# A BRIEF INTRODUCTION TO CATEGORY THEORY

WEIZIWEI
201900180143
CLASS 2 OF GRADE 2019

Abstract. Category theory is a toolset for describing the general abstract structures in mathematics and is intruduced in the study of algebraic topology by Samuel Eilenberg and Saunders Mac Lane. Now it plays an important role not only in algebraic topology but also in algebraic geometry, homological algebra, etc.In this note we expose the basic notions of the language of categories and functors. And the goal is given a proof of Yoneda lemma which is an elementary but deep and central result in category theory. In the end of this note we will try to give an application of it.

## 1. Basic Notions

**Definition 1.1.** *A category $\mathcal{C}$ consists of:*

1) *A set of "objects" $Ob(\mathcal{C})$.*

2) *A set of morphism $Mor(\mathcal{C})$, with a couple of maps* $Mor(\mathcal{C}) \overset{s}{\underset{t}{\rightrightarrows}} Ob(\mathcal{C})$ *, where $s$ and $t$ give the source and target of the morphism respectively. For $X, Y \in Ob(\mathcal{C})$, we denote the set of morphisms from $X$ to $Y$ by $Hom_{\mathcal{C}}(X,Y) := s^{-1}(X) \cap t^{-1}(Y)$*

3) *for each $X \in Ob(\mathcal{C})$, there exists $id_X \in Hom_{\mathcal{C}}(X,X)$ which is the identity morphism on $X$.*

4) *For any $X, Y, Z \in Ob(\mathcal{C})$, given the composition map*

$$\circ : Hom_{\mathcal{C}}(Y,Z) \times Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{C}}(X,Z)$$
$$(f,g) \mapsto f \circ g$$

*satisfying $(i)$ it is associative,*

*$(ii)$ for any morphism $f \in Hom_{\mathcal{C}}(X,Y)$, we have*

$$f \circ id_X = f = id_Y \circ f.$$

Note that $id_X$ is characterized by its property.

We have a similar definition of group isomorphism in category.

**Definition 1.2.** *Morphism $f : X \to Y$ is called a isomorphism if there exists a morphism $g : Y \to X$ such that $f \circ g = id_Y$, $g \circ f = id_X$.*

Now we give some notations.

**Notations**.

1) **0** is the category that the object set and morphism set are both empty,

2) $Isom_{\mathcal{C}}(X,X)$ is the set of isomorphisms from X to Y.

3) $End_{\mathcal{C}}(X) := Hom_{\mathcal{C}}(X,X)$, $Aut_{\mathcal{C}}(X) := Isom_{\mathcal{C}}(X,X)$. Actually $End_{\mathcal{C}}(X)$ is a monoid and $Aut_{\mathcal{C}}(X)$ is a group equiped with the composition.

**Definition 1.3.** *A category $\mathcal{C}'$ is called a subcategory of $\mathcal{C}$ if*

1) *$Ob(\mathcal{C}') \subset Ob(\mathcal{C})$,*

2) *$Mor(\mathcal{C}') \subset Mor(\mathcal{C})$ and the identity morphisms are induced by those in $\mathcal{C}$,*

3) *the source and target maps $Mor(\mathcal{C}') \overset{s}{\underset{t}{\rightrightarrows}} Ob(\mathcal{C}')$ and compositions are induced by those in $\mathcal{C}$*

*Say $\mathcal{C}'$ is a full subcategory if $Hom_{\mathcal{C}'}(X,Y) = Hom_{\mathcal{C}}(X,Y)$*

*Remark* 1.4. In this note we always assume that a university $\mathcal{U}$ has been chosen. We called a category $\mathcal{C}$ is a $\mathcal{U}-$category if for any $X, Y \in Ob(\mathcal{C})$ $Hom_{\mathcal{C}}(X,X)$ is $\mathcal{U}-$small. If $Mor(\mathcal{C})$ is also $\mathcal{U}-$small, then $\mathcal{C}$ itself is $\mathcal{U}-$small. And $\mathcal{C}$ is $\mathcal{U}-$small if and only if $Ob(\mathcal{C})$ is $\mathcal{U}-$small.

We are not going to introduce the Grothendieck universe in this note and will omit the symbol $\mathcal{U}$ and the categories in this note are all $\mathcal{U}-$category if there is no other claims.

