GRASSMANNIANS

ADAUS

1. Basic Notions

Recall that projective space is the set of 1-dim vector subspace. Now it is time to consider k-dim vector subspace.

Definition 1.1. *Let F be a field and V a finite dimension F-vec.space. We call* $Gr(d,V) := \{d-dim\ vec.subspace\ of\ V\}$ $(0 \le k \le n)$*the Grassmannians of V.*

We first consider a special case where $d = 1$.And We have a map

$$(V \setminus \{0\})/F^\times \xleftrightarrow{1-1} Gr(1,V)$$
$$F^\times \cdot v \mapsto Fv$$

Then we have the projective space $\mathbb{P}(V) = Gr(1,V)$.

Considering standard chart on $\mathbb{P}(V)$: $(U_i, \varphi_i : U_i \to F^n)$, we can give the structure of smooth manifold or complex manifold when $F = \mathbb{R}$ or $\mathbb{C}$. Now we go back to the general field k considered in algebraic geometry. And let $Gr(d,n) = \{d-dim\ vec.subspace\ of\ k^n\}$.

2. Plücker Embedding

Exterior product will be an excellent tool in our discussion. We first do some recall of it without proof that you can find in books of linear algebra.

Lemma 2.1. *Let* $v_1, \cdots, v_d \in k^n$ *for some* $d \le n$*. Then*

$$v_1 \wedge \cdots \wedge v_d = 0 \Leftrightarrow v_1, \cdots, v_d\ are\ linearly\ dependent.$$

Lemma 2.2. *Let* $v_1, \cdots, v_d \in k^n$ *and* $u_1, \cdots, u_d \in k^n$ *both be linear independent. Then*

$$v_1 \wedge \cdots \wedge v_d\ and\ u_1 \wedge \cdots \wedge u_d\ are\ linearly\ dependent \Leftrightarrow \langle v_1, \cdots, v_d \rangle = \langle u_1, \cdots, u_d \rangle$$

Definition-Theorem 2.1 (Plücker Embedding)**.** *Map*

$$\psi : Gr(d,n) \hookrightarrow \mathbb{P}(\bigwedge^d k^n) \simeq \mathbb{P}^{(\binom{n}{d}-1)}$$
$$v \mapsto k^\times \cdot v_1 \wedge \cdots \wedge v_d = \bigwedge^d V \setminus \{0\}$$

is injective where $\{v_i\}$ *is a basis of* V*.*

Proof. It is well-defined since $v_1 \wedge \cdots \wedge v_d \neq 0$ by **Lemma 2.1**, and representing the same subspace by a different basis does not change the image in $\mathbb{P}(\bigwedge^d k^n)$ by **Lemma 2.2**. Moreover, ψ is injective. $\square$

For a d-dim linear subspace $L \in Gr(d,n)$ the homogeneous coordinates of $\psi(L)$ are called the **Plücker coordinates** of L. They are just all the maximal minors of the matrix whose rows are $v_1, \cdots, v_d$.

However, the Plücker embedding is not enough to discribe the property of Grassmannian. Before we show that one can embedd the Grassmannian into a projective space, we give a key lemma as the following.

Lemma 2.3. *For a fixed non zero $\Lambda \in \bigwedge^d k^n$ with $d < n$ consider the k-linear map*

$$f(\Lambda) : k^n \to \bigwedge^{d+1} k^n$$
$$v \mapsto v \wedge \Lambda$$

Then we have rank $f(\Lambda) \geq$ n-d with the equality holding iff $\Lambda = v_1 \wedge \cdots \wedge v_d$ for some $v_1, \cdots, v_d \in k^n$

Proof. Let $e_1, \cdots, e_r$ be a basis of ker$f$ where $r = n-$ rank f. It can be extended to a basis $e_1, \cdots, e_n$ of k^n. Then we can write

$$\Lambda = \sum_{1 \leq i_1 < \cdots < i_d \leq n} \Lambda^{i_1 \cdots i_d} e_{i_1} \wedge \cdots \wedge e_{i_d}$$

For $j = 1, 2, \cdots, r$, we have

$$\sum_{1 \leq i_1 < \cdots < i_d \leq n} \Lambda^{i_1 \cdots i_d} e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_d} = 0$$

Now $\Lambda^{i_1 \cdots i_d}$ can only be non zero if $\{1, \cdots, r\} \subset \{i_1, \cdots, i_d\}$, but Λ is non zero, then we have $r \leq d$.

Moreover, $r = d \Rightarrow \Lambda$ is a scalar multiple of $e_1 \wedge \cdots \wedge e_d$.

Conversely, if $\Lambda = v_1 \wedge \cdots \wedge v_d$, then $v_1, \cdots, v_d \in$ ker$f$ are necessarily linearly independent. Therefore $\dim \ker f \geq d$, and we already have the "$\leq$". $\square$

Theorem 2.4. *The image of $Gr(d,n)$ under Plücker embedding is a projective variety.*

Proof. We may assmue that $d < n$ since $Gr(n,n)$ is just a point.

Considering the map $f(\Lambda)$ in **Lemma 2.3**, we have that $\forall \Lambda \in \mathbb{P}(\bigwedge^d k^n)$, Λ in the image of $Gr(d,n)$ under Plücker embedding iff the rank$f(\Lambda) \leq n-d$. And it is equivalent that all the $n-d+1$ minors of the corresponding matrix vanish. And the entries of the matrix are polynomials of Λ. Therefore, the image of $Gr(d,n)$ under Plücker embedding is a projective variety. $\square$

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book written by professor Wen Wei Li, which is really an excellent book and I enjoy reading it all the time.