STAT0041: Stochastic Calculus

Lecture 1 - Introduction & Preliminaries

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Key concepts:

- *Probability space;*
- *Random variables;*
- *Filtrations.*

1.1 Axioms of Probability Theory

We have learned probability, which is the theory studying the deterministic laws of random phenomena. In order to use mathematical tools, we first establish mathematical model of random phenomena.

In probability theory, a *random trial* is any procedure that can be infinitely repeated and has a well-defined set of possible outcomes, known as *sample space*.

Definition 1.1 (Sample Space) *The* ***sample point*** *is the possible outcome or result of a random trial, denoted as* ω*. the* ***sample space*** *of a random trial is the set of all possible outcomes or results of that trial, denoted as* Ω*.*

For example, if the random trial is tossing a single coin twice, the sample space is the set $\{HH, HT, TH, TT\}$, where the outcome H means that the coin is heads and the outcome T means that the coin is tails. In fact, we are interested in are some "things" that occur in the random trial. Such as tossing a single coin twice, we might interested in "two outcomes are same". That is, one of the sample points $\{HH, TT\}$ occurs. These things are the set of sample points, known as *event*.

Definition 1.2 (Event) *An* ***event*** *is a set of outcomes of a random trial, i.e. a subset of the sample space. An event is said to occur if and only if one of the sample points it contains occurs.*

Sample space Ω is also an event. In each trial, a sample point from Ω is always occur, meaning Ω certainly happens. Therefore, we refer to Ω as the *certain event*. Similarly, the empty set $\emptyset$ is also considered an event. In each trial, it never occurs, and therefore, $\emptyset$ is called the *impossible event*.

We need to describe complex event by simple event, which is realized through set theory.

- We say the occurrence of event A implies the occurrence of event B, if $A \subset B$.
  $A = B \iff A \subset B \;\&\; B \subset A$;
- We say the set composed of all the sample points not included in event A is the **complementary event** of A, denote as A^c;
- We use $A \cap B$ or AB denoting events A and B both occur;
- We use $A \cup B$ denoting at least one of events A and B occurs;

- We use $A \backslash B$ denoting event A occurs but B does not;

Now we need to figure out the set of the events and all events obtained through above calculation.

Definition 1.3 (Event field) *A set $\mathscr{F}$ composed of some subsets of sample space Ω is called a* ***event field*** *if it satisfies:*

(1) $\mathscr{F} \neq \emptyset$;

(2) $A \in \mathscr{F} \Longrightarrow A^c \in \mathscr{F}$;

(3) $A_n \in \mathscr{F}, n = 1, 2, \cdots \Longrightarrow \cup_{n=1}^{\infty} A_n \in \mathscr{F}$.

Also called σ-field or σ-algebra.

Next we need to measure the possibility of an event occurring, i.e. probability measure.

Definition 1.4 (Probability measure) *A set function P is called a probability measure on a event field $\mathscr{F}$ if:*

(1) $\mathrm{P}(A) \geq 0$, $\forall A \in \mathscr{F}$;

(2) $\mathrm{P}(\emptyset) = 0,\ \mathrm{P}(\Omega) = 1$;

(3) Countable additivity: for disjoint sets $A_n \in \mathscr{F}, n = 1, 2, \cdots$

$$\mathrm{P}(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathrm{P}(A_n).$$

To sum up above, we need three things for establishing mathematical model of random phenomena:

(1) Sample space of the random trial Ω. It is a nonempty set including all possible outcomes of this trial;

(2) Event field $\mathscr{F}$. It is the set of all interesting events and events obtained from some calculation;

(3) Probability measure P. It measures the possibility of an event occurring.

Definition 1.5 (Probability space) ***Probability space*** *is a mathematical triplet $(\Omega, \mathscr{F}, \mathrm{P})$.*

1.2 Random Variables

In practice, sample space can be very different which is difficult to use mathematical tools. We consider projecting outcomes of random trial to Euclid space, i.e. a function $\xi : \Omega \to \mathbb{R}$. Furthermore, there should be connection between ξ and event field $\mathscr{F}$.

Definition 1.6 (Random variable) *For given probability space $(\Omega, \mathscr{F}, \mathrm{P})$, a random variable (r.v.) is a function $\xi : \Omega \to \mathbb{R}$ satisfies for all $A \in \mathscr{B}(\mathbb{R})$[1], $\{\omega : \xi(\omega) \in A\} \in \mathscr{F}$.*

Followings are some basic concepts.

[1] Denote set of all open sets in $\mathbb{R}$ as $\mathscr{B}(\mathbb{R})$, we say it is a Borel σ-field

Definition 1.7 (Probability distribution) *Let ξ be a r.v. on probability space $(\Omega, \mathscr{F}, \mathrm{P})$, function:*

$$\mathrm{P}_\xi : \mathscr{B}(\mathbb{R}) \to [0,1], \quad \mathrm{P}_\xi(A) := \mathrm{P} \circ \xi^{-1}(A), \ \forall A \in \mathscr{B}(\mathbb{R}).$$

*is called **probability distribution** of ξ.*

*If two r.v. ξ and η share same probability distribution, we say ξ and η are **identically distributed**.*

*$F(x) = \mathrm{P}(\omega : \xi(\omega) \leq x), x \in \mathbb{R}$ is **distribution function** of ξ.*

