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SUBMISSION DATE / POSTED DATE

21-01-2020 / 23-01-2020

CITATION

Ren, Wei (2020): A Reduced Collatz Dynamics Maps to a Residue Class, and its Count of x/2 over Count of 3*x+1 is larger than ln3/ln2. TechRxiv. Preprint. https://doi.org/10.36227/techrxiv.11664567.v1

DOI

10.36227/techrxiv.11664567.v1

A Reduced Collatz Dynamics Maps to a Residue Class, and its Count of x/2 over Count of 3*x+1 is larger than ln3/ln2

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Abstract

We propose Reduced Collatz conjecture and prove that it is equivalent to Collatz conjecture but more primitive due to reduced dynamics. We study reduced dynamics (that consists of occurred computations from any starting integer to the first integer less than it), because it is the component of original dynamics (from any starting integer to 1). Reduced dynamics is denoted as a sequence of "I" that represents (3*x+1)/2 and "O" that represents x/2. Here 3*x+1 and x/2 are combined together because 3*x+1 is always even and thus followed by x/2. We discover and prove two key properties on reduced dynamics: (1) Reduced dynamics is invertible. That is, given a reduced dynamics, a residue class that presents such reduced dynamics, can be computed directly by our derived formula. (2) Reduced dynamics can be constructed algorithmically, instead of by computing concrete 3*x+1 and x/2 step by step. We discover the sufficient and necessary condition that guarantees a sequence consisting of "I" and "O" to be a reduced dynamics. Counting from the beginning of a sequence, if and only if the count of x/2 over the count of 3*x+1 is larger than ln3/ln2, reduced dynamics will be obtained (i.e., current integer will be less than starting integer).

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Preprint submitted to Elsevier *January 21, 2020*

Keywords: Collatz Conjecture, 3x+1 Problem, Arithmetics, Residue Class, Computational Number Theory
2000 MSC: 11Y55, 11B85, 11A07

1. Introduction

The Collatz conjecture can be stated simply as follows: Take any positive integer number x. If x is even, divide it by 2 to get $x/2$. If x is odd, multiply it by 3 and add 1 to get $3 * x + 1$. Repeat the process again and again. The Collatz conjecture is that no matter what the number (i.e., x) is taken, the process will always eventually reach 1.

The current known integers that have been verified are about 60 bits by T.O. Silva using normal personal computers [1, 2]. They verified all integers that are less than 60 bits.

Wei Ren et al. [3] verified $2^{100000} - 1$ can return to 1 after 481603 times of $3 * x + 1$ computation, and 863323 times of $x/2$ computation, which is the largest integer being verified in the world. Wei Ren [4] also pointed out a new approach for the possible proof of Collatz conjecture. Wei Ren [5] proposed to use a tree-based graph to observe two key inner properties in reduced Collatz dynamics: one is ratio of the count of $x/2$ over the count of $3 * x + 1$, and the other is partition (all positive integers are partitioned regularly corresponding to ongoing dynamics). Wei Ren et al. [6] also proposed an automata method for fast computing Collatz dynamics. All source code and output data by computer programs in those related papers can be accessed in public repository [7].

2. Preliminaries

Notation 2.1.

(1) $\mathbb{N}^*$*: positive integers;*

(2) $\mathbb{N} = \mathbb{N}^* \cup \{0\}$;

(3) $[1]_2 = \{x | x \equiv 1 \mod 2, x \in \mathbb{N}^*\}$; $[0]_2 = \{x | x \equiv 0 \mod 2, x \in \mathbb{N}^*\}$.

(4) $[i]_m = \{x | x \equiv i \mod m, x \in \mathbb{N}^*, m \geq 2, m \in \mathbb{N}^*, 0 \leq i \leq m - 1, i \in \mathbb{N}\}$.

Proposition 2.2. $x/2$ *always follows after* $3 * x + 1$.

Proof When $x \in [1]_2$, then next computation is $3{*}x{+}1$. Obviously, $3{*}x{+}1 \in [0]_2$, thus the next computation must be $x/2$ consequently. $\square$

We thus can represent required computation as $(3 * x + 1)/2$ and $x/2$, which are denoted by $I(x)$ and $O(x)$, respectively.

Notation 2.3. *$I(x) = (3 * x + 1)/2$, $O(x) = x/2$.*

Note that, $I(x)$ and $O(x)$ can be simply denoted as $I(\cdot)$ and $O(\cdot)$, or I and O, respectively. Obviously, $\forall x \in \mathbb{N}^*$, $I(x) = (3 * x + 1)/2 > x$, $O(x) = x/2 < x$. That is the reason of notation - I represents "Increase" and O represents "dOwn".

Definition 2.4. *Collatz transformation, denoted as $f(\cdot)$, where $f(\cdot) = I(\cdot) = (3 * x + 1)/2$ if $x \in [1]_2$, and $f(\cdot) = O(\cdot) = x/2$ if $x \in [0]_2$.*

Remark 2.5.

(1) We assume $f^0(x) = x$.

(2) Obviously, $f_1 \| f_2 \| ... \| f_n(x) = f_n(f_{n-1}(...f_2(f_1(x))))$, where $f_i(\cdot) \in \{I(\cdot), O(\cdot)\}$, $i = 1, 2, ..., n$, and "$\|$" is concatenation of Collatz transformations. For simplicity, we just denote $f_i(\cdot)$ as $f \in \{I, O\}$.

(3) $f^n(x) = \underbrace{f...f}_{n}(x)$, $f^n(x) = f(f^{n-1}(x)), n \in \mathbb{N}^*$. Note that, whether f is I or O in $f(f^{n-1}(x))$, is determined by $f^{n-1}(x) \in [1]_2$ or $[0]_2$.

Definition 2.6. *Collatz Conjecture. $\forall x \in \mathbb{N}^*, \exists L \in \mathbb{N}^*$, such that $f^L(x) = 1$ where $f \in \{I, O\}$.*

Obviously, Collatz conjecture is held when $x = 1$. In the following, we mainly concern $x \geq 2, x \in \mathbb{N}^*$.

Definition 2.7. *Reduced Collatz Conjecture. $\forall x \in \mathbb{N}^*, x \geq 2$, $\exists L \in \mathbb{N}^*$, such that $f^L(x) < x$ and $f^i(x) \nless x$, $i = 0, 1, ..., L - 1$, $f \in \{I, O\}$.*

Obviously, L must be the minimal positive integer such that $f^L(x) < x$.

Theorem 2.8. *Collatz Conjecture is equivalent to Reduced Collatz Conjecture.*

Proof $\forall x, L \in \mathbb{N}^*, x \geq 2$, it is obvious that $f^L(x) \in \mathbb{N}^*$, i.e., $f^L(x) \geq 1$.

(1) Suppose Collatz Conjecture is true. That is, $\forall x \in \mathbb{N}^*, x \geq 2$, $\exists L \in \mathbb{N}^*$, $f^L(x) = 1$. Thus, $f^L(x) < x$. Hence, Reduced Collatz Conjecture is true.

(2) Inversely, suppose Reduced Collatz Conjecture is true. That is, $\forall x \in \mathbb{N}^*, x \geq 2$, $\exists q_0 \in \mathbb{N}^*$, $f^{q_0}(x) < x$.

If $f^{q_0}(x) = 1$, then Collatz Conjecture is true.

If $f^{q_0}(x) > 1$, then let $y_1 = f^{q_0}(x)$. As Reduced Collatz Conjecture is true, $\exists q_1 \in \mathbb{N}^*$, $f^{q_1}(y_1) < y_1$.

For better notation, let $y_0 = x$. Iteratively, if $y_i = f^{q_{i-1}}(y_{i-1}) = 1, i \in \mathbb{N}^*$, then Collatz Conjecture is true. If $y_i = f^{q_{i-1}}(y_{i-1}) > 1$, then $\exists q_i \in \mathbb{N}^*$, $y_{i+1} = f^{q_i}(y_i) < y_i$.

Thus, $y_{i+1} < y_i < ... < y_1 < y_0 = x$. y_i ($i \in \mathbb{N}^*$) is a *strictly* decreasing sequence.

Besides, $y_{i+1} = f^{q_0+q_1+q_2+...+q_i}(x) \geq 1$.

Therefore, after finite times of iterations, $\exists n \in \mathbb{N}^*$, $y_n = 1$.

That is, $\exists q = q_0 + q_1 + ... + q_{n-1} = \Sigma_{i=0}^{n-1} q_i$, $q \in \mathbb{N}^*$, $f^q(x) = 1$.

Thus, Collatz Conjecture is true. $\square$

Remark 2.9.

(1) We call an ordered sequence $f^q \in \{I, O\}^q$ in above proof as original dynamics (referring to $f^q(x) = 1$), which consists of q occurred Collatz transformations during the computing procedure from a starting integer to 1. For example, the original dynamics of 5 is IOOO due to $5 \to 16 \to 8 \to 4 \to 2 \to 1$.

(2) In contrast, we call f^{q_0} in above proof as reduced dynamics (referring to $f^{q_0}(x) < x$), which is represented by a sequence of occurred Collatz transformations during the computing procedure from a starting integer (i.e., x) to the first transformed integer that is less than the starting integer (i.e., $f^{q_0}(x)$). For example, the reduced dynamics of 5 is IO due to $5 \to 16 \to 8 \to 4$.

(3) Obviously, reduced dynamics is more primitive than original dynamics, because original dynamics consists of reduced dynamics. Simply speaking, reduced dynamics are building blocks of original dynamics.

Due to above theorem, we concentrate on reduced dynamics.

Notation 2.10. $\mathsf{RD}[x]$. *It denotes reduced dynamics of x that are represented by $f \in \{I, O\}$. Formally, $\forall x \in \mathbb{N}^*$, $x \geq 2$, if $\exists L \in \mathbb{N}^*$ such that $f^L(x) < x$ and $f^i(x) \nless x, i = 0, 1, ..., L-1$, where $f \in \{I, O\}$, then let*

$s = f^L \in \{I, O\}^L$ and s is called as reduced dynamics of x, denoted as $\mathsf{RD}[x] = f^L = s$.

Remark 2.11.

(1) Simply speaking (or recall that), $\mathsf{RD}[x]$ *represents occurred Collatz transformations in terms of I and O during the computing process from starting integer x to the first transformed integer that is less than x.*

(2) Recall that, $f^L \in \{I, O\}^L$ is an ordered sequence consisting of I and O. Besides, $f^L = f^{L-1} \| f$, $f^L(x) = f(f^{L-1}(x))$, and $f^0(x) = x$. Furthermore, this sequence implicitly matches the parity *of all occurred intermediate transformed integers that are taken as input of $f(\cdot)$.*

(3) Roughly speaking, in $\mathsf{RD}[x] = f^L$*, x is called starting integer. $f^i(x), i = 1, 2, ..., L$ are called transformed integers. $f^L(x)$ is the first transformed integer that is less than the starting integer x. In other words, $f^i(x) \nless x, i = 0, 1, ..., L-1$, and $f^L(x) < x$. ($f^0(x) = x$.) Besides, the parity of $f^i(x)$ determines the selection of the intermediately next $f \in \{I, O\}$ after f^i.*

(4) Obviously, $\mathsf{RD}[x \in [0]_2] = O$.

(5) For example, $\mathsf{RD}[3] = IIOO$, $\mathsf{RD}[5] = IO$, $\mathsf{RD}[7] = IIIOIOO$, $\mathsf{RD}[9] = IO$, $\mathsf{RD}[11] = IIOIO$. *Indeed, we design computer programs that output all* $\mathsf{RD}[x], \forall x \in [1, 99999999]$ *[7]. From the data we discover the property - period and its relation to the number of computing $x/2$ in reduced dynamics - will be proved in the following of this paper.*

(6) In fact, we proved some results on $\mathsf{RD}[x]$ *for specific x, e.g.,* $\mathsf{RD}[x \in [1]_4] = IO$, $\mathsf{RD}[x \in [3]_{16}] = IIOO$, $\mathsf{RD}[x \in [11]_{32}] = IIOIO$*, and so on [5].*

(7) $IIOO$ can be denoted in short as I^2O^2. $IIIOIOO$ can be denoted in short as I^3OIO^2. In other words, we denote $\underbrace{I...I}_{n}$ as I^n, and we denote $\underbrace{O...O}_{n}$ as O^n where $n \in \mathbb{N}^*, n \geq 2$. We also assume $I^1 = I$, $O^1 = O$. $I^0 = O^0 = \emptyset$ means no transformation occurs.

(8) We will formally proved that the ratio exists in any reduced Collatz dynamics. That is, the count of $x/2$ over the count of $3 * x + 1$ is larger than $log_2 3$ in this paper. The ratio can also be observed and verified in my proposed tree-based graph [5].

Example 2.12. $\mathsf{RD}[5] = IO$*, if and only if*

(1) "I" is due to $5 \in [1]_2$*;*

(2) $I(5) = (3 * 5 + 1)/2 = 8 \not< 5$*, thus continue;*

(3) "O" is due to $I(5) = 8 \in [0]_2$*;*

(4) $IO(5) = O(I(5)) = O(8) = 8/2 = 4 < 5$*, thus end.*

To better present above the implicity in reduced dynamics, we introduce two functions as follows:

Definition 2.13. $IsMatched : x \times c \rightarrow bool$*. It takes as input* $x \in \mathbb{N}^*$ *and* $c \in \{I, O\}$*, and outputs* $bool \in \{True, False\}$*. If* $x \in [1]_2$ *and* $c = I$*, or if* $x \in [0]_2$ *and* $c = O$*, then output* $bool = True$*; Otherwise, output* $bool = False$*.*

Remark 2.14. *Simply speaking, this function checks whether the forthcoming Collatz transformation (i.e.,* $c \in \{I, O\}$*) matches with the current transformed integer* x*.*

Definition 2.15. $GetS : s \times i \times j \rightarrow s'$*. It takes as input* s, i, j*, where* $s \in \{I, O\}^{|s|}$*,* $1 \leq i \leq |s|$*,* $1 \leq j \leq |s| - (i - 1)$*, and outputs* s' *where* $s = s_a \| s' \| s_b$*,* $|s_a| = i - 1$*,* $|s'| = j$*,* $|s_b| = |s| - |s_a| - |s'|$ *and "*$| \cdot |$*" returns length in terms of the total count of* I *or* O*.*

Remark 2.16.

(1) For example, $GetS(IIOO, 1, 4) = IIOO, GetS(IIOO, 1, 3) = IIO$*.*

(2) Especially, $GetS(s, 1, |s|) = s$*.* $GetS(s, |s|, 1)$ *returns the last transformation in* s*.* $GetS(s, 1, 1)$ *returns the first transformation in* s*.* $GetS(s, j, 1)$ *returns the* j*-th transformation in* s*.*

(3) In other words, s' *is a selected segment in* s *that starts from the location* i *and has the length of* j*. Indeed, that is the reason we call this function as "Get Substring".*

(4) Simply speaking, this function can obtain the Collatz transforms from i *to* $i + j - 1$ *from a given inputting transform sequence (e.g., reduced dynamics) in terms of* $s \in \{I, O\}^{|s|}$*.*

(5) Note that, $GetS(\cdot)$ *itself is a function. In other words, it can be looked as* $GetS(\cdot)(\cdot)$*. E.g.,* $GetS(IIOO, 1, 1)(3) = I(3) = (3 * 3 + 1)/2 = 5$*,*
$GetS(IIOO, 1, 2)(3) = II(3) = I(I(3)) = I(5) = (3 * 5 + 1)/2 = 8$*,*
$GetS(IIOO, 1, 3)(3) = IIO(3) = O(II(3)) = O(8) = 8/2 = 4$*,*
$GetS(IIOO, 1, 4)(3) = IIOO(3) = O(IIO(3)) = O(4) = 4/2 = 2 < 3$*.*

(6) It is worth to stress that, although in above definition $j \geq 1$, it can be extended to $j \geq 0$ by assuming $GetS(\cdot, \cdot, 0)(x) = x$.

Proposition 2.17. *Suppose $x \in \mathbb{N}^*, x \geq 2$. If $\mathsf{RD}[x]$ exists, then*
(1) $s(x) < x$, where $s = \mathsf{RD}[x]$;
(2) $GetS(s, 1, i)(x) \nless x$, where $i = 1, 2, ..., |s| - 1$;
(3) $IsMatched(GetS(s, 1, j-1)(x), GetS(s, j, 1)) = True$, where $j = 1, 2, ..., |s|$.

Proof Straightforward by the definition of $\mathsf{RD}[x]$. $\square$

Remark 2.18.

(1) $s(x)$ is the last transformed integer, or the first transformed integer that is less than the starting integer.

(2) $GetS(s, 1, i)(x)$ ($i = 1, 2, ..., |s| - 1$) are all intermediate transformed integers.

(3) When $j = 1$, $GetS(s, 1, j-1)(x) = GetS(s, 1, 0)(x) = x$. $GetS(s, j, 1)$ is the first transformation.

(4) If $GetS(s, 1, j-1)(x)$ ($j = 2, ..., |s|$) is current transformed integer, then $GetS(s, j, 1)$ is the next intermediate Collatz transformation.

