CLASSIFICATION OF ROOT SYSTEMS

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1. Basic Notions

Definition 1.1. *Let $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$ be a base of the root system Φ. The matrix $(\langle \alpha_i, \alpha_j \rangle)$ is called the **Cartan matrix** of Φ. And its entries are called **Cartan integers***

The matrix obviously depends on the ordering of Δ, and it is nonsingular following the explicit forms of cartan integers immediately. But it is independent of the base. We have the following proposition.

Proposition 1.2. *The Cartan matrix is independent of the choice of Δ.*

Proof. Let Δ' be another base of Φ.
Thm 10.3(b) $\Rightarrow \sigma(\Delta') = \Delta$ for some $\sigma \in \mathscr{W}$.
Since $(\sigma(\alpha), \sigma(\beta)) = (\alpha, \beta)$ for all reflections and $\alpha, \beta \in E$. We have $\langle \sigma(\alpha_i), \sigma(\alpha_j) \rangle = \langle \alpha_i, \alpha_j \rangle$ for $1 \leq i, j \leq \ell, \sigma \in \mathscr{W}$.
$\square$

From Cartan matrix we can characterize the corresponding Φ completely.

Theorem 1.3. *Let $\Phi \subset E$ be a root system with base $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$. Let $\Phi' \subset E'$ be another root system with base $\Delta' = \{\alpha'_1, \cdots, \alpha'_\ell\}$.*
If $\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle$ for any $1 \leq i, j \leq \ell$ Then

$$\Delta \xleftrightarrow{1:1} \Delta'$$

extends to an isomorphism

$$\phi : E \to E'$$

uniquely that satisfies the following:
(1) $\phi(\Phi) = \Phi'$,
(2) $\langle \phi(\alpha), \phi(\beta) \rangle = \langle \alpha, \beta \rangle \ \forall \alpha, \beta \in \Phi$,
Therefore, the Cartan matrix of Φ determines Φ up to isomorphism.

Proof. (1) By definition, we have the unique vector space isomorphism

$$\phi : E \to E'$$
$$\sum_{i=1}^{\ell} \lambda_i \alpha_i \mapsto \sum_{i=1}^{\ell} \lambda_i \alpha'_i$$

For $\alpha, \beta \in \Delta$, Let $\alpha' = \phi(\alpha), \beta' = \phi(\beta)$. Then

$$\begin{aligned}\sigma_{\phi(\alpha)}(\phi(\beta)) = \sigma_{\alpha'}(\beta') &= \beta' - \langle \beta', \alpha' \rangle \alpha' \\ &= \phi(\beta) - \langle \phi(\beta), \phi(\alpha) \rangle \phi(\alpha) \\ &= \phi(\beta - \langle \beta, \alpha \rangle \alpha) \\ &= \phi(\sigma_\alpha(\beta))\end{aligned}$$

Then for each $\alpha \in \Delta$ the following diagram

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[d, "\sigma_\alpha"'] & E' \arrow[d, "\sigma_{\phi(\alpha)}"] \\ E \arrow[r, "\phi"'] & E' \end{tikzcd}$$

commutes.

It implies the map

$$\begin{aligned} \mathscr{W} &\to \mathscr{W}' \\ \sigma &\mapsto \sigma' = \phi \circ \sigma \circ \phi^{-1} \end{aligned}$$

is an isomorphism.

Since for $\beta \in \Phi$, $\exists \sigma \in \mathscr{W}, \alpha \in \Delta$, s.t. $\sigma(\alpha) = \beta$, we have $\phi(\beta) = \phi \circ \sigma \circ \phi^{-1}(\phi(\alpha))$. Hence $\phi(\Phi) = \Phi'$.

(2) For $\alpha \in \Phi$ and $\beta \in \Delta$, we have the $\langle \sigma(\alpha), \sigma(\beta) \rangle = \langle \alpha, \beta \rangle$ since the $\langle -, - \rangle$ is linear on the first coordinate.

For any $\beta \in \Phi$, there exists $\sigma \in \mathscr{W}$ s.t. $\sigma(\beta) \in \Delta$. Now assume that $\beta \in \Phi$ is arbitrary, then there exists $\sigma \in \mathscr{W}$ s.t. $\sigma(\alpha_i) = \beta$. Then

$$\phi(\beta) = \phi(\sigma(\alpha_i)) = \sigma'(\phi(\alpha_1)) = \sigma'(\alpha_i')$$

Hence we have

$$\begin{aligned}\langle \phi(\alpha), \phi(\beta) \rangle &= \langle \phi(\alpha), \sigma'(\alpha_i') \rangle = \langle \sigma'^{-1}(\phi(\alpha)), \alpha_i' \rangle \\ &= \langle \sigma^{-1}(\alpha), \alpha_i \rangle = \langle \alpha, \sigma(\alpha_i) \rangle \\ &= \langle \alpha, \beta \rangle\end{aligned}$$

□

Actually this turns to the following

$\{root\ systems\} \xleftrightarrow{1-1} \{Cartan\ matrices\}$

We'll see a slightly different version later.

2. Coxeter graph and Dynkin diagram

We first give some basic notions from Combinatorics.

