}$.

(b) Let $h(z) = \sum_{n\geq 0} h_n z^n \in \mathfrak{h}[[z]]$, *then* $\exp(\mathrm{ad}(h(z)))f_i = \exp\left(-\sum_{n\geq 0} \alpha_i(h_n)z^n\right) f_i$.

(c) Let $\phi_i(z) \in \mathbb{C}[[z]]^\times$, $i = 1, ..., l$, *there exists a unique* $H(z) \in H[[z]]$, *such that* $\mathrm{Ad}_{H(z)}\left(\sum_{i=1}^l \phi_i(z) f_i\right) = \sum_{i=1}^l f_i$.

Proof. (a) Let $f(z) = \sum_{n\geq 0} a_n z^n \in \mathbb{C}[[z]]$. The n^{th} coefficient of $\exp(f(z))$ only dependents on the coefficients of $f(z)$ with degree $\leq n$. So, Let $f_n(z) = \sum_{k=0}^n a_k z^k$, $e^{f_n(z)}$ is a holomorphic function over $\mathbb{C}$. Thus the n^{th} coefficient of $\exp(f(z))$ is the n^{th} coefficient of the Taylor expansion of $\exp(f_n(z))$ at $z = 0$, which is finite. Thus $f(z)$ is well-defined.

Next we prove the surjectivity. Let $g(z) = \sum_{n\geq 0} a_n z^n$, $a_0 \neq 0$. We prove that there exists a sequence of polynomials $f_0(z), f_1(z), ...,$ such that $f_{n+1}(z) \equiv f_n(z) \mod z^{n+1}$ and $\exp(f_n(z)) \equiv g(z) \mod z^{n+1}$. We construct $\{f_n(z)\}$ by induction. For $n = 0$, take $f_0(z) = \ln(a_0)$. Suppose we have constructed $f_0(z), ..., f_{k-1}(z)$. Assume $\exp(f_{k-1}(z)) = \sum_{i=1}^{k-1} a_i z^i + b_k z^k \mod z^{k+1}$. Then

$$f_k(z) = f_{k-1}(z) + \frac{a_k - b_k}{a_0} z^k$$

satisfies the condition. Thus, $\{f_n(z)\}$ exists. Take $f(z) = \lim_{n\to+\infty} f_n(z) \in \mathbb{C}[[z]]$, then $\exp(f(z)) = g(z)$. Thus, the exponential map is surjective.

At last, we compute the kernel. Suppose $f(z) = \sum_{n\geq} a_n z^n \in \mathbb{C}[[z]]$ such that $\exp(f(z)) = 1$. Let $m > 0$ be the least integer such that $a_m \neq 0$.

$$1 = \exp(f(z)) = e^{a_0}(1 + a_m z^m) \mod z^{m+1}$$

Contradiction! Thus, $f(z)$ has only constant term and $e^{a_0} = 1$. Therefore, the kernel is $\{2\pi n i : n \in \mathbb{Z}\}$.

(b) The calculation is similar to Lemma 2.1.

(c) By (a), there exists $\varphi_i(z) = \sum_{n\geq} a_{i,n} z^n$, such that $\exp(\varphi_i(z)) = \phi_i(z)$. Now take

$$h_n = \sum_{i=1}^l a_{i,n} w_i \qquad n \in \mathbb{Z}_{\geq 0}$$

Let $h(z) = \sum_{n\geq 0} h_n z^n$ and $H(z) = \exp(h(z))$. By (b), we get

$$\mathrm{Ad}_{H(z)}\left(\sum_{i=1}^l \phi_i(z) f_i\right) = \sum_{i=1}^l f_i$$

Suppose that $H'(z) = \exp(h'(z)) \in H[[z]]$ also satisfies the condition. Then

$$\exp(ad(h'(z) - h(z)))f_i = f_i$$

By (b), this implies $\exp\left(-\sum_{n\geq 0} \alpha_i(h'_n - h_n)z^n\right) = 1$. By (a), we have $\alpha_i(h'_n - h_n) = 0$ for $n \geq 1$ and $\alpha_i(h'_0 - h_0) \in 2\pi i\mathbb{Z}$, $i = 1, ..., l$. Thus $h'_n = h_n$ for $n \geq 1$ and $w = h'_0 - h_0 \in 2\pi i\Lambda_{coweight}$.

$$H'(z) = \exp(h'(z)) = \exp(w + h(z)) = \exp(w)\exp(h(z)) = H(z)$$

We proved the uniqueness.

□

Corollary 2.3. *For $\gamma \in \Lambda_{coweight}$, the cocharacter*

$$\widetilde{\gamma} : \mathbb{C}[[Z]]^{\times} \longrightarrow H[[z]] \qquad f(z) \longmapsto \exp(\ln(f(z))\gamma)$$

is well-defined.

Therefore, by Proposition 2.2, we conclude that

$$\mathrm{Op}_G(\mathcal{O}) = \widetilde{\mathrm{Op}}_G(\mathcal{O}) \Big/ N[[z]]\text{-gauge equivalence}$$

The action of $N[[z]]$ on $\widetilde{\mathrm{Op}}_G(\mathcal{O})$ is actually free. To prove this fact and find the equivalence classes, we need the following lemma. Denote $p_{-1} = \sum_{i=1}^{l} f_i$

Lemma 2.4. *1. There exists $p_0 \in \mathfrak{h}$ and $p_1 \in span\{e_1, ..., e_l\}$, such that $\{p_{-1}, p_0, p_1\}$ is a $\mathfrak{sl}_2$-triple.*

2. $\mathfrak{g}$ decomposes into $\bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$, where $\mathfrak{g}_n = \{\alpha \in \mathfrak{g} : [p_0, \alpha] = 2n\alpha\}$.

3. For $n \geq 0$, the map $ad_{p_{-1}} : \mathfrak{g}_{n+1} \to \mathfrak{g}_n$ is an injection. Assume $\mathfrak{g}_n = V_n \oplus ad_{p_{-1}}(\mathfrak{g}_{n+1})$, then $V_n \neq 0$ if and only if n is an exponent of $\mathfrak{g}$.

4. $V = \bigoplus_{n\in\mathbb{Z}} V_n$, then $\dim V = l$, and V is the invariant subspace of the action ad_{p_1}.

The proof in given in the appendix A.

Let $p_1, ..., p_l$ be a basis of V such that p_i is in $\mathfrak{g}_{d_i}$ and p_1 is just the element in the $\mathfrak{sl}_2$-triple.

Theorem 2.5. *The action of $N[[z]]$ on $\widetilde{\mathrm{Op}}_G(\mathcal{O})$ is free.*

Proof. We claim that for each element $\partial_z + p_{-1} + v(t) \in \widetilde{\mathrm{Op}}_G(\mathcal{O})$, there exists a unique $U(z) \in n[[z]]$ and $c(z) \in V[[z]]$, such that

$$\partial_z + p_{-1} + v(t) = \exp(ad(U(z))) \cdot (\partial_z + p_{-1} + c(z)) \tag{2.2}$$

Under the principal grading, assume $v(t) = \sum_{i\geq 0} v_i(z)$, $v_i(z) \in \mathfrak{g}_i[[z]]$. We use induction to construct $U(t) = \sum_{i\geq 0} U_i(z)$, $U_i(z) \in \mathfrak{g}_i[[z]]$, $c(t) = \sum_{i\geq 1} c_i(z)$, $c_i(z) \in V_i[[z]]$, such that the (2.2) holds. When $m = 0$, Equating the homogeneous components of degree 0 in both sides of (2.2) gives

$$v_0(z) = [U_1(z), p_{-1}]$$

since $ad_{p_{-1}} : \mathfrak{g}_1 \longrightarrow \mathfrak{g}_0$ is bijective, there exists a unique $U_1(z)$ such that the equality holds. Now suppose we have already constracted $U_1(z), ..., U_m(z)$ and $c_i(z)$, $i < m$. Now we construct $U_{m+1}(z)$ and $c_i(z)$ if $i = m$. Equating the homogeneous components of degree m in both sides of (2.2) gives

$$v_m(z) = c_m(z) + [U_{m+1}, p_{-1}] + \text{terms expressed in } U_i, i \leq m \text{ and } c_j, j < m$$

Since $\mathfrak{g}_m = V_m \oplus ad_{p_{-1}}(\mathfrak{g}_{m+1})$, and $ad_{p_{-1}} : \mathfrak{g}_{m+1} \longrightarrow \mathfrak{g}_m$ is injective, this determines $c_m(z)$ and $U_{m+1}(z)$ uniquely. Therefore, by induction, there exists a unique pair $(U(z), c(z))$, such that (2.2) holds. This implies the action of $N[[z]]$ on $\widetilde{\mathrm{Op}}_G(\mathcal{O})$ is free.

□

Corollary 2.6. *The set*

$$\left\{\partial_z + p_{-1} + \sum_{i=1}^{l} v_i(z)p_i : v_i(z) \in \mathbb{C}[[z]], i = 1, ..., l\right\}$$

is a complete representatives of of $\widetilde{\mathrm{Op}}_G(\mathcal{O})$ under the action of $N[[z]]$. We call it the gauge equivalence classes.

