# ALGEBRAIC GEOMETRY

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ABSTRACT. This is my personal note of course *Advanced Geometry 3* taught by Prof. Fabio Perroni in spring 2024 at University of Trieste. And as the title, this course is focused on algebraic geometry, especially on algebraic varieties over $\mathbb{C}$. The latest version can be found on my website:

This note is not completely following what's on the blackbord. The materials might be re-organized or have some slightly changes. I will also add some supplementary materials which will be marked by *.

Attention: there are definitely a considerable number of mistakes in this note, and all due to me. If you have any comments, corrections or suggestions, please send them to zwei@sissa.it. Any feedback is appreciated!

## CONTENTS

## 0. Preface

The main reference is [4].

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## 1. Affine algebraic varieties

### 1.1. Algebraic subsets.

All ring will be assumed as commutative ring with unit.

**Definition 1.1.1.** A **closed algebraic subset** $X \subset \mathbb{C}^n$ is the set of zeroes of a finite numbers of polynomials

$$X = \{a = (a_1, \dots, a_n) \mid f_i(a_1, \dots, a_n) = 0, \forall i = 1, \dots, m\}$$

where $f_i \in \mathbb{C}[x_1, \dots, x_n]$.

It is also denoted by $V(f_1, \dots, f_m)$.

*Remark* 1.1. The ideal generated by $f_1, \dots, f_m$ is

$$I = (f_1, \dots, f_m) = \{\sum g_i f_i \mid g_i \in \mathbb{C}[x_1, \dots, x_n]\}.$$

And the set of zeroes of $I$ is $X = V(I) = \{a \in \mathbb{C}^n \mid f(a) = 0, \forall f \in I\}$.

By Hilbert basis theorem, every ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ is f.g., i.e., $\exists f_1, \dots, f_m$ s.t. $I = (f_1, \dots, f_m)$. Hence we will talk about $V(I), I \subset \mathbb{C}[x_1, \dots, x_n]$.

**Proposition 1.1.1.** Let $I_1, I_2, \{I_\alpha\}_{\alpha \in A}$ be ideas of $\mathbb{C}[x_1, \dots, x_n]$. $a = (a_1, \dots, a_n) \in \mathbb{C}^n$. Then the following hold true.

(1) If $I_1 \subset I_2$, then $V(I_2) \subset V(I_1)$,
(2) $V(I_1 \cup I_2) = V(I_1 \cap I_2) = V(I_1 \cdot I_2)$ $(I_1 I_2 = \{fg \mid f \in I_1, g \in I_2\})$,
(3) $V(\sum_{\alpha \in A} I_\alpha) = \bigcap_{\alpha \in A} V(I_\alpha)$,
(4) If $\mathfrak{m}_a := (x_1 - a_1, \dots, x_n - a_n)$, then $V(\mathfrak{m}_a) = \{a\}$,
(5) $V(\sqrt{I}) = V(I)$
($\sqrt{I} = \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f^K \in I$ for some $K > 0\}$).

*Proof.* (1) evident.

(2) Since $I_1 I_2 \subset I_1 \cap I_2 \subset I_1, I_2$, (1) implies that

$$V(I_1 I_2) \supset V(I_1 \cap I_2) \supset V(I_1), V(I_2).$$

Conversely, let $a \in V(I_1 I_2)$. If $a \notin V(I_1 \cap I_2)$, then $\exists f \in I_1 \cap I_2$ s.t. $f(1) \neq 0$. Then $f^2(a) \neq 0$, but $f^2 \in I_1 I_2$. The remain is similar.

(3) $(\subset)$ $I_\alpha \subset \sum I_\alpha \forall \alpha$ , hence $V(\sum I_\alpha) \subset V(I_\alpha) \forall \alpha$.
$(\supset)$ Immediately.

(4) $b \in V_{\mathfrak{m}_a}$ iff $b_i - a_i = 0,\ \forall i$.

