ALGEBRAIC GEOMETRY

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Abstract. This is my personal note[1] of course *Advanced Geometry 3* taught by Prof. Fabio Perroni in spring 2024 at University of Trieste. And as the title, this course is focused on algebraic geometry, especially on algebraic varieties over $\mathbb{C}$. The latest version can be found on my website:

This note is not completely following what's on the blackbord. The materials might be re-organized or have some slightly changes. I will also add some supplementary materials which will be marked by *.

Attention: there are definitely a considerable number of mistakes in this note, and all due to me. If you have any comments, corrections or suggestions, please send them to zwei@sissa.it. Any feedback is appreciated!

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Date: March 15, 2024.

0. Preface

The main reference is [4].

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1. Affine algebraic varieties

1.1. Algebraic subsets. All ring will be assumed as commutative ring with unit.

Definition 1.1.1. A **closed algebraic subset** $X \subset \mathbb{C}^n$ is the set of zeroes of a finite numbers of polynomials

$$X = \{a = (a_1, \dots, a_n) \mid f_i(a_1, \dots, a_n) = 0, \forall i = 1, \dots, m\}$$

where $f_i \in \mathbb{C}[x_1, \dots, x_n]$.

It is also denoted by $V(f_1, \dots, f_m)$.

Remark 1.1. The ideal generated by $f_1, \dots, f_m$ is

$$I = (f_1, \dots, f_m) = \{\sum g_i f_i \mid g_i \in \mathbb{C}[x_1, \dots, x_n]\}.$$

And the set of zeroes of I is $X = V(I) = \{a \in \mathbb{C}^n \mid f(a) = 0, \forall f \in I\}$.

By Hilbert basis theorem, every ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ is f.g., i.e., $\exists f_1, \dots, f_m$ s.t. $I = (f_1, \dots, f_m)$. Hence we will talk about $V(I), I \subset \mathbb{C}[x_1, \dots, x_n]$.

Proposition 1.1.1. Let $I_1, I_2, \{I_\alpha\}_{\alpha \in A}$ be ideas of $\mathbb{C}[x_1, \dots, x_n]$. $a = (a_1, \dots, a_n) \in \mathbb{C}^n$. Then the following hold true.

(1) If $I_1 \subset I_2$, then $V(I_2) \subset V(I_1)$,
(2) $V(I_1 \cup I_2) = V(I_1 \cap I_2) = V(I_1 \cdot I_2)$ $(I_1 I_2 = \{fg | f \in I_1, g \in I_2\})$,
(3) $V(\sum_{\alpha \in A} I_\alpha) = \bigcap_{\alpha \in A} V(I_\alpha)$,
(4) If $\mathfrak{m}_a := (x_1 - a_1, \dots, x_n - a_n)$, then $V(\mathfrak{m}_a) = \{a\}$,
(5) $V(\sqrt{I}) = V(I)$
$(\sqrt{I} = \{f \in \mathbb{C}[x_1, \dots, x_n] \mid f^K \in I \text{ for some } K > 0\})$.

Proof. (1) evident.
(2) Since $I_1 I_2 \subset I_1 \cap I_2 \subset I_1, I_2$, (1) implies that

$$V(I_1 I_2) \supset V(I_1 \cap I_2) \supset V(I_1), V(I_2).$$

Conversely, let $a \in V(I_1 I_2)$. If $a \notin V(I_1 \cap I_2)$, then $\exists f \in I_1 \cap I_2$ s.t. $f(1) \neq 0$. Then $f^2(a) \neq 0$, but $f^2 \in I_1 I_2$. The remain is similar.
(3) $(\subset)$ $I_\alpha \subset \sum I_\alpha \forall \alpha$, hence $V(\sum I_\alpha) \subset V(I_\alpha) \forall \alpha$.
$(\supset)$ Immediately.
(4) $b \in V_{\mathfrak{m}_a}$ iff $b_i - a_i = 0,\ \forall i$.
(5) $(\subset)$ $\sqrt{I} \supset I$
$(\supset)$ Let $a \in V(I)$. If $a \notin V(\sqrt{I})$, then $\exists f \in \sqrt{I}$ s.t. $f(a) \neq 0$. Hence $f^K(a) \neq 0$. contradiction. ∎

Remark 1.2. (1) It can happen that $I_1 I_2 \subsetneqq I_1 \cap I_2$,
(2) $\sqrt{I}$ is an ideal and it is called the radical of I,

(3) **Proposition** 1.1.1(2), (3) implies that algebraic subsets of $\mathbb{C}^n$ satisfy the axiom of closed sets of a topology on $\mathbb{C}^n$ and it is called **Zariski topology**.

Remark 1.3. The Zariski topology is not Hausdorff unless the base field $\Bbbk$ is finite.

1.2. Affine varieties.

Definition 1.2.1. An **affine variety** is a closed algebraic set $X \subset \mathbb{C}^n$ of the form $X = V(P)$ with P prime ideal.

Example 1.2.1. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be irrd. Then $V(f) = V((f)) \subset \mathbb{C}^n$ is an affine variety and it is called an hypersurface of $\mathbb{C}^n$.

Note that if f is not irrd then (f) is not prime.

Example 1.2.2. Let $g_2, \ldots, g_n \in \mathbb{C}[x_1]$. Consider $X := \{(a, g_2(a), \ldots, g_n(a)) \in \mathbb{C}^n \mid a \in \mathbb{C}\}$.

It is a closed algebraic subset by $X = V(x_2 - g_2(x_1), \ldots, x_n - g_n(x_1))$. And since $\mathbb{C}[x_1, \cdots x_n]/(x_2 - g_2(x_1), \ldots, x_n - g_n(x_1)) \cong \mathbb{C}[x_1]$ which is a integral domain. Hence X is an affine variety and it is called rational space curve.

Exercise 1.2.1. Let $\varphi_1, \ldots, \varphi_k \subset \mathbb{C}[x_1, \ldots, x_n]$ be homogeneous polynomials of degree 1. Suppose that $\{\varphi_i\}$ are linearly independent as elements of $(\mathbb{C}^n)^*$. Then for any $b_1, \ldots, b_k \in \mathbb{C}$, fixed $X = V(\varphi_1 - b_1, \ldots, \varphi_k - b_k)$, which is the set of solutions of the linear system $\varphi_i = b_i$.

Prove that X is an affine variety. It is called a linear subspace of $\mathbb{C}^n$ of dimension $n - k$.

Now for any subset $S \subset \mathbb{C}^n$, we can define

$$I(S) := \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(a) = 0, \forall a \in S\}.$$

We have the following amazing theorem.