Thus, set-theoretically, there is a bijection

$$\mathrm{Op}_G(D) \cong \mathbb{C}[[z]]^{\oplus l} \tag{2.3}$$

2.3 Opers on the abstract disk

In last section, we find out a complete representatives for G-opers over a formal disk D, and thereby obtains an isomorphism $\mathrm{Op}_G(D) \cong \mathbb{C}[[z]]^{\oplus l}$. In this section, we give a description for the space of G-opers over an abstract disk D_x. Though there is a set-theoretical bijection $\mathrm{Op}_G(D_x) \cong \mathrm{Op}_G(D) \cong \mathbb{C}[[z]]^{\oplus l}$, this bijection is not canonical! To find out $\mathrm{Op}_G(D_x)$ is isomorphic to what kinds of geometric objects, we need to figure out how the gauge equivalence classes change under the coordinate changes.

Suppose, $D' = \mathrm{Spec}(C[[w]])$ is another chart of D_x, with coordinate change given by $z = \mu(w)$. Then $\partial_w = \mu'(w)\partial_z$. Choose a connection $\nabla_{\partial_z} = \partial_z + p_{-1} + \sum_{i=1}^{l} v_i(z)p_i$ on D. Replace ∂_z with $(\mu'(w))^{-1}\partial_w$ gives

$$\nabla_{\partial_w} = \partial_w + \mu'(w)p_{-1} + \mu'(w) \cdot \sum_{i=1}^{l} v_i(\mu(w))p_i \tag{2.4}$$

The connection above is not in the standard expression that the gauge equivalence classes have. Thus, we need to apply gauge transformations to find the gauge equivalence class of (2.4).

Let $\rho = \sum_{i=1}^{l} w_i$. $\alpha_i(\rho) = 1$ for $i = 1, ..., l$, thus $\rho = \frac{1}{2}p_0$. Let $\check{\rho}$ be the cocharacter $\mathbb{C}[[w]]^{\times} \longrightarrow H[[w]]$ defined by ρ, i.e.

$$\check{\rho}(f(w)) = \exp(\ln(f(w))\rho)$$

First, we apply the gauge transformation by $\check{\rho}(\mu'(w))$

$$\begin{aligned}&\check{\rho}(\mu'(w)) \cdot (\partial_w + \mu'(w)p_{-1} + \mu'(w) \cdot \sum_{i=1}^{l} v_i(\mu(w))p_i)\\ =&\partial_w + p_{-1} + \mu'(w) \cdot \sum_{i=1}^{l} v_i(\mu(w)) \cdot \mathrm{Ad}_{\check{\rho}(\mu'(w))}p_i + \partial_w(\check{\rho}(\mu'(w)))\check{\rho}(\mu'(w))^{-1}\end{aligned}$$

Let's compute this term by term. By Lemma 2.4, $p_i \in \mathfrak{g}_{d_i}$, thus $[p_0, p_i] = 2d_i p_i$, $\mathrm{ad}_\rho(p_i) = [\frac{1}{2}p_0, p_1] = d_i p_i$. Then

$$\mathrm{Ad}_{\check{\rho}(\mu'(w))}(p_i) = \exp(\ln \mu'(w)\mathrm{ad}_\rho) \cdot p_i = \exp(\ln \mu'(w) d_i) \cdot p_i = \mu'(z)^{d_i} p_i$$

And

$$\partial_w(\check{\rho}(\mu'(w)))\check{\rho}(\mu'(w))^{-1} = \partial_w(\ln \mu'(w) \cdot \rho) = \frac{\mu''(w)}{\mu'(w)} \cdot \rho$$

Thus, the connection now transforms to

$$\nabla_{\partial_w} = \partial_w + p_{-1} + \sum_{i=1}^{l} v_i(\mu(w))\mu'(w)^{d_i+1} p_i - \frac{\mu''(w)}{\mu'(w)} \cdot \rho$$

The connection above is still not in standard expression. Apply another gauge transformation given by

$$A = \exp\left(\frac{1}{2}\frac{\mu''(w)}{\mu'(w)} p_1\right)$$

Notice $\mathrm{ad}_{p_1}(p_{-1}) = p_0$, $(\mathrm{ad}_{p_1})^2(p_{-1}) = -2p_1$, $(\mathrm{ad}_{p_1})^n(p_{-1}) = 0$ for $n \geq 3$, so

$$\mathrm{Ad}_A(p_{-1}) = p_{-1} + \frac{1}{2}\frac{\mu''(w)}{\mu'(w)} p_0 - \frac{1}{4}\left(\frac{\mu''(w)}{\mu'(w)}\right)^2 p_1$$

$$\mathrm{Ad}_A\left(-\frac{\mu''(w)}{\mu'(w)} \cdot \rho\right) = -\frac{1}{2}\frac{\mu''(w)}{\mu'(w)} p_0 + \frac{1}{2}\left(\frac{\mu''(w)}{\mu'(w)}\right)^2 p_1$$

Recall that $p_1, ..., p_l$ are ad_{p_1}-invariant, so $\mathrm{Ad}_A(p_i) = p_i$, $i = 1, ..., l$.

$$-\partial_w(A)A^{-1} = -\partial_w\left(\frac{1}{2}\frac{\mu''(w)}{\mu'(w)} p_1\right) = -\frac{1}{2}\frac{\mu'''(w)}{\mu'(w)} p_1 + \frac{1}{2}\left(\frac{\mu''(w)}{\mu'(w)}\right)^2 p_1$$

Thus, the connection now transforms to

$$\nabla_{\partial_w} = \partial_w + p_{-1} + \left(v_1(\mu(w))(\mu'(w))^2 - \frac{1}{2}\{\mu, w\}\right) p_1 + \sum_{i=2}^{l} v_i(\mu(w))\mu'(w)^{d_i+1} p_i \tag{2.5}$$

The connection above is a representative in the gauge equivalence classes. Therefore, under the coordinate change $z = \mu(w)$, the connection $\partial_z + p_{-1} + \sum_{i=1}^{l} v_i(z) p_i$ transforms to $\partial_w + p_{-1} + \sum_{i=1}^{l} \overline{v}_i(w) p_i$, where

$$\overline{v}_1(w) = v_1(\mu(w))(\mu'(w))^2 - \frac{1}{2}\{\mu, w\} \tag{2.6}$$

$$\overline{v}_i(w) = v_i(\mu(w)) \cdot (\mu'(w))^{d_i+1} \qquad i = 2, ..., l \tag{2.7}$$

Compare the equations (2.6) and (1.1), one can find that, under the changes of coordinate, v_1 and projective connection on D_x transform in the same law. Moreover, v_i and the $(d_i + 1)$-differential on D_x transform in the same law. Therefore, we obtain an isomorphism

$$\mathrm{Op}_G(D_x) \simeq \mathit{Proj}(D_x) \times \bigoplus_{i=2}^{l} \Omega_{D_x}^{\otimes(d_i+1)}$$

By generalizing the local result to global, we get

$$\mathrm{Op}_G(X) \simeq \mathit{Proj}(X) \times \bigoplus_{i=2}^{l} \Omega_X^{\otimes(d_i+1)}$$

3 Opers and the center of affine Kac-Moody vertex algebra

In this chapter, we study a canonical isomorphism between the algebra of function space of ${}^LG$-opers on the disk and the center of affine Kac-Moody vertex algebra $\mathfrak{Z}(\widehat{\mathfrak{g}})$. We will briefly introduce the Gaudin model, which serves as a motivation to study the center of $U(\widehat{\mathfrak{g}}_-)$. Then we will introduce the affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ and it's associated vertex algebra $V_\kappa(\mathfrak{g})$. We will define the Segal-Sugawara operators and study their commutators. Then we will reconstruct the algebras above in a coordinate independent way, and see how Segal-Sugawara operators transform under the coordinate changes. At the end, we state our main theorems.

3.1 Gaudin model and center of $U(\widehat{\mathfrak{g}}_-)$

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$ and $U(\mathfrak{g})$ its universal enveloping algebra. Choose a basis J_a, $a = 1, ..., d$, of $\mathfrak{g}$, and let J_a be the dual basis with respect to a non-degenerate invariant bilinear form on $\mathfrak{g}$(such bilinear form exists and is unique up to a scalar).
Let $z_1, ..., z_N$ be distinct complex numbers. For $A \in U(\mathfrak{g})$, define $A^{(i)}$ to be $1 \otimes \cdots \otimes A \otimes \cdots \otimes 1 \in U(\mathfrak{g})^{\otimes N}$, where A is in the ith factor. The *Gaudin Hamiltonians* are the following elements of the algebra $U(\mathfrak{g})^{\otimes N}$:

$$H_i = \sum_{j \neq i} \sum_{a=1}^{d} \frac{J_a^{(i)} \cdot J^{a(j)}}{z_i - z_j}, \qquad i = 1, \ldots, N$$

This definition is canonical, which means it doesn't depend on the choice of basis $\{J_a\}$. These elements commute with each other. In order to prove it, we should consider a different element instead. For $A \in U(\mathfrak{g})$, denote $A(z) = \sum_{i=1}^{N} \frac{A^{(i)}}{z - z_i}$, a meromorphic function with values in $U(\mathfrak{g})^{\otimes N}$. Define

$$S(z) = \frac{1}{2} \sum_{a=1}^{d} J_a(z) J^a(z)$$

By expanding the right hand side, we get

$$S(z) = \sum_{i=1}^{N} \frac{Cas^{(i)}}{(z - z_i)^2} + \sum_{i=1}^{N} \frac{H_i}{z - z_i}$$

One can prove that $\{H_i\}_{i=1,\ldots,N}$ mutually commute if and only if $[S(z), \partial_z^n S(z)] = 0$ for all $n > 0$.