(5) $(\subset)$ $\sqrt{I} \supset I$
$(\supset)$ Let $a \in V(I)$. If $a \notin V(\sqrt{I})$, then $\exists f \in \sqrt{I}$ s.t. $f(a) \neq 0$. Hence $f^K(a) \neq 0$. contradiction. □

*Remark* 1.2. (1) It can happen that $I_1 I_2 \subsetneqq I_1 \cap I_2$,
(2) $\sqrt{I}$ is an ideal and it is called the radical of $I$,

(3) **Proposition** 1.1.1(2), (3) implies that algebraic subsets of $\mathbb{C}^n$ satisfy the axiom of closed sets of a topology on $\mathbb{C}^n$ and it is called **Zariski topology**.

*Remark* 1.3. The Zariski topology is not Hausdorff in general. However, if the base field $\Bbbk$ is finite, then the Zariski topology is Hausdorff.

## 1.2. **Affine varieties.**

**Definition 1.2.1.** An **affine variety** is a closed algebraic set $X \subset \mathbb{C}^n$ of the form $X = V(P)$ with $P$ prime ideal.

**Example 1.2.1.** Let $f \in \mathbb{C}[x_1, \dots, x_n]$ be irrd. Then $V(f) = V((f)) \subset \mathbb{C}^n$ is an affine variety and it is called an hypersurface of $\mathbb{C}^n$.

Note that if $f$ is not irrd then $(f)$ is not prime.

**Example 1.2.2.** Let $g_2, \dots, g_n \in \mathbb{C}[x_1]$. Consider $X := \{(a, g_2(a), \dots, g_n(a)) \in \mathbb{C}^n \mid a \in \mathbb{C}\}$.

It is a closed algebraic subset by $X = V(x_2 - g_2(x_1), \dots, x_n - g_n(x_1))$. And since $\mathbb{C}[x_1, \cdots x_n]/(x_2 - g_2(x_1), \dots, x_n - g_n(x_1)) \cong \mathbb{C}[x_1]$ which is a integral domain. Hence $X$ is an affine variety and it is called rational space curve.

**Exercise 1.2.1.** Let $\varphi_1, \dots, \varphi_k \subset \mathbb{C}[x_1, \dots, x_n]$ be homogeneous polynomials of degree 1. Suppose that $\{\varphi_i\}$ are linearly independent as elements of $(\mathbb{C}^n)^*$. Then for any $b_1, \dots, b_k \in \mathbb{C}$, fixed $X = V(\varphi_1 - b_1, \dots, \varphi_k - b_k)$, which is the set of solutions of the linear system $\varphi_i = b_i$.

Prove that $X$ is an affine variety. It is called a linear subspace of $\mathbb{C}^n$ of dimension $n - k$.

Now for any subset $S \subset \mathbb{C}^n$, we can define

$$I(S) := \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f(a) = 0, \forall a \in S\}.$$

We have the following amazing theorem.

**Theorem 1.2.1** (Hilbert's Nullstellensatz)**.** For any ideal $J \subset \mathbb{C}[x_1, \dots, x_n]$,

$$I(V(J)) = \sqrt{J}.$$

In particular, if the ideal $J$ is prime, then $I(V(J)) = J$.

*Remark* 1.4. (1) The theorem holds true for any algebraic closed field(See [1]).
(2) It fails if the field is not algebraic closed. For example, take $\Bbbk = \mathbb{R}$, $I(V(x^2 + y^2 + 1)) = \mathbb{R}[x, y]$ where $V(x^2 + y^2 + 1)$ is actually empty.
(3) (Study's lemma) Let $\Bbbk = \bar{\Bbbk}$. If $f \in \Bbbk[x_1, \dots, x_n]$ is irrd, then $I(V(f)) = (f)$

