Lecture on Initial Classification of Lie Algebra

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1 Rough Classification of Lie Algebras

Before beginning, we give three definitions straightforward. First, we define the center $Z(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ to be the subspace of $\mathfrak{g}$ of elements $X \in \mathfrak{g}$ such that $[X, Y] = 0$ for all $Y \in \mathfrak{g}$. Second, we say $\mathfrak{g}$ is abelian if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$. Third, the commutant of $\mathfrak{g}$ is the ideal $[g, g]$.

Next, we say that a Lie subalgebras $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is an ideal if it satisfies the condition

$$[X, Y] \in \mathfrak{h} \text{ for all } X \in \mathfrak{h}, Y \in \mathfrak{g}.$$

To attempt to classify Lie algebras, we introduce two descending chains of subalgebras. For a Lie algebra $\mathfrak{g}$ define its derived series by the formulas

$$D^0(\mathfrak{g}) = \mathfrak{g},\ D^{n+1}(\mathfrak{g}) = [D^n(\mathfrak{g}), D^n(\mathfrak{g})].$$

The lower central series of subalgebras is defined inductively by

$$D_0(\mathfrak{g}) = \mathfrak{g},\ D_{n+1}(\mathfrak{g}) = [\mathfrak{g}, D_n(\mathfrak{g})].$$

Both of these are descending sequences of ideals in $\mathfrak{g}$.

Theorem 1.1. *$D^k\mathfrak{g}$ is an ideal in $\mathfrak{g}$. More generally, if $\mathfrak{h}$ is an ideal in a Lie algebra $\mathfrak{g}$, then $[\mathfrak{h}, \mathfrak{h}]$ is also an ideal in $\mathfrak{g}$; hence all $D^k\mathfrak{h}$ are ideals in $\mathfrak{g}$.*

Proof. The Jacobi identity implies, for ideals I, J, K, that $[I, [J, K]] \subset [J, [K, I]] + [K, [I, J]]$. Thus,

$$[\mathfrak{g}, [\mathfrak{h}, \mathfrak{h}]] \subset [\mathfrak{h}, [\mathfrak{g}, \mathfrak{h}]] + [\mathfrak{h}, [\mathfrak{h}, \mathfrak{g}]] = [\mathfrak{h}, [\mathfrak{g}, \mathfrak{h}]] \subset [\mathfrak{h}, \mathfrak{h}],$$

which means that $[\mathfrak{h}, \mathfrak{h}]$ is an ideal in $\mathfrak{g}$. It follows by induction on k that $D^k\mathfrak{h}$ is an ideal. This applies in particular to $\mathfrak{h} = \mathfrak{g}$. $\square$

Observe that we have $D^k\mathfrak{g} \subset D_k\mathfrak{g}$ for all k, with equality when $k = 1$. We often write simply $D\mathfrak{g}$ for $D_1\mathfrak{g} = D^1\mathfrak{g}$ and call it the commutator subalgebra.

Definition 1.2. *We say that* $\mathfrak{g}$ *is solvable if* $D^k\mathfrak{g} = 0$ *for some* k*;* $\mathfrak{g}$ *is nilpotent if* $D_k\mathfrak{g} = 0$ *for some* k*;* $\mathfrak{g}$ *is perfect if* $D\mathfrak{g} = \mathfrak{g}$*;* $\mathfrak{g}$ *is semisimple if* $\mathfrak{g}$ *has no nonzero solvable ideals.*

Example 1.3. *Consider the space* $\mathfrak{b}_n\mathbb{R}$ *of upper-triangular* $n \times n$ *matrices, and the algebra* $\mathfrak{n}_n\mathbb{R}$ *of strictly upper-triangular* $n \times n$ *matrices. In this Lie algebra the commutator* $D\mathfrak{b}_n\mathbb{R}$ *is the algebra* $\mathfrak{n}_n\mathbb{R}$ *and the derived series is* $D^k\mathfrak{b}_n\mathbb{R} = \mathfrak{n}_{2^{k-1},n}\mathbb{R}$*. It follows that any subalgebra of the algebra* $\mathfrak{b}_n\mathbb{R}$ *is also solvable. Similarily, it is easy to check that any subalgebra of the Lie algebra* $\mathfrak{n}_n\mathbb{R}$ *is nilpotent.*