Let $\widehat{\mathfrak{g}}_-$ be the lie algebra $\mathfrak{g} \otimes (t - z)^{-1} \mathbb{C}[(t - z)^{-1}]$, with Lie bracket define by

$$[A \otimes f(t - z), B \otimes g(t - z)] = [A, B] \otimes f(t - z) g(t - z)$$

We can define a Lie algebra homomorphism $\phi : \widehat{\mathfrak{g}}_- \longrightarrow \mathfrak{g}^{\oplus N}$ by:

$$A \otimes f(t - z) \rightarrow (A \cdot f(z_1 - z), \ldots, A \cdot f(z_N - z))$$

The Lie algebra homomorphism ϕ induces an algebra homomorphism $\Phi : U(\widehat{\mathfrak{g}}_-) \longrightarrow U(\mathfrak{g})^{\otimes N}$. Now we consider z as an indeterminate, we actually define an algebra homomorphism :

$$\Phi : U(\widehat{\mathfrak{g}}_-) \longrightarrow U(\mathfrak{g})^{\otimes N}[(z-z_i)^{-1}]_{i=1,\dots,N}$$

For $A \in \mathfrak{g}$, denote $A_n = A \otimes (t-z)^n$. Then $\Phi(A_{-1}) = -A(z)$. Denote $\mathcal{S} = \sum_{a=1}^{d} J_{a,-1} J^a_{-1}$, since Φ is a homomorphism of algebra, $\Phi(\mathcal{S}) = S(z)$.

Define derivative ∂_t on algebra $U(\widehat{\mathfrak{g}}_-)$ and ∂_z on algebra $U(\mathfrak{g})^{\otimes N}[(z-z_i)^{-1}]_{i=1,\dots,N}$ in usual sense. It's easy to verify that $\partial_z \circ \Phi = \Phi \circ (-\partial_t)$. Denote $T = -\partial_t$. Recall that our goal is to prove that $[S(z), \partial_z^n S(z)] = 0$ for all $n > 0$. Since $\Phi(\mathcal{S}) = S(z)$, it suffices to prove that $[\mathcal{S}, T^n \mathcal{S}] = 0$ for all $n > 0$.

Actually, we have the following theorems:

Theorem 3.1. *$\{T^n S\}_{n \geq 0}$ are algebraically independent in $U(\widehat{\mathfrak{g}}_-)$, and they mutually commute.*

Theorem 3.2. *There exists elements $S_i \in U(\widehat{\mathfrak{g}}_-)$, ord$(S_i) = d_i + 1$, deg$(S_i) = d_i + 1$, $i = 1, ..., l$, with $S_1 = \mathcal{S}$, such that $\{T^{n_i} S_i\}_{n_i \geq 0, i=1,\dots,l}$ are algebraically independent in $U(\widehat{\mathfrak{g}}_-)$ and mutually commute.*

3.2 Affine Kac-Moody algebra and vertex algebra

Let $\mathfrak{g}$ be a finite–dimensional simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. We define the formal loop algebra of $\mathfrak{g}$,

$$L\mathfrak{g} = \mathfrak{g}((t)) = \mathfrak{g} \otimes C((t))$$

as a Lie algebra with commutator

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t)$$

The affine Kac-Moody algebra $\widehat{\mathfrak{g}}_\kappa$ is a central extension of $L\mathfrak{g}$

$$0 \longrightarrow \mathbb{C} \cdot \mathbf{1} \longrightarrow \widehat{\mathfrak{g}}_\kappa \longrightarrow L\mathfrak{g} \longrightarrow 0$$

with commutation relations: $[\mathbf{1}, \cdot] = 0$ (so $\mathbf{1}$ is central), and

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) - (\mathrm{Res}_{t=0} f dg)\kappa(A, B) \cdot \mathbf{1}$$

Here $\kappa(\cdot, \cdot)$ is an invariant bilinear form of $\mathfrak{g}$. Usually we take $\kappa(\cdot, \cdot)$ to be the canonical normalized invariant bilinear form $(\cdot, \cdot)$, which satisfies:

$$(\cdot, \cdot) = \frac{1}{2h^\vee} \kappa_{\mathfrak{g}}(\cdot, \cdot)$$

where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$ and $\kappa_{\mathfrak{g}}(\cdot, \cdot)$ is the Killing form.

$\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1}$ is a lie subalgebra of $\widehat{\mathfrak{g}}_\kappa$. Consider an one dimensional representation $\mathbb{C}v_\kappa$ of $\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1}$, on which $\mathfrak{g}[[t]]$ acts trivially and $\mathbf{1}$ acts as identity. We define the vacuum representation of level κ of $\widehat{\mathfrak{g}}_\kappa$ as

$$V_\kappa(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_\kappa} \mathbb{C}v_\kappa = U(\widehat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1})} \mathbb{C}v_\kappa$$

$\widehat{\mathfrak{g}}_\kappa \cong \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1}$ as vector space. By Poincare-Birkhoff-Witt Theorem, we have an isomorphism of vector space

$$U(\widehat{\mathfrak{g}}_\kappa) \cong U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \otimes_{\mathbb{C}} U(\mathfrak{g}[[t]] \oplus \mathbb{C}\mathbf{1})$$

Thus

$$V_\kappa(\mathfrak{g}) = U(\widehat{\mathfrak{g}}_\kappa) \otimes_{U(\mathfrak{g}[[t]]\oplus\mathbb{C}\mathbf{1})} \mathbb{C}v_\kappa \cong U(\mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}])v_\kappa$$

We can define a vertex algebra structure on $V_\kappa(\mathfrak{g})$. Let J^a be a basis of $\mathfrak{g}$. For any $A \in \mathfrak{g}$, denote $A_n = A \otimes t^n \in \widehat{\mathfrak{g}}$. Then J^a_n, $a = 1, ..., d, n \in \mathbb{Z}$ and $\mathbf{1}$ form a topological basis for $\widehat{\mathfrak{g}}_\kappa$. By Poincare-Birkhoff-Witt Theorem, $V_\kappa(\mathfrak{g})$ is a vector space spanned by monomials $J^{a_1}_{n_1}\cdots J^{a_m}_{n_m}v_\kappa$, $n_1, ..., n_m < 0$.

The $\mathbb{Z}_+$-graded vertex algebra structure on $V_\kappa(\mathfrak{g})$ is defined as follow:

1. Vacuum vector: $|0\rangle = v_\kappa$
2. Translation operator: $T : V \longrightarrow V$ such that $[T, J^a_n] = -nJ^a_{n-1}$. T is just $-\partial_t$.
3. Vertex operator: $Y(v_\kappa, z) = \mathrm{Id}$.

$$Y(J^a_{-1}v_\kappa, z) = J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1}$$

in general,

$$Y(J^{a_1}_{n_1}\dots J^{a_k}_{n_m}v_\kappa, z) = \frac{1}{(-n_1-1)!\cdots(-n_m-1)!} : \partial_z^{-n_1-1}J^{a_1}(z)\cdots\partial_z^{-n_m-1}J^{a_m}(z) :$$

4. $\mathbb{Z}_+$-gradation:

$$\deg J^{a_1}_{n_1}\dots J^{a_k}_{n_m}v_\kappa = -\sum_{i=1}^m n_i$$

3.3 The Segal-Sugawara operators

Let κ_0 be an invariant bilinear form on $\mathfrak{g}$, $\{J_a\}_{a=1,\dots,d}$ is a dual basis of $\{J^a\}_{a=1,\dots,d}$ with respect to κ_0. Recall that in the first section, we are especially interested in the element $\mathcal{S} = \sum_{a=1}^d J_{a,-1}J^a_{-1} \in U(\widehat{\mathfrak{g}}_-)$. Now we consider it as an element in the vertex algebra $V_k(\mathfrak{g})$ and study its associated vertex operator

$$S(z) = Y(\mathcal{S}v_\kappa, z) = \frac{1}{2}\sum_{a=1}^d : J_a(z)J^a(z) :$$

Assume $S(z) = \sum_{n\in\mathbb{Z}} S_n z^{-n-2}$.

Proposition 3.3. $[S_m, J^a_n] = -\frac{\kappa+\frac{1}{2}\kappa_{\mathfrak{g}}}{\kappa_0} nJ^a_{n+m}$

Proof. See [1] 3.1.2. □

Denote

$$\widetilde{S} = \frac{\kappa_0}{\kappa+\frac{1}{2}\kappa_{\mathfrak{g}}}\mathcal{S} \qquad \widetilde{S}(z) = Y(\widetilde{S}, z) = \frac{\kappa_0}{\kappa+\frac{1}{2}\kappa_{\mathfrak{g}}}S(z) \qquad \widetilde{S}(z) = \sum_{n\in\mathbb{Z}}\widetilde{S}_n z^{-n-2}$$

then

$$[\widetilde{S}_m, J^a_n] = -nJ^a_{n+m}$$

Therefore, the operator $\mathrm{ad}(\widetilde{S}_m)$ acting on $V_\kappa(\mathfrak{g})$ is the same as the operator $-\partial_t$.