Proposition 2.19. *$\forall x \in \mathbb{N}^*$, $x \geq 2$, if $\mathsf{RD}[x]$ exists, then $\mathsf{RD}[x]$ is unique.*

Proof Straightforward. Given x, either $I(x)$ or $O(x)$ is deterministic and unique. Similarly, given x, $s'(x)$ is deterministic and unique, where $s' = GetS(s, 1, i), s = \mathsf{RD}[x], i = 1, 2, ..., |s|$. Thus, s is unique for any given x. $\square$

Remark 2.20.

We assume $\mathsf{RD}[x = 1] = IO$, although $IO(1) = O((3 * 1 + 1)/2) = O(2) = 2/2 = 1 \nless x$. In other words, we assume the reduced dynamics of $x = 1$ is IO. In the following, we always concern $x \geq 2, x \in \mathbb{N}^*$.

Proposition 2.21. *Given $x \in \mathbb{N}^*$, if $\mathsf{RD}[x]$ exists, then $\mathsf{RD}[x]$ ends by O.*

Proof Straightforward due to $I(x) = (3 * x + 1)/2 > x$. Suppose $\exists x \in \mathbb{N}^*$, $x \geq 2$, $s(x) \nless x$, $\mathsf{RD}[x] = s \| I$. Then, $\{s \| I\}(x) = I(s(x)) > s(x)$, thus $\mathsf{RD}[x] = \{s \| I\}(x) \nless x$. Contradiction occurs. $\square$

Proposition 2.22. $\mathsf{RD}[x \in [0]_2] = O$, $\mathsf{RD}[x \in [1]_4] = IO$.

Proof (1) $x \in [0]_2$, thus O occurs. $x/2 < x$, thus $\mathsf{RD}[x] = O$.

(2) If $x = 1$, $\mathsf{RD}[1] = IO$ (by assumption).

If $x \geq 2$, $x = 4t + 1 \in [1]_2$, where $t \in \mathbb{N}^*$. Thus, I occurs. $I(x) = (3 * x + 1)/2 = (3 * (4t + 1) + 1)/2 = (12t + 4)/2 = 2 * (3t + 1) \in [0]_2$. $2 * (3t + 1) > x = 4t + 1$, thus further transformation occurs. $O(I(x)) = 2 * (3t + 1)/2 = 3t + 1 < 4t + 1 = x$ ($\because t \in \mathbb{N}^*$), thus $\mathsf{RD}[x] = IO$. $\square$

Proposition 2.23. *Given* $x \in [3]_4$, *if* $\mathsf{RD}[x]$ *exists, then* $\mathsf{RD}[x] \in I^pO\|\{I, O\}^{\geq 1}$, $p \geq 2$.

Proof Let $x = 4t + 3$, $t \in \mathbb{N}$. Obviously, $x \in [1]_2$.

$I(x) = (3x + 1)/2 = (12t + 10)/2 = 6t + 5 \in [1]_2$.

$I^2(x) = I(I(x)) = (3(6t + 5) + 1)/2 = 9t + 8$.

(1) If $t \in [0]_2 \cup \{0\}$, then $9t + 8 \in [0]_2$. Thus, the next transformation is "$x/2$". Thus, current occurred transformations are "IIO". Besides, $IIO(x) = (9t+8)/2 = 4.5t+4 > 4t+3 = x$. Further transformation thus occurs. Hence, if $\mathsf{RD}[x]$ exists, then $\mathsf{RD}[x] \in I^2O\|\{I, O\}^{\geq 1}$.

(2) If $t \in [1]_2$, then $9t + 8 \in [1]_2$. Thus, current occurred transformations are "III". Besides, $III(x) = (3(9t+8)+1)/2 = (27t+25)/2 = 13.5t+12.5 > 4t + 3 = x$. Further transformation occurs. Hence, if $\mathsf{RD}[x]$ exists, then $\mathsf{RD}[x] \in I^3\|\{I, O\}^{\geq 1}$.

If $III(x) \in [1]_2$, then more "I" occurs. Obviously, $IIII(x) > x$. Further transformation occurs. Hence, if $\mathsf{RD}[x]$ exists, then $\mathsf{RD}[x] \in I^4\|\{I, O\}^{\geq 1}$. If $III(x) \in [0]_2$, then $IIIO(x) = (13.5t + 12.5)/2 = 6.75t + 6.25 > 4t + 3 = x$. Further transformation occurs consequently. Hence, if $\mathsf{RD}[x]$ exists, then $\mathsf{RD}[x] \in I^3\|O\|\{I, O\}^{\geq 1}$.

Note that, suppose $\mathsf{RD}[x] = f^L \in \{I, O\}^L$. There exists at least one "O" in L transformations, otherwise $f^L(x) = I^L(x) > I^{L-1}(x) > ... > I(x) > x$, which contradicts with $f^L(x) < x$. Besides, $I^pO(x) = I^p(x)/2 > I^{p-1}(x)/2 = I^{p-1}O(x) > ... > IIO(x) > x$, $p \geq 3$, thus there exists further transformation after I^pO ($p \geq 3$).

In summary, if $\mathsf{RD}[x \in [3]_4]$ exists, then $\mathsf{RD}[x] \in I^pO\|\{I, O\}^{\geq 1}, p \geq 2$. $\square$

Theorem 2.24. ***(Format Theorem.)*** *Given* $x \in \mathbb{N}^*$, *if* $\mathsf{RD}[x]$ *exists, then* $\mathsf{RD}[x] \in I^p\|O\|\{I, O\}^q$, *where* $p \in \mathbb{N}$, $q = |\mathsf{RD}[x]| - p - 1$. *Besides,*

$$p = \begin{cases} 0 & x \in [0]_2, \\ 1 & x \in [1]_4, \\ 2 & x \in [3]_8, \\ \alpha + 2 \quad t+1 = 2^{\alpha} * A, A \in [1]_2, \alpha \in \mathbb{N}^* & x \in [7]_8. \end{cases} \tag{1}$$

$q = 0$ *when* $p = 0, 1$; $q \geq 1$ *when* $p \geq 2$.

Proof The range of q is straightforward due to Proposition 2.22 and Proposition 2.23, thus we mainly concern p.

By Proposition 2.22, if $x \in [0]_2$, then $\mathsf{RD}[x] = O$ and $p = 0$. If $x \in [1]_4$, then $\mathsf{RD}[x] = IO$ and $p = 1$.

Next, we concentrate on $x \in [3]_4$.

$\mathsf{RD}[3] = IIOO = I^2O^2$, which can be manually and easily verified.

Let $x = 4t + 3, t \in \mathbb{N}^*$.

(1) Case I: $t \in [0]_2$.

As $x = 4t + 3 \in [1]_2$, $I(\cdot)$ is conducted consequently. As $I(x) = (3x + 1)/2 = 1.5x + 1.5 > x$ and $II(x) > I(x) > x$, the checking on whether current transformed number is less than starting number may be omitted in some straightforward cases.

$I(x) = (3x+1)/2 = (3(4t+3)+1)/2 = (12t+10)/2 = 6t+5 \in [1]_2$, thus transformation $I(\cdot)$ is conducted consequently.

$I(I(x)) = II(x) = (3(6t+5)+1)/2 = (18t+16)/2 = 9t+8 \in [0]_2$. Thus, $O(\cdot)$ is conducted consequently.

$O(I(I(x))) = IIO(x) = (9t + 8)/2 = 4.5t + 4 > 4t + 3$. Thus, further transformation is conducted consequently.

Therefore, $\mathsf{RD}[x] \in I^2O\|\{I, O\}^{\geq 1}$.

(2) Case II: $t \in [1]_2$.

As $x = 4t + 3 \in [1]_2$, $I(\cdot)$ is conducted consequently.

$I(x) = (3x+1)/2 = (3(4t+3)+1)/2 = (12t+10)/2 = 6t+5 \in [1]_2$, thus $I(\cdot)$ is conducted consequently.

$II(x) = (3(6t + 5) + 1)/2 = (18t + 16)/2 = 9t + 8 \in [1]_2$. Thus, $I(\cdot)$ is conducted consequently.

$III(x) = (3(9t+8)+1)/2 = (27t+25)/2$. It depends on the partition of t (more specifically, $t \in [1]_4$ or $[3]_4$) whether $(27t+25)/2$ is even or odd.

(2.1) If $t \in [1]_4$, $III(x) = (27t + 25)/2 = (27 * (4 * k + 1) + 25)/2 = (108k + 52)/2 = 54k + 26 \in [0]_2$ ($k \in \mathbb{N}^*$), thus $O(\cdot)$ will occur consequently.

(2.2) If $t \in [3]_4$, $III(x) = (27t + 25)/2 = (27 * (4 * k + 3) + 25)/2 = (108k + 106)/2 = 54k + 53 \in [1]_2$, thus $I(\cdot)$ will occur consequently.

$IIIO(x) = (27t + 25)/2/2 = (54k + 26)/2 = 27k + 13$, whose parity depends on the parity of k.

$IIII(x) = (3(27t+25)/2+1)/2 = (3(54k+53)+1)/2 = (162k+160)/2 = 81k + 80$, whose parity depends on the parity of k.

In other words, the judgement on the parity of $IIIO(x)$ is undecidable, unless the domain ($t \in [1]_4$ or $t \in [3]_4$) is *partitioned* further.

For exploring more general results, we put it in another way as follows:

Suppose there exist at most p times of "I" at head (i.e., $I^p\|O\|...$) for $x \in [3]_4$. Observing following equation for $I^p(x)$ after consecutive $p \geq 2$ times of "I":

$$\begin{aligned}
I^p(x) &= (3(...(3(3x+1)/2)+1)/2...)+1)/2 \\
&= \frac{3}{2}(\frac{3}{2}(...\frac{3}{2}(\frac{3}{2}x+\frac{1}{2})+\frac{1}{2})+...+\frac{1}{2})+\frac{1}{2} \\
&= (\frac{3}{2})^p x + \frac{1}{2}((\frac{3}{2})^{p-1} + (\frac{3}{2})^{p-2} + ... + 1) \\
&= (\frac{3}{2})^p x + \frac{1}{2}(\frac{(\frac{3}{2})^p - 1}{\frac{3}{2} - 1}) \\
&= (\frac{3}{2})^p x + (\frac{3}{2})^p - 1 \\
&= (\frac{3}{2})^p (x+1) - 1 \ \ (\because x = 4t+3, t \in \mathbb{N}) \\
&= (\frac{3}{2})^p (4t+3+1) - 1 = (\frac{3^p}{2^{p-2}})(t+1) - 1 \in \mathbb{N}^*.
\end{aligned} \tag{2}$$

Note that, above computation implicitly includes two requirements due to p times of consecutive $I(\cdot)$ as follows:

(i) All intermediate transformed numbers during processes (i.e., computing p times of consecutive "I") satisfy $(\frac{3^i}{2^{i-2}})(t+1) - 1 \in [1]_2$, where $2 \leq i \leq p-1, i \in \mathbb{N}^*$.

(ii) Besides, $(\frac{3^i}{2^{i-2}})(t+1) - 1 \in [0]_2$, where $i = p$, as only (or at most) p consecutive $I(\cdot)$ occur.

In other words, p can also be looked as the minimal value to let current transformed number be in $[0]_2$. Thus, we need to explore the requirement on p for given t such that

$$\begin{cases} (\frac{3^i}{2^{i-2}})(t+1)-1 \in [1]_2 & 2 \le i \le p-1, \\ (\frac{3^i}{2^{i-2}})(t+1)-1 \in [0]_2 & i = p. \end{cases} \tag{3}$$

Represent $t+1$ as $2^\alpha * A, A \in [1]_2, \alpha \in \mathbb{N}^*$. That is, $t+1 = 2^\alpha * A$. Obviously, this representation is unique. We thus need to prove that the requirement in Eq. 3 is satisfied if and only if $p = \alpha + 2$. Note that, we will see that here p is indeed determined by α.

For $2 \le i < p = \alpha + 2, i \in \mathbb{N}^*$, we have $\alpha + 2 - i > 0$.

$$(\frac{3^i}{2^{i-2}})(t+1)-1 = (\frac{3^i}{2^{i-2}}) * 2^\alpha * A - 1 = 3^i * 2^{\alpha-i+2} * A - 1.$$

$\alpha + 2 - i > 0 \Rightarrow 2^{\alpha-i+2} \in [0]_2 \Rightarrow 3^i * 2^{\alpha-i+2} * A \in [0]_2$
$\Rightarrow 3^i * 2^{\alpha-i+2} * A - 1 \in [1]_2$.

When $i = p = \alpha + 2$, we have exactly

$$(\frac{3^p}{2^{p-2}})(t+1)-1 = (\frac{3^p}{2^{p-2}}) * 2^\alpha * A - 1$$
$$= 3^p * 2^{\alpha-p+2} * A - 1 = 3^p * A - 1 \in [0]_2. \quad \because A, 3^p \in [1]_2$$

It is easy to see that $p = \alpha + 2$ is the one and only one for the requirement in Eq. 3, as desired. □

Corollary 2.25. *(t determines p Corollary.) Given starting integer $x \in [3]_4$ (i.e., $x = 4t+3, t \in \mathbb{N}$), the count of consecutive "I" (denoted as p) is determined by t as follows:*
If $t \in [0]_2$, then $p = 2$; if $t \in [1]_2$, then $p = \alpha + 2$, where $\alpha = \log_2 \frac{t+1}{A} \in \mathbb{N}^*$, $A = \max(\{a \mid a \in [1]_2, a | (t+1)\})$.

Proof It is straightforward by Theorem 2.24. □

Corollary 2.26. *Given starting integer $x \in [3]_4, t = (x-3)/4 \in \mathbb{N}$, the first p count of Collatz transformations must be I^p and p can be determined by t, and the transformed integer after I^p transformations is*

$$I^p(x) = (\frac{3}{2})^p(x+1) - 1 = (\frac{3^p}{2^{p-2}})(t+1) - 1 \in \mathbb{N}^*, p \ge 2.$$

Proof It is straightforward by Theorem 2.24 and Corollary 2.25. □

Remark 2.27.

(1) Note that, due to Corollary 2.25, p for I^p in $\mathsf{RD}[x]$ *can be computed by* $t = (x-3)/4$ *and* $\log_2 \frac{t+1}{A}$ *directly without conducting concrete Collatz transformations, which can shorten the computation delay for reduced dynamics.*

(2) Besides, by Eq. 2 or Corollary 2.26, if $t = 0$, *we then have* $p = 2$, *because* $I^p(x) = (\frac{3^p}{2^{p-2}})(t+1) - 1 = (\frac{3^p}{2^{p-2}}) - 1 \in \mathbb{N}^*$, *which matches with the result* $\mathsf{RD}[3] = IIOO$ *by manually computing.*

(3) Indeed, Eq. 2 can be extended to include all cases (i.e., for $p = 0, 1$*). If* $p = 0$, *by assuming* $I^0(x) = x$, $I^0(x) = x = (\frac{3}{2})^0(x+1) - 1 = (\frac{3}{2})^p(x+1) - 1$; *If* $p = 1$, $I^1(x) = I(x) = (3 * x + 1)/2 = (\frac{3}{2})^1(x+1) - 1 = (\frac{3}{2})^p(x+1) - 1$. *Therefore,* $I^p(x) = (\frac{3}{2})^p(x+1) - 1$ *for* $p \in \mathbb{N}^*$.

Corollary 2.28. *Suppose* $x = 2^n - 1, n \in \mathbb{N}^*$. *If* $\mathsf{RD}[x]$ *exists, then* $\mathsf{RD}[x] \in I^n \| O \| \{I, O\}^{\geq 0}$.

Proof Straightforward. □

Remark 2.29. *Simply speaking, above Corollary states that if* $\mathsf{RD}[x = 2^n - 1, n \in \mathbb{N}^*]$ *exists, then the first* $n+1$ *Collatz transformations for* x *must be* I^nO. *Indeed, the resumption on the existence of the reduced dynamics of* x *can be omitted. That is, the first* $n+1$ *Collatz transformations in original dynamics of* x *must also be* I^nO.

Example 2.30. $\mathsf{RD}[7] = IIIOIOO = I^3 \| O \| IOO$, *as* $7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 < 7$.

Next corollary states that reduced dynamics consists of segment or segments with a unified form as $I^pO^q, p \geq 1, q \geq 1$.

Corollary 2.31. *Given* $x \in [3]_4$, *if* $\mathsf{RD}[x]$ *exists, then*

$$\mathsf{RD}[x] \in I^{p_0 \geq 2} \| O^{q_0 \geq 1} \| I^{p_1 \geq 1} \| O^{q_1 \geq 1} \| ... \| I^{p_n \geq 1} \| O^{q_n \geq 1}.$$

Proof Straightforward. $x \in [1]_2$, thus I occurs. After p times of I transformations, $I^p(x) \in [0]_2$ and thus O follows. After q times of O transformations, $I^pO^q(x) \in [1]_2$, thus I occurs. Indeed, q can be determined by $I^p(x)$ by $q = \log_2 \frac{I^p(x)}{B}$, $B = \max(\{b | b \in [1]_2, b | I^p(x)\})$.