2 Solvability

Lemma 2.1. *The quotient* $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ *is abelian. Moreover, if* $I \subset \mathfrak{g}$ *is an ideal such that* $\mathfrak{g}/I$ *is abelian, then* $I \supset [\mathfrak{g}, \mathfrak{g}]$*.*

Proposition 2.2. *The following conditions on* $\mathfrak{g}$ *are equivalent:*

(1) $\mathfrak{g}$ *is solvable.*

(2) There exists a sequence of ideals $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_m = 0$ *such that* $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ *is abelian.*

Proof. It is easy to check that (1) implies (2), since we can take $\mathfrak{g}_i = D^i\mathfrak{g}$. Conversely, for $i = 0$, it satisfies that $D^0\mathfrak{g} = \mathfrak{g} = \mathfrak{g}_0$. Assume that $D^i\mathfrak{g} \subset \mathfrak{g}_i$ for all $0 \leq i \leq n$. Then for $i = n + 1$, we desire to show that $D^{n+1}\mathfrak{g} \subset \mathfrak{g}_{n+1}$. Since $\mathfrak{g}_{n+1} \subset \mathfrak{g}_n$ and $\mathfrak{g}_n/\mathfrak{g}_{n+1}$ is abelian, $D^{n+1}\mathfrak{g} = [\mathfrak{g}_n, \mathfrak{g}_n] \subset \mathfrak{g}_{n+1}$ by Lemma 2.1. □

Proposition 2.3. *Let* L *be a Lie algebra.*

(1) If L *is solvable, then so are all subalgebras and homomorphic images of* L*.*

(2) If I is a solvable ideal of L such that L/I is solvable, then L is solvable.

(3) If I, J are solvable ideals of L, then so is $I + L$.

Proof. (1) Let K be a subalgebra of L, then $D^iK \subset D^iL$. Let $\varphi : L \to M$ is an epimorphism, it is easy to show that $\varphi(D^iK) = D^iM$ by induction on i.

(2) Let $D^n(L/I) = 0$. Applying (1) to the homomorphism $\pi : L \to L/I$, we get $D^nL \subset I = \mathrm{Ker}\pi$, or $D^nL \subset I = \mathrm{Ker}\pi$. If $D^mI = 0$, the fact that $D^j(D^iL) = D^{i+j}L$ implies that $D^{m+n}L = 0$.

(3) We have the fact that $(I + J)/J \cong I/(I \cap J)$ by homomorphism theorem. As a homomorphic image of I, the right side is solvable, then $(I + J)/J$ is solvable. Then so is $I + J$ by (2). $\square$

3 Nilpotency

Proposition 3.1. *The following conditions on $\mathfrak{g}$ are equivalent:*

(1) $\mathfrak{g}$ is nilpotent.

(2) There exists a sequence of ideals $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots \supset \mathfrak{g}_m = 0$ such that $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$.

Proof. Take $\mathfrak{g}_i = D_i\mathfrak{g}$, then it is easy to check that (1) implies (2). Conversely, by induction, $D_0\mathfrak{g} = \mathfrak{g} = \mathfrak{g}_0$. Assume that $D_i\mathfrak{g} \subset \mathfrak{g}_i$ for all $0 \leq i \leq n$, then we desire to show $D_{n+1}\mathfrak{g} \subset \mathfrak{g}_{n+1}$. Notice that any nilpotent Lie algebra is solvable since $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ implies $[\mathfrak{g}_i, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$, hence $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian. Thus, we can conclude that $D_{n+1}\mathfrak{g} \subset \mathfrak{g}_{n+1}$ by Lemma 2.1. $\square$

Proposition 3.2. *Let L be a Lie algebra.*

(1) If L is nilpotent, then so are all subalgebras and homomorphic images of L.

(2) If $L/Z(L)$ is nilpotent, then so is L.

(3) If L is nilpotent and nonzero, then $Z(L) \neq 0$.

Proof. (1) Let K be a subalgebra of L, then $D_iK \subset D_iL$. Similarly, if $\varphi : L \to M$ is an epimorphism, an easy induction on i shows that $\varphi(D_iL) = D_iM$.