Proposition 3.4. $[\widetilde{S}_n, \widetilde{S}_m] = (n-m)\widetilde{S}_{n+m} + \frac{n^3-n}{12}c_k \cdot \delta_{n,-m}$, *where* $c_\kappa = \frac{\kappa}{\kappa + \frac{1}{2}\kappa_{\mathfrak{g}}}$.

Notice that, if $\kappa = -\frac{1}{2}\kappa_{\mathfrak{g}}$, then S_m commutes with the action of $\widehat{\mathfrak{g}}_\kappa$, and for any $A \in V_\kappa(\mathfrak{g})$,

$$[Y(\mathcal{S}, z), Y(A, w)] = 0$$

We say $\kappa_c = -\frac{1}{2}\kappa_{\mathfrak{g}}$ is the critical value for $\mathfrak{g}$. The operators $\{S_m\}$, or more generally, the Fourier coefficients of vertex operators of $V_\kappa(\mathfrak{g})$, can be viewed as elements in $\widetilde{U}_\kappa(\mathfrak{g})$, the complete universal enveloping algebra of $\widehat{\mathfrak{g}}_\kappa$. Let $U(V_\kappa(\mathfrak{g}))$ be the associated Lie algebra spanned by Fourier coefficients of all vertex operators, then there is a Lie algebra homomorphism:

$$U(V_\kappa(\mathfrak{g})) \longrightarrow \widetilde{U}_\kappa(\mathfrak{g})$$

Therefore, when $\kappa = \kappa_c$, Segal-Sugawara operators are central elements in $\widetilde{U}_\kappa(\mathfrak{g})$.

Let V be a vertex algebra, we define the center of V to be

$$\mathfrak{Z}(V) = \{A \in V : [Y(A,z), Y(B,w)] = 0 \quad \forall B \in V\}$$

one can prove that

$$\begin{aligned} \mathfrak{Z}(V) &= \{A \in V : B_{(m)}A = 0 \quad \forall B \in V, m \geq 0\} \\ &= \{A \in V : A_{(m)}B = 0 \quad \forall B \in V, m \geq 0\} \end{aligned}$$

Lemma 3.5. *The center $\mathfrak{Z}(V)$ of vertex algebra V is a commutative algebra.*

Proof. See [1] □

The center of affine Kac-Moody algebra $V_\kappa(\mathfrak{g})$ is given by

$$\mathfrak{Z}(V_\kappa(\mathfrak{g})) = V_\kappa(\mathfrak{g})^{\mathfrak{g}[[t]]} \tag{3.1}$$

Denote $\mathfrak{Z}(V_\kappa(\mathfrak{g}))$ by $\mathfrak{Z}(\widehat{\mathfrak{g}})$. When $\kappa \neq \kappa_c$, one can prove that the center is trivial, that is, $\mathfrak{Z}(\widehat{\mathfrak{g}}) = \mathbb{C}v_\kappa$. Now we consider the case when $\kappa = \kappa_c$. Since $\mathcal{S} \in \mathfrak{Z}(\widehat{\mathfrak{g}})$, the center is no longer trivial. The identity (3.1) enables us to identify $\mathfrak{Z}(\widehat{\mathfrak{g}})$ with the algebra of $\widehat{\mathfrak{g}}_\kappa$-endomorphism of $V_\kappa(\mathfrak{g})$. Indeed, a $\widehat{\mathfrak{g}}_\kappa$-endomorphism φ of $V_\kappa(\mathfrak{g})$ is uniquely determined by $\varphi(v_\kappa)$. If $\varphi(v_\kappa) = \alpha$, then $0 = \varphi(J_a^n v_\kappa) = J_a^n \varphi(v_\kappa) = J_a^n \alpha$. Thus α is $\mathfrak{g}[[t]]$-invariant. Conversely, if $\alpha \in \mathfrak{g}[[t]]$, then the map $\varphi : V_\kappa(\mathfrak{g}) \longrightarrow V_\kappa(\mathfrak{g})$ sending v_κ to α induces a $\widehat{\mathfrak{g}}_\kappa$-linear transformation. Thus, we obtain an isomorphism

$$\Phi : V_\kappa(\mathfrak{g})^{\mathfrak{g}[[t]]} \longrightarrow \mathrm{End}_{\widehat{\mathfrak{g}}_\kappa}(V_\kappa(\mathfrak{g}))$$

Let $\alpha_1, \alpha_2 \in V_\kappa(\mathfrak{g})^{\mathfrak{g}[[t]]}$ e_1, e_2 are their corresponding endomorphisms. Then

$$e_1 \circ e_2(v_\kappa) = e_1(\alpha_2) = e_1((\alpha_2)_{(-1)}v_\kappa) = (\alpha_2)_{(-1)} \circ e_1(v_\kappa) = (\alpha_2)_{(-1)}\alpha_1$$

Since $V_\kappa(\mathfrak{g})^{\mathfrak{g}[[t]]} = \mathfrak{Z}(\widehat{\mathfrak{g}})$ is a commutative vertex algebra, $(\alpha_2)_{(-1)}\alpha_1 = (\alpha_1)_{(-1)}\alpha_2$. Thus $\Phi((\alpha_1)_{(-1)}\alpha_2) = \Phi(\alpha_1) \circ \Phi(\alpha_2)$, Φ is an isomorphism of commutative algebra. In particular, $\mathrm{End}_{\widehat{\mathfrak{g}}_\kappa}(V_\kappa(\mathfrak{g}))$ is a commutative algebra.

Finally, recall that $V_\kappa(\mathfrak{g})$ is isomorphic to $U(\widehat{\mathfrak{g}}_-)$. Thus we have an injection

$$\mathfrak{Z}(\widehat{\mathfrak{g}}) \hookrightarrow U(\widehat{\mathfrak{g}}_-)$$

This is an embedding of algebra. Thus, $\mathfrak{Z}(\widehat{\mathfrak{g}})$ can be viewed as a commutative subalgebra of $U(\widehat{\mathfrak{g}}_-)$.

Theorem 3.6. *The center $\mathfrak{Z}(\widehat{\mathfrak{sl}_2})$ of the vertex algebra $V_{\kappa_c}(\mathfrak{sl}_2)$ is equal to*

$$\mathbb{C}[S_n]_{n \leq -2}v_\kappa$$

Proof. See [1] 3.5.1. □

3.4 Transformation laws for Segal-Sugawara operators

So far, our discussion are coordinate dependent. We start with a Lie algebra $\mathfrak{g}$ and construct the Lie algebra $\widehat{\mathfrak{g}}_\kappa$ as the central extension of $\mathfrak{g} \otimes \mathbb{C}((t))$. If our goal is to attach a affine Kac-Moody algebra/vertex algebra to a point x of a smooth algebraic curve X, then what we have done are constructing these objects on a chart $\mathrm{Spec}(\mathbb{C}[[t]])$. The coordinate independent construction should be like the following.

Let $\mathcal{O}_x$ be the completion of local ring at x. $\mathcal{K}_x$ is the completion of the function field $K(X)$ with respect to the valuation given by $\mathcal{O}_x$, which is a discrete valuation ring. From the knowledge of algebraic geometry, we know that if we choose an isomorphism $\mathcal{O}_x \cong \mathbb{C}[[t]]$, then $\mathcal{K}_x \cong \mathbb{C}((t))$. The affine Kac-Moody algebra $\widehat{\mathfrak{g}}_{\kappa,x}$ is defined to be the central extension of $\mathfrak{g} \otimes \mathcal{K}_x$ (the residue is coordinate independent). The vacuum $\widehat{\mathfrak{g}}_{\kappa,x}$-module is

$$V_\kappa(\mathfrak{g})_x = \mathrm{Ind}_{\mathfrak{g}\otimes\mathcal{O}_x\oplus\mathbb{C}\mathbf{1}}^{\widehat{\mathfrak{g}}_{\kappa,x}} \mathbb{C}v_\kappa$$

The algebra $\mathfrak{Z}(\widehat{\mathfrak{g}})_x$ is defined to be

$$\mathrm{End}_{\widehat{\mathfrak{g}}_{\kappa,x}} V_\kappa(\mathfrak{g})_x = (V_\kappa(\mathfrak{g})_x)^{\mathfrak{g}\otimes\mathcal{O}_x}$$

Recall that for $\mathfrak{sl}_2(\mathbb{C})$, we have the isomorphism of algebra

$$\mathfrak{Z}(\widehat{\mathfrak{sl}}_2) \cong \mathbb{C}[S_n]_{n\leq -2}$$

Can we get a coordinate independent version of this isomorphism? For that we need to know how the Segal-Sugawara operators transform under the coordinate changes.