Iteratively, each segment has a unified form I^pO^q, where $p, q \in \mathbb{N}^*$.

The first segment is listed solely, because the distinction between the first segment and the other segments is that $p_0 \geq 2$ but $p_i \geq 1, i \in \mathbb{N}^*$. (In other words, when and only when an intermediate transformed number is in $[1]_4$ occurs, $p_i = 1$. Otherwise, $p_i \geq 2$.) $\square$

3. Derive x from RD[x]

3.1. Preparation

Notation 3.1. $Set_{\mathsf{RD}} = \{s | x \in \mathbb{N}^*, \exists \mathsf{RD}[x], s = \mathsf{RD}[x], s \in \{I, O\}^{\geq 1}\}$.

That is, $\forall x \in \mathbb{N}^*$, if $\mathsf{RD}[x]$ exists, then $\mathsf{RD}[x] = s$ will be included in Set_{RD}, which is a set of existing reduced dynamics.

In this section, we will study two problems as follows:

1. *Inverse* problem: Given $\forall s \in Set_{\mathsf{RD}}$, is it possible to derive starting integer x such that $\mathsf{RD}[x] = s$?
2. What is the *sufficient and necessary* conditions for any existing reduced dynamics. That is, given $s \in \{I, O\}^{\geq 1}$, how to decide whether $s \in Set_{\mathsf{RD}}$?

Before exploring general situations, we give two trivial cases.

Proposition 3.2. *$O \in Set_{\mathsf{RD}}$, $IO \in Set_{\mathsf{RD}}$.*

Proof Straightforward. $\mathsf{RD}[x \in [0]_2] = O$ and $\mathsf{RD}[x \in [1]_4] = IO$ by Proposition 2.22. $\square$

In the following, we thus mainly concentrate on $x \in [3]_4$ where their reduced dynamics presents the form like $I^{p \geq 2}O \| \{I, O\}^{q \geq 1}, p, q \in \mathbb{N}^*$ once it exists (recall Proposition 2.23).

Theorem 3.3. ***(Subset Theorem.)***

Suppose $s \in Set_{\mathsf{RD}}$, $|s| \geq 2$, $x \in \mathbb{N}^*$, $i = 0, 1, ..., |s| - 2$. We have

(1.1) $\{x | GetS(s, 1, i+1)(x) \in [1]_2\} \subset \{x | GetS(s, 1, i)(x) \in [0]_2\}$;

(1.2) $\{x | GetS(s, 1, i+1)(x) \in [0]_2\} \subset \{x | GetS(s, 1, i)(x) \in [0]_2\}$;

(2.1) $\{x | GetS(s, 1, i+1)(x) \in [1]_2\} \subset \{x | GetS(s, 1, i)(x) \in [1]_2\}$;

(2.2) $\{x | GetS(s, 1, i+1)(x) \in [0]_2\} \subset \{x | GetS(s, 1, i)(x) \in [1]_2\}$.

Proof When $i = 0$, there exists two and only two cases as follows:

(1) If $GetS(s, i + 1, 1) = O$, then $GetS(s, 1, i)(x) \in [0]_2$. There exists two subcases as follows:

(1.1) If $GetS(s, i + 2, 1) = I$, then

$O(GetS(s, 1, i)(x))) \in [1]_2$

$\Rightarrow GetS(s, 1, i)(x)/2 \in [1]_2$

$\Rightarrow GetS(s, 1, i)(x) \in [2]_4 \subset [0]_2$.

Thus, $\{x|GetS(s, 1, i + 1)(x) \in [1]_2\} \subset \{x|GetS(s, 1, i)(x) \in [0]_2\}$.

(1.2) If $GetS(s, i + 2, 1) = O$, then

$O(GetS(s, 1, i)(x))) \in [0]_2$

$\Rightarrow GetS(s, 1, i)(x)/2 \in [0]_2$

$\Rightarrow GetS(s, 1, i)(x) \in [0]_4 \subset [0]_2$.

Thus, $\{x|GetS(s, 1, i + 1)(x) \in [0]_2\} \subset \{x|GetS(s, 1, i)(x) \in [0]_2\}$.

(2) If $GetS(s, i + 1, 1) = I$ then $GetS(s, 1, i)(x) \in [1]_2$. There exists two subcases as follows:

(2.1) If $GetS(s, i + 2, 1) = I$, then

$I(GetS(s, 1, i)(x))) \in [1]_2$

$\Rightarrow (3 * GetS(s, 1, i)(x) + 1)/2 \in [1]_2$

$\Rightarrow 3 * GetS(s, 1, i)(x) + 1 \in [2]_4$

$\Rightarrow 3 * GetS(s, 1, i)(x) \in [1]_4$

$\Rightarrow GetS(s, 1, i)(x) \in [3]_4 \subset [1]_2$.

Thus, $\{x|GetS(s, 1, i + 1)(x) \in [1]_2\} \subset \{x|GetS(s, 1, i)(x) \in [1]_2\}$.

(2.2) If $GetS(s, i + 2, 1) = O$, then

$I(GetS(s, 1, i)(x))) \in [0]_2$

$\Rightarrow (3 * GetS(s, 1, i)(x) + 1)/2 \in [0]_2$

$\Rightarrow 3 * GetS(s, 1, i)(x) + 1 \in [0]_4$

$\Rightarrow 3 * GetS(s, 1, i)(x) \in [3]_4$

$\Rightarrow GetS(s, 1, i)(x) \in [1]_4 \subset [1]_2$.

Thus, $\{x|GetS(s, 1, i + 1)(x) \in [0]_2\} \subset \{x|GetS(s, 1, i)(x) \in [1]_2\}$.

We can prove similarly for $i = 1, 2, ..., |s| - 2$. □

Corollary 3.4. *Suppose $s \in Set_{\mathsf{RD}}$, $|s| \geq 2$, $i = 0, 1, ..., |s| - 2$, $x \in \mathbb{N}^*$. We have*

(1.1) $\{x|GetS(s, 1, i + 1)(x) \in [1]_2\} = \{x|GetS(s, 1, i)(x) \in [2]_4\}$;

(1.2) $\{x|GetS(s, 1, i + 1)(x) \in [0]_2\} = \{x|GetS(s, 1, i)(x) \in [0]_4\}$;

(2.1) $\{x|GetS(s, 1, i + 1)(x) \in [1]_2\} = \{x|GetS(s, 1, i)(x) \in [3]_4\}$;

(2.2) $\{x|GetS(s, 1, i + 1)(x) \in [0]_2\} = \{x|GetS(s, 1, i)(x) \in [1]_4\}$.

Proof Straightforward by Theorem 3.3. □

Remark 3.5.

(1) Corollary 3.4 states the residue classes are partitioned regularly into halves and either half will present either next intermediate transformation in terms of I or O. (Indeed, the partition property is thoroughly explored and proved in my another paper [9]. Note that, this paper is independent with cited paper.)

(2) Note that, Theorem 3.3 is not only guaranteed for reduced dynamics, but also for original dynamics.

Corollary 3.6. *Suppose $s \in Set_{\mathsf{RD}}$, $|s| \geq 2$, $i = 2, 3, ..., |s|$, $x \in \mathbb{N}^*$. We have*

$\{x | IsMatched(GetS(s, 1, i-1)(x), GetS(s, i, 1)) = True\}$
$\subset \{x | IsMatched(GetS(s, 1, i-2)(x), GetS(s, i-1, 1)) = True\}$.

Proof Straightforward by Theorem 3.3. □

Roughly speaking, above corollary asserts that the last parity requirement is sufficient to guarantee all previous parity requirements. That is, the residue equation for last parity requirement is sufficient for deriving the required residue class for satisfying all parity requirements.

In the following, notation $[i]_m$ is extended in that i could be larger than m or less than 0. That is, $[i]_m = \{x | x = k * m + i, k \in \mathbb{N}, i \in \mathbb{Z}, m \in \mathbb{N}^*, m \geq 2\}$. $\mathbb{Z}$ is integer.

Lemma 3.7. *$a \in \mathbb{N}, x, b, c \in \mathbb{N}^*, b \geq 2$, thus*
(1) $x + c \in [a]_b \Leftrightarrow x \in [a-c]_b$.
(2) $c * x \in [a]_b \Leftrightarrow x \in [c^{-1} * a]_b$, if $gcd(c, b) = 1$.
(3) $x/c \in [a]_b \Leftrightarrow x \in [a * c]_{b*c}$.

Proof (1) $x + c \in [a]_b \Leftrightarrow x + c = k * b + a, k \in \mathbb{N}$
$\Leftrightarrow x = k * b + a - c \Leftrightarrow x \in [a - c]_b$.

(2) $c * x \in [a]_b \Leftrightarrow c * x = k * b + a, k \in \mathbb{N}$
$\Leftrightarrow (k' * b + [c]_b) * x = k * b + a, \quad \because c = k' * b + [c]_b, k' \in \mathbb{N}$
$\Leftrightarrow [c]_b * x = (k - k' * x) * b + a$
$\Leftrightarrow [c]_b * x \in [a]_b$

$\Leftrightarrow [c^{-1}]_b * [c]_b * x \in [c^{-1}]_b * [a]_b \quad \because gcd(c,b) = 1, \exists [c^{-1}]_b,\ s.t.\ [c^{-1}]_b * [c]_b = [1]_b$
$\Leftrightarrow x \in [c^{-1} * a]_b$.

(3) $x/c \in [a]_b \Leftrightarrow x/c = k * b + a, k \in \mathbb{N}$
$\Leftrightarrow x = k * b * c + a * c, k \in \mathbb{N} \Leftrightarrow x \in [a * c]_{b*c}$. □

Indeed, if $gcd(c,b) = 1$, then $[c^{-1}]_b$ exists and is the inverse of $[c]_b$ in group $\mathbb{Z}/b\mathbb{Z}^* = \langle \mathbb{Z}/b\mathbb{Z} \setminus \{0\}, * \mod b\rangle$ such that $[c^{-1}]_b * [c]_b = [1]_b$.

3.2. Derive x if RD[x]=I^pO^q

In this subsection, we assume $p, q \in \mathbb{N}^*, p \geq 2,\ x \in [3]_4$ if no explicitly declaration exists.

Lemma 3.8. *Suppose $p, q \in \mathbb{N}^*, p \geq 2$.*
$\exists x \in [3]_4$ such that $\mathsf{RD}[x] = I^pO^q$, if and only if
Req-I) $I^p(x) \in [0]_{2^q}$;
Req-II) $I^pO^q(x) < x \wedge I^pO^{q-1}(x) \not< x$.

Proof By Theorem 3.3 or Corollary 3.6, all parity sequence can be guaranteed if the last parity requirement is guaranteed.

$I^pO^{q-1}(x) \in [0]_2 \Leftrightarrow I^p(x)/2^{q-1} \in [0]_2 \Leftrightarrow I^p(x) \in [0]_{2^q}$, by Lemma 3.7 (3).

Thus, Req-I is sufficient for the requirements for all parities.

Any x such that $I^pO^{q-1}(x) \not< x$ can guarantee $I^pO^i(x) > x, i = 1, 2, ..., q-2$, because $I^pO^i(x) = I^pO^{q-1}(x) * 2^{(q-1)-i}$. Besides, $I^i(x) > x, i = 1, 2, ..., p$, due to $I(x) = (3*x+1)/2 > x$. Thus, Req-II is sufficient for the requirements for all transformed integers. □

Next, we check the requirement for Req-I.

Lemma 3.9. $I^p(x) \in [0]_{2^q} \Leftrightarrow x \in [(3^p)^{-1} * 2^p - 1]_{2^{p+q}}$.

Proof $I^p(x) \in [0]_{2^q}$
$\Leftrightarrow (\frac{3}{2})^p(x+1) - 1 \in [0]_{2^q} \quad \because Eq.2$
$\Leftrightarrow (\frac{3}{2})^p(x+1) \in [1]_{2^q} \quad \because$ Lemma 3.7 (1)
$\Leftrightarrow 3^p(x+1) \in [2^p]_{2^{p+q}} \quad \because$ Lemma 3.7 (3)
$\Leftrightarrow [3^p]_{2^{p+q}}(x+1) \in [2^p]_{2^{p+q}} \quad \because$ Lemma 3.7 (2)
$\Leftrightarrow x + 1 \in [(3^p)^{-1} * 2^p]_{2^{p+q}} \quad \because$ Lemma 3.7 (2) and $gcd(3^p, 2^{p+q}) = 1$
$\Leftrightarrow x \in [(3^p)^{-1} * 2^p - 1]_{2^{p+q}}. \quad \because$ Lemma 3.7 (1) □

(Indeed, $[(3^p)^{-1}]_{2^{p+q}}$ is the inverse of $[3^p]_{2^{p+q}}$ in group $\mathbb{Z}/2^{p+q}\mathbb{Z}^* = \langle \mathbb{Z}/2^{p+q}\mathbb{Z} \setminus \{0\}, * \mod 2^{p+q}\rangle$. Besides, even if $p = 1$, above lemma still holds due to Corollary 2.26 Remark (3).)

Next, we explore Req-II, assuming that Req-I is already guaranteed - $x \in [(3^p)^{-1} * 2^p - 1]_{2^{p+q}}$.

Lemma 3.10. $I^pO^q(x) < x \wedge I^pO^{q-1}(x) \geq x \Leftrightarrow q = \lceil \lambda * p \rceil \ \wedge \ x > \frac{3^p - 2^p}{2^{p+q} - 3^p}$.

Proof (1) $I^pO^q(x) < x \Leftrightarrow ((\frac{3}{2})^p(x+1) - 1)/2^q < x \quad \because Eq.2$
$\Leftrightarrow 3^p x + 3^p < (2^q x + 1)2^p$
$\Leftrightarrow 3^p x + 3^p < 2^{p+q}x + 2^p$
$\Leftrightarrow 3^p - 2^p < (2^{p+q} - 3^p)x \quad \because x > 0, 3^p - 2^p > 0$
$\Leftrightarrow 2^{p+q} - 3^p > 0 \wedge x > \frac{3^p - 2^p}{2^{p+q} - 3^p}$.

(2) $2^{p+q} - 3^p > 0 \Leftrightarrow 2^q > 3^p/2^p \Leftrightarrow q > \log_2 1.5 * p \Leftrightarrow q \geq \lceil \log_2 1.5 * p \rceil$, since $q \in \mathbb{N}^*, \log_2 1.5 \notin \mathbb{Q}, \log_2 1.5 * p \notin \mathbb{N}^*$.

(3) $I^pO^q(x) < x \wedge I^pO^{q-1}(x) \geq x \Leftrightarrow q = \lceil \log_2 1.5 * p \rceil$, as q is the minimal one in $q \geq \lceil \log_2 1.5 * p \rceil$. □

Notation 3.11. $\lambda = \log_2 1.5$.

Next theorem proves that $q = \lceil \lambda * p \rceil$ is the necessary condition for $I^pO^q \in Set_{\mathsf{RD}}$.

Theorem 3.12. *If* $I^pO^q \in Set_{\mathsf{RD}}, p, q \in \mathbb{N}^*, p \geq 2$, *then*
(1) $\exists x \in [(3^p)^{-1} * 2^p - 1]_{2^{p+q}} \ \wedge \ x > \frac{3^p - 2^p}{2^{p+q} - 3^p}$ *such that* $\mathsf{RD}[x] = I^pO^q$;
(2) $q = \lceil \lambda * p \rceil$.

Proof It is straightforward due to Lemma 3.8, Lemma 3.9, Lemma 3.10. □

Next theorem proves that $q = \lceil \lambda * p \rceil$ is *also* the sufficient condition for $I^pO^q \in Set_{\mathsf{RD}}$.

Theorem 3.13. *If* $q = \lceil \lambda * p \rceil$, $p, q \in \mathbb{N}^*, p \geq 2$, *then*
(1) $\exists x \in [(3^p)^{-1} * 2^p - 1]_{2^{p+q}} \wedge x > \frac{3^p - 2^p}{2^{p+q} - 3^p}$ *such that* $\mathsf{RD}[x] = I^pO^q$;
(2) $I^pO^q \in Set_{\mathsf{RD}}$.

Proof It is due to Lemma 3.8, Lemma 3.9, Lemma 3.10. □

Next corollary claims sufficient and necessary condition for $I^pO^q \in Set_{\mathsf{RD}}$ $(p \geq 2)$.

Corollary 3.14. *$q = \lceil \lambda * p \rceil \Leftrightarrow I^pO^q \in Set_{\mathsf{RD}}$, where $p, q \in \mathbb{N}^*, p \geq 2$.*

Proof It is straightforward by Theorem 3.12 and Theorem 3.13. □

Next corollary extends above corollary by tackling the cases $p = 0$ and $p = 1$.

Corollary 3.15. *$q = \max(1, \lceil \lambda * p \rceil) \Leftrightarrow I^pO^q \in Set_{\mathsf{RD}}$, where $p \in \mathbb{N}, q \in \mathbb{N}^*$.*

Proof (1) $p = 0$, $q = \max(1, 0) = 1$. $I^0O^1 = O \in Set_{\mathsf{RD}}$, as $\mathsf{RD}[x \in [0]_2] = O$.