Theorem 1.2.1 (Hilbert's Nullstellensatz)**.** For any ideal $J \subset \mathbb{C}[x_1, \ldots, x_n]$,

$$I(V(J)) = \sqrt{J}.$$

In particular, if the ideal J is prime, then $I(V(J)) = J$.

Remark 1.4. (1) The theorem holds true for any algebraic closed field(See [1]).

(2) It fails if the field is not algebraic closed. For example, take $\Bbbk = \mathbb{R}$, $I(V(x^2 + y^2 + 1)) = \mathbb{R}[x, y]$ where $V(x^2 + y^2 + 1)$ is actually empty.

(3) (Study's lemma) Let $\Bbbk = \bar{\Bbbk}$. If $f \in \Bbbk[x_1, \ldots, x_n]$ is irrd, then $I(V(f)) = (f)$

Lemma* 1.2.1. If $Y_1 \subset Y_2$ are algebraic subsets of $\mathbb{C}^n$, then $I(Y_1) \supset I(Y_2)$.

Proposition* 1.2.1. $V(I(S)) = \bar{S}$

Proof. On the one hand we have $S \subset V(I(S))$ where by definition S is closed. Hence $\bar{S} \subset V(I(S))$. On the other hand, recall that the closure

$$\bar{S} = \bigcap W$$

where W runs over all algebraic subsets of $\mathbb{C}^n$ that contain S. And we can write $W = V(J)$ for some ideal J. Then $S \subset V(J)$ and by **Lemma* 1.2.1**, we have $I(S) \supset I(V(J)) \supset J$. Then by **Proposition 1.1.1**(1), $W = V(J) \subset V(I(S))$ for any such W. It follows the statement. □

Definition 1.2.2. Let $V(P) \subset \mathbb{C}^n$ be an affine variety. And let $\Bbbk \subset \mathbb{C}$ be a subfield. A point $a \in V(P)$ is called a **$\Bbbk$-generic point** if the following condition holds true: $\forall f \in \Bbbk[x_1, \dots, x_n]$, if $f(a) = 0$, then $f \in P$.

Example 1.2.3. Consider $g_2, \dots, g_n \in \mathbb{Q}[x_1]$ and let $X = V(x_2 - g_2(x_1), \dots, x_n - g_n(x_1))$ be the rational space curve. Let $a := (\pi, g_2(pi), \dots, g_n(\pi)) \in X$. Then a is $\mathbb{Q}$-generic.

Indeed, let $f \in \mathbb{Q}[x_1, \dots, x_n]$ is s.t. $f(a) = 0$. But $\varphi := f(x_1, x_2 - g_2(x_1), \dots, x_n - g_n(x_1)) \in \mathbb{Q}[x_1]$, hence $\varphi = 0$. It follows that $\varphi \in P$.

Proposition 1.2.1. Let $V(P)$ be an affine variety. Let $\Bbbk \subset \mathbb{C}$ be a subfield s.t. $\operatorname{tr.deg} \mathbb{C}|\Bbbk = \infty$. Then there exists $a \in V(P)$ a $\Bbbk$-generic point.

Proof. Let $P = (f_1, \dots, f_m)$ and, WLOG, assume that $f_1, \dots, f_m \in \Bbbk[x_1, \dots, x_n]$

(Otherwise let $\Bbbk'$ be the minimal subfield of $\mathbb{C}$ containing $\Bbbk$ and the coefficients of $f_1, \dots, f_m$. Then $\operatorname{tr.deg} \mathbb{C}|\Bbbk' = \infty$ and any $\Bbbk'$-generic point is also a $\Bbbk$-generic point).

Let $P_0 = P \cap \Bbbk[x_1, \dots, x_n]$, which is prime. And let K be the fraction field of $\Bbbk[x_1, \dots, x_n]/P_0$.

Since for any $f/g \in K$, it is a root of $gy - f \in \Bbbk(\bar{x}_1, \dots, \bar{x}_n)[y]$, where $\bar{x}_1, \dots, \bar{x}_n$ is the isomorphic class in $\Bbbk[x_1, \dots, x_n]/P_0$. We have that $K|\Bbbk(\bar{x}_1, \dots, \bar{x}_n)$ is algebraic. Hence $\operatorname{tr.deg} K|\Bbbk \leq n < \infty$.

In this situation, there exists a field homomorphism

$$\phi : K \to \mathbb{C}$$

s.t. $\phi|_K = \mathrm{id}_K$(Indeed, let $\lambda_1, \dots, \lambda_\delta \in K$ be a transcendence basis for $K|\Bbbk$. Let $z_1, \dots, z_\delta \in \mathbb{C}$ be algebraically independent over $\Bbbk$. The map $\lambda_i \mapsto z_i$, $\forall i$ extends to a unique field homomorphism from $K \to \mathbb{C}$. See [5] Ch.2 Thm 33).

Let $a_i := \phi(\bar{x}_i) \in \mathbb{C}$.

Claim. $a = (a_1, \dots, a_n) \in X$ is a $\Bbbk$-generic point.

Indeed. First we have that $f_i(\bar{x}_1, \dots, \bar{x}_n) = 0$ $i = 1, \dots, m$ in $\Bbbk[x_1, \dots, x_n]/P_0$. It follows that

$$0 = \phi(f_i(\bar{x}_1, \dots, \bar{x}_n)) = f_i(\phi(\bar{x}_1), \dots, \phi(\bar{x}_n)) = f_i(a_1, \dots, a_n) \ i = 1, \dots, m.$$

Hence $a \in X$.

Now let $f \in \Bbbk[x_1, \dots, x_n]$ s.t. $f(a) = 0$. If $f \notin P_0$, then $[f] \in \Bbbk[x_1, \dots, x_n]$ is nonzero. Applying ϕ to this class we get that $f(a) = 0$, which is contradiction. □

Remark 1.5. One could have defined $\Bbbk$-generic point for all $V(I)$ where $I \subset \mathbb{C}[x_1, \dots, x_n]$ is any ideal. But in the following case, it doesn't exist.

Let $I = (xy) \subset \mathbb{C}[x, y]$ be an ideal and $a = (a_1, a_2)$. If $a_2 = 0$, then for $y \subset \Bbbk[x, y]$, $\forall \Bbbk \subset \mathbb{C}$, $y(a) = a_2 = 0$, but $y \notin I$. It is similar when $a_1 = 0$.

Now we can give a proof of **Theorem** 1.2.1.