Fix a chart $\mathrm{Spec}\ \mathcal{O} = \mathrm{Spec}(\mathbb{C}[[t]])$ of D_x. Let $\mathcal{A}ut\mathcal{O}$ be the group of continuous automorphism of $\mathcal{O}$. Then $\mathcal{A}ut\mathcal{O}$ is isomorphic to the group $z(\mathbb{C}[[z]])^\times$ with multiplication by composition. Denote $\mathcal{A}ut_+\mathcal{O} = \{\mu(t) \in \mathcal{A}ut\mathcal{O} : \mu'(t) = 1\}$, then

$$\mathcal{A}ut\mathcal{O} = \mathcal{A}ut_+\mathcal{O} \rtimes \mathbb{C}^\times$$

The Lie algebra of $\mathcal{A}ut\mathcal{O}$ and $\mathcal{A}ut_+\mathcal{O}$ are $\mathrm{Der}_0\mathcal{O} = t\mathbb{C}[[t]]\partial_t$ and $\mathrm{Der}_+\mathcal{O} = t^2\mathbb{C}[[t]]\partial_t$. Denote $\mathrm{Der}\mathcal{O} = \mathbb{C}[[t]]\partial_t$, $L_n = -t^{n+1}\partial_t \in \mathrm{Der}\mathcal{O}$, $n \geq -1$.

Lemma 3.7. *i The exponential map $Der_+\mathcal{O} \longrightarrow \mathcal{A}ut_+\mathcal{O}$ is an isomorphism.*

ii The exponential map $Der_0\mathcal{O} \longrightarrow \mathcal{A}ut_0\mathcal{O}$ is surjection.

iii A $Der_0\mathcal{O}$-module V can be exponentiated to an $\mathcal{A}ut_0$-module if the following condition are satisfied:

(a) the action of $z\partial_z$ is diagonalizable with integral eigenvalues

(b) the action of $Der_+\mathcal{O}$ is locally nilpotent, i.e. for any $v \in V$ and $x \in Der_+\mathcal{O}$, there exists $N \in \mathbb{N}$, such that $x^n \cdot v = 0$ for all $n \geq N$.

Proof. See [2] 6.2.1. □

Now we need to study how Segal-Sugawara operators transform under the changes of coordinate by $\mathcal{A}ut\mathcal{O}$. In other word, we need to study $V_\kappa(\mathfrak{g})$ as a representation of $\mathcal{A}ut(\mathcal{O})$. By taking differential, we get a representation of $\mathrm{Der}_0\mathcal{O}$. For $L_0 \in \mathrm{Der}_0\mathcal{O}$, $J^{a_1}_{n_1}\dots J^{a_k}_{n_m}v_\kappa \in V_\kappa(\mathfrak{g})$ with degree $-\sum_{i=1}^m n_i$,

$$L_0(J^{a_1}_{n_1}\dots J^{a_k}_{n_m}v_\kappa) = \left(-\sum_{i=1}^m n_i\right) J^{a_1}_{n_1}\dots J^{a_k}_{n_m}v_\kappa$$

Thus, the action of L_0 is diagonalizable with integral eigenvalues. Generally, the action of L_n has degree $-n$ and $V_\kappa(\mathfrak{g})$ is $\mathbb{Z}_+$-graded. So, the action of $\mathrm{Der}_+\mathcal{O}$ is locally nilpotent. Therefore, the action of $\mathrm{Der}_0\mathcal{O}$ can be exponentiated to $\mathcal{A}ut\mathcal{O}$, the original action.

Recall that the operator $\widetilde{S}_n$ acts as same as $L_n = -t^{n+1}\partial_t$. Thus, for any $Xv_\kappa \in V_\kappa(\mathfrak{g})$, $L_n \cdot Xv_\kappa = [\widetilde{S}_n, X]v_k$. We are interesting in the case when $X = S_m$. When $\kappa \neq \kappa_c$,

$$L_n \cdot S_m = [\widetilde{S}_n, S_m] = (n-m)S_{n+m} + \frac{n^3-n}{12}\dim\mathfrak{g} \cdot \frac{\kappa}{\kappa_0}\delta_{n,-m} \tag{3.2}$$

When $\kappa = \kappa_c$, take the limit $\kappa \to \kappa_c$ in (3.2), we get

$$L_n \cdot S_m = [\widetilde{S}_n, S_m] = (n-m)S_{n+m} + \frac{n^3-n}{12}\dim\mathfrak{g} \cdot \frac{\kappa_c}{\kappa_0}\delta_{n,-m} \tag{3.3}$$

Specially, Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ and κ_0 be the canonical normalized invariant form, we have

$$L_n \cdot S_m = \begin{cases} (n-m)S_{n+m} & \text{if } n+m \leq -2 \\ -\frac{1}{2}(n^3-n) & \text{if } n+m = 0 \\ \quad 0 & \text{otherwise} \end{cases} \tag{3.4}$$

The equation (3.4) gives a complete description of the transformation laws of Segal-Sugawara operators. We will see these transformation laws for a different geometric object in the next section.

3.5 Canonical isomorphism for $\mathfrak{sl}_2$

In the first chapter, we have proved that there is a canonical one to one correspondence between projective connections and PGL_2-opers on an abstract disk, given by

$$\begin{array}{ccc} \text{Projective connections} & & \mathrm{PGL}_2(\mathbb{C})\text{-opers} \\ \partial_z^2 - v(z) & \longmapsto & \partial_z + p_{-1} + v(z)p_1 \end{array}$$

This correspondence is canonical because the transformation laws of $v(z)$ above are the same (compare the equation (1.1) with (2.7)). This transformation law actually makes $\mathbb{C}[[z]]$ a $\mathcal{A}ut_\mathcal{O}$-representation, with action given by

$$\mu(z) \cdot f(z) = f(\mu(z))(\mu'(z))^2 - \frac{1}{2}\{\mu, z\}$$

Let Fun $\mathbb{C}[[z]]$ be the space of continuous polynomial function on $\mathbb{C}[[z]]$. Then Fun $\mathbb{C}[[z]]$ is generated by

$$\widetilde{\pi}_n : \mathbb{C}[[z]] \longrightarrow \mathbb{C} \qquad \sum_{n\geq 0} c_n z^n \longmapsto c_n$$

When the group $\mathcal{A}ut_\mathcal{O}$ acting on $\mathbb{C}[[z]]$ (act as coordinate change), it also acts on Fun $\mathbb{C}[[z]]$. The action satisfies

$$< \mu(z) \cdot \varphi, \mu(z) \cdot f(z) > = < \varphi, f(z) >$$

and this induces a action of $\mathrm{Der}\mathcal{O}$ on Fun $\mathbb{C}[[z]]$.

Let's figure out how L_n acts on $\{\widetilde{\pi}_m\}_{m\geq 0}$. Let $\mu_\epsilon(z) = z + \epsilon z^{n+1}$, $\epsilon^2 = 0$. For $f(z) = \sum_{m\geq 0} c_m z^m$,

$$\begin{aligned}\mu_\epsilon(z)\cdot f(z) &= f(\mu_\epsilon(z))(\mu_\epsilon'(z))^2 - \frac{1}{2}\{\mu_\epsilon, z\}\\ &= f(z) + \left(\sum_{m\geq 0} c_m(2n+m+2)z^{n+m}\right)\epsilon - \frac{1}{2}\{\mu_\epsilon, z\}\end{aligned}$$

where $\frac{1}{2}\{\mu_\epsilon, z\} = \begin{cases}\frac{1}{2}(n^3-n)z^{n-2}\epsilon & n\geq 2\\ 0 & n<2\end{cases}$. By taking the differential, we get

$$L_n\left(\sum_{m\geq 0} c_m z^m\right) = \sum_{m\geq 0} -c_m(2n+m+2)z^{n+m} - \frac{1}{2}(n^3-n)z^{n-2}$$

the term z^{n-2} is zero if $n<2$. Thus,

$$L_n\cdot\widetilde{\pi}_m = \begin{cases}(m+n+2)\widetilde{\pi}_{m-n} & m\geq n\\ -\frac{1}{2}(n^3-n) & m=n-2 \text{ and } n\geq 2\\ 0 & \text{otherwise}\end{cases}$$

Let $\pi_m = \widetilde{\pi}_{-m-2}$, $m\leq -2$. Then

$$L_n\cdot\pi_m = \begin{cases}(n-m)\pi_{m+n} & m+n\leq -2\\ -\frac{1}{2}(n^3-n) & m+n=0\\ 0 & \text{otherwise}\end{cases} \tag{3.5}$$

Compare (3.5 with (3.4), we find that the transformation laws $\{\pi_m\}_{m\leq -2}$ and $\{S_m\}_{m\leq -2}$ are the same. So, the geometric meaning of $\mathfrak{Z}(\widehat{\mathfrak{sl}_2})$ is finally revealed:

Theorem 3.8. *The center $\mathfrak{Z}(\widehat{\mathfrak{sl}}_2)_x$ at critical level is isomorphic to the algebra of function on the space $\mathrm{Op}_{PGL_2}(D_x)$. The canonical isomorphism is given by*

$$\mathfrak{Z}(\widehat{\mathfrak{sl}}_2) \longrightarrow \text{Fun } \mathrm{Op}_{PGL_2}(D_x) \qquad S_n\cdot v_\kappa \longmapsto \pi_n$$

where

$$\pi_n\left(\partial_z + p_{-1} + \sum_{m\leq -2} c_m z^{-m-2}p_1\right) = c_n$$

3.6 Canonical isomorphism for general simple Lie algebra

In last section, we construct a natural isomorphism between $\mathfrak{Z}(\widehat{\mathfrak{sl}}_2)_x$ and Fun $\mathrm{Op}_{\mathrm{PGL}_2}(D_x)$. This is actually a special case of our general theorem. Now we state the main theorem.