**Lemma* 1.2.1.** If $Y_1 \subset Y_2$ are algebraic subsets of $\mathbb{C}^n$, then $I(Y_1) \supset I(Y_2)$.

**Proposition* 1.2.1.** $V(I(S)) = \bar{S}$

*Proof.* On the one hand we have $S \subset V(I(S))$ where by definition $S$ is closed. Hence $\bar{S} \subset V(I(S))$. On the other hand, recall that the closure

$$\bar{S} = \bigcap W$$

where $W$ runs over all algebraic subsets of $\mathbb{C}^n$ that contain $S$. And we can write $W = V(J)$ for some ideal $J$. Then $S \subset V(J)$ and by **Lemma* 1.2.1**, we have $I(S) \supset I(V(J)) \supset J$. Then by **Proposition 1.1.1**(1), $W = V(J) \subset V(I(S))$ for any such $W$. It follows the statement. □

**Definition 1.2.2.** Let $V(P) \subset \mathbb{C}^n$ be an affine variety. And let $\Bbbk \subset \mathbb{C}$ be a subfield. A point $a \in V(P)$ is called a **$\Bbbk$-generic point** if the following condition holds true: $\forall f \in \Bbbk[x_1, \ldots, x_n]$, if $f(a) = 0$, then $f \in P$.

**Example 1.2.3.** Consider $g_2, \ldots, g_n \in \mathbb{Q}[x_1]$ and let $X = V(x_2 - g_2(x_1), \ldots, x_n - g_n(x_1))$ be the rational space curve. Let $a := (\pi, g_2(pi), \ldots, g_n(\pi)) \in X$. Then $a$ is $\mathbb{Q}$-generic.

Indeed, let $f \in \mathbb{Q}[x_1, \ldots, x_n]$ is s.t. $f(a) = 0$. But $\varphi := f(x_1, x_2 - g_2(x_1), \ldots, x_n - g_n(x_1)) \in \mathbb{Q}[x_1]$, hence $\varphi = 0$. It follows that $\varphi \in P$.

**Proposition 1.2.1.** Let $V(P)$ be an affine variety. Let $\Bbbk \subset \mathbb{C}$ be a subfield s.t. $tr.\deg \mathbb{C}|\Bbbk = \infty$. Then there exists $a \in V(P)$ a $\Bbbk$-generic point.

*Proof.* Let $P = (f_1, \ldots, f_m)$ and, WLOG, assume that $f_1, \ldots, f_m \in \Bbbk[x_1, \ldots, x_n]$

(Otherwise let $\Bbbk'$ be the minimal subfield of $\mathbb{C}$ containing $\Bbbk$ and the coefficients of $f_1, \ldots, f_m$. Then $tr.\deg \mathbb{C}|\Bbbk' = \infty$ and any $\Bbbk'$-generic point is also a $\Bbbk$-generic point).

Let $P_0 = P \cap \Bbbk[x_1, \ldots, x_n]$, which is prime. And let $K$ be the fraction field of $\Bbbk[x_1, \ldots, x_n]/P_0$.

Since for any $f/g \in K$, it is a root of $gy - f \in \Bbbk(\bar{x}_1, \ldots, \bar{x}_n)[y]$, where $\bar{x}_1, \ldots, \bar{x}_n$ is the isomorphic class in $\Bbbk[x_1, \ldots, x_n]/P_0$. We have that $K|\Bbbk(\bar{x}_1, \ldots, \bar{x}_n)$ is algebraic. Hence $tr.\deg K|\Bbbk \leq n < \infty$.

In this situation, there exists a field homomorphism

$$\phi : K \to \mathbb{C}$$

s.t. $\phi|_K = \mathrm{id}_K$(Indeed, let $\lambda_1, \ldots, \lambda_\delta \in K$ be a transcendence basis for $K|\Bbbk$. Let $z_1, \ldots, z_\delta \in \mathbb{C}$ be algebraically independent over $\Bbbk$. The map $\lambda_i \mapsto z_i$, $\forall i$ extends to a unique field homomorphism from $K \to \mathbb{C}$. See [5] Thm33 p.102).

Let $a_i := \phi(\bar{x}_i) \in \mathbb{C}$.