Proof. **Step 1.** Let $J = P$ be prime. Let $f \in I(V(P))$ and $\Bbbk$ be the minimal subfield of $\mathbb{C}$ containing $\mathbb{Q}$ and the coefficients of f. Then $\operatorname{tr.deg} \mathbb{C}/\Bbbk = \infty$ and by **Proposition** 1.2.1, there exists a $\Bbbk$-generic point $a \in X$. And since $f \in I(X)$, $f(a) = 0$, then $f \in P$.

Step 2. Not let J be any ideal and $f \in I(V(J))$. Consider the primary rep

$$\sqrt{J} = P_1 \cap \cdots \cap P_N.$$

Then $V(J) = V(\sqrt{J}) = V(P_1) \cup \cdots \cup V(P_N)$. So $f \in I(V(P_i))$ $i = 1, \dots, N$. Then by **Step 1.**, $f \in P_i$ $i = 1, \dots, N$, and $f \in \sqrt{I}$. □

Corollary 1.2.1. There is an order-reversing correspondence

$$\{J \subset \mathbb{C}[x_1, \dots, x_n] \mid J = \sqrt{J}\} \leftrightarrow \{\text{closed algebraic subset of } \mathbb{C}^n\}$$
$$J \mapsto V(J)$$
$$I(X) \leftarrow\!\shortmid X$$

Definition 1.2.3. Let $X = V(P) \subset \mathbb{C}^n$ be an affine variety with $P \subset \mathbb{C}[x_1, \dots, x_n]$ prime ideal. The ring $R_X := \mathbb{C}[x_1, \dots, x_n]/P$ is the **affine coordinate ring** of X.

Corollary 1.2.2. In this situation, R_X is isomorphic to the ring of functions $X \to \mathbb{C}$ which are restrictions of polynomials in $\mathbb{C}[x_1, \dots, x_n]$.

Proof. Let $\mathcal{F}(X) := \{F : X \to \mathbb{C} \mid$ s.t. $\exists f \in \mathbb{C}[x_1, \dots, x_n]$ s.t. $F(a) = f(a),\ \forall a$.

Restriction yields an surjective homomorphism

$$\mathbb{C}[x_1, \dots, x_n] \to \mathcal{F}(X) \to 0$$

and its kernel is P. Then we have the isomorphism. □

1.3. **Tangent spaces of affine varieties.**

Definition 1.3.1. Let $X = V(P)$ be an affine variety with $P \in \mathbb{C}[x_1, \dots, x_n$ prime. Let $a \in X$, the **Zariski tangent space** of X at a is the linear subspace of $\mathbb{C}^n$ given by the equations

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) = 0, ,\ \forall f \in P$$

and denoted by $T_{X,a}$[1].

[1] I prefer $T_a X$ so I might change this symbol hereafter

Remark 1.6. (1) If $P = (f_1, \ldots, f_m)$, then

$$T_aX = V(\{\sum_{i=0}^{m} \frac{\partial f_j}{\partial x_i}(a)(x_i - a_i) = 0 \mid j = 1, \ldots, m\}).$$

Indeed, ($\subset$) is obvious. ($\supset$) If $(b_1, \ldots, b_n) \in \mathbb{C}^n$ is s.t.

$$\sum_{i=0}^{n} \frac{\partial f_j}{\partial x_i}(a)(b_i - a_i) = 0, \ \forall j = 1, \ldots, m.$$

Let $f \in P$, we can write $f = \sum_{i=1}^{m} f_i g_i$ for some $g_i \in \mathbb{C}[x_1, \ldots, x_n]$. Then

$$\sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(a)(b_i - a_i) = \sum_{i=0}^{n}\sum_{i=1}^{m} \frac{\partial f_i g_i}{\partial x_i}(a)(b_i - a_i) = 0.$$

(2) $T_aX \subset \mathbb{C}^n$ is an affine subspace passing through a.

1.4. **Tangent spaces and derivations.** Let $R := R_X$ be the affine coordinate ring of X.

Recall that a **derivation** of R (centered) at $a \in X$ is a $\mathbb{C}$-linear map

$$D : R \to \mathbb{C}$$

s.t.

(1) $D(fg) = f(a)D(g) + g(a)D(f), , \ \forall f, g \in R,$

(2) $D(\lambda) = 0, \ \forall \lambda \in \mathbb{C}.$

Let $Der_{R,a}$ be the set of such derivations.

Remark 1.7. $Der_{R,a}$ is a vector space over $\mathbb{C}$.

Proposition 1.4.1. Let $\bar{x}_1, \ldots, \bar{x}_n \in R$ be the classes of $x_1, \ldots, x_n$. Then the map

$$\varphi : Der_{R,a} \to \mathbb{C}^n$$
$$D \mapsto (D(\bar{x}_1), \ldots, D(\bar{x}_n))$$

is an injective linear map and its image is $T_aX - a$.

Proof. Exercise. □

1.5. **Dimension theory.** The Zariski tangent space we have defined before is an affine subspace of $\mathbb{C}^n$. As a vector space, it has dimension

$$\dim T_aX = n - \mathrm{rk}(\frac{\partial f_j}{\partial x_i(a)})_{i,j}.$$

For any $k \in \mathbb{N}$, we have

$$\begin{aligned}\{a \in X \mid \dim T_aX \geq k\} &= \{a \in X \mid \mathrm{rk}(\frac{\partial f_j}{\partial x_i(a)})_{i,j} \leq n-k\} \\ &= \{a \in X \mid \text{the determinants of all minors of ?} \\ &\quad (n-k+1) \times (n-k+1) \text{ of } \frac{\partial f_j}{\partial x_i(a)} \text{ are } 0\}.\end{aligned}$$

Hence $\{a \in X \mid \dim T_aX \geq k\}$ is a closed subset of X in the Zariski topology of X.

Remark 1.8. (1) $\{a \in X \mid \dim T_aX \geq k\} \subset \{a \in X \mid \dim T_aX \geq k-1\}$,
(2) Let $d := \min\{\dim T_aX \mid a \in X\}$. Observe that

$$U := \{a \in X \mid \dim T_aX = d\} = X - \{a \in X \mid \dim T_aX \geq d+1\}$$

is open and nonempty.

Proposition 1.5.1. Let $X = V(P)$ be an affine variety with $P \subset \mathbb{C}[x_1, \ldots, x_n]$ prime. Let $\mathbb{C}(X) = \mathrm{Frac}(R_X)$. ($\mathbb{C}(X)$ is called the field of rational functions of X) Then

$$d = \mathrm{tr.deg}(\mathbb{C}(X)/\mathbb{C}).$$

Definition 1.5.1. The dimension of an affine variety X is $\dim X := \mathrm{tr.deg}(\mathbb{C}(X)|X)$.