Theorem 3.9 (B.Feigin and E. Frenkel)**.** *For finite dimensional simple Lie algebra $\mathfrak{g}$, the algebra $\mathfrak{Z}(\widehat{\mathfrak{g}})_x$ is isomorphic to the algebra of functions on the space $\mathrm{Op}_{{}^LG}(D_x)$ of ${}^LG$-opers on the abstract disk D_x.*

Equivalently, we can state this theorem on a given chart $D = \text{Spec } \mathcal{O}$

Theorem 3.10. *For finite dimensional simple Lie algebra $\mathfrak{g}$, the algebra $\mathfrak{Z}(\widehat{\mathfrak{g}})$ is isomorphic to the algebra* $\mathrm{Fun}\ \mathrm{Op}_{^LG}(D)$ *in a* $(\mathrm{Der}\mathcal{O}, \mathcal{A}ut\mathcal{O})$*-equivalent way.*

From (2.3), we know that

$$\mathrm{Op}_{^LG}(D) \cong \mathbb{C}[[z]]^{\oplus l}$$

Thus, each oper is represented by $(v_1(z), ..., v_l(z))$,

$$v_i(t) = \sum_{n<0} v_{i,n} z^{-n-1}$$

Then we obtain an isomorphism

$$\mathrm{Fun}\ \mathrm{Op}_{^LG}(D) \cong \mathbb{C}[v_{i,n}]_{i=1,...,l,n<0}$$

Here $v_{i,n}$ is the linear functional on $\mathrm{Op}_{^LG}(D)$ that the takes the $(-n-1)^{\mathrm{th}}$ coefficient of $v_i(z)$. By calculation (same method we used in last section), we can show that the action of $\mathrm{Der}\mathcal{O}$ on $\mathbb{C}[v_{i,n}]_{i=1,...,l,n<0}$ is given by

$$L_n \cdot v_{1,m} = \begin{cases} (n-m+1)v_{1,m+n} & m+n \leq -1 \\ -\frac{1}{2}(n^3-n) & m+n=1 \\ 0 & \text{otherwise} \end{cases} \qquad L_n \cdot v_{i,m} = \begin{cases} (d_i(n+1)-m)v_{i,m+n} & m+n \leq -1 \\ 0 & \text{otherwise} \end{cases}$$

Specially, we have $L_{-1} \cdot v_{i,m} = -m \cdot v_{i,m}$ for all $i = 1, ..., l$ and $m \leq -1$.

For $\mathfrak{Z}(\widehat{\mathfrak{g}})$, the operator $L_{-1} = -\partial_z$ is the same as the translation operator T on $V_{\kappa_c}(\mathfrak{g})$. Since the isomorphism of $\mathfrak{Z}(\widehat{\mathfrak{g}})$ and $\mathbb{C}[v_{i,n}]_{i=1,...,l,n<0}$ is $\mathrm{Der}\mathcal{O}$-equivalent, we get

$$L_{-1}^m S_i \longmapsto L_{-1}^m v_{i,-1}$$

Assume $Y(S_i v_\kappa, z) = \sum_{n\leq 0} S_{i,(n)} z^{-n-1}$, then $L_{-1}^m S_i v_\kappa = T^m S_i v_\kappa = m! S_{i,(-m-1)}$ (recall that $\mathfrak{Z}(\widehat{\mathfrak{g}})$ is a commutative vertex algebra). Besides, $L_{-1}^m v_{i,-1} = m! v_{i,-m-1}$, thus

$$S_{i,(-m-1)} \longrightarrow v_{i,-m-1}$$

Therefore, we obtain that

$$\mathfrak{Z}(\widehat{\mathfrak{g}}) = \mathbb{C}[S_{i,(n)}]_{i=1,...,l;n<0}$$

A Lie Group and Lie Algebra

In this section, we briefly introduce some properties of finite dimensional simple Lie algebras over $\mathbb{C}$. A Lie algebra is simple if it has no non-trivial ideal. The finite dimensional simple Lie algebra over $\mathbb{C}$ have been completely classified. They are divided into four infinite family (A_n, B_n, C_n, D_n) and six exceptional ones. First, Let's give some general properties for simple Lie algebra.

A simple Lie algebra $\mathfrak{g}$ has a maximal commutative subalgebra $\mathfrak{h}$ such that all elements in $\mathfrak{h}$ are semi-simple. Such a subalgebra is called Cartan subalgebra. The dimension of $\mathfrak{h}$ is called the rank of $\mathfrak{g}$. Assume the rank of $\mathfrak{g}$ is l. The adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ is simultaneously diagonalizable. Thus, $\mathfrak{g}$ has a eigenspace decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$$

where $\alpha \in \mathfrak{h}^*$, and $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} : hx = \alpha(h)x, \forall h \in \mathfrak{h}\}$, $\Delta = \{\alpha \in \mathfrak{h}^* : \mathfrak{g}_\alpha \neq 0\}$. We can prove that $\mathfrak{g}_0$ is just $\mathfrak{h}$. This decomposition has following properties:

1. all $\mathfrak{g}_\alpha$ are 1-dimensional
2. for any $\alpha \in \Delta$, there exists $e_\alpha \in \mathfrak{g}_\alpha$, $h_\alpha \in \mathfrak{h}$, $f_\alpha \in \mathfrak{g}_{-\alpha}$, such that $\{e_\alpha, h_\alpha, f_\alpha\}$ is a $\mathfrak{sl}_2$-triple.

and the root space Δ has the following properties:

1. The root space Δ span $\mathfrak{h}^*$.
2. The real span of Δ together with the induced bilinear form (Killing form) on $\mathfrak{h}^*$ form a Euclidean space.
3. $\alpha \in \Delta$, then $\gamma\alpha \in \Delta$ if and only if $\gamma = \pm 1$;
4. for any $\alpha, \beta \in \Delta$, $\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$
5. for any $\alpha, \beta \in \Delta$, $\beta - \frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \Delta$

Moreover, we can find a basis $\alpha_1, ..., \alpha_l$ (called simple roots) of the space $\mathrm{Span}_{\mathbb{R}}\{\Delta\}$, such that for any $\alpha \in \Delta$, $\alpha = \sum_{i=1}^{l} k_i \alpha_i$

1. all $k_i \in \mathbb{Z}$;
2. $\{k_i\}_{i=1,...,l}$ are either all non-negative (positive root) or all non-positive (negative root).

Denote the set of all positive (negative) root by Δ_+ (Δ_-). Denote $\mathfrak{n}_\pm = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_\alpha$, then $\mathfrak{g}$ has a Cartan decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

Notice that $\mathfrak{n}_+$ is generated by $e_1, ..., e_l$, $\mathfrak{h}$ is generated by $h_1, ..., h_l$, $\mathfrak{n}_-$ is generated by $f_1, ..., f_l$ (here we we omit the α in our previous notion). The Cartan matrix is defined by $C = (a_{ij})_{l \times l} = (\alpha_j(h_i))_{l \times l}$, then the Cartan matrix satisfies the following properties

(1) $a_{ii} = 2$;

(2) for $i \neq j$, a_{ij} is an integer and $a_{ij} \leq 0$;

(3) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

(4) C can be written as $D \cdot S$, where D is a diagonal matrix and S is a symmetric matrix.

And the relation between the generators e_i, h_i, f_i, $i = 1, ..., l$ are given by

1. $[h_i, h_j] = 0$; **2.** $[h_i, e_j] = a_{ij} e_j$;
3. $[h_i, f_j] = -a_{ij} f_j$ **4.** $[e_i, f_j] = \delta_{i,j} h_i$
5. $(\mathrm{ad}_{e_i})^{1-a_{ij}}(e_j) = 0$, $i \neq j$ **6.** $(\mathrm{ad}_{f_i})^{1-a_{ij}}(f_j) = 0$, $i \neq j$

In fact, the converse also hold. If the Cartan matrix (that is a square matrix that satisfies (1), (2), (3), (4)) is positive defined, then the Lie algebra generated by $e_i, h_i, f_{i i=1,...,l}$ satisfies the relations **1** $\sim$ **6** is a finite dimensional simple Lie algebra.

Now we introduce the exponents of simple Lie algebra $\mathfrak{g}$. The exponents of a Lie algebra arise naturally in two different situations. First, it appears in the following theorem

Theorem A.1. *Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let $\mathfrak{Z}(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. Assume l is the rank of $\mathfrak{g}$. Then, there exists l elements $S_1, ..., S_l$ with order $m_1 \leq \cdots \leq m_l$ ($A \in U(\mathfrak{g})$ has order m if $A \in U_{\leq m}(\mathfrak{g})$ and $A \notin U_{\leq m-1}(\mathfrak{g})$) such that*

$$\mathfrak{Z}(\mathfrak{g}) = \mathbb{C}[S_1, ..., S_l]$$

We define the exponents of $\mathfrak{g}$ are the numbers $d_i = m_i - 1$, $i = 1, ..., l$.

Now let's discuss the second situation that the exponents appear. Recall that $f_1, ..., f_l$ are generators of $\mathfrak{n}_-$. Denote $p_{-1} = \sum_{i=1}^{l} f_i$

Lemma A.2. *1. There exists a unique element $p_0 \in \mathfrak{h}$, such that $\alpha_i(h) = 2$ for all simple roots.*

2. Assume $p_0 = c_1 h_1 + ... + c_l h_l$, take $p_1 = c_1 e_1 + ... + c_l e_l$, then $\{p_{-1}, p_0, p_1\}$ is a $\mathfrak{sl}_2$-triple.