(2) $p = 1$, $q = \max(1, \lceil \lambda * 1 \rceil) = \lceil \lambda * 1 \rceil = 1$. $I^1O^1 = IO \in Set_{\mathsf{RD}}$, as $\mathsf{RD}[x \in [1]_4] = IO$.

(3) $p \geq 2$, $q = \max(1, \lceil \lambda * p \rceil) = \lceil \lambda * p \rceil$. $I^pO^q \in Set_{\mathsf{RD}}$ due to Corollary 3.14.

Inverse direction can be proved similarly. □

Corollary 3.16. *$\|Set_{\mathsf{RD}}\| = +\infty$.*

Proof $\forall p \in \mathbb{N}^*, p \geq 2$, let $q = \lceil \lambda * p \rceil$. Thus, $2^{p+q} > 3^p$. $\exists x \in [(3^p)^{-1} * 2^p - 1]_{2^{p+q}} \wedge x > \frac{3^p - 2^p}{2^{p+q} - 3^p}$ such that $\mathsf{RD}[x] = I^pO^q$. Hence, $\|Set_{\mathsf{RD}}\| = +\infty$. □

3.3. Derive x if RD[x]=$I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n}$

In this subsection, we assume $p_i, q_i \in \mathbb{N}^*$, $p_1 \geq 2$, $n \geq 2$, and $x \in [3]_4$, if there exists no explicitly declaration.

Lemma 3.17. *Suppose $s = I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n}$, $k_j = \sum_{i=1}^{j}(p_i + q_i)$, $j = 1, 2, ..., n-1$.*

(1) $GetS(s, 1, k_j)(x) \geq x \Rightarrow GetS(s, 1, k_j - m)(x) > x$, $m = 1, 2, ..., q_j$.

(2) $GetS(s, 1, k_j)(x) \geq x \Rightarrow GetS(s, 1, k_j + m)(x) > x$, $m = 1, 2, ..., p_{j+1}$.

Proof Straightforward.

(1) Because $GetS(s, 1, k_j)(x) \geq x$, and
$GetS(s, 1, k_j - m)(x) = GetS(s, 1, k_j)(x) * 2^m > GetS(s, 1, k_j)(x)$, we have

$GetS(s, 1, k_j - m)(x) > x$, $m = 1, 2, ..., q_j$.

(2) Because $GetS(s, 1, k_j)(x) \geq x$, and
$GetS(s, 1, k_j + m)(x) = I^m(GetS(s, 1, k_j)(x)) > GetS(s, 1, k_j)(x)$ (due to $I(x) > x$), we have

$GetS(s, 1, k_j + m)(x) > x$, $m = 1, 2, ..., p_{j+1}$. □

Remark 3.18. *For better understanding above lemma, we can explain the conclusion as follows: if "going Down" stage ends with a transformed integer that is not less than the starting integer, then all transformed integers in its "left" and "right" will be larger than the starting integer.*

Lemma 3.19. *Suppose $p_i, q_i \in \mathbb{N}^*, p_1 \geq 2, n \geq 2$.*
$\exists x \in [3]_4$ *such that* $\mathsf{RD}[x] = I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n}$, *if and only if*
Req-I): $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}(x) \in [0]_{2^{q_n}}$;
Req-II): $(I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n}(x) < x) \wedge (I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n-1}(x) \geq x)$;
Req-III): $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_j}O^{q_j}(x) \geq x, j = 1, 2, ..., n-1$.

Proof Roughly speaking, Req-I is a residue equation to specify the requirement for parity sequence, Req-II and Req-III are two inequalities to specify the requirements for transformed integers.

(1) Req-I is sufficient and necessary because of Theorem 3.3.

(2) Req-II.
$I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{i}(x) = I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n-1}(x) * 2^{(q_n-1)-i}$. Thus,
$I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n-1}(x) \geq x$
$\Rightarrow I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{i}(x) > x$, $i = 1, 2, ..., q_n - 2$.

Together with $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n}(x) < x$, q_n is thus the minimal one to let $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{i}(x) < x$.

(3) Req-III. Obviously, $I^i(x) > x, i = 1, 2, ..., p_1$. Besides, let $s = \mathsf{RD}[x]$. By Lemma 3.17, we have
$GetS(s, 1, \sum_{i=1}^{j} p_i + q_i)(x) \geq x$, $j = 1, 2, ..., n-1$
$\Rightarrow GetS(s, 1, k)(x) > x, k = p_1 + 1, p_1 + 2, ..., \sum_{i=1}^{n-1}(p_i + q_i) + p_n$. □

Lemma 3.20. $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}(x) \in [0]_{2^{q_n}}, n, p_i, q_i \in \mathbb{N}^*, p_1 \geq 2, i = 1, 2, ..., n$, $\Leftrightarrow x \in [-\sum_{i=1}^{n} A_i B_i C_{i-1}]_{C_n}$, *where* $A_i = 3^{p_i} - 2^{p_i}$, $B_i = (3^{\sum_{j=1}^{i} p_j})^{-1} \mod C_n$, $C_i = 2^{\sum_{j=1}^{i}(p_j+q_j)}, C_0 = 1$.

The proof is somewhat straightforward but the computation is tedious; we thus move them to Appendix (section 4.1).

Remark 3.21.

(1) The conclusion for $n = 1$ is identical with that in Theorem 3.13.

(2) When $p_1 = q_1 = 1$, the conclusion $[3^{-p_1}2^{p_1} - 1]_{2^{p_1+q_1}} = [3^{-1} * 2 - 1]_4 = [3*2-1]_4 = [1]_4$, which is identical with that in Proposition 3.2. Thus, $p_1 \geq 2$ in Lemma 3.20 can be extended to $p_1 \geq 1$.

Next lemma is stated only for $x \in [-\sum_{i=1}^{n} A_i B_i C_{i-1}]_{C_n}$, since Req-I should be guaranteed for ordered parity sequence firstly.

Lemma 3.22. $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n}(x) < x, n, p_i, q_i \in \mathbb{N}^*, p_1 \geq 2, i = 1, 2, ..., n$
$\Leftrightarrow x > \Psi * \frac{2^{\sum_{i=1}^{n}(p_i+q_i)}}{2^{\sum_{i=1}^{n}(p_i+q_i)} - 3^{\sum_{i=1}^{n} p_i}} \wedge 2^{\sum_i^n (p_i+q_i)} - 3^{\sum_i^n p_i} > 0.$

Proof $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n}(x) < x$
$\Leftrightarrow \prod_{i=1}^{n} a_i x + \prod_{i=2}^{n} a_i b_1 + \prod_{i=3}^{n} a_i b_2 + ... + \prod_{i=n-1}^{n} a_i b_{n-2} + \prod_{i=n}^{n} a_i b_{n-1} + b_n < x$
(due to the equation in the proof of Lemma 3.20 (7))
$\Leftrightarrow (1 - \prod_{i=1}^{n} a_i)x > \Psi$
(Let $\Psi = \prod_{i=2}^{n} a_i b_1 + \prod_{i=3}^{n} a_i b_2 + ... + \prod_{i=n-1}^{n} a_i b_{n-2} + \prod_{i=n}^{n} a_i b_{n-1} + b_n$)
$\Leftrightarrow (1 - \prod_{i=1}^{n} \frac{3^{p_i}}{2^{p_i+q_i}}) * x > \Psi$
$\Leftrightarrow (1 - \frac{3^{\sum_{i=1}^{n} p_i}}{2^{\sum_{i=1}^{n}(p_i+q_i)}}) * x > \Psi$
$\Leftrightarrow x > \Psi * \frac{2^{\sum_{i=1}^{n}(p_i+q_i)}}{2^{\sum_{i=1}^{n}(p_i+q_i)} - 3^{\sum_{i=1}^{n} p_i}} \wedge 2^{\sum_i^n (p_i+q_i)} - 3^{\sum_i^n p_i} > 0. \quad \because x > 0, \Psi > 0$

Indeed, $\Psi = \prod_{i=2}^{n} a_i b_1 + \prod_{i=3}^{n} a_i b_2 + ... + \prod_{i=n-1}^{n} a_i b_{n-2} + \prod_{i=n}^{n} a_i b_{n-1} + b_n$
$= \frac{(3^{p_1} - 2^{p_1})3^{\sum_{i=2}^{n} p_i}}{2^{\sum_{i=1}^{n}(p_i+q_i)}} + \frac{(3^{p_2} - 2^{p_2})3^{\sum_{i=3}^{n} p_i}}{2^{\sum_{i=2}^{n}(p_i+q_i)}} + ... + a_n b_{n-1} + b_n$
$= \frac{(3^{p_1} - 2^{p_1})3^{\sum_{i=2}^{n} p_i}}{2^{\sum_{i=1}^{n}(p_i+q_i)}} + \frac{(3^{p_2} - 2^{p_2})3^{\sum_{i=3}^{n} p_i}}{2^{\sum_{i=2}^{n}(p_i+q_i)}} + ... + \frac{3^{p_n}}{2^{p_n+q_n}} \frac{3^{p_{n-1}} - 2^{p_{n-1}}}{2^{p_{n-1}+q_{n-1}}} + \frac{3^{p_n} - 2^{p_n}}{2^{p_n+q_n}}$
$= \frac{(3^{p_1} - 2^{p_1})3^{\sum_{i=2}^{n} p_i}}{2^{\sum_{i=1}^{n}(p_i+q_i)}} + \frac{(3^{p_2} - 2^{p_2})3^{\sum_{i=3}^{n} p_i}}{2^{\sum_{i=2}^{n}(p_i+q_i)}} + ... + \frac{(3^{p_{n-1}} - 2^{p_{n-1}})3^{p_n}}{2^{p_n+q_n+p_{n-1}+q_{n-1}}} + \frac{3^{p_n} - 2^{p_n}}{2^{p_n+q_n}}.$ □

Remark 3.23.
(1) When $n = 1$, $\Psi = \frac{3^{p_n} - 2^{p_n}}{2^{p_n+q_n}} = \frac{3^{p_1} - 2^{p_1}}{2^{p_1+q_1}}$.
$I^{p_1}O^{q_1}(x) < x$
$\Leftrightarrow x > \Psi * \frac{2^{\sum_{i=1}^{n}(p_i+q_i)}}{2^{\sum_{i=1}^{n}(p_i+q_i)} - 3^{\sum_{i=1}^{n} p_i}} \wedge 2^{\sum_i^n (p_i+q_i)} - 3^{\sum_i^n p_i} > 0$
$\Leftrightarrow x > \Psi * \frac{2^{p_1+q_1}}{2^{p_1+q_1} - 3^{p_1}} \wedge 2^{p_1+q_1} - 3^{p_1} > 0$
$\Leftrightarrow x > \frac{3^{p_1} - 2^{p_1}}{2^{p_1+q_1}} * \frac{2^{p_1+q_1}}{2^{p_1+q_1} - 3^{p_1}} \wedge 2^{p_1+q_1} - 3^{p_1} > 0$
$\Leftrightarrow x > \frac{3^{p_1} - 2^{p_1}}{2^{p_1+q_1} - 3^{p_1}} \wedge 2^{p_1+q_1} - 3^{p_1} > 0$, *which is exactly identical with that of Theorem 3.13. Therefore, the conclusion can be extended to* $n = 1$.

(2) When $n = 1$ *with* $p_1 = 1, q_1 = 1$. $x > \Psi * \frac{2^{\sum_{i=1}^{n}(p_i+q_i)}}{2^{\sum_{i=1}^{n}(p_i+q_i)} - 3^{\sum_{i=1}^{n} p_i}} = \frac{3^{p_1} - 2^{p_1}}{2^{p_1+q_1} - 3^{p_1}} = \frac{3-2}{2^2 - 3^1} = 1/1 = 1$. *It is consistent with the conclusion in Proposition 3.2 for* $\mathsf{RD}[x \in [1]_4] = IO$.

Note that, here $2^{\sum_i^n (p_i+q_i)} - 3^{\sum_i^n p_i} > 0$ is a *necessary* condition for $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n}(x) < x$.

Next, we check the requirement Req-III. Again, next lemma is discussed only for $x \in [-\sum_{i=1}^{n} A_i B_i C_{i-1}]_{C_n}$, since Req-I should be guaranteed for designated parity sequence as a prerequisite.

Lemma 3.24. $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_j}O^{q_j}(x) \geq x, j = 1, 2, ..., n-1, n \geq 2$
$\Leftrightarrow 2^{\sum_i^j (p_i+q_i)} - 3^{\sum_i^j p_i} < 0$.

Proof $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_j}O^{q_j}(x) \geq x, j = 1, 2, ..., n-1$
$\Leftrightarrow \prod_{i=1}^{j} a_i x + \prod_{i=2}^{j} a_i b_1 + \prod_{i=3}^{j} a_i b_2 + ... + \prod_{i=j-1}^{j} a_i b_{j-2} + \prod_{i=j}^{j} a_i b_{j-1} + b_j \geq x$
(due to the equation in the proof of Lemma 3.20 (7))
$\Leftrightarrow (1 - \prod_{i=1}^{j} a_i)x \leq \Psi_j$
Let $\Psi_j = \prod_{i=2}^{j} a_i b_1 + \prod_{i=3}^{j} a_i b_2 + ... + \prod_{i=j-1}^{j} a_i b_{j-2} + \prod_{i=j}^{n} a_i b_{j-1} + b_j$
$\Leftrightarrow (1 - \prod_{i=1}^{j} \frac{3^{p_i}}{2^{p_i+q_i}}) * x \leq \Psi_j$
$\Leftrightarrow (1 - \frac{3^{\sum_{i=1}^{j} p_i}}{2^{\sum_{i=1}^{j}(p_i+q_i)}}) * x \leq \Psi_j$
$\Leftrightarrow (x \leq \Psi_j * \frac{2^{\sum_{i=1}^{j}(p_i+q_i)}}{2^{\sum_{i=1}^{j}(p_i+q_i)} - 3^{\sum_{i=1}^{j} p_i}} \ \wedge \ 2^{\sum_i^j (p_i+q_i)} - 3^{\sum_i^j p_i} > 0)$
$\vee\ (2^{\sum_i^j (p_i+q_i)} - 3^{\sum_i^j p_i} < 0)$.

Next, we will prove that $2^{\sum_i^j (p_i+q_i)} - 3^{\sum_i^j p_i} > 0, j = 1, 2, ..., n-1$ is impossible by contradiction.

Suppose $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_j}(x) \in [0]_{2^{q_j}} \Leftrightarrow x \in S_j, j = 1, 2, ..., n$. By Lemma 3.19 (1) or Theorem 3.3, $S_{j+1} \subset S_j, j = n-1, n-2, ..., 1$.

Suppose $\exists j \in [1, n-1]$ such that $2^{\sum_i^j (p_i+q_i)} - 3^{\sum_i^j p_i} > 0$. Select the minimal one in them. (Thus, for $1 \leq k \leq j-1$, $2^{\sum_i^k (p_i+q_i)} - 3^{\sum_i^k p_i} < 0$.) Because the generality of n in q_n in Lemma 3.22, we have $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_j}O^{q_j}(x) < x$, when $x > \Psi'_j = \Psi_j * \frac{2^{\sum_{i=1}^{j}(p_i+q_i)}}{2^{\sum_{i=1}^{j}(p_i+q_i)} - 3^{\sum_{i=1}^{j} p_i}}$, $j \in [1, n-1]$.

Let $\Psi'_n = \Psi * \frac{2^{\sum_{i=1}^{n}(p_i+q_i)}}{2^{\sum_{i=1}^{n}(p_i+q_i)} - 3^{\sum_{i=1}^{n} p_i}}$.

Therefore, if $x > \max(\Psi'_j, \Psi'_n)$ and $x \in S_n$, then $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_j}(x) \in [0]_{2^{q_j}}$, $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_j}O^{q_j}(x) < x$, and $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_k}O^{q_k}(x) \geq x, k = 1, 2, ..., j-1$. Thus, $\mathsf{RD}[x \in S_n] = I^{p_1}O^{q_1}I^{p_2}O^{q_2}...\ I^{p_j}O^{q_j}$, which contradicts with $\mathsf{RD}[x \in S_n] = I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n}$ in assumption. Thus, $\nexists j \in [1, n-1], 2^{\sum_i^j (p_i+q_i)} - 3^{\sum_i^j p_i} > 0$. That is, $2^{\sum_i^j (p_i+q_i)} - 3^{\sum_i^j p_i} < 0, j = 1, 2, ..., n-1$. $\square$

Lemma 3.25. $2^{\sum_i^n (p_i+q_i)} - 3^{\sum_i^n p_i} > 0 \Leftrightarrow \sum_{i=1}^{n} q_i \geq \lceil \lambda * \sum_{i=1}^{n} p_i \rceil$.

Proof Let $U = \sum_i^n p_i, D = \sum_i^n (p_i + q_i)$.