Definition 1.8 (Expectation) *Let ξ be a r.v. on probability space $(\Omega, \mathscr{F}, \mathrm{P})$, if $\int_\Omega |\xi(\omega)| \mathrm{dP}(\omega) < \infty$, we say mathematical expectation of ξ is exist,*

$$\mathbb{E}[\xi] = \int_\Omega \xi(\omega) \mathrm{dP}(\omega)$$

*is called the **mathematical expectation** of ξ.*

Proposition 1.9 *Expectation has following properties*

(1) $\mathbb{E}[\mathbf{1}_A] = \mathrm{P}(A)$*;*

(2) *If $\xi \geq \eta$, then $\mathbb{E}[\xi] \geq \mathbb{E}[\eta]$;*

(3) $\mathbb{E}[a\xi + b\eta] = a\mathbb{E}[\xi] + b\mathbb{E}[\eta]$*;*

(4) $|\mathbb{E}[\xi]| \geq \mathbb{E}[|\xi|]$*;*

(5) (Cauchy-Schwarz inequality) *Assume $\mathbb{E}[\xi^2] < \infty, \mathbb{E}[\eta^2] < \infty$, then*

$$(\mathbb{E}(\xi\eta))^2 \leq \mathbb{E}\xi^2 \mathrm{E}\eta^2.$$

"=" hold if and only if exist a, b at least one not equal to 0, such that

$$\mathrm{P}(\{\omega : a\xi(\omega) + b\eta(\omega) = 0\}) = 1.$$

1.3 Convergence of Random Variables

In order to characterize the limit of a sequence of random variables (X_n), we define the following kinds of convergence.

Definition 1.10 *Let $(\Omega, \mathscr{F}, \mathrm{P})$ be a probability space, (X_n) be a sequence of random variables on $(\Omega, \mathscr{F}, \mathrm{P})$.*

(1) Convergence in probability: *A sequence (X_n) of random variables converges in probability towards the random variable X if for all $\epsilon > 0$*

$$\mathrm{P}(\omega : |X_n(\omega) - X(\omega)| > \epsilon) \to 0, n \to \infty.$$

Denote as $X_n \xrightarrow{\mathrm{P}} X$.

(2) Almost sure convergence: *To say that the sequence (X_n) converges almost surely (almost everywhere or with probability 1) towards X means that*

$$\mathrm{P}(\omega : \lim_{n \to \infty} X_n(\omega) = X(\omega)) = 1.$$

Denote as $X_n \to X$, *a.s.*.

(3) Convergence in distribution: *A sequence* (X_n) *of random variables, with probability distribution* P_{X_n}, *is said to converge in distribution (or converge weakly, or converge in law) to a random variable* X *with probability distribution* P_X *if for all bounded continuous function* f

$$\int f \mathrm{dP}_{X_n} \to \int f \mathrm{dP}_X,$$

Denote as $X_n \xrightarrow{d} X$.

(4) Convergence in L^p: *We say that the sequence* (X_n) *converges in the* L^p*-norm towards the random variable* X, *if*

$$\|X_n - X\|_p = (\mathbb{E}|X_n - X|^p)^{1/p} \to 0.$$

Denote as $X_n \xrightarrow{L^p} X$.

Proposition 1.11 *Four kinds of convergence satisfy following descriptions*

(1) *If* $X_n \to X$, *a.s., then* $X_n \xrightarrow{\mathrm{P}} X$;

(2) *If* $X_n \xrightarrow{\mathrm{P}} X$, *then there exists subsequence* n_k, $X_{n_k} \to X$, *a.s.;*

(3) *If* $X_n \xrightarrow{L^p} X$, *then* $X_n \xrightarrow{\mathrm{P}} X$;

1.4 Filtration

Since randomness, people can not predict "future path" clearly. However, one always hopes using "previous and present knowledge". In probability, we introduce *filtration*.

Definition 1.12 (Filtration) *A filtration on* $(\Omega, \mathscr{F}, \mathrm{P})$ *is a collection* $(\mathscr{F}_t)$ *indexed by* $\mathbb{R}^+ \cup \{0, \infty\} / \mathbb{Z}^+ \cup \{0, \infty\}$ *of sub-*σ*-fields of* $\mathscr{F}$, *such that* $\mathscr{F}_s \subset \mathscr{F}_t$ *for every* $s \le t \le \infty$. *We also say that* $(\Omega, \mathscr{F}, (\mathscr{F}_t), \mathrm{P})$ *is a filtered probability space.*

Definition 1.13 (Adapted process) *A random process* (X_t) *with values in a measurable space* $(E, \mathscr{E})$ *is called adapted if, for every* $t \ge 0$, X_t *is* $\mathscr{F}_t$*-measurable.*

Example 1.14 (Natural filtration) *if* $X = (X_t, t \ge 0)$ *is any random process indexed by* $\mathbb{R}^+ \cup \{0, \infty\} / \mathbb{Z}^+ \cup \{0, \infty\}$, *the natural filtration of* X *is defined by*

$$\mathscr{F}_t^X = \sigma(X_s, 0 \le s \le t), \quad \mathscr{F}_\infty^X = \sigma(X_s, s \ge 0). \tag{1.1}$$

Natural filtration is the minimum filtration that X *is adapted.*

Definition 1.15 (Usual condition) *We say the filtered probability space* $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t \ge 0}, \mathrm{P})$ *satisfies usual condition, if*

(1) Right-continuity: $\forall t \ge 0, \mathscr{F}_t = \mathscr{F}_{t+} := \bigcap_{\delta \downarrow 0} \mathscr{F}_{t+\delta}$.

(2) Completeness: $\mathscr{F}_0$ *contains all* P*-null set.*