We have natural extension of the monomorphism and epimorphism in category theory.

**Definition 1.5.** *Let $X, Y \in Ob(\mathcal{C})$, $f \in Hom_{\mathcal{C}}(X,Y)$*

1) *we say $f$ is a monomorphism if for any object $Z$ and any couple of morphisms $g, h : Z \to X$, $f \circ g = f \circ h \Leftrightarrow g = h$,*

2) *we say $f$ is an epimorphism if for any object $Z$ and any couple of morphisms $g, h : Y \to Z$, $g \circ h = h \circ f \Leftrightarrow g = h$.*

**Definition 1.6.** *For any category $\mathcal{C}$, the opposite category $\mathcal{C}^{op}$ is given by*

1) *$Ob(\mathcal{C}^{op}) = Ob(\mathcal{C})$,*

2) *for any objects $X$, $Y$, $Hom_{\mathcal{C}^{op}}(X,Y) := Hom_{\mathcal{C}}(Y,X)$,*

3) *identity morphism is given the same as $\mathcal{C}$ case,*

4) *the composition of $f \in Hom_{\mathcal{C}^{op}}(Y,Z), g \in Hom_{\mathcal{C}^{op}}(X,Y)$ is given by $f \circ^{op} g := g \circ f$ in $\mathcal{C}$.*

## 2. Functor and Natural transformations

**Definition 2.1.** *Let $\mathcal{C}, \mathcal{C}'$ be two categories. A functor $F : \mathcal{C}' \to \mathcal{C}$ consists of*

1) *map between objects $F : Ob(\mathcal{C}') \to Ob(\mathcal{C})$*

2) *map between morphisms $F : Mor(\mathcal{C}') \to Mor(\mathcal{C})$, such that $sF = Fs, tF = Ft$ and*

$$F(g \circ f) = F(g) \circ F(f), F(id_X) = id_{FX}$$

The definition of composition of functors is given obviously.

**Definition 2.2.** *For functor $F : \mathcal{C}' \to \mathcal{C}$*

1) *we call $F$ is faithful(resp. full) if for any $X, Y \in Ob(\mathcal{C}')$, the map $Hom_{\mathcal{C}'}(X, Y) \to Hom_{\mathcal{C}}(FX, FY)$ is injective(resp. surjective),*

2) *we call $F$ is essentially surjective if for any $Y \in \mathcal{C}$, there exists $X \in \mathcal{C}'$ such that $Y \simeq FX$*

**Definition 2.3** (natural transfomation , or morphism between functors)**.** *Let $F, G : \mathcal{C}' \to \mathcal{C}$ be two functors. A natural transformation $\theta : F \to G$ between them is a family of morphisms*

$$\theta_X \in Hom_{\mathcal{C}}(FX, GX), X \in Ob(\mathcal{C}')$$

*such that for all morphism $f : X \to Y$ in $\mathcal{C}'$, the diagram*

$$\begin{tikzcd} FX \arrow[r, "\theta_X"] \arrow[d, "Ff"'] & GX \arrow[d, "Gf"] \\ FY \arrow[r, "\theta_Y"'] & GY \end{tikzcd}$$

$$\begin{tikzcd} \widetilde{X} \arrow[r, "\exists! \widetilde{T}"] & \widetilde{Y} \\ X \arrow[u, "J_x"] \arrow[r, "T"'] & Y \arrow[u, "J_Y"'] \end{tikzcd}$$

*commutes.*

*It is also visualized by the below diagram:*

$$\begin{tikzcd} \mathcal{C}' \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \theta"] & \mathcal{C} \end{tikzcd}$$

The study of natural transformation is actually the source of category theory.

With the morphism between we can talk about the category of functors, but before that we will give some useful results.

**Definition 2.4.** *If for a couple of functors $\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left, "F"] & \mathcal{C}_2 \arrow[l, bend left, "G"] \end{tikzcd}$, there exist isomorphisms between functors*

$$\theta : FG \simeq id_{\mathcal{C}_2},\ \psi : GF \simeq id_{\mathcal{C}_1}$$

*then we say $G$ is the quasi-inverse functor of $F$ and $F$ is called the equivalence of categories $\mathcal{C}_1$ and $\mathcal{C}_2$*

Then we have the following important result.