$2^{\sum_i^n (p_i+q_i)} - 3^{\sum_i^n p_i} > 0 \Leftrightarrow 2^D > 3^U \Leftrightarrow D > \log_2 3 * U$
$\Leftrightarrow D \geq \lceil \log_2 3 * U \rceil \quad \because U, D \in \mathbb{N}^*, \log_2 3 \notin \mathbb{Q}^*, \log_2 3 * U \notin \mathbb{N}^*$
$\Leftrightarrow D - U \geq \lceil \log_2 3 * U \rceil - U$
$\Leftrightarrow D - U \geq \lceil \log_2 3 * U - U \rceil$
$\Leftrightarrow D - U \geq \lceil \log_2 1.5 * U \rceil$
$\Leftrightarrow \sum_{i=1}^n q_i \geq \lceil \lambda * \sum_{i=1}^n p_i \rceil$. □

Lemma 3.26. $2^{\sum_i^j (p_i+q_i)} - 3^{\sum_i^n p_i} < 0 \Leftrightarrow \sum_{i=1}^j q_i < \lceil \lambda * \sum_{i=1}^j p_i \rceil$, $j = 1, 2, ..., n-1$.

Proof Let $U' = \sum_i^j p_i, D' = \sum_i^j (p_i + q_i)$.
$2^{\sum_i^j (p_i+q_i)} - 3^{\sum_i^j p_i} < 0 \Leftrightarrow 2^{D'} < 3^{U'} \Leftrightarrow D' < \log_2 3 * U'$
$\Leftrightarrow D' \leq \lceil \log_2 3 * U' \rceil - 1 \quad \because U', D' \in \mathbb{N}^*, \log_2 3 \notin \mathbb{Q}^*, \log_2 3 * U' \notin \mathbb{N}^*$
$\Leftrightarrow D' - U' \leq \lceil \log_2 3 * U' \rceil - U' - 1$
$\Leftrightarrow D' - U' \leq \lceil \log_2 3 * U' - U' \rceil - 1$
$\Leftrightarrow D' - U' \leq \lceil \log_2 1.5 * U' \rceil - 1$
$\Leftrightarrow \sum_{i=1}^j q_i < \lceil \lambda * \sum_{i=1}^j p_i \rceil$. □

Lemma 3.27. $I^{p_1} O^{q_1} I^{p_2} O^{q_2} ... I^{p_n} O^{q_n}(x) < x \ \wedge$
$I^{p_1} O^{q_1} I^{p_2} O^{q_2} ... I^{p_n} O^{q_n - 1}(x) \geq x$
$\Leftrightarrow q_n = \lceil \lambda * \sum_{i=1}^n p_i \rceil - \sum_{i=1}^{n-1} q_i \ \wedge \ x > \Psi * \frac{2^{\sum_{i=1}^n (p_i+q_i)}}{2^{\sum_{i=1}^n (p_i+q_i)} - 3^{\sum_{i=1}^n p_i}}$.

Proof By Lemma 3.25, $\sum_{i=1}^n q_i \geq \lceil \lambda * \sum_{i=1}^n p_i \rceil$.
$I^{p_1} O^{q_1} I^{p_2} O^{q_2} ... I^{p_n} O^{q_n}(x) < x \ \wedge \ I^{p_1} O^{q_1} I^{p_2} O^{q_2} ... I^{p_n} O^{q_n - 1}(x) \geq x$, thus q_n is the minimal one. Therefore, $q_n = \lceil \lambda * \sum_{i=1}^n p_i \rceil - \sum_{i=1}^{n-1} q_i$. □

Theorem 3.28. *If* $I^{p_1} O^{q_1} I^{p_2} O^{q_2} ... I^{p_n} O^{q_n} \in Set_{\mathsf{RD}}$, $p_i, q_i \in \mathbb{N}^*, p_1 \geq 2$, $i = 1, 2, ..., n$, $n \in \mathbb{N}^*, n \geq 2$, *then*

(1) $\exists x \in [-\sum_{i=1}^n A_i B_i C_{i-1}]_{C_n} \ \wedge \ x > \Psi * \frac{2^{\sum_{i=1}^n (p_i+q_i)}}{2^{\sum_{i=1}^n (p_i+q_i)} - 3^{\sum_{i=1}^n p_i}}$ *such that* $\mathsf{RD}[x] = I^{p_1} O^{q_1} I^{p_2} O^{q_2} ... I^{p_n} O^{q_n}$, *where* $A_i = 3^{p_i} - 2^{p_i}$, $B_i = (3^{\sum_{j=1}^i p_j})^{-1} \bmod C_n$, $C_i = 2^{\sum_{j=1}^i (p_j+q_j)}, C_0 = 1$, $\Psi = \prod_{i=2}^n a_i b_1 + \prod_{i=3}^n a_i b_2 + ... + \prod_{i=n-1}^n a_i b_{n-2} + \prod_{i=n}^n a_i b_{n-1} + b_n$, $a_i = \frac{3^{p_i}}{2^{p_i+q_i}}, b_i = \frac{3^{p_i} - 2^{p_i}}{2^{p_i+q_i}}, i = 1, 2, ...n;$

(2) $\sum_{i=1}^n q_i = \lceil \lambda * \sum_{i=1}^n p_i \rceil \ \wedge \ \sum_{i=1}^j q_i < \lceil \lambda * \sum_{i=1}^j p_i \rceil, j = 1, 2, ..., n-1, n \geq 2$.

Proof It is straightforward due to Lemma 3.20, Lemma 3.22, Lemma 3.24, Lemma 3.25, Lemma 3.26, and Lemma 3.27. □

Next theorem proves that $\sum_{i=1}^{n} q_i = \lceil\lambda * \sum_{i=1}^{n} p_i\rceil$ and $\sum_{i=1}^{j} q_i < \lceil\lambda * \sum_{i=1}^{j} p_i\rceil, j = 1, 2, ..., n-1, n \geq 2$ is *also* the sufficient condition for $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n} \in Set_{\mathsf{RD}}$.

Theorem 3.29. *If* $\sum_{i=1}^{n} q_i = \lceil\lambda * \sum_{i=1}^{n} p_i\rceil$, $\sum_{i=1}^{j} q_i < \lceil\lambda * \sum_{i=1}^{j} p_i\rceil, j = 1, 2, ..., n-1$, $p_i, q_i \in \mathbb{N}^*, p_1 \geq 2$. $i = 1, 2, ..., n$, $n \in \mathbb{N}^*, n \geq 2$, *then*

(1) $\exists x \in [-\sum_{i=1}^{n} A_i B_i C_{i-1}]_{C_n} \ \wedge \ x > \Psi * \frac{2^{\sum_{i=1}^{n}(p_i+q_i)}}{2^{\sum_{i=1}^{n}(p_i+q_i)} - 3^{\sum_{i=1}^{n} p_i}}$ *such that* $\mathsf{RD}[x] = I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n}$ *where* $A_i = 3^{p_i} - 2^{p_i}$, $B_i = (3^{\sum_{j=1}^{i} p_j})^{-1}$ mod C_n, $C_i = 2^{\sum_{j=1}^{i}(p_j+q_j)}$, $C_0 = 1$, $\Psi = \prod_{i=2}^{n} a_i b_1 + \prod_{i=3}^{n} a_i b_2 + ... + \prod_{i=n-1}^{n} a_i b_{n-2} + \prod_{i=n}^{n} a_i b_{n-1} + b_n$, $a_i = \frac{3^{p_i}}{2^{p_i+q_i}}, b_i = \frac{3^{p_i}-2^{p_i}}{2^{p_i+q_i}}, i = 1, 2, ..., n;$

(2) $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n} \in Set_{\mathsf{RD}}$.

Proof By Lemma 3.27,

$\sum_{i=1}^{n} q_i = \lceil\lambda * \sum_{i=1}^{n} p_i\rceil \wedge x > \Psi * \frac{2^{\sum_{i=1}^{n}(p_i+q_i)}}{2^{\sum_{i=1}^{n}(p_i+q_i)} - 3^{\sum_{i=1}^{n} p_i}}$

$\Rightarrow I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n}(x) < x \wedge I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n-1}(x) \geq x.$

Together with $x \in [-\sum_{i=1}^{n} A_i B_i C_{i-1}]_{C_n}$, we have

$I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}(x) \in [0]_{2^{q_n}}$ due to Lemma 3.20.

Besides, $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_j}O^{q_j}(x) \geq x, j = 1, 2, ..., n-1$ by Lemma 3.24 and Lemma 3.26. Therefore, by Lemma 3.19,

$\exists x \in \mathbb{N}^*, \mathsf{RD}[x] = I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n}$. □

Note that, above theorem indeed implies that $\mathsf{RD}[x] = \mathsf{RD}[x + P] = I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n}$ where $P = 2^{p_1+q_1+p_2+q_2+...p_n+q_n} = 2^{\sum_{i=1}^{n}(p_i+q_i)} = 2^{|\mathsf{RD}[x]|}$ when x is sufficient large (i.e., $x > \Psi * \frac{2^{\sum_{i=1}^{n}(p_i+q_i)}}{2^{\sum_{i=1}^{n}(p_i+q_i)} - 3^{\sum_{i=1}^{n} p_i}}$).

(Indeed, the requirement for being sufficient large can be omitted and proved by us in another paper [8]. Note that, this paper is independent with cited paper.)

Following corollary states the necessary and sufficient condition for guaranteeing that any $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n}$ where $p_i, q_i \in \mathbb{N}^*, p_1 \geq 2$ and $i = 1, 2, ..., n, n \in \mathbb{N}^*, n \geq 2$, is a reduced dynamics of some $x \in \mathbb{N}^*$.

Corollary 3.30. $\sum_{i=1}^{n} q_i = \lceil\lambda * \sum_{i=1}^{n} p_i\rceil$, $p_i, q_i \in \mathbb{N}^*, p_1 \geq 2$ $i = 1, 2, ..., n, n \in \mathbb{N}^*, n \geq 2 \wedge \sum_{i=1}^{j} q_i < \lceil\lambda * \sum_{i=1}^{j} p_i\rceil, j = 1, 2, ..., n-1$
$\Leftrightarrow I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n} \in Set_{\mathsf{RD}}$.

Proof Straightforward due to Theorem 3.28 and Theorem 3.29. □

Theorem 3.31. ***(Prefix-free Theorem.)*** *Suppose* $p_i, q_i \in \mathbb{N}^*, i = 1, 2, ..., n,$ $p_1 \geq 2, n \geq 2$. *If* $s = I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n} \in Set_{\mathsf{RD}}$, *then* $GetS(s, 1, \sum_{i=1}^{j}(p_i + q_i)) \notin Set_{\mathsf{RD}}, j = 1, 2, ..., n-1.$

Proof $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n} \in Set_{\mathsf{RD}}$
$\Rightarrow \forall x \in [-\sum_{i=1}^{n} A_i B_i C_{i-1}]_{C_n}, I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_j}O^{q_j}(x) \geq x, j = 1, 2, ..., n-1$
$\Rightarrow 2^{\sum_i^j (p_i+q_i)} - 3^{\sum_i^j p_i} < 0, j = 1, 2, ..., n-1 \quad \because$ Lemma 3.24
$\Rightarrow \nexists x \in \mathbb{N}^*, GetS(s, 1, \sum_{i=1}^{j}(p_i + q_i))(x) < x \quad \because$ Lemma 3.22
$\Rightarrow GetS(s, 1, \sum_{i=1}^{j}(p_i + q_i)) \notin Set_{\mathsf{RD}}, j = 1, 2, ..., n-1.$ □

Simply speaking, Theorem 3.31 shows that Set_{RD} is *prefix-free*. Regarding more general conclusion by including O or IO (i.e., for $x \notin [3]_4$) and $n = 1$ (i.e., for I^pO^q), we have following corollary.

Corollary 3.32. $s \in Set_{\mathsf{RD}} \Rightarrow s \| \{I, O\}^{\geq 1} \not\subset Set_{\mathsf{RD}}$.

Proof Straightforward. □

Next, we give more general results by introducing two functions as follows:

Definition 3.33. *Function* $CntI(\cdot)$. $CntI : c \rightarrow y$ *takes as input* $c \in \{I, O\}^{\geq 1}$, *and outputs* $y \in \mathbb{N}$ *that is the count of "I" in c.*

Definition 3.34. *Function* $CntO(\cdot)$. $CntO : c \rightarrow y$ *takes as input* $c \in \{I, O\}^{\geq 1}$, *and outputs* $y \in \mathbb{N}^*$ *that is the count of "O" in c.*

E.g., $CntI(IIOO) = 2, CntO(IIOO) = 2, CntI(IIOIO) = 3, CntO(IIOIO) = 2$.

Corollary 3.30 states the *sufficient and necessary* condition for $s = I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n} \in Set_{\mathsf{RD}}$, $n \geq 2$. It can be extended by using $GetS(s, 1, k), k = 1, 2, ..., |s|-1$ instead of $GetS(s, 1, \sum_{i=1}^{j}(p_i+q_i))$ as follows:

Corollary 3.35. $s = I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n} \in Set_{\mathsf{RD}}$, $p_i, q_i \in \mathbb{N}^*$, $p_1 \geq 2$, $i = 1, 2, ..., n$, $n \in \mathbb{N}^*$, $n \geq 2$, *if and only if*

(1) $CntO(s) = \lceil \lambda * CntI(s) \rceil$, *and*

(2) $CntO(GetS(s, 1, k)) < \lceil \lambda * CntI(GetS(s, 1, k)) \rceil, k = 1, 2, ..., |s| - 1.$

Proof Straightforward. □

If $n = 1$ and $p = 0, 1$ for I^pO^q are both included, above corollary can be extended to following corollary.

Corollary 3.36. *$s = I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n} \in Set_{\mathsf{RD}}$, $p_i \in \mathbb{N}$, $q_i \in \mathbb{N}^*$, $i = 1, 2, ..., n, n \in \mathbb{N}^*$, if and only if*
(1) $p_1 = 0, n = 1$. $s = O$; Or,
(2) $p_1 \geq 1$. $CntO(s) = \lceil \lambda * CntI(s) \rceil$, and
$CntO(GetS(s, 1, k)) < \lceil \lambda * CntI(GetS(s, 1, k)) \rceil, k = 1, 2, ..., |s| - 1.$

Proof (1) It is trivial $s = O \Rightarrow s \in Set_{\mathsf{RD}}$. Inversely, $n = 1, p_1 = 0, s = I^{p_1}O^{q_1} = O^{q_1} \in Set_{\mathsf{RD}} \Rightarrow s = O$.

(2) $n = 1$, $s = I^{p_1}O^{q_1}, p_1, q_1 \in \mathbb{N}^*$.

(2.1) $p_1 = 1$. $GetS(s, 1, 1) = I$. Obviously, $CntO(GetS(s, 1, 1)) = 0 < \lceil \lambda * CntI(GetS(s, 1, 1)) \rceil = \lceil \lambda * 1 \rceil = \lceil log_2 1.5 \rceil = 1$.

$s = IO^{q_1}$, thus $GetS(s, 1, 2) = IO$. Then, $CntO(GetS(s, 1, 2)) = 1$, $\lceil \lambda * CntI(GetS(s, 1, 2)) \rceil = \lceil \lambda * 1 \rceil = 1$. Thus, $|s| = 2$ and $s = IO$. Obviously, $IO \in Set_{\mathsf{RD}}$.

Inversely,

$s = IO \in Set_{\mathsf{RD}}$
$\Rightarrow (CntO(s) = 1 = \lceil \lambda * 1 \rceil = \lceil \lambda * CntI(s) \rceil) \wedge$
$(CntO(GetS(s, 1, k)) = 0 < \lceil \lambda * CntI(GetS(s, 1, k)) \rceil = 1, k = |s| - 1 = 1.)$

(2.2) $p_1 \geq 2$. $q_1 = \lceil \lambda * p_1 \rceil \Leftrightarrow I^{p_1}O^{q_1} \in Set_{\mathsf{RD}}$ by Corollary 3.14. Thus, $q_1 = \lceil \lambda * p_1 \rceil \Leftrightarrow CntO(s) = \lceil \lambda * CntI(s) \rceil$ because $CntO(s) = q_1, CntI(s) = p_1$.

Besides, $q_1 = \lceil \lambda * p_1 \rceil \Rightarrow CntO(GetS(s, 1, k)) < \lceil \lambda * CntI(GetS(s, 1, k)) \rceil$, $k = 1, 2, ..., |s| - 1$. The reason is as follows:

When $k = 1, 2, ..., p_1$,
$CntO(GetS(s, 1, k)) = 0 < \lceil \lambda * CntI(GetS(s, 1, k)) \rceil \in \mathbb{N}^*$ is trivial;

When $k = p_1 + 1, p_1 + 2, ..., p_1 + q_1 - 1$,
$q_1 = \lceil \lambda * p_1 \rceil \Rightarrow CntO(GetS(s, 1, k)) = k - p_1 \leq q_1 - 1 < q_1 = \lceil \lambda * p_1 \rceil = \lceil \lambda * CntI(GetS(c, 1, k)) \rceil$ as $p_1 = CntI(GetS(s, 1, k))$.