And a point $a \in X$ is smooth if $\dim T_aX = \dim X$. $a \in X$ is singular if $\dim T_aX > \dim X$.

Remark 1.9. Let $\bar{x}_1, \ldots, \bar{x}_n \in R_X$ be the classes of $x_1, \ldots, x_n$. Then $\mathbb{C}(X) = \mathbb{C}[\bar{x}_1, \ldots, \bar{x}_n]$.

Indeed. ($\subset$) is clear.

($\supset$) Let $\frac{\bar{f}}{\bar{g}} \in \mathbb{C}(X)$ where $\bar{f}, \bar{g} \in R_X$ and $\bar{g} \neq 0$. And $\bar{f}, \bar{g}$ are the classes of f, g respectively. Then $\bar{f}.\bar{g}$ are polynomials in $\bar{x}_1, \ldots, \bar{x}_n$. Then $\frac{\bar{f}}{\bar{g}} \in bC[\bar{x}_1, \ldots, \bar{x}_n]$.

It implies that $\mathrm{tr.deg}(\mathbb{C}(X)|\mathbb{C}) < \infty$.

Example 1.5.1. (1) $\dim \mathbb{C}^n = n$
(2) $\forall a \in \mathbb{C}^n$, $\dim\{a\} = 0$ (Jacobian is the identity)
(3) Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be irrd($f \notin \mathbb{C}$). Let $X = V(f)$.

$$0 \leq \mathrm{rk}(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_1}) \leq 1$$

Notice that there exists $a \in X$ s.t. $\mathrm{rk}(\frac{\partial f}{\partial x_i}) = 1$.

Indeed. If $\mathrm{rk}(\frac{\partial f}{\partial x_i}) = 0, \forall a \in X$, then $\frac{\partial f}{\partial x_i} \in I(X) = (f)$. Hence $f | \frac{\partial f}{\partial x_i}, \forall i$. It follows that $\frac{\partial f}{\partial x_i} = 0, \forall i$ sicne $\deg \frac{\partial f}{\partial x_i} < \deg f$ if $\frac{\partial f}{\partial x_i} \neq 0$. Then $f \in \mathbb{C}$ contradiction.

Therefore, $\dim X = n-1$.

(4) Consider the rational space curve $X = V(x_2 - g_2(x_1, \ldots, x_n - g_n(x_1))$. Its Jacobian is

$$\begin{pmatrix} -\frac{\partial g_2}{\partial x_1} & 1 & 0 & \cdots & 0 \\ -\frac{\partial g_2}{\partial x_1} & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\partial g_n}{\partial x_1} & 0 & 0 & \cdots & 1 \end{pmatrix}$$

of rank $n-1$. Hence $\dim X = n-1$.

(5) Consider the cuspidal cubic curve $X = V(x^2 - y^3) \subset \mathbb{C}^2$. Its Zariski tangent space at $p = (a^3, a^2)$ is

$$T_pX = \{(x,y) \in \mathbb{C}^2 \mid 2a^3(x-a^3) - 3a^4(y-a^2) = 0\}$$
$$= \begin{cases} \mathbb{C}^2, & a = 0, \\ 2a^3(x-a^3) - 3a^4(y-a^2) = 0, & a \neq 0. \end{cases}$$

Then $\min\{\dim T_pX\} = 1$ and $\dim X = n-1$. X is singular at $(0,0)$

Lemma 1.5.1. Let R be an integral domain over field $\Bbbk$ and $P \subset R$ a prime ideal. Let $K := \operatorname{Frac}(R)$ and $K' = \operatorname{Frac}(R/P)$. Assume $\operatorname{tr.deg} K|\Bbbk < \infty$. Then

$$\operatorname{tr.deg} K|\Bbbk \geq \operatorname{tr.deg} K'|\Bbbk$$

and the equality holds iff $P = (0)$.

Proof. If $P = (0)$ everything is clear. Assume $P \neq (0)$ and assume by contradiction that

$$\operatorname{tr.deg} K|\Bbbk < \operatorname{tr.deg} K'|\Bbbk$$

By Ch.II, Sec 12, Thm 27 of [5], there exist $\varphi_1, \ldots, \varphi_n \in R/P$ that are algebraically independent over $\Bbbk$ where $n = \operatorname{tr.deg} K|\Bbbk$. Let $f_1, \ldots, f_n \in R$ s.t. their classes in R/P are $\varphi_1, \ldots, \varphi_n$ respectively. Let $p \in P$, $p \neq 0$. Then $p, f_1, \ldots, f_n$ are algebraically dependent. Hence there exists a polynomial $\Phi \in \Bbbk[y, x_1, \ldots, x_n] \setminus 0$ s.t. $\Phi(p, f_1, \ldots, f_n) = 0$. WLOG, we can assume Φ is irrd (since R is an integral domain). Moreover $\Phi \neq \alpha y,\ \alpha \in \Bbbk$ since $p \neq 0$. Hence $\Phi(0, x_1, \ldots, x_n) \neq 0$. And passing to R/P, $\Phi(0, \varphi_1, \ldots, \varphi)n) = 0$, contradiction. $\square$

Proposition 1.5.2. Let X, Y be two affine varieties with $X \subsetneqq Y$. Then $\dim X < \dim Y$.

Proof. Let $X = V(P)$, $Y = V(Q)$ with $P, Q \subset \mathbb{C}[x_1, \ldots, x_n]$ prime. Then $Q \subsetneqq P$. We have

$$0 \to \bar{P} \to R_Y \to R_X \to 0$$

where $\bar{P} = P/Q$. Then $R_X = R_Y/\bar{P}$.

By **Lemma 1.5.1**, $\operatorname{tr.deg}(\mathbb{C}(Y)|\mathbb{C}) \geq \operatorname{tr.deg}(\mathbb{C}(X))|\mathbb{C}$ and the equality holds iff $\bar{P} = (0)$, which is $P = Q$. $\square$

Corollary 1.5.1. Let $X \subset \mathbb{C}^n$ be an affine variety of dimension $n-1$. Then X is a hypersurface(i.e. $\exists f \in \mathbb{C}[x_1, \ldots, x_n]$ irrd s.t. $X = V(f)$).