**Claim.** $a = (a_1, \ldots, a_n) \in X$ is a $\Bbbk$-generic point.

**Indeed.** First we have that $f_i(\bar{x}_1, \ldots, \bar{x}_n) = 0$ $i = 1, \ldots, m$ in $\Bbbk[x_1, \ldots, x_n]/P_0$. It follows that

$$0 = \phi(f_i(\bar{x}_1, \ldots, \bar{x}_n)) = f_i(\phi(\bar{x}_1), \ldots, \phi(\bar{x}_n)) = f_i(a_1, \ldots, a_n) \; i = 1, \ldots, m.$$

Hence $a \in X$.

Now let $f \in \Bbbk[x_1, \ldots, x_n]$ s.t. $f(a) = 0$. If $f \notin P_0$, then $[f] \in \Bbbk[x_1, \ldots, x_n]$ is nonzero. Applying $\phi$ to this class we get that $f(a) = 0$, which is contradiction. □

*Remark* 1.5. One could have defined $\Bbbk$-generic point for all $V(I)$ where $I \subset \mathbb{C}[x_1, \ldots, x_n]$ is any ideal. But in the following case, it doesn't exist.

Let $I = (xy) \subset \mathbb{C}[x, y]$ be an ideal and $a = (a_1, a_2)$. If $a_2 = 0$, then for $y \subset \Bbbk[x, y]$, $\forall \Bbbk \subset \mathbb{C}$, $y(a) = a_2 = 0$, but $y \notin I$. It is similar when $a_1 = 0$.

Now we can give a proof of **Theorem** 1.2.1.

*Proof.* **Step 1.** Let $J = P$ be prime. Let $f \in I(V(P))$ and $\Bbbk$ be the minimal subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and the coefficients of $f$. Then $tr.\deg \mathbb{C}/\Bbbk = \infty$ and by **Proposition** 1.2.1, there exists a $\Bbbk$-generic point $a \in X$. And since $f \in I(X)$, $f(a) = 0$, then $f \in P$.

**Step 2.** Not let $J$ be any ideal and $f \in I(V(J))$. Consider the primary rep

$$\sqrt{J} = P_1 \cap \cdots \cap P_N.$$

Then $V(J) = V(\sqrt{J}) = V(P_1) \cup \cdots \cup V(P_N)$. So $f \in I(V(P_i))$ $i = 1, \ldots, N$. Then by **Step 1.**, $f \in P_i$ $i = 1, \ldots, N$, and $f \in \sqrt{I}$. □

**Corollary 1.2.1.** There is an order-reversing correspondence

$$\{J \subset \mathbb{C}[x_1, \ldots, x_n] \mid J = \sqrt{J}\} \leftrightarrow \{\text{closed algebraic subset of } \mathbb{C}^n\}$$
$$J \mapsto V(J)$$
$$I(X) \leftarrow\!\shortmid X$$

**Definition 1.2.3.** Let $X = V(P) \subset \mathbb{C}^n$ be an affine variety with $P \subset \mathbb{C}[x_1, \ldots, x_n]$ prime ideal. The ring $R_X := \mathbb{C}[x_1, \ldots, x_n]/P$ is the **affine coordinate ring** of $X$.

**Corollary 1.2.2.** In this situation, $R_X$ is isomorphic to the ring of functions $X \to \mathbb{C}$ which are restrictions of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$.

*Proof.* Let $\mathcal{F}(X) := \{F : X \to \mathbb{C} \mid$ s.t. $\exists f \in \mathbb{C}[x_1, \ldots, x_n]$ s.t. $F(a) = f(a), \forall a$.

Restriction yields an surjective homomorphism

$$\mathbb{C}[x_1, \ldots, x_n] \to \mathcal{F}(X) \to 0$$

and its kernel is $P$. Then we have the isomorphism. □

## 1.3. Tangent spaces of affine varieties.

**Definition 1.3.1.** Let $X = V(P)$ be an affine variety with $P \in \mathbb{C}[x_1, \ldots, x_n$ prime. Let $a \in X$, the **Zariski tangent space** of $X$ at $a$ is the linear subspace of $\mathbb{C}^n$ given by the equations

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) = 0, , \ \forall f \in P$$

and denoted by $T_{X,a}$[1].