(3) $n \geq 2$. $p_1 \geq 2$. By Corollary 3.35,
$CntO(c) = \lceil \lambda * CntI(c) \rceil \wedge$
$CntO(GetS(s, 1, k)) < \lceil \lambda * CntI(GetS(s, 1, k)) \rceil, k = 1, 2, ..., |s| - 1.$
$\Leftrightarrow s \in Set_{\mathsf{RD}}$. □

Next, we use $CntO(\cdot)$ and $CntI(\cdot)$ to restate *sufficient and necessary* condition for $s \in Set_{\mathsf{RD}}$ as follows:

Corollary 3.37. ***(Form Corollary.)*** *$s \in \{I, O\}^{\geq 1}$ is a reduced dynamics, if and only if*

(1) $|s| = 1, s = O$; Or,

(2) $|s| \geq 2$,

$$\begin{cases} CntO(s) = \lceil \lambda * CntI(s) \rceil, \\ CntO(s') < \lceil \lambda * CntI(s') \rceil, \quad s' = GetS(s, 1, i), i = 1, 2, ..., |s| - 1. \end{cases} \tag{4}$$

Proof It is straightforward due to Corollary 3.30, Theorem 3.31, Corollary 3.32, Corollary 3.35 and Corollary 3.36. $\square$

Finally, we obtain following Inverse Theorem as a summary.

Theorem 3.38. ***(Inverse Theorem.)***
(1) If $|s| = 1, s = O$, then $\exists x \in [0]_2$ such that $\mathsf{RD}[x] = O$.
(2) If $|s| \geq 2$, $s = I^{p_1} O^{q_1} I^{p_2} O^{q_2} ... I^{p_n} O^{q_n}$, $p_i, q_i \in \mathbb{N}^*$, $i = 1, 2, ..., n, n \in \mathbb{N}^*$,
$CntO(s) = \lceil \lambda * CntI(s) \rceil$,
$CntO(s') < \lceil \lambda * CntI(s') \rceil$, $s' = GetS(s, 1, k), k = 1, 2, ..., |s| - 1$.
then $\exists x \in [-\sum_{i=1}^{n} A_i B_i C_{i-1}]_{C_n} \wedge x > \Psi * \frac{2^{\sum_{i=1}^{n}(p_i+q_i)}}{2^{\sum_{i=1}^{n}(p_i+q_i)} - 3^{\sum_{i=1}^{n} p_i}}$
such that $\mathsf{RD}[x] = s$
where $A_i = 3^{p_i} - 2^{p_i}$, $B_i = (3^{\sum_{j=1}^{i} p_j})^{-1} \mod C_n$, $C_i = 2^{\sum_{j=1}^{i}(p_j+q_j)}, C_0 = 1$, $\Psi = \prod_{i=2}^{n} a_i b_1 + \prod_{i=3}^{n} a_i b_2 + ... + \prod_{i=n-1}^{n} a_i b_{n-2} + \prod_{i=n}^{n} a_i b_{n-1} + b_n$, $a_i = \frac{3^{p_i}}{2^{p_i+q_i}}, b_i = \frac{3^{p_i} - 2^{p_i}}{2^{p_i+q_i}}$.

Proof It is straightforward due to Theorem 3.29 and Form Corollary (Corollary 3.37). $\square$

Following corollary states that Set_{RD} can be constructed by generating $s \in \{I, O\}^{\geq 1}$ that satisfies requirements algorithmically instead of by conducting concrete Collatz transformations for enumerated starting integers.

Corollary 3.39. *$Set_{\mathsf{RD}} = \{O\} \cup \{s | s \in \{I, O\}^L, L \in \mathbb{N}^*, L \geq 2,$*
$CntO(s) = \lceil \lambda * CntI(s) \rceil$,
$CntO(s') < \lceil \lambda * CntI(s') \rceil, s' = GetS(s, 1, i), i = 1, 2, ..., L - 1\}$.

Proof It is straightforward due to Corollary 3.37. $\square$

Following corollary states the relations between $CntI(s)$ and $CntO(s) + CntI(s) = |s|$, $s \in Set_{\mathsf{RD}}$. Indeed, $CntI(s)$ is the count of $(3*x+1)/2$ that equals the count of $3*x+1$ computation, and $CntO(s)+CntI(s)$ equals the total count of $x/2$ computation in reduced dynamics.

Corollary 3.40. *If $s \in Set_{\mathsf{RD}}$, then*
(1) $D \geq \lceil \log_2 3 * U \rceil$, where $U = CntI(s)$, $D = |s|$;
(2) $3^U < 2^D$.

Proof (1) When $s = O$, $U = 0$, $D = 1$. $1 > \lceil \log_2 3 * 0 \rceil = 0$ by Corollary 3.39.

When $s \neq O$, $s \in Set_{\mathsf{RD}} \Rightarrow D = CntO(s) + CntI(s) = \lceil \log_2 1.5 * CntI(s) \rceil + CntI(s) = \lceil \log_2 1.5 * U \rceil + U = \lceil \log_2 3 * U \rceil$ by Corollary 3.39.

In summary, $s \in Set_{\mathsf{RD}} \Rightarrow D \geq \lceil \log_2 3 * U \rceil$, and note that "$>$" is obtained *when and only when* $s = O$.

(2) $D \geq \lceil \log_2 3 * U \rceil \Rightarrow D \geq \log_2 3^U \Rightarrow 3^U \leq 2^D \Rightarrow 3^U < 2^D$. □

Corollary 3.41. $s \in Set_{\mathsf{RD}}, |s| \geq 2$
$\Rightarrow 3^{CntI(GetS(s,1,i))} > 2^i$, $i = 1, 2, ..., |s| - 1$.

Proof Let $s' = GetS(s, 1, i), i = 1, 2, ..., |s| - 1$. Obviously, $|s'| = i$.
$s \in Set_{\mathsf{RD}}$
$\Rightarrow CntO(s') < \lceil \log_2 1.5 * CntI(s') \rceil \quad \because$ Corollary 3.37
$\Rightarrow CntO(s') < \log_2 1.5 * CntI(s') \quad \because \log_2 1.5 \notin \mathbb{Q}, CntI(s), CntO(s) \in \mathbb{N}^*$
$\Rightarrow CntO(s') + CntI(s') < \log_2 3 * CntI(s')$
$\Rightarrow |s'| < \log_2 3 * CntI(s') \Rightarrow 2^{|s'|} < 3^{CntI(s')}$
$\Rightarrow 3^{CntI(GetS(s,1,i))} > 2^i$. □

Example 3.42. *By using the conclusion in Form Corollary (Corollary 3.37), $CntO(s)$ can be computed from $CntI(s)$* directly *as follows:*

(1) $s \in \{I, O\}^{\geq 1}, s \in Set_{\mathsf{RD}}$, $CntI(s) = 1 \Rightarrow CntO(s) = \lceil \lambda * CntI(s) \rceil = \lceil \lambda * 1 \rceil = \lceil \log_2 1.5 * 1 \rceil = \lceil 0.58496250 * 1 \rceil = 1$.

(2) $s \in \{I, O\}^{\geq 1}, s \in Set_{\mathsf{RD}}$, $CntI(s) = 2 \Rightarrow CntO(s) = \lceil \lambda * CntI(s) \rceil = \lceil \lambda * 2 \rceil = \lceil \log_2 1.5 * 2 \rceil = \lceil 0.58496250 * 2 \rceil = 2$.

(3) $s \in \{I, O\}^{\geq 1}, s \in Set_{\mathsf{RD}}$, $CntI(s) = 3 \Rightarrow CntO(s) = \lceil \lambda * CntI(s) \rceil = \lceil \lambda * 3 \rceil = \lceil \log_2 1.5 * 3 \rceil = \lceil 0.58496250 * 3 \rceil = 2$.

(4) $s \in \{I, O\}^{\geq 1}, s \in Set_{\mathsf{RD}}$, $CntI(s) = 4 \Rightarrow CntO(s) = \lceil \lambda * CntI(s) \rceil = \lceil \lambda * 4 \rceil = \lceil \log_2 1.5 * 4 \rceil = \lceil 0.58496250 * 4 \rceil = 3$.

Due to Inverse Theorem (Theorem 3.38), each $s \in Set_{\mathsf{RD}}$ cannot map to more than one residue class with the module $2^{|s|}$. Thus, a function called $Invrs$ can be defined as follows:

Definition 3.43. *Function* $Invrs(\cdot)$. $Invrs : s \rightarrow rs$ *takes as input*
$s = O$ *or*
$s = I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}O^{q_n} \in \{I, O\}^{\geq 2}$, $p_i, q_i \in \mathbb{N}^*, i = 1, 2, ..., n, n \in \mathbb{N}^*$
$CntO(s) = \lceil \log_2 1.5 * CntI(s) \rceil$,
$CntO(s') < \lceil \log_2 1.5 * CntI(s') \rceil, s' = GetS(s, 1, k), k = 1, 2, ..., |s| - 1$,
and outputs
$r = [0]_2$ *when* $s = O$, *or*
$r = ([-\sum_{i=1}^{n} A_i B_i C_{i-1}]_{2^{|c|}} \cap \{x | x > \Psi * \frac{2^{\sum_{i=1}^{n}(p_i+q_i)}}{2^{\sum_{i=1}^{n}(p_i+q_i)} - 3^{\sum_{i=1}^{n} p_i}}\})$ *when* $|s| \geq 2$,
where $A_i = 3^{p_i} - 2^{p_i}$, $B_i = (3^{\sum_{j=1}^{i} p_j})^{-1} \mod C_n$, $C_i = 2^{\sum_{j=1}^{i}(p_j+q_j)}, C_0 = 1$,
$\Psi = \prod_{i=2}^{n} a_i b_1 + \prod_{i=3}^{n} a_i b_2 + ... + \prod_{i=n-1}^{n} a_i b_{n-2} + \prod_{i=n}^{n} a_i b_{n-1} + b_n$,
$a_i = \frac{3^{p_i}}{2^{p_i+q_i}}, b_i = \frac{3^{p_i}-2^{p_i}}{2^{p_i+q_i}}$.

The following corollary states that $Invrs(\cdot)$ is injective. It is straightforward because the reduced dynamics of a starting integer is unique (Proposition 2.19).

Corollary 3.44. $\forall s_1, s_2 \in Set_{\mathsf{RD}}, s_1 \neq s_2 \Rightarrow r_1 \neq r_2$, *where* $r_1 = Invrs(s_1)$ *and* $r_2 = Invrs(s_2)$.

Proof It is straightforward by proving converse-negative proposition.
$r_1 = r_2 \Rightarrow \mathsf{RD}[x \in r_1] = \mathsf{RD}[x \in r_2] \Rightarrow s_1 = s_2$. □

If $Invrs(\cdot)$ is surjective, then Collatz conjecture will be true. That is, if $\bigcup_{s \in Set_{\mathsf{RD}}} Invrs(s) = \mathbb{N}^*$, then Collatz conjecture will be true. However, it is difficult to be proved because each residue class is partial due to the requirement of $\{x | x > \Psi * \frac{2^{\sum_{i=1}^{n}(p_i+q_i)}}{2^{\sum_{i=1}^{n}(p_i+q_i)} - 3^{\sum_{i=1}^{n} p_i}}\}$.

(Thanks to the Period Theorem that is proved in my another paper [8], this requirement can be omitted. Besides, all above conclusions, e.g., Subset Theorem (Theorem 3.3), Prefix-free Theorem (Theorem 3.31), Form Corollary (Corollary 3.37), and Inverse Theorem (Theorem 3.38), can be observed in my proposed tree-based graph to present reduced dynamics [5].)

4. Conclusion

We use $\mathsf{RD}[x]$ to denote the reduced dynamics of x, which is represented by a sequence of computation of "I" or "O", where "I" denotes $(3*x+1)/2$ and "O" denotes $x/2$.

This paper discovered and proved two major facts as follows:

(1) The relation between the count of $x/2$ (i.e., "O") and the count of $(3*x+1)/2$ (i.e., "I") in any reduced dynamics is proved by Form Corollary (Corollary 3.37). That is, given any $s \in \{I,O\}^{\geq 1}$, the sufficient and necessary condition for $s \in Set_{\mathsf{RD}}$ is as follows:

(i) $|s| = 1, s = O$;

(ii) $|s| \geq 2$, $CntO(s) = \lceil \lambda * CntI(s) \rceil, \lambda = \log_2 1.5$
$CntO(s') < \lceil \lambda * CntI(s') \rceil, s' = GetS(s,1,i), i = 1,2,...,|s|-1$.

Therefore, any reduced dynamics can be generated by guaranteeing sufficient and necessary condition in Form Corollary, instead of by enumerating all integers one by one and computing concrete Collatz transformations step by step. That is,

$Set_{\mathsf{RD}} = \{O\} \cup \{s | s \in \{I,O\}^L, L \in \mathbb{N}^*, L \geq 2, CntO(s) = \lceil \lambda * CntI(s) \rceil,$
$CntO(s') < \lceil \lambda * CntI(s') \rceil, s' = GetS(s,1,i), i = 1,2,...,L-1\}$.

If $CntO(s) = \lceil \lambda * CntI(s) \rceil$, then reduced dynamics will be available. That is, if and only if the count of "O" is larger than $\lambda = \log_2 1.5$ times the count of "I", then reduced dynamics will be available (i.e., transformed integer will be less than starting integer immediately).

The count of $x/2$ equals $CntO(s)+CntI(s)$ and the count of $3*x+1$ equals $CntI(s)$. $CntO(s) + CntI(s) = \lceil \lambda * CntI(s) \rceil + CntI(s) = \lceil \lambda * CntI(s) + CntI(s) \rceil = \lceil (\lambda+1) * CntI(s) \rceil$. In other words, if and only if the count of $x/2$ computation over the count of $3*x+1$ computation is larger than $\lambda+1 = \log_2 1.5 + 1 = \log_2 3 = \ln 3 / \ln 2$, then transformed integer will be less than starting integer.

(2) Given a reduced dynamics, a residue class of starting integer can be derived by Inverse Theorem (Theorem 3.38). That is, given $s \in Set_{\mathsf{RD}}$, a residue class r can be calculated from s directly by proposed formula (algorithm), such that $\mathsf{RD}[x \in r] = s$ where the module of r is $2^{|s|}$ (for sufficient large x).

Acknowledgement

The research was financially supported by National Natural Science Foundation of China (No.61972366), Major Scientific and Technological Special

Project of Guizhou Province (No. 20183001), the Foundation of Key Laboratory of Network Assessment Technology, Chinese Academy of Sciences (No. KFKT2019-003), and the Foundation of Guizhou Provincial Key Laboratory of Public Big Data (No. 2018BDKFJJ009, No. 2019BDKFJJ003, No. 2019BDKFJJ011).

References

[1] Tomas Oliveira e Silva, *Maximum excursion and stopping time record-holders for the 3x+1 problem: computational results*, *Mathematics of Computation*, vol. 68, no. 225, pp. 371-384, 1999.

[2] Tomas Oliveira e Silva, *Empirical Verification of the 3x+1 and Related Conjectures*. In *The Ultimate Challenge: The 3x+1 Problem*, (book edited by Jeffrey C. Lagarias), pp. 189-207, AMS, 2010.

[3] Wei Ren, Simin Li, Ruiyang Xiao and Wei Bi, *Collatz Conjecture for* $2^{100000} - 1$ *is True - Algorithms for Verifying Extremely Large Numbers*, Proc. of IEEE UIC, Oct. 2018, Guangzhou, China, 411-416, 2018

[4] Wei Ren, *A New Approach on Proving Collatz Conjecture*, Journal of Mathsmatics, Hindawi, April 2019, ID 6129836, https://www.hindawi.com/journals/jmath/2019/6129836/.

[5] Wei Ren, *Ratio and Partition are Revealed in Proposed Graph on Reduced Collatz Dynamics*, Proc. of IEEE ISPA, pp. 474-483, 16-28 Dec. 2019, Ximen, China

[6] Wei Ren, Ruiyang Xiao, *How to Fast Verify Collatz Conjecture by Automata*, Proc. of IEEE HPCC, pp. 2720-2729, 10-12 Aug. 2019, Zhangjiajie, China

[7] Wei Ren, *Reduced Collatz Dynamics Data Reveals Properties for the Future Proof of Collatz Conjecture*, Data, MDPI, 2019, 4, 89.

[8] Wei Ren, *Reduced Collatz Dynamics is Periodical and the Period Equals 2 to the Power of the Count of x/2*, Journal of Number Theory, Elsevier, Submitted, 2020.

[9] Wei Ren, *Collatz Dynamics is Partitioned by Residue Class Regularly*, Information and Computation, Elsevier, Submitted, 2019.

Appendix

4.1. The Proof of Lemma 3.20

Proof Due to Lemma 3.19, only the last requirement is needed to check. That is, $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}(x) \in [0]_{2^{q_n}}$.