**Theorem 2.5.** *For functor $F : \mathcal{C}_1 \to \mathcal{C}_2$, $F$ is an equivalence of categories if and onlf $F$ is fully faithful and essentially surjective.*

To prove this theorem we need some useful concepts.

**Definition 2.6.** *A full subcategory $\mathcal{C}' \subset \mathcal{C}$ is a skeleton of $\mathcal{C}$ if for any $X \in \mathcal{C}$, there exists an isomorphism $X \simeq Y \in Ob(\mathcal{C}')$ where $Y$ is unique. Category that is a skeleton of itself is called a skeleton category.*

**Lemma 2.7.** *For any category $\mathcal{C}$ there exists a skeleton $\mathcal{C}'$ and the inclusion functor $\iota : \mathcal{C}' \hookrightarrow \mathcal{C}$ is an equivalence of categories. Full faithful and essentially surjective functors between skeleton categories are isomorphisms.*

*proof of* **Lemma 2.7**.

**Part I** Choose the representative from every equivalence class in $Ob(\mathcal{C})$ by Axiom of Choice, where the class is given by isomorphisms. Denote the full subcategory consists of these representatives by $\mathcal{C}'$ then we get a skeleton.

For each $X \in Ob(\mathcal{C})$, we can choose isomorphism $\theta_X : X \simeq \kappa(X) \in Ob(\mathcal{C}')$ since $\mathcal{C}'$ is a skeleton itself. Assume that for any $Y \in Ob(\mathcal{C}')$, $\theta_Y = id_Y$. Then,

$$\kappa(f) := \theta_Y \circ f \circ \theta_X^{-1} \in Hom_{\mathcal{C}'}(\kappa(X), \kappa(Y)),\ f \in Hom_{\mathcal{C}}(X, Y)$$

gives the only way to extend $\kappa : Ob(\mathcal{C}) \to Ob(\mathcal{C}')$ to a functor and hence $\theta : id_{\mathcal{C}} \simeq \iota\kappa$ is an isomorphism. Therefore $\iota$ is an equivalence of categories.

**Part II** Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ is the fully faithful and essentially surjective functor between skeleton categories. For any $Z \in \mathcal{C}_2$, there exists $X \in \mathcal{C}_1$ such that $Z \simeq FX$. Hence $Z = FX$. And $FX' \simeq FX$ implies to $X' \simeq X$ for $X, X' \in Ob(\mathcal{C}_1)$ since F is fully faithfull. Therefore F as a map between objects is bijective. Then we can define its inverse. $\square$

*proof of* **Theorem 2.5**.

**Part I** Assume that F is an equivalence of categories and G is the quasi-inverse functor of F.

Then we have $\phi : FG \simeq id_{\mathcal{C}_2},\ \psi : GF \simeq id_{\mathcal{C}_1}$, for any $Z \in \mathcal{C}_2$, $\phi_Z : FGZ \simeq Z$. Hence F is essentially surjective. G isessentially surjective for the same reason.

Consider composition map

$$\begin{array}{ccccccc} Hom(X,Y) & \xrightarrow{F} & Hom(FX,FY) & \xrightarrow{G} & Hom(GFX,GFY) & \xrightarrow{\sim} & Hom(X,Y) \\ f & \longmapsto & Ff & \longmapsto & GFf & \longmapsto & \psi_Y GFf\psi_X^{-1} \end{array}$$

which is a identity map. Hence F has left inverse and G has right inverse. For each $X, Y \in \mathrm{Im}(G) \subset Ob(\mathcal{C}_1)$ where $G : Ob(\mathcal{C}_2) \to Ob(\mathcal{C}_1)$ is seen as a map, map $F : Hom_{\mathcal{C}_1}(X,Y) \to Hom_{\mathcal{C}_2}(FX,FY)$ has right inverse.

Therefore F is fully faithful since any object in $\mathcal{C}_1$ is isomorphic to an image of G.