[1] I prefer $T_a X$ so I might change this symbol hereafter

*Remark* 1.6. (1) If $P = (f_1, \dots, f_m)$, then

$$T_a X = V(\{\sum_{i=0}^{m} \frac{\partial f_j}{\partial x_i}(a)(x_i - a_i) = 0 \mid j = 1, \dots, m\}).$$

Indeed, ($\subset$) is obvious. ($\supset$) If $(b_1, \dots, b_n) \in \mathbb{C}^n$ is s.t.

$$\sum_{i=0}^{n} \frac{\partial f_j}{\partial x_i}(a)(b_i - a_i) = 0, \ \forall j = 1, \dots, m.$$

Let $f \in P$, we can write $f = \sum_{i=1}^{m} f_i g_i$ for some $g_i \in \mathbb{C}[x_1, \dots, x_n]$. Then

$$\sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(a)(b_i - a_i) = \sum_{i=0}^{n} \sum_{i=1}^{m} \frac{\partial f_i g_i}{\partial x_i}(a)(b_i - a_i) = 0.$$

(2) $T_a X \subset \mathbb{C}^n$ is an affine subspace passing through $a$.

1.4. **Tangent spaces and derivations.** Let $R := R_X$ be the affine coordinate ring of $X$.

Recall that a **derivation** of $R$ (centered) at $a \in X$ is a $\mathbb{C}$-linear map

$$D : R \to \mathbb{C}$$

s.t.

(1) $D(fg) = f(a)D(g) + g(a)D(f),, \ \forall f, g \in R,$
(2) $D(\lambda) = 0, \ \forall \lambda \in \mathbb{C}.$

Let $Der_{R,a}$ be the set of such derivations.

*Remark* 1.7. $Der_{R,a}$ is a vector space over $\mathbb{C}$.

**Proposition 1.4.1.** Let $\bar{x}_1, \dots, \bar{x}_n \in R$ be the classes of $x_1, \dots, x_n$. Then the map

$$\varphi : Der_{R,a} \to \mathbb{C}^n$$
$$D \mapsto (D(\bar{x}_1), \dots, D(\bar{x}_n))$$

is an injective linear map and its image is $T_a X - a$.

*Proof.* (exer.)

$\square$

1.5. **Appendix. primary decomposition.** Now We recall some commutative algebra [5].

**Definition 1.5.1.** Let $R$ be a ring and $I \subset R$ be an ideal of $R$. $I$ is called **primary** if whenever $a, b \in R$ are such that $ab \in I$ and $a \notin I$, then $b \in \sqrt{I}$.

We have immediately that the radical of a primary ideal is prime.

**Theorem 1.5.1** (Lasker-Noether decomposition theorem)**.**

(1) Let $R$ be a Noetherian ring, then every ideal $I \subset R$ admits the so called primary representation as
$$I = Q_1 \cap \cdots \cap Q_N$$
where $Q_i$'s are primary ideals of $R$.
Moreover, we can find $Q_1, \ldots, Q_N$ s.t.no $Q_i$ contains $\bigcap_{j \neq i} Q_j$ and the associated prime ideals $\sqrt{Q_1}, \ldots, \sqrt{Q_n}$ are distinct. In this case it is called irredundant primary representation.

(2) Let $R$ be a ring and $I \subset R$ be an ideal that admits an irredundant primary representation
$$I = Q_1 \cap \cdots \cap Q_N.$$
Then $I = \sqrt{I}$ iff $Q_1, \ldots, Q_N$ are prime.

**Theorem 1.5.2.** Let $R$ be a ring and $I \subset R$ be an ideal admitting an irreduandant primary representation
$$I = Q_1 \cap \cdots \cap Q_N.$$
Then the prime ideals $P_i := \sqrt{Q_i}$ are uniquely determined by $I$. And they are called the associated primes of $I$.