When $n = 1$, $c = I^{p_1}O^{q_1}$. By Lemma 3.9, $I^p(x) \in [0]_{2^q} \Leftrightarrow x \in [(3^p)^{-1} * 2^p - 1]_{2^{p+q}}$, which is exactly identical with $x \in [-A_1B_1C_0]_{C_1}$ where
$A_1 = 3^{p_1} - 2^{p_1}$, $C_1 = 2^{\sum_{j=1}^{1}(p_j+q_j)} = 2^{p_1+q_1}$,
$B_1 = (3^{\sum_{j=1}^{1}p_j})^{-1} \mod C_1 = (3^{p_1})^{-1} \mod 2^{p_1+q_1}$, $C_0 = 1$, because
$x \in [-A_1B_1C_0]_{C_1} = [-(3^{p_1} - 2^{p_1}) * (3^{p_1})^{-1}]_{2^{p_1+q_1}} = [(3^{p_1})^{-1}2^{p_1} - 1]_{2^{p_1+q_1}}$.

Next, suppose $n \geq 2$.

(1) $I^{p_1}O^{q_1}(x) = O^{q_1}(I^{p_1}(x))$
$= ((\frac{3}{2})^{p_1}(x+1) - 1)/2^{q_1} \quad \because Eq.2$
$= \frac{3^{p_1}}{2^{p_1+q_1}}(x+1) - \frac{1}{2^{q_1}} = \frac{3^{p_1}}{2^{p_1+q_1}}x + \frac{3^{p_1}}{2^{p_1+q_1}} - \frac{1}{2^{q_1}} = \frac{3^{p_1}}{2^{p_1+q_1}}x + \frac{3^{p_1}-2^{p_1}}{2^{p_1+q_1}}$
$= a_1x + b_1$, where $a_1 = \frac{3^{p_1}}{2^{p_1+q_1}}, b_1 = \frac{3^{p_1}-2^{p_1}}{2^{p_1+q_1}}$.

(2) Similarly, $I^{p_2}O^{q_2}(x) = a_2x + b_2$, where $a_2 = \frac{3^{p_2}}{2^{p_2+q_2}}, b_2 = \frac{3^{p_2}-2^{p_2}}{2^{p_2+q_2}}$.

(3) Similarly, $I^{p_i}O^{q_i}(x) = a_ix + b_i$, where $a_i = \frac{3^{p_i}}{2^{p_i+q_i}}, b_i = \frac{3^{p_i}-2^{p_i}}{2^{p_i+q_i}}, i = 1, 2, ...n$. Obviously, $a_i > 0, b_i > 0$. Indeed, $x \in \mathbb{N}^*$ and $I^{p_i}O^{q_i}(x) \in \mathbb{N}^*$, thus, $a_i, b_i \in \mathbb{N}^*$.

(4) $I^{p_1}O^{q_1}I^{p_2}O^{q_2}(x) = I^{p_2}O^{q_2}(I^{p_1}O^{q_1}(x))$
$= a_2X_c + b_2 \quad$ (Let $X_c = a_1x + b_1$)
$= a_2(a_1x + b_1) + b_2 = a_2a_1x + a_2b_1 + b_2$.

(5) Observe $I^{p_i}O^{q_i}I^{p_{i+1}}O^{q_{i+1}}(x) = a_{i+1}(a_ix + b_i) + b_{i+1}$
$= a_{i+1}a_ix + a_{i+1}b_i + b_{i+1}, i = 1, 2, ..., n-1$.
$I^{p_i}O^{q_i}I^{p_{i+1}}O^{q_{i+1}}I^{p_{i+2}}O^{q_{i+2}}(x) = a_{i+2}(a_{i+1}a_ix + a_{i+1}b_i + b_{i+1}) + b_{i+2}$
$= a_{i+2}a_{i+1}a_ix + a_{i+2}a_{i+1}b_i + a_{i+2}b_{i+1} + b_{i+2}, i = 1, ..., n-2$.

(6) Next, we prove following result by induction:
$I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_{n-1}}O^{q_{n-1}}I^{p_n}O^{q_n}(x)$
$= a_n...a_1x + a_n...a_2b_1 + a_n...a_3b_2 + ... + a_nb_{n-1} + b_n$

(6.1) $n = 1$. $I^{p_1}O^{q_1}(x) = a_1x + b_1$, as desired in (1).

(6.2) (Proof from $n = i$ to $n = i + 1$.)

Suppose $n = i$, $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_i}O^{q_i}(x)$
$= a_ia_{i-1}...a_1x + a_ia_{i-1}...a_2b_1 + a_ia_{i-1}...a_3b_2 + ... + a_ib_{i-1} + b_i$.

Let's check $n = i + 1$,
$I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_k}O^{q_i}I^{p_{i+1}}O^{q_{i+1}}(x)$
$= I^{p_{i+1}}O^{q_{i+1}}(a_ia_{i-1}...a_1x + a_ia_{i-1}...a_2b_1 + a_ia_{i-1}...a_3b_2 + ... + a_ib_{i-1} + b_i)$
$= a_{i+1}(a_ia_{i-1}...a_1x + a_ia_{i-1}...a_2b_1 + a_ia_{i-1}...a_3b_2 + ... + a_ib_{i-1} + b_i) + b_{i+1}$
$= a_{i+1}a_i...a_1x + a_{i+1}a_i...a_2b_1 + a_{i+1}a_i...a_3b_2 + ... + a_{i+1}a_ib_{i-1} + a_{i+1}b_i + b_{i+1}$.

Therefore, $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_{n-1}}O^{q_{n-1}}I^{p_n}O^{q_n}(x)$
$= a_n...a_1x + a_n...a_2b_1 + a_n...a_3b_2 + ... + a_nb_{n-1} + b_n$.

(7) Due to (6), $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_{n-1}}O^{q_{n-1}}(x)$
$= a_{n-1}...a_1x + a_{n-1}...a_2b_1 + a_{n-1}...a_3b_2 + ... + a_{n-1}b_{n-2} + b_{n-1}$
$= \prod_{i=1}^{n-1} a_ix + \prod_{i=2}^{n-1} a_ib_1 + \prod_{i=3}^{n-1} a_ib_2 + ... + \prod_{i=n-1}^{n-1} a_ib_{n-2} + b_{n-1}$.

(8) $I^{p_n}(x) = (\frac{3}{2})^{p_n}(x+1) - 1$, due to Eq. 2.

(9) $I^{p_1}O^{q_1}I^{p_2}O^{q_2}...I^{p_n}(x) \in [0]_{2^{q_n}}$
$\Leftrightarrow I^{p_n}(\prod_{i=1}^{n-1} a_ix + \prod_{i=2}^{n-1} a_ib_1 + \prod_{i=3}^{n-1} a_ib_2 + ... + \prod_{i=n-1}^{n-1} a_ib_{n-2} + b_{n-1}) \in [0]_{2^{q_n}}$
$\Leftrightarrow (\frac{3}{2})^{p_n}(\prod_{i=1}^{n-1} a_ix + \prod_{i=2}^{n-1} a_ib_1 + \prod_{i=3}^{n-1} a_ib_2 + ... + \prod_{i=n-1}^{n-1} a_ib_{n-2} + b_{n-1} + 1) - 1 \in [0]_{2^{q_n}}$
$\Leftrightarrow$
$(\frac{3}{2})^{p_n}(\prod_{i=1}^{n-1} a_ix + \prod_{i=2}^{n-1} a_ib_1 + \prod_{i=3}^{n-1} a_ib_2 + ... + \prod_{i=n-1}^{n-1} a_ib_{n-2} + b_{n-1} + 1) \in [1]_{2^{q_n}}$
$\Leftrightarrow$
$3^{p_n}(\prod_{i=1}^{n-1} a_ix + \prod_{i=2}^{n-1} a_ib_1 + \prod_{i=3}^{n-1} a_ib_2 + ... + \prod_{i=n-1}^{n-1} a_ib_{n-2} + b_{n-1} + 1) \in [2^{p_n}]_{2^{q_n+p_n}}$
$\Leftrightarrow$
$\prod_{i=1}^{n-1} a_ix + \prod_{i=2}^{n-1} a_ib_1 + \prod_{i=3}^{n-1} a_ib_2 + ... + \prod_{i=n-1}^{n-1} a_ib_{n-2} + b_{n-1} + 1 \in [(3^{p_n})^{-1}2^{p_n}]_{2^{q_n+p_n}}$
$\Leftrightarrow$
$\prod_{i=1}^{n-1} a_ix \in [-(\prod_{i=2}^{n-1} a_ib_1 + \prod_{i=3}^{n-1} a_ib_2 + ... + \prod_{i=n-1}^{n-1} a_ib_{n-2} + b_{n-1} + 1) + (3^{p_n})^{-1}2^{p_n}]_{2^{q_n+p_n}}$.

Besides, $(3^{p_n}, 2^{q_n+p_n}) = 1$, thus $(3^{p_n})^{-1} \mod 2^{q_n+p_n}$ exists.

Calculate following results one by one as preparations.

$$\begin{cases} \prod_{i=1}^{n-1} a_i = \prod_{i=1}^{n-1} \frac{3^{p_i}}{2^{p_i+q_i}} = \frac{3^{\sum_{i=1}^{n-1} p_i}}{2^{\sum_{i=1}^{n-1}(p_i+q_i)}}, \\ (\prod_{i=2}^{n-1} a_i)b_1 = (\prod_{i=2}^{n-1} \frac{3^{p_i}}{2^{p_i+q_i}}) * \frac{3^{p_1}-2^{p_1}}{2^{p_1+q_1}} = \frac{3^{\sum_{i=2}^{n-1} p_i}}{2^{\sum_{i=2}^{n-1}(p_i+q_i)}} * \frac{3^{p_1}-2^{p_1}}{2^{p_1+q_1}} \\ = \frac{(3^{p_1}-2^{p_1})3^{\sum_{i=2}^{n-1} p_i}}{2^{\sum_{i=1}^{n-1}(p_i+q_i)}}, \\ (\prod_{i=3}^{n-1} a_i)b_2 = (\prod_{i=3}^{n-1} \frac{3^{p_i}}{2^{p_i+q_i}}) * \frac{3^{p_2}-2^{p_2}}{2^{p_2+q_2}} = \frac{3^{\sum_{i=3}^{n-1} p_i}}{2^{\sum_{i=3}^{n-1}(p_i+q_i)}} * \frac{3^{p_2}-2^{p_2}}{2^{p_2+q_2}} \\ = \frac{(3^{p_2}-2^{p_2})3^{\sum_{i=3}^{n-1} p_i}}{2^{\sum_{i=2}^{n-1}(p_i+q_i)}}, \\ (\prod_{i=k}^{n-1} a_i)b_{k-1} = (\prod_{i=k}^{n-1} \frac{3^{p_i}}{2^{p_i+q_i}}) * \frac{3^{p_{k-1}}-2^{p_{k-1}}}{2^{p_{k-1}+q_{k-1}}} = \frac{3^{\sum_{i=k}^{n-1} p_i}}{2^{\sum_{i=k}^{n-1}(p_i+q_i)}} * \frac{3^{p_{k-1}}-2^{p_{k-1}}}{2^{p_{k-1}+q_{k-1}}} \\ = \frac{(3^{p_{k-1}}-2^{p_{k-1}})3^{\sum_{i=k}^{n-1} p_i}}{2^{\sum_{i=k-1}^{n-1}(p_i+q_i)}}, k = 2, 3, ..., n-1. \end{cases} \tag{5}$$

Especially, when $k = n-1$, $\prod_{i=k}^{n-1} a_i b_{k-1} = \prod_{i=n-1}^{n-1} a_i b_{n-1-1} = a_{n-1}b_{n-2}$ $= \frac{3^{p_{n-1}}}{2^{p_{n-1}+q_{n-1}}} * \frac{3^{p_{n-2}}-2^{p_{n-2}}}{2^{p_{n-2}+q_{n-2}}} = \frac{(3^{p_{n-2}}-2^{p_{n-2}})3^{p_{n-1}}}{2^{p_{n-1}+q_{n-1}+p_{n-2}+q_{n-2}}}$.

Recall b_i in (3), $b_{n-1} = \frac{3^{p_{n-1}}-2^{p_{n-1}}}{2^{p_{n-1}+q_{n-1}}}$.

Thanks to above preparations, we can continue the calculation as follows:

$$\prod_{i=1}^{n-1} a_i x \in [-(\prod_{i=2}^{n-1} a_i b_1 + \prod_{i=3}^{n-1} a_i b_2 + ... + \prod_{i=n-1}^{n-1} a_i b_{n-2} + b_{n-1} + 1) + (3^{p_n})^{-1} 2^{p_n}]_{2^{q_n+p_n}},$$

$$\Leftrightarrow$$

$$(\frac{3^{\sum_{i=1}^{n-1} p_i}}{2^{\sum_{i=1}^{n-1}(p_i+q_i)}}) * x \in [-(\frac{(3^{p_1} - 2^{p_1})3^{\sum_{i=2}^{n-1} p_i}}{2^{\sum_{i=1}^{n-1}(p_i+q_i)}} + \frac{(3^{p_2} - 2^{p_2})3^{\sum_{i=3}^{n-1} p_i}}{2^{\sum_{i=2}^{n-1}(p_i+q_i)}} + ...$$
$$+ \frac{(3^{p_{n-2}} - 2^{p_{n-2}})3^{p_{n-1}}}{2^{p_{n-1}+q_{n-1}+p_{n-2}+q_{n-2}}} + b_{n-1} + 1) + (3^{p_n})^{-1} 2^{p_n}]_{2^{q_n+p_n}},$$

$$\Leftrightarrow$$

$$(3^{\sum_{i=1}^{n-1} p_i}) * x \in [(-\frac{(3^{p_1} - 2^{p_1})3^{\sum_{i=2}^{n-1} p_i}}{2^{\sum_{i=1}^{n-1}(p_i+q_i)}} - \frac{(3^{p_2} - 2^{p_2})3^{\sum_{i=3}^{n-1} p_i}}{2^{\sum_{i=2}^{n-1}(p_i+q_i)}} - ...$$
$$-\frac{(3^{p_{n-2}} - 2^{p_{n-2}})3^{p_{n-1}}}{2^{p_{n-1}+q_{n-1}+p_{n-2}+q_{n-2}}} - b_{n-1} - 1 + (3^{p_n})^{-1} 2^{p_n}) 2^{\sum_{i=1}^{n-1}(p_i+q_i)}]_{2^{q_n+p_n} * 2^{\sum_{i=1}^{n-1}(p_i+q_i)}},$$

$$\Leftrightarrow$$

$$(3^{\sum_{i=1}^{n-1} p_i}) * x \in [-(3^{p_1} - 2^{p_1})3^{\sum_{i=2}^{n-1} p_i} - (3^{p_2} - 2^{p_2})3^{\sum_{i=3}^{n-1} p_i} 2^{p_1+q_1} - ...$$
$$-(3^{p_{n-2}} - 2^{p_{n-2}})3^{p_{n-1}} 2^{p_1+q_1+...+p_{n-3}+q_{n-3}} - \frac{3^{p_{n-1}} - 2^{p_{n-1}}}{2^{p_{n-1}+q_{n-1}}} 2^{\sum_{i=1}^{n-1}(p_i+q_i)}$$
$$-2^{\sum_{i=1}^{n-1}(p_i+q_i)} + (3^{p_n})^{-1} 2^{p_n} 2^{\sum_{i=1}^{n-1}(p_i+q_i)}]_{2^{\sum_{i=1}^{n}(p_i+q_i)}},$$

$$\Leftrightarrow$$

$$x \in [-(3^{p_1} - 2^{p_1})3^{\sum_{i=2}^{n-1} p_i}(3^{\sum_{i=1}^{n-1} p_i})^{-1} - (3^{p_2} - 2^{p_2})3^{\sum_{i=3}^{n-1} p_i} 2^{p_1+q_1}(3^{\sum_{i=1}^{n-1} p_i})^{-1} - ...$$
$$-(3^{p_{n-2}} - 2^{p_{n-2}})3^{p_{n-1}} 2^{p_1+q_1+...+p_{n-3}+q_{n-3}}(3^{\sum_{i=1}^{n-1} p_i})^{-1}$$
$$-(3^{p_{n-1}} - 2^{p_{n-1}})2^{\sum_{i=1}^{n-2}(p_i+q_i)}(3^{\sum_{i=1}^{n-1} p_i})^{-1}$$
$$-2^{\sum_{i=1}^{n-1}(p_i+q_i)}(3^{\sum_{i=1}^{n-1} p_i})^{-1} + (3^{p_n})^{-1} 2^{p_n} 2^{\sum_{i=1}^{n-1}(p_i+q_i)}(3^{\sum_{i=1}^{n-1} p_i})^{-1}]_{2^{\sum_{i=1}^{n}(p_i+q_i)}},$$

$$\Leftrightarrow$$

$$x \in [-(3^{p_1} - 2^{p_1})(3^{p_1})^{-1} - (3^{p_2} - 2^{p_2})(3^{p_1+p_2})^{-1} 2^{p_1+q_1} - ...$$
$$-(3^{p_{n-2}} - 2^{p_{n-2}})(3^{p_1+p_2+...+p_{n-2}})^{-1} 2^{p_1+q_1+...+p_{n-3}+q_{n-3}}$$
$$-(3^{p_{n-1}} - 2^{p_{n-1}})(3^{\sum_{i=1}^{n-1} p_i})^{-1} 2^{\sum_{i=1}^{n-2}(p_i+q_i)}$$
$$-(3^{\sum_{i=1}^{n-1} p_i})^{-1} 2^{\sum_{i=1}^{n-1}(p_i+q_i)} + (3^{p_n})^{-1} 2^{p_n}(3^{\sum_{i=1}^{n-1} p_i})^{-1} 2^{\sum_{i=1}^{n-1}(p_i+q_i)}]_{2^{\sum_{i=1}^{n}(p_i+q_i)}}. \quad (6)$$

$(3^{\sum_{i=1}^{n-1} p_i})^{-1} \mod 2^{\sum_{i=1}^{n}(p_i+q_i)}$ exists, since $((3^{\sum_{i=1}^{n-1} p_i})^{-1}, 2^{\sum_{i=1}^{n}(p_i+q_i)}) = 1$.