Therefore, with the adjoint action of p_{-1}, p_0, p_1, $\mathfrak{g}$ can be viewed as a $\mathfrak{sl}_2$-module. This gives us two decompositions of $\mathfrak{g}$. First, We can decompose $\mathfrak{g}$ as eigenspace of p_0

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n$$

where $\mathfrak{g}_n = x \in \mathfrak{g} : [p_0, x] = 2nx$. This decomposition also gives a graded structure on $\mathfrak{g}$, called principal gradation. It's easy to see that $\mathfrak{g}_0 = \mathfrak{h}$, $\bigoplus_{i>0} \mathfrak{g}_i = \mathfrak{n}_+$, $\bigoplus_{i<0} \mathfrak{g}_i = n_-$. Second, we cna also decompose $\mathfrak{g}$ to the direct sum of irreducible subspace

$$\mathfrak{g} = \bigoplus_{i=1}^{t} W_{n_i}$$

where W_{n_i} is the irreducible representation of $\mathfrak{sl}_2$ with highest weight n_i, $n_1 \leq \cdots \leq n_t$.

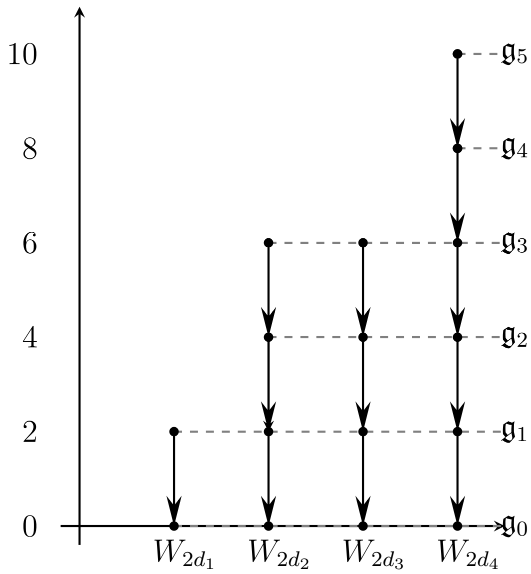
Lemma A.3. *$t = l$, and $n_i = 2d_i$, where d_i is the i^{th} exponent of $\mathfrak{g}$, $i = 1, ..., l$.*

Now we state the main result we want:

Theorem A.4. *Let E be the multi set of exponents of $\mathfrak{g}$.*

1. *For $n \geq 0$, the map $\mathrm{ad}_{-1} : g_{n+1} \longrightarrow \mathfrak{g}_n$ is injective.*
2. *Denote $\mathfrak{g}_n = V_n \oplus \mathrm{ad}_{p_{-1}}(\mathfrak{g}_{n+1})$, then $V_n \neq 0$ if and only if n is an exponent of $\mathfrak{g}$. Moreover, there exists a basis $\{p_1, ..., p_l\}$of V_n, such that $p_i \in \mathfrak{g}_{d_i}$, $i = 1, ..., l$.*
3. *Let $V = \bigoplus_{i \in E} V_i$, then $V = (\mathfrak{n}_+)^{\mathrm{ad}_{p_1}}$, i.e. V is the ad_{p_1}-invariant space*

The proof can be easily understand by the picture below. In the picture, every points represents a one dimensional eigenspace of ad_{p_0} with weight given by the value of y-axis. Each column of points represents an irreducible representation W_{2d_i} $i = 1, ..., l$ from left to right. Then the direct sum of points on the n^{th} row represents $\mathfrak{g}_n$. The adjoint action of $\mathrm{ad}_{p_{-1}}$ from $\mathfrak{g}_{n+1}$ to $\mathfrak{g}_n$ is the same as projecting the $(n+1)^{th}$ row to n^{th} row vertically, which is obviously injective. The dimension of $\mathfrak{g}_n$ is equal to $\#\{a \in E : a \geq n\}$. So, the projection is not surjective if and only if $n \in E$. Notice that $V = \bigoplus_{i \in E} V_i$ is also the space of highest weights, thus, it is the ad_{p_1}-invariant space.



B Fiber Bundle and Connection

Definition A fiber bundle is a structure (E, B, π, F), where E, B, F are topological spaces and $\pi : E \longrightarrow B$ is a continuous surjection that satisfies local triviality condition. The local triviality condition says that there exists an open cover $\{U_i\}_{i \in I}$ of B, homeomorphisms $\varphi_i : \pi^{-1}(U_i) \longrightarrow U \times F$, and transition map $\varphi_{ij} : U_i \cap U_j \times F \longrightarrow U_i \cap U_j \times F$ such that

$$\begin{tikzcd} \pi^{-1}(U_i) \arrow[rr, "\varphi_i"] \arrow[dr, "\pi"'] & & U_i \times F \arrow[dl, "proj"] \\ & U_i & \end{tikzcd} \qquad \begin{tikzcd} U_i \cap U_j \times F \arrow[rr, "\varphi_{ij}"] & & U_i \cap U_j \times S \\ & \pi^{-1}(U_i \cap U_j) \arrow[ul, "\varphi_i"] \arrow[ur, "\varphi_j"'] & \end{tikzcd}$$

It's obvious that for any U_i, U_j, U_k,

$$\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij} \text{ on } U_i \cap U_j \cap U_k \text{ for any } i, j, k \in I$$

Conversely, If we have the datum $\{U_i, \varphi_i, \varphi_{ij}\}$ that satisfying the above condition, then we can construct a fiber bundle with the given trivialization and transition laws.

For F in a category $\mathcal{C}$, we request φ_{ij} to be a morphism in this category at each fiber. In this case, the transition map can be also written as

$$\widetilde{\varphi} : U_i \cap U_j \longrightarrow \mathcal{A}ut_{\mathcal{C}}(F)$$

Here are two kinds of fiber bundles:

Vector Bundle: F is in the category of vector space. Then φ_{ij} are linear maps at each fiber. The transition map can be written as $\varphi_{ij} : U_i \cap U_j \longrightarrow \mathrm{GL}(V)$.

Principal G-bundle: F is in the category of right G-torsor.Then φ_{ij} are right G-equivalent maps at each fiber, i.e. $\varphi_{ij}((x,s)) \cdot g = \varphi_{ij}((x,sg))$ for any $x \in U_i \cap U_j$ and $s \in F, g \in G$. Since all G-torsor are isomorphic to G, we can replace F by G. Then the transition map is given by $\widetilde{\varphi_{ij}} : U_i \cap U_j \longrightarrow G$. However, this map should act on the left, i.e. $\varphi_{ij}((x,g)) = (x, \widetilde{\varphi_{ij}}(x)g)$.

Now, let (V,ρ) be a representation of G, $\pi : E \to B$ is a G-bundle, we define the associated vector bundle E_V as

$$E_V = E \times_G V$$

Suppose the G-bundle has local trivialization datum $\{U_i, \varphi_i : \pi^{-1}(U_i) \to U_i \times G, \varphi_{ij} : U_i \cap U_j \to G\}$. Then we can construct a local trivialization datum for E_V, given by

$$\begin{aligned}\widetilde{\varphi_i} : \pi_V^{-1}(U_i) = \pi^{-1}(U_i) \times_G V \xrightarrow{\varphi_i \times 1} U_i \times G \times_G V &\longrightarrow U_i \times V \\ (u,g,v) &\longmapsto (u, \rho(g)v)\end{aligned}$$

$$\begin{tikzcd} U_i \cap U_j \times V \arrow[r, "\widetilde{\varphi}_{ij}"'] & U_i \cap U_j \times V \\ U_i \cap U_j \times G \times_G V \arrow[u, "\sim"'] \arrow[r, "\varphi_{ij} \times 1"] & U_i \cap U_j \times G \times_G V \arrow[u, "\sim"'] \end{tikzcd} \qquad \begin{tikzcd} (u, \rho(g)v) \arrow[r, dashed] & (u, \rho(\varphi_{ij}(u))\rho(g)v) \\ (u,g,v) \arrow[u] \arrow[r] & (u, \varphi_{ij}(u)g, v) \arrow[u] \end{tikzcd}$$

Therefore, if φ_{ij} is the transition map for G-bundle E, then $\rho \circ \varphi_{ij}$ is the transition map for E_V.

Now we discuss about the connections on fiber bundles. All topological space should be at least smooth now. Let $\pi : E \longrightarrow M$ be a bundle with fiber F. For each $x \in M$, denote E_x to be the fiber at x. For distinct point $x, y \in M$, though E_x and E_y are isomorphic to F, there is no canonical isomorphism between E_x and E_y. One can see the need for such an isomorphism when defining the covariant derivatives. Therefore, we need an extra piece of structure to "connect" these fibers, at least in a small neighborhood.

This leads to the idea of parallel transport. A parallel transport on the bundle E is an assignment to each pair $(x, \gamma(t))$, where $x \in M$ and γ is a curve on M such that $\gamma(0) = x$, a collection of isomorphism

$$P_\gamma^t : E_x \longrightarrow E_{\gamma(t)}$$

with $P_\gamma^0 = \mathrm{Id}$.