**Part II** Assume that F is fully faithful and essentially surjective. Choose skeleton given by inclusion functor $\iota_i : \mathcal{C}_i' \to \mathcal{C}_i$ and its quasi-inverse $\kappa_i$ ($i = 1, 2$). Then functor $F' := \kappa_2 \circ F \circ \iota_1 : \mathcal{C}_1' \to \mathcal{C}_2'$ is fully faithful and essentially surjective, moreover, it is an equivalence. Now let $G := \iota_1 \circ F'^{-1} \circ \kappa_2$, then

$$GF = \iota_1 F'^{-1} \kappa_2 F \simeq \iota_1 \kappa_1 \simeq id_{\mathcal{C}_1}$$
$$FG = F \iota_1 F'^{-1} \kappa_2 \simeq \iota_2 \kappa_2 \simeq id_{\mathcal{C}_2}$$

We have the conclusion we need. □

**Definition 2.8.** *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be $\mathcal{U}-$category, we define functor category $Fct(\mathcal{C}_1, \mathcal{C}_2)$ with*
1) *objects are functors between $\mathcal{C}_1$ and $\mathcal{C}_2$,*
2) *morphisms of any object F, G is natural transformation $\theta F \to G$,*
3) *the composition is given by the **Case I** composition of natural transformation.*

We introduce the Hom functor which will be used in the later of the application part.

**Example 2.1** (Hom functor)**.** *Given category $\mathcal{C}$, $(X, Y) \mapsto Hom_{\mathcal{C}}(X, Y)$ define the functor*

$$Hom_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \boldsymbol{Set}$$

.

*Therefore, for any couple of morphisms in $\mathcal{C}$*

$$f : X' \to X, \ \ g : Y' \to Y$$

*we have*

$$Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{C}}(X', Y')$$
$$\phi \mapsto g\phi f$$

We will see such a functor in algebraic topology more often.

## 3. Yoneda Lemma and its Application

**Definition 3.1.** *Let **Set** be the category of $\mathcal{U}-$set. One defines the categories*

$$\mathcal{C}^{\vee} := Fct(C^{op}, \boldsymbol{Set}), \ \mathcal{C}^{\wedge} := Fct(C^{op}, \boldsymbol{Set}^{op}) = Fct(C, \boldsymbol{Set})^{op}$$

*and the functors*

$$h_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^{\wedge}$$
$$S \mapsto Hom_{\mathcal{C}}(\cdot, S)$$
$$k_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^{\vee}$$
$$S \mapsto Hom_{\mathcal{C}}(S, \cdot)$$

*Respectively we have the evaluation functors*

$$\begin{aligned} ev^{\wedge} : \mathcal{C}^{op} \times \mathcal{C}^{\wedge} &\to \boldsymbol{Set} \\ (S, A) &\mapsto A(S) \\ ev^{\vee} : (\mathcal{C}^{\vee})^{op} \times \mathcal{C} &\to \boldsymbol{Set} \\ (B, S) &\mapsto B(S) \end{aligned}$$

$\mathcal{C}^{\vee}$ and $\mathcal{C}^{\wedge}$ are both $\mathcal{U}-$categories if $\mathcal{C}$ is $\mathcal{U}-$small. Sometimes $\mathcal{C}^{\vee}$ is called the presheaf category on $\mathcal{C}$.

Now we give the Yoneda lemma.

**Theorem 3.2** (Yoneda Lemma)**.** *For $S \in Ob(\mathcal{C})$ and $A \in \mathcal{C}^{\vee}$, the map*

$$Hom_{\mathcal{C}^{\wedge}}(h_{\mathcal{C}}(S), A) \to A(S)$$

$$[Hom_{\mathcal{C}}(\cdot, S) \xrightarrow{\phi} A(\cdot)] \mapsto \phi_S(id_S)$$

*is bijective. And it gives the isomorphism between functors*

$$Hom_{\mathcal{C}^{\wedge}}(h_{\mathcal{C}}(\cdot), \cdot) \xrightarrow{\sim} ev^{\wedge}.$$

*The functor $h_{\mathcal{C}}$ is full faithful.*

*Similarly we have the isomorphism between functors*

$$Hom_{\mathcal{C}^{\vee}}(\cdot, k_{\mathcal{C}}(\cdot)) \xrightarrow{\sim} ev^{\vee}.$$

*The functor $k_{\mathcal{C}}$ is full faithful. $h_{\mathcal{C}}$ and $k_{\mathcal{C}}$ are usually calle Yoneda embedding.*

*Proof.* We prove the first part since $(C^{\vee})^{op} = (C^{op})^{\wedge}$ gives the other case.