Proof. Let $X = V(P)$ with P prime. Let $f \in P$, $f \neq 0$. Then $X \subset V(f)$. And there exist $f_1, \ldots, f_N \in \mathbb{C}[x_1, \ldots, x_n]$ irrd s.t.

$$f = f_1 \cdots f_N \in P.$$

Since P is prime, there exists $i \in \{1, \ldots, N\}$ s.t. $f_i \in P$. Hence $X \subset V(f_i)$. And since $\dim X = n - 1 = \dim V(f_i)$, by **Proposition 1.5.2**, we have $X = V(f_i)$. $\square$

Corollary 1.5.2. Let $X \subset \mathbb{C}^n$ be an affine variety. Then $\dim X = 0 \Leftrightarrow X = \{a\}$ for some $a \in \mathbb{C}^n$

Proof. ($\Leftarrow$) Clear. ($\Rightarrow$) If $\exists a \in X$ and $\{a\} \neq X$, then $0 = \dim\{a\} < \dim X = 0$, contradiction. $\square$

Remark 1.10. Let $X = V(P) \subset \mathbb{C}^n$ be an affine variety with P prime. And $\dim X = n - r$. In general, there are no $f_1, \dots, f_r \in \mathbb{C}[x_1, \dots, x_n]$ s.t. $P = (f_1, \dots, f_r)$.

For example, Let $X \subset \mathbb{C}^3$ be an affine variety with $\dim X = 1$. If $P = I(X)$, the minimal number of generators of P is 3. Consider the map

$$\begin{aligned} \varphi : \mathbb{C} &\to \mathbb{C}^3 \\ a &\mapsto (a^3, a^4, a^5) \end{aligned}$$

Let $X := \{(a^3, a^4, a^5) \mid a \in \mathbb{C}\} \subset \mathbb{C}^3$. Then clearly we have that $X \subset V(I)$ where $I = (xz - y^2, x^3 - yz, x^2y - z^2)$. Conversely, let $(x, y, z) \in V(I)$, set $a := \frac{y}{x}$ if $x \neq 0$ (if $x = 0$ then $y = z = 0$). Then we have

$$\begin{aligned} a^3 &= \frac{y^3}{x^3} = \frac{xzy}{x^3} = x, \\ a^4 &= xa = y, \\ a^3 &= ya = z. \end{aligned}$$

Therefore, $X = V(I)$.

Moreover, I is a prime ideal and it cannot be generated by 2 polynomials.

Claim. $\sqrt{I}$ is prime.

Indeed. By **Theorem** 1.2.1, $\sqrt{I} = I(X)$. If $\exists f_1, f_2 \in \mathbb{C}[x, y, z]$ s.t. $f_1 f_2 \in \sqrt{I}$ but $f_1, f_2 \notin \sqrt{I}$. Then $f_1 \circ \varphi, f_2 \circ \varphi \in \mathbb{C}[t] \setminus 0$ but $(f_1 f_2) \circ \varphi = (f_1 \circ \varphi)(f_2 \circ \varphi) = 0$, contradiction. $\square$

Claim. $\sqrt{I}$ cannot be generated by 2 polynomials.

Indeed. Let $f \in \sqrt{I}$. It can be written as

$$f = \sum c_{ijk} x^i y^j z^k$$

s.t. $\sum c_{ijk} t^{3i+4j+5k} = 0,\ \forall t$. i.e., $\forall m \geq 0$, $\forall (i, j, k)$ s.t. $3i + 4j + 5k = m$,

$$\sum_{\substack{(i,j,k) \\ 3i+4j+5k=m}} c_{ijk} = 0,\ \forall m \geq 0.$$

(1) $m = 0$. $c_{000} = 0$.
(2) $m = 1, 2$. None.
(3) $m = 3$. $c_{100} = 0$.
(4) $m = 4$. $c_{010} = 0$.
(5) $m = 5$. $c_{001} = 0$.
(6) $m = 6$. $c_{200} = 0$.
(7) $m = 7$. $c_{110} = 0$.
(8) $m = 8$. $c_{101} + c_{020} = 0$. We get $\mathbb{C}(xz - y^2)$.

(9) $m = 9$. $c_{300} + c_{011} = 0$. We get $\mathbb{C}(x^3 - yz)$.
(10) $m = 10$. $c_{210} + c_{002} = 0$. We get $\mathbb{C}(x^2y - z^2)$.

In conclusion, f has the form

$$f = \alpha(xz - y^2) + \beta(x^3 - yz) + \gamma(x^2y - z^2) + \tilde{f},\ \alpha, \beta, \gamma \in \mathbb{C}.$$

If $\sqrt{I} = (f, g)$, then

$$g = \alpha'(xz - y^2) + \beta'(x^3 - yz) + \gamma'(x^2y - z^2) + \tilde{g}.$$

and we can express $xz - y^2, x^3 - yz, x^2y - z^2$ as a linear cpmbination of f, g. But they are linearly indenpendent. Contradiction. $\square$

To prove **Proposition** 1.5.1, we need the following lemmas.

Lemma 1.5.2. Let $U_1, U_2 \subset X$ be nonempty Zariski open subsets. Then $U_1 \cap U_2 \neq \varnothing$.

Proof. Let $X = V(P)$ with P prime. We can write the open sets as

$$U_i = X \cap (\mathbb{C}^n \setminus V(I_i)),\ i = 1, 2.$$

Nonempty implies that there exists $a_i \in X$ and $f_i \in I_i$ s.t. $f_i(a_i) \neq 0$, and hence $f_i \notin P$ for $i = 1, 2$. If $U_1 \cap U_2 = \varnothing$, then

$$X \cap (\mathbb{C}^n \setminus V(I_1)) \cap (\mathbb{C}^n \setminus V(I_2)) = X \cap (\mathbb{C}^n \setminus (V(I_1) \cup V(I_2))) = X \cap (\mathbb{C}^n \setminus V(I_1 I_2)) = \varnothing.$$

It implies that $X \subset V(I_1 I_2)$ and then $f_1 f_2 \in P$, Contradiction. $\square$

Definition 1.5.2. Let S be a ring and $R \subset S$ be a subring. A map $R \to S$ is said to be a **derivation of R (with values in S)** if

(1) $D(x + y) = D(x) + D(y)$, $\forall x, y \in R$,
(2) $D(xy) = xD(y) + yD(x)$,$\forall x, y \in R$.

Definition 1.5.3. Let S be a ring and $R \subset S$ be a subring. Let $R' \subset R$ be a subring. A derivation $D : R \to S$ is called a **R'-derivation** if $D(x) = 0$, $\forall x \in R'$. We denote $\mathcal{D}_{R/R'}(S)$ the set of all R'-derivation of R. If $S = R$, we write $\mathcal{D}_{R/R'} = \mathcal{D}_{R/R'}(S)$

Remark 1.11. (1) $\mathcal{D}_{R/R'}(S)$ is an S-module. In particular, if S is a field, then $\mathcal{D}_{R/R'}(S)$ is an S-vector space.