Definition 2.1. *A **graph** is a pair (V, E), where V is a set and $E \subset \mathcal{P}(V)$ consists of set with two element.*

*We call elements of V **vertices** and elements of E **edges**.*

Definition 2.2. ***Coxeter graph*** *of Φ is a graph with ℓ vertices, the ith joined to the jth by $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle$ edges*

The Coxeter graph determines the number $\langle \alpha_i, \alpha_j \rangle$ if all roots have the equal length. It fails if there is more than one root length. Hence we introduce the **Dynkin diagram** by adding arrow from long root to the short root in the corresponfing Coxeter graph.

Then by the definition of Dynkin diagram we can extend the corresponding relation to the following

$$\{\textit{root systems}\} \xleftrightarrow{1-1} \{\textit{Cartan matrices}\} \xleftrightarrow{1-1} \{\textit{Dynkin diagrams}\}$$

3. CLASSIFICATION OF ROOT SYSTEMS

Now it turns to the main theorem of this note, we will use the Dynkin diagrams to classify the root system. It is amazing that one can use point, line and arrow to construct the objects which seems to be very abstract.

Since it is quite difficult to deal with the general root systems, we may try to make it simpler. Actually the following theorem shows that it is sufficient to consider the irreducible case.

Theorem 3.1. *For root system Φ of E there exists subspaces $E_1, \cdots E_t$ of E and unique decomposition $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_t$ where $\Phi_i \subset E_i$ are irreducible root systems of E_i such that $E = E_1 \oplus \cdots \oplus E_t$ is an orthogonal direct sum.*

Proof. Let $\Delta = \{\alpha_1, \cdots, \alpha_\ell\}$ be a base of Φ. Let $\Delta = \Delta_1 \sqcup \cdots \sqcup \Delta_t$ where $(\Delta_i, \Delta_j) = 0$ for all $i \neq j$. The corresponding Coxeter graphs of these Δ_i are the connected components of the Coxeter graphs of Φ.

Then let $E_i = \mathrm{Span}_{\mathbb{R}}(\Delta_i)$ and $\Phi_i = \mathrm{Span}_{\mathbb{Z}}(\Delta_i) \cap \Phi$. Φ_i is a root system of E_i, by definition.

Since Δ_i is irreducible, Φ_i is irreducible. And $(\Delta_i, \Delta_j) = 0$ Hence $E = E_1 \oplus \cdots \oplus E_t$ is an orthogonal direct sum.

The Weyl group of each E_i is the subgroup generated by $\sigma_\alpha, \alpha \in \Delta_i$. If $\alpha \in \Delta_i$, we have that $\sigma_\alpha(E_i) = E_i$. And if $\alpha \notin \Delta_i$, $\forall \beta \in E_i$ $\sigma_\alpha(\beta) = \beta$, i.e., $\sigma_\alpha|_{E_i} = 1$. Hence E_i is $\mathscr{W}-$invariant.

Claim: Let E' be a subspace of E. If $\sigma_\alpha(E') = E'$, then either $\alpha \in E'$ or $E' \subset P_\alpha$

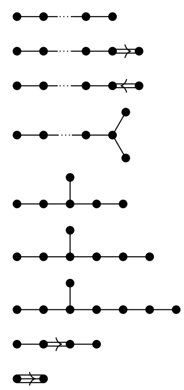
Indeed, $\forall \beta \in E'$, we have $\sigma_\alpha(\beta) \in E'$. Then $2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in E'$. Therefore, either $\alpha \in E'$ or $(\beta, \alpha) = 0$. We prove the claim.

Finally we have $\Phi = \Phi_1 \sqcup \cdots \sqcup \Phi_t$ □

Now we give the classification theorems.

Theorem 3.2. *If Φ is an irreducible root system of rank n, its Dynkin diagram if one of the following (n vertices in each case)*

A_n
B_n
C_n
D_n
E_6
E_7
E_8
F_4
G_2



where we set restrictions below
$n \geq 1$ *for* A_n
$n \geq 2$ *for* B_n
$n \geq 3$ *for* C_n
$n \geq 4$ *for* D_n.

The restrictions are imposed to avoid duplication. For example, one can see that $A_1 = B_1 = C_1$ and $B_2 = C_2$

Theorem 3.3. *For each Dynkin diagram or Cartan matrix of type A-G, there exists an irreducible root system having the given diagram.*

I won't give the proofs which is not difficult but quite complex. And I recommend the reader to learn the proofs from the book written by Bourbaki[1].

Until now we can see the relation

$$\{irrd.\ root\ systems\} \xleftrightarrow{1-1} \{indecomposable\ Cartan\ matrices\}$$

$$\xleftrightarrow{\qquad 1-1 \qquad} \{connected\ Dynkin\ diagrams\}$$

as a map. For certain root systems we can write its Cartan matrix and Dynkin diagram and from certain Dynkin diagram we can construct its root system. I also recommend the reader to check these from [1].

References

[1] Bourbaki, N.: Lie Groups and Lie Algebras: Chapters 4-6 *Springer Berlin Heidelberg* (2008).