The primary utility of the family P_γ^t is that, once chosen, it enables us to differentiate sections along paths. Let $x \in M$, $X \in T_x M$, and γ is a curve on M such that $\gamma(0) = x$. For a smooth section $s \in \Gamma(E)$, one can define the covariant derivative of s with direction X to be

$$\nabla_X s\big|_x = \frac{d}{dt}\bigg|_{t=0} (P_\gamma^t)^{-1}(s(\gamma(t))) \in T_{s(x)}E$$

In order to let the covariant derivative ∇ to be well-defined, we should require the definition of parallel transport to satisfy two condition:

1. The definition of $\nabla_X s\big|_x$ is independent of the choice of γ with given direction.
2. The map $\nabla_{(-)} s\big|_x : T_x M \longrightarrow T_{s(x)} E$ is linear.

Notice that the value of covariant derivative at $x \in M$ always lies in the kernel of map $\pi^* : T_{s(x)} E \longrightarrow T_x M$. The vertical bundle $VE \longrightarrow E$ is the subbundle of $TE \longrightarrow E$ defined by $VE = \{X \in TE : \pi^*(X) = 0\}$. We denote the fiber of VE at $p \in E$ by $V_p E$. Then the covariant derivative is a linear map

$$\nabla_{(-)} s\big|_x : T_x M \longrightarrow V_{s(x)} E$$

Connections on vector bundles For connections on vector bundles, we request that the induced parallel transports from connections are always linear. For vector space, the tanget space at each point is naturally isomorphic to the vector space itself. Therefore, the covariant derivative, now can be identified as a map

$$\nabla : \Gamma(TM) \times \Gamma(E) \longrightarrow \Gamma(E)$$

that satisfies two condition: Let $s \in \Gamma(E)$, $X \in \Gamma(TM)$, $f \in C^\infty(M)$, then

1. $\nabla_{fX} s = f \nabla_X s$
2. $\nabla_X fs = f \nabla_X s + X(f) \cdot s$

On the local trivialization $\varphi_i : \pi^{-1}(U) \longrightarrow U \times \mathbb{F}^n$, we can express the connection as a differential operator. For any point $x \in U$, let $X \in \Gamma(TM)$, $s \in \Gamma(E)$, s is equivalent to a map $sU \longrightarrow \mathbb{F}^n$. γ is a curve such that $\gamma(0) = x$, $\gamma'(0) = X_x$. Let $A(t) = P_\gamma^t : \mathbb{F}^n \longrightarrow \mathbb{F}^n$, then

$$\begin{aligned}
\nabla_X s\big|_x =& \frac{d}{dt}\bigg|_{t=0} (A(t))^{-1}(s(\gamma(t))) \\
=& \frac{d}{dt}\bigg|_{t=0} ((A(t))^{-1}) \cdot s(x) + \frac{d}{dt}\bigg|_{t=0} (s(\gamma(t))) \\
=& (X + A_X) \cdot s(x)
\end{aligned}$$

where X acts on each component of s, and A_X is a matrix that linearly depend on X.

If we change the local trivialization to $\varphi' : \pi^{-1}(U) \longrightarrow U \times \mathbb{F}^n$ with transition map $\psi : U \longrightarrow \mathrm{GL}_n(\mathbb{F})$, then the parallel transport map now becomes $\widetilde{A}(t) = \psi(\gamma(t)) A(t) \psi(x)^{-1}$

$$\begin{tikzcd}
\mathbb{F}^n \arrow[r, dashed, "\widetilde{A}(t)"] & \mathbb{F}^n \\
\mathbb{F}^n \arrow[u, "\psi(x)"] \arrow[r, "A(t)"'] & \mathbb{F}^n \arrow[u, "\psi(\gamma(t))"']
\end{tikzcd}$$

$$\begin{aligned}
\nabla_X s\big|_x =& \frac{d}{dt}\bigg|_{t=0} (\widetilde{A}(t))^{-1} s(\gamma(t)) \\
=& \frac{d}{dt}\bigg|_{t=0} \psi(x) A(t)^{-1} \psi^{-1}(\gamma(t)) \cdot s(\gamma(t)) \\
=& \psi(x) \frac{d}{dt}\bigg|_{t=0} A(t)^{-1} \psi^{-1}(x) \cdot s(x) + \psi(x) \frac{d}{dt}\bigg|_{t=0} \psi^{-1}(\gamma(t)) \cdot s(x) + \frac{d}{dt}\bigg|_{t=0} s(\gamma(t)) \\
=& (X + \psi A_X \psi^{-1} + \psi \cdot X(\psi^{-1})) s(x)
\end{aligned}$$

Notice that $\psi \cdot X(\psi^{-1}) + X(\psi) \cdot \psi^{-1} = 0$. Thus, under the change of coordinate, the connection changes by gauge transformation:

$$X + A_X \longmapsto X + \psi A_X \psi^{-1} - X(\psi) \cdot \psi$$

Connections on principal G-bundles For connections on principal G-bundles, we request that the induced parallel transport from connections are always right G-equivalent map. Let $\varphi : \pi^{-1}(U) \longrightarrow U \times G$ be a local trivialization for G-bundle $\pi : E \longrightarrow M$. We can still express the connection on E as a differential operator

$$\nabla_X = X + A_X \text{ , where } A \text{ takes value in } \mathfrak{g}.$$

This differential operator doesn't make sense on E itself. Instead, it make sense on all the associated vector bundle of E.

Let (V, ρ) be a representation of G. Given a connection ∇ on E, we can define a associated connection on the associated bundle $E_V = E \times_G V$ as follow. Let $x \in M$ and γ is a curve with $\gamma(0) = x$. Assume P_γ^t is the parallel transport on the G-bundle E. Then the parallel transport on V is given by

$$\widetilde{P}_\gamma^t : (E_V)_x = E_x \times_G V \longrightarrow (E_V)_{\gamma(t)} = E_{\gamma(t)} \times_G V \qquad (p, v) \longmapsto (P_\gamma^t(p), v)$$

This defines a connection on E_V.

The parallel transport P_γ^t can be expressed as $P_\gamma : (-\epsilon, \epsilon) \longrightarrow G$, such that $P_\gamma^t(g) = P_\gamma(t) \cdot g$. Then, the parallel transport on E_V is given by

$$\widetilde{P}_\gamma^t : V \longrightarrow V \qquad v \longmapsto \rho(P_\gamma(t))v$$

Now choose a local trivialization U of E, $\varphi : \pi^{-1} \longrightarrow U \times G$. Let $s \in \Gamma(E)$, $X = \gamma'(0)$, then

$$\begin{aligned}
\nabla_X s\big|_x &= \frac{d}{dt}\bigg|_{t=0} (\widetilde{P}_\gamma^t)^{-1} s(\gamma(t)) \\
&= \frac{d}{dt}\bigg|_{t=0} \rho(P_\gamma(t))^{-1} s(\gamma(t)) \\
&= X \cdot s(x) + \frac{d}{dt}\bigg|_{t=0} \rho(P_\gamma(t))^{-1} s(x) \\
&= X \cdot s(x) + \widetilde{\rho}\left(\frac{d}{dt}\bigg|_{t=0} P_\gamma(t)^{-1}\right) s(x)
\end{aligned}$$

Notice that $\widetilde{\gamma}(t) = P_\gamma(t)^{-1}$ is a curve in G with $\widetilde{\gamma}(0) = 1$. Therefore, $\frac{d}{dt}\Big|_{t=0} P_\gamma(t)^{-1}$ is an element in Lie algebra $\mathfrak{g}$. Therefore, on the localization, we can write the connection as

$$\nabla_X = X + A_X \text{ , where } A \text{ takes value in } \mathfrak{g}.$$

In the sense that, for any representation $\rho : G \longrightarrow \mathrm{GL}(V)$ with differential map $\widetilde{\rho} : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$, the associated connection $\widetilde{\nabla}$ on associated vector bundle E_V can be written as

$$\widetilde{\nabla}_X = X + \widetilde{\rho}(A_X)$$

Similarly, let $\varphi' : \pi^{-1}(U) \longrightarrow U \times G$ with transition map $\psi : U \longrightarrow G$. Then the transition of connection can be expressed as

$$\begin{aligned} X + A_X \longmapsto & \quad X + \psi A_X \psi^{-1} - X(\psi) \cdot \psi^{-1} \\ = & \quad X + \mathrm{Ad}_\psi(A_X) - (R_\psi)_*^{-1}(X\psi) \end{aligned}$$

In the sense that, the transition of associated connection on associated vector bundle is given by

$$\begin{aligned} X + \widetilde{\rho}(A_X) \longmapsto & \quad X + (\rho \circ \psi)\widetilde{\rho}(A_X)(\rho \circ \psi)^{-1} - X((\rho \circ \psi)) \cdot (\rho \circ \psi)^{-1} \\ = & \quad X + \widetilde{\rho}(\mathrm{Ad}_\psi(A_X)) - \widetilde{\rho}((R_\psi)_*^{-1}(X\psi)) \end{aligned}$$

References

[1] E. Frenkel, *Langlands correspondence for loop groups*, Cambridge studies in advanced mathematics, vol. 103, Cambridge University Press, New York, 2007.

[2] E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves*, Second Edition, Mathematical Surveys and Monographs, vol. 88. AMS 2004.