(2) Assume that R is an integral domain. Let $K = \mathrm{Frac}(R)$. Then any derivation D of R with values in K can be extended uniquely to a derivation of K. Moreover, we have $\mathcal{D}_R(K) \cong \mathcal{D}_K(K)$.

Indeed. Let $x, y \in R$ and $y \neq 0$. Define $D(\frac{x}{y}) := \frac{yD(x) - xD(y)}{y^2}$. Observe that if $\frac{x}{y} = \frac{x'}{y'}$, by definition we have $D(\frac{x}{y}) = D(\frac{x'}{y'})$. It is easy to see that the map $D : K \to K$ is a derivation. Uniqueness is immediately.

Example 1.5.2. (1) Let R be a ring and D be a derivation on R. Let $A = R[x_1, \ldots, x_n]$. For any

$$f = \sum c_{i_1, \ldots, i_n} x_1^{i_1} \cdots x_n^{i_n},$$

define

$$f^D = \sum D(c_{i_1,\dots,i_n})x_1^{i_1}\cdots x_n^{i_n}.$$

It gives a derivation of A.

(2) Let R' be a ring and $R = R'[x_1,\dots,x_n]$. Define

$$D_i = \frac{\partial}{\partial x_i} : R \to R.$$

with

$$D_i(c) = 0,\ \forall c \in R',$$

$$D_i(\sum c_{k_1,\dots,k_n} x_1^{k_1}\cdots x_n^{k_n}) = \sum c_{k_1,\dots,k_n} k_i x_1^{k_1}\cdots x_n^{k_i-1})\cdots x_n^{k_n}).$$

D_i is a R'-derivation.

D_i is uniquelly determinde by $(1), (2)$ in **Definition** 1.5.2 and $D_i(c) = 0,\ \forall c \in R',\ D_i(x_j) = \delta_{ij}$.

(3) If $R' = \Bbbk$ is a field and $K = \Bbbk(x_1,\dots,x_n)$. Then $\dim_K \mathcal{D}_{K|\Bbbk} = n$ and $D_1,\dots,D_n$ form a basis for $\mathcal{D}_{K|\Bbbk}$.

Indeed. Let $D \in \mathcal{D}_{K|\Bbbk}$, we consider $D' := \sum_{i=1}^n D(x_i)D_i \in \mathcal{D}_{K|\Bbbk}$. It is easy to see that $D = D'$. Hence $\mathcal{D}_{K|\Bbbk} = \operatorname{span}(D_1,\dots,D_n)$. It remains to show that $D_1,\dots,D_n$ are linearly independent. Let $\lambda_i \in K$ be such that

$$\sum \lambda_i D_i = 0.$$

Then

$$\lambda_j = (\sum \lambda_i D_i)(x_j) = 0.$$

In fact, we have the following theorems.

Theorem 1.5.1 ([5] Ch.2, Sec.17, Thm41)**.** Let K be a field, $\operatorname{char} K = 0$. Let $F = K(x_1,\dots,x_n)$ by any f.g.extension of K. Then

$$\operatorname{tr.deg}(F|K) = \dim_F(\mathcal{D}_{F|K}).$$

Corollary 1.5.3 ([5] Ch.2, Sec.17, Cor2$'$)**.** Let K be a field. Let $F|K$ by a separable algebraic extension. Then any derivation of K can be extended to a derivation of F in a unique way

Example 1.5.3. Consider the polynomial ring $K[x_1,\dots,x_n]$ and its field of fraction $F = K(x_1,\dots,x_n)$. Then $\mathcal{D}_{F|K}(F)$ as vector space over F hase basis $D_1,\dots,D_n$.

Lemma 1.5.3. There exists a nonempty Zariski open subset $\tilde{U} \subset X$ s.t. $\forall a \in \tilde{U}$, $\dim T_aX = \operatorname{tr.deg}(\mathbb{C}(X)|\mathbb{C})$.

Proof. Let $\bar{x}_1,\dots,\bar{x}_n$ be the classes of $x_1,\dots,x_n$ in $\mathbb{C}(X)$ Then $\mathbb{C}(X) = \mathbb{C}(\bar{x}_1,\dots,\bar{x}_n)$ and it is f.g.over $\mathbb{C}$. Then by **Theorem** 1.5.1,

$$\operatorname{tr.deg}(\mathbb{C}(X)|\mathbb{C}) = \dim_{\mathbb{C}(X)} \mathcal{D}_{R_X|\mathbb{C}}(\mathbb{C}(X)) = \dim_{\mathbb{C}(X)} \mathcal{D}_{\mathbb{C}[x_1,\dots,x_n]/(P+\mathbb{C})}(\mathbb{C}(X))$$

where

$$\mathcal{D}_{\mathbb{C}[x_1,\dots,x_n]/(P+\mathbb{C})}(\mathbb{C}(X)) = \{(\lambda_1,\dots,\lambda_n)\in\mathbb{C}(X) \mid \sum_{i=1}^{n}\lambda_i D_i(f)=0,\ \forall f\in P\}$$
$$= \{(\lambda_1,\dots,\lambda_n)\in\mathbb{C}(X) \mid \sum_{i=1}^{n}\lambda_i \frac{\partial}{\partial x_i}(f)=0,\ \forall f\in P\}$$

And then the dimension of this set is $n-\mathrm{rk}_{\mathbb{C}(X)}(\frac{\partial f_j}{\partial x_i})$.

Claim. There exists a nonempty Zariski open subset $\tilde{U}\subset X$ s.t.

$$\mathrm{rk}_{\mathbb{C}(X)}(\frac{\partial f_j}{\partial x_i}) = \mathrm{rk}_{\mathbb{C}}(\frac{\partial f_j}{\partial x_i}(a)),\ \forall a\in\tilde{U}.$$

$r := \mathrm{rk}_{\mathbb{C}(X)}(\frac{\partial f_j}{\partial x_i})$

Indeed. By linear algebra we know that there exist $A\in\mathrm{GL}_m(\mathbb{C}(X))$ and $B\in\mathrm{GL}_n(\mathbb{C}(X))$ s.t.

$$A(\frac{\partial f_j}{\partial x_i})B = \begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix}$$

But we can write $A=\frac{1}{\alpha}A_0,\ B=\frac{1}{\beta}B_0$ for some $\alpha,\beta\in R_X$ and $A_0\in\mathrm{Mat}_m(R_X)$ and $B_0\in\mathrm{Mat}_n(R_X)$.