(2) Since $D_n(L) \subset Z(L)$, we have $D_{n+1}L = [L, D_nL] \subset [L, Z(L)] = 0$.

(3) The last nonzero term of the descending central series is central. □

4 Semisimple Lie Algebra

Let $\mathfrak{g}$ be a finite dimensional Lie algebra over a field $\mathbb{F}$.

Proposition 4.1. *$\mathfrak{g}$ contains the largest solvable ideal which contains all solvable ideals of $\mathfrak{g}$. This ideal is called the radical of g and denoted rad$(\mathfrak{g})$.*

Proof. Let I, J be solvable ideals of $\mathfrak{g}$. Then $I + J \subset \mathfrak{g}$ is an ideal, and $(I + J)/I \cong J/(I \cap J)$ is solvable, so $I + J$ is solvable. Thus, the sum of finitely many solvable. Hence the sum of all solvable ideals in $\mathfrak{g}$ is a solvable ideal, as desire. □

Definition 4.2. *(1) $\mathfrak{g}$ is called semisimple if rad$(\mathfrak{g}) = 0$, i.e., $\mathfrak{g}$ does not contain nonzero solvable ideals.*

(2) A non-abelian $\mathfrak{g}$ is called simple if it contains no ideals other than 0, $\mathfrak{g}$. In other words, a non-abelian $\mathfrak{g}$ is simple if its adjoint representation is irreducible (=simple).

If $\mathfrak{g}$ is both solvable and semisimple then $\mathfrak{g} = 0$.

Proposition 4.3. *(1) rad$(\mathfrak{g} \oplus \mathfrak{h}) =$ rad$(\mathfrak{g}) \oplus$ rad$(\mathfrak{h})$.*

(2) A simple Lie algebra is semisimple.

Proof. (1) The images of $\mathrm{rad}(\mathfrak{g} \oplus \mathfrak{h})$ in $\mathfrak{g}$ and in $\mathfrak{h}$ are solvable, hence contained in $\mathrm{rad}(\mathfrak{g})$, respectively $\mathrm{rad}(\mathfrak{h})$. Thus

$$\mathrm{rad}(\mathfrak{g} \oplus \mathfrak{h}) \subset \mathrm{rad}(\mathfrak{g}) \oplus \mathrm{rad}(\mathfrak{h}).$$

And $\mathrm{rad}(\mathfrak{g}) \oplus \mathrm{rad}(\mathfrak{h})$ is solvable ideal in $\mathfrak{g} \oplus \mathfrak{h}$, so

$$\mathrm{rad}(\mathfrak{g} \oplus \mathfrak{h}) \supset \mathrm{rad}(\mathfrak{g}) \oplus \mathrm{rad}(\mathfrak{h}).$$

(2) The only nonzero ideal in $\mathfrak{g}$ is $\mathfrak{g}$, and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ since $\mathfrak{g}$ is not abelian. Hence $\mathfrak{g}$ is not solvable. Thus, $\mathfrak{g}$ is semisimple. $\square$

Theorem 4.4 (weak Levi decomposition)**.** *The quotient* $\mathfrak{g}/rad(\mathfrak{g})$ *is semisimple. Any Lie algebra* $\mathfrak{g}$ *fits into an exact sequence*

$$0 \to rad(\mathfrak{g}) \to \mathfrak{g} \to \mathfrak{g}/rad(\mathfrak{g}) \to 0$$

where $rad(\mathfrak{g})$ *is a solvable ideal and* $\mathfrak{g}/rad(\mathfrak{g})$ *is semisimple. Moreover, if* $\mathfrak{h} \subset \mathfrak{g}$ *is a solvable ideal such that* $\mathfrak{g}/\mathfrak{h}$ *is semisimple then* $\mathfrak{h} = rad(\mathfrak{g})$*.*

Proof. Let $I \subset \mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ be a solvable ideal, and let $\widetilde{I}$ be its preimage in $\mathfrak{g}$. Then $\widetilde{I}$ is a solvable ideal in $\mathfrak{g}$. Thus $\widetilde{I} = \mathrm{rad}(\mathfrak{g})$ and $I = 0$. $\square$

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