Recall that $(3^{p_n})^{-1}$ is the inverse module $2^{q_n+p_n}$. However,

$(3^{p_n})^{-1} * 3^{p_n} \equiv 1 \mod 2^{\sum_{i=1}^{n}(p_i+q_i)}$

$\Rightarrow (3^{p_n})^{-1} * 3^{p_n} = k * 2^{\sum_{i=1}^{n}(p_i+q_i)} + 1, k \in \mathbb{N}$

$\Rightarrow (3^{p_n})^{-1} * 3^{p_n} = k * 2^{q_n+p_n} * 2^{\sum_{i=1}^{n-1}(p_i+q_i)} + 1, k \in \mathbb{N}$

$\Rightarrow (3^{p_n})^{-1} * 3^{p_n} \equiv 1 \mod 2^{q_n+p_n}$. Thus, when $(3^{p_n})^{-1}$ is the inverse module $2^{\sum_{i=1}^{n}(p_i+q_i)}$, it is also the inverse module $2^{q_n+p_n}$. Formally, $\forall a, k, k' \in \mathbb{N}^*, k'|k$, $a^{-1}*a \equiv 1 \mod k \Rightarrow k|(a^{-1}*a-1) \Rightarrow k'|k|(a^{-1}*a-1) \Rightarrow a^{-1}*a \equiv 1 \mod k'$. Therefore, all inverse in above last equation can be module $2^{\sum_{i=1}^{n}(p_i+q_i)}$.

Let $A_i = 3^{p_i} - 2^{p_i}$, $B_i = (3^{\sum_{j=1}^{i} p_j})^{-1} \mod C_n$, $C_i = 2^{\sum_{j=1}^{i}(p_j+q_j)}$, $i = 1, 2, ..., n$, $n, i, j \in \mathbb{N}^*$.

Besides, it is obvious that $(3^{p_i})^{-1} * B_{i-1} = B_i, (2^{p_i+q_i}) * C_{i-1} = C_i, i = 2, 3, ..., n$.

Therefore,

$x \in [-A_1B_1 - A_2B_2C_1 - A_3B_3C_2 - ... - A_{n-1}B_{n-1}C_{n-2} - B_{n-1}C_{n-1} + (3^{p_n})^{-1}B_{n-1}2^{p_n}C_{n-1}]_{C_n}$

$= [-A_1B_1 - A_2B_2C_1 - A_3B_3C_2 - ... - A_{n-1}B_{n-1}C_{n-2} - B_{n-1}C_{n-1} + 2^{p_n}B_nC_{n-1}]_{C_n}$

$= [-A_1B_1C_0 - A_2B_2C_1 - A_3B_3C_2 - ... - A_{n-1}B_{n-1}C_{n-2} - 3^{p_n}B_nC_{n-1} + 2^{p_n}B_nC_{n-1}]_{C_n}$

$= [-A_1B_1C_0 - A_2B_2C_1 - A_3B_3C_2 - ... - A_{n-1}B_{n-1}C_{n-2} - A_nB_nC_{n-1}]_{C_n}$

$= [-\sum_{i=1}^{n} A_iB_iC_{i-1}]_{C_n}, C_0 = 1.$

In summary,

$$\begin{cases} x \in [-\sum_{i=1}^{n} A_iB_iC_{i-1}]_{C_n}, \\ A_i = 3^{p_i} - 2^{p_i}, \\ B_i = (3^{\sum_{j=1}^{i} p_j})^{-1} \mod C_n, \\ C_i = 2^{\sum_{j=1}^{i}(p_j+q_j)}, C_0 = 1. \end{cases} \tag{7}$$

$\square$

4.2. Examples for Deriving x from RD[x]

Example 4.1.

(1) $\mathsf{RD}[x] = I^2O^2$. $p = 2, q = 2$.

$x \in [(3^2)^{-1}*2^2-1]_{2^{2+2}} \Rightarrow x \in [9^{-1}*4-1]_{16} \Rightarrow x \in [9*4-1]_{16} \Rightarrow x \in [3]_{16}$.

Obviously, $3 > \frac{3^2-2^2}{2^{2+2}-3^2} = \frac{9-4}{16-9} = 5/7$.

(2) $\mathsf{RD}[x] = I^3O^2$. $p = 3, q = 2$.

$x \in [(3^3)^{-1} * 2^3 - 1]_{2^{3+2}} \Rightarrow x \in [27^{-1} * 8 - 1]_{32} \Rightarrow x \in [19 * 8 - 1]_{32} \Rightarrow x \in [23]_{32}$.

Obviously, $23 > \frac{3^3-2^3}{2^{3+2}-3^3} = \frac{27-8}{32-27} = 19/5 = 3.8$.

Therefore, $x \in [23]_{32}$.

(3) $\mathsf{RD}[x] = I^5O^3$. $p = 5, q = 3$.

$x \in [(3^5)^{-1} * 2^5 - 1]_{2^{5+3}} \Rightarrow x \in [243^{-1} * 32 - 1]_{256}$
$\Rightarrow x \in [32 * 59 - 1]_{256} \Rightarrow x \in [95]_{256}$.

Obviously, $95 > \frac{3^5-2^5}{2^{5+3}-3^5} = \frac{243-32}{256-243} = 211/13 \approx 16.23$.

Example 4.2.

(1) $\mathsf{RD}[x] = I^4O^2I^2O^2$. *Thus,* $p_1 = 4, q_1 = 2, p_2 = 2, q_2 = 2$. $n = 2$.

$A_i = 3^{p_i} - 2^{p_i}$, $B_i = (3^{\sum_{j=1}^{i} p_j})^{-1} \mod C_n$, $C_i = 2^{\sum_{j=1}^{i}(p_j+q_j)}$,

$A_1 = 3^{p_1} - 2^{p_1} = 3^4 - 2^4 = 81 - 16 = 65$,
$A_2 = 3^{p_2} - 2^{p_2} = 3^2 - 2^2 = 9 - 4 = 5$,
$C_1 = 2^{p_1+q_1} = 2^{4+2} = 2^6 = 64$,
$C_2 = 2^{p_1+q_1+p_2+q_2} = 2^{4+2+2+2} = 2^{10} = 1024$,
$B_1 = (3^{p_1})^{-1} \mod C_2 = (3^4)^{-1} \mod 1024 = 81^{-1} \mod 1024 = 177$,
$B_2 = (3^{p_1+p_2})^{-1} \mod C_2 = (3^{4+2})^{-1} \mod 1024 = (3^6)^{-1} \mod 1024 = 729^{-1} \mod 1024 = 361$,
$[-A_1B_1C_0 - A_2B_2C_1]_{C_2} = [-65 * 177 * 1 - 5 * 361 * 64]_{1024}$
$= [-11505 - 115520]_{1024} = [-127025]_{1024} = [975]_{1024}$.

Obviously,

$\Psi = \frac{(3^{p_1}-2^{p_1})3^{\sum_{i=2}^{n} p_i}}{2^{\sum_{i=1}^{n}(p_i+q_i)}} + \frac{(3^{p_2}-2^{p_2})3^{\sum_{i=3}^{n} p_i}}{2^{\sum_{i=2}^{n}(p_i+q_i)}} + ... + \frac{(3^{p_{n-1}}-2^{p_{n-1}})3^{p_n}}{2^{p_n+q_n+p_{n-1}+q_{n-1}}} + \frac{3^{p_n}-2^{p_n}}{2^{p_n+q_n}}$
$= \frac{(3^4-2^4)3^2}{2^{4+2+2+2}} + \frac{3^2-2^2}{2^{2+2}} = \frac{(81-16)*9}{1024} + \frac{5}{16} = \frac{585}{1024} + \frac{5}{16} = \frac{905}{1024}$.

$975 > \Psi * \frac{2^{\sum_{i=1}^{n}(p_i+q_i)}}{2^{\sum_{i=1}^{n}(p_i+q_i)} - 3^{\sum_{i=1}^{n} p_i}}$
$= \Psi * \frac{2^{4+2+2+2}}{2^{4+2+2+2}-3^{4+2}} = \Psi * \frac{1024}{1024-729} = \frac{905}{1024} * \frac{1024}{295} = \frac{905}{295}$.

(2) $\mathsf{RD}[x] = I^4OIOIO^2$. *Thus,* $p_1 = 4, q_1 = 1, p_2 = 1, q_2 = 1, p_3 = 1, q_3 = 2$. $n = 3$.

$A_i = 3^{p_i} - 2^{p_i}$, $B_i = (3^{\sum_{j=1}^{i} p_j})^{-1} \mod C_n$, $C_i = 2^{\sum_{j=1}^{i}(p_j+q_j)}$,

$A_1 = 3^{p_1} - 2^{p_1} = 3^4 - 2^4 = 81 - 16 = 65$,
$A_2 = 3^{p_2} - 2^{p_2} = 3^1 - 2^1 = 1$,
$A_3 = 3^{p_3} - 2^{p_3} = 3^1 - 2^1 = 1$,
$C_1 = 2^{p_1+q_1} = 2^{4+1} = 2^5 = 32$,
$C_2 = 2^{p_1+q_1+p_2+q_2} = 2^{4+1+1+1} = 2^7 = 128$,
$C_3 = 2^{p_1+q_1+p_2+q_2+p_3+q_3} = 2^{4+1+1+1+1+2} = 2^{10} = 1024$,
$B_1 = (3^{p_1})^{-1} \mod C_3 = (3^4)^{-1} \mod 1024 = 81^{-1} \mod 1024 = 177$,

$B_2 = (3^{p_1+p_2})^{-1} \mod C_3 = (3^{4+1})^{-1} \mod 1024 = (3^5)^{-1} \mod 1024 = 243^{-1} \mod 1024 = 59$,
$B_3 = (3^{p_1+p_2+p_3})^{-1} \mod C_3 = (3^{4+1+1})^{-1} \mod 1024 = (3^6)^{-1} \mod 1024 = 729^{-1} \mod 1024 = 361$,
$[-A_1B_1C_0 - A_2B_2C_1 - A_3B_3C_2]_{C_3} = [-65*177*1 - 1*59*32 - 1*361*128]_{1024}$
$= [-59601]_{1024} = [815]_{1024}$.

Obviously,

$\Psi = \frac{(3^{p_1}-2^{p_1})3^{\sum_{i=2}^{n} p_i}}{2^{\sum_{i=1}^{n}(p_i+q_i)}} + \frac{(3^{p_2}-2^{p_2})3^{\sum_{i=3}^{n} p_i}}{2^{\sum_{i=2}^{n}(p_i+q_i)}} + ... + \frac{(3^{p_{n-1}}-2^{p_{n-1}})3^{p_n}}{2^{p_n+q_n+p_{n-1}+q_{n-1}}} + \frac{3^{p_n}-2^{p_n}}{2^{p_n+q_n}}$
$= \frac{(3^4-2^4)3^{1+1}}{2^{4+1+1+1+1+2}} + \frac{(3^1-2^1)3^1}{2^{1+1+1+2}} + \frac{3^1-2^1}{2^{1+2}} = \frac{(81-16)9}{1024} + \frac{1*3}{32} + \frac{1}{8}$
$= \frac{585}{1024} + \frac{3}{32} + \frac{1}{8} = \frac{585+96+128}{1024} = \frac{809}{1024}$.

$815 > \Psi * \frac{2^{\sum_{i=1}^{n}(p_i+q_i)}}{2^{\sum_{i=1}^{n}(p_i+q_i)} - 3^{\sum_{i=1}^{n} p_i}} = \Psi * \frac{2^{4+1+1+1+1+2}}{2^{4+1+1+1+1+2} - 3^{4+1+1}}$
$= \Psi * \frac{1024}{1024-729} = \frac{809}{1024} * \frac{1024}{295} = \frac{809}{295}$.

(3) $\mathsf{RD}[x] = I^4OIOIOIO^2$. *Thus,*

$p_1 = 4, q_1 = 1; p_2 = 1, q_2 = 1; p_3 = 1, q_3 = 1; p_4 = 1, q_4 = 2; n = 4$.

$A_i = 3^{p_i} - 2^{p_i}$, $B_i = (3^{\sum_{j=1}^{i} p_j})^{-1} \mod C_n$, $C_i = 2^{\sum_{j=1}^{i}(p_j+q_j)}$,

$A_1 = 3^{p_1} - 2^{p_1} = 3^4 - 2^4 = 81 - 16 = 65$,
$A_2 = 3^{p_2} - 2^{p_2} = 3^1 - 2^1 = 1$,
$A_3 = 3^{p_3} - 2^{p_3} = 3^1 - 2^1 = 1$,
$A_4 = 3^{p_4} - 2^{p_4} = 3^1 - 2^1 = 1$,
$C_1 = 2^{p_1+q_1} = 2^{4+1} = 2^5 = 32$,
$C_2 = 2^{p_1+q_1+p_2+q_2} = 2^{4+1+1+1} = 2^7 = 128$,
$C_3 = 2^{p_1+q_1+p_2+q_2+p_3+q_3} = 2^{4+1+1+1+1+1} = 2^9 = 512$,
$C_4 = 2^{p_1+q_1+p_2+q_2+p_3+q_3+p_4+q_4} = 2^{4+1+1+1+1+1+1+2} = 2^{12} = 4096$,
$B_1 = (3^{p_1})^{-1} \mod C_4 = (3^4)^{-1} \mod 4096 = 81^{-1} \mod 4096 = 2225$,
$B_2 = (3^{p_1+p_2})^{-1} \mod C_4 = (3^{4+1})^{-1} \mod 4096 = (3^5)^{-1} \mod 4096 = 243^{-1} \mod 4096 = 2107$,
$B_3 = (3^{p_1+p_2+p_3})^{-1} \mod C_4 = (3^{4+1+1})^{-1} \mod 4096 = (3^6)^{-1} \mod 4096 = 729^{-1} \mod 4096 = 3433$,
$B_4 = (3^{p_1+p_2+p_3+p_4})^{-1} \mod C_4 = (3^{4+1+1+1})^{-1} \mod 4096 = (3^7)^{-1} \mod 4096 = 2187^{-1} \mod 4096 = 3875$,
$[-A_1B_1C_0 - A_2B_2C_1 - A_3B_3C_2 - A_4B_4C_3]_{C_4}$
$= [-65 * 2225 * 1 - 1 * 2107 * 32 - 1 * 3433 * 128 - 1 * 3875 * 512]_{4096}$
$= [-2635473]_{4096} = [2351]_{4096}$.

Obviously,

$\Psi = \frac{(3^{p_1}-2^{p_1})3^{\sum_{i=2}^{n} p_i}}{2^{\sum_{i=1}^{n}(p_i+q_i)}} + \frac{(3^{p_2}-2^{p_2})3^{\sum_{i=3}^{n} p_i}}{2^{\sum_{i=2}^{n}(p_i+q_i)}} + ... + \frac{(3^{p_{n-1}}-2^{p_{n-1}})3^{p_n}}{2^{p_n+q_n+p_{n-1}+q_{n-1}}} + \frac{3^{p_n}-2^{p_n}}{2^{p_n+q_n}}$

$$= \frac{(3^4-2^4)3^{1+1+1}}{2^{4+1+1+1+1+1+1+1+2}} + \frac{(3^1-2^1)3^{1+1}}{2^{1+1+1+1+1+2}} + \frac{(3^1-2^1)3^1}{2^{1+1+1+2}} + \frac{3^1-2^1}{2^{1+2}}$$
$$= \frac{(81-16)27}{4096} + \frac{1*9}{128} + \frac{1*3}{32} + \frac{1}{8}$$
$$= \frac{1775}{4096} + \frac{9}{128} + \frac{3}{32} + \frac{1}{8} = \frac{1775+288+384+512}{4096} = \frac{2959}{4096}.$$
$$2351 > \Psi * \frac{2^{\sum_{i=1}^{n}(p_i+q_i)}}{2^{\sum_{i=1}^{n}(p_i+q_i)} - 3^{\sum_{i=1}^{n} p_i}}$$
$$= \Psi * \frac{2^{4+1+1+1+1+1+1+1+2}}{2^{4+1+1+1+1+1+1+1+2} - 3^{4+1+1+1}} = \Psi * \frac{4096}{4096-2187} = \frac{2959}{4096} * \frac{4096}{1909} = \frac{2959}{1909}.$$