Let $\tilde{U} := \{a\in X \mid \alpha(a)\beta(a)\det(A_0(a))\det(B_0 a)\neq 0\}$, which is a nonempty Zariski open set. And for any $a\in\tilde{U}$,

$$\frac{1}{\alpha(a)}A_0(a)(\frac{\partial f_j}{\partial x_i}(a))\frac{1}{\beta(a)}B_0(a) = \begin{pmatrix} I_r & 0\\ 0 & 0\end{pmatrix}$$

Hence $r=\mathrm{rk}_{\mathbb{C}}(\frac{\partial f_j}{\partial x_i}(a)),\ \forall a\in\tilde{U}$. □

Now we give the proof of **Proposition** 1.5.1.

Proof. We have seen that there exists a nonempty Zariski open subset $U\subset X$ s.t. $\forall a\in U$, $\dim T_aX = \min\{\dim T_bX \mid b\in X\}$. Then by **Lemma** 1.5.2, $U\cap\tilde{U}\neq\varnothing$, where $\tilde{U}$ is as in the **Lemma** 1.5.3. □

1.6. Structure of affine varieties at smooth points.

Theorem 1.6.1 ([4], Thm 1.16, Cor 1.20)**.**

(1) Let $f_1,\dots,f_r\in\mathbb{C}[x_1,\dots,x_n]$ without constant terms ($f_j(0)=0,\ j=1,\dots,r$) and s.t. the linear parts are linearly independent ($\frac{\partial f_j}{\partial x_1}(0),\dots,\frac{\partial f_j}{\partial x_n}(0),\ j=1,\dots,r$, are linearly independent.) Define

$$P := \{g\in\mathbb{C}[x_1,\dots,x_n] \mid \frac{\sum_{j=1}^{r}h_jf_j}{K}=g,\ h_j,K\in\mathbb{C}[x_1,\dots,x_n],\ K(0)\neq 0\}.$$

Then P is a prime ideal and $X:=V(P)$ is a variety of dimension $n-r$ and $0\in X$ is a smooth point.

Moreover, $V(f_1, \dots, f_r) = X \cup Y$ where Y is a closed algebraic set s,t, $0 \notin Y$.

(2) Conversely, if $X = V(P) \subset \mathbb{C}^n$ is an affine variety of dimension $n-r$ and $a \in X$ is smooth. Then there exist $f_1, \dots, f_r \in P$ s.t.

$$\operatorname{rk}\left(\frac{\partial f_j}{\partial x_i}(a)\right) = r$$

and

$$P = \{g \in \mathbb{C}[x_1, \dots, x_n] \mid \frac{\sum_{j=1}^r h_j f_j}{K} = g,\ h_j, K \in \mathbb{C}[x_1, \dots, x_n],\ K(0) \neq 0\}.$$

Example 1.6.1. Again consider $X = \{(a^3, a^4, a^5) \mid a \in \mathbb{C}\} = V(P) \subset \mathbb{C}^3$ where $P = (xz-y^2, x^3-yz, x^2y-z^2)$. And it is easy to see that $(1,1,1) \in X$ is a smooth point. One can check that it satisfies (2) in **Theorem** 1.6.1.

1.7. **The local ring of a point.** Let $R = \mathbb{C}[x_1, \dots, x_n]$, $P = (x_1 - a_1, \dots, x_n - a_n)$ where $a = (a_1, \dots, a_n) \in \mathbb{C}^n$. Here P is a maximal ideal. $\mathcal{O}_{\mathbb{C}^n, a} = R_P$ is called the **local ring** a whose elements are rational functions defined in some neighborhood of a.

Remark 1.12. If $g \in \mathbb{C}[x_1, \dots, x_n]$ is s.t. $g(a) \neq 0$, then we can consider $\tilde{g}(y_1, \dots, y_n) := g(a_1 + y_1, \dots, a_n + y_n) \in \mathbb{C}[y_1, \dots, y_n]$ and then $\tilde{g}(0) \neq 0$. Then it as a inverse in the ring of formal power series

$$\frac{1}{\tilde{g}(y)} = \sum_{i_1, \dots, i_n = 0}^{\infty} c_{i_1, \dots, i_n} y_1^{i_1} \cdots y_n^{i_n} \in \mathbb{C}[[y_1, \dots, y_n]].$$

For example,

$$\frac{1}{1 - \sum c_i y_i} = 1 + \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n} c_i y_i \right)^k$$

Hence we have $\mathcal{O}_{\mathbb{C}^n, a} \subset \mathbb{C}[[y_1, \dots, y_n]]$. Then in a neighborhood of the smooth point a, it is also a complex manifold in the Euclidean topology.

Now we consider the case of affine variety. Let $X = V(P) \in \mathbb{C}^n$ be an affine variety, $a \in X$. And let $\mathfrak{m}_a := (\bar{x}_1 - a_1, \dots, \bar{x}_n - a_n)$. We can also define $\mathcal{O}_{X,a} := (R_X)_{\mathfrak{m}_a}$ the local ring of $a \in X$.

Remark 1.13. Note that $\operatorname{Frac}(\mathcal{O}_{X,a}) = \mathbb{C}(X)$.

Proposition 1.7.1. $R_X = \cap_{a \in X} \mathcal{O}_{X,a}$ in $\mathbb{C}(X)$.

Proof. ($\subset$) We have the map $f \mapsto \frac{f}{1} \in \mathcal{O}_{X,a},\ \forall a \in X$.

($\supset$) Let $u \in \cap_{a \in X} \mathcal{O}_{X,a}$. Let $I := \{h \in \mathbb{C}[x_1, \dots, x_n] \mid \bar{h}u \in R_X\}$ where $\bar{h}$ is the class of h in R_X. Note that I is an ideal and $P \in I$.

For any $a \in X$, since $u \in \cap_{a \in X} \mathcal{O}_{X,a}$ can be expressed as

$$u = \frac{f}{g},\ g(a) \neq 0.$$

Hence $g \in I$. But $g(a) \neq 0$, if follows that $a \notin V(I)$. And since $P \subset I$, we have $V(I) \subset X$ and $V(I) = \varnothing$. By **Theorem** 1.2.1, $1 \in \sqrt{I}$. Therefore, $1 \in I$ and $u = 1 \cdot u \in R_X$ $\square$

1.8. **Appendix. primary decomposition.** Now We recall some commutative algebra [5].

Definition 1.8.1. Let R be a ring and $I \subset R$ be an ideal of R. I is called **primary** if whenever $a, b \in R$ are such that $ab \in I$ and $a \notin I$, then $b \in \sqrt{I}$.

We have immediately that the radical of a primary ideal is prime.