**Example 1.5.1.** Let $I = (x^2, y) \subset \mathbb{C}[x, y]$ be an ideal. It has an irreduandant primary representation
$$I = (x^2) \cap (y).$$
And its radical is
$$\sqrt{I} = (x) \cap (y).$$

More generally, let $f \in \mathbb{C}[x_1, \ldots, x_n]$ and write $f = g_1^{k_1} \cdots g_N^{k_N}$ where $g_i$ are irrd and not associated to each other. Then we have
$$(f) = (g_1^{k_1}) \cap \cdots \cap (g_N^{k_N}).$$
And its radical is
$$\sqrt{(f)} = (g_1) \cap \cdots \cap (g_N) = (g_1 \cdot g_N).$$

**Example 1.5.2.** Let $\Bbbk$ be any field. Consider the polynomial ring $\Bbbk[x, y]$ and ideal $I = (x^2, xy)$. Then for any $c \in \Bbbk$,
$$I = (x) \cap (y - cx, x^2)$$
is an irredundant primary representation of $I$.

**Question 1.5.1.** What are the associated primes of $I$?

**Corollary 1.5.1.** Let $I \subset \mathbb{C}[x_1 \ldots, x_n]$ be a radical ideal. Then there exists unique prime ideals $P_1, \ldots, P_N \subset \mathbb{C}[x_1 \ldots, x_n]$ s.t.

$$I = P_1 \cap \cdots \cap P_N$$

and $P_i \neq P_j,\ \forall i \neq j$.

### 1.6. **Appendix. transcendental extension.**

**Definition 1.6.1.** An extension $K|\mathbb{k}$ is **transcendental** if it is not algebraic(i.e. if $\exists \alpha \in K$ not algebraic over $\mathbb{k}$).

**Example 1.6.1.** (1) $\mathbb{Q}(\pi)|\mathbb{Q}$ is transcendental.
(2) $\mathbb{Q}(i)|\mathbb{Q}$ is algebraic.
(3) Let $\mathbb{k}$ be any field and $K$ be the fraction field of $\mathbb{k}[x_1, \ldots, x_n]$, which is $K = \mathbb{k}(x_1, \ldots, x_n)$. Then $K|\mathbb{k}$ is transcendental.

**Definition 1.6.2.** Let $K|\mathbb{k}$ be a field extension. Let $L \subset K$. The elements of $L$ are said to be **algebraically independent** over $\mathbb{k}$ if $\forall \alpha_1, \ldots, \alpha_N \in L$, there is no $f \in \mathbb{k}[x_1, \ldots, x_n]$ s.t. $f(\alpha_1, \ldots, \alpha_N) = 0$. In this case, $L$ is called a **transcendental set** over $\mathbb{k}$.

**Definition 1.6.3.** A **transcendental basis** for $K|\mathbb{k}$ is a transcendental set $L \subset K$ over $\mathbb{k}$ that is not contained in any bigger transcendental set.

*Remark* 1.8. $L \subset K$ is a transcendental basis for $K|\mathbb{k}$ iff $K|\mathbb{k}(L)$ is algebraic.

**Example 1.6.2.** $\{x_1, \ldots, x_n\} \in \mathbb{k}(x_1, \ldots, x_n)$ form a transcendental basis for $\mathbb{k}(x_1, \ldots, x_n)|\mathbb{k}$.

**Theorem 1.6.1.** There exists a transcendental basis for any field extension. Moreover, any two transcendental basis have the same cardinality.

See Chapter II Sec.12 in [5] for the proof.

**Definition 1.6.4.** The cardinality of any transcendental basis for $K|\mathbb{k}$ is called the **transcendental degree** of $K|\mathbb{k}$, denoted by $tr.\deg(K|\mathbb{k})$.

*Remark* 1.9. $tr.\deg \mathbb{R}|\mathbb{Q} = tr.\deg \mathbb{C}|\mathbb{Q} = \infty$

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