Let $\varphi$ and $\psi$ be constructed by two chains of morphisms:

$$\varphi : Hom_{\mathcal{C}^{\wedge}}(h_{\mathcal{C}}(S), A) \to Hom_{\mathbf{Set}}(Hom_{\mathcal{C}}(S, S), A(S)) \to A(S)$$

$$\psi : Hom_{\mathcal{C}}(T, S) \to Hom_{\mathbf{Set}}(A(S), A(T)) \to A(T)$$

Let $u_s := \phi_S(id_S) \in A(S)$, diagram

$$\begin{tikzcd} id_S \arrow[d, mapsto] & \in & Hom_{\mathcal{C}}(S,S) \arrow[r, "\phi_S"] \arrow[d, "f^*"'] & A(S) \arrow[d, "A(f)"] & \ni & u_S \arrow[d, mapsto] \\ f & \in & Hom_{\mathcal{C}}(T,S) \arrow[r, "\phi_T"'] & A(T) & \ni & \phi_T(f) \end{tikzcd}$$

is commutative. Hence $\varphi$ and $\psi$ are inverse to each other.

Let $A = h_{\mathcal{C}}(T)$, then

$$Hom_{\mathcal{C}^{\wedge}}(h_{\mathcal{C}}(S), h_{\mathcal{C}}(T)) \to h_{\mathcal{C}}(T)(S) = Hom_{\mathcal{C}}(S, T)$$

is bijective. $\square$

At the end of this note we give an example as an application to Yoneda Lemma.

**Proposition 3.3.** *Consider a morphism of distinguished triangles:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[r, "g"] \arrow[d, "\beta"'] & Z \arrow[r, "h"] \arrow[d, "\gamma"'] & T(X) \arrow[d, "T(\alpha)"'] \\
X' \arrow[r, "f'"] & Y' \arrow[r, "g'"] & Z' \arrow[r, "h'"] & T(X')
\end{tikzcd}$$

*If $\alpha$ and $\beta$ are isomorphisms, then so is $\gamma$.*

*Proof.* Apply $Hom(W,\cdot)$ to the diagram. Let $\widetilde{X}$ instead of $Hom(W,X)$ and $\widetilde{\alpha}$ instead of $Hom(W,\alpha)$. We have

$$\begin{tikzcd}
\widetilde{X} \arrow[r, "\widetilde{f}"] \arrow[d, "\widetilde{\alpha}"'] & \widetilde{Y} \arrow[r, "\widetilde{g}"] \arrow[d, "\widetilde{\beta}"'] & \widetilde{Z} \arrow[r, "\widetilde{h}"] \arrow[d, "\widetilde{\gamma}"'] & \widetilde{T(X)} \arrow[d, "\widetilde{T(\alpha)}"'] \\
\widetilde{X'} \arrow[r, "\widetilde{f'}"] & \widetilde{Y'} \arrow[r, "\widetilde{g'}"] & \widetilde{Z'} \arrow[r, "\widetilde{h'}"] & \widetilde{T(X)'}
\end{tikzcd}$$

The rows are exact in view of the proceeding proposition, and$\widetilde{\alpha}, \widetilde{\beta}, \widetilde{T(\alpha)}, \widetilde{T(\beta)}$ are isomorphisms. Therefore $\widetilde{\gamma}$ is an isomorphism. Then $\gamma$ is an isomorphism by the Yoneda Lemma □

**Acknowledgement** The LaTeX template is downloaded on the website of professor Zuoqin Wang. And the note is based on the book written by professor Wen Wei Li, which is the first one in references. It is really a excellent book and I enjoy it all the time.

## References

[1] : ( ), . . , , 2019.
[2] Pierre Schapira: Categories and Homological Algebra. Online Lecture.
[3] J. P. May: A Concise Course in Algebraic Topology. Chicago Lectures in Mathematics. *The University of Chicago Press, Chicago, IL*, 1999.

School of Mathematical Sciences, Shandong University, JiNan, 250100, P.R. China,
*Email address*: `weiziwei@mail.sdu.edu.cn`