Theorem 1.8.1 (Lasker-Noether decomposition theorem)**.**

(1) Let R be a Noetherian ring, then every ideal $I \subset R$ admits the so called primary representation as
$$I = Q_1 \cap \cdots \cap Q_N$$
where Q_i's are primary ideals of R.

Moreover, we can find $Q_1, \ldots, Q_N$ s.t.no Q_i contains $\bigcap_{j \neq i} Q_j$ and the associated prime ideals $\sqrt{Q_1}, \ldots, \sqrt{Q_n}$ are distinct. In this case it is called irredundant primary representation.

(2) Let R be a ring and $I \subset R$ be an ideal that admits an irredundant primary representation
$$I = Q_1 \cap \cdots \cap Q_N.$$
Then $I = \sqrt{I}$ iff $Q_1, \ldots, Q_N$ are prime.

Theorem 1.8.2. Let R be a ring and $I \subset R$ be an ideal admitting an irreduandant primary representation
$$I = Q_1 \cap \cdots \cap Q_N.$$
Then the prime ideals $P_i := \sqrt{Q_i}$ are uniquely determined by I. And they are called the associated primes of I.

Example 1.8.1. Let $I = (x^2, y) \subset \mathbb{C}[x, y]$ be an ideal. It has an irredundant primary representation
$$I = (x^2) \cap (y).$$
And its radical is
$$\sqrt{I} = (x) \cap (y).$$

More generally, let $f \in \mathbb{C}[x_1, \ldots, x_n]$ and write $f = g_1^{k_1} \cdots g_N^{k_N}$ where g_i are irrd and not associated to each other. Then we have
$$(f) = (g_1^{k_1}) \cap \cdots \cap (g_N^{k_N}).$$
And its radical is
$$\sqrt{(f)} = (g_1) \cap \cdots \cap (g_N) = (g_1 \cdots g_N).$$

Example 1.8.2. Let $\Bbbk$ be any field. Consider the polynomial ring $\Bbbk[x, y]$ and ideal $I = (x^2, xy)$. Then for any $c \in \Bbbk$,
$$I = (x) \cap (y - cx, x^2)$$
is an irredundant primary representation of I.

Question 1.8.1. What are the associated primes of I?

Corollary 1.8.1. Let $I \subset \mathbb{C}[x_1 \ldots, x_n]$ be a radical ideal. Then there exists unique prime ideals $P_1, \ldots, P_N \subset \mathbb{C}[x_1 \ldots, x_n]$ s.t.

$$I = P_1 \cap \cdots \cap P_N$$

and $P_i \neq P_j$, $\forall i \neq j$.

1.9. **Appendix. transcendental extension.**

Definition 1.9.1. An extension $K|\Bbbk$ is **transcendental** if it is not algebraic(i.e. if $\exists \alpha \in K$ not algebraic over $\Bbbk$).

Example 1.9.1. (1) $\mathbb{Q}(\pi)|\mathbb{Q}$ is transcendental.
(2) $\mathbb{Q}(i)|\mathbb{Q}$ is algebraic.
(3) Let $\Bbbk$ be any field and K be the fraction field of $\Bbbk[x_1, \ldots, x_n]$, which is $K = \Bbbk(x_1, \ldots, x_n)$. Then $K|\Bbbk$ is transcendental.

Definition 1.9.2. Let $K|\Bbbk$ be a field extension. Let $L \subset K$. The elements of L are said to be **algebraically independent** over $\Bbbk$ if $\forall \alpha_1, \ldots, \alpha_N \in L$, there is no $f \in \Bbbk[x_1, \ldots, x_n]$ s.t. $f(\alpha_1, \ldots, \alpha_N) = 0$. In this case, L is called a **transcendental set** over $\Bbbk$.

Definition 1.9.3. A **transcendental basis** for $K|\Bbbk$ is a transcendental set $L \subset K$ over $\Bbbk$ that is not contained in any bigger transcendental set.

Remark 1.14. $L \subset K$ is a transcendental basis for $K|\Bbbk$ iff $K|\Bbbk(L)$ is algebraic.

Example 1.9.2. $\{x_1, \ldots, x_n\} \in \Bbbk(x_1, \ldots, x_n)$ form a transcendental basis for $\Bbbk(x_1, \ldots, x_n)|\Bbbk$.

Theorem 1.9.1. There exists a transcendental basis for any field extension. Moreover, any two transcendental basis have the same cardinality.

See Chapter II Sec.12 in [5] for the proof.

Definition 1.9.4. The cardinality of any transcendental basis for $K|\Bbbk$ is called the **transcendental degree** of $K|\Bbbk$, denoted by $\operatorname{tr.deg}(K|\Bbbk)$.

Remark 1.15. $\operatorname{tr.deg} \mathbb{R}|\mathbb{Q} = \operatorname{tr.deg} \mathbb{C}|\mathbb{Q} = \infty$

1.10. **Appendix. Localization.**

Definition 1.10.1. Let R be a ring and $P \subset R$ be a prime ideal. The **localization** of R at P is

$$R_P := \{(f, g) \in R \times R \mid g \notin P\}/$$

where (f, g) (f', g') iff $\exists h \notin P$ s.t. $(fg' - gf')h = 0$.

One may view the element $(f, g) \in R_P$ as "$\frac{f}{g}$".
We have a morphism

$$\begin{aligned} \varphi : R &\to R_P \\ f &\mapsto \frac{f}{1}. \end{aligned}$$

And $\forall f \in R \setminus P$, $\varphi(f)$ is invertible. More generally, $\frac{f}{g}$ is invertible in R_P if $f \notin P$ and $(\frac{f}{g})^{-1} = \frac{g}{f}$.

Let $\mathfrak{m} := \{\frac{f}{g} \mid f \in P\}$. It is a (unique) maximal ideal of R_P and $(R_P, \mathfrak{m})$ is a local ring.

Remark 1.16. If R is an integral domain then so is R_P.

Proposition 1.10.1. If R is Noetherian, then so is R_P.

Proof. Let $I \subset R_P$ be an ideal and $\bar{I} := \varphi^{-1}(I) \subset R$. Since R is Noetherian, $\bar{I} = (\bar{f}_1, \dots, \bar{f}_m)$ for some $\bar{f}_i \in R$.

Let $u \in I$, then $u = \frac{f}{g}$ and $gu = f \in I$. Then $gu = \varphi(f)$. Hence $f \in \bar{I}$. It follows that $f = \sum h_i \bar{f}_i$. Then $gu = \sum \varphi(h_i)\varphi(\bar{f}_i)$. Hence $u \in (\varphi(\bar{f}_1), \dots, \varphi(\bar{f}_m))$. $\square$

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